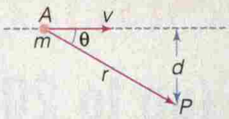
Angular M

Angular momentum for a body rotating about a fixed axis is defined in following two ways:

1. Angular Momentum of Point Objects

The magnitude of angular momentum is evaluated by moment of linear momentum. It is always evaluated with respect to a given point (*or axis of rotation*). So its value can be evaluated by multiplying the linear momentum with the shortest distance of the point from the line of momentum. For example, consider the situation as shown in the figure. A particle A of mass m is moving with a linear speed v along a straight line. If we find the angular momentum of this particle with respect to a point P shown in figure, it is given as



$$L = mv \times d$$

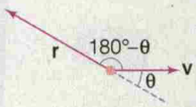
If position vector of A from P is $\mathbf{r}$, then

$$L = mv \times r \sin\theta$$

In vector form, we can write as

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) \qquad \ldots\text{(i)}$$

It states that the direction of angular momentum is perpendicular to the plane containing vectors $\mathbf{r}$ and $\mathbf{v}$, given by the right hand thumb rule.

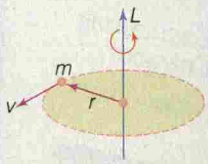


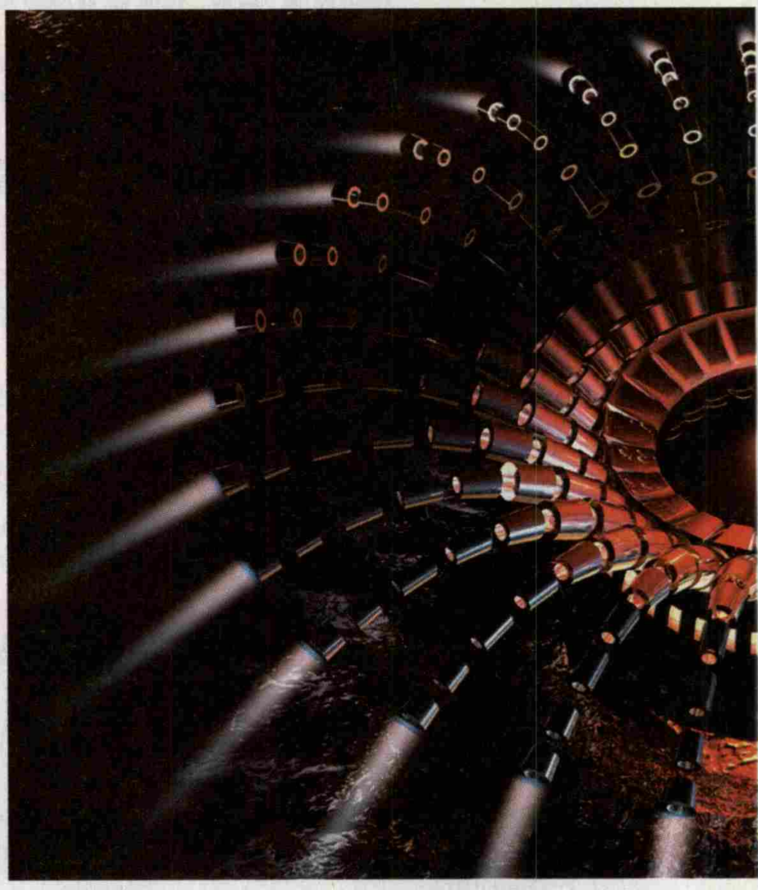
Note that in the case discussed above angle between $\mathbf{r}$ and $\mathbf{v}$ is $180° - \theta$.

$$\therefore \quad L = mrv \sin(180° - \theta) = mrv \sin\theta$$

$$[\because \sin(180° - \theta) = \sin\theta]$$

If a particle is revolving in a circular path as shown in figure, here the shortest distance of linear momentum from the centre is its radius, thus the angular momentum of the particle about the centre of circle is mvr. Here the direction of angular momentum, given by right hand thumb rule, is in upward direction along the axis of circular motion.





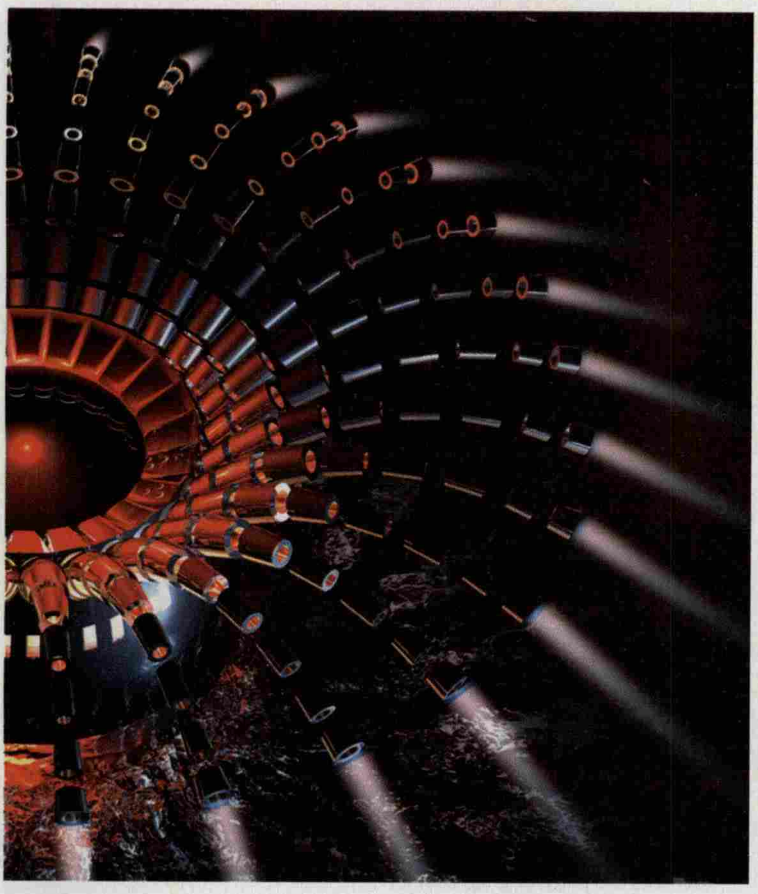
2. Angular Momentum of a Rigid Body in Rotation

Consider an extended body in rotational motion with an angular velocity ω. This body does not have any linear momentum, but different particles of the body have linear momenta. The particles which are far away from the axis of

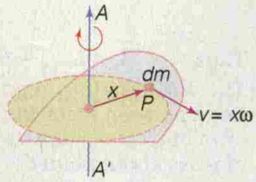
"Improvizing things is always Changing. A lot of momentum."
—Ikue Mori

Momentum

by PRS Murthy



rotation have larger speed and the particles near the axis have the smaller speed. Let us consider a small element of mass dm at a distance x from the axis of rotation shown in the figure. During rotation of the body, this dm is in circular motion of radius x and it will have a linear speed $x\omega$, tangential to that circle.



The angular momentum of this dm is

$$dL = (dm)\,(x\omega)\,(x)$$

The angular momentum of the whole body can be given by

$$L = \int dL = \int dm\, x^2\omega$$

Angular velocity ω is same for all particles of a body in rotation. Thus,

$$L = \omega\int x^2\, dm$$

$$\Rightarrow \qquad L = I\omega \qquad \ldots\text{(ii)}$$

Here $I = \int x^2\, dm$ is the moment of inertia in the extended body.

The vector relation $\mathbf{L} = I\omega$ is not correct in the above case because $\mathbf{L}$ and ω don't point in the same direction, but we can write $L_{AA'} = I\omega$. If however the body is symmetric about the axis of rotation $\mathbf{L}$ and ω are parallel and we can write ($\mathbf{L} = I\omega$) in vector form as $\mathbf{L} = I\omega$. By symmetric we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation. Thus, remember that $\mathbf{L} = I\omega$ applies only to bodies that have symmetry about the (*fixed*) rotational axis. However, the relation $L_{AA'} = I\omega$ holds for any rigid body, symmetrical or not, that is rotating about a fixed axis.

Above discussion reveals that the angular momentum of moving bodies can be obtained in two ways. Eq. (i) gives the angular momentum of point objects moving in translational or circular motion and Eq. (ii) gives the angular momentum of extended bodies having rotational motion. If a body has both translational and rotational motions, the angular momentum of the body is evaluated as follows:

Angular Momentum of a Rigid Body in Combined Rotation and Translation

Let O be a fixed point in an inertial frame of reference. Angular momentum of the body (*see figure*) about O is

$$\mathbf{L} = \sum_i m_i\,(\mathbf{r}_i \times \mathbf{v}_i)$$

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$= \Sigma m_i (\mathbf{r}_{i,\,CM} + \mathbf{r}_0) \times (\mathbf{v}_{i,\,CM} + \mathbf{v}_0)$

Here, $\mathbf{r}_0$ is the position vector of centre of mass and $\mathbf{v}_0$ is its velocity.

Thus, $\mathbf{L} = \sum_i m_i (\mathbf{r}_{i,\,CM} \times \mathbf{v}_{i,\,CM}) + \{\sum_i m_i \mathbf{r}_{i,\,CM}\} \times \mathbf{v}_0 + \mathbf{r}_0$

$$\times \{\sum_i m_i \mathbf{v}_{i,\,CM}\} + \{\sum_i m_i\} \mathbf{r}_0 \times \mathbf{v}_0$$

Now, $\sum_i m_i \mathbf{r}_{i,\,CM} = M\,\mathbf{R}_{CM,\,CM} = 0,$

and similarly, $\sum_i m_i \mathbf{v}_{i,\,CM} = M\mathbf{v}_{CM,\,CM} = 0.$

Therefore, $\mathbf{L} = \sum_i m_i (\mathbf{r}_{i,\,CM} \times \mathbf{v}_{i,\,CM})$

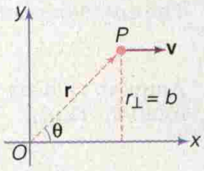
$$+ M\mathbf{r}_0 \times \mathbf{v}_0 = \mathbf{L}_{CM} + M(\mathbf{r}_0 \times \mathbf{v}_0)$$

Thus, $\mathbf{L} = \mathbf{L}_{CM} + M(\mathbf{r}_0 \times \mathbf{v}_0)$

The first term $\mathbf{L}_{CM}$ represents the angular momentum of the body as seen from the centre of mass frame. The second term $M(\mathbf{r}_0 \times \mathbf{v}_0)$ equals to the angular momentum of centre of mass about point O.

Example 1 *A particle of mass m is moving along the line y = b, z = 0 with constant speed v. State whether the angular momentum of particle about origin is increasing, decreasing or constant.*

Solution. $|\mathbf{L}| = mvr \sin\theta = mvr_\perp = mvb$



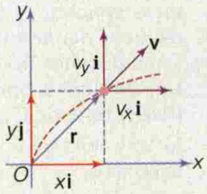
$\therefore$ $|\mathbf{L}|$ is constant as m, v and b all are constants.

Direction of $\mathbf{r} \times \mathbf{v}$ also remains the same. Therefore, angular momentum of the particle about origin remains constant with due course of time.

In this problem $|\mathbf{r}|$ is increasing, θ is decreasing but $r \sin\theta$, *i.e.*, b remains constant.

Example 2 *A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of the particle at any time t about O before the particle strikes the ground again.*

Solution. At any time t the position of the body as shown in the diagram.



Its position vector is (at this instant)

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

and velocity is (at this instant)

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$

or $\mathbf{r} = (u\cos\theta)\,t\,\mathbf{i} + \left\{(u\sin\theta)t - \frac{1}{2}gt^2\right\}\mathbf{j}$

and $\mathbf{v} = (u\cos\theta)\,\mathbf{i} + (u\sin\theta - gt)\,\mathbf{j}$

$\therefore$ $\mathbf{L} = m[\mathbf{r} \times \mathbf{v}]$

$$= m\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (u\cos\theta)t & (u\sin\theta)\,t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

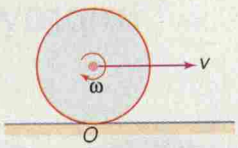
$$= m\,[\mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,\{u^2 \sin\theta\cos\theta\, t$$

$$- (u\cos\theta)\,gt^2 - (u^2\sin\theta\cos\theta\, t - (u\cos\theta)\,1/2\,gt^2)\}]$$

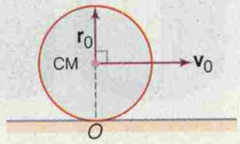
$$= m\left[-\frac{1}{2}(u\cos\theta)\,gt^2\right]\mathbf{k}$$

$$= -\frac{1}{2}m\,(u\cos\theta)\,gt^2\,\mathbf{k}$$

Example 3 *A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = \frac{v}{R}$ in clockwise direction as shown in the figure. Find the magnitude of the total angular momentum of the disc about the bottommost point O of the disc.*



Solution. $\mathbf{L} = \mathbf{L}_{CM} + m\,(\mathbf{r}_0 \times \mathbf{v}_0)$ ...(i)



Here, $\mathbf{L}_{CM} = I\omega = \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)$

$$= \frac{1}{2}mvR$$

(perpendicular to the paper inwards).

and $m\,(\mathbf{r}_0 \times \mathbf{v}_0) = mRv$ (perpendicular to the paper inwards).

Since, both the terms of right hand side of Eq. (i) are in the same direction

$\therefore$ $$|\mathbf{L}| = \frac{1}{2}mvR + mvR = \frac{3}{2}mvR$$

Example 4 *In the above problem suppose the disc starts rotating anticlockwise with the same angular velocity $\omega = \frac{v}{R}$, then what will be the angular momentum of the disc about the bottommost point in this new situation?*

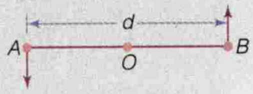
Solution. In this case the angular momentum of the rigid body about centre of mass $\mathbf{L}_{CM}$ will be perpendicular to the paper outwards. If we take this direction as negative and $m\,(\mathbf{r}_0 \times \mathbf{v}_0) = mRv$ (perpendicular to paper inwards)

$$\mathbf{L}_{CM} = -\frac{1}{2}mvR$$

$\therefore$ $$|\mathbf{L}| = -\frac{1}{2}mvR + mvR = \frac{1}{2}mvR$$

Example 5 *Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.*

Solution. Consider the situation as shown in the figure. The velocity of ball A with respect to centre O is $v = \omega\frac{d}{2}$.



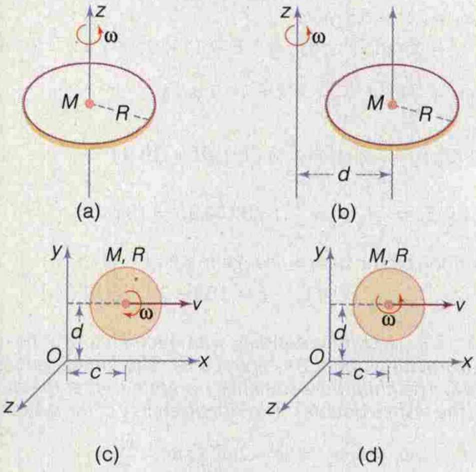
The angular momentum of ball A wrt axis is $L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}m\omega d^2$. The same is the angular momentum L_2 of the second ball. The angular momentum of the system is equal to the sum of these two angular momenta *i.e.*, $L = \frac{1}{2}m\omega d^2$.

Example 6 *Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end. The rod is fixed at the other end and is rotated in a plane at angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.*

Solution. The situation is as shown in figure. The velocity of A wrt O is $v_A = \omega\left(\frac{d}{2}\right)$ and that of B wrt O is $v_B = \omega d$.

Hence, velocity of B wrt A is $v_B - v_A = \omega d - \frac{\omega d}{2} = \frac{\omega d}{2}$. The angular momentum of B wrt A is, therefore $L = mvr = m\left(\frac{\omega d}{2}\right)\frac{d}{2} = \frac{1}{4}m\omega d^2$ along the direction perpendicular to the plane of rotation.

Example 7 *Find the angular momentum of a disc about the axis {z-axis in parts (a) and (b) and about origin in parts (c) and (d)} shown in figures in the following situations.*



Solution. (a) $\mathbf{L} = I\omega = \left(\frac{MR^2}{2}\right)\cdot\omega\mathbf{k}$ kg-m^{2}s^{-1}

(b) $\mathbf{L} = I\omega = (I_{CM} + Md^2)\,\omega\mathbf{k} = \left(\frac{MR^2}{2} + Md^2\right)\omega\mathbf{k}$

$$= M\left(\frac{R^2}{2} + d^2\right)\omega\mathbf{k} \text{ kg-m}^2\text{ s}^{-1}$$

(c) $\mathbf{L} = \mathbf{L}_{CM} + M(\mathbf{r}_0 \times \mathbf{v}_0) = \frac{MR^2}{2}\omega(-\mathbf{k})$

$$+ M\{(d\mathbf{j} + c\mathbf{i}) \times v\mathbf{i}\} = -\frac{MR^2}{2}\omega\mathbf{k} - M\,vd\,\mathbf{k}$$

$$= \left(\frac{MR^2}{2}\omega + Mvd\right)(-\mathbf{k}) \text{ kg-m}^2\text{ s}^{-1}$$

(d) $\mathbf{L} = \mathbf{L}_{CM} + M(\mathbf{r}_0 + \mathbf{v}_0) = \frac{MR^2}{2}\omega(\mathbf{k}) + M\{(d\mathbf{j} + c\mathbf{i}) \times v\mathbf{i}\}$

$$= \left(Mvd - \frac{MR^2}{2}\omega\right)(-\mathbf{k}) \text{ kg-m}^2\text{ s}^{-1}$$

Conservation of Angular Momentum

As we have seen earlier, the angular momentum of a particle about some reference point O is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \ldots\text{(i)}$$

Here, $\mathbf{p}$ is the linear momentum of the particle and $\mathbf{r}$ is its position vector with respect to the reference point O. Differentiating Eq. (i) wrt time, we get

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad \ldots\text{(ii)}$$

Here, $\frac{d\mathbf{p}}{dt} = \mathbf{F}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ (*velocity of particle*)

Hence, Eq. (ii) can be written as

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} + \mathbf{v} \times \mathbf{p}$$

Now, $\mathbf{v} \times \mathbf{p} = 0$, because $\mathbf{v}$ and $\mathbf{p}$ are parallel to each other and the cross product of two parallel vectors is zero. Thus,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \tau$$

or

$$\tau = \frac{d\mathbf{L}}{dt} \qquad \ldots\text{(iii)}$$

which states that the time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torque acting on it. This result is rotational analog of the equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, which states that the rate of change of the linear momentum of a particle is equal to the force acting on it.

The same equation can be generalized for a system of particles, as $\tau_{ext} = \frac{d\mathbf{L}}{dt}$, according to which "the time rate of change of the total angular momentum of a system of particles about some reference point in an inertial frame of reference is equal to the sum of all external torques (*of course the vector sum*) acting on, the system about the same reference point."

Now, suppose that $\tau_{ext} = 0$ then, $\frac{d\mathbf{L}}{dt} = 0 \Rightarrow \mathbf{L}$ is constant.

Thus, "when the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant." This is the principle of conservation of angular momentum.

For a rigid body rotating about an axis (*the z-axis, say*) that is fixed in an inertial reference frame, we have $L_z = I\omega$. It is possible for the moment of inertia I of rotating body to change by rearrangement of its parts. If no external torque acts, then L_z must remain constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as $I\omega$ = constant.

or

$$I_1\omega_1 = I_2\omega_2 \qquad \ldots\text{(iv)}$$

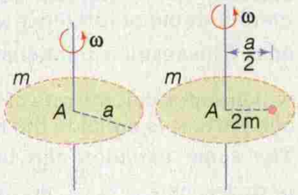
1. In case of motion for a single particle acted on by a torque $\tau = \frac{d\mathbf{L}}{dt}$ holds only if τ and $\mathbf{L}$ are measured with respect to any point O fixed in an inertial frame.
2. In case of motion of a system of particles acted on by an external torque, $\tau_{ext} = \frac{d\mathbf{L}}{dt}$ holds only if τ_{ext} and $\mathbf{L}$ are measured with respect to (a) any point O fixed in an inertial frame and (b) the centre of mass of the system.

Example 8 *A wheel is rotating at an angular speed ω about its axis which is kept vertical. An identical wheel initially at rest is gently dropped into the same axle and the two wheels start rotating with a common angular speed. Find this common angular speed.*

Solution. Let the moment of inertia of the wheel about the axis be I. Initially the first wheel is rotating at the angular speed ω about the axle and the second wheel is at rest. Take both the wheels together as the system. The total angular momentum of the system before the coupling is $I\omega + 0 = I\omega$. When the second wheel is dropped into the axle, the two wheels slip on each other and exert forces of friction. The forces of friction have torques about the axis of rotation but these are torques of internal forces. No external torque is applied on the two wheel system and hence the angular momentum of the system remains unchanged. If the common angular speed is ω', the total angular momentum of the two wheel system is $2I\omega'$ after the coupling. This $I\omega = 2I\omega' \Rightarrow \omega' = \frac{\omega}{2}$.

Example 9 *A uniform circular disc of mass m and radius a is rotating with constant angular velocity ω in a horizontal plane about a vertical axis through its centre A · A particle P of mass 2m is placed gently on the disc at a point distant $\frac{1}{2}$ a from A. If the particle does not slip on the disc, find the new angular velocity of the rotating system.*

Solution. Since there is no external torque this process we can apply the principle of conservation of angular momentum.



$$I\omega = I'\omega'$$

$$\Rightarrow \frac{ma^2}{2}\cdot\omega = \left(\frac{ma^2}{2} + 2m\cdot\frac{a^2}{4}\right)\omega'$$

$$\Rightarrow \omega' = \frac{\omega}{2}$$

Example 10 *The polar ice caps contain about 2.3×10^{19} kg of ice. This contributes essentially nothing to the moment of inertia of the earth because it is located at the poles, close to the axis of rotation. Estimate the change in length of the day to be expected if the polar ice caps melt, distributing the water uniformly over the surface of the earth. Take mass of the earth = 5.94×10^{24} kg, and the radius of the earth = 6.37×10^{6} m.*

Solution. If we take ice caps + earth as the system, the external torque on the system, in the process of ice caps melting and water distributing uniformly over the surface of the earth, is zero.

∴ Angular momentum is conserved *i.e.*,

$$I_1\omega_1 = I_2\omega_2 \quad \text{...(i)}$$

$$I_1\cdot\frac{2\pi}{T_1} = I_2\cdot\frac{2\pi}{T_2} \Rightarrow \frac{I_1}{I_2} = \frac{T_1}{T_2}$$

Let I_1 be the moment of inertia before the ice melts and I_2 be the moment of inertia after the ice melts.

$$T_1 = 24 \text{ h and } T_2 = ?$$

Let M be the mass of earth and m be the mass of ice. Thus $I_1 = \frac{2}{5}MR^2$ and $I_2 = \frac{2}{5}MR^2 + \frac{2}{3}MR^2$ (∵ *Water of total mass m is distributed on the surface of the earth. We can assume it as a hollow sphere of radius R and mass m*).

From Eq. (i) $I_1\omega_1 = I_2\,\omega_2$,

or $$I_1\cdot\frac{2\pi}{T_1} = (I_1 + \Delta I)\left(\frac{2\pi}{T_2}\right)$$

or $$\frac{T_2}{T_1} = \frac{I_1 + \Delta I}{I_1} = 1 + \frac{\Delta I}{I_1}$$

∴ $$\Delta T = T_2 - T_1 = T_1\left(\frac{\Delta I}{I_1}\right)$$

Here, $I_1 = \frac{2}{5}MR^2$ and $\Delta I = \frac{2}{3}mR^2$.

Substituting the values

$$\Delta T = \frac{(24\times3600)\left(\frac{2}{3}\times2.3\times10^{19}R^2\right)}{\frac{2}{5}\times5.94\times10^{24}R^2} \simeq 0.6\text{s}$$

∴ Day time increases by about 0.6s.

Example 11 *A man stands at the centre of a circular platform holding his arms extended horizontally with 4 kg block in each hand. He is set rotating about a vertical axis at 0.5 rev s^{-1}. The moment of inertia of the man plus platform is 1.6 kg-m^2, assumed constant. The blocks are 90 cm from axis of rotation. He now pulls the blocks in towards his body until they are 15 cm from the axis of rotation. Find (a) his new angular velocity, (b) The initial and final kinetic energy of the man and platform, and (c) How much work must the man do to pull in the blocks?*

Solution. Since there is no external torque in the entire process, $L_i = L_f$.

(a) $I_1\omega_1 = I_2\omega_2$

Here, $I_1 = 1.6 + (4\times0.9)^2\times2 = 8.08$ kg-m^2 and

$$\omega_1 = 2\pi\times0.5 \text{ rad/s}$$
$$= \pi \text{ rad s}^{-1} \text{ and } I_2 = 1.6 + (4\times0.15^2)\times2 = 1.78 \text{ kg m}^2$$

$$\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{8.08}{1.78}\times\pi = 14.3 \text{ rad s}^{-1}$$

(b) Initial KE, $E_i = \frac{1}{2}I_1\,\omega_1^2 = \frac{1}{2}(8.08)(\pi)^2 = 39.9$ J

Final KE, $E_f = \frac{1}{2}I_2\omega_2^2 = \frac{1}{2}(1.78)(14.3)^2 = 182$ J

(c) Work done by the man = change in KE of system

$$\Rightarrow \quad W = E_f - E_i = 182 - 39.9 = 142.1 \text{ J}$$

Example 12 *A child is standing with folded hands at the centre of a platform rotating about its central axis. The kinetic energy of the system is K. The child now stretches his arms so that the moment of inertia of the system doubles. The kinetic energy of the system now, is*

Solution. $I_1\omega_1 = I_2\omega_2 \Rightarrow I\omega = 2I\omega' \Rightarrow \omega' = \frac{\omega}{2}$

$$K = \frac{1}{2}I\omega^2,\ K' = \frac{1}{2}(2I)\omega'^2 = \frac{1}{2}(2I)\left(\frac{\omega}{2}\right)^2 = \frac{1}{2}I\omega^2\cdot\frac{1}{2} = \frac{K}{2}$$

Example 13 *A horizontally oriented uniform disc of mass M and radius R rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass m. A light thread running down through the hollow axle of the disc is tied up to the body. Initially, the body was located at the edge of the disc and the whole system rotated with an angular velocity $\omega_0 > 0$. Then by means of a force F applied to the lower end of the thread, the body was slowly pulled to the rotation axis. Find :*

(a) the angular velocity of the system in its final state.

(b) the work performed by the force F.

Solution. Here, the external force F is applied on the system along the axis of rotation. So torque due to external force (τ_{ext}) is zero and therefore angular momentum is conserved.

(a) $I_1 = \frac{MR^2}{2} + mR^2,\ \omega_1 = \omega_0,\ I_2 = \frac{MR^2}{2},\ \omega_2 = ?$

$$I_1\omega_1 = I_2\omega_2 \Rightarrow \omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{(MR^2/2 + mR^2)}{MR^2/2}\omega_0 = \left(1 + \frac{2m}{M}\right)\omega_0$$

(b) The work performed by force F = change in KE of the system

$$\therefore \quad W = E_f - E_i = \frac{1}{2}I_2\,\omega_2^2 - \frac{1}{2}I_1\,\omega_1^2$$

$$= \frac{1}{2}\left(\frac{MR^2}{2}\right)\left(1 + \frac{2m}{M}\right)^2\omega_0^2 - \frac{1}{2}\left(\frac{MR^2}{2} + mR^2\right)\omega_0^2$$

$$= \frac{1}{2}m\omega_0^2R^2\left(1 + \frac{2m}{M}\right)$$

Example 14 *A boy is standing on a platform which is free to rotate about its axis. The boy holds an open umbrella in his hand. The axis of the umbrella coincides with that of the platform. The moment of inertia of "the platform + the boy system" is 3×10^{-3} kg m^2 and that of the umbrella is 2×10^{-3} kg m^2. The boy starts spinning the umbrella about the axis at an angular speed of 2 rev s^{-1} with respect to himself. Find the angular velocity imparted to the platform. The values in this problem are not practical.*

Solution. Let the boy rotated the umbrella in clockwise sense. Initial angular momentum of the system is zero. To make the final total angular momentum zero the platform should rotate in anticlockwise sense. We take here anticlockwise +ve and clockwise –ve.

Let ω be the angular velocity imparted to the system (*platform + umbrella*).

$$\omega_{up} = \omega_u - \omega_p \Rightarrow \omega_u = \omega_{up} + \omega_p = -2 + \omega$$

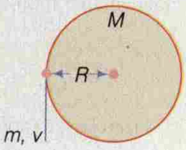
$$L_i = 0;\ L_f = (3\times10^{-3})\omega + (2\times10^{-3})(\omega - 2)$$

$$= 5\times10^{-3}\omega - 4\times10^{-3}$$

$$\therefore\quad 5\times10^{-3}\omega - 4\times10^{-3} = 0 \Rightarrow \omega = 0.8 \text{ rev s}^{-1}$$

Example 15 *A kid of mass M stands at the edge of a platform of radius R which can be freely rotated about its axis. The moment of inertia of the platform is I. The system is at rest when a friend throws a ball of mass m and the kid catches it. If the velocity of the ball is horizontally along the tangent to the edge of the platform when it was caught by the kid, find the angular speed of the platform after the event.*

Solution. Take the kid + ball + platform as the system. In the process of boy catching the ball there is no external torque on the system about the axis of rotation. So, angular momentum is conserved.



$$L_i = L_{ball} + L_{kid + platform}$$

$$= mvR + 0 = mvR$$

$$L_f = [I_{kid} + I_{ball} + I_{platform}]\,\omega$$

$$= (mR^2 + MR^2 + I)\omega = [I + R^2(m + M)]\,\omega$$

$$L_i = L_f \Rightarrow mvR = [I + R^2(m + M)]\,\omega$$

$$\Rightarrow\quad \omega = \frac{mvR}{I + R^2(m + M)}$$

Example 16 *Suppose the platform of the previous problem is brought to rest with the ball in the hand of the kid standing on the rim. The kid now throws the ball horizontally to his friend in a direction tangential to the rim with a speed v as seen by his friend. Find the angular velocity with which the platform will start rotating.*

Solution. Let ω be the angular velocity (*boy + platform*) just after he throws the ball. Just before throwing the ball, $L_i = 0$.

Just after throwing,

$$L_f = mvR + (MR^2 + I)\omega$$

$$L_f = L_i \Rightarrow mvR + (MR^2 + I)\omega = 0 \Rightarrow \omega = -\frac{mvR}{MR^2 + I}$$

Negative sign signifies that the sense of rotation is opposite to that of velocity of the ball.

Example 17 *Suppose the platform with the kid in the previous problem is rotating in anticlockwise direction at an angular speed ω. The kid starts walking along the rim with a speed v relative to the platform also in the anticlockwise direction. Find the new angular speed of the platform.*

Solution. Let ω' be the new angular speed of the platform.

$$v_{bp} = v_b - v_p$$

$$\Rightarrow\quad v_b = v_{bp} + v_p = v + R\omega' \qquad (= v',\ say)$$

$$L_i = (MR^2 + I)\omega$$

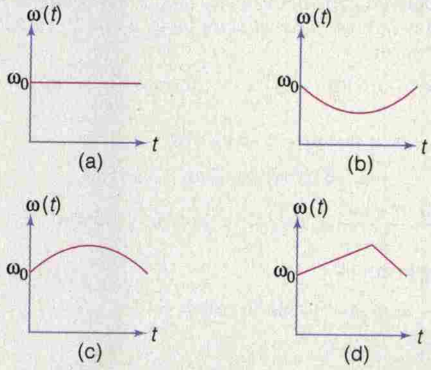
$$L_f = L_{boy} + L_{platform} = Mv'R + I\omega'$$

$$\text{or}\quad L_f = M(v + R\omega')R + I\omega' = MvR + (MR^2 + I)\omega'$$

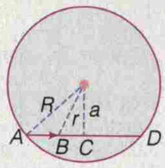
$$L_f = L_i \Rightarrow MvR + (I + MR^2)\omega' = (I + MR^2)\omega$$

$$\Rightarrow\quad \omega' = \omega - \frac{MvR}{I + MR^2}$$

Example 18 *A circular platform is free to rotate in a horizontal plane about a vertical axis passing through its centre. A tortoise is sitting at the edge of the platform. Now the platform is given an angular velocity ω. When the tortoise moves along a chord of the platform with a constant velocity (with respect to the platform), the angular velocity of the platform ω (t) will vary with time t as*



Solution. Since, there is no external torque, angular momentum will remain conserved. The moment of inertia will first decrease till the tortoise moves from A to C and then increase as it moves from C to D. Therefore, ω will initially increase and then decrease.



Let R be the radius of platform, m be the mass of tortoise and M be the mass of platform.

Moment of inertia when the tortoise is at A be $I_1 = mR^2 + \frac{MR^2}{2}$, and

Moment of inertia when the tortoise is at B be $I_2 = mr^2 + \frac{MR^2}{2}$.

Here, $\quad r^2 = a^2 + [\sqrt{R^2 - a^2} - vt]^2$

$$L_i = L_f \Rightarrow I_1\omega_0 = I_2\,\omega(t)$$

Substituting the values we can see that variation of ω (t) is non-linear.

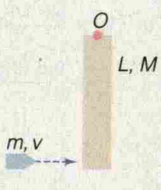
$\therefore$ Option (c) is the correct answer.

1. Suppose a rod is lying on a smooth horizontal surface, when a particle strikes it at some point.

   Then angular momentum of rod + particle remains conserved about any point fixed to an inertial frame of reference before and after collision, as $\tau_{ext} = 0$, about any point.

2. Suppose a rod is suspended from a support at O and a particle strikes the rod at some point, then the angular momentum of rod + particle system remains conserved only about point of suspension or point 'O'. Because in this case τ_{ext} on the system is zero only about O, because if we take particle + rod as the system. The support will be external to the system. To make the torque of force by support we should calculate torque about O.

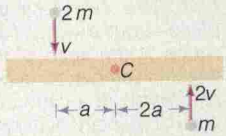
Example 19 *A rod of length L and mass M is hinged at point O. A small bullet of mass m hits the rod as shown in the figure. The bullet gets embedded in the rod. Find the angular velocity of the system just after impact.*



Solution. Angular momentum of the system about point O will remain conserved.

$$L_i = L_f \Rightarrow mvL = I\omega \Rightarrow mvL = \left(mL^2 + \frac{ML^2}{3}\right)\omega \text{ or } \omega = \frac{3mv}{L(3m + M)}.$$

Example 20 *A uniform bar of length 6a and mass 8 m lies on a smooth horizontal table. Two point masses m and 2m moving in the same horizontal plane with speed 2v and v respectively, strike the bar (as shown in the figure) and stick to the bar after collision. Denoting angular velocity (about the centre of mass), total energy and centre of mass velocity by ω, E and V_c respectively, we have after collision*



(a) $v_C = 0$ (b) $\omega = \frac{3v}{5a}$ (c) $\omega = \frac{v}{5a}$ (d) $E = \frac{3}{5}mv^2$

Solution. $p_i = 0 \therefore p_f = 0 \Rightarrow v_C = 0$

$$L_i = L_f \Rightarrow (2mv)a + (2mv)(2a) = I\omega \quad ...(i)$$

Here, $I = \frac{(8\ m)(6a)^2}{12} + m(2a)^2 + 2m(a)^2 = 30\ ma^2$

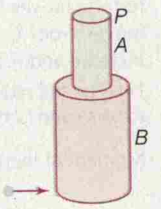
Substituting in Eq. (i)

$$6\ mva = 30\ ma^2\omega \Rightarrow \omega = \frac{v}{5a}$$

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2}(30\ ma^2)\left(\frac{v}{5\ a}\right)^2 = \frac{3}{5}mv^2$$

∴ Option (a), (c) and (d) are correct.

Example 21 *Two uniform rods A and B of length 0.6 m each and of masses 0.01 kg and 0.02 kg respectively are rigidly joined end to end. The combination is pivoted at the lighter end P, shown in figure, such that it can freely rotate about point P in a vertical plane. A small object of mass 0.05 kg, moving horizontally, hits the lower end of the combination and sticks to it. What should be the velocity of the object, so that the system could just be raised to the horizontal position?*



Solution. System is free to rotate but not free to translate. During collision, net torque on the system (*rod A + rod B + mass m*) about point *P* is zero.

∴ Angular momentum of the System before collision

= Angular momentum of system just after collision (*About P*)

Let ω be the angular velocity of system just after collision, then

$$L_i = L_f \Rightarrow mv(2l) = I\omega \quad ...(i)$$

Here, I = moment of inertia of system about *P*

$$= m(2l)^2 + m_A\frac{l^2}{3} + m_B\left[\frac{l^2}{12} + \left(\frac{l}{2} + l\right)^2\right]$$

Given, $l = 0.6$ m, $m = 0.05$ kg, $m_A = 0.01$ kg and $m_B = 0.02$ kg

Substituting the values, we get $I = 0.09$ kg-m^2.

∴ From Eq. (i) $\omega = \frac{2mvl}{I} = \frac{2(0.05)(v)(0.06)}{0.09} \Rightarrow \omega = 0.67\ v$

Now, after collision mechanical energy will be conserved.

∴ Decrease in rotational KE = Increase in gravitational PE

or $\frac{1}{2}I\omega^2 = mg(2l) + m_A g\left(\frac{l}{2}\right) + m_B g\left(l + \frac{l}{2}\right)$

or $\omega^2 = \frac{gl(4m + m_A + 3m_B)}{I}$

$$= \frac{9.8\ (0.6)\ (4\times 0.05 + 0.01 + 3\times 0.02)}{0.09}$$

$$\omega^2 = 17.64 \Rightarrow \omega = \sqrt{17.64} = 4.2\ \text{rad s}^{-1}$$

$$\therefore \quad v = \frac{\omega}{0.67} = \frac{4.2}{0.67} = 6.3\ \text{ms}^{-1}$$

Example 22 *A uniform rod of length L lies on a smooth horizontal table. A particle moving on the table strikes the rod perpendicularly at an end and then stops. Find the distance travelled by the centre of the rod by the time it turns through a right angle. Show that if the mass of the rod is four times that of the particle, the collision is elastic.*

Solution. (a) Let *M* be the mass of the rod and *m* be the mass of the particle. Let *v* be the velocity with which the particle strikes the rod.

Let *v′* be the velocity of centre of mass just after collision and ω be the angular velocity of the rod just after collision.

If we take the rod + particle as the system the force exerted by the particle on the rod will be the internal force and torque due to that force about any axis will be internal torque. Since no external force is acting on the system linear momentum is conserved and no external torque is acting on the system. About any axis angular momentum is conserved.

$$p_i = mv;\ p_f = Mv';\ mv = Mv' \quad ...(i)$$

Initial angular momentum of the system about CM $= mv\left(\frac{L}{2}\right)$.

Final angular momentum of the system about CM $= I\omega = \left(\frac{ML^2}{12}\right)\omega$.

$$mv\left(\frac{L}{2}\right) = M\frac{L^2}{12}\omega \quad ...(ii)$$

From Eqs. (i) and (ii), we have

$$Mv'\left(\frac{L}{2}\right) = M\frac{L^2}{12}\omega$$

$$\Rightarrow \quad \omega = \frac{6v'}{L} \quad ...(iii)$$

Time taken by the rod to turn through an angle $\frac{\pi}{2}$ is

$$t = \frac{\theta}{\omega} = \frac{\pi}{2}\Big/\frac{6v'}{L}$$

or $t = \frac{\pi L}{12v'};\ s = v't = v'\left[\frac{\pi L}{12v'}\right] = \frac{\pi L}{12}$

(b) Velocity of approach = *v*

Velocity of separation

= Velocity of point *A* of the rod

$$= v' + \frac{L}{2}\omega \quad [\because \text{vel} = v_{CM} + R\omega]$$

Since collision is elastic, *e* = 1, and velocity of separation = velocity of approach.

$$\therefore \quad v' + \left(\frac{L}{2}\right)\omega = v \quad ...(iv)$$

From Eq. (i) $\frac{m}{M} = \frac{v'}{v}$;

From Eq. (iv)

$$\frac{v}{v'} = 1 + \frac{L\omega}{2v'}$$

From Eq. (iii) and (i)

$$\frac{M}{m} = 1 + \frac{L}{2v'}\cdot\frac{6v'}{L} = 1 + 3 = 4 \Rightarrow M = 4\ m$$

In other words, when *M* = 4 *m*, the collision is elastic.

Example 23 *Suppose the particle of the previous problem has a mass m and a speed v before the collision and it sticks to the rod after collision. The rod has a mass M.*

(a) *Find the velocity of centre of mass, C of the system constituting "the rod plus particle".*

(b) *Find the velocity of the particle with respect to C before the collision.*

(c) *Find the velocity of the rod with respect to C before the collision.*

(d) *Find the angular momentum of the particle and of the rod about centre of mass C before the collision.*

(e) *Find the moment of inertia of the system about vertical axis through the centre of mass C after the collision.*

(f) *Find the velocity of centre of mass C and the angular velocity of the system about the centre of mass after the collision.*

Solution. (a) Velocity of centre of mass of the system (rod+ particle) is

$$\mathbf{v}_{CM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

Here, $v_1 = v$ and $v_2 = 0$, $m_1 = m$ and $m_2 = M$

$\therefore \mathbf{v}_{CM} = \dfrac{mv}{M+m}$ in the same direction as that of the particle.

(b) Velocity of the particle with respect to COM is

$$= \mathbf{v}_{pc} = \mathbf{v}_p - \mathbf{v}_c = v - \frac{mv}{m+M}$$

$$\mathbf{v}_{pc} = \frac{(m+M)v - mv}{m+M} = \frac{Mv}{m+M}$$

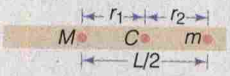
in the same direction as that of C.

(c) The velocity of rod with respect to C is

$$\mathbf{v}_{rod,\,C} = \mathbf{v}_{rod} - \mathbf{v}_C = 0 - \frac{mv}{M+m} = -\frac{mv}{M+m}$$

(the negative sign signifies that the direction of this velocity is opposite to that of the particle)

(d) First we have to locate the centre of mass C of the system.



From the definition of centre of mass

$$r_1 = \frac{m\left(\frac{L}{2}\right)}{m+M} = \frac{mL}{2(m+M)} \text{ and } r_2 = \frac{M\left(\frac{L}{2}\right)}{m+M} = \frac{ML}{2(m+M)}$$

$\therefore$ Angular momentum of the particle about the centre of mass C of the system before collision $= mv_{pc}r_2$.

[Here, v_{pc} = *velocity of particle with respect to C before collision*]

$$= m\left(\frac{Mv}{m+M}\right)\left(\frac{ML}{2(m+M)}\right) = \frac{M^2mvL}{2(m+M)^2}$$

Angular momentum of the rod about C just before collision

$$= M(v_{rod,\,C})r_1$$

[Here, $v_{rod,\,C}$ = *velocity of rod wrt C* $= \dfrac{mv}{m+M}$]

$$\therefore \quad L_{rod,\,C} = M\left(\frac{mv}{m+M}\right)\left(\frac{mL}{2(m+M)}\right) = \frac{Mm^2vL}{2(m+M)^2}$$

(e) Moment of inertia of the system about the vertical axis through centre of mass C after the collision is $I = I_{rod} + I_{particle}$

$$I_{rod} = \frac{ML^2}{12} + Mr_1^2 = \frac{ML^2}{12} + M\left[\frac{mL}{2(m+M)}\right]^2$$

$$= \frac{ML^2}{12} + \frac{m^2L^2}{4(m+M)^2}M$$

$$I_{particle} = mr_2^2 = \frac{mM^2L^2}{4(m+M)^2}$$

$$\therefore \quad I = \frac{ML^2}{12} + \frac{mML^2(m+M)}{4(m+M)^2} = \frac{M(M+4m)L^2}{12(m+M)}$$

(f) (i) Since there is no external force is acting on the system. Velocity of CM before collision

$$= \text{velocity of CM after collision} = \frac{mv}{m+M}$$

(ii) Since there is no external torque about any point on the system, angular momentum is conserved about any point.

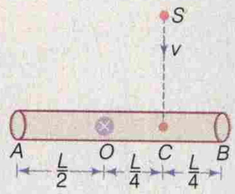
Angular momentum of the system about C of system before collision is [*From part (d) of the question*]

$$L_i = \frac{M^2mvL}{2(m+M)^2} + \frac{Mm^2vL}{2(m+M)^2} = \frac{mMvL}{2(m+M)}$$

Angular momentum of the system after collision is $L_f = I\omega$, where ω is the angular velocity of the system after collision

$$L_i = L_f \Rightarrow \frac{mMvL}{2(m+M)} = \frac{M(M+4m)L^2}{12(m+M)}\omega \Rightarrow \omega = \frac{6mv}{(M+4m)L}$$

Example 24 *A homogeneous rod AB of length $L = 1.8$ m and mass M is pivoted at the centre O in such a way that it can rotate freely in a vertical plane (see figure). The rod is initially in the horizontal position. An insect S of the same mass M falls vertically with speed v on the point C midway between O and B. Immediately after falling, the insect moves towards the end B such that the rod rotates with a constant angular velocity ω.*



(a) *Determine angular velocity in terms of v and L.*

(b) *If the insect reaches the end B when the rod has turned through an angle of 90°, determine v. ($g = 10\ ms^{-2}$)*

Solution. In this problem we will write K for the angular momentum because L has been used for length of the rod.

(a) $(K_i)_{\text{of rod + insect about }O} = (K_f)_{\text{of rod + insect about }O}$

$$0 + Mv\frac{L}{4} = \left\{\frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2\right\}\omega \Rightarrow \omega = \frac{12}{7}\frac{v}{L}$$

(b) Due to the torque of weight of insect about O, angular momentum of the system will not remain conserved (*although angular velocity ω is constant*).

As the insect moves towards B, moment of inertia of the system increases, hence, the angular momentum of the system will increase. Let at time t_1 the insect be at a distance x from O and by then the rod has rotated through an angle θ. Then, the angular momentum at that moment

$$K = \left[\frac{ML^2}{12} + Mx^2\right]\omega$$

$$\Rightarrow \quad \frac{dK}{dt} = 2M\omega x\frac{dx}{dt} \qquad (\omega = \text{Constant})$$

$$\tau = 2M\omega x\frac{dx}{dt} \qquad \left[\because \frac{dK}{dt} = \tau\right]$$

$$\text{or} \quad Mgx\cos\theta = 2M\omega x\frac{dx}{dt}$$

$$\Rightarrow \quad dx = \frac{g}{2\omega}\cos\omega t\ dt \qquad [\because \theta = \omega t]$$

At time $t = 0$, $x = \dfrac{L}{4}$ and at time $t = \dfrac{T}{4}$ or $\dfrac{\pi}{2\omega}$, $x = \dfrac{L}{2}$

Thus, $\displaystyle\int_{L/4}^{L/2} dx = \frac{g}{2\omega}\int_0^{\pi/2\omega}\cos\omega t dt$

$$\Rightarrow \quad \frac{L}{2} - \frac{L}{4} = \frac{g}{2\omega^2}\left[\sin\frac{\pi}{2} - \sin 0°\right]$$

$$\Rightarrow \quad \frac{L}{4} = \frac{g}{2\omega^2} \text{ or } \omega = \sqrt{\frac{2g}{L}}$$

$$\sqrt{\frac{2g}{L}} = \frac{12}{7}\cdot\frac{v}{L} \qquad \left[\because \omega = \frac{12v}{7L}\right]$$

$$v = \frac{7}{12}\sqrt{2gL} = \frac{7}{12}\sqrt{2\times 10\times 1.8} = 3.5\ \text{ms}^{-1}$$