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PHYSICS

FOR JEE MAIN AND ADVANCED

MODULE - 1

Units and Dimensions ◆ Vector and Calculus ◆ Kinematics
Constrained, NLM, Friction ◆ Circular Motion, Work, Power and Energy
Centre of Mass ◆ Rotational Dynamics

JEE Class Companion PHYSICS MODULE 1

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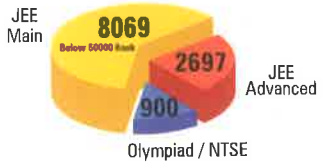

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for JEE Main and Advanced

Module-1

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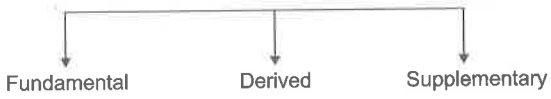
CHAPTER 1

Units and Dimensions

PHYSICAL QUANTITIES

The quantities which can be measured by an instrument and by means of which we can describe the laws of physics are called physical quantities.

Types of physical quantities:



Fundamental Quantities

Although the number of physical quantities that we measure is very large, we need only a limited number of units for expressing all the physical quantities since they are interrelated with one another. So, certain physical quantities have been chosen arbitrarily and their units are used for expressing all the physical quantities, such quantities are known as **Fundamental, Absolute** or **Base Quantities** (such as length, time and mass in mechanics).

1. All other quantities may be expressed in terms of the fundamental quantities.
2. They are independent of each other and cannot be obtained from one another.

An international body named General Conference on Weights and Measures chose seven physical quantities as fundamental:

1. length
2. mass
3. time
4. electric current
5. thermodynamic temperature
6. amount of substance
7. luminous intensity

Note

These are also called as absolute or base quantities.

In mechanics, we treat length, mass and time as the three basic or fundamental quantities.

Derived Quantities

Physical quantities which can be expressed as a combination of the base quantities are called as derived quantities.

For example: Speed, velocity, acceleration, force, momentum, pressure, energy, etc.

SOLVED EXAMPLE

EXAMPLE 1

$$\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{\text{length}}{\text{time}}$$ ■

Supplementary Quantities

Beside the seven fundamental physical quantities, two supplementary quantities are also defined, they are (1) Plane angle and (2) Solid angle.

Note

The supplementary quantities have only units but no dimensions.

MAGNITUDE

Magnitude of physical quantity = (numerical value) × (unit)

Magnitude of a physical quantity is always constant. It is independent of the type of unit.

$$\Rightarrow \quad \text{Numerical value} \propto \frac{1}{\text{unit}}.$$

or $$n_1 u_1 = n_2 u_2 = \text{constant}$$

SOLVED EXAMPLE

EXAMPLE 2

Length of a metal rod bar is unchanged whether it is measured as 2 m or 200 cm. Observe the change in the numerical value (from 2 to 200) as the unit is changed from metre to centimetre. ■

UNITS

Measurement of any physical quantity is expressed in terms of an internationally accepted certain basic reference standard called unit.

The units for the fundamental or base quantities are called fundamental or base units. Other physical quantities are expressed as a combination of these base units and, hence, are called derived units.

A complete set of units, including both fundamental and derived, is called a system of unit.

Principle Systems of Units

There are various systems in use over the world: CGS, FPS, SI (MKS), etc.

Supplementary Units

1. Plane angle: radian (rad)
2. Solid angle: steradian (sr)

* The SI system is at present widely used throughout the world. In IIT JEE, only the SI system is followed.

Definitions of Some Important SI Units

(i) **Metre:** 1 m = 1,650,763.73 wavelengths in vacuum of radiation corresponding to organ-red light emitted by Krypton-86.

(ii) **Second:** 1 s = 9,192,631,770 time periods of a particular transition from Cesium-133 atom.

(iii) **Kilogram:** 1 kg = mass of 1 litre volume of water at 4°C.

(iv) **Ampere:** It is the current which when flows through two infinitely long straight conductors of negligible cross-section placed at a distance of 1 m in vacuum produces a force of 2×10^{-7} N/m between them.

(v) **Kelvin:** 1 K = 1/273.16 part of the thermodynamic temperature of triple point of water.

(vi) **Mole:** It is the amount of substance of a system which contains as many elementary particles (atoms, molecules, ions, etc.) as there are atoms in 12 g of Carbon-12.

(vii) **Candela:** It is the luminous intensity in a perpendicular direction of a surface of $\left(\frac{1}{600000}\right)$ m^2 of a black body at the temperature of freezing point under a pressure of 1.013×10^5 N/m^2.

(viii) **Radian:** It is the plane angle between two radii of a circle which cut-off on the circumference an arc equal in length to the radius.

(ix) **Steradian:** The steradian is the solid angle which has its vertex at the centre of the sphere and cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere.

Table 1.1 Units of some physical quantities in different systems.

	Physical quantity	System		
		CGS (Gaussian)	MKS (SI)	FPS (British)
Fundamental	Length	centimetre	metre	foot
	Mass	gram	kilogram	pound
	Time	second	second	second
Derived	Force	dyne	newton → N	poundal
	Work or energy	erg	joule → J	ft-poundal
	Power	erg/s	watt → W	ft-poundal/s

SOLVED EXAMPLE

EXAMPLE 3

Find the SI unit of speed and acceleration.

SOLUTION

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{\text{metre(m)}}{\text{second(s)}} = \text{m/s}$$

(called as metre per second)

$$\text{Acceleration} = \frac{\text{velocity}}{\text{time}}$$

$$= \frac{\text{displacement/time}}{\text{time}}$$

$$= \frac{\text{displacement}}{(\text{time})^2}$$

$$= \frac{\text{metre}}{\text{second}^2} = \text{m/s}^2$$

(called as metre per second square) ■

SI PREFIXES

The magnitudes of physical quantities vary over a wide range. The CGPM recommends standard prefixes for expressing magnitudes that are too large or too small in a compact manner by using certain powers of 10.

Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10^{1}	deca	da	10^{-18}	atto	a

GENERAL GUIDELINES FOR USING SYMBOLS FOR SI UNITS, SOME OTHER UNITS AND SI PREFIXES

1. Symbols for units of physical quantities are printed/written in Roman (upright type) and not in italics.

   For example: 1 N is correct but 1 *N* is incorrect.

2. **(a)** Unit is never written in the expanded form with a capital initial letter even if it is named after a scientist.

   For example: SI unit of force is newton (correct) (but Newton is incorrect).

   (b) For a unit named after a scientist, the symbol is a capital letter.
   But for other units, the symbol is NOT a capital letter.

   For example:

force	newton (N)
energy	joule (J)
electric current	ampere (A)
temperature	kelvin (K)
frequency	hertz (Hz)

   For example:

length	metre (m)
mass	kilogram (kg)
luminous intensity	candela (cd)
time	second (s)

Note

The exception is *L*, for the unit litre.

   (c) Symbols for units do not contain any final full stop at the end of the recommended letter and remain unaltered in the plural form, using only the singular form of the unit.

   For example:

Quantity	Correct	Incorrect
25 centimetres	25 cm	25 c m
		25 cms.

(d) Use of solidus (/) is recommended only for indicating a division of one letter unit symbol by another unit symbol. Not more than one solidus is used.

For example:

Correct	Incorrect
m/s²	m/s/s
N s/m²	N s/m/m
J/K mol	J/K/mol
Kg/m s	kg/m/s

(e) Prefix symbols are printed in roman (upright) type without spacing between the prefix symbol and the unit symbol. Thus, certain approved prefixes written very close to the unit symbol are used to indicate decimal fractions or multiples of an SI unit, when it is inconveniently small or large.

For example:

megawatt	1 MW = 10^6 W
centimetre	1 cm = 10^{-2} m
kilometre	1 km = 10^3 m
millivolt	1 mV = 10^{-3} V
kilowatt-hour	1 kW h = 10^3 W h = 3.6 M J = 3.6 × 10^6 J
microampere	1 μA = 10^{-6} A
angstrom	1 Å = 0.1 nm = 10^{-10} m
nanosecond	1 ns = 10^{-9} s
picofarad	1 pF = 10^{-12} F
microsecond	1 μs = 10^{-6} s
gigahertz	1 GHz = 10^9 Hz
micron	1 μm = 10^{-6} m

The unit 'fermi', equal to a femtometre or 10^{-15} m, has been used as the convenient length unit in nuclear studies.

(f) When a prefix is placed before the symbol of a unit, the combination of prefix and symbol is considered as a new symbol, for the unit, which can be raised to a positive or negative power without using brackets. These can be combined with other unit symbols to form compound unit.

For example:

Quantity	Correct	Incorrect
cm^3	$(cm)^3 = (0.01\ m)^3 = (10^{-2}\ m)^3 = 10^{-6}\ m^3$	0.01 m^3 or $10^{-2}\ m^3$
mA^2	$(mA)^2 = (0.001\ A)^2 = (10^{-3}\ A)^2 = 10^{-6}\ A^2$	0.001 A^2

(g) A prefix is never used alone. It is always attached to a unit symbol and written or fixed before the unit symbol.

For example:

$10^3/m^3 = 1000/m^3$ or 1000 m^{-3}, but not k/m^3 or k m^{-3}.

(h) Prefix symbol is written very close to the unit symbol without space between them, while unit symbols are written separately with space with units being multiplied together.

For example:

Quantity	Correct	Incorrect
1 ms^{-1}	1 metre per second	1 milli per second
1 ms	1 millisecond	1 metre second
1 C m	1 coulomb metre	1 centimetre
1 cm	1 centimetre	1 coulomb metre

(i) The use of double prefixes is avoided when single prefixes are available.

For example:

Quantity	Correct	Incorrect
10^{-9} m	1 nm (nanometre)	1 mμ m (milli micrometre)
10^{-6} m	1 μm (micron)	1 m m m (milli millimetre)
10^{-12} F	1 pF (picofarad)	1 $\mu\mu$ F (micro microfarad)
10^9 F	1 GW (giga watt)	1 kM W (kilo megawatt)

(j) The use of a combination of unit and the symbols for unit is avoided when the physical quantity is expressed by combining two or more units.

Quantity	Correct	Incorrect
joule per mole Kelvin	J/mol K or J mol^{-1} K^{-1}	Joule / mole K or J/mol Kelvin or J/mole K
newton metre second	N m s	newton m second or N m second or N metre s or newton metre s

Characteristics of Base Units or Standards

1. Well defined
2. Accessibility
3. Invariability
4. Convenience in use

Some Special Types of Units

1.	1 Micron (1μ) = 10^{-4} cm = 10^{-6} m	(length)
2.	1 Angstrom (1 Å) = 10^{-8} cm = 10^{-10} m	(length)
3.	1 fermi (1f) = 10^{-13} cm = 10^{-15} m	(length)
4.	1 inch = 2.54 cm	(length)
5.	1 mile = 5280 feet = 1.609 km	(length)
6.	1 atmosphere = 10^5 N/m^2 = 76 torr = 76 mm of Hg pressure	(pressure)
7.	1 litre = 10^{-3} m^3 = 1000 cm^3	(volume)
8.	1 carat = 0.0002 kg	(weight)
9.	1 pound (lb) = 0.4536 kg	(weight)

DIMENSIONS

Dimensions of a physical quantity are the power to which the fundamental quantities must be raised to represent the given physical quantity.

For example,

$$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{\text{mass}}{(\text{length})^3}$$

or $$\text{density} = (\text{mass})\,(\text{length})^{-3} \quad \text{(i)}$$

Thus, the dimensions of density are 1 in mass and -3 in length. The dimensions of all other fundamental quantities are zero.

For convenience, the fundamental quantities are represented by one-letter symbols. Generally, mass is denoted by M, length by L, time by T and electric current by A.

The thermodynamic temperature, the amount of substance and the luminous intensity are denoted by the symbols of their units K, mol and cd, respectively. The physical quantity that is expressed in terms of the base quantities is enclosed in square brackets.

$$[\sin\theta] = [\cos\theta] = [\tan\theta] = [e^x] = [M^0L^0T^0]$$

DIMENSIONAL FORMULA

It is an expression which shows how and which of the fundamental units are required to represent the unit of physical quantity.

Different Quantities with Units, Symbol and Dimensional Formula

Quantity	Symbol	Formula	SI Unit	DF
Displacement	s	ℓ	Metre or m	M^0LT^0
Area	A	$\ell \times b$	(Metre)^2 or m^2	$M^0L^2T^0$
Volume	V	$\ell \times b \times h$	(Metre)^3 or m^3	$M^0L^3T^0$
Velocity	v	$v = \frac{\Delta s}{\Delta t}$	m/s	M^0LT^{-1}
Momentum	p	$p = mv$	kgm/s	MLT^{-1}
Acceleration	a	$a = \frac{\Delta v}{\Delta t}$	m/s^2	M^0LT^{-2}
Force	F	$F = ma$	Newton or N	MLT^{-2}
Impulse	-	$F \times t$	N·sec	MLT^{-1}
Work	W	$F \cdot d$	N·m	ML^2T^{-2}
Energy P.E. = mgh	KE or U	$K.E. = \frac{1}{2}mv^2$	Joule or J	ML^2T^{-2}
Power	P	$P = \frac{W}{t}$	watt or W	ML^2T^{-3}
Density	d	d = mass/volume	kg/m^3	$ML^{-3}T^0$

Quantity	Symbol	Formula	SI Unit	DF
Pressure	P	$P = F/A$	Pascal or Pa	$ML^{-1}T^{-2}$
Torque	τ	$\tau = r \times F$	N.m.	ML^2T^{-2}
Angular displacement	θ	$\theta = \frac{\text{arc}}{\text{radius}}$	radian or rad	$M^0L^0T^0$
Angular velocity	ω	$\omega = \frac{\theta}{t}$	rad/sec	$M^0L^0T^{-1}$
Angular acceleration	α	$\alpha = \frac{\Delta\omega}{\Delta t}$	rad/sec^2	$M^0L^0T^{-2}$
Moment of inertia	I	$I = mr^2$	kg-m^2	ML^2T^0
Frequency	v or f	$f = \frac{1}{T}$	hertz or Hz	$M^0L^0T^{-1}$
Stress	-	F/A	N/m^2	$ML^{-1}T^{-2}$
Strain	-	$\frac{\Delta\ell}{\ell}; \frac{\Delta A}{A}; \frac{\Delta V}{V}$	-	$M^0L^0T^0$
Youngs modulus (Bulk modulus of rigidity)	Y	$Y = \frac{F/A}{\Delta\ell/\ell}$	N/m^2	$ML^{-1}T^{-2}$
Surface tension	T	$\frac{F}{\ell}$ or $\frac{W}{A}$	$\frac{\text{N}}{\text{m}}$; $\frac{\text{J}}{\text{m}^2}$	ML^0T^{-2}
Force constant (spring)	k	$F = kx$	N/m	ML^0T^{-2}
Coefficient of viscosity	η	$F = \eta\left(\frac{dv}{dx}\right)A$	kg/ms (poise in C.G.S.)	$ML^{-1}T^{-1}$
Gravitation constant	G	$F = \frac{Gm_1m_2}{r^2}$	$\frac{\text{N-m}^2}{\text{kg}^2}$	$M^{-1}L^3T^{-2}$
Gravitational potential	V_g	$V_g = \frac{PE}{m}$	$\frac{\text{J}}{\text{kg}}$	$M^0L^2T^{-2}$
Temperature	θ	-	Kelvin or K	$M^0L^0T^0\theta^{+1}$
Heat	Q	$Q = m \times S \times \Delta t$	Joule or Calorie	ML^2T^{-2}
Specific heat	S	$Q = m \times S \times \Delta t$	$\frac{\text{Joule}}{\text{kg.Kelvin}}$	$M^0L^2T^{-2}\theta^{-1}$
Latent heat	L	$Q = mL$	$\frac{\text{Joule}}{\text{kg}}$	$M^0L^2T^{-2}$

Quantity	Symbol	Formula	SI Unit	DF
Coefficient of thermal conductivity	K	$Q=\frac{KA(\theta_1-\theta_2)t}{d}$	$\frac{\text{Joule}}{\text{m sec K}}$	$MLT^{-3}\theta^{-1}$
Universal gas constant	R	$PV=nRT$	$\frac{\text{Joule}}{\text{mol}\cdot\text{K}}$	$ML^2T^{-2}\theta^{-1}$
Mechanical equivalent of heat	J	$W=JH$	-	$M^0L^0T^0$
Charge	Q or q	$I=\frac{Q}{t}$	Coulomb or C	M^0L^0TA
Current	I	-	Ampere or A	$M^0L^0T^0A$
Electric permittivity	ε_0	$\varepsilon_0=\frac{1}{4\pi F}\cdot\frac{q_1q_2}{r^2}$	$\frac{(\text{coul.})^2}{\text{N}\cdot\text{m}^2}$ or $\frac{\text{C}^2}{\text{N-m}^2}$	$M^{-1}L^{-3}T^4A^2$
Electric potential	V	$V=\frac{\Delta W}{q}$	Joule/coulomb	$ML^2T^{-3}A^{-1}$
Intensity of electric field	E	$E=\frac{F}{q}$	N/coulomb	$MLT^{-3}A^{-1}$
Capacitance	C	$Q=CV$	Farad	$M^{-1}L^{-2}T^4A^2$
Dielectric constant or relative permittivity	ε_r	$\varepsilon_r=\frac{\varepsilon}{\varepsilon_0}$	-	$M^0L^0T^0$
Resistance	R	$V=IR$	Ohm	$ML^2T^{-3}A^{-2}$
Conductance	S	$S=\frac{1}{R}$	Mho	$M^{-1}L^{-2}T^{-3}A^2$
Specific resistance or resistivity	ρ	$\rho=\frac{RA}{\ell}$	Ohm × metre	$ML^3T^{-3}A^{-2}$
Conductivity or specific conductance	s	$\sigma=\frac{1}{\rho}$	Mho/metre	$M^{-1}L^{-3}T^3A^2$
Magnetic induction	B	$F=qvB\sin\theta$ or $F=BIL$	Tesla or weber/m^2	$MT^{-2}A^{-1}$
Magnetic flux	ϕ	$e=\frac{d\phi}{dt}$	Weber	$ML^2T^{-2}A^{-1}$
Magnetic intensity	H	$B=\mu H$	A/m	$M^0L^{-1}T^0A$
Magnetic permeability of free space or medium	μ_0	$B=\frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}$	$\frac{\text{N}}{\text{amp}^2}$	$MLT^{-2}A^{-2}$

Quantity	Symbol	Formula	SI Unit	DF
Coefficient of self or mutual inductance	L	$e = L.\frac{dI}{dt}$	Henry	$ML^2T^{-2}A^{-2}$
Electric dipole moment	p	$p = q \times 2\ell$	C.m.	M^0LTA
Magnetic dipole moment	M	$M = NIA$	amp.m^2	$M^0L^2AT^0$

USE OF DIMENSIONS

Theory of dimensions have the following main uses:

Conversion of Units

This is based on the fact that the product of the numerical value (n) and its corresponding unit (u) is a constant, i.e.,

$$n[u] = \text{constant}$$

or

$$n_1[u_1] = n_2[u_2]$$

Suppose the dimensions of a physical quantity are a in mass, b in length and c in time. If the fundamental units in one system are M_1, L_1 and T_1 and in the other system are M_2, L_2 and T_2, respectively. Then we can write.

$$n_1\left[M_1^a L_1^b T_1^c\right] = n_2\left[M_2^a L_2^b T_2^c\right] \quad \text{(i)}$$

Here n_1 and n_2 are the numerical values in two systems of units, respectively. Using Eq. (i), we can convert the numerical value of a physical quantity from one system of units into the other system.

SOLVED EXAMPLE

EXAMPLE 4

The value of gravitation constant is $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 in SI units. Convert it into CGS system of units.

SOLUTION

The dimensional formula of G is $[M^{-1}L^3T^{-2}]$.
Using Eq. (i), i.e.,

$$n_1\left[M_1^{-1}L_1^3T_1^{-2}\right] = n_2\left[M_2^{-1}L_2^3T_2^{-2}\right]$$

$$n_2 = n_2\left[\frac{M_1}{M_2}\right]^{-1}\left[\frac{L_1}{L_2}\right]^3\left[\frac{T_1}{T_2}\right]^{-2}$$

Here,

$$n_1 = 6.67 \times 10^{-11}$$
$$M_1 = 1 \text{ kg},$$
$$M_2 = 1 \text{ g} = 10^{-3} \text{ kg}$$
$$L_1 = 1 \text{ m},$$
$$L_2 = 1 \text{ cm} = 10^{-2} \text{ m},$$
$$T_1 = T_2 = 1\text{s}$$

Substituting in the above equation, we get

$$n_2 = 6.67 \times 10^{-11}\left[\frac{1\text{kg}}{10^{-3}\text{kg}}\right]^{-1}\left[\frac{1\text{m}}{10^{-2}\text{m}}\right]^3\left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

or

$$n_2 = 6.67 \times 10^{-8}$$

Thus, the value of G in CGS system of units is 6.67×10^{-8} dyne cm^2/g^2. ■

To Check the Dimensional Correctness of a Given Physical Equation

Every physical equation should be dimensionally balanced. This is called the 'Principle of Homogeneity'. The dimensions of each term on both sides of an equation must be the same. On this basis, we can judge whether a given equation is correct. But a dimensionally correct equation may or may not be physically correct.

SOLVED EXAMPLE

EXAMPLE 5

Show that the expression of the time period T of a simple pendulum of length l given by $T = 2\pi\sqrt{\frac{l}{g}}$ is dimensionally correct.

SOLUTION

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Dimensionally, $$[T]=\sqrt{\frac{[L]}{[LT^{-2}]}}=[T]$$

As in the above equation, the dimensions of both sides are the same. The given formula is dimensionally correct. ■

Principle of Homogeneity of Dimensions

This principle states that the dimensions of all the terms in a physical expression should be the same. For example, in the physical expression $s = ut + \frac{1}{2}at^2$, the dimensions of s, ut and $\frac{1}{2}at^2$ all are same.

Note

The physical quantities separated by the symbols $+$, $-$, $=$, $>$, $<$, etc., have the same dimensions.

SOLVED EXAMPLE

EXAMPLE 6

The velocity v of a particle depends upon the time t according to the equation $v = a + bt + \frac{c}{d+t}$. Write the dimensions of a, b, c and d.

SOLUTION

From the principle of homogeneity

$$[a] = [v]$$

or $$[a] = [LT^{-1}]$$

$$[bt] = [v]$$

or $$[b]=\frac{[v]}{[t]}=\frac{[LT^{-1}]}{[T]}$$

or $$[b] = [LT^{-2}]$$

Similarly, $$[d] = [t] = [T]$$

Further, $$\frac{[c]}{[d+t]}=[v]$$

or $$[c] = [v]\,[d + t]$$

or $$[c] = [LT^{-1}]\,[T]$$

or $$[c] = [L]$$ ■

To Establish the Relation Among Various Physical Quantities

If we know the factors on which a given physical quantity may depend, we can find a formula relating the quantity with those factors. Let us take an example.

SOLVED EXAMPLE

EXAMPLE 7

The frequency (f) of a stretched string depends upon the tension F (dimensions of force), length l of the string and the mass per unit length μ of the string. Derive the formula for frequency.

SOLUTION

Suppose, that the frequency f depends on the tension raised to the power a, length raised to the power b and mass per unit length raised to the power c. Then.

$$f \propto [F]^a\,[l]^b\,[\mu]^c$$

or $$f = k[F]^a\,[l]^b\,[\mu]^c$$

Here, k is a dimensionless constant. Thus,

$$[f] = [F]^0\,[l]^b\,[\mu]^c$$

or $$[M^0L^0T^{-1}] = [MLT^{-2}]^a\,[L]^b\,[ML^{-1}]^c$$

or $$[M^0L^0T^{-1}] = [M^{a+c}L^{a+b-c}T^{-2a}]$$

For dimensional balance, the dimension on both sides should be the same.

Thus, $$a + c = 0 \quad (1)$$

$$a + b - c = 0 \quad (2)$$

and $$-2a = -1 \quad (3)$$

Solving these three equations, we get

$$a=\frac{1}{2},\ c=-\frac{1}{2}$$

and $$b = -1$$

Substituting these values in equation (i), we get

$$F = k(F)^{1/2}\,(l)^{-1}\,(\mu)^{-1/2}$$

or $$f=\frac{k}{l}\sqrt{\frac{F}{\mu}}$$

Experimentally, the value of k is found to be $\frac{1}{2}$.

Hence, $f = \frac{1}{2l}\sqrt{\frac{F}{\mu}}$. ■

Limitations of Dimensional Analysis

The method of dimensions has the following limitations:

1. The value of a dimensionless constant cannot be calculated.
2. The equation containing trigonometrical, exponential and logarithmic terms cannot be analyzed.
3. If a physical quantity depends on more than three factors, then the relation among them cannot be established because we can have only three equations by equalizing the powers of M, L and T.

BASIC MATHEMATICS

MENSURATION FORMULAS

r = radius $\quad$ d = diameter

V = volume $\quad$ SA = surface area

1. Circle

Perimeter: $2\pi r = \pi d$

Area: $\pi r^2 = \frac{1}{4}\pi d^2$

2. Sphere

$$\text{Surface area} = 4\pi r^2 = \pi d^2$$

$$\text{Volume} = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$$

3. Spherical shell (Hollow sphere)

$$\text{Surface area} = 4\pi r^2 = \pi d^2$$

Volume of material used $= (4\pi r^2)(dr)$,

dr = thickness

4. Cylinder

$$\text{Lateral area} = 2\pi rh$$

$$V = \pi r^2 h$$

$$\text{Total area} = 2\pi rh + 2\pi r^2$$

$$= 2\pi r\,(h + r)$$

5. Cone

$$\text{Lateral area} = \pi r\sqrt{r^2 + h^2}$$

h = height

$$\text{Total area} = \pi r\left(\sqrt{r^2 + h^2} + r\right)$$

$$V = \frac{1}{3}\pi r^2 h$$

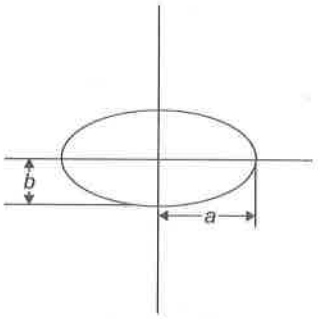
6. Ellipse

$$\text{Circumference} \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$$

Area $= \pi ab$

a = semi-major axis

b = semi-minor axis



7. Parallelogram

$A = bh = ab\sin\theta$

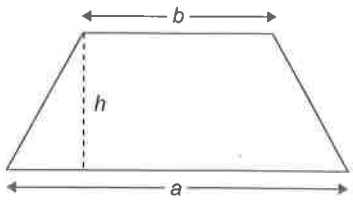
a = side; h = height;

b = base

θ = angle between sides a and b

8. Trapezoid

$$\text{Area} = \frac{h}{2}(a + b)$$



a and b parallel sides

h = height

9. Triangle

$$\text{Area} = \frac{bh}{2} = \frac{ab}{2}\sin\gamma$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

a, b, c sides are opposite to angles α, β, γ.

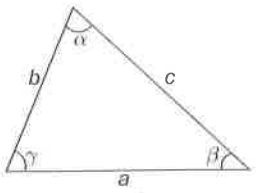
b = base; h = height

$$s = \frac{1}{2}(a+b+c)$$

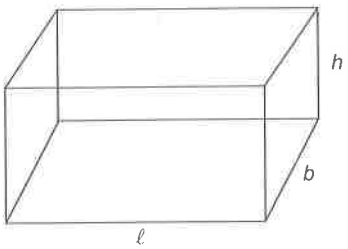
10. Rectangular container

$$\text{Lateral area} = 2(\ell b + bh + h\ell)$$

$$V = \ell bh$$



Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics.



Side ℓ, b, h

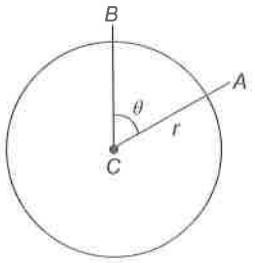
LOGARITHMS

1. $e \approx 2.7183$
2. If $e^x = y$, then $x = \log_e y = \ln y$
3. If $10^x = y$, then $x = \log_{10} y$
4. $\log_{10} y = 0.4343 \quad \log_e y = 2.303 \log_{10} y$
5. $\log(ab) = \log(a) + \log(b)$
6. $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
7. $\log a^n = n \log(a)$

TRIGONOMETRIC PROPERTIES

(i) Measurement of Angle and Relationship Between Degrees and Radian

In navigation and astronomy, angles are measured in degrees, but in calculus, it is best to use units called radians because of they simplify later calculations.



Let ACB be a central angle in **circle** of radius r, as in the figure. Then the angle ACB or θ is defined in radius as

$$\theta = \frac{\text{Arc length}}{\text{Radius}}$$

$$\Rightarrow \qquad \theta = \frac{\overset{\frown}{AB}}{r}$$

If $r = 1$, then $\theta = AB$

The **radian measure** for a circle of unit radius of angle ABC is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation:

$$\pi \text{ radians} = 180°$$

Angle Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180°} \ (\approx 0.02) \text{ radian}$$

Degrees to radians: multiply by $\frac{\pi}{180°}$.

$$1 \text{ radian} \approx 57 \text{ degrees}$$

Radians to degrees: multiply by $\frac{180°}{\pi}$.

SOLVED EXAMPLES

EXAMPLE 15

Covert 45° to radians:

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad}$$

Convert $\frac{\pi}{6}$ rad to degrees:

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°$$

■

EXAMPLE 16

Convert 30° to radians.

SOLUTION

$$30° \times \frac{\pi}{180°} = \frac{\pi}{6} \text{ rad}$$ ■

EXAMPLE 17

Convert $\frac{\pi}{3}$ rad to degrees.

SOLUTION

$$\frac{\pi}{3} \times \frac{180}{\pi} = 60°$$ ■

Standard values

(a) $30° = \frac{\pi}{6}$ rad

(b) $45° = \frac{\pi}{4}$ rad

(c) $60° = \frac{\pi}{3}$ rad

(d) $90° = \frac{\pi}{2}$ rad

(e) $120° = \frac{2\pi}{3}$ rad

(f) $135° = \frac{3\pi}{4}$ rad

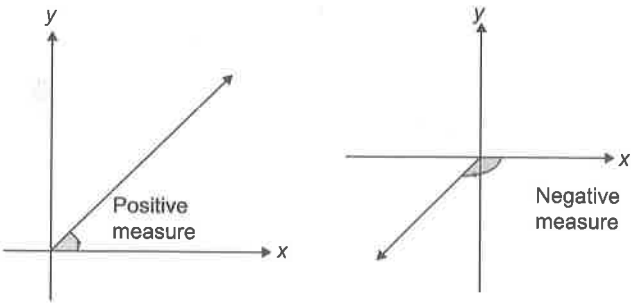
(g) $150° = \frac{5\pi}{6}$ rad

(h) $180° = \pi$ rad

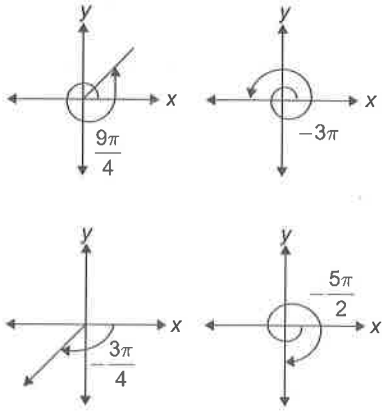
(i) $360° = 2\pi$ rad

(Check these values yourself to see that they satisfy the conversion formulas.)

(ii) Measurement of Positive and Negative Angles

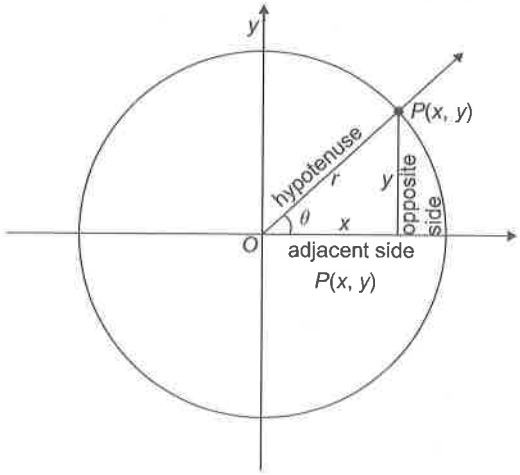


An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure). Angles measured counter-clockwise from the positive x-axis are assigned positive measures and angles measured clockwise are assigned negative measures.



(iii) Six Basic Trigonometric Functions

The trigonometric functions of a general angle θ are defined in terms of x, y and r.



Sine: $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$

Cosecant: $\text{cosec}\,\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y}$

Cosine: $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$

Secant: $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x}$

Tangent: $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

Cotangent: $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$

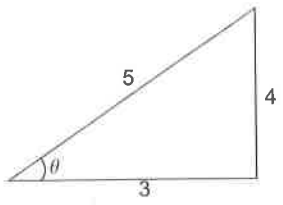
Values of Trigonometric Functions

If the circle in the above figure has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x,$$

$$\sin\theta = y$$

We can then calculate the values of the cosine and sine directly from the coordinates of P.



SOLVED EXAMPLES

EXAMPLE 18

Find the six trigonometric ratios from the given figure (see above).

SOLUTION

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$$

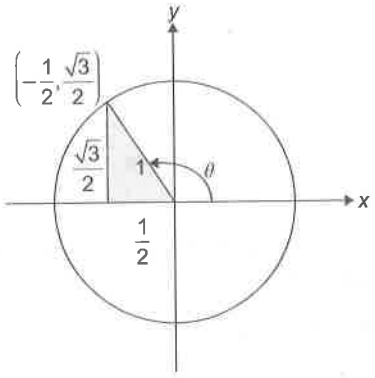
$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

$$\sec\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{3}$$

$$\text{cosec}\,\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

EXAMPLE 19

Find the sine and cosine of angle θ shown in the unit circle if coordinate of point p are as shown.



SOLUTION

$$\cos\theta = x\text{-coordinate of } P$$

$$= -\frac{1}{2}$$

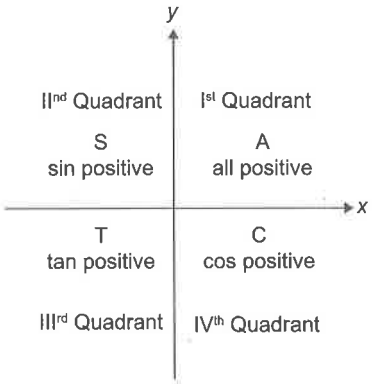
$$\sin\theta = y\text{-coordinate of } P$$

$$= \frac{\sqrt{3}}{2}.$$

VALUES OF SIN θ, COS θ AND TAN θ FOR SOME STANDARD ANGLES

Degree	0	30	37	45	53	60	90	120	135	180
Radians	0	$\pi/6$	$37\pi/180$	$\pi/4$	$53\pi/180$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$\sin\theta$	0	1/2	3/5	$1/\sqrt{2}$	4/5	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	0
$\cos\theta$	1	$\sqrt{3}/2$	4/5	$1/\sqrt{2}$	3/5	1/2	0	−1/2	$-1/\sqrt{2}$	−1
$\tan\theta$	0	$1/\sqrt{3}$	3/4	1	4/3	$\sqrt{3}$	∞	$-\sqrt{3}$	−1	0

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule. If you are not very enthusiastic about CAST, you can remember it as ASTC (after school to college).



The CAST rule

Rules for Finding Trigonometric Ratio of Angles Greater than 90°

Step 1: Identify the quadrant in which the angle lies.

Step 2: (a) If the angle $= (n\pi \pm \theta)$, where n is an integer, then the trigonometric function of $(n\pi \pm \theta) =$ same trigonometric function of θ and the sign will be decided by CAST rule.

(b) If the angle $= \left[(2n+1)\frac{\pi}{2} \pm \theta\right]$, where n is an integer, then the trigonometric function of $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ = complimentary trigonometric function of θ and the sign will be decided by CAST rule.

SOLVED EXAMPLES

EXAMPLE 20

Evaluate sin 120°

SOLUTION

$$\sin 120° = \sin(90° + 30°)$$
$$= \cos 30° = \frac{\sqrt{3}}{2}$$

Aliter

$$\sin 120° = \sin(180° - 60°)$$
$$= \sin 60° = \frac{\sqrt{3}}{2}$$

EXAMPLE 21

Evaluate cos 210°

SOLUTION

$$\cos 210° = \cos(180° + 30°)$$
$$= -\cos 30° = -\frac{\sqrt{3}}{2}$$

EXAMPLE 22

$$\tan 210° = \tan(180° + 30°)$$
$$= \tan 30° = \frac{1}{\sqrt{3}}$$

IMPORTANT FORMULAS

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
4. $\sin 2\theta = 2\sin\theta\cos\theta$
5. $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$
6. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
7. $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$
8. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
9. $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$
10. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
11. $\cos C - \cos D = 2\sin\left(\frac{D-C}{2}\right)\sin\left(\frac{C+D}{2}\right)$
12. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
13. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$
14. $\sin(90° + \theta) = \cos\theta$
15. $\cos(90° + \theta) = -\sin\theta$
16. $\tan(90° + \theta) = -\cot\theta$
17. $\sin(90° - \theta) = \cos\theta$
18. $\cos(90° - \theta) = \sin\theta$
19. $\cos(180° - \theta) = -\cos\theta$

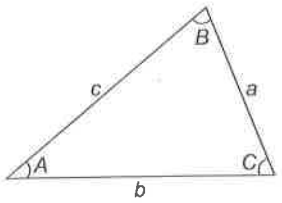
20. $\sin(180° - \theta) = \sin\theta$
21. $\cos(180° + \theta) = -\cos\theta$
22. $\tan(180° + \theta) = \tan\theta$
23. $\sin(-\theta) = -\sin\theta$
24. $\cos(-\theta) = \cos\theta$
25. $\tan(-\theta) = -\tan\theta$

- **Sine rule**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

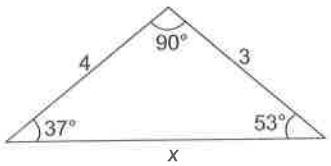
- **Cosine rule**

$$a^2 = b^2 + c^2 - 2bc\cos A$$



SOLVED EXAMPLES

EXAMPLE 23



Find x.

SOLUTION

$$\frac{\sin 90°}{x} = \frac{\sin 53°}{4}$$

$$x = 5$$

■

SMALL-ANGLE APPROXIMATION

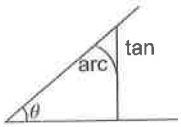
It is a useful simplification which is only approximately true for finite angles. It involves linearization of the trigonometric functions so that the angle θ is measured in radians.

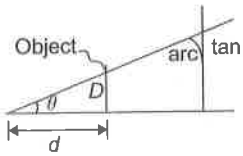
$$\sin\theta \approx \theta$$

$\cos\theta \approx 1$ or $\cos\theta \approx 1 - \frac{\theta^2}{2}$ for the second-order approximation

$$\tan\theta \approx \theta$$

Geometric justification





Small-angle approximation. The value of the small angle X in radians is approximately equal to its tangent.

- When one angle of a right triangle is small, the hypotenuse is approximately equal in length to the leg adjacent to the small angle, so the cosine is approximately 1.
- The short leg is approximately equal to the arc from the long leg to the hypotenuse, so the sine and tangent are both approximated by the value of the angle in radians.

BINOMIAL THEOREM

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$(1 \pm x)^{-n} = 1 \pm nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

If $x \ll 1$, then

$$(1 \pm x)^n = 1 \pm nx$$

(neglecting higher terms)

$$(1 \pm x)^{-n} = 1 \pm (-n)x = 1 \pm nx$$

$$(1 + x)^2 = 1 + 2x + x^2$$

$$(1 + x)^3 = 1 + 3x + x^3 - 3x^2$$

$$(1 + x)^n = 1 + nx + \cdots$$

if $x \ll 1$

Note

1. When n is a positive integer, then the expansion will have $(n + 1)$ terms.
2. When n is a negative integer, the expansion will have infinite terms.
3. When n is a fraction, the expansion will have infinite terms.

SOLVED EXAMPLES

EXAMPLE 24

Calculate $(1001)^{1/3}$.

SOLUTION

We can write 1001 as

$$1001 = 1000\left(1+\frac{1}{1000}\right),$$

so that we have

$$(1001)^{1/3} = \left[1000\left(1+\frac{1}{1000}\right)\right]^{1/3}$$

$$= 10\left[1+\frac{1}{1000}\right]^{1/3}$$

$$= 10(1+0.001)^{1/3}$$

$$= 10\left(1+\frac{1}{3}\times 0.001\right)$$

$$= 10.003333$$ ■

EXAMPLE 25

Expand $(1+x)^{-3}$.

$$(1+x)^{-3} = 1+(-3)x+\frac{(-3)(-3-1)x^2}{2!}$$

$$+\frac{(-3)(-3-1)(-3-2)}{3!}x^3+\cdots$$

SOLUTION

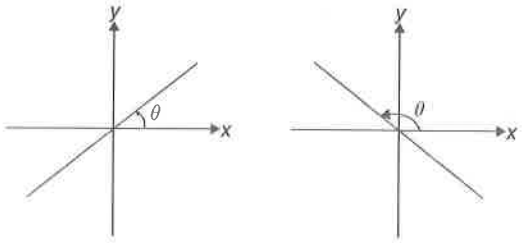
$$= 1-3x+\frac{12}{2}x^2-\frac{60}{3\times 2}x^3+\cdots$$

$$= 1-3x+6x^2-10x^3+\cdots$$ ■

GRAPHS

The following graphs and their corresponding equations are frequently used in Physics.

(i) $y = mx$, represents a straight line passing through the origin. Here, $m = \tan\theta$ is also called the slope of line, where θ is the angle that the line makes with positive x-axis, when drawn in anticlockwise direction from the positive x-axis towards the line.



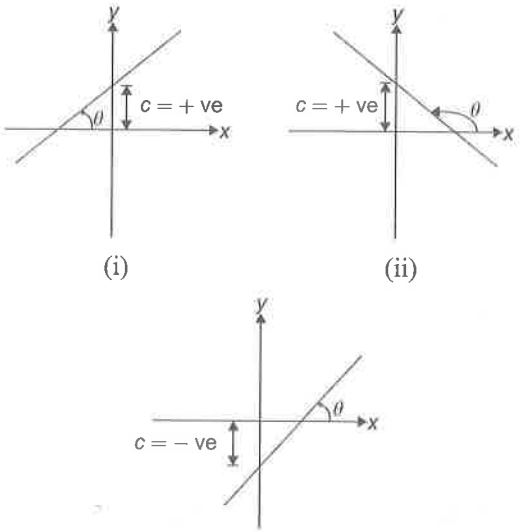
(i) (ii)

Fig. 1

The two possible cases are shown in Fig. 1(i), $\theta < 90°$. Therefore, $\tan\theta$ or slope of the line is positive. In Fig. 1(ii), $90° < \theta < 180°$. Therefore, $\tan\theta$ or slope of the line is negative.

Note

It should be noted that $y = mx$ of $y \propto x$ also means that the value of y becomes double if x is doubled, or it becomes $\frac{1}{4}$th if x becomes $\frac{x}{4}$ and c the intercept on y-axis.



(i) (ii)

(iii)

Fig. 2

In Fig. 2(i), both the slope and the intercept are positive.

In Fig. 2(ii), the slope is negative but the intercept is positive and

In Fig. 2(iii), the slope is positive but the intercept is negative.

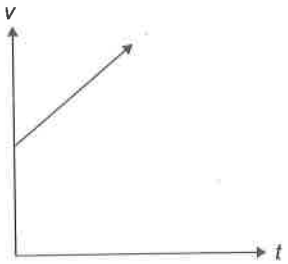
Note

It should be noted that in $y = mx + c$, y does not become two times if x is doubled.

SOLVED EXAMPLES

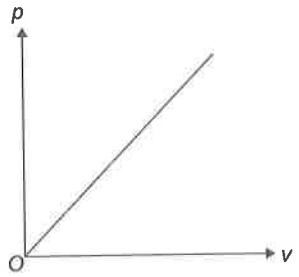
EXAMPLE 26

$v = u + at$



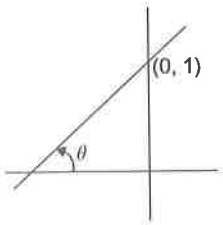
EXAMPLE 27

$P = mv$



EXAMPLE 28

Draw the graph for the equation: $2y = 3x + 2$



$\tan\theta = \frac{3}{2}$

SOLUTION

$$2y = 3x + 2$$

$$\Rightarrow \quad y = \frac{3}{2}x + 1$$

$$m = \frac{3}{2} > 0$$

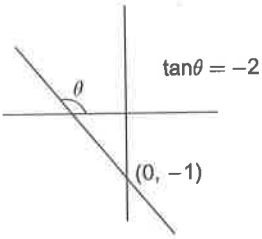
$$\Rightarrow \quad \theta < 90°$$

$$c = +1 > 0$$

$\Rightarrow$ The line will pass through (0, 1).

EXAMPLE 29

Draw the graph for the equation: $2y + 4x + 2 = 0$



SOLUTION

$$2y + 4x + 2 = 0$$

$$\Rightarrow \quad y = -2x - 1$$

$$m = -2 < 0,$$

i.e., $\quad \theta > 90°$

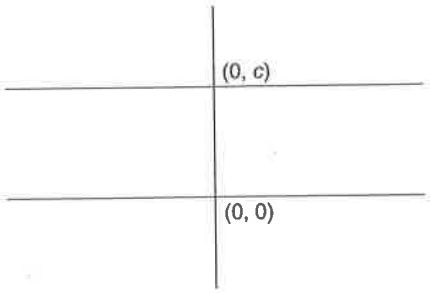
$$c = -1, \text{ i.e.,}$$

The line will pass through (0, −1).

Note

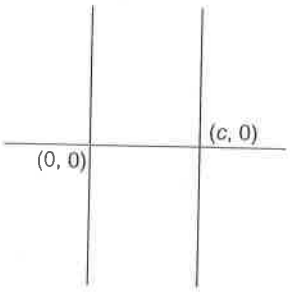
(i) If $c = 0$, the line will pass through the origin.

(ii) $y = c$ will be a line parallel to the x-axis.



Note (contd)

(iii) $x = c$ will be a line perpendicular to the y-axis



(ii) Parabola

A general quadratic equation represents a parabola.

$$y = ax^2 + bx + c$$

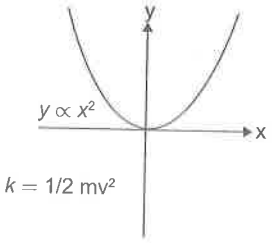
$$a \neq 0$$

if $a > 0$, it will be an upward-opening parabola.

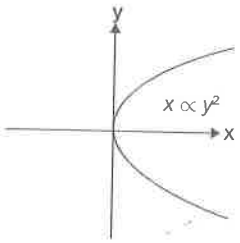
if $a < 0$, it will be a downward-opening parabola.

if $c = 0$, it will pass through the origin.

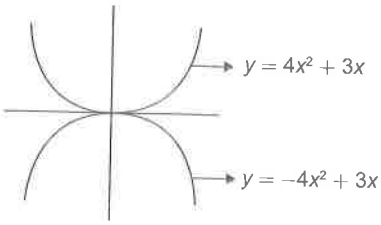
$y \propto x^2$ or $y = 2x^2$, etc., represents a parabola passing through the origin as shown in the figure.

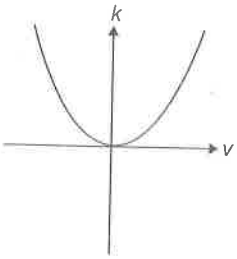


e.g., $y = 4x^2 + 3x$



e.g., $k = \frac{1}{2}mv^2$





Note

It should be noted that in the parabola $y = 2x^2$ or $y \propto x^2$, if x is doubled, y will increase four times.

Graph $x \propto y^2$ or $x = 4y^2$ is again a parabola passing through the origin as shown in the figure. In this case, if y is doubled, x will increase four times.

$y = x^2 + 4$ or $x = y^2 - 6$ will represent a parabola that does not pass through the origin. In the first equation, $y = x^2 + 4$, if x is doubled, y will not increase by four times.

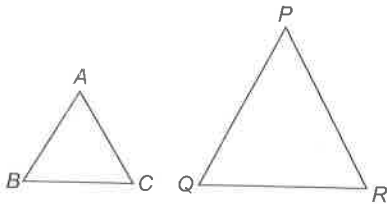
SIMILAR TRIANGLE

Two given triangles are said to be similar if

(1) All the respective angles are the same

or

(2) All the respective side ratios are the same.



As examples, ABC and PQR are two triangles as shown in the figure.

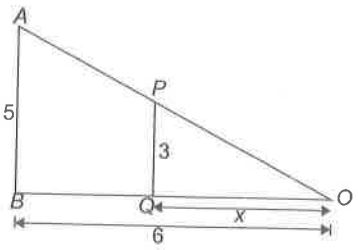
If they are similar triangles, then

(1)
$$\angle A = \angle P$$
$$\angle B = \angle Q$$
$$\angle C = \angle R$$

OR

(2)
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

EXAMPLE 30



Find x.

SOLUTION

By similar triangle concept

$$\frac{AB}{PQ} = \frac{OB}{OQ}$$

$$\frac{5}{3} = \frac{6}{x}$$

$$\Rightarrow \quad x = \frac{18}{5}.$$

EXERCISES

JEE Main

Section A: Units

1. Which of the following is not the name of a physical quantity?
   (A) Kilogram (B) Impulse
   (C) Energy (D) Density

2. Light year is the unit of
   (A) speed (B) mass
   (C) distance (D) time

3. PARSEC is a unit of
   (A) time (B) angle
   (C) distance (D) velocity

4. Which of the following system of units is NOT based on the unit of mass, length and time alone?
   (A) FPS (B) SI
   (C) CGS (D) MKS

5. In the SI system, the unit of energy is
   (A) erg (B) calorie
   (C) Joule (D) electron volt

6. The unit of pressure in the SI system is
   (A) atmosphere (B) dynes per square cm
   (C) pascal (D) bar

7. In SI unit, the angular acceleration has the unit of
   (A) $Nmkg^{-1}$ (B) ms^{-2}
   (C) $rad \cdot s^{-2}$ (D) Nkg^{-1}

8. The SI unit of the universal gravitational constant G is
   (A) $Nm\ kg^{-2}$ (B) Nm^2kg^{-2}
   (C) $Nm^2\ kg^{-1}$ (D) $Nmkg^{-1}$

9. The surface tension has the unit of
   (A) Joule·m^2 (B) Joule·m^{-2}
   (C) Joule·m (D) Joule·m^3

10. The unit of intensity of magnetization is
    (A) Amp m^2 (B) Amp m^{-2}
    (C) Amp m (D) Amp m^{-1}

11. The MKS unit of coefficient of viscosity is
    (A) $kg\ m^{-1}s^{-1}$ (B) $kg\ m\ s^{-2}$
    (C) $kg\ m^2\ s^{-1}$ (D) $kg^{-1}\ m^{-1}\ s^2$

12. Specific resistance has the unit of
    (A) ohm/m (B) ohm/m^2
    (C) ohm·m^2 (D) ohm·m

13. Mutual inductance has the unit of
    (A) Gauss (B) Weber
    (C) Farad (D) Henry

14. The unit of magnetic moment is
    (A) amp m^2 (B) amp m^{-2}
    (C) amp m (D) amp m^{-1}

15. The SI unit of the universal gas constant R is
    (A) erg K^{-1} mol^{-1} (B) watt K^{-1} mol^{-1}
    (C) newton K^{-1} mol^{-1} (D) joule K^{-1} mol^{-1}

16. A physical quantity is measured and the result is expressed as nu where u is the unit used and n is the numerical value. If the result is expressed in various units, then
    (A) $n \propto$ size of u (B) $n \propto u^2$
    (C) $n \propto \sqrt{u}$ (D) $n \propto 1/u$

17. One watt-hour is equivalent to
    (A) 6.3×10^3 Joule (B) 6.3×10^{-7} Joule
    (C) 3.6×10^3 Joule (D) 3.6×10^{-3} Joule

18. Which of the following statement is wrong?
    (A) Unit of K.E. is Newton-metre
    (B) Unit of viscosity is poise
    (C) Work and energy have the same dimensions
    (D) Unit of surface tension is Newton metre

Section B: Dimensions

19. What are the dimensions of length in force × displacement/time?
    (A) −2 (B) 0
    (C) 2 (D) none of these

20. The angular frequency is measured in rad s^{-1}. Its dimension in length is

(A) -2 (B) -1
(C) 0 (D) 2

21. $[MLT^{-1}]$ are the dimensions of
(A) power (B) momentum
(C) force (D) couple

22. The dimensional formula for angular momentum is
(A) ML^2T^{-2} (B) ML^2T^{-1}
(C) MLT^{-1} (D) $M^0L^2T^{-2}$

23. The dimensions of universal gravitational constant are
(A) $M^{-1}L^3T^{-2}$ (B) $M^{-1}L^3T^{-1}$
(C) $M^{-1}L^{-1}T^{-2}$ (D) $M^{-2}L^2T^{-2}$

24. The SI unit of Stefan's constant is
(A) $Ws^{-1}\ m^{-2}\ K^{-4}$ (B) $J\ s\ m^{-1}\ K^{-1}$
(C) $J\ s^{-1}\ m^{-2}\ K^{-1}$ (D) $W\ m^{-2}\ K^{-4}$

25. What are the dimensions of Boltzmann's constant?
(A) $MLT^{-2}K^{-1}$ (B) $ML^2T^{-2}K^{-1}$
(C) M^0LT^{-2} (D) $M^0L^2T^{-2}K^{-1}$

26. The dimensions of magnetic flux density is
(A) $M^1L^0T^{-1}A^{-1}$ (B) $M^1L^0T^{-2}A^{-1}$
(C) $M^1L^1T^{-2}A^{-1}$ (D) $M^1L^0T^{-1}A^{-2}$

27. A pair of physical quantities having the same dimensional formula is
(A) angular momentum and torque
(B) torque and energy
(C) force and power
(D) power and angular momentum

28. The dimensions of pressure are the same as that of
(A) force per unit volume
(B) energy per unit volume
(C) force
(D) energy

29. Which one of the following has the dimensions $ML^{-1}T^{-2}$?
(A) torque (B) surface tension
(C) viscosity (D) stress

30. The dimensions of the quantity $\frac{L}{RCV}$ are
(A) $M^0L^0T^1A^1$ (B) $M^0L^0T^{-1}A^{-1}$
(C) $M^0L^0T^0A^1$ (D) $M^0L^0T^0A^{-1}$

31. For $10^{(at+3)}$, the dimension of a is
(A) $M^0L^0T^0$ (B) $M^0L^0T^1$
(C) $M^0L^0T^{-1}$ (D) none of these

32. The pressure of 10^6 dyne/cm^2 is equivalent to
(A) 10^5 N/m^2 (B) 10^6 N/m^2
(C) 10^7 N/m^2 (D) 10^8 N/m^2

33. The SI unit of length is metre. Suppose we adopt a new unit of length which equals to x metres. The area 1 m^2 expressed in terms of the new unit has a magnitude
(A) x (B) x^2
(C) $\frac{1}{x}$ (D) $\frac{1}{x^2}$

34. Given $\rho = 2$ g/cm^3, convert it into the MKS system
(A) $2 \times 10^{-3}\ \frac{kg}{m^3}$ (B) $2 \times 10^{3}\ \frac{kg}{m^3}$
(C) $4 \times 10^{3}\ \frac{kg}{m^3}$ (D) $2 \times 10^{6}\ \frac{kg}{m^3}$

35. Given that v is the speed, r is the radius and g is the acceleration due to gravity, which of the following is dimensionless?
(A) $\frac{v^2g}{r}$ (B) v^2rg
(C) vr^2g (D) $\frac{v^2}{rg}$

36. Choose the correct statement(s):
(A) A dimensionally correct equation must be correct.
(B) A dimensionally correct equation may be correct.
(C) A dimensionally incorrect equation may be correct.
(D) A dimensionally incorrect equation may be incorrect.

37. The value of $G = 6.67 \times 10^{-11}\ N{\cdot}m^2(kg)^{-2}$. Its numerical value in CGS system will be
(A) 6.67×10^{-8} (B) 6.67×10^{-6}
(C) 6.67 (D) 6.67×10^{-5}

38. The density of mercury is 13,600 kg·m^{-3}. Its value in the CGS system will be
(A) 13.6 g cm^{-3} (B) 1360 g cm^{-3}
(C) 136 g cm^{-3} (D) 1.36 g cm^{-3}

Basic Mathematics

39. The radii of two circles are r and $4r$. What will be the ratio of their area and perimeter.

40. The internal radius of a ball is 3 cm and the external radius is 4 cm. What will be the volume of the material used.

JEE Advanced

1. Which of the following sets cannot enter into the list of fundamental quantities in any system of units?
(A) Length, mass and velocity
(B) Length, time and velocity
(C) Mass, time and velocity
(D) Length, time and mass

2. Which of the following is not the unit of time?
(A) Solar day (B) Parallactic second
(C) Leap year (D) Lunar month

3. The unit of impulse is the same as that of
(A) moment force
(B) linear momentum
(C) rate of change of linear momentum
(D) force

4. Which of the following is not the unit of energy?
(A) watt-hour (B) electron-volt
(C) N × m (D) kg × m/sec^2

5. If the unit of length is micrometre and the unit of time is microsecond, the unit of velocity will be
(A) 100 m/s (B) 10 m/s
(C) micrometres (D) m/s

6. A dimensionless quantity
(A) never has a unit (B) always has a unit
(C) may have a unit (D) does not exist

7. A unitless quantity
(A) never has a non-zero dimension
(B) always has a non-zero dimension
(C) may have a non-zero dimension
(D) does not exist

8. If a and b are two physical quantities having different dimensions, then which of the following can denote a new physical quantity
(A) $a+b$ (B) $a-b$
(C) a/b (D) $e^{a/b}$
(E) $\sin(a/b)$

9. Two physical quantities whose dimensions are not the same cannot be
(A) multiplied with each other
(B) divided
(C) added or subtracted in the same expression
(D) added together

10. Choose the correct statement(s):
(A) All quantities may be represented dimensionally in terms of the base quantities.
(B) A base quantity cannot be represented dimensionally in terms of the rest of the base quantities.
(C) The dimension of a base quantity in other base quantities is always zero.
(D) The dimension of a derived quantity is never zero in any base quantity.

11. Planck's constant has the dimensions of
(A) force (B) energy
(C) linear momentum (D) angular momentum

12. The velocity v (in cm/s) of a particle is given in terms of time t (in s) by the equation

$$v = at + \frac{b}{t+c}$$

The dimensions of a, b and c are

	a	b	c		a	b	c
(A)	L^2	T	LT^2	(B)	LT^2	LT	L
(C)	LT^{-2}	L	T	(D)	L	LT	T^2

13. The position of a particle at time t is given by the relation

$$x(t) = \frac{V_0}{\alpha}[1 - e^{-\alpha t}]$$

where V_0 is a constant and $\alpha > 0$. The dimensions of V_0 and α are, respectively.
(A) $M^0L^1T^0$ and T^{-1} (B) $M^0L^1T^0$ and T^{-2}
(C) $M^0L^1T^{-1}$ and T^{-1} (D) $M^0L^1T^{-1}$ and T^{-2}

14. The time dependence of a physical quantity is given as

$$P = P_0 \exp(-\alpha t^2)$$

where α is a constant and t is time
The constant α
(A) will be dimensionless
(B) will have the dimensions of T^{-2}
(C) will have the dimensions as that of P
(D) will have the dimension equal to the dimension of P multiplied by T^{-2}

15. Force F is given in terms of time t and distance x by

$$F = A \sin Ct + B \cos Dx$$

Then the dimensions of $\frac{A}{B}$ and $\frac{C}{D}$ are given by

(A) MLT^{-2}, $M^0L^0T^{-1}$ (B) MLT^{-2}, $M^0L^{-1}T^0$
(C) $M^0L^0T^0$, $M^0L^1T^{-1}$ (D) $M^0L^1T^{-1}$, $M^0L^0T^0$

16. The Van der Waal equation for 1 mole of a real gas is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

where P is the pressure, V is the volume, T is the absolute temperature, R is the molar gas constant and a, b are Van dar Waal constants. The dimensions of a are the same as those of
(A) PV (B) PV^2
(C) P^2V (D) P/V

17. In question 16, the dimensions of b are the same as those of
(A) P (B) V
(C) PV (D) nRT

18. In question 16, the dimensions of nRT are the same as those of
(A) energy (B) force
(C) pressure (D) specific heat

19. In question 16, the dimensional formula for ab is
(A) ML^2T^{-2} (B) ML^4T^{-2}
(C) ML^6T^{-2} (D) ML^8T^{-2}

20. Which pair of the following quantities has dimensions different from each other?
(A) Impulse and linear momentum
(B) Plank's constant and angular momentum
(C) Moment of inertia and moment of force
(D) Young's modulus and pressure

21. Force (F) is given by $F = Pt^{-1} + \alpha t$, where t is the time. The unit of P is the same as that of
(A) velocity (B) displacement
(C) acceleration (D) momentum

22. The product of energy and time is called action. The dimensional formula for action is the same as that of
(A) power (B) angular energy
(C) force × velocity (D) impulse × distance

23. When a wave traverses a medium, the displacement of a particle located at x at time t is given by

$$y = a \sin(bt - cx)$$

where a, b and c are constants of the wave. The dimensions of b are the same as those of
(A) wave velocity (B) amplitude
(C) wavelength (D) wave frequency

24. In the above question dimensions of $\frac{b}{c}$ are the same as those of
(A) wave velocity (B) wavelength
(C) wave amplitude (D) wave frequency

25. What is the physical quantity whose dimensions are ML^2T^{-2}?
(A) kinetic energy (B) pressure
(C) momentum (D) power

26. If force, acceleration and time are taken as fundamental quantities, then the dimensions of length will be
(A) FT^2 (B) $F^{-1}A^2T^{-1}$
(C) FA^2T (D) AT^2

27. The dimensions $ML^{-1}T^{-2}$ can correspond to
(A) moment of a force or torque
(B) surface tension
(C) pressure
(D) co-efficient of viscosity

(useful relation are $\vec{\tau} = \vec{r} \times \vec{F}$, $S = F/l$, $F = 6phrv$, where symbols have usual meaning)

28. Which of the following can be a set of fundamental quantities?
(A) length, velocity, time
(B) momentum, mass, velocity
(C) force, mass, velocity
(D) momentum, time, frequency

29. If area (A) velocity (v) and density (ρ) are base units, then the dimensional formula of force can be represented as
(A) $Av\rho$ (B) $Av^2\rho$
(C) $Av\rho^2$ (D) $A^2v\rho$

30. In a certain system of units, 1 unit of time is 5 s, 1 unit of mass is 20 kg and unit of length is 10 m. In this system, one unit of power will correspond to
(A) 16 watts (B) 1/16 watts
(C) 25 watts (D) none of these

31. In a book, the answer for a particular question is expressed as $b = \frac{ma}{k}\left[\sqrt{1+\frac{2kl}{ma}}\right]$, here m represents mass, a represents accelerations and l represents length. The unit of b should be
(A) m/s (B) m/s^2
(C) metre (D) /sec

32. $\alpha = \frac{F}{V^2}\sin(\beta t)$ (here V = velocity, F = force, t = time): Find the dimensions of α and β.
(A) $\alpha = [M^1L^1T^0]$, $\beta = [T^{-1}]$
(B) $\alpha = [M^1L^1T^{-1}]$, $\beta = [T^1]$
(C) $\alpha = [M^1L^1T^{-1}]$, $\beta = [T^{-1}]$
(D) $\alpha = [M^1L^{-1}T^0]$, $\beta = [T^{-1}]$

33. If E, M, J and G denote energy, mass, angular momentum and gravitational constant, respectively, then $\frac{EJ^2}{M^5G^2}$ has the dimensions of
(A) length (B) angle
(C) mass (D) time

34. The dimensions $ML^{-1}T^{-2}$ may correspond to
(A) work done by a force (B) linear momentum
(C) pressure (D) energy per unit volume

35. The velocity of water waves may depend on their wavelength λ, the density of water ρ and the acceleration due to gravity g. The method of dimensions gives the relation between these quantities as
(A) $v^2 = k\lambda^{-1}g^{-1}\rho^{-1}$
(B) $v^2 = kg\lambda$
(C) $v^2 = kg\lambda\,\rho$
(D) $v^2 = k\lambda^3g^{-1}\rho^{-1}$, where k is a dimensionless constant

36. If the unit of force is 1 kilo newton, the length is 1 km and time is 100 second, what will be the unit of mass?
(A) 1000 kg (B) 10 kg
(C) 10,000 kg (D) 100 kg

37. A body moving through air at a high speed v experiences a retarding force F given by $F = KA\,d\,v^x$, where A is the surface area of the body, d is the density of air and K is a numerical constant. The value of x is
(A) 1 (B) 2
(C) 3 (D) 4

38. The velocity of a freely falling body changes as g^ph^q, where g is the acceleration due to gravity and h is the height. The values of p and q are
(A) $1, \frac{1}{2}$ (B) $\frac{1}{2}, \frac{1}{2}$
(C) $\frac{1}{2}, 1$ (D) 1, 1

39. If the acceleration due to gravity is 10 ms^{-2} and the units of length and time are changed to kilometre and hour, respectively, the numerical value of the acceleration is
(A) 360,000 (B) 72,000
(C) 36,000 (D) 129,600

40. If c, the velocity of light, g, the acceleration due to gravity, and P, the atmospheric pressure are fundamental units, then the dimensions of length will be
(A) c/g (B) $P \times c \times g$
(C) c/P (D) c^2/g

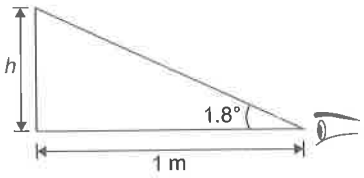
41. The units of length, velocity and force are doubled. Which of the following is the correct change in the other units?
(A) Unit of time is doubled
(B) Unit of mass is doubled
(C) Unit of momentum is doubled
(D) Unit of energy is doubled

42. If the units of force and that of length are doubled, the unit of energy will be
(A) 1/4 times (B) 1/2 times
(C) 2 times (D) 4 times

43. If the units of M, L are doubled, then the unit of kinetic energy will become
(A) 2 times (B) 4 times
(C) 8 times (D) 16 times

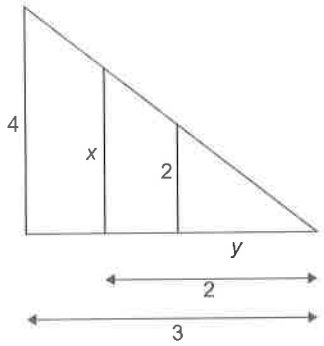
44. Binomial
(A) $(99)^{1/2}$ (B) $(120)^{1/2}$
(C) $(126)^{1/3}$

45. A normal human eye can see an object making an angle of 1.8° at the eye. What is the approximate height of object which can be seen by an eye placed at a distance of 1 m from the eye?



46. Draw the graph for the following equations:
(i) $v = v_0 - at$ (ii) $x = 4t - 3$
(iii) $x = 4at^2$ (iv) $v = -gt$

47.



Find x and y.

48. Which of the following is not the unit of length?
(A) Micron (B) Light year
(C) Angstrom (D) Radian

49. A particle is in a unidirectional potential field where the potential energy (U) of the particle depends on the x-coordinate given by $U_x = k(1 - \cos ax)$ and k and a are constants. Find the physical dimensions of a and k.

50. The time period (T) of a spring mass system depends upon mass (m), spring constant (k) and length of the spring (l) $\left[k = \frac{\text{Force}}{\text{length}}\right]$. Find the relation among (T), (m), (l) and (k) using the dimensional method.

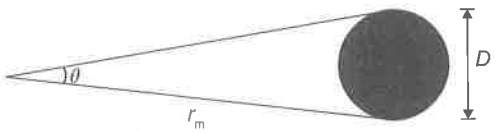
51. The equation of state for a real gas at high temperature is given by $P = \frac{nRT}{V-b} - \frac{a}{T^{1/2}V(V+b)}$, where n, P, V and T are the number of moles, pressure, volume and temperature, respectively, and R is the universal gas constant. Find the dimensions of the constant a in the above equation.

52. The distance moved by a particle in time t from the centre of a ring under the influence of its gravity is given by $x = a \sin \omega t$, where a and ω are constants. If ω is found to depend on the radius of the ring (r), its mass (m) and universal gravitational constant (G), find using dimensional analysis an expression for ω in terms of r, m and G.

53. If the velocity of light c, gravitational constant G and Plank's constant h be chosen as fundamental units, find the dimension of mass, length and time in the new system.

54. A satellite is orbiting around a planet. Its orbital velocity (v_0) is found to depend upon
(A) radius of orbit (R)
(B) mass of planet (M)
(C) universal gravitation constant (G)
Using dimensional analysis, find an expression relating orbital velocity (v_0) to the above physical quantities.

55. The angle subtended by the moon's diameter at a point on the earth is about 0.50°. Use this and the fact that the moon is about 384,000 km away to find the approximate diameter of the moon.



(A) 192,000 km (B) 3350 km
(C) 1600 km (D) 1920 km

56. Use the approximation $(1 + x)^n \approx 1 + nx$, $|x| \ll 1$, to find the approximate value for
(A) $\sqrt{99}$ and (B) $\frac{1}{1.10}$.

57. Use the small-angle approximations to find approximate values for
(A) sin 8° and (B) tan 5°.

Previous Year Questions

JEE Main

1. Identify the pair whose dimensions are equal. **[AIEEE-2002]**
(A) Torque and work (B) Stress and energy
(C) Force and stress (D) Force and work

2. The physical quantities not having the same dimensions are **[AIEEE-2003]**
(A) torque and work
(B) momentum and Planck's constant
(C) stress and Young's modulus
(D) speed and $(\mu_0\varepsilon_0)^{-1/2}$

3. The dimensions of $\frac{1}{\mu_0\varepsilon_0}$, where symbols have their usual meaning, are **[AIEEE-2003]**
(A) $[L^{-1}T]$ (B) $[L^2T^2]$
(C) $[L^2T^{-2}]$ (D) $[LT^{-1}]$

4. Which one of the following represents the correct dimensions of the coefficient of viscosity? **[AIEEE-2004]**
(A) $[ML^{-1}T^{-2}]$ (B) $[MLT^{-1}]$
(C) $[ML^{-1}T^{-1}]$ (D) $[ML^{-2}T^{-2}]$

5. Out of the following pairs, which one does not have identical dimensions? **[AIEEE-2005]**
(A) Angular momentum and Planck's constant
(B) Impulse and momentum
(C) Moment of inertia and moment of a force
(D) Work and torque

6. The 'rad' is the correct unit used to report the measurement of **[AIEEE-2006]**
(A) the ability of a beam of gamma ray photons to produce ions in a target
(B) the energy delivered by radiation to a target
(C) the biological effect of radiation
(D) the rate of decay of a radioactive source

7. Which of the following units denotes the dimensions $[ML^2/Q^2]$, where Q denotes the electric charges? **[AIEEE-2006]**
(A) Wb/m^2 (B) henry (H)
(C) H/m^2 (D) weber (Wb)

8. A body of mass $m = 3.513$ kg is moving along the x-axis with a speed of 5.00 ms^{-1}. The magnitude of its momentum is recorded as **[AIEEE-2008]**
(A) 17.6 kg ms^{-1} (B) 17.565 kg ms^{-1}
(C) 17.56 kg ms^{-1} (D) 17.57 kg ms^{-1}

9. The dimensions of magnetic field in M, L, T and C (coulomb) is given as **[AIEEE-2008]**
(A) $[MLT^{-2}C^{-1}]$ (B) $[MT^2C^{-2}]$
(C) $[MT^{-1}C^{-1}]$ (D) $[MT^2C^{-1}]$

10. In an experiment, the angles are required to be measured using an instrument. 29 divisions of the main scale exactly coincide with the 30 divisions of the Vernier scale. If the smallest division of the main scale is half-a-degree (= 0.5°), then the least count of the instrument is **[AIEEE-2009]**
(A) one minute (B) half minute
(C) one degree (D) half degree

11. The respective number of significant figures for the numbers 23.023, 0.0003 and 21×10^{-3} are **[AIEEE-2010]**
(A) 5, 1, 2 (B) 5, 1, 5
(C) 5, 5, 2 (D) 4, 4, 2

JEE Advanced

Note: (*) means not for speed batch

1*. The pairs of physical quantities that have the same dimensions are
(A) Reynolds number and coefficient of friction
(B) latent heat and gravitational potential
(C) Curie and frequency of light wave
(D) Planck's constant and torque **[JEE-1995' 2/100]**

2*. In the formula $X = 3YZ^2$, X and Z have dimensions of capacitance and magnetic induction, respectively. What are the dimensions of Y in $MKSQ$ system? **[JEE-1995' 2/100]**
(A) $[M^{-3}L^{-1}T^3Q^4]$ (B) $[M^{-3}L^{-2}T^4Q^4]$
(C) $[M^{-2}L^{-2}T^4Q^4]$ (D) $[M^0L^{-2}T^4Q^1]$

3*. Which of the following pairs have the same dimensions?
(A) Torque and work
(B) Angular momentum and work
(C) Energy and young's modulus
(D) Light year and wavelength **[JEE-1996' 2/100]**

27

4*. The SI unit of inductance, the henry can be written as **[JEE-1998' 2/200]**

(A) weber/ampere
(B) volt-second/ampere
(C) joule/(ampere)^2
(D) ohm-second

5*. Let $[\varepsilon_0]$ denote the dimensional formula of the permittivity of the vacuum and $[\mu_0]$ that of the permeability of the vacuum. If M = mass, L = length, T = time and I = electric current **[JEE-1998' 2/200]**

(A) $[\varepsilon_0] = M^{-1}L^{-3}T^2I$
(B) $[\varepsilon_0] = M^{-1}L^{-3}T^4I^2$
(C) $[\mu_0] = MLT^{-2}I^{-2}$
(D) $[\mu_0] = ML^2T^{-1}I$

6*. The dimension of $\left(\frac{1}{2}\right)\varepsilon_0E^2$ (ε_0: permittivity of free space, E: electric field) is **[JEE Sc-2000' 2/200]**

(A) MLT^{-1}
(B) ML^2T^{-2}
(C) $ML^{-1}T^{-2}$
(D) ML^2T^{-1}

7*. A quantity X is given by $\varepsilon_0L\frac{\Delta V}{\Delta t}$, where ε_0 is the permittivity of free space, L is length, ΔV is potential difference and Δt is the time interval. The dimensional formula for X is the same as that of **[JEE Sc.-2000'3/105]**

(A) resistance
(B) charge
(C) voltage
(D) current

8*. Pressure depends on distance as, $P = \frac{\alpha}{\beta}\exp\left(-\frac{\alpha z}{k\beta}\right)$, where α, β are constants, z is distance, k is Boltzmann's constant and θ is temperature. The dimension of β is **[JEE-2004s '3/84]**

(A) $M^0L^0T^0$
(B) $M^{-1}L^{-1}T^{-1}$
(C) $M^0L^2T^0$
(D) $M^{-1}L^1T^2$

9*. Which of the following set has different dimensions? **[JEE-2005s '3/60]**

(A) Pressure, Young's modulus, stress
(B) EMF, potential difference, electric potential
(C) Heat, work done, energy
(D) Dipole moment, electric flux, electric field

10. To find the distance d over which a signal can be seen clearly in foggy conditions, a railways engineer uses dimensional analysis and assumes that the distance depends on the mass density ρ of the fog, intensity (power/area) S of the light from the signal and its frequency f. The engineer finds that d is proportional to $S^{1/n}$. The value of n is **[JEE Main-2014]**

ANSWER KEYS

Exercises

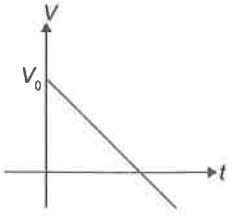
JEE Main

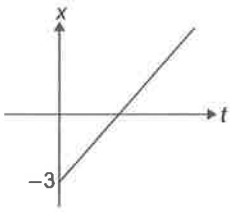
1. A	**2.** C	**3.** C	**4.** B	**5.** C	**6.** C	**7.** C	**8.** B	**9.** B	**10.** D
11. A	**12.** D	**13.** D	**14.** A	**15.** D	**16.** D	**17.** C	**18.** D	**19.** C	**20.** C
21. B	**22.** B	**23.** A	**24.** D	**25.** B	**26.** B	**27.** B	**28.** B	**29.** D	**30.** D
31. C	**32.** A	**33.** D	**34.** B	**35.** D	**36.** ABD	**37.** A	**38.** A	**39.** $\frac{1}{16}, \frac{1}{4}$	

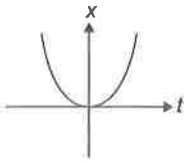
40. $\frac{148\pi}{3}$

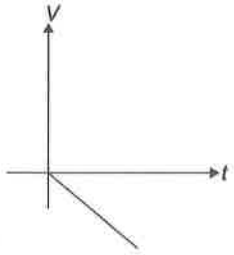
JEE Advanced

1. B	**2.** B	**3.** B	**4.** D	**5.** D	**6.** C	**7.** A	**8.** C	**9.** C	**10.** ABC
11. D	**12.** C	**13.** C	**14.** B	**15.** C	**16.** B	**17.** B	**18.** A	**19.** D	**20.** C
21. D	**22.** D	**23.** D	**24.** A	**25.** A	**26.** D	**27.** C	**28.** C	**29.** B	**30.** A
31. C	**32.** D	**33.** B	**34.** CD	**35.** B	**36.** C	**37.** B	**38.** B	**39.** D	**40.** D
41. C	**42.** D	**43.** C	**44.** (A) 9.9498 (B) 10.954 (C) 5.0132				**45.** π cm		

46. (i) 

(ii) 

(iii) 

(iv) 

47. $x = \frac{8}{3}, y = \frac{3}{2}$ **48.** D **49.** L^{-1}, ML^2T^{-2} **50.** $T = a\sqrt{\frac{m}{k}}$ **51.** $ML^5T^{-2}K^{1/2}$

52. $\omega = K\sqrt{\frac{Gm}{r^3}}$ **53.** $[M] = [h^{1/2}.c^{1/2}.G^{-1/2}]$; $[L] = [h^{1/2}.c^{-3/2}.G^{1/2}]$; $[T] = [h^{1/2}.c^{-5/2}.G^{1/2}]$ **54.** $v_0 = K\sqrt{\frac{Gm}{R}}$

55. B **56.** (a) 9.95, (b) 0.99 **57.** 0.14, 0.09

Previous Year Questions

JEE Main

1. A **2.** B **3.** C **4.** C **5.** C **6.** C **7.** B **8.** A **9.** C **10.** A
11. A

JEE Advanced

1. A, B, C **2.** B **3.** A, D **4.** A, B, C, D **5.** B, C **6.** C **7.** D **8.** C **9.** D **10.** 3

CHAPTER 2

Vector and Calculus

SCALAR

In physics, we deal with two types of physical quantities: one is scalar and the other is vector. Each scalar quantity has a magnitude and a unit.

For example, mass = 4 kg

Magnitude of mass = 4

and unit of mass = kg

Examples of scalar quantities: mass, speed, distance, etc.

Scalar quantities can be added, subtracted and multiplied following simple laws of algebra.

VECTOR

Vectors are physical quantities that have magnitude as well as a specified direction.

For example:

Speed = 4 m/s (is a scalar)

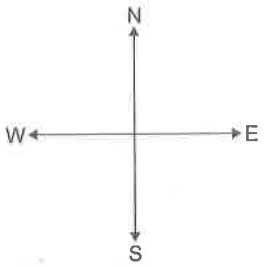
Velocity = 4 m/s towards north (is a vector)

If someone wants to reach some location, then it is not sufficient to provide information about the distance of that location, it is also essential to tell them about the proper direction from the initial location to the destination.

The magnitude of a vector $(\vec{A})$ is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.

Examples of a vector quantity: displacement, velocity, acceleration, force, etc.

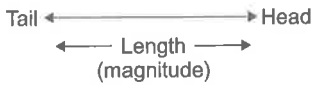
Knowledge of Direction

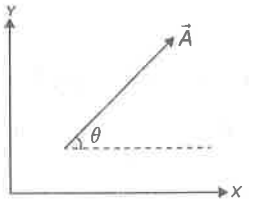


GENERAL POINTS REGARDING VECTORS

Representation of a Vector

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as





Mathematically, vector is represented by $\vec{A}$.

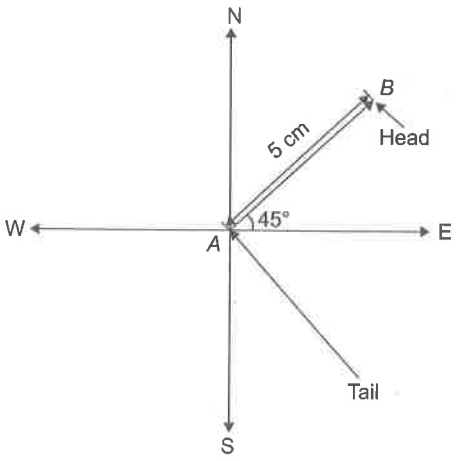
Sometimes, it is represented by a bold letter $\boldsymbol{A}$.

Thus, the arrow in the above figure represents a vector $\vec{A}$ in the xy-plane making an angle θ with the x-axis.

A representation of vector will be complete if it gives us direction and magnitude.

Symbolic form: $\vec{v}, \vec{a}, \vec{F}, \vec{s}$ used to separate a vector quantity from scalar quantities (u, i, m).

Graphical form: A vector is represented by a directed straight line, having the magnitude and direction of the quantity represented by it.



1 cm ≡ 1 N

For example, if we want to represent a force of 5 N acting 45° N of E

1. We choose direction co-ordinates.
2. We choose a convenient scale like 1 cm ≡ 1 N
3. We draw a line of length equal in magnitude and in the direction of vector to the chosen quantity.
4. We put arrow in the direction of vector.

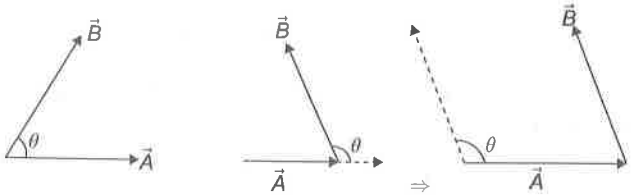
$$\overrightarrow{AB}$$

Magnitude of a vector:

$$|\overrightarrow{AB}| = 5\text{N}$$

Angle Between Two Vectors (θ)

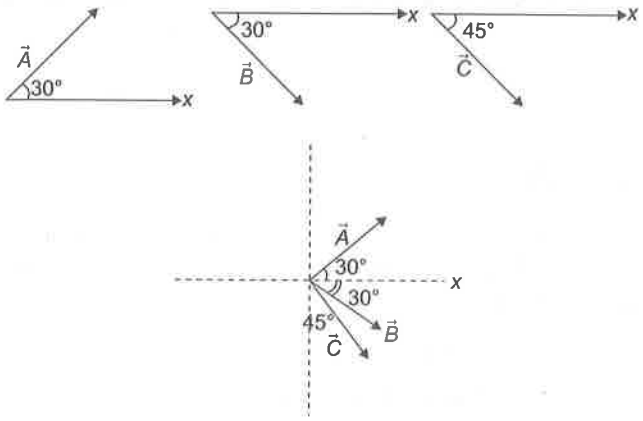
Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e., $0 \leq \theta \leq \pi$).



SOLVED EXAMPLE

EXAMPLE 1

Three vectors $\vec{A}, \vec{B}, \vec{C}$ are shown in the figure. Find the angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$ and (iii) $\vec{A}$ and $\vec{C}$.



SOLUTION

To find the angle between two vectors, we connect the tails of the two vectors. We can shift $\vec{B}$ and $\vec{C}$ such that the tails of $\vec{A}, \vec{B}$ and $\vec{C}$ are connected as shown in the figure.

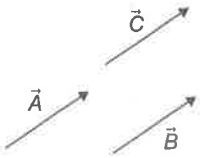
Now we can easily observe that the angle between $\vec{A}$ and $\vec{B}$ is 60°, $\vec{B}$ and $\vec{C}$ is 15° and between $\vec{A}$ and $\vec{C}$ is 75°. ■

Negative of a Vector

It implies that the vectors are of the same magnitude but are opposite in direction.

Equality of Vectors

Vectors having equal magnitude and same direction are called equal vectors.



If $|\vec{A}| = |\vec{B}| = |\vec{C}|$

and $\hat{A} = \hat{B} = \hat{C}$

then $\vec{A} = \vec{B} = \vec{C}$

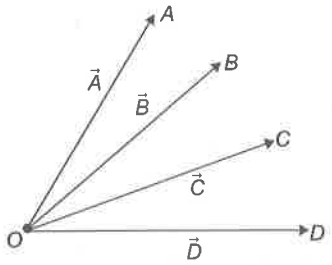
Collinear Vectors

If any two vectors are co-linear, then one can be expressed in terms of the other.

$\vec{a} = \lambda \vec{b}$ (where λ is a constant).

Co-initial Vector

If two or more vectors start from the same point, then they are called co-initial vectors.



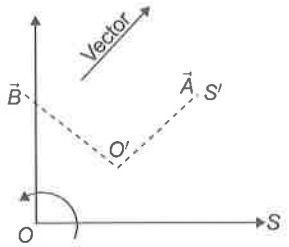
For example, here A, B, C and D are co-initial vectors.

Coplanar Vectors

Three (or more) vectors are called coplanar vectors if they lie in the same plane or are parallel to the same plane. Two (free) vectors are always coplanar.

Important Points

- If the frame of reference is translated or rotated, the vector does not change (though its components may change).



- Two vectors are called equal if their magnitudes and directions are same, and they represent values of the same physical quantity.

Multiplication and Division of a Vector by a Scalar

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $(\vec{B} = \lambda \vec{A})$ whose magnitude become λ times but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times $|\vec{A}|$.

The division of vector $\vec{A}$ by a non-zero scalar m is defined as multiplication of $\vec{A}$ by $\frac{1}{m}$.

Here, $\vec{A}$ and $\vec{B}$ are co-linear vectors.

SOLVED EXAMPLE

EXAMPLE 2

A physical quantity ($m = 3$ kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

1. $\vec{a} = 3$ m/s^2 Eastwards
2. $\vec{a} = -4$ m/s^2 Northwards

SOLUTION

1. $\vec{F} = m\vec{a} = 3 \times 3$ ms^{-2} Eastwards
   $= 9$ N Eastwards
2. $\vec{F} = m\vec{a} = 3 \times (-4)$ N Northwards
   $= -12$ N Northwards
   $= 12$ N Southwards ■

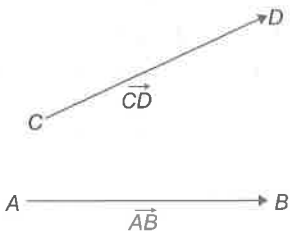
LAWS OF ADDITION AND SUBTRACTION OF VECTORS

Triangle Rule of Addition

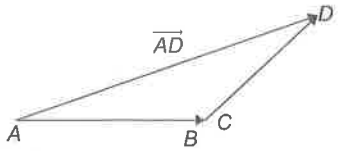
The steps for addition of two vectors representing the same physical quantity by triangle law are given below.

1. Place the vectors such that the tail of one vector coincides with the head of the other.
2. Join the tail of the first vector to the head of the other vector by a line with an arrow at the head of the second.
3. This new vector is the sum of two vectors (also called a resultant).

(i)



(ii)



(iii) $\overrightarrow{AB}+\overrightarrow{CD}=\overrightarrow{AD}$

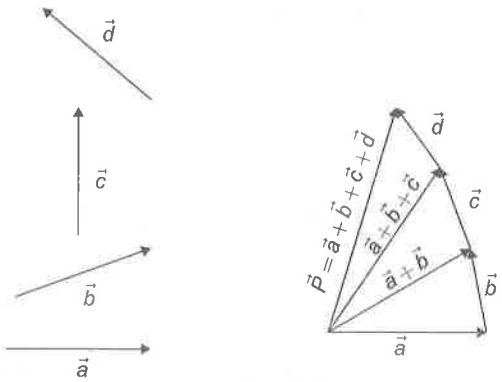
Consider the following example.

Q. A boy moves 4 m south and then 5 m in direction 37° E of N. Find the resultant displacement.

Polygon Law of Addition

This law is used for adding more than two vectors. This is an extension of the triangle law of addition. We keep on arranging vectors such that the tail of the next vector lies on the head of the former.

When we connect the tail of the first vector to the head of the last, we get the resultant of all the vectors.



Note

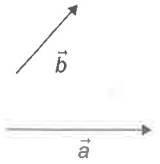
$\vec{P}=\left(\left(\vec{a}+\vec{b}\right)+\vec{c}\right)+\vec{d}=\left(\left(\vec{c}+\vec{a}+\vec{d}\right)\right)+\vec{d}$

[Associative Law]

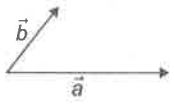
Parallelogram Law of Addition

Steps:

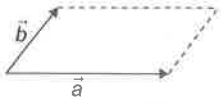
1. Place the two vectors such that their tails coincide.
2. Draw parallel vectors to both these considering both these as sides of a parallelogram.
3. Then the diagonal drawn from the point where the tails coincide represents the sum of the two vectors, with its tail at the point of coincidence of the two vectors.



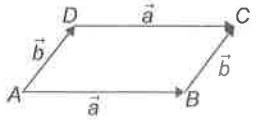
(i)



(ii)



(iii)

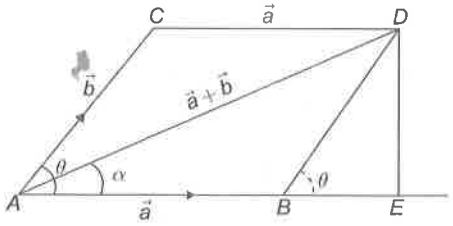


$\overrightarrow{AC}=\vec{a}+\vec{b}$

Note

- $\overrightarrow{AC}=\vec{a}+\vec{b}$ and $\overrightarrow{AC}=\vec{b}+\vec{a}$. Thus, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ [Commutative Law].
- The angle between two vectors is the angle between their positive directions.

Suppose the angle between these two vectors is θ, and $|\vec{a}|=a, |\vec{b}|=b$.



$$(AD)^2=(AE)^2+(DE)^2$$
$$=(AB+BE)^2+(DE)^2$$
$$=(a+b\cos\theta)^2+(b\sin\theta)^2$$
$$=a^2+b^2\cos^2\theta+2ab\cos\theta+b^2\sin^2\theta$$
$$=a^2+b^2+2ab\cos\theta$$

Thus, $AD=\sqrt{a^2+b^2+2ab\cos\theta}$

or $|\vec{a}+\vec{b}|=\sqrt{a^2+b^2+2ab\cos\theta}$

Angle α with vector a is

$$\tan\alpha = \frac{DE}{AE} = \frac{b\sin\theta}{(a+b\cos\theta)}$$

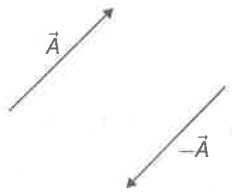
Important Points

- To a vector, only a vector of the same type can be added that represents the same physical quantity and the resultant is also a vector of the same type.
- As $R = [A^2 + B^2 + 2AB\cos\theta]^{1/2}$, so R will be maximum when, $\cos\theta = \max = 1$,
  i.e., $\theta = 0°$, i.e., vectors are like or parallel and $R_{max} = A + B$.
- $|\vec{A}| = |\vec{B}|$ and the angle between them is θ, then $R = 2A\cos\theta/2$.
- $|\vec{A}| = |\vec{B}|$ and the angle between them is $\pi - \theta$, then $R = 2A\sin\theta/2$.
- The resultant will be minimum if, $\cos\theta = \min = -1$, i.e., $\theta = 180°$, i.e., vectors are anti-parallel and $R_{min} = A - B$.
- If the vectors A and B are orthogonal, i.e., $\theta = 90°$, $R = \sqrt{A^2 + B^2}$.
- As previously mentioned, the resultant of two vectors can have any value from $(A - B)$ to $(A + B)$ depending on the angle between them, and the magnitude of the resultant decreases as θ increases from 0° to 180°.
- The minimum number of unequal coplanar vectors whose sum can be zero is three.
- The resultant of three non-coplanar vectors can never be zero, or the minimum number of non-coplanar vectors whose sum can be zero is four.

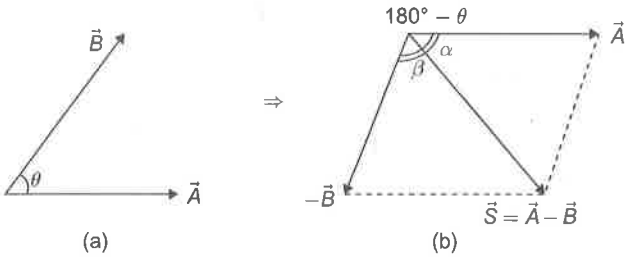
SUBTRACTION OF VECTORS

Negative of a vector say $-\vec{A}$ is a vector of the same magnitude as vector $\vec{A}$, but points in a direction opposite to that of $\vec{A}$.

Thus, $\vec{A} - \vec{B}$ can be written as $\vec{A} + (-\vec{B})$ or $\vec{A} - \vec{B}$ is really the vector addition of $\vec{A}$ and $-\vec{B}$.



Suppose the angle between two vectors $\vec{A}$ and $\vec{B}$ is θ. Then, the angle between $\vec{A}$ and $-\vec{B}$ will be $180° - \theta$ as shown in the figure.



Magnitude of $\vec{S} = \vec{A} - \vec{B}$ will be thus given by

$$S = |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos(180° - \theta)}$$

or $$S = \sqrt{A^2 + B^2 - 2AB\cos\theta} \quad \text{(i)}$$

For the direction of $\vec{S}$, we will either calculate angle α or β, where

$$\tan\alpha = \frac{B\sin(180° - \theta)}{A + B\cos(180° - \theta)} = \frac{B\sin\theta}{A - B\cos\theta} \quad \text{(ii)}$$

or $$\tan\beta = \frac{A\sin(180° - \theta)}{B + A\cos(180° - \theta)} = \frac{A\sin\theta}{B - A\cos\theta} \quad \text{(iii)}$$

SOLVED EXAMPLES

EXAMPLE 3

Two vectors of 10 units and 5 units make an angle of 120° with each other. Find the magnitude and angle of the resultant with the vector of 10 unit magnitude.

SOLUTION

$$|\vec{a} + \vec{b}| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$
$$= \sqrt{100 + 25 + 2\times10\times5(-1/2)} = 5\sqrt{3}$$

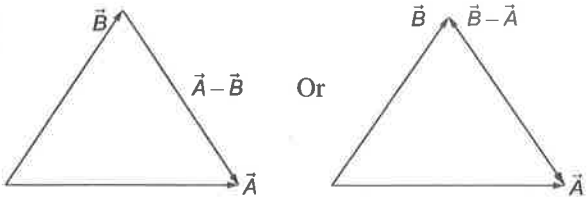
$$\tan\alpha = \frac{5\sin120°}{10 + 5\cos120°} = \frac{5\sqrt{3}}{20 - 5}$$
$$= \frac{5\sqrt{3}}{5\times3} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad \alpha = 30°$

[Here it is seen that the angle between both the vectors = 120° and not 60°.] ■

Note

$\vec{A}-\vec{B}$ or $\vec{B}-\vec{A}$ can also be found by making triangles as shown in figure (a) and (b).



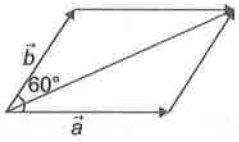
(a) (b)

EXAMPLE 4

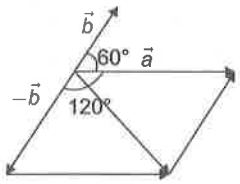
Two vectors of equal magnitude 2 are at an angle of 60° to each other. Find the magnitude of their sum and difference.

SOLUTION

$$|\vec{a}+\vec{b}| = \sqrt{2^2+2^2+2\times2\times2\cos60^\circ}$$
$$= \sqrt{4+4+4} = 2\sqrt{3}$$



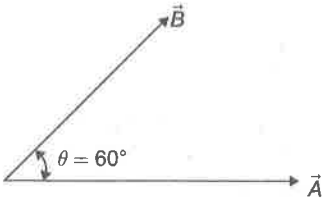
$$|\vec{a}-\vec{b}| = \sqrt{2^2+2^2+2\times2\times2\cos120^\circ}$$
$$= \sqrt{4+4-4} = 2$$



■

EXAMPLE 5

Find $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ in the figure, given $A = 4$ units and $B = 3$ units.

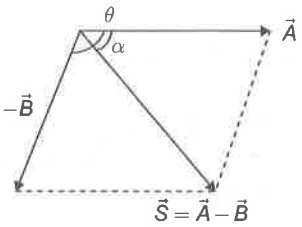


SOLUTION

Addition

$$R = \sqrt{A^2+B^2+2AB\cos\theta}$$
$$= \sqrt{16+9+2\times4\times3\cos60^\circ} = \sqrt{37} \text{ units}$$
$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \frac{3\sin60^\circ}{4+3\cos60^\circ}$$
$$= 0.472$$
$$\therefore \quad \alpha = \tan^{-1}(0.472) = 25.3^\circ$$

Thus, the resultant of $\vec{A}$ and $\vec{B}$ is $\sqrt{37}$ units at an angle 25.3° from $\vec{A}$ in the direction shown in the figure.



Subtraction

$$S = \sqrt{A^2+B^2-2AB\cos\theta}$$
$$= \sqrt{16+9-2\times4\times3\cos60^\circ}$$
$$= \sqrt{13} \text{ units}$$

and $$\tan\theta = \frac{B\sin}{A-B\cos\theta}$$
$$= \frac{3\sin60^\circ}{4-3\cos60^\circ} = 1.04$$
$$\therefore \quad \alpha = \tan^{-1}(1.04) = 46.1^\circ$$

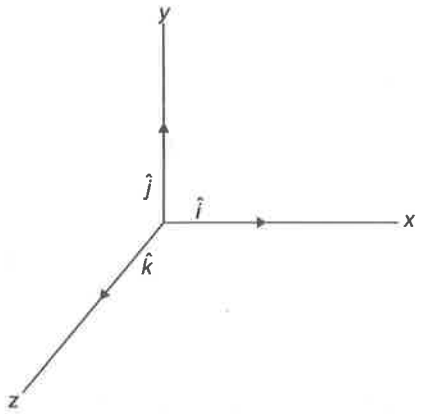
Thus, $\vec{A}-\vec{B}$ is $\sqrt{13}$ units at 46.1° from $\vec{A}$ in the direction as shown in the figure. ■

UNIT VECTOR AND ZERO VECTOR

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector $(\vec{A})$ can be written as the product of the unit vector $(\hat{A})$ in that direction and the magnitude of the given vector.

$$\vec{A} = A\hat{A} \text{ or } \hat{A} = \frac{\vec{A}}{A}$$

A unit vector has no dimensions and unit. Unit vectors along the positive x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, respectively, such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$.



A vector of zero magnitude is called a **zero or a null vector**. Its direction is arbitrary.

SOLVED EXAMPLE

EXAMPLE 6

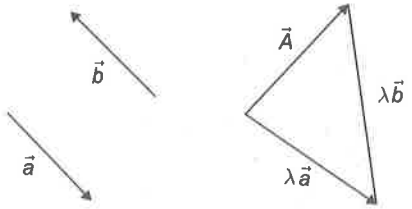
A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts westwards. Represent the force in terms of $\hat{i}$.

SOLUTION

$$\vec{F} = -10^5 \hat{i} \text{ dynes}$$ ■

RESOLUTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two non-zero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors-one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.



$\vec{A} = \lambda\vec{a} + \mu\vec{b}$, where λ and μ are real numbers

We say that $\vec{A}$ has been resolved into two component vectors, namely

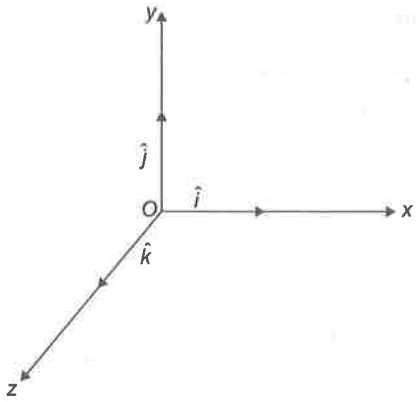
$\vec{A} = \lambda\vec{a} + \mu\vec{b}$, where$\lambda$ and μ are real numbers

We say that $\vec{A}$ has been resolved into two component vectors, namely $\lambda\vec{a}$ and $\mu\vec{b}$.

$\lambda\vec{a}$ and $\mu\vec{b}$ along with $\vec{a}$ and $\vec{b}$, respectively. Hence, one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane.

Resolution Along Rectangular Components

It is convenient to resolve a general vector along the axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x-, y- and z-axes as shown in the figure below:



Resolution in Two Dimensions

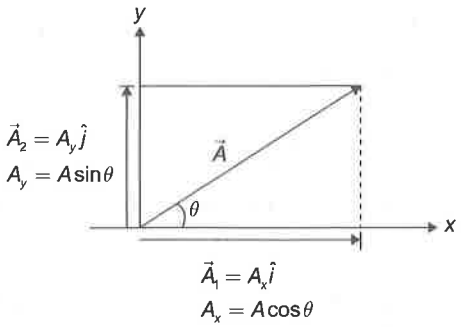
Consider a vector $\vec{A}$ that lies in the xy-plane as shown in the figure

$$\vec{A} = \vec{A_1} + \vec{A_2}$$

$$\vec{A_1} = A_x\hat{i}, \vec{A_2} = A_y\hat{j}$$

$$\Rightarrow \quad \vec{A} = A_x\hat{i} + A_y\hat{j}$$

The quantities A_x and A_y are called the x- and y-components of the vector $\vec{A}$.



A_x is itself not a vector, but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$.

$$A_x = A\cos\theta$$

and $$A_y = A\sin\theta$$

It is clear from the above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways:

(a) its magnitude A and the direction θ it makes with the x-axis or

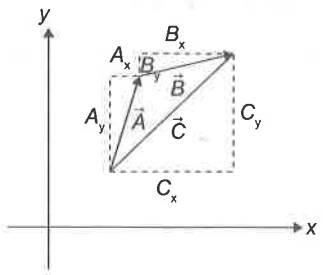
(b) its components A_x and A_y $A = \sqrt{A_x^2 + A_y^2}$,

$\theta = \tan^{-1}\dfrac{A_y}{A_x}$

Note

If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$, i.e.,

Components of a vector perpendicular to itself are always zero. The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in the figure. We saw that



(a)

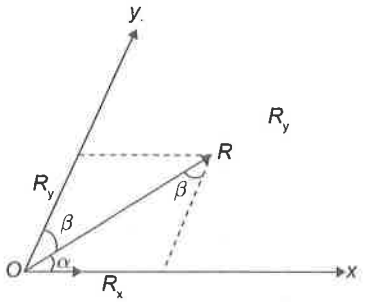
$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both

$C_x = A_x + B_x$

Note (contd)

and $$C_y = A_y + B_y$$

Refer Fig. (b)



(b)

Vector $\vec{R}$ has been resolved in two axes x and y not perpendicular to each other. Applying sine law in the triangle shown, we have

$$\frac{R}{\sin[180° - (\alpha+\beta)]} = \frac{R_x}{\sin\beta} = \frac{R_y}{\sin\alpha}$$

or $$R_x = \frac{R\sin\beta}{\sin(\alpha+\beta)}$$

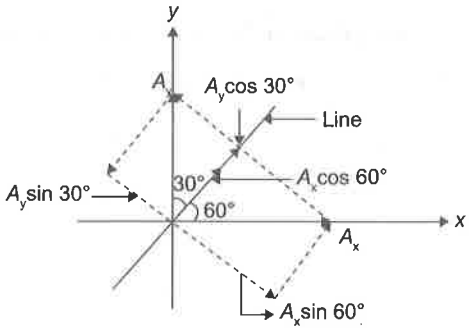
and $$R_y = \frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

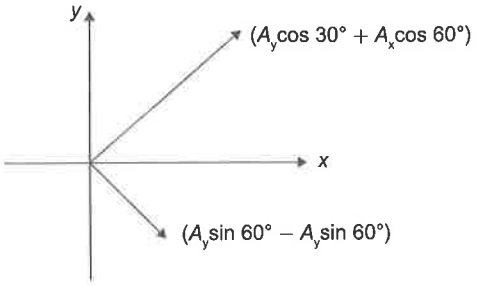
If $\alpha + \beta = 90°$, $R_x = R\sin\beta$ and $R_y = R\sin\alpha$.

SOLVED EXAMPLES

EXAMPLE 7

Resolve the vector $A = A_x\hat{i} + A_y\hat{j}$ along a direction perpendicular to the line, which makes an angle 60° with the x-axis.



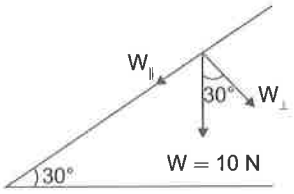


SOLUTION

So the component along line $= |A_y \cos 30° + A_x \cos 60°|$
and perpendicular to the line $= |A_x \sin 60° - A_y \sin 30°|$ ■

EXAMPLE 8

Resolve a weight of 10 N in two directions, which are parallel and perpendicular to a slope inclined at 30° to the horizontal.



SOLUTION

The component perpendicular to the plane

$$W_{\perp} = W\cos 30°$$

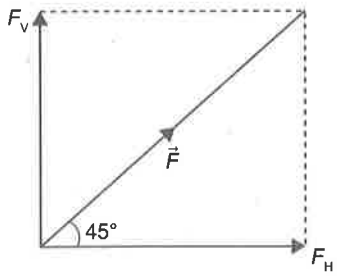
$$= (10)\frac{\sqrt{3}}{2} = 5\sqrt{3}\ \text{N} \qquad \textbf{Ans.}$$

The component parallel to the plane

$$W_{\parallel} = W\sin 30° = (10)\left(\frac{1}{2}\right)$$
$$= 5\ \text{N}$$ ■

EXAMPLE 9

Resolve horizontally and vertically a force $F = 8$ N, which makes an angle of 45° with the horizontal.



SOLUTION

Horizontal component of $\vec{F}$ is

$$F_H = F\cos 45° = (8)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 4\sqrt{2}\ \text{N}$$

The vertical component of $\vec{F}$ is

$$F_v = F\sin 45° = (8)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 4\sqrt{2}\,\text{N} \qquad \textbf{Ans.}$$

■

PROCEDURE TO SOLVE THE VECTOR EQUATION

$$\vec{A} = \vec{B} + \vec{C} \qquad (1)$$

1. There are six variables in this equation, which are as follows:
   - **(a)** Magnitude of $\vec{A}$ and its direction
   - **(b)** Magnitude of $\vec{B}$ and its direction
   - **(c)** Magnitude of $\vec{C}$ and its direction.
2. We can solve this equation if we know the value of the four variables. [Note: two of them must be directions.]
3. If we know the directions of any two vectors, then we will plot them on the same side and the other on a different side.
   For example
   If we know the directions of $\vec{A}$ and $\vec{B}$ and the direction of $\vec{C}$ is unknown, then we form the equation as follows:

$$\vec{C} = \vec{A} - \vec{B}$$

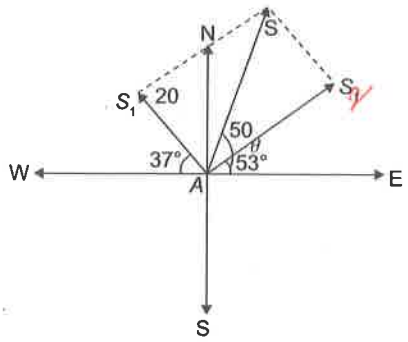
(d) Then, we draw the vector diagram according to the equation and resolve the vectors to determine the unknown values.

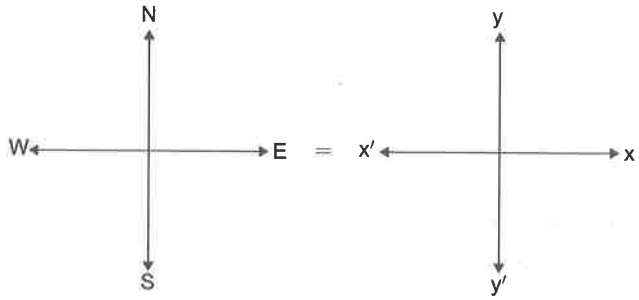
SOLVED EXAMPLES

EXAMPLE 10

Find the net displacement of a particle from its starting point if it undergoes two successive displacements given by $\vec{S}_1 = 20\,\text{m}$, 37° North of West, $\vec{S}_2 = 50\,\text{m}$, 53° North of East.

SOLUTION





$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$S_x = S_{1x} + S_{2x} \qquad S_y = S_{1y} + S_{2y}$$

$$= -20\cos 37° + 50\cos 53° \qquad = 20\sin 37° + 50\sin 53°$$

$$= 14 \qquad = 52$$

$$S = \sqrt{S_x^2 + S_y^2}$$

$$= \sqrt{(14)^2 + (52)^2}$$

$$= 53.85$$

Angle from the west–east axis (x-axis)

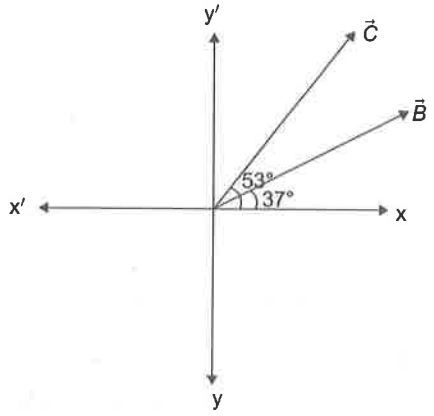
$$\tan\theta = \frac{S_y}{S_x} = \frac{52}{14}$$

$$= \frac{26}{7}$$

$$\theta = \tan^{-1}\left(\frac{26}{7}\right)$$

EXAMPLE 11

Find the magnitude of $\vec{B}$ and the direction of $\vec{A}$. If $\vec{B}$ makes an angle 37° and $\vec{C}$ makes an angle 53° with the x-axis and $\vec{A}$ has a magnitude equal to 10 and $\vec{C}$ has a magnitude of 5 (given $\vec{A}+\vec{B}+\vec{C}=0$).



SOLUTION

$$-\vec{A} = \vec{C} + \vec{B}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

$$\Rightarrow \quad -\vec{A} = -A_x\hat{i} + -A_y\hat{j}$$

$$A_x = -(B\cos 37° + C\cos 53°)$$

$$A_y = -(B\sin 37° + C\cos 53°)$$

$$|\vec{A}|^2 = A_x^2 + A_y^2$$

$$A^2 = \left(B\times\frac{4}{5} + C\times\frac{3}{5}\right)^2 + \left(B\times\frac{3}{5} + C\times\frac{4}{5}\right)^2$$

$$10^2 = \left(\frac{4B}{5} + 3\right)^2 + \left(\frac{3B}{5} + 4\right)^2$$

$$\Rightarrow \quad 100 = \frac{16}{25}B^2 + \frac{9}{25}B^2 + 25 + 2\left(\frac{3\times 4}{5} + \frac{4\times 3}{5}\right)B$$

$$\Rightarrow \quad B^2 + \frac{48}{5}B - 75 = 0$$

$B = 5$ (magnitude cannot be negative) and the angle made by A

$$\Rightarrow \quad A_x = -\left(\frac{20}{5} + 3\right) = -12$$

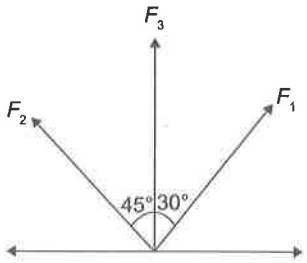
$$A_y = -\left(\frac{15}{5} + 4\right) = -7$$

$$\tan\theta = \frac{A_y}{A_x} = \frac{-7}{-12}$$

$$\theta = 180° + 25° = 205°$$ ■

EXAMPLE 12

Find the magnitude of F_1 and F_2. If F_1, F_2 make angles 30° and 45° with F_3 and magnitude of F_3 is 10 N (given $\vec{F}_1 + \vec{F}_2 = \vec{F}_3$).



SOLUTION

$$|\vec{F}_3| = F_1\cos 30° + F_2\cos 45°$$

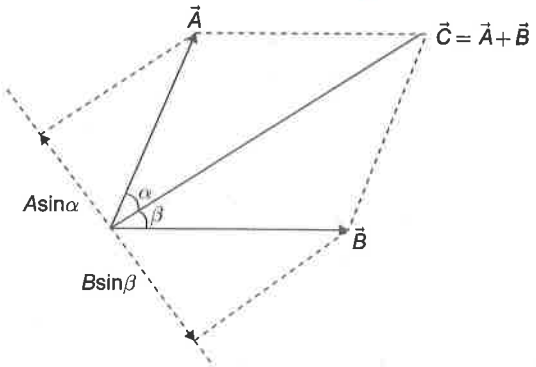
and $$F_2\sin 45° = F_1\sin 30°$$

$\Rightarrow$ $$10 = \frac{\sqrt{3}F_1}{2} + \frac{F_2}{\sqrt{2}}, \frac{F_2}{\sqrt{2}} = \frac{F_1}{2}$$

$\Rightarrow$ $$F_1 = \frac{20}{\sqrt{3}+1}$$

and $$F_2 = \frac{20\sqrt{2}}{\sqrt{3}+1}$$ ■

SHORT METHOD



If there are two vectors $\vec{A}$ and $\vec{B}$ and their resultant makes an angle α with $\vec{A}$ and β with $\vec{B}$.

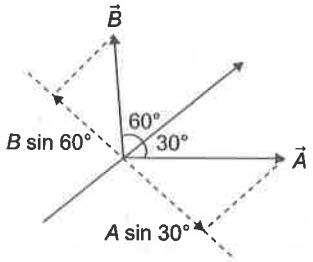
Then $A\sin\alpha = \beta\sin\beta$

Means component of $\vec{A}$ perpendicular to resultant is equal in magnitude to the component of $\vec{B}$ perpendicular to resultant.

SOLVED EXAMPLES

EXAMPLE 13

If two vectors $\vec{A}$ and $\vec{B}$ make angles 30° and 45° with their resultant and $\vec{B}$ has magnitude equal to 10, then find the magnitude of $\vec{A}$.



SOLUTION

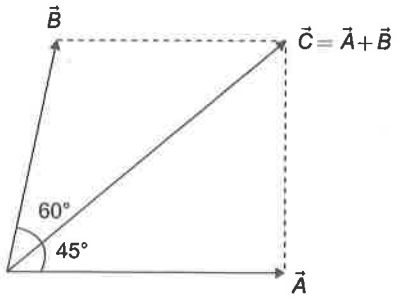
So $$B\sin 60° = A\sin 30°$$

$\Rightarrow$ $$10\sin 60° = A\sin 30°$$

$\Rightarrow$ $$A = 10\sqrt{3}$$ ■

EXAMPLE 14

If the angle between $\vec{A}$ and $\vec{B}$ equals 60° and their resultant make an angle 45° with $\vec{A}$ and $\vec{A}$ has a magnitude equal to 10. Then find the magnitude of $\vec{B}$.



SOLUTION

Here, $\alpha = 45°$ and $\beta = 60° - 45° = 15°$

So $A \sin\alpha = B \sin\beta$

$10 \sin 45° = B \sin 45°$

So $$B = \frac{10}{\sqrt{2}} \sin 15° = \frac{10}{\sqrt{2}} \sqrt{\frac{1-\cos(2\times 15)}{2}}$$

$$= \frac{5}{\sqrt{2}} \sqrt{2-\sqrt{3}}$$ ■

ADDITION AND SUBTRACTION IN COMPONENT FORM

Suppose there are two vectors in the component form. Then the addition and subtraction between these two are as given below:

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$A \pm B = (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}$$

Also if we are having a third vector present in the component form and this vector is added or subtracted from the addition or subtraction of the above two vectors, then

$$C = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

$$A \pm B \pm C = (A_x \pm B_x \pm C_x)\hat{i} + (A_y \pm B_y \pm C_y)\hat{j} + (A_z \pm B_z \pm C_z)\hat{k}$$

Note

Modulus of a vector A is given by

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

SOLVED EXAMPLES

EXAMPLE 15

Obtain the magnitude of $2\vec{A} - 3\vec{B}$ if $\vec{A} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + \hat{k}$.

SOLUTION

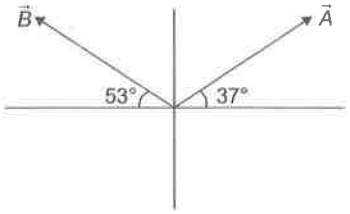
$$2\vec{A} - 3\vec{B} = 2(\hat{i} + \hat{j} - 2\hat{k}) - 3(2\hat{i} - \hat{j} + \hat{k})$$

$\therefore$ Magnitude of $2\vec{A} - 3\vec{B} = \sqrt{(-4)^2 + (5)^2 + (-7)^2}$

$$= \sqrt{16 + 25 + 49} = \sqrt{90}$$ ■

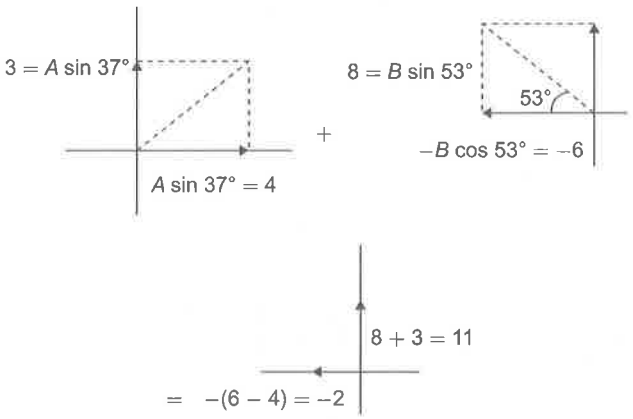
EXAMPLE 16

Find $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$ if $\vec{A}$ makes an angle 37° with the positive x-axis and $\vec{B}$ makes an angle 53° with the negative x-axis as shown and magnitude of $\vec{A}$ is 5 and that of B is 10.



SOLUTION

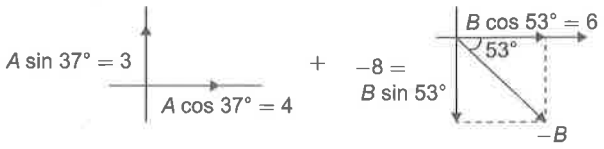
For $\vec{A} + \vec{B}$

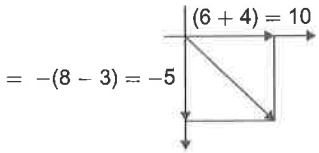


So the magnitude of the resultant will be

$$= \sqrt{11^2 + (-2)^2} = 5\sqrt{5}$$

It will have an angle $\theta = \tan^{-1}\left(\frac{11}{2}\right)$ from the negative x-axis towards up for $\vec{A} - \vec{B}$.





So the magnitude of the resultant will be

$$= \sqrt{10^2 + (-5)^2} = 5\sqrt{5}$$

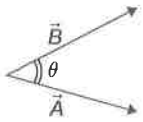
It will have an angle $\theta = \tan^{-1}\left(\frac{5}{10}\right)$ from the positive x-axis towards down. ■

MULTIPLICATION OF VECTORS (THE SCALAR AND VECTOR PRODUCTS)

Scalar product

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of the angle between them. Thus,

$$\vec{A}.\vec{B} = AB\cos\theta$$



(here θ is the angle between the vectors)

Properties:

- It is always a scalar which is positive if the angle between the vectors is acute (i.e., $< 90°$) and negative if the angle between them is obtuse (i.e., $90° < q \leq 180°$).
- It is commutative, i.e.,

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

- It is distributive, i.e.,

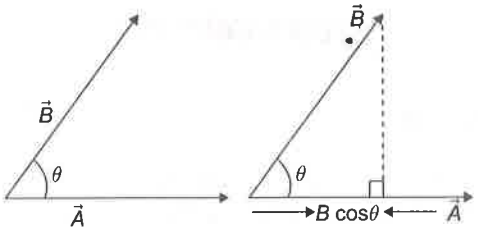
$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

- As by definition $\vec{A}.\vec{B} = AB\cos\theta$. The angle between the vectors

$$\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$$

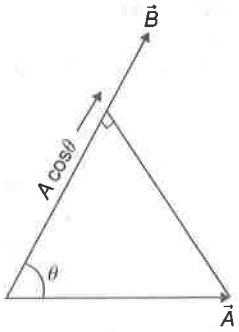
- $\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta)$.

Geometrically, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ and vice versa.



Component of $\vec{B}$ along $\vec{A}$ = $B\cos\theta = \frac{\vec{A}\cdot\vec{B}}{A} = \hat{A}\cdot\vec{B}$ (Projection of $\vec{B}$ on $\vec{A}$).

Component of $\vec{A}$ along $\vec{B}$ = $A\cos\theta = \frac{\vec{A}\cdot\vec{B}}{B} = \vec{A}\cdot\hat{B}$ (Projection of $\vec{A}$ on $\vec{B}$).



- Scalar product of two vectors will be maximum when $\cos\theta = \max = 1$, i.e., $\theta = 0°$, i.e., vectors are parallel

$$\Rightarrow \quad (\vec{A}\cdot\vec{B})_{max} = AB\,.$$

- If the scalar product of two non-zero vectors vanishes, then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by

$$(\vec{A})^2 = \vec{A}\cdot\vec{A} = AA\cos\theta$$
$$= A^2$$

$$\Rightarrow \quad A = \sqrt{\vec{A}\cdot\vec{A}}$$

- In case of unit vector $\hat{n}$,

$$\hat{n}\cdot\hat{n} = 1\times 1\times\cos 0° = 1$$

$$\Rightarrow \quad \hat{n}\cdot\hat{n} = \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j}$$
$$= \hat{k}\cdot\hat{k} = 1$$

In case of orthogonal unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$;

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$$
$$= 0$$

$$\vec{A}\cdot\vec{B} = \left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right).\left(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z\right)$$

$$= [A_xB_x + A_yB_y + A_zB_z]$$

SOLVED EXAMPLES

EXAMPLE 17

If the vectors $\vec{P}=a\hat{i}+a\hat{j}+3\hat{k}$ and $\vec{Q}=a\hat{i}-2\hat{j}-\hat{k}$ are perpendicular to each other, find the value of a.

SOLUTION

If the vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$$\Rightarrow \quad \vec{P}\cdot\vec{Q}=0$$

$$\Rightarrow \quad (a\hat{i}+a\hat{j}+3\hat{k})\cdot(a\hat{i}-2\hat{j}-\hat{k})=0$$

$$\Rightarrow \quad a^2-2a-3=0$$

$$\Rightarrow \quad a^2-3a+a-3=0$$

$$\Rightarrow \quad a(a-3)+1(a-3)=0$$

$$\Rightarrow \quad a=-1,3$$

■

EXAMPLE 18

Find the component of $3\hat{i}+4\hat{j}$ along $\hat{i}+\hat{j}$.

SOLUTION

Component of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}.\vec{B}}{B}$. Hence, the required component

$$=\frac{(3\hat{i}+4\hat{j}).(\hat{i}+\hat{j})}{\sqrt{2}}=\frac{7}{\sqrt{2}}$$

■

EXAMPLE 19

Find the angle between $\vec{A}=3\hat{i}+4\hat{j}$ and $\vec{B}=12\hat{i}+5\hat{j}$.

SOLUTION

We have $\cos\theta=\frac{\vec{A}.\vec{B}}{AB}=\frac{(3\hat{i}+4\hat{j}).(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$$\cos\theta=\frac{36+20}{5\times13}=\frac{56}{65}$$

$$\theta=\cos^{-1}\left(\frac{56}{65}\right)$$

■

EXAMPLE 20

(i) For what value of m the vector $\vec{A}=2\hat{i}+3\hat{j}-6\hat{k}$ is perpendicular to $\vec{B}=3\hat{i}-m\hat{j}+6\hat{k}$.

(ii) Find the component of vector $\vec{A}=2\hat{i}+3\hat{j}$ along the direction of $\hat{i}+\hat{j}$.

SOLUTION

(i) $m=-10$ (ii) $\frac{5}{\sqrt{2}}$

■

Important Note:

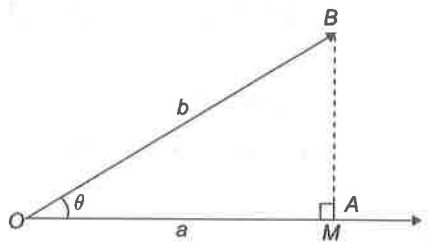
Components of ***b*** along and perpendicular to ***a***.

Let $\overrightarrow{OA}.\overrightarrow{OB}$ represent two (non-zero) given vectors ***a***, ***b***, respectively. Draw BM perpendicular to $\overrightarrow{OA}$.

From ΔOMB, $\overrightarrow{OB}=\overrightarrow{OM}+\overrightarrow{BM}$

$$\Rightarrow \quad b=\overrightarrow{OM}+\overrightarrow{BM}.$$

Thus, $\overrightarrow{OM}$ and $\overrightarrow{BM}$ are components of b along a and perpendicular to a.



Now $\overrightarrow{OM}=(\overrightarrow{OM})\hat{a}=(OB\cos\theta)\hat{a}$

$$=|b|\cos\theta\,\hat{a}=|b|\cdot a\cdot b/|a||b|\cdot\hat{a}$$

$$=a\cdot b/|a|\cdot a/|a|=(a\cdot b)a/|a|^2$$

$$=(a\cdot b)a/a^2$$

$$\overrightarrow{BM}=b-\overrightarrow{OM}$$

$$=b-(a\cdot b/|a|^2)\cdot a$$

Hence, the components of b along a perpendicular to ***a*** are

$(a\cdot b/|a|^2)a$ and $b-(a\cdot b/|a|^2)a$, respectively.

EXAMPLE 21

The velocity of a particle is given by $\vec{v}=3\hat{i}+2\hat{j}+3\hat{k}$. Find the vector component of its velocity parallel to the line $\hat{l}=\hat{i}-\hat{j}+\hat{k}$.

SOLUTION

Component of $\vec{v}$ along $\vec{l}$

$$= v\cos\theta\hat{l} = v\frac{\vec{v}\cdot\vec{l}}{vl}\hat{l}$$

$$= \frac{\vec{v}\cdot\vec{l}}{l^2}\vec{l}$$

$$= \frac{(3\hat{i}+2\hat{j}+3\hat{k})(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|^2}$$

$$= \frac{4}{3}(\hat{i}-\hat{j}+\hat{k})$$ ■

Vector Product

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as

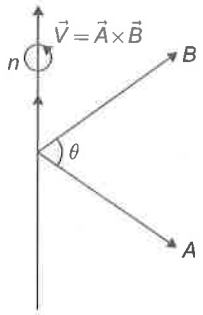
$\vec{A}\times\vec{B}$ (read as $\vec{A}$ cross $\vec{B}$) is defined as

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand thumb rule.

Right-hand Thumb Rule

To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$. The direction of the thumb gives the direction of $\hat{n}$.



Properties

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors, i.e., orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.
- Vector product of two vectors is not commutative,

  i.e., $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$.

  But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$.

- The vector product is distributive when the order of the vectors is strictly maintained,

  i.e., $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$.

- The magnitude of the vector product of two vectors will be maximum when $\sin\theta = \max = 1$, i.e., $\theta = 90°$.

  $$|\vec{A}\times\vec{B}|_{max} = AB$$

- The magnitude of the vector product of two non-zero vectors will be minimum when $|\sin\theta|$ = minimum = 0, i.e., $\theta = 0°$ or $180°$ and $|\vec{A}\times\vec{B}|_{min} = 0$, i.e., if the vector product of two non-zero vectors vanishes, the vectors are collinear.
- The self-cross product, i.e., product of a vector by itself vanishes, i.e. is a null vector.

  $$\vec{A}\times\vec{A} = A\sin 0°\,\hat{n} = \vec{0}.$$

- In case of unit vector $\hat{n}$,

  $$\hat{n}\times\hat{n} = \vec{0}$$

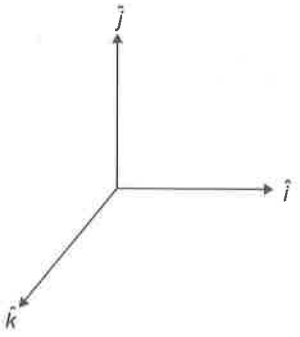
  $$\Rightarrow \quad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

- In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,

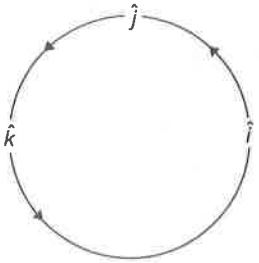
  $$\hat{i}\times\hat{j} = \hat{k}$$

  $$\hat{j}\times\hat{k} = \hat{i}$$

  $$\hat{k}\times\hat{i} = \hat{j}$$



(A)



(B)

- In terms of components, $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$.

$\vec{A}\times\vec{B}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$.

SOLVED EXAMPLES

EXAMPLE 22

$\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A}\times\vec{B}$.

SOLUTION

Applying right-hand thumb rule, we find that $\vec{A}\times\vec{B}$ is along the North. ■

EXAMPLE 23

If $\vec{A}\cdot\vec{B}=|\vec{A}\times\vec{B}|$, find the angle between $\vec{A}$ and $\vec{B}$.

SOLUTION

$$\vec{A}\cdot\vec{B}=|\vec{A}\times\vec{B}|$$

$$AB\cos\theta=AB\sin\theta$$

$$\tan\theta=1$$

$$\Rightarrow\quad \theta=45°.$$ ■

EXAMPLE 24

$\vec{A}\times\vec{B}=AB\sin\theta\,\hat{n}\Rightarrow\hat{n}=\dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$. ■

EXAMPLE 25

Find $\vec{A}\times\vec{B}$ if $\vec{A}=\hat{i}-2\hat{j}+4\hat{k}$ and $\vec{B}=2\hat{i}-\hat{j}+2\hat{k}$.

SOLUTION

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-2&4\\3&-1&2\end{vmatrix}$$

$$=\hat{i}\left(-4-(-4)-\hat{j}(2-12)+\hat{k}(-1-(-6))\right)$$

$$=10\hat{j}+5\hat{k}$$ ■

EXAMPLE 26

(i) $\vec{A}$ is North-East and $\vec{B}$ is downwards. Find the direction of $\vec{A}\times\vec{B}$.

(ii) Find $\vec{B}\times\vec{A}$ if $\vec{A}=3\hat{i}-2\hat{j}+6\hat{k}$ and $\vec{B}=\hat{i}-\hat{j}+\hat{k}$.

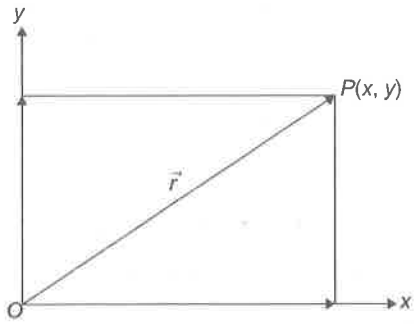
SOLUTION

(i) North-west. **(ii)** $-4\hat{i}-3\hat{j}+\hat{k}$. ■

POSITION VECTOR

Position vector for a point is a vector for which the tail is origin and head is the given point itself.

Position vector of a point defines the position of the point with reference to the origin.

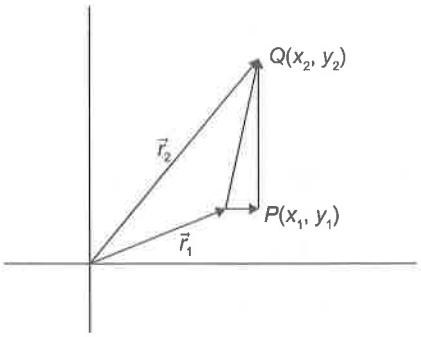


$$\overrightarrow{OP}=\vec{r}$$

$$\vec{r}=x\hat{i}+y\hat{j}$$

DISPLACEMENT VECTOR

Change in position vector of particle is known as displacement vector.



$$\overrightarrow{OP}=\vec{r}_1=x_1\hat{i}+y_1\hat{j}$$

$$\vec{Q}=\vec{r}_2=x_2\hat{i}+y_2\hat{j}$$

$$\overrightarrow{PQ}=\vec{r}_2-\vec{r}_1=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}$$

Thus, we can represent a vector in space starting from (x_1, y_1) and ending at (x_2, y_2) as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$.

CALCULUS

CONSTANTS

They are fixed real numbers, the values of which do not change.

For example, 3, e, a, -1, etc.

VARIABLE

Something that is likely to vary, something that is subject to variation.

or

A quantity that can assume any of a set of values.

Types of variables

1. Independent variable: Independent variable is typically the variable that can be manipulated or changed.
2. Dependent variable: The dependent variable is the object result of the independent variable being manipulated.

For example, $y = x^2$

Here y is the dependent variable and x is the independent variable.

FUNCTION

Function is a rule of relationship between two variables in which one is assumed to be a dependent variable and the other is an independent variable.

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent variable and temperature is the dependent variable.

The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent variable and interest is the dependent variable.

In each case, the value of one variable quantity (dependent variable), which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as $y = f(x)$.

$$\underset{\text{Input}}{\underset{\text{(Domain)}}{x}} \xrightarrow{\quad f \quad} \underset{\text{Output}}{\underset{\text{(Range)}}{f(x)}}$$

All possible values of independent variables (x) are called domain of function.

All possible values of dependent variable (y) are called range of function.

Think of function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usually call the area A and the radius r. Since area depends on radius, we say that A is a function of r, $A = f(r)$. The equation $A = \pi r^2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(x) = \pi r^2$. (Here the rule of relationship which describes the function may be described as square and multiply by π.)

If $r = 1$ $A = \pi$.

If $r = 2$ $A = 4\pi$.

If $r = 3$ $A = 9\pi$.

The set of all possible input values for the radius is called the domain of the function. The set of all output values of the area is the range of the function.

We usually denote functions in one of the two ways:

1. By giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function.
2. By giving a formula such as $f(x) = x^2$ that defines a functions symbol f to name the function.

Strictly speaking, we should call the function f and not $f(x)$.

$y = \sin x$. Here the function is y since x is the independent variable.

SOLVED EXAMPLES

EXAMPLE 27

The volume V of a ball (solid sphere) of radius r is given by the function $V(r) = \frac{4}{3}\pi(r)^3$. Find the volume of a ball of radius 3 m.

SOLUTION

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\,\pi\text{m}^3.$$ ■

EXAMPLE 28

Suppose that the function F is defined for all real numbers r by the formula

$$F(r) = 2(r - 1) + 3.$$

Evaluate F at the input values 0, $2x + 2$ and $F(2)$.

SOLUTION

In each case, we substitute the given input value for r into the formula for F:

$$F(0) = 2(0 - 1) + 3 = -2 + 3 = 1$$

$$F(2) = 2(2 - 1) + 3 = 2 + 3 = 5$$

$$F(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$$

$$F(F(2)) = F(5) = 2(5 - 1) + 3 = 11$$ ■

EXAMPLE 29

A function $f(x)$ is defined as $f(x) = x^2 + 3$. Find (0), $f(1)$, $f(x^2)$, $f(x + 1)$ and $f(f(1))$.

SOLUTION

$$f(0) = 0^2 + 3 = 3$$

$$f(1) = 1^2 + 3 = 4$$

$$f(x^2) = (x^2)^2 + 3 = x^4 + 4$$

$$f(x + 1) = (x + 1)^2 + 3 = x^2 + 2x + 4$$

$$f(f(1)) = f(4) = 4^2 + 3 = 19$$ ■

DIFFERENTIATION

Finite Difference

The finite difference between two values of a physical is represented by Δ notation.

For example

Difference in two values of y is written as Δy as given in the table below.

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

Infinitely Small Difference

The infinitely small difference means very-very small difference. And this difference is represented by d notation instead of 'Δ'.

For example, infinitely small difference in the values of y is written as 'dy'.

If $y_2 = 100$ and $y_1 = 99.9999999999999\ldots$

Then $dy = 0.00000000000000\ldots00001$.

Definition of Differentiation

Another name of differentiation is derivative. Suppose y is a function of x or $y = f(x)$.

Differentiation of y with respect to x is denoted by the symbol $f'(x)$, where $f'(x) = \frac{dy}{dx}$; dx is a very small change in x and dy is a corresponding very small change in y.

Notation: There are many ways to denote the derivative of function $y = f(x)$. The most common notations are as given below:

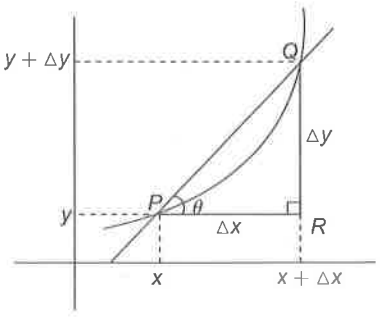
y'	y prime	Nice and brief and does not name the independent variable
$\frac{dy}{dx}$	dy by dx	Names the variable and uses d for derivative
$\frac{df}{dx}$	df by dx	Emphasizes the function's name
$\frac{d}{dx}f(x)$	d by dx of f	Emphasizes the idea that differentiation is an operation performed on f.
$D_x f$	dx of f	A common operator notation
$\dot{y}$	y dot	One of Newton's notations, now common for time derivative, i.e., dy/dt

Average Rates of Change

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $(x, x + \Delta x)$ by dividing the change in value of y, i.e., $\Delta y = f(x + \Delta x) - f(x)$, by the length of the interval Δx over which the change occurred.

The average rate of change of y with respect to x over the interval $[x, x + \Delta x]$

$$= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Geometrically,

$$\frac{\Delta y}{\Delta x} = \frac{QR}{PR} = \tan\theta$$

$$= \text{Slope of the line PQ}$$

In triangle QPR, $\tan\theta = \frac{\Delta y}{\Delta x}$; therefore, we can say that the average rate of change of y with respect to x is equal to the slope of the line joining P and Q.

The Derivative of a Function

We know that the average rate of change of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If the limit of this ratio exists as $\Delta x \to 0$, then it is called the derivative of given function $f(x)$ and is denoted as

$$f'(x) = \frac{dy}{dx}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

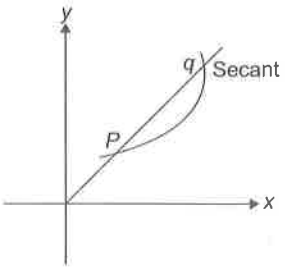
GEOMETRICAL MEANING OF DIFFERENTIATION

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives, we should have knowledge of secant and tangent to a curve.

Secant and Tangent to a Curve

Secant

A secant to a curve is a straight line, which intersects the curve at any two points.



Tangent

A tangent is straight line, which touches the curve a particular point. Tangent is limiting case of secant which intersects the curve at two overlapping points.

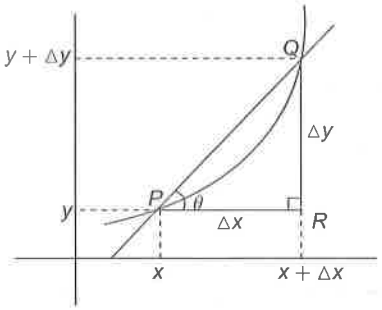


Fig. 1

In the Fig. 1, if the value of Δx is gradually reduced, then the point Q will move nearer to the point P. If the process is continuously repeated (Fig. 2), the value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P.

Therefore, $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = \tan\theta$.

We can say that the differentiation of y with respect to x, i.e., $\left(\frac{dy}{dx}\right)$ is equal to the slope of the tangent at point $P(x, y)$.

or $$\tan\theta = \frac{dy}{dx}$$

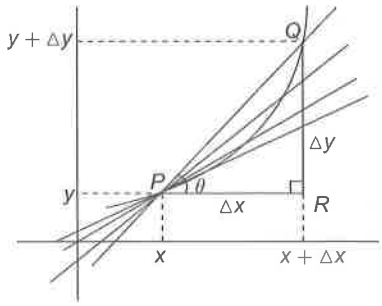


Fig. 2

(From Fig. 1, the average rate of change of y from x to $x + \Delta x$ is identical with the slope of secant PQ.)

Rule No. 1: Derivative of a Constant

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\frac{d}{dx}c = 0$.

SOLVED EXAMPLES

EXAMPLE 30

$$\frac{d}{dx}(8) = 0, \quad \frac{d}{dx}\left(-\frac{1}{2}\right) = 0, \quad \frac{d}{dx}(\sqrt{3}) = 0.$$ ■

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Rule No. 2: Power Rule

If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$.

To apply the power rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 31

1	x	x^2	x^3	x^4	...
f'	1	$2x$	$3x^2$	$4x^3$	...

■

EXAMPLE 32

1. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$
2. $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$ ■

EXAMPLE 33

1. $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

   Function defined for $x \geq 0$ derivative defined only for $x > 0$.
2. $\frac{d}{dx}(x^{1/5}) = \frac{1}{5}x^{-4/5}$

   Function defined for $x \geq 0$ derivative not defined at $x = 0$. ■

Rule No. 3: The Constant Multiple Rule

If u is a differentiable function of x and c is a constant,

then $$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

EXAMPLE 34

The derivative formula

$$\frac{d}{dx}(3x^2) = 3(2x) = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3. ■

EXAMPLE 35

A useful special case

The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\cdot\frac{d}{dx}(u)$$
$$= -\frac{d}{dx}(u)$$ ■

Rule No. 4: The Sum Rule

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable functions in their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx}$$
$$= \frac{du}{dx} - \frac{dv}{dx}$$

The sum rule also extends to sums of more than two functions, as long as there are only finite functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so if $u_1 + u_2 + \cdots + u_n$, then

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$$

EXAMPLE 36

1. $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$
$$= 4x^3 + 12$$

2. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$
$$= 3x^2 + \frac{4}{3}.2x - 5 + 0$$
$$= 3x^2 + \frac{8}{3}x - 5$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in the above example. ■

Rule No. 5: The Product Rule

If u and v are differentiable at x, then if their product uv is considered, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In prime notation

$$(uv)' = uv' + vu'$$

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \text{ while } \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1,$$

which is wrong.

EXAMPLE 37

Find the derivatives of $y = (x^2 + 1)(x^3 + 3)$.

SOLUTION

Using the product rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}(x^2+1)(x^3+3) &= (x^2+1)(3x^2) + (x^3+3)(2x)\\ &= 3x^4 + 3x^2 + 2x^4 + 6x\\ &= 5x^4 + 3x^2 + 6x\end{aligned}$$

Example can be done as well (perhaps better) by multiplying the original expression for y and differentiating the resulting polynomial. We now check:

$$\begin{aligned}y &= (x^2+1)(x^3+3)\\ &= x^5 + x^3 + 3x^2 + 3\\ \frac{dy}{dx} &= 5x^4 + 3x^2 + 6x\end{aligned}$$

This is in agreement with our first calculation.

There are times, however, when the product rule must be used. In the following examples, we have only numerical values to work with. ■

EXAMPLE 38

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$.

SOLUTION

From the product rule, in the form

$$y' = (uv)' = uv' + vu',$$

We have

$$\begin{aligned}y'(2) &= u(2)v'(2) + v(2)u'(2)\\ &= (3)(2) + (1)(-4)\\ &= 6 - 4 = 2\end{aligned}$$ ■

Rule No. 6: The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

EXAMPLE 39

Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$.

SOLUTION

We apply the quotient rule with $u = t^2 - 1$ and $v = t^2 + 1$.

$$\frac{dy}{dt} = \frac{(t^2+1)2t - (t^2-1).2t}{(t^2+1)^2}$$

$$\left[As\ \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\right]$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

Rule No. 7: Derivative of Sine Function

$$\frac{d}{dx}(\sin x) = \cos x$$

EXAMPLE 40

1. $y = x^2 - \sin x$; $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$ Difference rule

2. $y = x^2 \sin x$; $\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + 2x\sin x$ Product rule
$= x^2\cos x + 2x\sin x$

3. $y = \frac{\sin x}{x}$; $\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$ Quotient rule

$$= \frac{x\cos x - \sin x}{x^2}$$

Rule No. 8: Derivative of Cosine Function

$$\frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 41

(a) $y = 5x + \cos x$ Sum rule

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x$$

(b) $y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

Product rule

$$= \sin x(-\sin x) + \cos x\,(\cos x)$$

$$= \cos^2 x - \sin^2 x = \cos 2x$$ ■

Rule No. 9: Derivatives of Other Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}; \quad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}; \quad \operatorname{cosec} x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the quotient rule, are given by the following formulas.

$$\frac{d}{dx}(\tan x) = \sec^2 x; \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x; \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

EXAMPLE 42

Find dy/dx if $y = \tan x$.

SOLUTION

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$ ■

EXAMPLE 43

1. $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \operatorname{cosec}^2 x$

2. $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\operatorname{cosec} x) = 2\frac{d}{dx}(\operatorname{cosec} x)$

$$= 2(-\operatorname{cosec} x \cot x)$$

$$= -2\operatorname{cosec} x \cot x$$ ■

Rule No. 10: Derivative of Logarithm and Exponential Functions

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}, \quad \frac{d}{dx}(e^x) = e^x$$

EXAMPLE 44

$$y = e^x \cdot \log_e(x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) \cdot \log(x) + \frac{d}{dx}[\log_e(x)]e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x \cdot \log_e(x) + \frac{e^x}{x}$$ ■

Rule No. 11: Chain Rule or 'Outside Inside' Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

It sometime helps to think about the chain rule in the following way. If $y = f(g(x))$,

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

In words: To find dy/dx, differentiate the 'outside' function f and leave the 'inside' $g(x)$ alone, then multiply by the derivative of the inside.

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$?

The answer is, with the chain rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The chain rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

EXAMPLE 45

The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

SOLUTION

We have $\frac{dy}{dx} = 6, \frac{dy}{du} = 2, \frac{du}{dx} = 3$

Since $6 = 2 \times 3$ $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Is it an accident that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. ■

EXAMPLE 46

Let us try this again on another function.

$$y = 9x^4 + 6x^2 + 1$$
$$= (3x^2 + 1)^2$$

is the composite $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x = 2(3x^2 + 1) \cdot 6x$$
$$= 36x^3 + 12x$$

and

$$\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1)$$
$$= 36x^3 + 12x$$

Once again, $\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}$.

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. ■

EXAMPLE 47

Find the derivation of $y = \sqrt{x^2 + 1}$.

SOLUTION

Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \text{ and } g'(x) = 2x,$$

The chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$
$$= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2 + 1}} (2x)$$
$$= \frac{x}{\sqrt{x^2 + 1}}.$$ ■

EXAMPLE 48

Outside — Derivative of the outside

$$\frac{d}{dx}\sin(x^2 + x) = \cos(x^2 + x) \cdot (2x + 1)$$

Inside — Inside left along — Derivative of the inside ■

EXAMPLE 49

(a) $\frac{d}{dx}(1 - x^2)^{1/4} = \frac{1}{4}(1 - x^2)^{-3/4}(-2x)$

$u = 1 - x^2$ and $n = 1/4$.

(Function defined) on $[-1, 1]$.

$$= \frac{-x}{2(1 - x^2)^{3/4}}$$ (derivative defined only on $(-1, 1)$)

(b) $\frac{d}{dx}\sin 2x = \cos 2x \frac{d}{dx} 2x$

$$= \cos 2x \cdot 2 = 2 \cos 2x$$

(c) $\frac{d}{dt}(A \sin(\omega t + \phi)) = A \cos(\omega t + \phi) \frac{d}{dt}(\omega t + \phi)$

$$= A \cos(\omega t + \phi) \cdot \omega$$
$$= A\omega \cos(\omega t + \phi).$$ ■

Rule No. 12: Power Chain Rule

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}$$

EXAMPLE 50

$$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$
$$= -1(3x - 2)^{-2} \frac{d}{dx}(3x - 2)$$
$$= -1(3x - 2)^{-2}(3) = -\frac{3}{(3x - 2)^2}$$

In part (*d*), we could also have found the derivation with the quotient rule. ■

EXAMPLE 51

(a) $\frac{d}{dx}(Ax+B)^n$

SOLUTION

Here $u = Ax + B, \frac{du}{dx} = A$

$\therefore \frac{d}{dx}(Ax+B)^n = n(Ax+B)^{n-1} \cdot A$

1. $\frac{d}{dx}\sin(Ax+B) = \cos(Ax+B)\cdot A$
2. $\frac{d}{dx}\log(Ax+B) = \frac{1}{Ax+B}\cdot A$
3. $\frac{d}{dx}\tan(Ax+B) = \sec^2(Ax+B)\cdot A$
4. $\frac{d}{dx}e^{(Ax+B)} = e^{(Ax+B)}\cdot A$ ■

Note

These results are important.

DOUBLE DIFFERENTIATION

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f') = f''$. This new function f'' is called the second derivative of because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D_2 f(x)$.

SOLVED EXAMPLE

EXAMPLE 52

If $f(x) = x\cos x$, find $f''(x)$.

SOLUTION

Using the product rule, we have

$$f'(x) = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)$$

$$= -x\sin x + \cos x.$$

To find $f''(x)$, we differentiate $f'(x)$:

$$f''(x) = \frac{d}{dx}(-x\sin x + \cos x)$$

$$= -x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(-x) + \frac{d}{dx}(\cos x)$$

$$= -x\cos x - \sin x - \sin x = -x\cos x - 2\sin x.$$ ■

APPLICATION OF DERIVATIVE DIFFERENTIATION AS A RATE OF CHANGE

$\frac{dy}{dx}$ is the rate of change of y with respect to x.

For example:

1. $v = \frac{dx}{dt}$. This means velocity v is the rate of change of displacement x with respect to time t.
2. $a = \frac{dv}{dt}$. This means acceleration a is the rate of change of velocity v with respect to time t.
3. $F = \frac{dp}{dt}$. This means force F is the rate of change of momentum p with respect to time t.
4. $\tau = \frac{dL}{dt}$. This means torque τ is the rate of change of angular momentum L with respect to time t.
5. Power $= \frac{dW}{dt}$. This means power P is the rate of change of work W with respect to time t.

SOLVED EXAMPLES

EXAMPLE 53

The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$.

How fast is the area changing with respect to the diameter when the diameter is 10 m?

SOLUTION

The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4}2D = \frac{\pi D}{2}$$

When $D = 10$ m, the area is changing at rate $(\pi/2) = 5\pi$ m^2/m. This means that a small change ΔD m in the diameter would result in a change of about $5\pi\,\Delta D$ m^2 in the area of the circle. ■

PHYSICAL EXAMPLES

EXAMPLE 54

Boyle's law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$. Find the rate of change of volume with respect to pressure.

SOLUTION

$$\frac{dV}{dP} = -\frac{C}{P^2}$$ ■

EXAMPLE 55

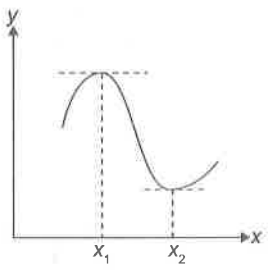
1. Find the average rate of change of the area of a circle with respect to its radius r as r changes from (i) 2 to 3 (ii) 2 to 2.5, and (iii) 2 to 2.1.
2. Find the instantaneous rate of change when $r = 2$.
3. Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically when this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

SOLUTION

(a) (i) 5π (ii) $4.5\,\pi$ (iii) $4.1\,\pi$
(b) $4\,\pi$
(c) $\Delta A \approx 2\,\pi r \Delta r$

MAXIMA AND MINIMA

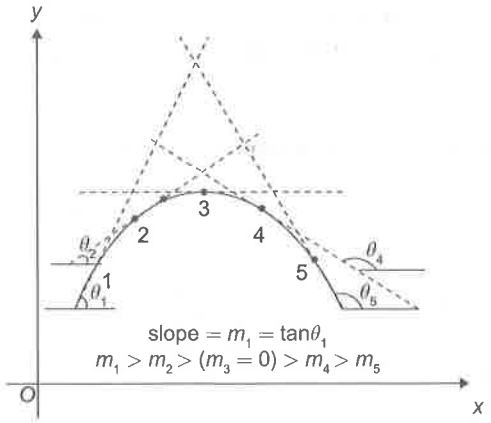
Suppose a quantity y depends on another quantity x in a manner as shown in figure. It becomes maximum at x_1 and minimum at x_2. At these points, the tangent to the curve is parallel to the x-axis and hence its slope is $\tan\theta = 0$. Thus, at a maxima or a minima slope



$$\Rightarrow \qquad \frac{dy}{dx} = 0$$

Maxima

Just before the maximum, the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum, i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at maximum. The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$. Conditions for maxima are (a) $\frac{dy}{dx} = 0$ and (b) $\frac{d^2y}{dx^2} < 0$.

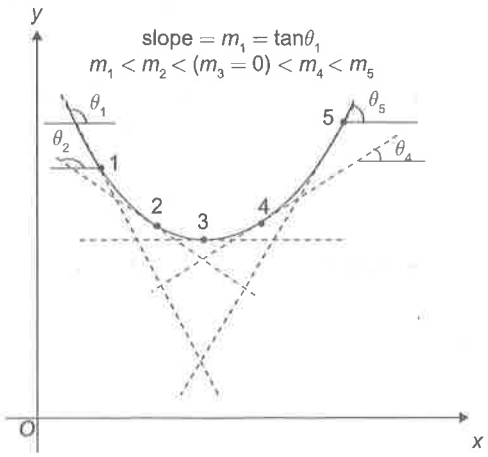


Minima

Similarly, at a minimum, the slope changes from negative to positive,

Hence with an increase in x, the slope is increasing, which means the rate of change of the slope with respect to x is positive.

Hence, $$\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0.$$



Conditions for minima are as follows:

(a) $\frac{dy}{dx}=0$, **(b)** $\frac{d^2y}{dx^2}>0$

Quite often, it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

SOLVED EXAMPLE

EXAMPLE 56

Find the maximum or minimum values of the functions:
(a) $y=25x^2+5-10x$ **(b)** $y=9-(x-3)^2$

SOLUTION

(a) For maximum and minimum value, we can put $\frac{dy}{dx}=0$

or $$\frac{dy}{dx}=50\times-10=0$$

$\therefore$ $$x=\frac{1}{5}.$$

Further, $$\frac{d^2y}{dx^2}=50$$

or $\frac{d^2y}{dx^2}$ has positive value at $x=\frac{1}{5}$. Therefore, y has a minimum value at $x=\frac{1}{5}$. Therefore, y has minimum value at $x=\frac{1}{5}$. Substituting $x=\frac{1}{5}$ in the given equation, we get

$$y_{\min}=25\left(\frac{1}{5}\right)^2+5-10\left(\frac{1}{5}\right)=4$$

(b) $$y=9-(x-3)^2$$
$$=9-x^2-9+6x$$

or $$y=6x-x^2$$

$\therefore$ $$\frac{dy}{dx}=6-2x$$

For minimum or maximum value of y, we will substitute $\frac{dy}{dx}=0$

or $$6-2x=0$$
$$x=3$$

To check whether the value of y is maximum or minimum at $x=3$, we will have to check whether $\frac{d^2y}{dx^2}$ is positive or negative.

$$\frac{d^2y}{dx^2}=-2$$

or $\frac{d^2y}{dx^2}$ is negative at $x=3$. Hence, the value of y is maximum. This maximum value of y is

$$y_{\max}=9-(3-3)^2=9$$ ■

INTEGRATION

Definition

A function $F(x)$ is an anti-derivative of a function $f(x)$ if

$$F'(x)=f(x)$$

for all x in the domain of f. The set of all anti-derivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of integration.

For example, $f(x)=x^3$ then $f'(x)=3x^2$,

So the integral of $3x^2$ is x^3.

Similarly, if $f(x)=x^3+4$.

Therefore, the general integral of $3x^2$ is x^3+c, where c is a constant.

One is the anti-derivative F of a function f, the other anti-derivatives of f differ from F by a constant. We indicate this in integral notation in the following way:

$$\int f(x)\,dx=F(x)+C \qquad \text{(i)}$$

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, 'The indefinite integral of f with respect to x is $F(x)+C$.' When we find $F(x)+C$, we say that we have integrated f and evaluated the integral.

SOLVED EXAMPLES

EXAMPLE 57

Evaluate $\int 2x\,dx$.

$$\int 2x\,dx = x^2 + C$$

an antiderivative of $2x$; the arbitrary constant

SOLUTION

The formula $x^2 + C$ generates all the anti-derivatives of the function $2x$. The function $x^2 + 1$, $x^2 - \pi$ and $x^2 + \sqrt{2}$ are all anti-derivatives of the function $2x$, as you can check by differentiation. Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

Integral Formulas

Indefinite integral	Reversed derivated formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n is rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1\,dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin kx\,dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx\,dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x\,dx = \tan x + C$	$\frac{d}{dx}\tan x = \sec^2 x$
5. $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$
6. $\int \sec x \tan x\,dx = \sec x + C$	$\frac{d}{dx}\sec x = \sec x \tan x$
7. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$	$\frac{d}{dx}(-\csc x) = \operatorname{cosec} x \cot x$

■

EXAMPLE 58

Examples based on the above formulas:

(a) $\int dx = x + c$

(b) $\int x^5 dx = \frac{x^6}{6} + C$ Formula 1 with $n = 5$

(c) $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = 2x^{1/2} + C = 2\sqrt{x} + C$ Formula 1 with $n = -\frac{1}{2}$

(d) $\int \sin 2x\,dx = \frac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(e) $\int \cos\frac{x}{2} dx = \int \cos\frac{1}{2}x\,dx = \frac{\sin(1/2)x}{1/2} + C = \int 2\sin\frac{x}{2} + C$ Formula 3 with $k = \frac{1}{2}$

■

EXAMPLE 59

Right: $\int x\cos x\,dx = x\sin x + \cos x + C$

Reason: The derivative of the right-hand side is the integrand.

Check: $\frac{d}{dx}(x\sin x + \cos x + C) = x\cos x + \sin x - \sin x + 0 = x\cos x.$

Wrong: $\int x\cos x\,dx = x\sin x + C$

Reason: The derivative of the right-hand side is not the integrand.

Check: $\frac{d}{dx}(x\sin x + C) = x\cos x + \sin x + 0 \neq x\cos x$

■

Rule No. 1: Constant Multiple Rule

A function is an anti-derivative of a constant multiple k of a function f if and only if it is k times an anti-derivative of f.

EXAMPLE 60

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int 5x^n dx = 5\int x^n dx = \frac{5(x)^{n+1}}{n+1} + c$$

Rule No. 2: Sum and Difference Rule

A function is an anti-derivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an anti-derivative of f and an anti-derivative of g.

$$\int [f(x) \pm g(x)\,dx] = \int f(x)\,dx \pm \int g(x)\,dx$$ ■

EXAMPLE 61

Term-by-term integration

Evaluate: $\int (x^2 - 2x + 5)\,dx$.

SOLUTION

If we recognize that $(x^3/3) - x^2 + 5x$ is an anti-derivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{an antiderivative}} + \overbrace{C}^{\text{the arbitrary constant}}$$

If we do not recognize the anti-derivative right away, we can generate it term by term with the sum and difference rule:

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x\,dx + \int 5dx$$
$$= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3$$

This formula is more complicated than it needs to be. If we combine C_1, C_2 and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the anti-derivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)\,dx = \int x^2 dx - \int 2x dx + \int 5dx$$
$$= \frac{x^3}{3} - x^2 + 5x + C$$

Find the simplest anti-derivative you can. For each part, add the constant at the end. ■

EXAMPLE 62

We can sometimes use trigonometric identities to transform integrals if we do not know how to evaluate into integrals. The integral formulas for sin 2x and cos 2x arise frequently in applications.

(a) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx \quad \sin^2 x = \frac{1-\cos 2x}{2}$

$$= \frac{1}{2}\int (1 - \cos 2x)\,dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$\frac{x}{2} + \left(-\frac{1}{2}\right)\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $\int \cos^2 x dx = \int \frac{1+\cos 2x}{2}dx \quad \cos^2 x = \frac{1+\cos 2x}{2}$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C,$$

as in part (a), but with a sign change. ■

SOME INDEFINITE INTEGRALS (AN ARBITRARY CONSTANT SHOULD BE ADDED TO EACH OF THESE INTEGRALS)

1. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}$ (provided $n \neq -1$) $+ C$
2. $\int \frac{1}{x}dh = lx + C$
3. $\int \frac{dx}{a+bx} = \frac{1}{b}ln\,(a+bx) + C$
4. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$
5. $\int \sin(ax+b) = \frac{-1}{a}\cos(ax+b) + C$
6. $\int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + C$

SOLVED EXAMPLE

EXAMPLE 63

(a) $\int (3x+2)^3\,dx = \frac{(3x+2)^4}{4\times 3} + C = \frac{(3x+2)^4}{12} + C$

(b) $\int \frac{2dx}{x} = 2\,ln\,x + C$

(c) $\int \frac{dx}{5+2x} = \frac{1}{2}ln(5+2x) + C$

(d) $\int \frac{dx}{3-5x} = -\frac{1}{5}ln(3-5x) + C$

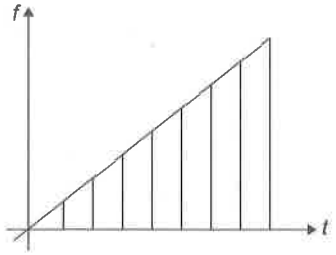
(e) $\int e^{3x}dx = \frac{1}{3}e^{3x} + C$

(f) $\int e^{-x/2}dx = -2e^{-x/2} + C$

EXAMPLE 67

If $F = kt$, then find impulse at $t = 3$ s.

So impulse will be area under f-t curve.

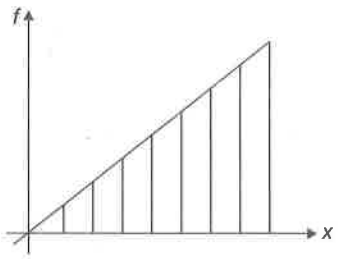


$$I = \int_0^3 kt\,dt = k\left[\frac{t^2}{2}\right]_0^3$$

$$\Rightarrow \quad I = \frac{9k}{2}$$

■

To calculate the work done by force:



$$w = \int \vec{f}.d\vec{x}$$

So area under the f-x curve will give the value of work done.

EXERCISES

JEE Main

Section A: Function

1. $f(x) = \cos x + \sin x$. Find $f(\pi/2)$.
2. $f(x) = 4x + 3$. Find $f(f(2))$.
3. $f(x) = \log x^3$ and $g(x) = \log x$.

   Which of the following statement(s) is/are true?

   (a) $f(x) = g(x)$ (b) $3f(x) = g(x)$

   (c) $f(x) = 3g(x)$ (d) $f(x) = (g(x))^3$

Section B: Differentiation of Elementry Functions

Find the derivative of the given function with respect to the corresponding independent variable:

4. $y = x^2 + x + 8$
5. $s = 5t^2 - 3t^5$
6. $y = 5 \sin x$
7. $y = x^2 + \sin x$
8. $y = \tan x + \cot x$

Find the first derivative and second derivative of the given functions with respect to the corresponding independent variable:

9. $y = 6x^2 - 10x - 5x^{-2}$
10. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$
11. $\omega = 3z^7 - 7z^3 + 12z^2$
12. $y = \sin x + \cos x$
13. $y = ln\, x + e^x$

Section C: Differentiation by Product Rule

Find the derivative of the given functions with respect to the independent variable x:

14. $x \sin x$
15. $y = e^x\, ln\, x$
16. $y = (x - 1)(x^2 + x + 1)$
17. $y = \left(x^2 + 1\right)\left(x + 5 + \frac{1}{x}\right)$
18. $y = \sin x \cos x$

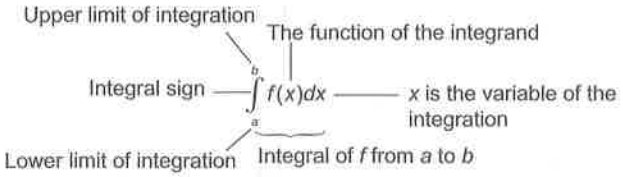
Section D: Differentiation by Quotient Rule

Find the derivative of the given function with respect to the independent variable:

(g) $\int \sin(3x+5)\,dx = -\frac{1}{3}\cos(3x+5)+C$

(h) $\int \cos(2x-5)\,dx = \frac{1}{2}\sin(2x-5)+C$ ■

DEFINITE INTEGRATION OR INTEGRATION WITH LIMITS



$$\int_a^b f(x)\,dx = \left[g(x)\right]_a^b = g(b)-g(a)$$

SOLVED EXAMPLE

EXAMPLE 64

$$\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3[x]_{-1}^{4} = 3\,[4-(-1)]$$
$$= (3)(5) = 15$$
$$\int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0)$$
$$= -0+1 = 1$$ ■

EXAMPLE 65

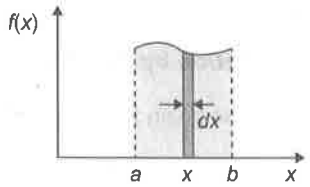
(a) $\int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

(b) $\int_3^5 x dx = \left[\frac{x^2}{2}\right]_3^5 = \frac{5^2-3^2}{2} = 8$

(c) $\int_0^b x^{3/2} dx = \left[\frac{x^{5/2}}{5/2}\right]_0^b = \frac{2}{5}b^{5/2}$ ■

APPLICATION OF DEFINITE INTEGRAL

Calculation of area of a curve



From graph shown in figure if we divide whole area in infinitely small strips of dx width.

We take a strip at x position of dx width.

Small area of this strip $dA = f(x)\,dx$

So, the total area between the curve and x-axis = sum of area of all strips $= \int_a^b f(x)dx$.

Let $f(x) \geq 0$ be continuous on $[a,b]$. The area of the region between the graph of f and the x-axis is

$$A = \int_a^b f(x)dx$$

SOLVED EXAMPLE

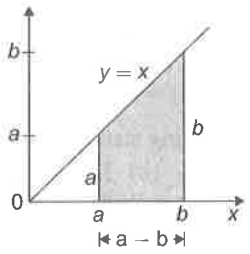
EXAMPLE 66

Using an area to evaluate a definite integral.

Evaluate $\int_a^b x dx \quad 0 < a < b.$

SOLUTION

We sketch the region under the curve $y = x$, $a \leq x \leq b$ (figure) and see that it is a trapezoid with height $(b - a)$ and bases a and b. The value of the integral is the area of this trapezoid:



Thus, $\int_a^b x dx = (b-a)\cdot\frac{a+b}{2} = \frac{b^2}{2} - \frac{a^2}{2}$

$$\int_1^{\sqrt{5}} x dx = \frac{(\sqrt{5})^2}{2} - \frac{(1)^2}{2} = 2$$

and so on.

Notice that $x^2/2$ is an anti-derivative of x, further evidence of a connection between anti-derivatives and summation.

To find impulse

$dF = \frac{dp}{dt}$. Thus, impulse $= \int F \cdot dt$ ■

19. $y = \dfrac{\sin x}{\cos x}$

20. $y = \dfrac{2x+5}{3x-2}$

21. $y = \dfrac{\ell nx}{x}$

22. $f(t) = \dfrac{t^2-1}{t^2+t-2}$, find $f'(t)$.

23. $z = \dfrac{2x+1}{x^2-1}$

24. $y = x^2 \cot x$

Section E: Differentiation by Chain Rule

Find $\dfrac{dy}{dx}$ as a function of x:

25. $y = (2x + 1)^5$

26. $y = (4 - 3x)^9$

27. $y = \left(1 - \dfrac{x}{7}\right)^{-7}$

28. $y = \left(\dfrac{x}{2} - 1\right)^{-10}$

29. $y = \sin 5x$

30. $y = \sin(x) + ln(x^2) + e^{2x}$

31. $y = 2\sin(\omega x + \phi)$, where ω and ϕ are constants.

Section F: Differentiation as a Rate Measurement

32. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

33. Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates $\dfrac{ds}{dt}$ to $\dfrac{dr}{dt}$.

Section G: Maxima and Minima

34. A particle's position as a function of time is given by $x = -t^2 + 4t + 4$. Find the maximum value of the position coordinate of the particle.

35. Find the maximum and minimum values of the function $2x^3 - 15x^2 + 36x + 1$.

Section H

Given $y = f(u)$ and $u = g(x)$. Find $\dfrac{dy}{dx}$ for the following functions:

36. $y = 2u^3, u = 8x - 1$

37. $y = \sin u, u = 3x + 1$

38. $y = 6u - 9, u = (1/2)x^4$

39. $y = \cos u,\ u = -\dfrac{x}{3}$

Part II: Integration

Find the integrals of the given functions:

1. (a) $2x$

   (b) x^2

   (c) $x^2 - 2x + 1$

2. (a) $-3x^{-4}$

   (b) x^{-4}

   (c) $x^{-4} + 2x + 3$

3. (a) $\dfrac{1}{x^2}$

   (b) $\dfrac{5}{x^2}$

   (c) $2 - \dfrac{5}{x^2}$

4. (a) $\dfrac{3}{2}\sqrt{x}$

   (b) $\dfrac{3}{2\sqrt{x}}$

   (c) $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

5. (a) $\dfrac{4}{3}\sqrt[3]{x}$

   (b) $\dfrac{1}{3\sqrt[3]{x}}$

   (c) $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

6. (a) $\frac{1}{2}x^{-1/2}$

   (b) $-\frac{1}{2}x^{-3/2}$

   (c) $-\frac{3}{2}x^{-5/2}$

7. $(1 - x^2 - 3x^5)$

8. $3 \sin x$

9. $\frac{1}{3x}$

Integrate by using the substitution suggested in bracket.

10. $\int \sin 3x dx$ (use, $u = 3x$)

11. $\int \sec 2t \tan 2t\, dt$ (use, $u = 2t$)

12. $\int_{-2}^{1} 5dx$

13. $\int_{-4}^{-1} \frac{\pi}{2} d\theta$

14. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

15. $\int_{\sqrt{2}}^{5\sqrt{2}} r dr$

16. $\int_{0}^{2\pi} \sin\theta d\theta$

17. $\int_{0}^{1} e^x dx$

Use a definite integral to find the area of the region between the given curve and the x-axis in the interval $[0, b]$.

18. $y = 2x$

19. $y = \frac{x}{2} + 1$

Use a definite integral to find the area of the region between the given curve and the x-axis in the interval $[0, \pi]$.

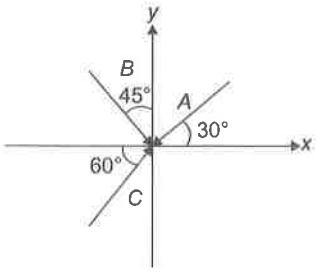
20. $y = \sin x$

Part III: Vector

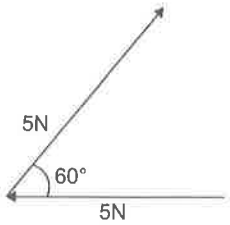
Section A: Definition of a Vector and Angle Between Vectors

1. Vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are shown in the figure. Find the angle between:

   (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{A}$ and $\vec{C}$, (iii) $\vec{B}$ and $\vec{C}$.



2. The forces, each numerically equal to 5 N, are acting as shown in the figure. Find the angle between the forces?



3. Rain is falling vertically downwards with a speed of 5 m/s. If the unit vector along the upward direction is defined as $\hat{j}$, represent the velocity of rain in the vector form.

4. The vector joining the points $A(1, 1, -1)$ and $B(2, -3, 4)$ and pointing from A to B is

   (a) $-\hat{i} + 4\hat{j} - 5\hat{k}$ (b) $\hat{i} + 4\hat{j} + 5\hat{k}$

   (c) $\hat{i} - 4\hat{j} + 5\hat{k}$ (d) $-\hat{i} - 4\hat{j} - 5\hat{k}$

Section B: Addition of Vectors

5. A man walks 40 m North, then 30 m East and then 40 m South. Find the displacement from the starting point.

6. Two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are acting at right angles to each other. Find their resultant.

7. A vector of magnitude 30 and direction eastwards is added with another vector of magnitude 40 and

direction northwards. Find the magnitude and direction of the resultant with the east.

8. Two force of $\overrightarrow{F_1} = 500\,\text{N}$ is due east and $\overrightarrow{F_1} = 250\,\text{N}$ is due north. Find $\overrightarrow{F_2} - \overrightarrow{F_1}$.

9. Two vectors $\vec{a}$ and $\vec{b}$ inclined at an angle θ with respect to each other have a resultant $\vec{c}$, which makes an angle β with $\vec{a}$. If the directions of $\vec{a}$ and $\vec{b}$ are interchanged, then the resultant will have the same
(A) magnitude
(B) direction
(C) magnitude as well as direction
(D) neither magnitude nor direction.

10. Two vectors $\vec{A}$ and $\vec{B}$ lie in a plane. Another vector $\vec{C}$ lies outside this plane. The resultant $\vec{A}+\vec{B}+\vec{C}$ of these three vectors
(A) can be zero
(B) cannot be zero
(C) lies in the plane of $\vec{A}+\vec{B}$
(D) lies in the plane of $\vec{A}-\vec{B}$

11. The vector sum of the forces of 10 N and 6 N can be
(A) 2 N (B) 8 N
(C) 18 N (D) 20 N

12. A set of vectors taken in a given order give a closed polygon. Then the resultant of these vectors is a
(A) scalar quantity (B) pseudo vector
(C) unit vector (D) null vector

13. The vector sum of two forces P and Q is minimum when the angle θ between their positive directions is
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$ (D) π

14. If the vector sum of two vectors $\vec{A}$ and $\vec{B}$ is maximum, then the angle θ between two vectors is
(A) 0° (B) 30°
(C) 45° (D) 60°

15. Given $\vec{C} = \vec{A} + \vec{B}$. Also, the magnitude of $\vec{A}, \vec{B}$ and $\vec{C}$ are 12, 5 and 13 units, respectively. The angle between $\vec{A}$ and $\vec{B}$ is
(A) 0° (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$ (D) π

16. If $\vec{P}+\vec{Q} = \vec{P}-\vec{Q}$ and θ is the angle between $\vec{P}$ and $\vec{Q}$, then
(A) $\theta = 0°$ (B) $\theta = 90°$
(C) $P = 0$ (D) $Q = 0$

17. The sum and difference of two perpendicular vectors of equal lengths are
(A) of equal lengths and have an acute angle between them
(B) of equal lengths and have an obtuse angle between them
(C) also perpendicular to each other and are of different lengths
(D) also perpendicular to each other and are of equal lengths

Section C: Resolution of Vectors

18. Find the magnitude of $3\hat{i} + 2\hat{j} + \hat{k}$.

19. If $\vec{A} = 3\hat{i} + 4\hat{j}$, then find $\hat{A}$.

20. What are the x and the y components of a 25 m displacement at an angle of 210° with the x-axis (clockwise)?

21. One of the rectangular components of a velocity of 60 km h^{-1} is 30 km h^{-1}. Find the other rectangular component.

22. If $0.5\hat{i} + 0.8\hat{j} + C\hat{k}$ is a unit vector, find the value of C.

23. The rectangular components of a vector are (2, 2). The corresponding rectangular components of another vector are $(1, \sqrt{3})$. Find the angle between the two vectors.

24. The x and y components of a force are 2 N and −3 N. The force is
(A) $2\hat{i} - 3\hat{j}$ (B) $2\hat{i} + 3\hat{j}$
(C) $-2\hat{i} - 3\hat{j}$ (D) $3\hat{i} + 2\hat{j}$

Section D: Product of Vectors

25. If $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j}$, find:
(A) $\vec{A} \cdot \vec{B}$, (B) $\vec{A} \times \vec{B}$.

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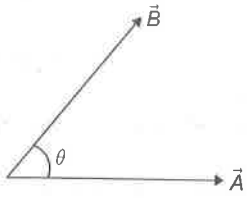
26. If $|\vec{A}| = 4$, $|\vec{B}| = 3$ and $\theta = 60°$ in the figure, find:
(a) $\vec{A} \cdot \vec{B}$, (b) $|\vec{A} \times \vec{B}|$.



27. Three non-zero vectors $\vec{A}, \vec{B}$ and $\vec{C}$ satisfy the relation $\vec{A} \cdot \vec{B} = 0$ & $\vec{A} \cdot \vec{C} = 0$. Then $\vec{A}$ can be parallel to:
(A) $\vec{B}$ (B) $\vec{C}$
(C) $\vec{B} \cdot \vec{C}$ (D) $\vec{B} \times \vec{C}$

28. The magnitude of the scalar product of two vectors is 8 and that of the vector product is $8\sqrt{3}$. The angle between them is
(A) 30° (B) 60°
(C) 120° (D) 150°

JEE Advanced

Section A: Function

1. If $f(x) = \dfrac{x-1}{x+1}$, then find $f\{f(x)\}$.

2. If $f(x) = \begin{cases} x+2, & x < 2 \\ 2x-1, & x \geq 2 \end{cases}$, evaluate $f(2)$, $f(1)$ and $f(3)$.

Section B: Differentiation of Elementry Functions

Find the first derivative and second derivative of the given functions with respect to the independent variable x:

3. $y = \ln x^2 + \sin x$

4. $y = \sqrt[7]{x} + \tan x$

Section C: Differentiation by Product Rule

Find the derivative of the given functions with respect to the corresponding independent variable:

5. $y = e^x \tan x$

6. $y = x^2 \sin^4 x + x \cos^{-2} x$

7. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$

8. $y = x^2 \sin x + 2x \cos x - 2\sin x$

9. $y = x^2 \cos x - 2x \sin x - 2\cos x$

10. $r = (1 + \sec\theta)\sin\theta$

Section D: Differentiation by Quotient Rule

Find the derivative of the given functions with respect to the respective independent variable:

11. $y = \dfrac{\sin x + \cos x}{\cos x}$

12. $y = \dfrac{\cot x}{1 + \cot x}$

13. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

14. $\pi = \dfrac{\tan\theta}{1 + \tan\theta}$

Section E: Differentiation by Chain Rule

Find $\dfrac{dy}{dx}$ as a function of x:

15. $y = \sin^3 x + \sin 3x$

16. $\sin^2(x^2 + 1)$

17. $y = x(x^2 + 1)^{-1/2}$

18. $\theta = \sqrt{2r - r^2}$, find $\dfrac{d\theta}{dr}$.

19. $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$

Section F: Differentiation as a Rate Measurement

20. The radius r and height h of a circular cylinder are related to the cylinder's volume V by the formula $v = \pi r^2 h$.
(a) If the height is increasing at a rate of 5 m/s while radius is constant, find the rate of increase of the volume of the cylinder.
(b) If the radius is increasing at a rate of 5 m/s while height is constant, find the rate of increase of the volume of the cylinder.
(c) If the height is increasing at a rate of 5 m/s and the radius is increasing at a rate of 5 m/s. Find the rate of increase of volume of the cylinder.

Section G: Maxima and Minima

21. Find two positive numbers x and y such that $x + y = 60$ and xy is maximum.

22. A sheet of area 40 m^2 is used to make an open tank with a square base. Find the dimensions of the base such that volume of this tank is maximum.

Section H: Miscellaneous

23. Find y'' if
(a) $y = \cos x$, (b) $y = \sec x$

24. $y = \cos u$, $u = \sin x$

25. $y = \sin u$, $u = x - \cos x$

Part II: Integration

Find the integrals of given functions:

1. $\int (2x^3 - 5x + 7)\,dx$

2. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

3. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

4. $\int x^{-3}(x+1)\,dx$

5. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$

6. $\int \frac{4 + \sqrt{t}}{t^3}\,dt$

7. $\int \cos\theta(\tan\theta + \sec\theta)\,d\theta$

8. $\int_{\pi}^{2\pi} \theta\,d\theta$

9. $\int_{0}^{\sqrt[3]{7}} x^2\,dx$

10. $\int_{0}^{\pi} \cos x\,dx$

11. $\int_{0}^{1} \frac{dx}{3x+2}$

Use a definite integral to find the area of the region between the given curve and the x-axis in the interval $[0, b]$.

12. $y = 3x^2$

13. $y = \sqrt{b^2 - x^2}$

Part III: Vector

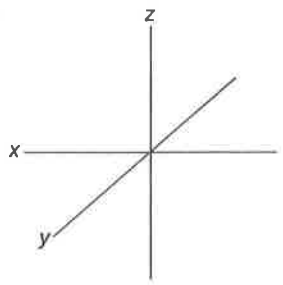
Section A: Definition of a Vector and the Angle Between Vectors

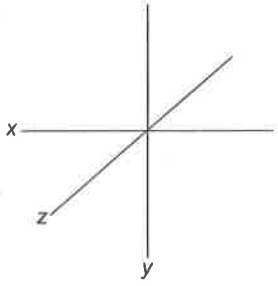
1. Vector $\vec{A}$ points N–E and its magnitude is 3 kg ms^{-1}. It is multiplied by the scalar λ such that $\lambda = -4$ s. Find the direction and magnitude of the new vector quantity. Does it represent the same physical quantity or not?

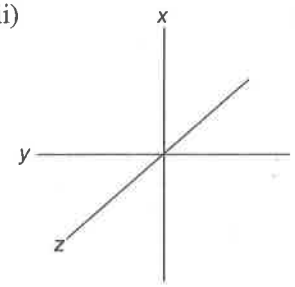
2. A hall has the dimensions 10 m × 12 m × 14 m. A fly starting at one corner ends up at a diametrically opposite corner. The magnitude of its displacement is nearly
(A) 16 m (B) 17 m
(C) 18 m (D) 21 m

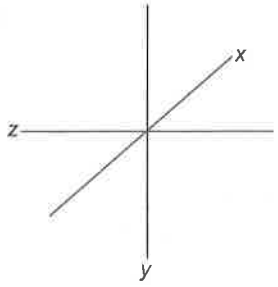
3. A vector is not changed if
(A) it is displaced parallel to itself
(B) it is rotated through an arbitrary angle
(C) it is cross-multiplied by a unit vector
(D) it is multiplied by an arbitrary scalar

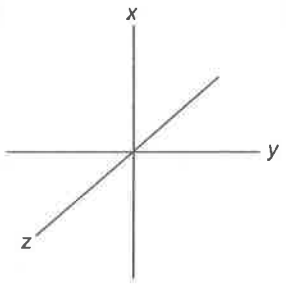
4. Which of the arrangement of axes in the figure can be labelled a 'right-handed coordinate system'? As usual, each axis label indicates the positive side of the axis.

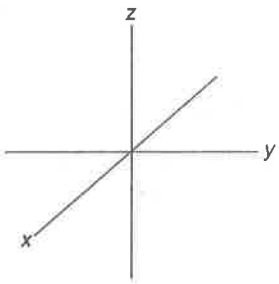
(i) 

(ii) 

(iii) 

(iv) 

(v) 

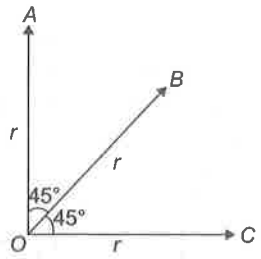
(vi) 

(A) (i), (ii)
(B) (iii) (iv)
(C) (vi)
(D) (v)

Section B: Addition of Vectors

5. The angle θ between the directions of forces $\vec{A}$ and $\vec{B}$ is 90° where $A = 8$ dyne and $B = 6$ dyne. If the resultant $\vec{R}$ makes an angle α with $\vec{A}$, then find the value of 'α'.

6. Find the resultant of three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ each of magnitude r as shown in the figure.



7. If the angle between two forces increases, the magnitude of their resultant
(A) decreases
(B) increases
(C) remains unchanged
(D) first decreases and then increases

8. A car is moving on a straight road due north with a uniform speed of 50 km h^{-1} when it turns left through 90°. If the speed remains unchanged after turning, the change in the velocity of the car in the turning process is
(A) zero
(B) $50\sqrt{2}$ kmh^{-1} S – W direction
(C) $50\sqrt{2}$ kmh^{-1} N – W direction
(D) 50 kmh^{-1} due West

9. Which of the following sets of displacements might be capable of bringing a car to its returning point?
(A) 5, 10, 30 and 50 km
(B) 5, 9, 9 and 16 km
(C) 40, 40, 90 and 200 km
(D) 10, 20, 40 and 90 km

10. When two vectors $\vec{a}$ and $\vec{b}$ are added, the magnitude of the resultant vector is always
(A) greater than $(a + b)$
(B) less than or equal to $(a + b)$
(C) less than $(a + b)$
(D) equal to $(a + b)$

11. Given $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 5\hat{i} - 6\hat{j}$. The magnitude of $\vec{A} + \vec{B}$ is
(A) 4 units
(B) 10 units
(C) $\sqrt{58}$ units
(D) $\sqrt{61}$ units

12. Given $\vec{A} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{B} = -\hat{i} - \hat{j} + \hat{k}$. The unit vector of $\vec{A} - \vec{B}$ is
(A) $\frac{3\hat{i} + \hat{k}}{\sqrt{10}}$
(B) $\frac{3\hat{i}}{\sqrt{10}}$
(C) $\frac{\hat{k}}{\sqrt{10}}$
(D) $\frac{-3\hat{i} - \hat{k}}{\sqrt{10}}$

13. If $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$, then the angle between $\vec{A}$ and $\vec{B}$ is
(A) 0
(B) 60°
(C) 90°
(D) 120°

14. Given $\vec{a}+\vec{b}+\vec{c}=0$. Of the three vectors $\vec{a},\vec{b}$ and $\vec{c}$, two are equal in magnitude. The magnitude of the third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. The angles between the vectors are
(A) 90°, 135°, 135° (B) 30°, 60°, 90°
(C) 45°, 45°, 90° (D) 45°, 60°, 90°

15. Which of the following is a true statement?
(A) A vector cannot be divided by another vector
(B) Angular displacement can either be a scalar or a vector
(C) Since the addition of vectors is commutative, vector subtraction is also commutative
(D) The resultant of two equal forces of magnitude F acting at a point is F if the angle between the two forces is 120°.

Section C: Resolution of Vectors

16. If $\vec{A}=3\hat{i}+4\hat{j}$ and $\vec{B}=\hat{i}+\hat{j}+2\hat{k}$, then find out a unit vector along $\vec{A}+\vec{B}$.

17. Vector $\vec{A}$ is of length 2 cm and is 60° above the x-axis in the first quadrant. Vector $\vec{B}$ is of length 2 cm and 60° below the x-axis in the fourth quadrant. The sum $\vec{A}+\vec{B}$ is a vector of magnitude
(A) 2 along $+y$ axis (B) 2 along $+x$-axis
(C) 1 along $-x$-axis (D) 2 along $-x$-axis

18. Six forces, 9.81 N each, acting at a point are coplanar. If the angles between the neighbouring forces are equal, then the resultant is
(A) 0 N (B) 9.81 N
(C) 2×9.81 N (D) 3×9.81 N

Section D: Product of Vectors

19. If $\vec{a}=x_1\hat{i}+y_1\hat{j}$ and $\vec{b}=x_2\hat{i}+y_2\hat{j}$. The condition that would make $\vec{a}$ and $\vec{b}$ parallel to each other is _____.

20. A vector $\vec{A}$ points vertically downward and $\vec{B}$ points towards east, then the vector product $\vec{A}\times\vec{B}$ is
(A) along west (B) along east
(C) zero (D) none of these

JEE Advanced

Level I

1. Match the statements given in column I with the statements given in column II

Column I	Column II
(A) If $\lvert\vec{A}\rvert=\lvert\vec{B}\rvert$ and $\lvert\vec{A}+\vec{B}\rvert=\lvert\vec{A}\rvert$, then angle between $\vec{A}$ and $\vec{B}$ is	(p) 90°
(B) Magnitude of resultant of two forces $\lvert\vec{F}_1\rvert=8$ N and $\lvert\vec{F}_2\rvert=4$ N may be	(q) 120°
(C) Angle between $\vec{A}=2\hat{i}+2\hat{j}$ and $\vec{B}=3\hat{k}$ is	(r) 12 N
(D) Magnitude of resultant of vectors $\vec{A}=2\hat{i}+\hat{j}$ and $\vec{B}=3\hat{k}$ is	(s) $\sqrt{14}$

2. The position of particle is given by $S=t^3-2t^2+5t+4$.
(a) Find the position of the particle at $t=1$ s.
(b) Find the first derivative of S at $t=1$ s.
(c) Find the second derivative of S at $t=1$ s.

3. Two forces $\vec{F}_1=2\hat{i}+2\hat{j}$ N and $\vec{F}_2=3\hat{i}+4\hat{k}$ N are acting on a particle.
(a) Find the resultant force acting on particle.
(b) Find the angle between $\vec{F}_1$ and $\vec{F}_2$.
(c) Find the component of force $\vec{F}_1$ along force $\vec{F}_2$.

4. **Statement 1:** A vector is a quantity that has both magnitude and direction and obeys the triangle law of addition.
Statement 2: The magnitude of the resultant vector of two given vectors can never be less than the magnitude of any of the given vectors.
(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.

(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

5. **Statement 1:** If the rectangular components of a force are 8 N and 6 N, then the magnitude of the force is 10 N.

Statement 2: If $|\vec{A}| = |\vec{B}| = 1$, then $|\vec{A} \times \vec{B}|^2 + |\vec{A} \cdot \vec{B}|^2 = 1$.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

6. **Statement 1:** If three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ satisfy the relation $\vec{A}.\vec{B} = 0$ and $\vec{A}.\vec{C} = 0$, then the vector $\vec{A}$ is parallel to $\vec{B} \times \vec{C}$.

Statement 2: $\vec{A} \perp \vec{B}$ and $\vec{A} \perp \vec{C}$; hence, A is perpendicular to the plane formed by $\vec{B}$ and $\vec{C}$.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

7. **Statement 1:** The minimum number of vectors of unequal magnitude required to produce zero resultant is three.

Statement 2: Three vectors of unequal magnitude which can be represented by the three sides of a triangle taken in order produce zero resultant.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

8. **Statement 1:** The angle between the two vectors $(\hat{i} + \hat{j})$ and $(\hat{k})$ is $\frac{\pi}{2}$ radian.

Statement 2: The angle between the two vectors $\vec{A}$ and $\vec{B}$ is given by $\theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

9. **Statement 1:** Distance is a scalar quantity.
Statement 2: Distance is the length of the path traversed.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

10. State whether the below statements are true or false

(i) If $\vec{A}$ and $\vec{B}$ are two force vectors, then $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

(ii) If $\vec{A}$ and $\vec{B}$ two force vectors, then $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$.

(iii) If the vector product of two non-zero vectors vanishes, the vectors are collinear.

(iv) If a function has maximum value at point P, the slope of the tangent drawn on the function at point P is zero.

11. Fill in the blanks

(i) The scalar product of the vectors $\vec{A} = 2\hat{i} + 5\hat{k}$ and $\vec{B} = 3\hat{j} + 5\hat{k}$ is _____.

(ii) If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 7\hat{i} + 24\hat{j}$, then the vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$ is _____.

(iii) If $\vec{A} \mid \vec{B}$, then $\vec{A} \times \vec{B}$ = _____.

(iv) The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A} = 3\hat{i} + 2\hat{j}$ and $\vec{B} = 2\hat{i} - 2\hat{k}$ is _____.

(v) A force is represented by $2\hat{i} + 3\hat{j} + 6\hat{k}$. The magnitude of the force is _____.

(vi) The unit vector along the vector $\hat{i} + \hat{j} + \hat{k}$ is _____.

(vii) If $\vec{A}$ is _____ to $\vec{B}$, then $\vec{A} \cdot \vec{B} = 0$.

(viii) The vector $\vec{A} = \hat{i} + \hat{j}$, where $\vec{i}$ and $\hat{j}$ are unit vectors along the x-axis and y-axis respectively, makes an angle of _____ degree with the x-axis.

(ix) If $\vec{A} + \vec{B} + \vec{C} = \vec{0}$, then $\vec{A} \cdot (\vec{B} \times \vec{C})$ = _____.

Level II

1. If the resultant of two forces of magnitudes P and Q acting at a point at an angle of 60° is $\sqrt{7}Q$, then P/Q is
   (A) 1 (B) 3/2
   (C) 2 (D) 4

2. The resultant of two forces F_1 and F_2 is P. If F_2 is reversed, then the resultant is Q. Then the value of $(P^2 + Q^2)$ in term of F_1 and F_2 is
   (A) $2(F_1^2 + F_2^2)$ (B) $F_1^2 + F_2^2$
   (C) $(F_1 + F_2)^2$ (D) none of these

3. A man moves 3 m towards north then 4 m towards east and finally 5 m towards 37° south of west. His displacement from the origin is
   (A) $5\sqrt{2}$ m (B) 0 m
   (C) 1 m (D) 12 m

4. Three forces P, Q and R are acting at a point in the plane. The angle between P and Q and Q and R are 150° and 120°, respectively, then for equilibrium, forces P, Q and R are in the ratio
   (A) 1:2:3 (B) 1:2:$\sqrt{3}$
   (C) 3:2:1 (D) $\sqrt{3}$:2:1

5. A man rows a boat with a speed of 18 km/hr in the northwest direction. The shoreline makes an angle of 15° south of west. Obtain the component of the velocity of the boat along the shoreline.
   (A) 9 km/hr (B) $18\frac{\sqrt{3}}{2}$ km/hr
   (C) 18 cos 15° km/hr (D) 18 cos 75° km/hr

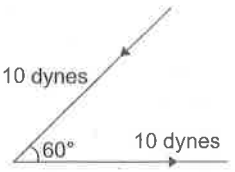
6. A bird moves from point (1, −2, 3) to (4, 2, 3). If the speed of the bird is 10 m/sec, then the velocity vector of the bird is
   (A) $5(\hat{i} - 2\hat{j} + 3\hat{k})$ (B) $5(4\hat{i} + 2\hat{j} + 3\hat{k})$
   (C) $0.6\hat{i} + 0.8\hat{j}$ (D) $6\hat{i} + 8\hat{j}$

7. The resultant of two forces, one double the other in magnitude, is perpendicular to the smaller of the two forces. The angle between the two forces is
   (A) 150° (B) 90°
   (C) 60° (D) 120°

8. If the angle between the unit vectors $\hat{a}$ and $\hat{b}$ is 60°, then $|\hat{a} - \hat{b}|$ is
   (A) 0 (B) 1
   (C) 2 (D) 4

9. For a particle moving in a straight line, the position of the particle at time (t) is given by $x = t^3 - 6t^2 + 3t + 7$. What is the velocity of the particle when its acceleration is zero?
   (A) −9 ms^{-1} (B) −12 ms^{-1}
   (C) 3 ms^{-1} (D) 42 ms^{-1}

10. Two forces, each numerically equal to 10 dynes, are acting as shown in the following figure, then their resultant is



    (A) 10 dynes (B) 20 dynes
    (C) $10\sqrt{3}$ dynes (D) 5 dynes

11. Two vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. The angle between the vectors $\vec{A}$ and $\vec{B}$ is
    (A) 0 (B) $\pi/3$
    (C) $\pi/2$ (D) π

12. A particle moves through angular displacement θ on a circular path of radius r. The linear displacement will be
    (A) $2r \sin(\theta/2)$ (B) $2r \cos(\theta/2)$
    (C) $2r \tan(\theta/2)$ (D) $2r \cot(\theta/2)$

13. The vector $\vec{P}$ makes 120° with the x-axis and vector $\vec{Q}$ makes 30° with the y-axis. What is their resultant?
    (A) $P + Q$ (B) $P - Q$
    (C) $\sqrt{P^2 + Q^2}$ (D) $\sqrt{P^2 - Q^2}$

14. A man travels 1 mile due east, then 5 miles due south, then 2 miles due east and finally 9 miles due north. How far is he from the starting point.
    (A) 3 miles (B) 5 miles
    (C) 4 miles (D) between 5 and 9 miles

15. The angle that the vector $\vec{A} = 2\hat{i} + 3\hat{j}$ makes with y-axis is

(A) $\tan^{-1}(3/2)$ (B) $\tan^{-1}(2/3)$
(C) $\sin^{-1}(2/3)$ (D) $\cos^{-1}(3/2)$

16. A man moves towards 3 m north then 4 m towards east and finally 5 m towards 37° south of west. His displacement from origin is
(A) $5\sqrt{2}$ m (B) 0 m
(C) 12 m (D) 5 m

17. If $3\hat{i}+2\hat{j}+8\hat{k}$ and $2\hat{i}+x\hat{j}+\hat{k}$ are at right angles, then $x =$
(A) 7 (B) -7
(C) 5 (D) -4

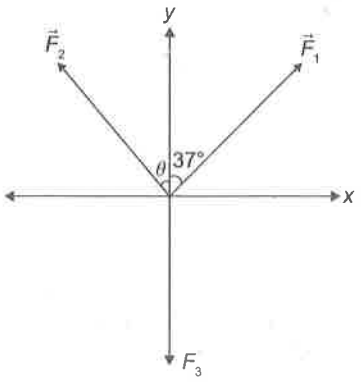
18. $a_1\hat{i}+a_2\hat{j}$ is a unit vector perpendicular to $4\hat{i}-3\hat{j}$ if
(A) $a_1=0.6, a_2=0.8$ (B) $a_1=3, a_2=4$
(C) $a_1=0.8, a_2=0.6$ (D) $a_1=4, a_2=3$

19. If $\vec{a}$ is a vector and x is a non-zero scalar, then
(A) $x\vec{a}$ is a vector in the direction of $\vec{a}$
(B) $x\vec{a}$ is a vector collinear to $\vec{a}$
(C) $x\vec{a}$ and $\vec{a}$ have independent directions
(D) none of these

20. Two vectors have magnitudes 3 units and 4 units, respectively. What should be the angle between them if the magnitude of the resultant is
(a) 1 unit, (b) 5 unit, (c) 7 unit

21. When two forces of magnitude P and Q are perpendicular to each other, their resultant is of magnitude R. When they are at an angle of 180° to each other, their resultant is of magnitude $\frac{R}{\sqrt{2}}$. Find the ratio of P and Q.

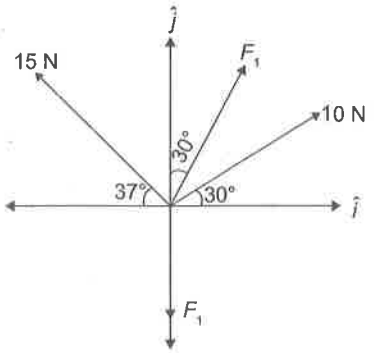
22. A body acted upon by three given forces is under equilibrium.



(a) If $|\vec{F}_1|=10Nt., |\vec{F}_2|=6Nt.$ Find the values of $|\vec{F}_3|$ and angle (θ).

(b) Express $|\vec{F}_2|$ in the unit vector form.

23. If the four forces as shown are in equilibrium. Express $|\vec{F}_1|$ and $|\vec{F}_2|$ in the unit vector form.



24. A particle is acted upon by the forces
$\vec{F}_1=2\hat{i}+a\hat{j}-3\hat{k}$, $\vec{F}_2=5\hat{i}+c\hat{j}-b\hat{k}$,
$\vec{F}_3=b\hat{i}+5\hat{j}-7\hat{k}$, $\vec{F}_4=c\hat{i}+6\hat{j}-a\hat{k}$.
Find the values of the constants a, b, c in order that the particle will be in equilibrium.

25. A plane body has perpendicular axes OX and OY marked on it and is acted on by following forces: $5P$ in the direction OY
$4P$ in the direction OX
$10P$ in the direction OA where A is the point $(3a, 4a)$
$15P$ in the direction AB where B is the point $(-a, a)$
Express each force in the unit vector from and calculate the magnitude and direction of the sum of the vector of these forces.

26. A vector $\vec{A}$ of length 10 units makes an angle of 60° with the vector $\vec{B}$ of length 6 units. Find the magnitude of the vector difference $\vec{A}-\vec{B}$ and the angle it makes with vector $\vec{A}$.

27. (a) Calculate $\vec{r}=\vec{a}-\vec{b}+\vec{c}$, where $\vec{a}=5\hat{i}+4\hat{j}-6\hat{k}$, $\vec{b}=-2\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{c}=4\hat{i}+3\hat{j}+2\hat{k}$.
(b) Calculate the angle between $\vec{r}$ and the z-axis.
(c) Find the angle between $\vec{a}$ and $\vec{b}$.

ANSWER KEYS

Exercises

JEE Main

Part I

Section A

1. 1 **2.** 47 **3.** (c)

Section B

4. $\frac{dy}{dx}=2x+1$ **5.** $\frac{ds}{dt}=15t^2-15t^4$ **6.** $\frac{dy}{dx}=5\cos x$ **7.** $\frac{dy}{dx}=2x+\cos x$ **8.** $\sec^2 x-\operatorname{cosec}^2 x$

9. $\frac{dy}{dx}=12x-10+10x^{-3}, \frac{d^2y}{dx^2}=12-30x^{-4}$ **10.** $\frac{dr}{d\theta}=-12\theta^{-2}+12\theta^{-4}-4\theta^{-5}, \frac{d^2r}{d\theta^2}=24\theta^{-3}-48\theta^{-5}+20\theta^{-6}$

11. $\frac{d\omega}{dz}=12z^6-12z^2+24z, \frac{d^2\omega}{dz^2}=126z^5-42z+42$ **12.** $\frac{dy}{dx}=\cos x-\sin x, \frac{d^2y}{dx^2}=-\sin x-\cos x$

13. $\frac{dy}{dx}=\frac{1}{x}+e^x, \frac{d^2y}{dx^2}=-\frac{1}{x^2}+e^x$

Section C

14. $\sin x+x\cos x$ **15.** $e^x\ell nx+\frac{e^x}{x}$ **16.** $\frac{dy}{dx}=3x^2$ **17.** $y'=3x^2+10x+2-\frac{1}{x^2}$ **18.** $\cos^2 x-\sin^2 x$

Section D

19. $\sec^2 x$ **20.** $y'=\frac{-19}{(3x-2)^2}$ **21.** $\frac{1}{x^2}-\frac{\ell nx}{x^2}$ **22.** $f'(t)=\frac{t^2-2t+1}{(t^2+t-2)^2}$ **23.** $\frac{dz}{dx}=\frac{-2x^2-2x-2}{(x^2-1)^2}$

24. $\frac{dy}{dx}=-x^2\operatorname{cosec}^2 x+2x\cot x$

Section E

25. With $u=(2x+1)$ $y=u^5: \frac{dy}{dx}=\frac{dy}{du}=5u^4\cdot 2=10(2x+1)^4$

26. $\frac{dy}{dx}=-27(4-3x)^8$

27. With $u=\left(1-\left(\frac{x}{7}\right)\right) y=u^{-7}: \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-7u^{-8}.\left(-\frac{1}{7}\right)=\left(1-\frac{x}{7}\right)^{-8}$

28. $\frac{dy}{dx}=-5\left(\frac{x}{2}-1\right)^{-1}$ **29.** $5\cos 5x$ **30.** $\cos(x)+\frac{2}{x}+2e^{2x}$ **31.** $2\omega\cos(\omega x+\phi)$

Section F

32. $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$ **33.** $\frac{ds}{dt}=8\pi t\frac{dr}{dt}$

Section G

34. 8 **35.** $y_{max}=39, y_{min}=38$

Section H

36. $\frac{dy}{dx}=48(8x-1)^2$ **37.** $3\cos(3x+1)$ **38.** $12x^3$ **39.** $\frac{dy}{dx}=-\frac{1}{3}\sin\frac{x}{3}$

Part II

1. (a) x^2 (b) $\frac{x^3}{3}$ (c) $\frac{x^3}{3}-x^2+x$ **2.** (a) x^{-3} (b) $-\frac{1}{3}x^{-3}$ (c) $-\frac{1}{3}x^{-3}+x^2+3x$

3. (a) $-\frac{1}{x}$ (b) $-\frac{5}{x}$ (c) $2x+\frac{5}{x}$ **4.** (a) $\sqrt{x^3}$ (b) $3\sqrt{x}$ (c) $\frac{2\sqrt{x^3}}{3}+2\sqrt{x}$

5. (a) $x^{4/3}$ (b) $\frac{x^{\frac{2}{3}}}{2}$ (c) $\frac{3x^{\frac{4}{3}}}{4}+\frac{3x^{\frac{2}{3}}}{2}$ **6.** (a) $x^{1/2}$ (b) $x^{-1/2}$ (c) $x^{-3/2}$ **7.** $x-\frac{x^3}{3}-\frac{x^6}{2}+C$

8. $-3\cos x$ **9.** $\frac{1}{3}\ell nx$ **10.** $-\frac{1}{3}\cos 3x+C$ **11.** $\frac{1}{2}\sec 2t+C$ **12.** 15 **13.** $\frac{3\pi}{2}$ **14.** Area = 21

15. 24 **16.** 0 **17.** $e-1$ **18.** Using n subintervals of length $\Delta x=\frac{b}{n}$ and right-endpoint values: Area $=\int_0^b 2x\,dx=b^2$

19. $\frac{b^2}{4}+b=\frac{b(4+b)}{4}$ **20.** 2

Part III

Section A

1. (i) 105°, (ii) 150°, (iii) 105° **2.** 120° **3.** $\vec{V}_R=-5\hat{j}$ **4.** (C)

Section B

5. 30 m East **6.** $\sqrt{F_1^2+F_2^2}$ **7.** 50, 53° with East **8.** $250\sqrt{5}$ N, $\tan^{-1}(2)$ w of N

9. A **10.** B **11.** B **12.** D **13.** D **14.** A **15.** C **16.** B **17.** D

Section C

18. $\sqrt{14}$ **19.** $\frac{3\hat{i}+4\hat{i}}{5}$ **20.** $-25\cos 30°$ and $+25\sin 30°$ **21.** $30\sqrt{3}$ km h^{-1} **22.** $\sqrt{0.1}$ **23.** 15° **24.** A

Section D

25. (a) 3 (b) $-\hat{i}+2\hat{j}-\hat{k}$ **26.** (a) 6 (b) $6\sqrt{3}$ **27.** D **28.** B

JEE Advanced

Part I

Section A

1. $-\frac{1}{x}$ **2.** $f(2)=3, f(1)=3, f(3)=5$

Section B

3. $\frac{dy}{dx}=\frac{2}{x}+\cos x$, $\frac{d^2y}{dx^2}=\frac{-2}{x^2}-\sin x$ 4. $\frac{dy}{dx}=\frac{x^{-\frac{6}{7}}}{7}+\sec^2 x \Rightarrow \frac{d^2y}{dx^2}-\frac{-6}{49}x^{-\frac{13}{7}}+2\tan x\sec^2 x$

Section C

5. $e^x(\tan x+\sec^2 x)$ 6. $2x\sin^4 x+4x^2\sin^3 x\cos x+\cos^{-2}x+2x\cos^{-3}x\sin x$ 7. $\frac{dy}{dx}=1+2x+\frac{2}{x^3}-\frac{1}{x^2}$

8. $x^2\cos x$ 9. $\frac{dy}{dx}=-x^2\sin x$ 10. $\frac{dr}{d\theta}=\cos\theta+\sec^2\theta$

Section D

11. $\frac{dy}{dx}=\sec^2 x$ 12. $\frac{-\cos^2 x}{(1+\cot x)^2}$ 13. $\frac{dy}{dx}=\frac{-x\sin x-\cos x}{x^2}+\frac{x\sin x+\cos x}{\cos^2 x}$ 14. $\frac{\sec^2\theta}{(1+\tan\theta)^2}$

Section E

15. $3\sin^2 x\cos x+3\cos 3x$ 16. $4x\sin(x^2+1)\cos(x^2+1)$ 17. $\frac{1}{(x^2+1)^{3/2}}$ 18. $\frac{1-r}{\sqrt{2r-r^2}}$

19. With $u=\left(\frac{x^2}{8}\right)+x-\left(\frac{1}{x}\right)$, $y=u^4$; $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=4u^3\cdot\left(\frac{x}{4}+1+\frac{1}{x^2}\right)=4$, $4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3\left(\frac{x}{4}+1+\frac{1}{x^2}\right)$

Section F

20. (a) $\frac{dv}{dt}=\pi r^2$, $\frac{dh}{dt}=5\pi r^2$ (b) $\frac{dv}{dt}=2\pi hr$, $\frac{dr}{dt}=10\pi rh$ (c) $\frac{dv}{dt}=\pi r^2\frac{dh}{dt}+2\pi hr\frac{dr}{dt}=5\pi r^2+10\pi rh$

Section G

21. $x=30$ and $y=30$ 22. $x=\sqrt{\frac{40}{3}}$ m

Section H

23. (a) $-\cos x$, (b) $2\sec^3 x-\sec x$

24. $-\sin(\sin x)\cos x$ 25. $\frac{dy}{dx}=\cos(x-\cos x)(1+\sin x)$

Given $y=f(u)$ and $u=g(x)$, find $\frac{dy}{dx}$.

Part II

1. $\frac{x^4}{2}-\frac{5x^2}{2}+7x+C$ 2. $\frac{x}{5}+\frac{1}{x^2}+x^2+C$ 3. $\frac{2}{3}x^{3/2}+\frac{3}{4}x^{4/3}+C$ 4. $-\frac{1}{x}-\frac{1}{2x^2}+C$

5. $2\sqrt{t}-\frac{2}{\sqrt{t}}+C$ 6. $-2t^{-2}-\frac{2}{3}t^{-3/2}+C$ 7. $-\cos\theta+\theta+C$ 8. $\frac{3\pi^2}{2}$

9. $\frac{7}{3}$ 10. 0 11. $\frac{1}{3}\ell n\frac{5}{2}$

12. Using n subintervals of length $\Delta x=\frac{b}{n}$ and right-end point values: Area $=\int_0^b 3x^2dx=b^3$ 13. $\frac{\pi b^2}{4}$

Part III

Section A

1. $\vec{B}=\lambda\vec{A}-4\times 3$ N-E $=12$ S-W

No it does not represent the same physical quantity.

2. D 3. A 4. A, B, C

Section B

5. 37° **6.** $r(1+\sqrt{2})$ **7.** A **8.** B **9.** B **10.** B **11.** C
12. A **13.** D **14.** A **15.** A, B, D

Section C

16. $\frac{4\hat{i}+5\hat{j}+2\hat{k}}{\sqrt{45}}$ **17.** B **18.** A

Section D

19. $\frac{x_1}{x_2}=\frac{y_1}{y_2}$ **20.** D

JEE Advanced

Level I

1. A → Q, B → R, C → P, D → S

2. (a) 8, (b) 4, (c) 2 (b) $\cos\theta=\frac{\vec{F}_1.\vec{F}_2}{|\vec{F}_1||\vec{F}_2|}\Rightarrow\theta=\cos^{-1}\left(\frac{3}{5\sqrt{2}}\right)$ (c) $F_1\cos\theta=\frac{\vec{F}_1.\vec{F}_2}{|\vec{F}_2|}=\frac{6}{5}$

3. (a) $\vec{F}_R=\vec{F}_1+\vec{F}_2=2\hat{i}+5\hat{j}+4\hat{k}$ **4.** C **5.** B **6.** A **7.** A **8.** A **9.** B

10. (i) True (ii) False (iii) True (iv) True

11. (i) 25 units (ii) $15\hat{i}+20\hat{j}$ (iii) Null vector (iv) $\sqrt{224}$ units (v) 7 units (vi) $\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$
(vii) Perpendicular (viii) 45 (ix) Zero

Level II

1. C **2.** A **3.** B **4.** D **5.** A **6.** D **7.** D **8.** B **9.** A **10.** A
11. C **12.** A **13.** A **14.** B **15.** B **16.** B **17.** B **18.** A **19.** B
20. (a) 180°, (b) 90°, (c) 0 **21.** $2\pm\sqrt{3}$ **22.** (a) $|\vec{F}_3|=8$ N, $\theta=90°$ (b) $\vec{F}_2=-6\hat{i}$
23. $\vec{F}_1=-(12\sqrt{3}-1)\hat{j}$ and $\vec{F}_2=(12-5\sqrt{3})\hat{i}+(12\sqrt{3}-15)\hat{j}$ **24.** $a=-7, b=-3, c=-4$
25. $5P\hat{j}, 4P\hat{i}, 6P\hat{i}+8P\hat{j}$, $-12P\hat{i}-9P\hat{j}$, $\sqrt{20}P$ $\tan^{-1}[-2]$ with the +ve x-axis **26.** $2\sqrt{19}$; $\cos^{-1}\frac{7}{2\sqrt{19}}$
27. (a) $11\hat{i}+5\hat{j}-7\hat{k}$, (b) $\cos^{-1}\left(\frac{-7}{\sqrt{195}}\right)$, (c) $\cos^{-1}\left(\frac{-20}{\sqrt{1309}}\right)$

CHAPTER 3

Kinematics

REST AND MOTION

- An object is said to be in motion with respect to a frame of reference S_1, when its location is changing with time in the same frame of reference S_1.
- Rest and motion are relative terms.
- Absolute rest and absolute motion have no meaning.

Motion is broadly classified into three categories.

Rectilinear and translatory motion	Circular and rotatory motion	Oscillatory and vibratory motion

Rectilinear or 1-D Motion

When a particle is moving along a straight line, then its motion is a rectilinear motion.

Parameters of rectilinear motion or translatory motion or plane motion:

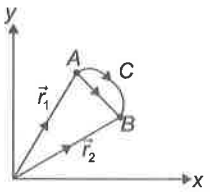
(A) *Time*

- It is a scalar quantity and its SI unit is second(s).
- At a particular instant of time, a physical object can be present at one location only.
- Time can never decrease.

(B) *Position or Location*

It is defined with respect to some reference point (origin) of given frame of reference.

Consider a particle which moves from location $\vec{r}_1$ (at time t_1) to location $\vec{r}_2$ (at time t_2) as shown in the figure below, following path ACB.



(C) *Distance*

The length of the actual path traversed by the particle is termed as its distance.

Distance = length of path ACB.

- Its SI unit is metre and it is a scalar quantity.
- It can never decrease with time.

(D) *Displacement*

The change in position vector of the particle for a given time interval is known as its displacement.

$$\overrightarrow{AB} = \vec{r} = \vec{r}_2 - \vec{r}_1$$

- Displacement is a vector quantity and its SI unit is metre.
- It can decrease with time.

For a moving particle in a given interval of time

- Displacement can be +ve, −ve or 0, but distance would be always +ve.
- Distance ≥ magnitude of displacement.
- Distance is always equal to displacement only and only if the particle is moving along a straight line without any change in direction.

(E) *Average Speed and Average Velocity*

Average speed and average velocity are always defined for a time interval.

$$\text{Average speed } (v_{av}) = \frac{\text{Total distance travelled}}{\text{Time interval}} = \frac{\Delta s}{\Delta t}$$

$$\text{Average velocity } (\vec{v}_{av}) = \frac{\text{Displacement}}{\text{Time interval}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

- Average speed is a scalar quantity, while average velocity is a vector quantity. Both have the same SI units, i.e., m/s.

For a moving particle in a given interval of time

- Average speed can be a many-valued function but average velocity would be always a single-valued function.
- Average velocity can be positive, negative or 0 but average speed would be always positive.

(F) *Instantaneous Speed and Instantaneous Velocity*

Instantaneous speed is also defined exactly like average speed, i.e., it is equal to the ratio of total distance and time interval, but with one qualification that time interval is extremely (infinitesimally) small. The instantaneous speed is the speed at a particular instant of time and may have entirely different value than that of average speed. Mathematically,

$$v = \lim_{\Delta s \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

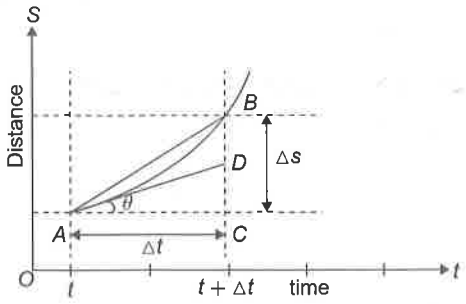
where Δs is the distance travelled in time Δt.

As Δt tends to zero, the ratio defining speed becomes finite and equals to the first derivative of the distance. The speed at the moment t is called the instantaneous speed at time t.

On the distance–time plot, the speed is equal to the slope of the tangent to the curve at the time instant t. Let the points A and B on the plot correspond to the time t and $t + \Delta t$ during the motion. As Δt approaches zero, the chord AB becomes the tangent AC at A. The slope of the tangent equal ds/dt, which is equal to the instantaneous speed at t.

$$v = \tan\theta = \frac{DC}{AC}$$

$$= \frac{ds}{dt}$$

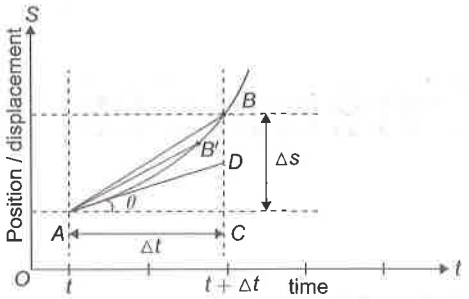


Instantaneous speed is equal to the slope of the tangent at given instant.

(G) *Instantaneous Velocity*

Instantaneous velocity is defined exactly like speed. It is equal to the ratio of the total displacement and time interval, but with one qualification that time interval is extremely (infinitesimally) small. Thus, instantaneous velocity can be termed as the average velocity at a particular instant of time when Δt tends to zero and may have entirely different value that of average velocity. Mathematically,

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}$$



Instantaneous velocity is equal to the slope of the tangent at given instant.

As Δt tends to zero, the ratio defining velocity becomes finite and equals to the first derivative of the position vector. The velocity at the moment t is called the instantaneous velocity or simply velocity at time t.

Note

The magnitude of average velocity $|v_{avg}|$ and average speed v_{avg} may not be equal, but the magnitude of instantaneous velocity $|v|$ is always equal to instantaneous speed v.

SOLVED EXAMPLES

EXAMPLE 1

In 1.0 s a particle goes from point A to point B moving in a semicircle of radius 1.0 m. The magnitude of average velocity is

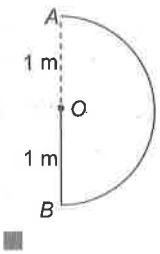
(A) 3.14 m/s (B) 2.0 m/s
(C) 1.0 m/s (D) zero

SOLUTION

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{d}{t}$$

$$D = AO + OB$$
$$= 1 + 1 = 2 \text{ m}$$
$$T = 1 \text{ s (given)}$$

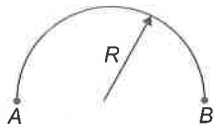
$$\Rightarrow \quad mg \text{ of } v \text{ of } \frac{2}{1} = 2 \text{ m/s}$$



EXAMPLE 2

A particle moves along a semi-circular path of radius R in time t with constant speed. For the particle calculate

(a) distance travelled
(b) displacement
(c) average speed
(d) average velocity



SOLUTION

(a) Distance = length of path of particle = $\overset{\frown}{AB} = \pi R$

(b) Displacement = minimum distance between initial and final point
$= AB = 2R$

(c) Average speed, $v = \dfrac{\text{total distance}}{\text{time}} = \dfrac{\pi R}{t}$

(d) Average velocity $= \dfrac{2R}{t}$ ■

EXAMPLE 3

A body travels the first half of the total distance with velocity v_1 and the second half with velocity v_2. Calculate the average velocity.

SOLUTION

Let total distance $= 2x$. Then

$$\text{total time taken} = \frac{x}{v_1} + \frac{x}{v_2} = x\left(\frac{v_1 + v_2}{v_1 v_2}\right)$$

$$\therefore \quad \text{Average speed} = \frac{2x}{x\left(\dfrac{v_1 + v_2}{v_1 v_2}\right)} = \frac{2v_1 v_2}{v_1 + v_2}$$ ■

(G-1) *When Velocity is Given as a Function of t*

EXAMPLE 4

Velocity–time equation of a particle moving in a straight line is $v = (10 + 2t + 3t^2)$.
Find:

(a) the displacement of the particle from the origin of time $t = 1$ s, if it is given that displacement is 20 m at time $t = 0$
(b) acceleration–time equation.

SOLUTION

(a) The given equation can be written as,

$$v = \frac{ds}{dt} = (10 + 2t + 3t^2)$$

$$ds = (10 + 2t + 3t^2)\, dt$$

or $$\int_{20}^{s} ds = \int_{0}^{t} (10 + 2t + 3t^2) dt$$

or $$s - 20 = [10t + t^2 + t^3]_0^1$$

or $$s = 20 + 12 = 32 \text{ m}$$

(b) Acceleration–time equation can be obtained by differentiating the given equation with respect to time. Thus,

$$a = \frac{dv}{dt} = \frac{d}{dt}(10 + 2t + 3t^2)$$

or $$a = 2 + 6t$$ ■

SPECIMEN PROBLEM

A. *When the Equation of Displacement is Given and Speed has to be Determined*

EXAMPLE 5

If displacement is dependent on time such that $x = 2t - 2$, then find out the average speed up to 4 s.

SOLUTION

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

for total distance
at $t = 0$ it is at $x = -2$
at $t = 1$ it is at 0 m
at $t = 4$ it is at 6 m.

Total distance $= |-2| + 0 + 6 = 8$ m

Average speed $= 8/4 = 2$ m/s ■

B. *When Velocity is Given as a Function of Time and Distance is to be Determined*

In this type of question, first find out at what instant the velocity is zero. If this instant falls within our time limit, then distance can be calculated by breaking the integration into two parts with modules.

EXAMPLE 6

If velocity is dependent on time such that $v = 4 - 2t$, find out the distance travelled by the particle from 1 to 3 s.

SOLUTION

Velocity is zero $(4 - 2t = 0)$ at $t = 2$ s
So for distance

$$\frac{dx}{dt} = 4 - 2t$$

$$\Rightarrow \quad dx = dx = \left|\int_1^2 (4-2t)\,dt\right| + \left|\int_2^3 (4-2t)\,dt\right|$$

$$dx = 1 + 1 = 2 \text{ m}$$

(G-2) *When Velocity is Given as a Function of x*

$$v = f(x)$$

$$\Rightarrow \quad \frac{dx}{dt} = f(x)$$

$$\Rightarrow \quad \int \frac{dx}{f(x)} = \int dt$$

EXAMPLE 7

If velocity is given by the function $V = x^2$, then find out the relation between x and t (assume $x = 1$ m at $t = 0$).

SOLUTION

Relation between v and x is

$$v = x^2$$

we know that $\quad v = \frac{dx}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = x^2$$

$$\frac{dx}{x^2} = dt$$

$$\Rightarrow \quad \int_1^x \frac{dx}{x^2} = \int_0^t dt$$

$$\Rightarrow \quad \left[-\frac{1}{x}\right]_1^x = t$$

$$\Rightarrow \quad -\frac{1}{x} + 1 = t$$

$$\Rightarrow \quad x = \frac{1}{1-t}$$

(E) *Average and Instantaneous Acceleration*

When the velocity of a moving object/particle changes with time, we can say that it is accelerated.
Average acceleration,

$$(\vec{a}_{av}) = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

$$= \frac{\Delta \vec{v}}{\Delta t}$$

$$= \frac{\text{Change in velocity}}{\text{Time interval}}$$

Instantaneous acceleration,

$$(\vec{a}) = \lim_{\Delta t \to 0} \vec{a}_{av} = \frac{d\vec{v}}{dt}$$

= Rate of change of velocity

- Acceleration is a vector quantity whose direction is the same as that of change in velocity vector. Its SI unit is m/s^2.
- When the direction of acceleration and velocity are opposite to each other, then acceleration is termed as retardation.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$= \vec{v}\frac{d\vec{v}}{d\vec{r}}$$

(E-1) *When Acceleration is Given as a Function of x*

EXAMPLE 8

Let $a = 2x$; initially, the particle is at $x = 2$ m and is moving with 3 ms^{-1}. Then find out v at $x = 5$ m.

SOLUTION

Given $\quad a = 2x$

$$\Rightarrow \quad \frac{v\,dv}{dx} = 2x$$

$$\Rightarrow \quad \int_3^v v\,dv = \int_2^5 2x\,dx$$

$$\Rightarrow \quad \frac{v^2}{2} - \frac{9}{2} = 25 - 4$$

$$\Rightarrow \quad v^2 - 9 = 21 \times 2$$

$$\Rightarrow \quad v = \sqrt{51} \text{ ms}^{-1}$$

(E-2) When Acceleration is Given as a Function of Velocity

EXAMPLE 9

If a is dependent on v as follows:
$a = v$ and at $t = 0$; $x = 1$ m, $v = 1$ m/s.
(a) Find out its velocity at $t = 2$ s.
(b) Find out its velocity at $x = 3$ m.

SOLUTION

(a) Given $a = v$

$$\Rightarrow \quad \frac{dv}{dt} = v$$

$$\Rightarrow \quad \int_1^v \frac{dv}{v} = \int_0^2 dt$$

$$\Rightarrow \quad \ln v = 2$$

$$\Rightarrow \quad v = e^2 \text{ ms}^{-1}$$

(b) Given $a = v$

$$\Rightarrow \quad \frac{v\,dv}{dx} = v$$

$$\Rightarrow \quad \int_1^v dv = \int_1^3 dx$$

$$\Rightarrow \quad v - 1 = 2$$

$$\Rightarrow \quad v = 3 \text{ ms}^{-1}$$ ■

(E-3) When Acceleration is Given as a Function of t

EXAMPLE 10

The acceleration of a particle which is dependent on time is given by the following function

$$a = 2t + 1$$

and at time $t = 0$, $x = 1$ m and $u_i = 2$ m/s.
Then find out the displacement of the particle at $t = 3$ s.

SOLUTION

We know that $a = \frac{dv}{dt}$

$$\Rightarrow \quad \frac{dv}{dt} = 2t + 1$$

$$\Rightarrow \quad dv = (2t + 1)\,dt$$

$$\int_2^{v_f} dv = \int_0^t (2t+1)dt$$

$$v_f - 2 = t^2 + t$$

$$\Rightarrow \quad v_f = t^2 + t + 2$$

Now $v = \frac{dx}{dt}$

$$\Rightarrow \quad \frac{dx}{dt} = t^2 + t + 2$$

$$\Rightarrow \quad \int_1^{x_f} dx = \int_0^t (t^2 + t + 2)$$

$$x_f = \frac{t^3}{3} + \frac{t^2}{2} + 2t + 1$$

Thus, x_f, at $t = 3$ s is

$$= \frac{(3)^3}{3} + \frac{(3)^2}{2} + 2(3) + 1$$

$\Rightarrow 20.5$ m

So, after $t = 3$ s, the position of the particle is 20.5 m, but the displacement of the particle is

$$= 20.5 - 1 = 19.5 \text{ m}$$ ■

(E-4) Constant Acceleration Format

Deduce the following equations for uniform acceleration motion by using integration method.

(A) $v = u + at$ (B) $s = ut + \frac{1}{2}at^2$

(C) $v^2 - u^2 = 2as$ (D) $s_{nth} = u + \frac{a}{2}(2n - 1)$

First equation of motion: Acceleration is defined as

$$a = \frac{dv}{dt}$$

or $dv = a\,dt$ (1)

When time = 0, velocity = u (say)

When time = t, velocity = v (say)

Integrating Eq. (1) within the above limits of time and velocity, we get

$$\int_u^v dv = \int_0^t a\,dt$$

or $[v]_u^v = a\int_0^t dt = a[t]_0^t$

or $v - u = a(t - 0)$

or $v = u + at$ (2)

Second equation of motion: Velocity is defined as

$$v = \frac{ds}{dt}$$

or $ds = v\,dt = (u + at)\,dt$ (iii)

When time = 0, displacement travelled = 0

When time = t, displacement travelled = s (say).

Integrating Eq. (3) within the above limits of time and distance, we get

$$\int_0^s ds = \int_0^t (u = at)\,dt$$

$$= u\int_0^t dt + a\int_0^t t\,dt$$

or $$[s]_0^s = u[t]_0^t + a\left[\frac{t^2}{2}\right]_0^t$$

or $$s - 0 = u(t - 0) + a\left[\frac{t^2}{2} - 0\right]$$

or $$s = ut + \frac{1}{2}at^2 \quad (4)$$

Third equation of motion: By the definitions of acceleration and velocity,

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt}$$

$$= \frac{dv}{ds} \times v$$

or $ads = vdv$ (5)

When time = 0, velocity = u, displacement travelled = 0

When time = t, velocity = v, displacement travelled = s (say)

Integrating Eq. (5) within the above limits of velocity and displacement, we get

$$\int_0^s a\,ds = \int_u^v v\,dv$$

or $$a\int_0^s ds = \int_u^v v\,dv$$

or $$a[s]_0^s = \left[\frac{v^2}{2}\right]_u^v$$

or $$a[s - 0] = \frac{v^2}{2} - \frac{u^2}{2}$$

or $$2as = v^2 - u^2$$

or $$v^2 - u^2 = 2as \quad (6)$$

Fourth equation of motion: By definition of velocity,

$$v = \frac{ds}{dt}$$

or $ds = vdt = (u + at)\,dt$ (7)

When time = $(n - 1)$ second, displacement travelled = s_{n-1} (say).

When time = n second, displacement travelled = s_n (say)

Integrating Eq. (7) within the above limits of time and distance, we get

$$\int_{s_{n-1}}^{s_n} ds = \int_{n-1}^{n} (u + at)dt$$

or $$[s]_{s_{n-1}}^{s_n} = u\int_{n-1}^{n} dt + a\int_{n-1}^{n} t\,dt$$

or $$s_n - s_{n-1} = u[t]_{n-1}^{n} + a\left[\frac{t^2}{2}\right]_{n-1}^{n}$$

$$= u[n - (n - 1)] + \frac{a}{2}[n^2 - (n - 1)^2]$$

$$= u + \frac{a}{2}[n^2 - (n^2 - 2n + 1)]$$

$$s_{nth} = u + \frac{a}{2}(2n - 1) \quad (8)$$

where $s_{nth} = s_n - s_{n-1}$ = displacement in nth second.

SOLVED EXAMPLE

EXAMPLE 11

A car starts from rest and accelerates uniformly for 20 s to a velocity of 72 km h^{-1}. It then runs at a constant velocity and is finally brought to rest in 200 m with a constant retardation. The total distance covered is 600 m. Find the acceleration, retardation and the total time taken.

SOLUTION

(a) Motion with uniform acceleration

Here, $u = 0$; $t_1 = 20$ s;

$$v = 72 \times \frac{5}{18} = 20 \text{ ms}^{-1}$$

$\therefore$ $v = u + at_1$

$20 = 0 + a \times 20$

or $a = 1 \text{ ms}^{-2}$

Distance travelled by the car in this time (20 s),

$$S_1 = ut + \frac{1}{2} at^2$$

$$= 0 + \frac{1}{2} \times 1 \times (20)^2$$

$$= 200 \text{ m}$$

(b) Motion with uniform velocity

As given, total distance = 600 m

we have calculated $S_1 = 200$ m (with uniform acc.)

and $S_2 = 200$ m (with retardation)

$\therefore$ Net distance for which the body moves with uniform velocity,

$$S = 600 - S_1 - S_2$$

$$= 600 - 200 - 200 = 200 \text{ m}$$

$\therefore$ Time taken, $t = \dfrac{\text{distance}}{\text{uniform velocity}} = \dfrac{200}{20}$

$$= 10 \text{ s}$$

$\therefore$ Total time of journey,

$$t = (20 + 10 + 20) \text{ s}$$

$$t = 50 \text{ s}$$

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total Time}} = \frac{600}{50}$$

$$= 12 \text{ m/s}.$$

(c) Motion with uniform retardation

For this motion, initial velocity, $u = 20 \text{ m s}^{-1}$ and final velocity $v = 0$; $S_2 = 200$ m

Acceleration $a' = ?$

Using, $v^2 - u^2 = 2a'S_2$

$$(0)^2 - (20)^2 = 2(a') \times 200$$

$$a' = -1 \text{ ms}^{-2}$$

Let t' = time for which the body comes to rest.

$\therefore$ $v = u + a't'$

$0 = 20 - 1t'$

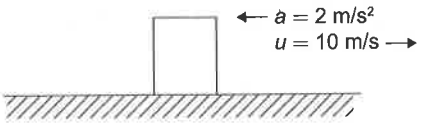
$\therefore$ $t' = 20$ s. ■

SPECIMEN PROBLEM

C

EXAMPLE 12

Find out the distance travelled by the block in 10 s. for a given situation.



SOLUTION

First find out what instant velocity of the block becomes zero.

$$v = u + at$$

Given: $u = 10$ m/s, $a = -2$ m/s^2

$\Rightarrow$ $0 = 10 - 2t$

$\Rightarrow$ $t = 5$ s

So we calculate the distance for the two time intervals. For the first 5 s.

$$S_1 = ut - \frac{1}{2} at^2$$

$\Rightarrow$ $S_1 = (10)(5) - \frac{1}{2}(2)(5)^2 = 25$ m

For the next 5 s (block is travelling towards the starting point)

$$S_2 = ut + \frac{1}{2} at^2$$

$u = 0$

$S_2 = \frac{1}{2} \times 2 \times 25 = 25$ m

$S = S_1 + S_2 = 50$ m ■

(D) Reaction Time

When a particular situation demands our immediate action, it takes some time before we really respond. Reaction time is the time a person takes to observe, think and act. For example, if a person is driving and suddenly a boy appears on the road, then the time lapse before he applies the breaks of the car is the reaction time. Reaction time depends on the complexity of the situation and on an individual.

One can measure one's reaction time by a simple experiment. Take a rule and ask your friend to drop it vertically through the gap between your thumb and forefinger. As soon as it is dropped, note the time elapsed t_s before you catch it and the distance d travelled by the ruler. (In a particular case, y was found to be 21.0 cm. Estimate the reaction time.)

SOLUTION

As the ruler drops under free fall, so $u = 0$, and $g = 9.8\ \text{ms}^{-2}$. The distance travelled d and the reaction time t_r are related by

$$d = \frac{1}{2}gt_r^2 = 4.9t_r^2$$

or, $$t_r = \sqrt{\frac{d}{4.9}}$$

Here $$d = 21.0\ \text{cm} = 0.21\ \text{m}$$

$\therefore$ $$t_r = \sqrt{\frac{0.21}{4.9}} \sim 0.2\ \text{s}$$

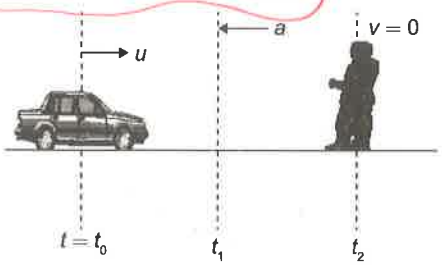
Note

Definition: Time taken by a driver to react to a situation

Reaction time of the driver is

$$\Delta t = t_1 - t_0$$

Total distance covered by the car before stopping = distance covered in uniform motion during t_0 to t_1 + distance covered in de-accelerated motion during t_1 to t_2 = Total distance $= u(\Delta t) + \frac{u^2}{2a}$



(brakes applied)

SOLVED EXAMPLES

EXAMPLE 13

Assume that a car is able to stop with a retardation of 8 ms^{-2} and that a driver can react to an emergency in 0.5 s. Calculate the overall stopping distance of the car for a speed of 60 km^{-1} of the car.

SOLUTION

Here, $$u = 60\ \text{km h}^{-1}$$

$$= 60 \times \frac{5}{18} = \frac{50}{3}\ \text{ms}^{-1}$$

Since the application of brakes takes 0.5 s, before this the car was moving with uniform speed of $\frac{50}{3}$ ms^{-1}.

$\therefore$ Distance covered in 0.5 s, with a uniform speed is

$$S_1 = u \times t = \frac{50}{3} \times 0.5$$

$$= \frac{25}{3}\ \text{m} = 8.33\ \text{m}$$

Now the car begins to move with a retardation of 8 ms^{-2}

$\therefore$ Distance covered before coming to rest,

$$2aS_2 = v^2 - u^2$$

or $$S_2 = \frac{v^2 - u^2}{2a} = \frac{0 - \left(\frac{50}{3}\right)^2}{-2 \times 8}$$

$$= \frac{50 \times 50}{9 \times 2 \times 8}$$

$$= \frac{2500}{144}$$

$$= 17.36\ \text{m}$$

$\therefore$ Total (overall) distance $= S_1 + S_2 = 8.33 + 17.36$

$\Rightarrow$ $$S = 25.69\ \text{m}.$$

EXAMPLE 14

Two buses A and B are at positions 50 m and 100 m from the origin at time $t = 0$. They start moving in the same direction simultaneously with uniform velocity of 10 ms^{-1} and 5 ms^{-1}. Determine the time and position at which A overtakes B.

SOLUTION

Here, we use equation of motion for constant velocity in the Cartesian form.

Given $x_1(0) = 50$ m, $x_2(0) = 100$ m,

$v_1 = 10$ ms^{-1}, $v_2 = 5$ ms^{-1}

The positions of the two buses at any instant t are

$$x_1(t) = x_1(0) + v_1 t = 50 + 10t$$

$$x_2(t) = x_2(0) + v_2 t = 100 + 10t$$

When A overtakes B,

$$x_1(t) = x_2(t)$$

$$50 + 10t = 100 + 5t$$

or $$5t = 50$$

$$t = 10 \text{ s}$$

$$x_1(10) = x_2(10) = 150 \text{ m}$$

Thus, A overtakes B at a position of 150 m from the origin at time $t = 10$ s. ■

EXAMPLE 15

A bus starts from rest with constant acceleration of 5 ms^{-2}. At the same time a car travelling with a constant velocity of 50 ms^{-1} overtakes and passes the bus. (a) Find at what distance will the bus overtake the car? (b) How fast will the bus be travelling then?

SOLUTION

(a) Suppose the bus overtakes the car after covering distance s.

When the two meet, time taken t is same.

For bus, $$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 5t^2$$

For car, $$s = 50t$$

$$\frac{5}{2}t^2 = 50t$$

or $$t = 20 \text{ s}$$

Hence, $$s = 50t = 50 \times 20$$

$$= 1000 \text{ m}.$$

(b) $v^2 = u^2 + 2as = 0 + 2 \times 5 \times 1000$

$= 10{,}000$

or $$v = 100 \text{ ms}^{-1}$$ ■

SPECIMEN PROBLEM

(E) *Maximum Separation*

EXAMPLE 16

What is the maximum separation between the car and the scooter?

SOLUTION

Initially, separation between the car and scooter increases and then decreases.

Separation between them will be maximum at an instant at which velocity of the car is equal to the velocity of scooter.

Velocity car = Velocity scooter = 40 m/s

from $$v = u + at$$

$$40 = 0 + 4t$$

$\Rightarrow$ $$t = 10 \text{ s}$$

The distance travelled by the scooter in 10 s. is

$$S_1 = 10 \times 40 = 400 \text{ m}$$

The distance travelled by the car in 10 s. is

$$S_2 = ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2} \times 4 \times (10)^2$$

$$= 200 \text{ m}$$

So maximum separation $= S_1 - S_2 = 400 - 200$

$$= 200 \text{ m}$$

- When they meet both will travel the same distance. ■

Notes

- Questions 14, 15, 16 can be solved in a relatively easier way using the concept of relative motion which is explained later.
- Now you can try Questions 1–13 in Exercise I and 1–6 in Exercise II.

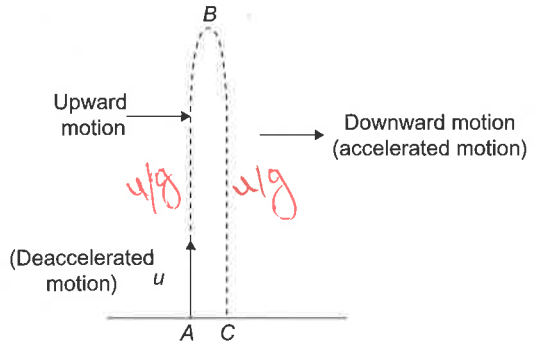
MOTION UNDER GRAVITY

I Format (When a Body is Thrown Vertically Upward)

Assume that the effect of air friction and wind resistance is negligible and the value of acceleration due to gravity ($\vec{g}$) is constant.

It includes two types of motion

1. Deaccelerated motion from A to B because the direction of velocity and acceleration is opposite. So speed decreases
2. Accelerated motion from B to C because the direction of velocity and acceleration is the same (downward). Thus, speed increases.



(a) *Time of Flight*

It is the time taken by the particle to reach the ground. If the particle is thrown vertically upward with initial velocity u, then

$$u_i = u$$

$a = -g$ (considering downward direction as negative)

from equation

$$S = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad S_{net} = 0$$

(when particle again reaches the ground)

$$t = T \text{ (time of flight)}$$

$$0 = uT - \frac{1}{2}gT^2$$

$$\Rightarrow \quad T = \frac{2u}{g}$$

(b) *Maximum Height*

From $\quad v^2 = u^2 + 2as$

at maximum height $v = 0$, $s = H_{max}$

$$\Rightarrow \quad 0 = u^2 - 2gH_{max}$$

$$\Rightarrow \quad H_{max} = \frac{u^2}{2g}$$

(c) *Final Velocity*

From $\quad v = u + at$

$$v = v_f$$

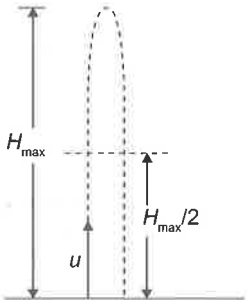
$$a = -g$$

$$t = T = \frac{2u}{g}$$

$$\Rightarrow \quad v_f = u - g\frac{2u}{g}$$

$$v_f = -u$$

i.e., the body reaches the ground with the same speed with which it was thrown vertically upwards.



(d) *Time to Reach Half of the Maximum Height*

$$u_i = u$$

$$a = -g$$

From $\quad S = ut + \frac{1}{2}at^2$

$$\frac{H}{2} = ut - \frac{1}{2}gt^2$$

$$H = 2ut - gt^2$$

$$\Rightarrow \quad gt^2 - 2ut + H = 0$$

$$\Rightarrow \quad t = \frac{2u \pm \sqrt{4u^2 - 4gH}}{2g}$$

$$\Rightarrow \quad t = \frac{2u \pm \sqrt{4u^2 - 4g \times \frac{u^2}{2g}}}{2g}$$

$$\left\{ \because H_{max} = \frac{u^2}{2g} \right.$$

$$t = \frac{u(2 \pm \sqrt{2})}{2g} \qquad (1)$$

Equation (1) gives the value of time, which corresponds to

$$t_1 = \frac{u(2 - \sqrt{2})}{2g}$$

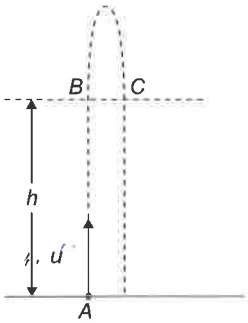
(from ground to $H_{max}/2$ in upward motion)

$$t_2 = \frac{u(2 + \sqrt{2})}{2g}$$

(from ground to $H_{max}/2$ in downward motion)

(e) *Time to Reach any General Height h*

Let us assume that the particle reaches from A to B in time t_1 and from A to C, in time t_2.



So from $\quad S = ut + \frac{1}{2}at^2$

$$\Rightarrow \quad h = ut - \frac{1}{2}gt^2$$

$$gt^2 - 2ut + 2h = 0$$

$$\Rightarrow \quad t = \frac{2u \pm \sqrt{4u^2 - 8gh}}{2g}$$

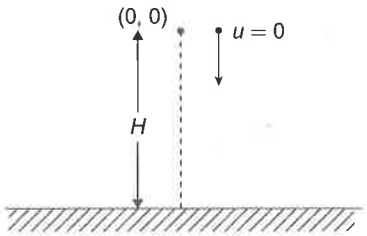
Thus, $\quad t_1 = \frac{u - \sqrt{u^2 - 2gh}}{g}$,

$$t_2 = \frac{u + \sqrt{u^2 - 2gh}}{g}$$

$$\Rightarrow \quad t_1 + t_2 = T \text{ (Time of flight)}$$

II Format (Free Fall)

A body released near the surface of the earth is accelerated downward under the influence of force of gravity.



(a) *Time of Flight*

From the equation $\quad S = ut + \frac{1}{2}at^2$

$$S = -H, u = 0,$$

$$a = -g$$

$$t = T \text{ (Let us assume)}$$

$$\Rightarrow \quad -H = (0)T - \frac{1}{2}gt^2$$

$$\Rightarrow \quad T = \sqrt{\frac{2H}{g}}$$

(b) *Final Velocity When the Body Reaches the Ground*

From $\quad v^2 - u^2 = 2as$

$$s = -H$$

$$v = v_f$$

$$u = 0$$

$$a = -g$$

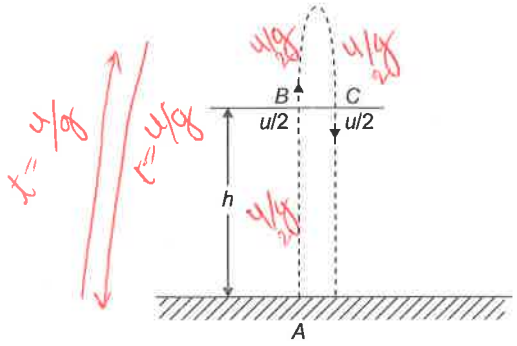
$\Rightarrow \quad v_f^2 - 0 = 2(-g)(-H)$

$\Rightarrow \quad v_f = \sqrt{2gH}$

SOLVED EXAMPLES

EXAMPLE 17

A ball is thrown vertically upwards with a velocity u from the ground. The ball attains a maximum height H_{max}. Find out the time and displacement at which the ball has half the maximum speed.



SOLUTION

Maximum speed of the ball is u

At point B and C, the ball has a speed $u/2$, but direction is opposite. Thus, consider the equation

$$v = u + at$$

Let t_1 is the time taken by the ball to move from point A to B and t_2 is the time taken by the ball to move from point A to C

From A to B, $\quad \frac{u}{2} = u - gt_1 \qquad (1)$

From A to C, $\quad -\frac{u}{2} = u - gt_2 \qquad (2)$

From (1) $\quad t_1 = \frac{u}{2g}$,

From (2) $\quad t_2 = \frac{3u}{2g}$

From the equation $\quad v^2 - u^2 = 2as$

$\Rightarrow \quad v = \pm u/2,\ u = u,$

$a = -g$

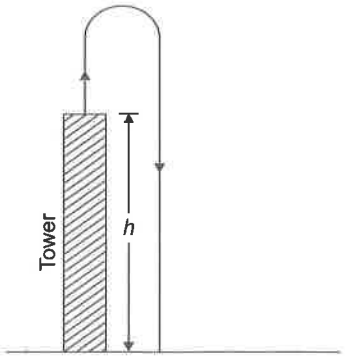
$\Rightarrow \quad \left(\frac{u}{2}\right)^2 - u^2 = -2gh$

$h = \frac{3u^2}{8g} \qquad \left\{\frac{u^2}{2g} = H_{max}\right\}$

$\therefore \quad h = \frac{3}{4} H_{max}$ ■

EXAMPLE 18

A ball is thrown vertically upwards with a speed of 19.6 ms^{-1} from the top of a tower and returns to the earth in 6 s. Find the height of the tower.



SOLUTION

Here $\quad u = 19.6\ ms^{-1}$

$g = -9.8\ ms^{-2}$

Net displacement, $\quad s = -h$

Negative sign is taken because displacement is in the opposite direction of the initial velocity.

As $\quad s = ut + \frac{1}{2}gt^2$

$\therefore \quad -h = 19.6 \times 6 + \frac{1}{2} \times (-9.8) \times 6^2$

$= 117.6 - 176.4 = -58.8$

or $\quad h = 58.8\ m$ ■

EXAMPLE 19

A ball is thrown vertically upwards with a velocity of 20 ms^{-1} from the top of a multi-storeyed building. The height

of the point from where the ball is thrown is 25 m from the ground. (i) How high will the ball rise and (ii) how long will it be before the ball hits the ground?

SOLUTION

(i) Here $u = +20\ \text{ms}^{-1}$, $g = -10\ \text{ms}^{-2}$
At the highest point, $v = 0$
Suppose the ball rises to the height h from the point of projection.

As $v^2 - u^2 = 2gs$

$\therefore$ $0^2 - 20^2 = 2 \times (-10) \times h$

or $h = +20$ m.

(ii) Net displacement, $s = -25$ m
Negative sign is taken because displacement is in the opposite direction of initial velocity.

As $s = ut + \frac{1}{2}gt^2$

$\therefore$ $-25 = 20t + \frac{1}{2} \times (-10) \times t^2$

or $5t^2 - 20t - 25 = 0$

or $t^2 - 4t - 5 = 0$

or $(t + 1)(t - 5) = 0$

As $t \neq -1$; thus, $t = 5$ s. ■

EXAMPLE 20

A ball is thrown up and is caught by the thrower after 4 s. How high did it go and with what velocity was it thrown? How far was it below the highest point 3 s after it was thrown?

SOLUTION

As time of ascent = time of descent
$\therefore$ Time taken by the ball to reach the highest point = 2 s
For upward motion of the ball: $u = ?$, $v = 0$,

$t = 2$s, $g = -9.8\ \text{ms}^{-2}$

As $v = u + gt$

$\therefore$ $0 = u - 9.8 \times 2$

or $u = 19.6\ \text{ms}^{-1}$

Maximum height attained by the ball is given by

$$s = ut + \frac{1}{2}gt^2$$

$$= 19.6 \times 2 + \frac{1}{2} \times (9.8) \times 2^2$$

$$= 19.6 \text{ m}.$$

Displacement of the ball in 3 s,

$$s = 19.6 \times 3 + \frac{1}{2} \times (-9.8) \times 3^2$$

$$= 58.8 - 44.1$$

$$= 14.7 \text{ m}$$

Distance of the ball from the highest point 3 s after it was thrown

$$= 19.6 - 4.7 = 4.9 \text{ m}.$$ ■

EXAMPLE 21

A balloon is ascending at the rate of $9.8\ \text{ms}^{-1}$ at a height of 39.2 m above the ground when a food packet is dropped from the balloon. After how much time and with what velocity does it reach the ground?
Take $g = 9.8\ \text{ms}^{-2}$.

SOLUTION

Initially, the food packet attains the upward velocity of the balloon, so

$$u = 9.8\ \text{ms}^{-1},\ g = 9.8\ \text{ms}^{-2},$$

$$s = -39.2 \text{ m}$$

Here s is considered negative because it is in the opposite direction of initial velocity.

Using, $s = ut + \frac{1}{2}gt^2$, we get

$$-39.2 = 9.8t - \frac{1}{2} \times 9.8t^2$$

or $4.9t^2 - 9.8t - 39.2 = 0$

or $t^2 - 2t - 8 = 0$

or $(t - 4)(t + 2) = 0$

or $t = 4$ s

or -2 s

As time is never negative, so $t = 4$ s.

Velocity with which the food packet reaches the ground is

$$v = u + gt$$

$$= 9.8 - 9.8 \times 4$$

$$= -29.4 \text{ ms}^{-1}.$$

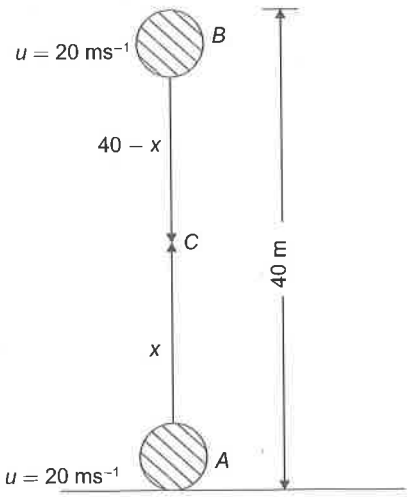
Negative sign shows that the velocity is directed vertically downwards. ■

Note

When a particle is dropped, then it will automatically attain the velocity of the frame at that time.

EXAMPLE 22

Two balls are thrown simultaneously. A vertically upwards with a speed of 20 ms^{-1} from the ground and B vertically downwards from a height of 40 m with the same speed and along the same line of motion. At what point do the two balls collide? Take $g = 9.8 \text{ ms}^{-2}$.



SOLUTION

Suppose the two balls meet at a height of x from the ground after time t from the start.

For upward motion of balls A:

$$u = 20 \text{ ms}^{-1}, g = -9.8 \text{ ms}^{-2}$$

$$s = ut + \frac{1}{2}gt^2$$

$$x = 20t - \frac{1}{2} \times 9.8t^2$$

$$= 20t - 4.9t^2 \quad (1)$$

For downward motion of ball B,

$$40 - x = 20 \times t + \frac{1}{2} \times 9.8t^2$$

$$= 20t + 4.9t^2 \quad (2)$$

Adding (1) and (2), $40 = 40t$

or $t = 1$ s

From (i), $x = 20 \times 1 - 4.9 \times (1)^2$

$$= 15.1 \text{ m}$$

Hence, the two balls will collide after 1 s at a height of 15.1 m from the ground. ■

GRAPHS

(i) Straight line

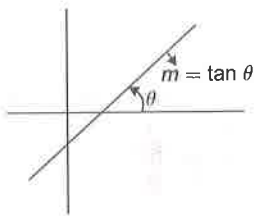
A linear relation between y and x represents a straight line.

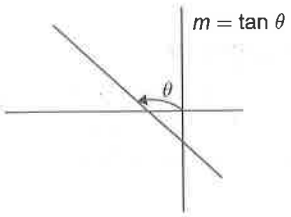
General equation of a straight line

$$y = mx + c$$

$m \equiv$ slope of line

$c \equiv y$ intercept, i.e. where the line cuts the y-axis.





Slope is defined as the **tan** of the angle made by the straight line with positive x-axis in the anti-clockwise direction $m < 0$

$\Rightarrow$ $\theta > 90°$

$m > 0$

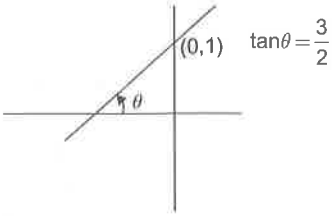
$\Rightarrow$ $\theta < 90°$

$0° \leq \theta < 180°$

SOLVED EXAMPLES

EXAMPLE 23

Draw the graph for the equation: $2y = 3x + 2$.



SOLUTION

$$2y = 3x + 2$$

$$\Rightarrow \quad y = \frac{3}{2}x + 1$$

$$m = \frac{3}{2} > 0$$

$$\Rightarrow \quad \theta < 90°$$

$$c = +1 > 0$$

$$\Rightarrow$$ The line will pass through (0, 1). ■

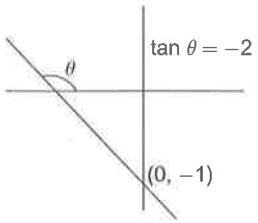
EXAMPLE 24

Draw the graph for the equation: $2y + 4x + 2 = 0$.

SOLUTION

$$2y + 4x + 2 = 0$$

$$\Rightarrow \quad y = -2x - 1$$



$$m = -2 < 0,$$

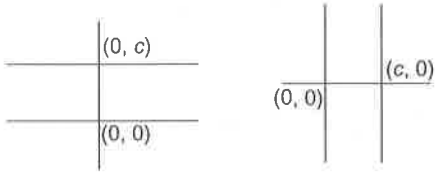
i.e., $\theta > 90°$

$$c = -1,$$

i.e., line will pass through (0, −1). ■

Note

(i) If $c = 0$, line will pass through origin.
(ii) $y = c$ will be a line parallel to the x axis.
(iii) $x = c$ will be a line perpendicular to the y axis



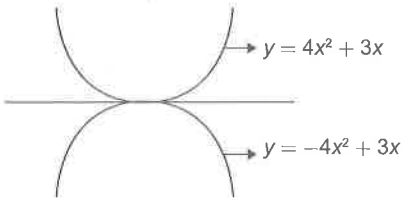
(ii) Parabola

A general quadratic equation represents a parabola.

$$y = ax^2 + bx + c$$

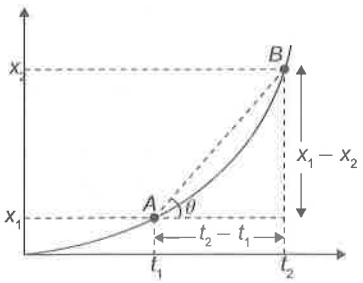
$$a \neq 0$$

If $a > 0$, it will be a opening upwards parabola.
If $a < 0$, it will be a opening downwards parabola.
If $c = 0$, it will pass through origin.



e.g. $y = 4x^2 + 3x$

Average Velocity and Instantaneous Velocity from Position Versus Time Graph



Average velocity from t_1 to t_2

$$= \frac{\text{displacement}}{\text{time taken}} = \frac{x_2 - x_1}{t_2 - t_1}$$

$= \tan\theta =$ slope of the chord AB

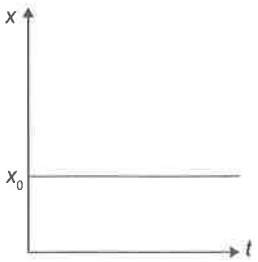
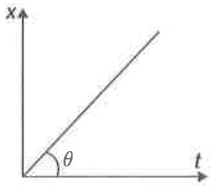
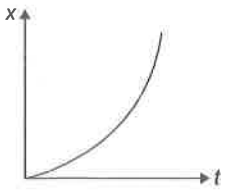
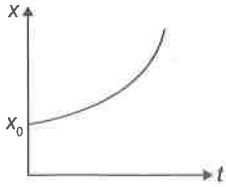
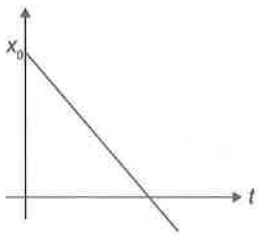
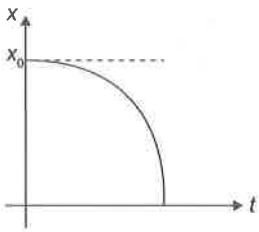
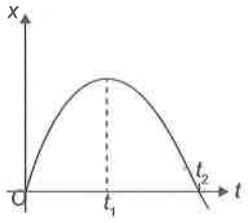
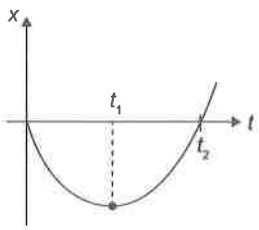
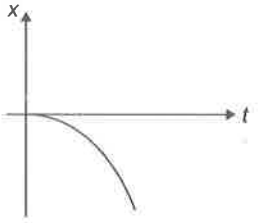
$$v_{\text{instantaneous}} = \text{as } \lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1}$$

when t_2 approaches t_1, point B approaches point A and the chord AB becomes tangent to the curve. Therefore,

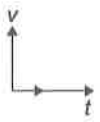
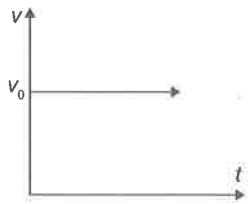
$$v_{\text{instantaneous}} = \text{Slope of the tangent } x\text{–}t \text{ curve}$$

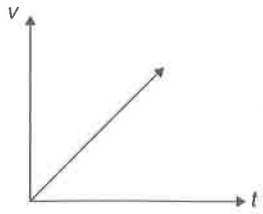
(iii) Reading the graph

(A) Reading x Versus t Graphs

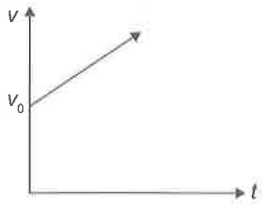
	Explanation
(1) 	Body is at rest at x_0.
(2) 	Body starts from origin and is moving with speed tan θ away from the origin.
(3) 	Body starts from rest from origin and moves away from the origin with increasing speed velocity and positive acceleration.
(4) 	Body starts from rest from $x = x_0$ and moves away from the origin with increasing velocity or +ve acceleration.
(5) 	Body starts from $x = x_0$ and is moving towards the origin with constant velocity and passes through the origin after some time and continues to move away from origin.
(6) 	Body starts from rest at $x = x_0$ and then moves with increasing speed towards origin. ∴ acceleration is −ve.
(7) 	Body starts moving away from the origin with some initial speed. Speed of the body is decreasing till t_1 and it becomes 0 momentarily at $t = t_1$, and at this instant, it reverses its direction and moves towards the origin with increasing speed.
(8) 	Body starts from the origin and moves away from the origin in the −ve x-axis at $t = t_1$ with decreasing speed and at $t = t_1$, it comes at rest momentarily and reverses its direction and moves towards the origin with increasing speed. It crosses the origin at $t = t_2$.
(9) 	Body starts from origin from rest and moves away from the origin with increasing speed.

(B) v-t Graphs

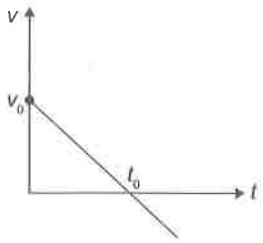
(1) 	Body is always at rest.
(2) 	Body is moving with constant velocity v_0

(3) 

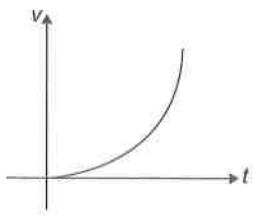
Body is at rest initially then it starts moving with its velocity increasing at a constant rate, i.e., body is moving with constant acceleration.

(4) 

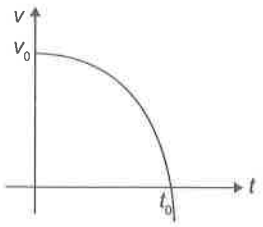
Body starts its motion with initial velocity v_0 and continues to move with its velocity increasing at a constant rate i.e., acceleration of the body is constant.

(5) 

Body starts its motion with initial velocity v_0. Then it continues to move with its velocity decreasing at a constant rate, i.e. acceleration of the body is negative and constant. At $t = t_0$ the body comes to rest instantaneously and reverses its direction of motion and then continues to move with decreasing velocity or increasing speed.
For $0 < t < t_0$, the motion of the body is de-accelerated ($\because$ speed is decreasing).
For $t > t_0$, the motion of the body is accelerated ($\because$ speed is increasing).

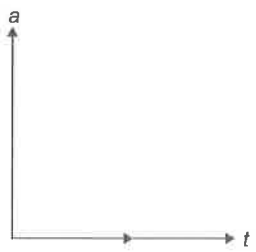
(6) 

Body is at rest initially. Then it starts moving with increasing velocity. As time increases, its velocity increases more rapidly, i.e. moving with increasing acceleration.

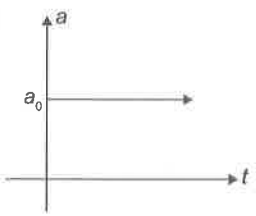
(7) 

Body starts its motion with initial velocity v_0. Its velocity is decreasing with time and at $t = t_0$, it becomes zero. After this body reverses its direction of motion and continues to move with decreasing velocity or increasing speed. Since velocity of the body is decreasing for whole motion, its acceleration is negative. For $0 < t < t_0$, the motion of the body is deaccelerated (speed is decreasing) and for $t > t_0$ motion of the body is accelerated ($\because$ speed is increasing)

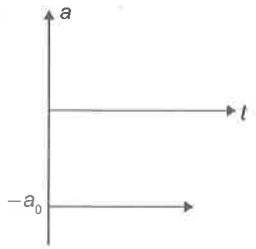
(C) Reading of *a*–*t* Graphs

(1) 

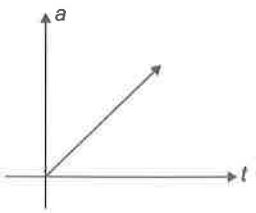
Acceleration of the body is zero indicates that the body is moving a constant velocity.

(2) 

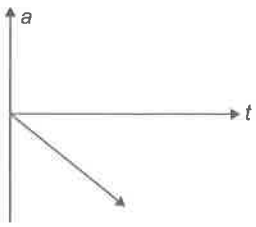
Acceleration of the body is constant and positive.

(3) 

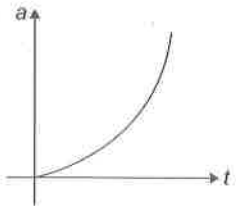
Acceleration of the body is constant and negative.

(4) 

Initially, the acceleration of the body is zero. Then its acceleration increases at a constant rate.

(5) 

The body starts accelerating (initial acceleration zero) at $t = 0$. Its acceleration is negative for its entire motion and decreases at a constant rate.

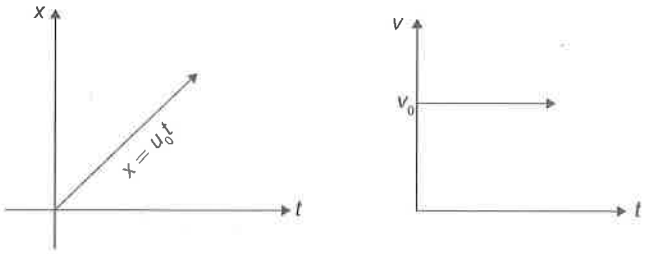
(6) 

Initially, acceleration of the body is zero. Its acceleration is positive for its entire motion. Its acceleration is increasing for the whole of its motion.

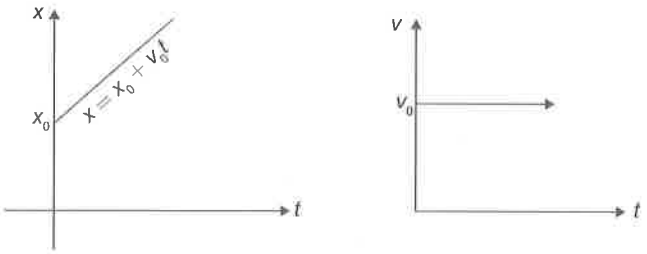
(iv) Drawing of Graphs on the Basis of the Given Information

(a) *If Acceleration of the Body is Zero*

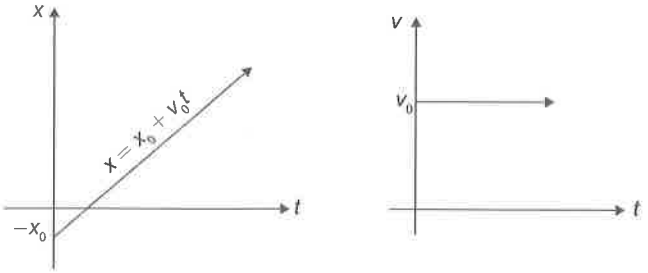
(i) If the velocity of the body is v_0 and it starts from origin:



(ii) If at $t = 0, x = x_0$ then,



(iii) If at $t = 0, x = -x_0$ then,



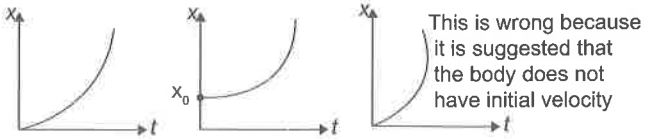
(b) *If a Body has Constant Acceleration*

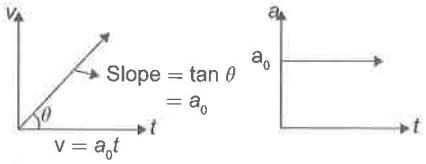
For this section

(i) u_0, x_0 and a_0 are positive constants.
(ii) $u \equiv$ initial velocity.
(iii) $v \equiv$ velocity at any time t.
(iv) $x \equiv$ Position at any time t.
$x_i \equiv$ initial position

(i) If $u = 0, a = a_0$

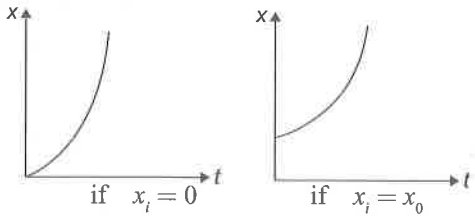
If $x_i = 0, x = \frac{1}{2}at^2$ if $x_i = x_0, x = x_0 + (1/2)at^2$

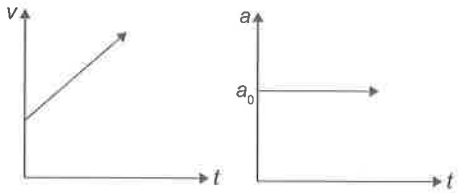




(ii) If $u = u_0, a = a_0$

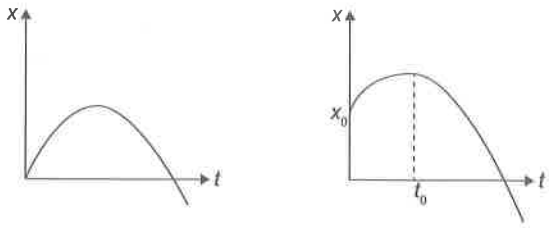
$$x = x_i + u_0t + \frac{1}{2}a_0t^2 \qquad v = u_0 + a_0t$$

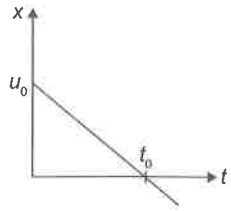


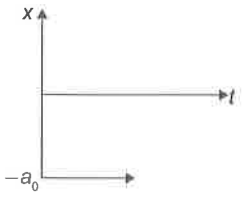


(iii) If $u = u_0, a = -a_0$

$$x = x_i + u_0t - \frac{1}{2}a_0t^2$$

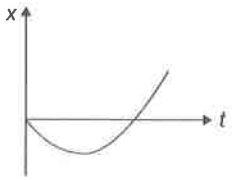




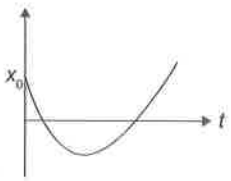


(iv) If $u = -u_0$, $a = +a_0$

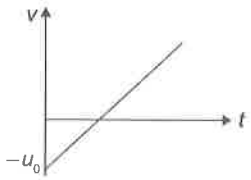
$$x = x_i - u_0 t + \frac{1}{2} a_0 t^2$$

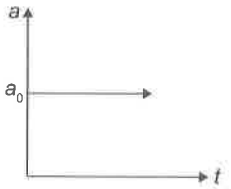


if $x_i = 0$



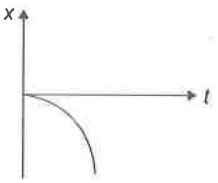
if $x_i = x_0$



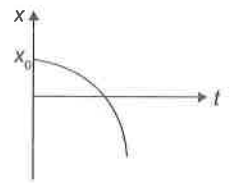


(v) If $u = u_0$, $a = -a_0$

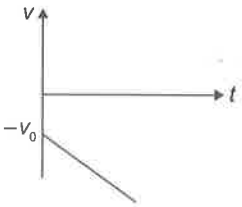
$$x = x_i - u_0 t - \frac{1}{2} a_0 t^2$$

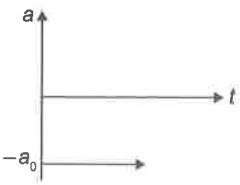


if $x_i = 0$



if $x_i = x_0$





SOLVED EXAMPLE

EXAMPLE 25

Draw the following:

(a) position versus time graph

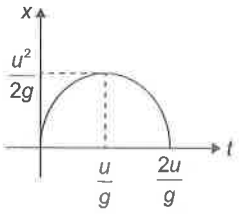
(b) velocity versus time graph

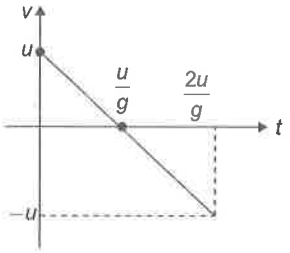
(c) acceleration versus time graph

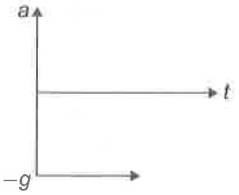
for the given cases.

(i) If a body is projected vertically upwards with initial velocity u. Take the projection point to be the origin and upward direction as positive.

$$x = ut - \frac{1}{2} gt^2$$

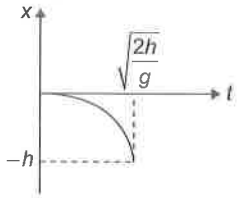


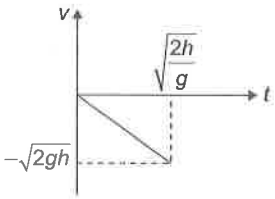




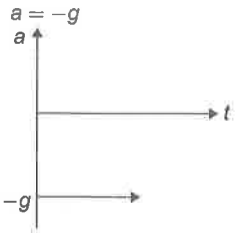
(ii) If a body is dropped from a height h above the ground. Take the dropping point to be the origin and upward direction as positive.

$$x = -\frac{1}{2} gt^2$$

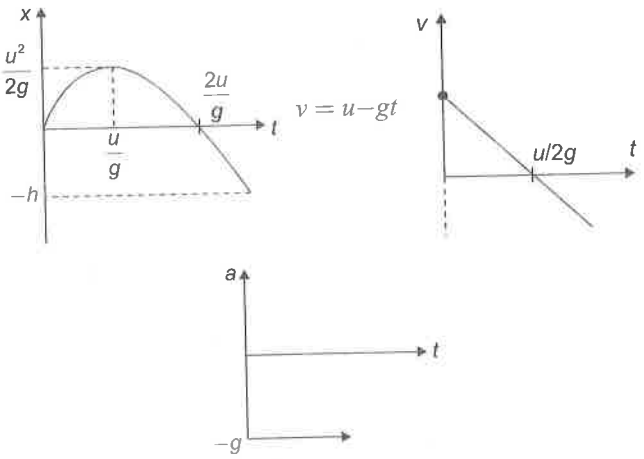




$v = -gt$



(iii) If a body is projected vertically upwards from a tower of height h with initial velocity u. Take the projection point to be the origin and upward direction as positive.



(iv) A car starting from rest accelerates uniformly at 2 ms^{-2} for 5 s and then moves with constant speed acquired for the next 5 s and then comes to rest retarding at 2 ms^{-2}.

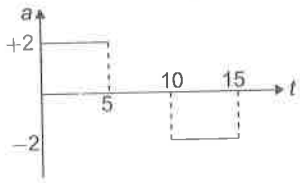
Draw its

(a) position versus time graph

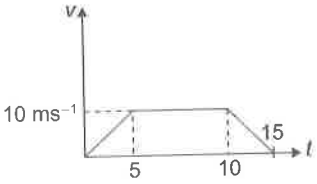
(b) velocity versus time graph

(c) acceleration versus time graph

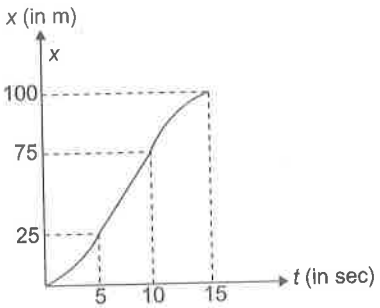
Acceleration versus time graph



acceleration versus time graph

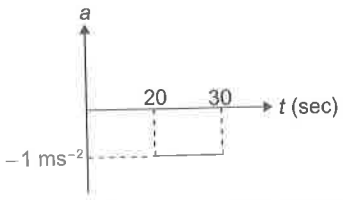


velocity versus time graph

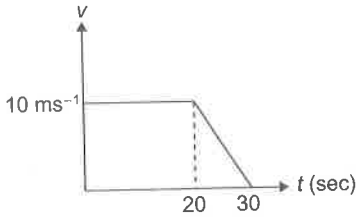


Position versus time graph

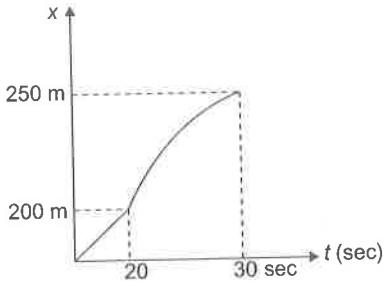
(v) A particle starts from $x = 0$ and initial speed 10 ms^{-1} and moves with constant speed 10 ms^{-1} for 20 s. and then retarding uniformly comes to rest in next 10 s.



Acceleration versus time graph



Velocity versus time graph



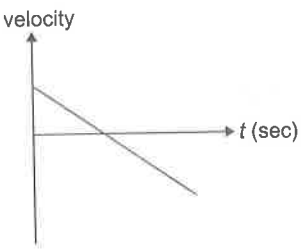
Position versus time graph

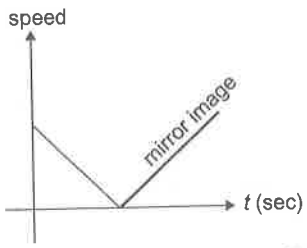
(v) Conversion of Velocity *Versus* Time Graph to Speed Versus Time Graph

We know that magnitude of velocity represents speed; therefore, whenever velocity becomes negative, take its mirror image about the time axis.

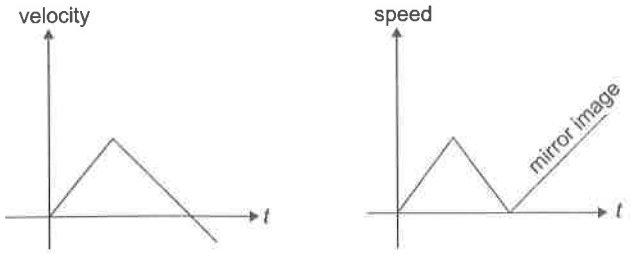
SOLVED EXAMPLES

EXAMPLE 26

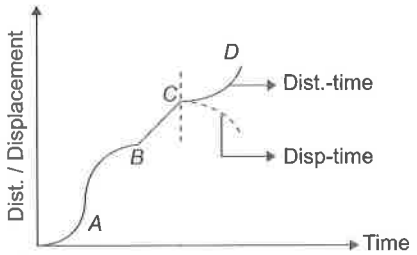




EXAMPLE 27



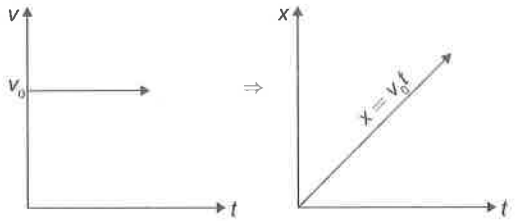
(vi) Conversion of Displacement Versus Time Graph to Distance Versus Time Graph



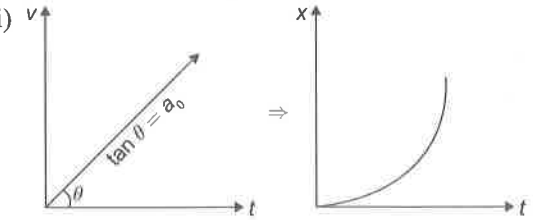
For the distance–time graph, just make the mirror image of the displacement–time graph from the point of zero velocity onwards.

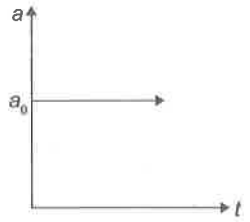
(vii) Conversion of *v–t* Graphs into *x–t* and *a–t* Graphs

(i)



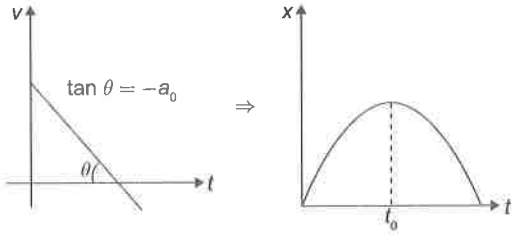
(ii)

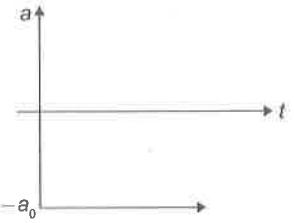




a–t graph

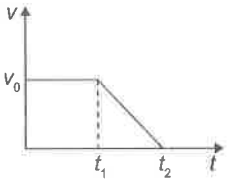
(iii)





at $t = t_0$, velocity reverses its direction.

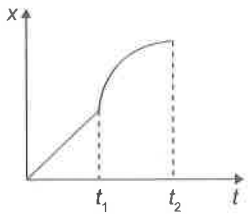
(iv) $x-t$ graph



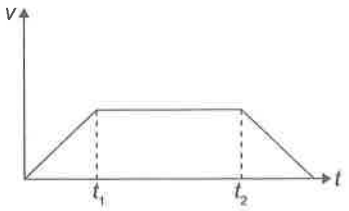
From $t = 0$ to $t = t_1$, acceleration $= 0$. Therefore, from $t = 0$ to $t = t_1$, the $x-t$ graph will be a straight line.

From $t = t_1$ to t_2, acceleration is negative.

$\therefore$ It will be an opening downward parabola.

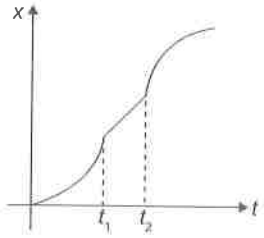


(v) Up to $t = t_1$, acceleration is positive.



For $t_1 < t < t_2$, acceleration is zero.

For $t > t_2$, acceleration is negative.



x–t graph

Some Important Points

$$a = \frac{dv}{dt}$$

$$\Rightarrow \quad \int dv = \int a\,dt$$

$\Rightarrow$ Δv = area under the a–t curve

$$v = \frac{dx}{dt}$$

$$\Rightarrow \quad \int dx = \int v\,dt$$

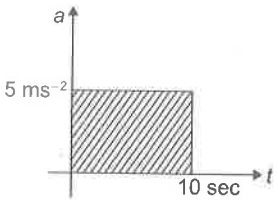
$\Rightarrow$ Δx = area under the $v - t$ curve

$\Rightarrow$ displacement = area under the $v - t$ curve.

EXAMPLE 28

If at $t = 0$, $u = 5\ \text{ms}^{-1}$, then velocity at $t = 10$ s

SOLUTION



$= u +$ change in velocity

$= 5 +$ area of the shaded part

$= 5 + 10 \times 5$

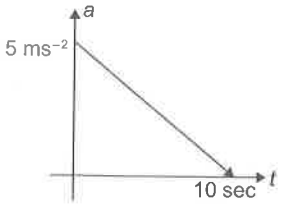
$= 55\ \text{ms}^{-1}$ ■

EXAMPLE 29

If at $t = 0$, $u = 2\ \text{ms}^{-2}$, find out its maximum velocity.

SOLUTION

Since the entire motion is accelerating, the velocity will be maximum at the end of the motion, which will be

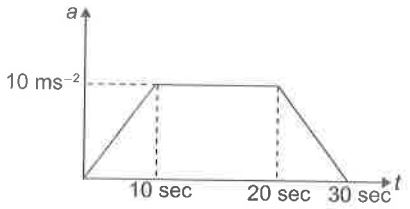


$$= 2 + \frac{1}{2} \times 5 \times 10 = 27\ \text{ms}^{-1}$$ ■

EXAMPLE 30

If at $t = 0$, $u = 4\ \text{ms}^{-1}$, find out v at $t = 10$ s, $t = 20$ s and $t = 30$ s.

SOLUTION



Since for the entire motion acceleration of the body is positive,

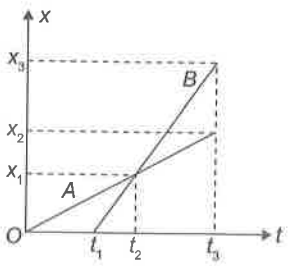
$$v_{t=10\,\text{s}} = 4 + \frac{1}{2} \times 10 \times 10 = 54\ \text{ms}^{-1}$$

$$v_{t=20\,\text{s}} = 4 + \frac{1}{2} \times 10 \times 10 + 10 \times 10 = 154\ \text{ms}^{-1}$$

$$v_{t=30\,\text{s}} = 154 + \frac{1}{2} \times 10 \times 10 = 204\ \text{ms}^{-1}$$ ■

(viii) Reading of Graphs if the Motion of Two Bodies is Sketched on the Same Axes

Reading of the x-t Graphs



Conclusion:

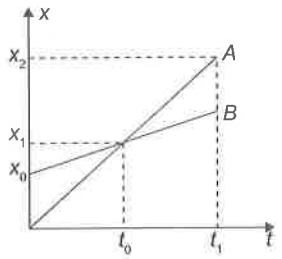
1. Body A starts its motion at $t = 0$ from the origin and is moving away from the origin with constant velocity. Finally, it ends its motion at a distance of x_2m from origin at $t = t_3$.

2. Body B starts its motion at $t = t_1$ from the origin and is moving away from the origin with constant velocity. Finally, it ends its motion at a distance of x_3m from the origin at $t = t_3$.
3. Since slope of B is greater than slope of A, the velocity of B is greater than the velocity of A.
4. A $t = t_2$, both A and B are at the same distance from the starting point, which means that B overtakes A at $t = t_2$.
5. Since velocity of both A and B are constant, acceleration of both the bodies are zero.
6. Therefore, $x_3 > x_2$.

$\therefore$ At the end of the motion, B is at a greater distance from the starting point.

SOLVED EXAMPLES

EXAMPLE 31



Conclusion:

(i) Body A starts its motion at $t = 0$ from the origin and is moving away from the origin with a constant velocity. Finally, its motion ends at $t = t_1$ at $x = x_2$ m.

(ii) Body B starts its motion at $t = 0$ from $x = x_0$ and then moves with a constant velocity away from the origin. Finally, it ends its motion at $t = t_1$.

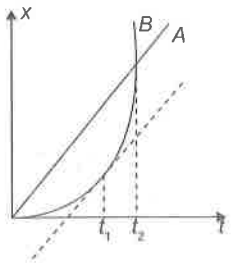
(iii) Velocity of A is greater than that of B.

(iv) At $t = t_0$, A overtakes B.

(v) Acceleration of both A and B is zero.

(vi) Since $x_2 > x_1$, at the end of the motion, A is at a greater distance from the starting point than B.

EXAMPLE 32



Conclusion:

(i) Both A and B start their motion at the same time $t = 0$ and from the same point $x = 0$.

(ii) Both are moving away from the starting point.

(iii) A is moving with a constant velocity while B starts its motion from rest and its velocity is increasing with time, i.e., it has some positive acceleration.

(iv) At $t = t_1$ the tangent on B's graph becomes parallel to the A's graphs.
At $t = t_1$ velocity of both A and B is same.

(v) For $t < t_1$, velocity of A is greater than the velocity of B. Therefore, up to $t = t_1$, separation between A and B increases with time.

(vi) For $t > t_1$, velocity of B is greater than the velocity of A. Therefore, after $t = t_1$, separation between A and B starts decreasing and it becomes zero at $t = t_2$ where B overtakes A.

Note

Now you can try Questions 14–38 in Exercise I and Questions 7–11 in Exercise II.

TWO-DIMENSIONAL MOTION OR MOTION IN A PLANE

Motion in a plane can be described by the vector sum of two independent 1-***D*** motions along two mutually perpendicular directions (as motions along two mutual directions do not affect each other).

Consider a particle moving in the X–Y plane. Its equations of motions for X and Y axes are

$$v_x = u_x + a_x t,$$

$$v_y = u_y + a_y t$$

$$x = u_x t + \frac{1}{2} a_x t^2, y = u_y t + \frac{1}{2} a_y t^2,$$

and $$v_x^2 = u_x^2 + 2a_x x$$

$$v_y^2 = u_y^2 + 2a_y y,$$

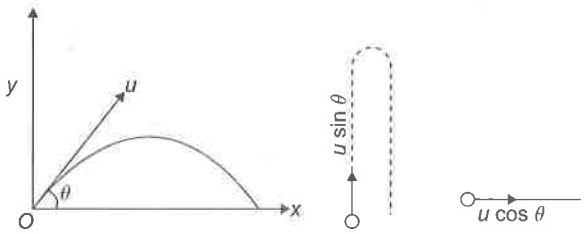
where the symbols have their usual meanings. Thus, the resultant motion would be described by the equations

$$\vec{r} = x\hat{i} + y\hat{j}$$

and $$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

Projectile Motion

It is the best example to understand the motion in a plane. If we project a particle obliquely from the surface of the earth, as shown in the figure below, then it can be considered as two perpendicular 1-D motions—one along the horizontal and the other along the vertical directions.



Assume that the effect of air friction and wind resistance is negligible and the value of acceleration due to gravity ($\vec{g}$) is constant.

Take the point of projection as the origin and the horizontal and vertical directions as +ve x- and y-axes, respectively.

For x-axis	For y-axis
$u_x = u\cos\theta$,	$u_y = u\sin\theta$
$a_x = 0$,	$a_y = -g$,
$v_x = u\cos\theta$ and	$v_y = u\sin\theta - gt$ and
$x = u\cos\theta \times t$	$y = u\sin\theta t - \frac{1}{2}gt^2$

It is clear from the above equations that the horizontal component of velocity of the particle remains constant while the vertical component of velocity is first decreasing, becomes zero at the highest point of trajectory and then increases in the opposite direction. At the highest point, speed of the particle is minimum.

The time that the projectile takes to come back to the same (initial) level is called the time of flight (T).

At the initial and final points, $y = 0$.

Thus, $u\sin\theta t - \frac{1}{2}gt^2 = 0.$

$\Rightarrow$ $t = 0$

and $t = \frac{2u\sin\theta}{g}.$

Thus, $T = \frac{2u\sin\theta}{g}.$

Range (R): The horizontal distance covered by the projectile during its motion is said to be the range of the projectile.

$$R = u\cos\theta \times T = \frac{u^2\sin 2\theta}{g}.$$

For a given projection speed, the range would be maximum for $\theta = 45°$.

Maximum height attained by the projectile is

$$H = \frac{u^2\sin^2\theta}{2g}.$$

At maximum height, the vertical component of velocity is 0.

Time of ascent = Time of descent

$$= \frac{u\sin\theta}{g}$$

$$= \frac{T}{2}.$$

Notes

- Speed, kinetic energy and momentum of the particle initially decrease in a projectile motion and attain a minimum value (not equal to zero) and then again increase.
- θ is the angle between $\vec{v}$ and the horizontal, which decreases to zero (at the topmost point) and again increases in the negative direction.

SOLVED EXAMPLES

EXAMPLE 33

A body is projected with a velocity of 30 ms^{-1} at an angle of 30° with the vertical. Find the maximum height, time of flight and the horizontal range.

SOLUTION

Here $u = 30\ ms^{-1}$,

Angle of projection,

$\theta = 90 - 30 = 60°$

Maximum height,

$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{30^2 \sin^2 60^\circ}{2\times 9.8}$$

$$= 34.44 \text{ m}$$

Time of flight,

$$T = \frac{2u\sin\theta}{g} = \frac{2\times 30 \sin 60^\circ}{9.8}$$

$$= 5.3 \text{ s}$$

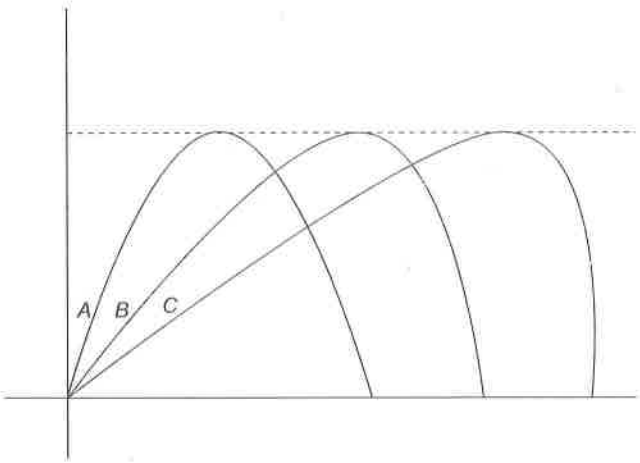
Horizontal range,

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{30^\circ \sin 120^\circ}{9.8}$$

$$= \frac{30^2 \sin 60^\circ}{9.8} = 79.53 \text{ m}.$$

EXAMPLE 34

Find out the relation between u_A, u_B and u_C (where u_A, u_B and u_C are the initial velocities of the particles A, B and C, respectively).



SOLUTION

H_{max} is the same for all three particle A, B, C.

$$\Rightarrow \quad H_{max} = \frac{u_y^2}{2g}$$

$\Rightarrow$ u_y is the same for all the three particles

$$\therefore \quad u_{yA} = u_{yB} = u_{yC}$$

$$\Rightarrow \quad T_A = T_B = T_C \left(\frac{2u_y}{g}\right)$$

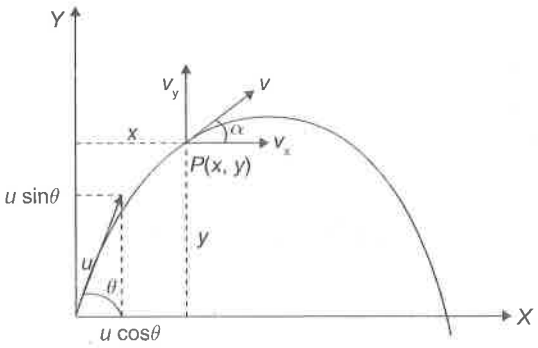
From the figure, $R_C > R_B > R_A$

$$\because \quad R = \frac{2u_x u}{g}$$

$$\Rightarrow \quad u_{xC} > u_{xB} > u_{xA}$$

$$\Rightarrow \quad u_A < u_B < u_C$$

Coordinates of a Particle after a Given Time t



The particle reaches point P after time t. Then,

$$x = u\cos\theta t$$

$$y = u\sin\theta t - \frac{1}{2}gt^2$$

Position vector

$$\vec{r} = (u\cos\theta \times t)\hat{i} + \left((u\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}.$$

Velocity and Direction of Motion after a Given Time

After time t, $v_x = u\cos\theta$

and $\quad v_y = u\sin\theta - gt.$

Hence, the resultant velocity $v = \sqrt{v_x^2 + v_y^2}$

$$= \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$$

$$\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{u\sin\theta - gt}{u\cos\theta}\right)$$

Velocity and Direction of Motion at a Given Height

At a height h, $v_x = u\cos\theta$

and $v_y = \sqrt{u^2 \sin^2\theta - 2gh}$

$\therefore$ The resultant velocity is given as

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{(u\cos\theta)^2 + u^2\sin^2\theta - 2gh}$$

$$v = \sqrt{u^2 - 2gh}$$

Note that this is the velocity that a particle would have at height h if it is projected vertically from the ground with velocity u.

SOLVED EXAMPLE

EXAMPLE 35

A body is projected with a velocity of 20 ms^{-1} in a direction making an angle of 60° with the horizontal. Calculate its (a) position after 0.5 s and (b) velocity after 0.5 s.

SOLUTION

Here $u = 20\ ms^{-1}, \theta = 60°, t = 0.5\ s$

(a) $x = (u\cos\theta)t = (20\cos 60°) \times 0.5$

$= 5$ m

$$y = (u\sin\theta)\,t - \frac{1}{2}gt^2$$

$$= (20 \times \sin 60°) \times 0.5$$

$$-\frac{1}{2} \times 9.8 \times (0.5)^2 = 7.43 \text{ m}$$

(b) $v_x = u\cos\theta = 20\cos 60° = 10\ ms^{-1}$

$$v_y = u\sin\theta - gt$$

$$= 20\sin 60° - 9.8 \times 0.5$$

$$= 12.42\ ms^{-1}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{(10)^2 + (12.42)^2}$$

$$= 15.95\ ms^{-1}$$

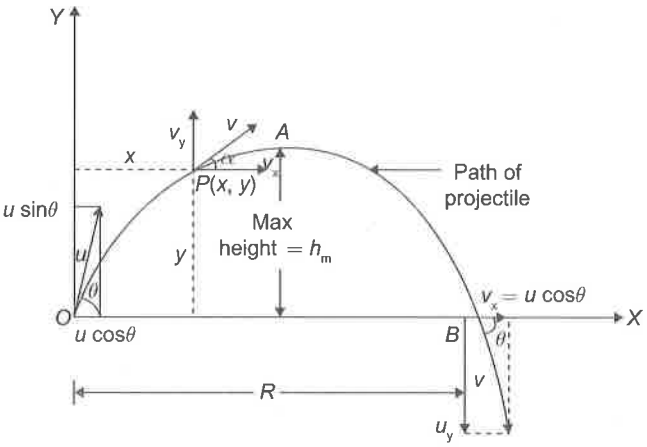
$$\tan\beta = \frac{v_y}{v_x} = \frac{12.42}{10}$$

$$= 1.242$$

$\therefore$ $\beta = \tan^{-1} 1.242 = 51.16°$. ■

Equation of Trajectory of a Projectile

Suppose the body reaches the point $P(x, y)$ after time t.



$\because$ The horizontal distance covered by the body in time t,

X = Horizontal velocity × time

$$= u\cos\theta \times t$$

or $$t = \frac{x}{u\cos\theta}$$

For vertical motion, $u = u\sin\theta$, $a = -g$, so the vertical distance covered in time t is given by

$$s = ut + \frac{1}{2}at^2$$

or $$y = u\sin\theta \cdot \frac{x}{u\cos\theta} - \frac{1}{2}g \cdot \frac{x^2}{u^2\cos^2\theta}$$

or $$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta} \qquad (1)$$

or $y = px - qx^2$, where p and q are constants.

Thus y is a quadratic function of x. Hence, the trajectory of a projectile is a parabola.

From Eq. (1),

$$y = x\tan\theta\left[1 - \frac{gx\cos\theta}{2u^2\cos^2\theta\sin\theta}\right]$$

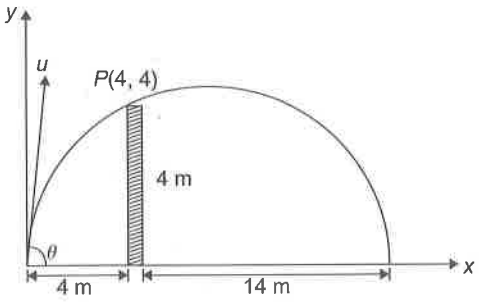
$\Rightarrow$ $$y = x\tan\theta\left[1 - \frac{gx}{2u^2\cos\theta\sin\theta}\right]$$

$$y = x\tan\theta\left[1 - \frac{x}{R}\right] \qquad (2)$$

Equation (2) is another form of trajectory equation of a projectile.

EXAMPLE 36

A ball is thrown from the ground level so as to just clear a wall 4 m high at a distance of 4 m and falls at a distance of 14 m from the wall. Find the magnitude and direction of the velocity.



SOLUTION

The ball passes through the point $P(4, 4)$. Thus, its range $= 4 + 14 = 18$ m.

The trajectory of the ball is estimated as follows:

Now $x = 4$ m, $y = 4$ m and $R = 18$ m.

$$\therefore \quad 4 = 4\tan\theta\left[1-\frac{4}{18}\right]$$

$$= 4\tan\theta\frac{7}{9}$$

or $$\tan\theta = \frac{9}{7},$$

$$\sin\theta = \frac{9}{\sqrt{130}},$$

$$\cos\theta = \frac{7}{\sqrt{130}}$$

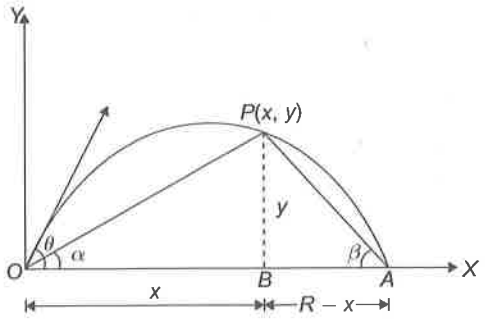
or $$u^2 = \frac{18\times 9.8\times 130}{2\times 9\times 7} = 182$$

or $$u = \sqrt{182} = 13.5 \text{ ms}^{-1}$$

Also, $$\theta = \tan^{-1}(9/7) = 52.1°$$ ■

EXAMPLE 37

A particle is projected over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.



SOLUTION

If R is the range of the particle, then from the figure we have

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R-x} = \frac{y(R-x)+xy}{x(R-x)}$$

or $$\tan\alpha + \tan\beta = \frac{y}{x}\times\frac{R}{(R-x)} \qquad (1)$$

Also, the trajectory of the particle is

$$y = x\tan\theta\left[1-\frac{x}{R}\right]$$

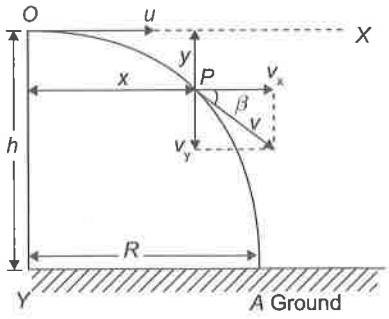
or $$\tan\theta = \frac{y}{x}\times\frac{R}{(R-x)}$$

From Eqs. (1) and (2), we get

$$\tan\theta = \tan\alpha + \tan\beta.$$ ■

Projectile Fired Parallel to Horizontal

As shown in the figure, suppose the body is projected horizontally with velocity u from a point O at a certain height h above the ground level. The body is under the influence of two simultaneous independent motions:



1. Uniform horizontal velocity u.
2. Vertically downward accelerated motion with constant acceleration g.

Under the combined effect of the above two motions, the body moves along the path OPA.

Trajectory of the Projectile

After the time t, suppose the body reaches the point $P(x, y)$. The horizontal distance covered by the body in time t is

$$x = ut$$

$$\therefore \quad t = \frac{x}{u}$$

The vertical distance travelled by the body in time t is given by

$$s = ut + \frac{1}{2}at^2$$

or $$y = 0 \times 1 + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

For vertical motion, $u = 0$. Thus,

$$y = \frac{1}{2}g\left(\frac{x}{u}\right)^2 = \left(\frac{g}{2u^2}\right)x^2$$

$$\left[\because t = \frac{x}{u}\right]$$

or $$y = kx^2$$

$$[\text{Here } k = \frac{g}{2u^2} = a \text{ constant.}]$$

As y is a quadratic function of x, the trajectory of the projectile is a parabola.

Time of Flight

It is the total time for which the projectile remains in its flight (from 0 to A). Let T be its time of flight.

For the vertical downward motion of the body, we use

$$s = ut + \frac{1}{2}at^2$$

or $$h = 0 \times T + \frac{1}{2}gT^2$$

or $$T = \sqrt{\frac{2h}{g}}$$

Horizontal Range

It is the horizontal distance covered by the projectile during its time of flight. It is equal to $OA = R$. Thus, R = horizontal velocity $\times$ time of flight $= u \times T$

or $$R = u\sqrt{\frac{2h}{g}}$$

Velocity of the Projectile at any Instant

At the instant t (when the body is at point P), let the velocity of the projectile be v. The velocity v has two rectangular components:

Horizontal component of velocity, $v_x = u$.

Vertical component of velocity, $v_y = 0 + gt = gt$.

$\therefore$ The resultant velocity at point P is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + g^2t^2}$$

If the velocity v makes an angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{gt}{u}$$

or $$\beta = \tan^{-1}\left(\frac{gt}{u}\right)$$

SOLVED EXAMPLES

EXAMPLE 38

A body is thrown horizontally from the top of a tower and strikes the ground after 3 s at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take $g = 9.8\ \text{ms}^{-2}$.

SOLUTION

As shown in the figure, suppose the body is thrown horizontally from the top O of a tower of height y with velocity u. The body hits the ground after 3 s. Considering vertically downward motion of the body,

$$y = u_y t + \frac{1}{2}gt^2$$

$$= 0 \times 3 + \frac{1}{2} \times 9.8 \times (3)^2 = 44.1 \text{ m}$$

$[\because$ Initial vertical velocity, $u_y = 0]$

Final vertical velocity,

$$v_y = u_y + gt = 0 + 9.8 \times 3$$

$$= 29.4\ \text{ms}^{-1}$$

Final horizontal velocity, $v_x = u$

As the resultant velocity u makes an angle of 45° with the horizontal,

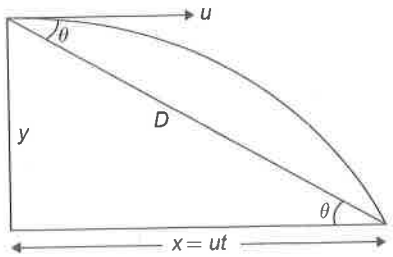
$$\tan 45° = \frac{v_y}{v_x}$$

or $$1 = \frac{29.4}{x}$$

or $$u = 29.4 \text{ ms}^{-1}.$$ ■

EXAMPLE 39

A particle is projected horizontally with a speed u from the top of a plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane?



SOLUTION

The horizontal distance covered in time t,

$$x = ut$$

or $$t = \frac{x}{u}.$$

The vertical distance covered in time t,

$$y = 0 + \frac{1}{2}gt^2$$

$$= \frac{1}{2}g \times \frac{x^2}{u^2} \qquad \text{[using (1)]}$$

Also, $$\frac{y}{x} = \tan\theta$$

or $$y = x \tan\theta$$

$$\therefore \quad \frac{gx^2}{2u^2} = x\tan\theta$$

or $$x\left(\frac{gx}{2u^2} - \tan\theta\right) = 0$$

As $x = 0$ is not possible, so $x = \frac{2u^2 \tan\theta}{g}$.

The distance of the point of strike from the point of projection is

$$D = \sqrt{x^2 + y^2}$$

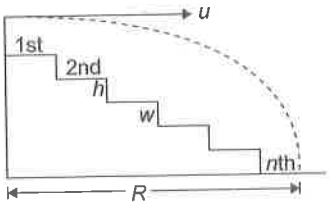
$$= \sqrt{x^2 + (x\tan\theta)^2}$$

$$= x\sqrt{1 + \tan^2\theta} = x \sec\theta$$

or $$D = \frac{2u^2}{g}\tan\theta\sec\theta$$ ■

EXAMPLE 40

A ball rolls off the top of a stairway with a constant horizontal velocity u. If the steps are h metre high and w metre wide, show that the ball will just hit the edge of the nth step if $n = \frac{2hu^2}{gw^2}$.



SOLUTION

Refer to the figure. For the nth step,

net vertical displacement $= nh$

net horizontal displacement $= n\omega$.

Let t be the time taken by the ball to reach the nth step. Then

$$R = ut$$

or $$n\omega = ut$$

or $$t = \frac{n\omega}{u}$$

Also, $$y = u_y t + \frac{1}{2}gt^2$$

or $$nh = 0 + \frac{1}{2}gt^2$$

$$= \frac{1}{2}g\left(\frac{n\omega}{u}\right)^2$$

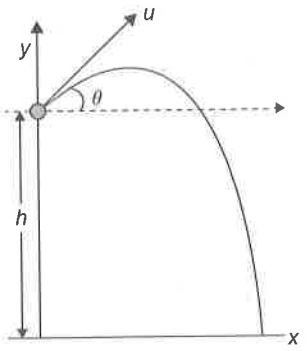
or $$n = \frac{2hu^2}{g\omega^2}.$$

Projectile at an Angle θ from Height h

Consider the projectile as shown in the figure.

Take the point of projection as the origin on the x- and y-axes as shown in the figure.

For x-axis,



$$u_x = u\cos\theta$$

$$a_x = 0$$

$$v_x = u\cos\theta,$$

and $$x = u\cos\theta \times t$$

For y-axis,

$$u_y = u\sin\theta,$$

$$a_y = -g,$$

$$v_y = u\sin\theta - gt,$$

and $$y = u\sin\theta t - \frac{gt^2}{2}$$

SOLVED EXAMPLE

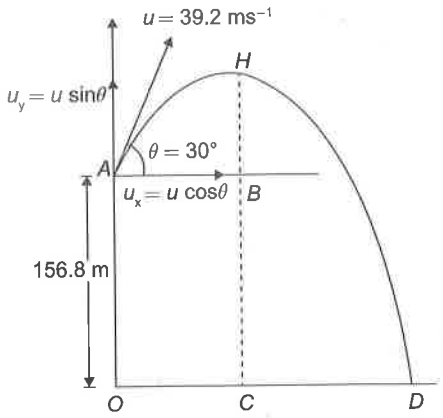
EXAMPLE 41

From the top of a tower 156.8 m high, a projectile is projected with a velocity of 39.2 ms^{-1} in a direction making an angle 30° with horizontal. Find the distance from the foot of tower where it strikes the ground and the time taken to do so.

SOLUTION

The situation is depicted in the figure.

Here, height of the tower



$$OA = 156.8 \text{ m}$$

$$u = 39.2 \text{ ms}^{-1}$$

$$\theta = 30°$$

Time for which projectile remain is air $= t = ?$

Horizontal distance covered $R = OD = ?$

Let $u_x = u\cos\theta$ and $u_y = u\sin\theta$ be the components of velocity $\vec{u}$.

Motion of the projectile from O to H to D:

Using equation $y = u_y t + \frac{1}{2}a_y t^2$

Here $y = 156.8$ m; $u_y = -u\sin\theta = 39.2\sin 30°$

$a_y = 9.8$ m/s^2; $t = ?$

$$156.8 = -39.2 \times 0.5t + 4.9t^2$$

$$156.8 = -19.6t + 4.9t^2$$

or $$4.9t^2 - 19.6t - 156.8 = 0$$

or $$t^2 - 4t - 32 = 0$$

$\Rightarrow$ $$(t-8)(t+4) = 0$$

We get $$t = 8 \text{ s}; t = -4 \text{ s}.$$

$t = -4$ s is not possible, Thus, we take $t = 8$ s.

Now the horizontal distance covered in this time

$$R = u_x \times t = u\cos\theta \times t$$

$$= 39.2 \times \cos 30° \times t$$

$$R = 271.57 \text{ m}$$

Projectile Motion in an Inclined Plane

Here, two cases arise. One is up the plane and the other is down the plane. Let us discuss both the cases separately.

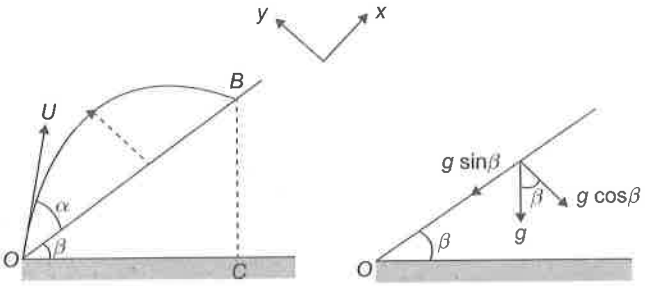
1. **Up the inclined plane:** In this case, direction x is chosen up the plane and direction y is chosen perpendicular to the plane. Hence,

$$u_x = u\cos\alpha$$

and $$a_x = -g\sin\beta$$

$$u_y = u\sin\alpha$$

and $$a_y = -g\cos\beta$$



Now, let us derive the expressions for time of flight (T) and range (R) along the plane.

Time of Flight

At point B, displacement along the y-direction is zero. So, substituting the proper values in $s_y = u_y t + \frac{1}{2}a_y t^2$, we get

$$0 = ut\sin\alpha + \frac{1}{2}(-g\cos\beta)\,t^2$$

$$\therefore \quad t = 0$$

and $$\frac{2u\sin\alpha}{g\cos\beta}.$$

$t = 0$ corresponds to point O and $t = \dfrac{2u\sin\alpha}{g\cos\beta}$ corresponds to point B. Thus,

$$T = \frac{2u\sin\alpha}{g\cos\beta}$$

Range

Range (R) or the distance OB is also equal to thee displacement of the projectile along the x-direction at $t = T$.

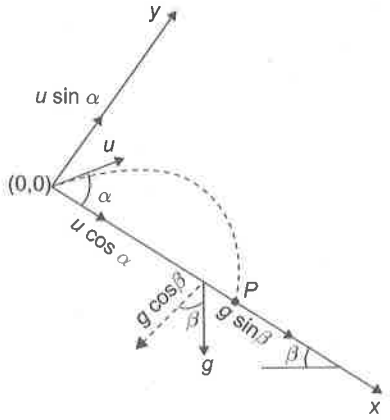
Therefore,

$$R = s_a = u_x T + \frac{1}{2}a_x T^2$$

$$\Rightarrow \qquad R = u\cos\alpha\, T - \frac{1}{2}\sin\beta\, T^2$$

2. **Down the inclined plane:**

Along the x-axis	y-axis
(1) $u_x = u\cos\alpha$	(1) $u_y = u\sin\alpha$
(2) $a_x = g\sin\beta$	(2) $a_y = g\cos\beta$



Velocity at P

$$v_y = u_y + a_y T$$

$$v_x = u_x + a_x T$$

Time of flight $T = \dfrac{2u_y}{a_y}$

$$= \frac{2u\sin\alpha}{g\cos\beta}$$

Range $$S_x = u_x T + \frac{1}{2}a_x T^2$$

$$= u\cos\alpha\, T + \frac{1}{2}g\sin\beta.T^2$$

SOLVED EXAMPLES

EXAMPLE 42

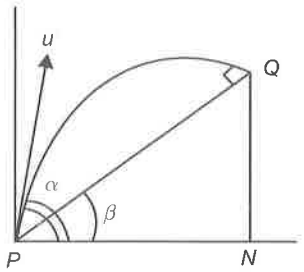
A particle is projected at an angle α with the horizontal from the foot of a plane whose inclination to the horizon-

tal is β. Show that it will strike the plane at right angles if $\cot\beta = 2\tan(\alpha - \beta)$.

SOLUTION

Let u be the velocity of projection so that $u\cos(\alpha-\beta)$ and $u\sin(\alpha-\beta)$ are the initial velocities, respectively, parallel and perpendicular to the inclined plane. The acceleration in these two directions are $(-g\sin\beta)$ and $(-g\cos\beta)$.

The initial component of velocity perpendicular to PQ is $u\sin(\alpha-\beta)$ and the acceleration in this direction is $(-g\cos\beta)$. If T is the time the particle takes to go from P to Q, then in time T, the space described in a direction perpendicular to PQ is zero.



$$0 = u\sin(\alpha-\beta)T - \frac{1}{2}g\cos\beta\cdot T^2$$

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

If the direction of motion at the instant when the particle hits the plane is perpendicular to the plane, then the velocity at that instant parallel to the plane must be zero.

$\therefore$ $\quad u\cos(\alpha-\beta) - g\sin\beta T = 0.$

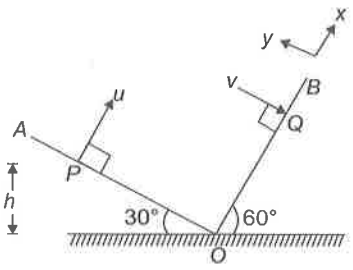
$$\frac{u\cos(\alpha-\beta)}{g\sin\beta} = T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$\therefore$ $\quad \cos\beta = 2\tan(\alpha-\beta).$ ■

EXAMPLE 43

Two inclined planes OA and OB having inclinations 30° and 60° with the horizontal, respectively, intersect each other at O, as shown in the figure. A particle is projected from point P with velocity $u = 10\sqrt{3}$ m/s along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicular to the direction of flight, then calculate.

(a) time of flight
(b) velocity with which the particle strikes the plane OB,
(c) height h of point P from point O
(d) distance PQ. (Take $g = 10$ m/s^2.)



SOLUTION

Let us choose the x and y directions along OB and OA, respectively. Then,

$$u_x = u = 10\sqrt{3}\ \text{m/s},\ u_y = 0$$

$$a_x = -g\sin 60° = -5\sqrt{3}\ \text{m/s}^2$$

and $\quad a_y = -g\cos 60° = -5\ \text{m/s}^2$

(a) At point Q, the x-component of velocity is zero. Hence, substituting in

$$v_x = u_x + a_x t$$

$$0 = 10\sqrt{3} - 5\sqrt{3}t$$

$\Rightarrow$ $\quad t = \dfrac{10\sqrt{3}}{5\sqrt{3}} = 2s$

(b) At point Q, $\quad v = v_y = u_y + a_y t$

$\therefore$ $\quad v = 0 - (5)(2) = -10$ m/s

Here, negative sign implies that velocity of particle at Q is along the negative y direction.

(c) Distance PO = |displacement of the particle along the y-direction| = $|s_y|$

Here, $\quad s_y = u_y t + \frac{1}{2}a_y t^2 = 0 - \frac{1}{2}(5)(2)^2$

$$= -10\ \text{m}$$

$\therefore$ $\quad PO = 10$ m

Therefore, $h = PO\sin 30° = (10)\left(\frac{1}{2}\right)$

or $\quad h = 5$ m

(d) Distance OQ = displacement of the particle along the x-direction = s_x

Here, $\quad s_x = u_x t + \frac{1}{2}a_x t^2$

$$= (10\sqrt{3})(2) - \frac{1}{2}(5\sqrt{3})(2)^2$$

$$= 10\sqrt{3} \text{ m}$$

or $$OQ = 10\sqrt{3}$$

$$PQ = \sqrt{(PO)^2 + (OQ)^2}$$

$$= \sqrt{(10)^2 + (10\sqrt{3})^2}$$

$$= \sqrt{100 + 300} = \sqrt{400}$$

$$PQ = 20 \text{ m}$$ ■

Note

Now you can try Questions 45–68 in Exercise I and Questions 12–20 in Exercise II.

RELATIVE MOTION

The word 'relative' is a very general term, which can be applied to physical, nonphysical, scalar or vector quantities. For example, my height is 5 feet and 6 inches while my wife's height is 5 feet and 4 inches. If I ask you how high I am relative to my wife, your answer will be 2 inches. What you did? You simply subtracted my wife's height from my height. The same concept is applied everywhere, whether it is a relative velocity, relative acceleration or anything else. So, from the above discussion, we may now conclude that the relative velocity of A with respect to B (written as $\vec{v}_{AB}$) is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

Similarly, relative acceleration of A with respect of B is

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B$$

If it is a one-dimensional motion, we can treat the vectors as scalars just by assigning positive sign to one direction and negative sign to the other. So, in case of a one-dimensional motion, the above equations can be written as

$$v_{AB} = v_A - v_B$$

and $$a_{AB} = a_A - a_B$$

Further, we can see that

$$\vec{v}_{AB} = -\vec{v}_{BA} \text{ or } \vec{a}_{BA} = -\vec{a}_{AB}$$

SOLVED EXAMPLE

EXAMPLE 44

Seeta is moving due east with a velocity of 1 m/s and Geeta is moving due west with a velocity of 2 m/s. What is the velocity of Seeta with respect to Geeta?

SOLUTION

It is a one-dimensional motion. So, let us choose the east direction as positive and the west as negative. Now, given that

$$v_s = \text{velocity of Seeta}$$
$$= 1 \text{ m/s}$$

and $$v_G = \text{velocity of Geeta}$$
$$= -2 \text{ m/s}$$

Thus, v_{SG} = velocity of Seeta with respect to Geeta

$$= v_S - v_G = 1 - (-2)$$
$$= 3 \text{ m/s}$$

Hence, velocity of Seeta with respect to Geeta is 3 m/s due east. ■

Important Note

Procedure to Solve the Vector Equation

$$\vec{A} = \vec{B} + \vec{C} \qquad (1)$$

1. There are six variables in this equation, which are as follows:
   (a) Magnitude of $\vec{A}$ and its direction
   (b) Magnitude of $\vec{B}$ and its direction
   (c) Magnitude of $\vec{C}$ and its direction.
2. We can solve this equation if we know the value of four variables [Note: two of them must be directions]
3. If we know the two directions of any two vectors, then we will put them on the same side and other on the different side.

   For example

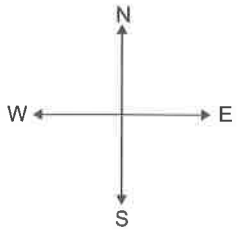
   If we know the directions of $\vec{A}$ and $\vec{B}$ and $\vec{C}$'s direction is unknown, then we write the equation as follows:

$$\vec{C} = \vec{A} - \vec{B}$$

4. We can then draw the vector diagram according to the equation and resolve the vectors to determine the unknown values.

EXAMPLE 45

Car A has an acceleration of 2 m/s^2 due east and car B 4 m/s^2 due north. What is the acceleration of car B with respect to car A?



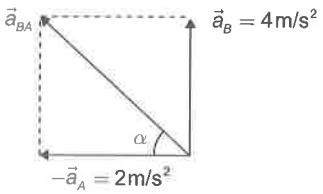
SOLUTION

It is a two-dimensional motion. Therefore,

$$\vec{a}_{BA} = \text{acceleration of car } B \text{ with respect to car } A$$
$$= \vec{a}_B - \vec{a}_A$$

Here, $\vec{a}_B$ = acceleration of car

$B = 4$ m/s^2 (due north)



and $\vec{a}_A$ = acceleration of car A

$= 2$ m/s^2 (due east)

$$|\vec{a}_{BA}| = \sqrt{(4)^2 + (2)^2}$$
$$= 2\sqrt{5}\text{m/s}^2$$

and $$\alpha = \tan^{-1}\left(\frac{4}{2}\right)$$
$$= \tan^{-1}(2)$$

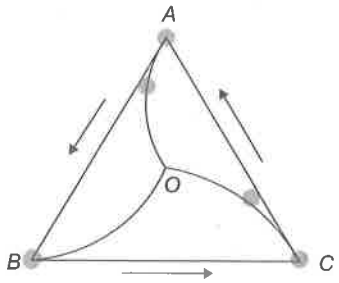
Thus, $\vec{a}_{BA}$ is $2\sqrt{5}$ m/s^2 at an angle of $\alpha = \tan^{-1}(2)$ from west towards north. ■

EXAMPLE 46

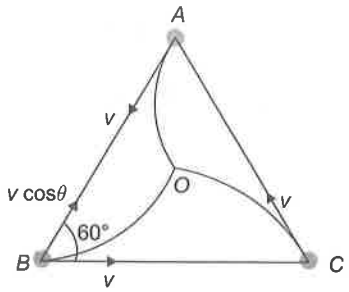
Three particle A, B and C situated at the vertices of an equilateral triangle start moving simultaneously at a constant speed v in the direction of the adjacent particle, which falls ahead in the anti-clockwise direction. If a be the side of the triangle, then find the time when they meet.

SOLUTION

Here, particle A follows B, B follows C and C follows A. The direction of motion of each particle keeps changing as the motion of each particle is always directed towards other particle. The situation after time t is shown in the below figure with a possible outline of path followed by the particles before they meet.



This problem appears to be complex as the path of motion is difficult to be defined. But, it has a simple solution in component analysis. Let us consider the pair A and B. The initial component of velocities in the direction of the line joining the initial position of the two particles is v and $v\cos\theta$ as shown in the figure below.



The component velocities are directed towards each other. Now, considering the linear (one-dimensional) motion in the direction of AB, the relative velocity of A with respect to B is

$$v_{AB} = v_A - v_B$$
$$v_{AB} = v - (-v\cos\theta)$$
$$= v + v\cos\theta$$

In equilateral triangle, $\theta = 60°$

$$v_{AB} = v + v\cos 60° = v + \frac{v}{2}$$

$$= \frac{3v}{2}$$

The time taken to cover the displacement a, i.e., the side of the triangle

$$t = \frac{2a}{3v}.$$ ■

Questions based on relative motion are usually of the following four types:

1. Minimum distance between two bodies in motion
2. River–boat problems
3. Aircraft–wind problems
4. Rain problems

Minimum Distance Between Two Bodies in Motion

When two bodies are in motion, the questions such as the minimum distance between them or the time when one body overtakes the other can be solved easily by the principle of relative motion. In these types of problems, one body is assumed to be at rest and the relative motion of the other body is considered. By converting the two-body problem into one-body problem, and the solution becomes easy. The following example will illustrate the statement.

SOLVED EXAMPLES

EXAMPLE 47

Car A and car B start moving simultaneously in the same direction along the line joining them. Car A moves with a constant acceleration $a = 4$ m/s², while car B moves with a constant velocity $v = 1$ m/s. At time $t = 0$, car A is 10 m behind car B. Find the time when car A overtakes car B.

SOLUTION

Given: $u_A = 0$, $u_B = 1$ m/s,

$$a_A = 4 \text{ m/s}^2 \text{ and } a_B = 0.$$

Assuming car B to be at rest, we have

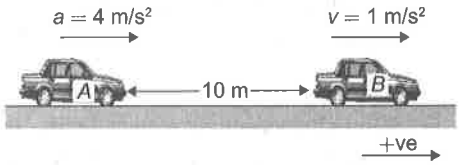
$$u_{AB} = u_A - u_B = 0 - 1$$

$$= -1 \text{ m/s}$$

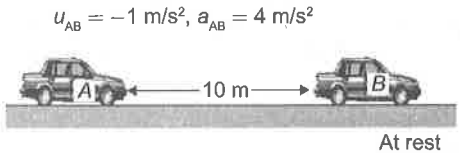
$$a_{AB} = a_A - a_B = 4 - 0$$

$$= 4 \text{ m/s}^2$$

Now, the problem can be assumed in simplified form as follows:



Substituting the proper values in the equation



$$s = ut + \frac{1}{2}at^2$$

We get $$10 = -t + \frac{1}{2}(4)(t^2)$$

or $$2t^2 - t - 10 = 0$$

or $$t = \frac{1 \pm \sqrt{1+80}}{4} = \frac{1 \pm \sqrt{81}}{4}$$

$$= \frac{1 \pm 9}{4}$$

or $$t = 2.5 \text{ s}$$

and $$-2 \text{ s}$$

Ignoring the negative value, the desired time is 2.5 s. ■

EXAMPLE 48

Note

The above problem can also be solved without using the concept of relative motion as under. At the time when A overtakes B,

$$s_A = s_B + 10$$

$$\therefore \quad \frac{1}{2} \times 4 \times t^2 = 1 \times t \times 10$$

or $$2t^2 - t - 10 = 0$$

which on solving gives $t = 2.5$ s and -2 s, the same as we found above.

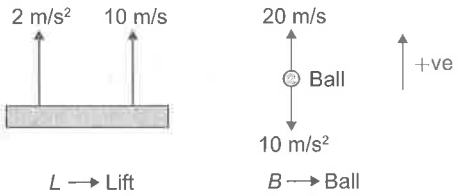
As per my opinion, this approach (by taking absolute values) is more suitable in case of two-body problem in one-dimensional motion. Let us see one more example in support of it.

An open lift is moving upwards with velocity 10 m/s. It has an upward acceleration of 2 m/s^2. A ball is projected upwards with velocity 20 m/s relative to the ground. Find:

(a) time when the ball again meets the lift
(b) displacement of the lift and ball at that instant
(c) distance travelled by the ball up to that instant. Take $g = 10$ m/s^2,

SOLUTION

(a) At the time when the ball again meets the lift, $s_L = s_B$.



$$\therefore \quad 10t + \frac{1}{2} \times 2 \times t^2 = 20t - \frac{1}{2} \times 10t^2$$

Solving this equation, we get

$$t = 0$$

and

$$t = \frac{5}{3}\text{ s}$$

$\therefore$ The ball will again meet the lift after $\frac{5}{3}$ s.

(b) At this instant,

$$s_L = s_B = 10 \times \frac{5}{3} \times \frac{1}{2} \times \left(\frac{5}{3}\right)^2$$

$$= \frac{175}{9}\text{ m} = 19.4\text{ m}$$

(c) For the ball, $u \uparrow\downarrow a$. Therefore, we will first find t_0, the time when its velocity becomes zero.

$$t_0 = \left|\frac{u}{a}\right| = \frac{20}{10}$$

$$= 2\text{s}$$

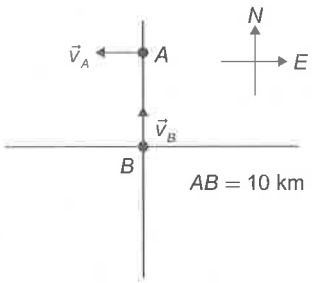
As $t\left(=\frac{5}{3}s\right) < t_0$, distance and displacement are equal

or $$d = 19.4\text{ m}$$

The concept of relative motion is more useful in two-body problem in two- (or three-) dimensional motion. This can be understood by the following example. ■

EXAMPLE 49

Two ships A and B are 10 km apart on a line running south to north. Ship A further north is streaming west at 20 km/h and ship B is streaming north at 20 km/h. What is their distance of the closest approach and how long do they take to reach it?



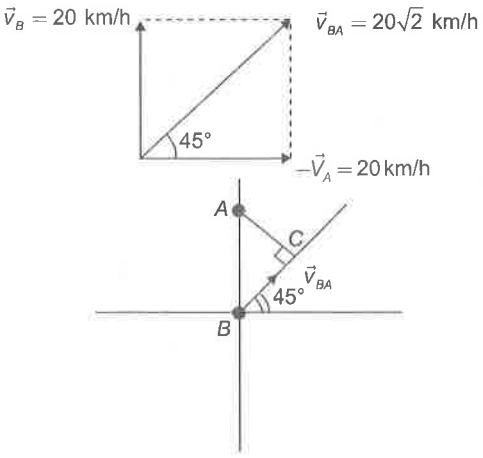
SOLUTION

Ships A and B are moving with the same speed 20 km/h in the directions shown in figure. It is a two-dimensional, two-body problem with zero acceleration. Let us find $\vec{v}_{BA}$

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

Here, $$|\vec{v}_{BA}| = \sqrt{(20)^2 + (20)^2}$$

$$= 20\sqrt{2}\text{ km/h}$$



i.e., $\vec{v}_{BA}$ is $20\sqrt{2}$ km/h at an angle of 45° from east towards north. Thus, the given problem can be simplified as:

A is at rest and B is moving with $\vec{v}_{BA}$ in the direction shown in the above figure.

Therefore, the minimum distance between the two is

$$s_{\min} = AC$$
$$= AB \sin 45°$$
$$= 10\left(\frac{1}{\sqrt{2}}\right) \text{ km} = 5\sqrt{2} \text{ km}$$

and the desired time is

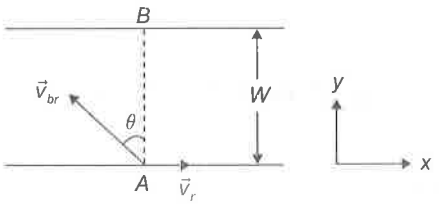
$$t = \frac{BC}{|\vec{v}_{BA}|} = \frac{5\sqrt{2}}{20\sqrt{2}}$$

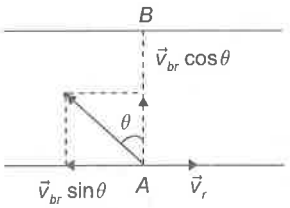
(BC = AC = $5\sqrt{2}$ km)

$$\frac{1}{4} \text{ h} = 15 \text{ min}$$ ■

River-Boat Problems

In river–boat problems, we come across the following three terms:





$\vec{v}_r$ = absolute velocity of river

$\vec{v}_{br}$ = velocity of boatman with respect to river or velocity of boatman is still water

and $\vec{v}_b$ = absolute velocity of boatman.

Here, it is important to note that $\vec{v}_{br}$ is the velocity of the boatman with which he steers and $\vec{v}_b$ is the actual velocity of the boatman relative to ground.

Furthermore, $\vec{v}_b = \vec{v}_{br} + \vec{v}_r$.

Now, let us derive some standard results and their special cases.

A boatman starts from point A from one bank of a river with velocity $\vec{v}_{br}$ in the direction shown in the above figure. The river is flowing along the positive x-direction with velocity $\vec{v}_r$. Width of the river is w, then

$$\vec{v}_b = \vec{v}_{br} + \vec{v}_r$$

Therefore, $v_{bx} = v_{rx} + v_{brx}$

$$= v_r - v_{br} \sin\theta$$

and $v_{by} = v_{ry} + v_{bry}$

$$= 0 + v_{br}\cos\theta = v_{br}\cos\theta$$

Now, time taken by the boatman to cross the river is

$$t = \frac{w}{v_{by}} = \frac{w}{v_{br}\cos\theta}$$

or
$$t = \frac{w}{v_{br}\cos\theta} \quad (1)$$

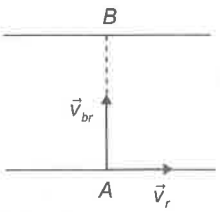
Furthermore, displacement along the x-axis when he reaches the other bank (also called drift) is

$$x = v_{bx}$$
$$t = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta}$$

or
$$x = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta} \quad (2)$$

Three specials to be considered are as follows.

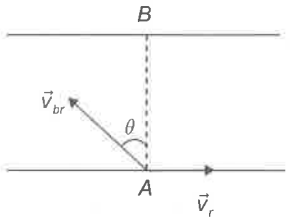
(i) Condition When the Boatman Crosses the River in the Shortest Interval of Time



From Eq. (1), we can see that time (t) will be minimum when $\theta = 0°$, i.e., the boatman should steer his boat perpendicular to the river current.

Also, $t_{min} = \dfrac{w}{v_{br}}$ as $\cos\theta = 1$.

(ii) Condition When the Boatman Wants to Reach Point *B*, i.e., at a Point Just Opposite from Where he Started



In this case, the drift (x) should be zero.

$\therefore \quad x = 0$

or $(v_r - v_{br}\sin\theta)\dfrac{w}{v_{br}\cos\theta} = 0$

or $v_r = v_{br}\sin\theta$

or $\sin\theta = \dfrac{v_r}{v_{br}}$

or $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$.

Hence, to reach point B, the boatman should row at an angle $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$ upstream from AB.

Furthermore, $\sin\theta$ is not greater than 1.

So, if $v_r \geq v_{br}$, the boatman can never reach point B. Because if $v_r = v_{br}$, $\sin\theta = 1$ or $\theta = 90°$ and it is just impossible to reach B if $\theta = 90°$. Moreover, it can be seen that $v_b = 0$ if $v_r = v_{br}$ and $\theta = 90°$. Similarly, if $v_r > v_{br}$, $\sin\theta > 1$, i.e., no such angle exists. Practically, it can be realized in this manner that it is not possible to reach B if river velocity (v_r) is too high.

(iii) Shortest Path

Path length travelled by the boatman when he reaches the opposite shore is

$$s = \sqrt{w^2 + x^2}$$

Here, w = width of the river is constant. So for s to be minimum, modulus of x (drift) should be minimum. Now two cases are possible.

When $v_r < v_{br}$: In this case, $x = 0$,

when $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$

or $s_{min} = w$ at $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$

When $v_r > v_{br}$: In this case, x is minimum,

where $\dfrac{dx}{d\theta} = 0$

or $\dfrac{d}{d\theta}\left\{\dfrac{w}{v_{br}\cos\theta}(v_r - v_{br}\sin\theta)\right\} = 0$

or $-v_{br}\cos^2\theta - (v_r - v_{br}\sin\theta)(-\sin\theta) = 0$

or $-v_{br} + v_r\sin\theta = 0$

or $\theta = \sin^{-1}\left(\dfrac{v_{br}}{v_r}\right)$

Now, at this angle, we can find x_{min} and then s_{min}, which is

$$s_{min} = w\left(\frac{v_r}{v_{br}}\right) \text{ at } \theta = \sin^{-1}\left(\frac{v_{br}}{v_r}\right)$$

SOLVED EXAMPLES

EXAMPLE 50

A man can row a boat with 4 km/h in still water. Assume he is crossing a river where the current is 2 km/h.

(a) In what direction will his boat be headed, if he wants to reach a point on the other bank, directly opposite to starting point?

(b) If width of the river is 4 km, how long will the man take to cross the river, with the condition in part (a)?

(c) In what direction should he head the boat if he wants to cross the river in the shortest time and what is this minimum time?

(d) How long will it take for him to row 2 km up the stream and then back to his starting point?

SOLUTION

(a) Given, that $v_{br} = 4$ km/h and $v_r = 2$ km/h

$\therefore \quad \theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right) = \sin^{-1}\left(\dfrac{2}{4}\right)$

$$= \sin^{-1}\left(\frac{1}{2}\right) = 30°.$$

Hence, to reach the point directly opposite to the starting point, he should head the boat at an angle of 30° with AB or 90° + 30°= 120° with the river flow.

(b) Time taken by the boatman to cross the river

$$w = \text{width of the river} = 4 \text{ km}$$

$$v_{br} = 4 \text{ km/h and } \theta = 30°$$

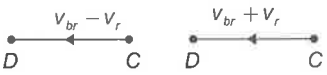
$$\therefore \quad t = \frac{4}{4\cos 30°} = \frac{2}{\sqrt{3}}h$$

(c) For the shortest time, $\theta = 0°$

and
$$t_{\min} = \frac{w}{v_{br}\cos 0°} = \frac{4}{4}$$

$$= 1 \text{ h}.$$

Hence, he should head his boat perpendicular to the river current for crossing the river in shortest time and this shortest time is 1 h.



(d) $t = t_{CD} + t_{DC}$

or
$$t = \frac{CD}{v_{db} - v_r} + \frac{DC}{v_{br} + v_r}$$

$$= \frac{2}{4-2} + \frac{2}{4+2}$$

$$= 1 + \frac{1}{3} = \frac{4}{3}\text{h}$$

EXAMPLE 51

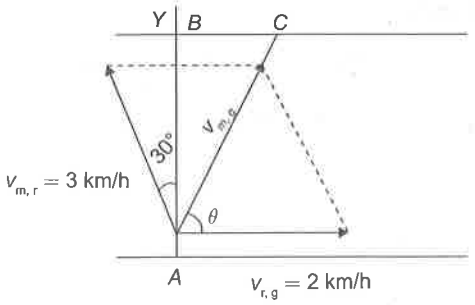
A man can swim at a speed of 3 km/h in still water. He wants to cross a 500-m-wide river flowing at 2 km/h. He keeps himself always at an angle of 120° with the river flow while swimming.

(a) Find the time he takes to cross the river.

(b) At what point on the opposite bank will he arrive?

SOLUTION

The situation is shown in the following figure.



Here, $\vec{v}_{r,g}$ = velocity of the river with respect to the ground

$\vec{v}_{m,r}$ = velocity of the man with respect to the river

$\vec{v}_{m,g}$ = velocity of the man with respect to the ground.

(a) We have

$$\vec{v}_{m,g} = \vec{v}_{m,r} + \vec{v}_{r,g} \qquad (1)$$

Hence, the velocity with respect to the ground is along AC.

Taking y-components in Eq. (1),

$$\vec{v}_{m,g}\sin\theta = 3 \text{ km/h} \cos 30° + 2 \text{ km/h} \cos 90°$$

$$= \frac{3\sqrt{3}}{2} \text{ km/h}$$

Time taken to cross the river

$$= \frac{\text{displacement along the } y\text{-axis}}{\text{velocity along the } y\text{-axis}}$$

$$= \frac{1/2 \text{ km}}{3\sqrt{3}/2 \text{ km/h}}$$

$$= \frac{1}{3\sqrt{3}}\text{h}$$

(b) Taking x-components in equation (i),

$$\vec{v}_{m,g}\cos\theta = -3\text{km/h} \sin 30° + 2 \text{ km/h}$$

$$= \frac{1}{2} \text{ km/h}$$

Displacement along the x-axis as the man crosses the river

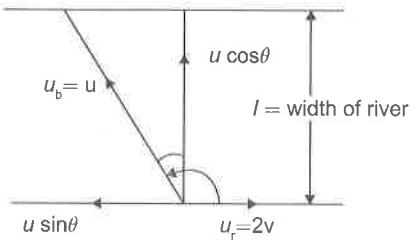
$$= (\text{velocity along the } x\text{-axis})(\text{time})$$

$$= \left(\frac{1 \text{ km}}{2 \text{ h}}\right) \times \left(\frac{1}{3\sqrt{3}} \text{ h}\right) = \frac{1}{6\sqrt{3}} \text{ km}$$

EXAMPLE 52

A boat moves relative to water with a velocity v and the river is flowing with $2v$. At what angle the boat shall move with the stream to have minimum drift?

(a) 30° (b) 60°
(c) 90° (d) 120°



SOLUTION

(d) Let the boat move at angle θ to the normal as shown in the figure. Then time to cross the river $= \dfrac{1}{v\cos\theta}$

drift $x = (2v - v\sin\theta)\dfrac{1}{v\cos\theta}$ for x to be minimum

$$\frac{dx}{d\theta} = 0$$

$$= 1\,(2\sec\theta\tan\theta - \sec^2\theta)$$

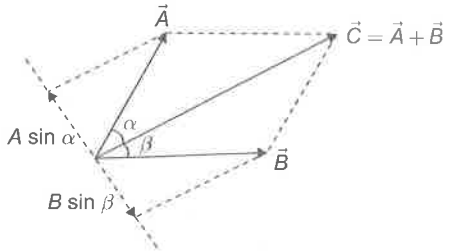
or $\sin\theta = 1/2$

or $\theta = 30°$

and $\phi = 90 + 30 = 120°$ ■

Aircraft-Wind Problems

This is similar to the river–boat problem. The only difference is that $\vec{v}_{br}$ is replaced by $\vec{v}_{aw}$ (velocity of aircraft with respect to wind or velocity of aircraft in still air), $\vec{v}_r$ is replaced by $\vec{v}_w$ (velocity of wind) and $\vec{v}_b$ is replaced by $\vec{v}_a$ (absolute velocity of aircraft). Further, $\vec{v}_a = \vec{v}_{aw} + \vec{v}_w$. The following example will illustrate the theory.



Note

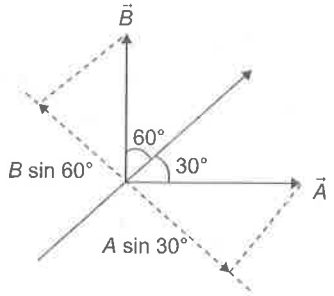
Short trick

If there are two vectors $\vec{A}$ and $\vec{B}$ and their resultant makes an angle α with $\vec{A}$ and β with $\vec{B}$ then $A\sin\alpha = \beta\sin\beta$.

This means component of $\vec{A}$ perpendicular to the resultant is equal in magnitude to the component of $\vec{B}$ also perpendicular to resultant.

EXAMPLE 53

If two vectors $\vec{A}$ and $\vec{B}$ make angle 30° and 60° with their resultant and $\vec{B}$ has magnitude equal to 10, then find the magnitude of $\vec{A}$.



So $B\sin 60° = A\sin 30°$

$\Rightarrow$ $10\sin 60° = A\sin 30°$

$\Rightarrow$ $A = 10\sqrt{3}$ ■

EXAMPLE 54

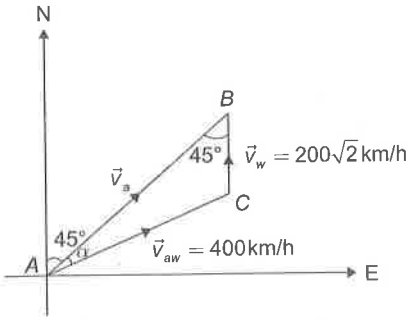
An aircraft flies at 400 km/h in still air. A wind of $200\sqrt{2}$ km/h is blowing from the south. The pilot wishes to travel from A to a point B north-east of A. Find the direction he must steer and time of his journey if $AB = 1000$ km.

SOLUTION

Given that $v_w = 200\sqrt{2}$ km/h.

$v_{aw} = 400$ km/h and $\vec{v}_a$ should be along AB or in the north-east direction. Thus, the direction of $\vec{v}_{aw}$ should be such that the resultant of $\vec{v}_w$ and $\vec{v}_{aw}$ is along AB or in the north-east direction.

Let $\vec{v}_{aw}$ make an angle α with AB as shown in the below figure.



Applying sine law in triangle ABC, we get

$$\frac{AC}{\sin 45^\circ} = \frac{BC}{\sin \alpha}$$

or $$\sin \alpha = \left(\frac{BC}{AC}\right) \sin 45^\circ$$

$$= \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}$$

$\therefore$ $$\alpha = 30^\circ$$

Therefore, the pilot should steer in a direction at an angle of $(45^\circ + \alpha)$ or 75° from north towards east.

Further, $$\frac{|\vec{v}_a|}{\sin(180^\circ - 45^\circ - 30^\circ)} = \frac{400}{\sin 45^\circ}$$

or $$|\vec{v}_a| = \frac{\sin 105^\circ}{\sin 45^\circ} \times (400)\frac{\text{km}}{\text{h}}$$

$$= \left(\frac{\cos 15^\circ}{\sin 45^\circ}\right)(400)\frac{\text{km}}{\text{h}} = \left(\frac{0.9659}{0.707}\right)(400)\frac{\text{km}}{\text{h}}$$

$$= 546.47 \text{ km/h}$$

$\therefore$ The time of journey from A to B is

$$t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47}\text{h}$$

$\Rightarrow$ $$t = 1.83 \text{ h}$$ ■

(D) Rain problems

In these type of problems, we again come across the three terms $\overrightarrow{v_r}$, $\overrightarrow{v_m}$ and $\overrightarrow{v_{rm}}$, Here,

$\overrightarrow{v_r}$ = velocity of rain

$\overrightarrow{v_m}$ = velocity of man

(it may be velocity of cyclist or velocity of motorist also)

and $\overrightarrow{v_{rm}}$ = velocity of rain with respect to man.

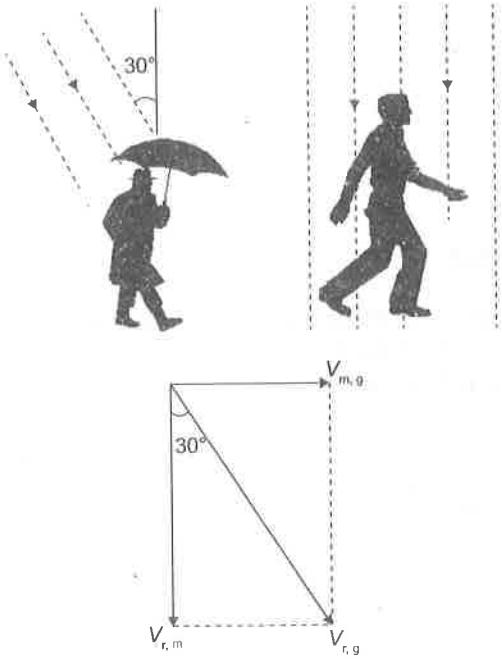
Here, $\overrightarrow{v_{rm}}$ is the velocity of rain which appears to the man. Now, let us take an example of this kind.

EXAMPLE 55

A man standing on a road has to hold his umbrella at 30° with the vertical to keep the rain away. He throws the umbrella and starts running at 10 km/h. He finds that raindrops are hitting his head vertically. Find the speed of raindrops with respect to (a) the road and (b) the moving man.

SOLUTION

When the man is at rest with respect to the ground, the rain hits him at an angle 30° with the vertical. This is the direction of the velocity of raindrops with respect to the ground. The situation when the man runs is shown in the below figure.



Here $\vec{v}_{r,g}$ = velocity of rain with respect to the ground,

$\vec{v}_{m,g}$ = velocity of the man with respect to the ground and

$\vec{v}_{r,m}$ = velocity of rain with respect to the man.

We have, $\vec{v}_{r,g} = \vec{v}_{r,m} + \vec{v}_{m,g}$ (1)

Taking horizontal components, Eq. (1) gives

$$v_{r,g} \sin 30° = u_{m,g} = 10 \text{ km/h or,}$$

$$v, g = \frac{10 \text{ km/h}}{\sin 30°} = 20 \text{ km/h}$$

Taking vertical components, Eq. (1) gives

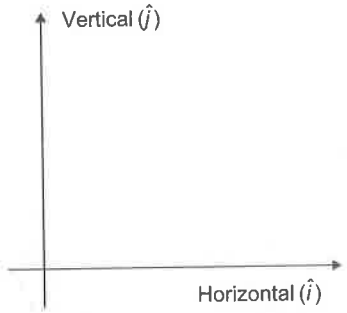
$$v_{r,g} \cos 30° = v_{r,m} \text{ or,}$$

$$v_{r,m} = (20 \text{ km/h}) \frac{\sqrt{3}}{2} = 10\sqrt{3} \text{ km/h.}$$

■

EXAMPLE 56

To a man walking at a rate of 3 km/h, the rain appears to fall vertically. When he increases his speed to 6 km/h, it appears to meet him at an angle of 45° with vertical. Find the speed of rain.



SOLUTION

Let $\hat{i}$ and $\hat{j}$ be the unit vectors in the horizontal and vertical directions, respectively.

Let the velocity of rain

$$\vec{v}_r = a\hat{i} + b\hat{j} \quad \text{(i)}$$

Then, speed of rain will be

$$|\vec{v}_r| = \sqrt{a^2 + b^2}$$

In the first case, $\vec{v}_m$ = velocity of man = $3\hat{i}$

$$\therefore \quad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (a-3)\hat{i} + b\hat{j}$$

It seems to be in the vertical direction. Hence,

$$a - 3 = 0 \text{ or } a = 3$$

In the second case, $\vec{v}_m = 6\hat{i}$

$$\therefore \quad \vec{v}_{rm} = (a-6)\hat{i} + b\hat{j}$$

$$= -3\hat{i} + b\hat{j}$$

This seems to be at 45° with vertical.

Hence, $|b| = 3$

Therefore, from Equation (ii), the speed of rain is

$$|\vec{v}_r| = \sqrt{(3)^2 + (3)^2}$$

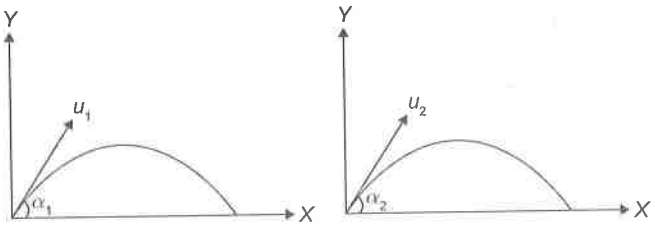
$$= 3\sqrt{2} \text{ km/h}$$

■

Relative Motion between Two Projectiles

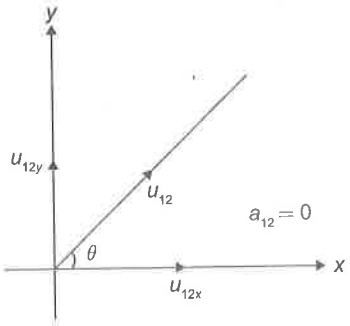
Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in Figs. A and B. Acceleration of both the particles is g downwards. So, the relative acceleration between them is zero because

$$a_{12} = a_1 - a_2 = g - g = 0$$



(A) (B)

i.e., the relative motion between the two particles is uniform. Now



$$u_{1x} = u_1 \cos \alpha_1, u_{2x} = u_2 \cos \alpha_2$$

$$u_{1y} = u_1 \sin \alpha_1$$

and $$u_{2y} = u_2 \sin \alpha_2$$

Therefore, $u_{12x} = u_{1x} - u_{2x} = u_1 \cos \alpha_1 - u_2 \cos \alpha_2$

and $u_{12y} = u_{1y} - u_{2y} = u_1 \sin \alpha_1 - u_2 \sin \alpha_2$

u_{12x} and u_{12y} are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straight line at an angle $\theta = \tan^{-1}\left(\frac{u_{12y}}{u_{12x}}\right)$ with positive x-axis.

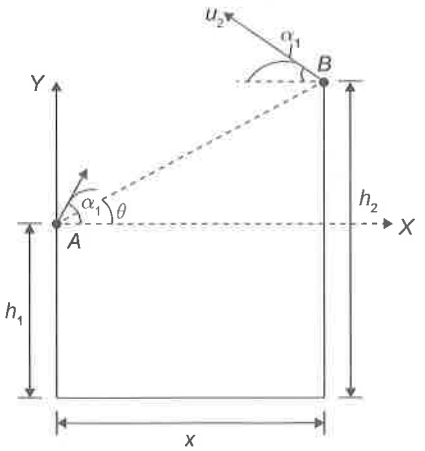
Now, if $u_{12x} = 0$ or $u_1 \cos \alpha_1 = u_2 \cos \alpha_2$, the relative motion is along the y-axis or in vertical direction (as $\theta = 90°$). Similarly, if $u_{12y} = 0$ or $u_1 \sin \alpha_1 = u_2 \sin \alpha_2$, the relative motion is along the x-axis or in the horizontal direction (as $\theta = 0°$).

Note

Relative acceleration between the two projectiles is zero. Relative motion between them is uniform. Therefore, condition of collision of two particles in air is that relative velocity of one with respect to the other should be a long line joining them, i.e., if two projectiles A and B collide in mid-air, then $\vec{V}_{AB}$ should be along AB or $\vec{V}_{BA}$ along BA.

Condition for Collision of Two Projectiles

Consider the situation shown in the figure. For the projectiles to collide, direction of velocity of A with respect to B has to be along the line AB.



Here, $v_{ABx} = u_1 \cos \alpha_1 + u_2 \cos \alpha_2$

$v_{ABy} = u_1 \sin \alpha_1 - u_2 \sin \alpha_2$

Let the direction of velocity vector of A (with respect to B) is making an angle β with the positive x-axis, which is given by

$$\tan \beta = \frac{v_{ABy}}{v_{ABx}} = \frac{u_1 \sin \alpha_1 - u_2 \sin \alpha_2}{u_1 \cos \alpha_1 + u_2 \cos \alpha_2}$$

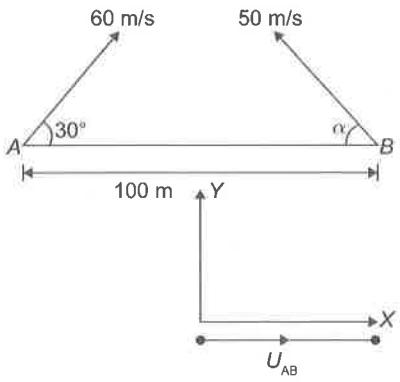
For collision to take place,

$$\tan \beta = \tan \theta = \frac{h_2 - h_1}{x}$$

SOLVED EXAMPLES

EXAMPLE 57

A particle A is projected with an initial velocity of 60 m/s at an angle 30° to the horizontal. At the same time, a second particle B is projected in the opposite direction with initial speed of 50 m/s from a point at a distance of 100 m from A. If the particles collide in air, find (a) the angle of projection α of particle B, (b) time when the collision takes place and (c) the distance of P from A, where collision occurs (g = 10 m/s^2).



SOLUTION

(a) Taking x and y directions as shown in the figure.

Here, $\vec{a}_A = -g\hat{j}$, $\vec{a}_B = -g\hat{j}$

$u_{Ax} = 60 \cos 30° = 30\sqrt{3}$ m/s

$u_{Ay} = 60 \sin 30° = 30$ m/s

$u_{Bx} = -50 \cos \alpha$

and $u_{By} = 50 \sin \alpha$

Relative acceleration between the two is zero as $\vec{a}_A = \vec{a}_B$. Hence, the relative motion between the two is uniform. It can be assumed that B is at rest and A is moving with $\vec{u}_{AB}$. Hence, the two particles will collide, if $\vec{u}_{AB}$ is along AB. This is possible only when

$$u_{Ay} = u_{By}$$

i.e., the component of relative velocity along the y-axis should be zero.

or $$30 = 50 \sin\alpha$$

$\therefore$ $$\alpha = \sin^{-1}(3/5)$$

(b) Now, $$\left|\vec{u}_{AB}\right| = u_{Ax} - u_{Bx} = (30\sqrt{3} + 50\cos\alpha)\ \text{m/s}$$

$$= \left(30\sqrt{3} + 50 \times \frac{4}{5}\right)\ \text{m/s}$$

$$= (30\sqrt{3} + 40)\ \text{m/s}.$$

Therefore, time of collision is

$$t = \frac{AB}{\left|\vec{u}_{AB}\right|} = \frac{100}{30\sqrt{3} + 40}$$

or $$t = 1.09\ \text{s}$$

(c) Distance of point P from A where collision takes place is

$$s = \sqrt{(u_{Ax}t)^2 + \left(u_{Ay}t - \frac{1}{2}gt^2\right)^2}$$

$$= \sqrt{\left(30\sqrt{3} \times 1.09\right)^2 + \left(30 \times 1.09 - \frac{1}{2} \times 10 \times 1.09 \times 1.09\right)^2}$$

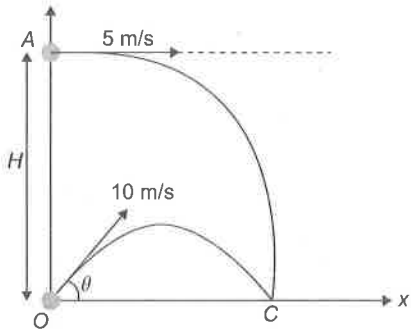
or $$s = 62.64\ \text{m}$$ ■

EXAMPLE 58

Two projectiles are projected simultaneously from a point on the ground O and an elevated position A, respectively, as shown in the figure. If collision occurs at the point of return of two projectiles on the horizontal surface, then find the height of A above the ground and the angle at which the projectile O at the ground should be projected.

SOLUTION

There is no initial separation between the two projectile in the x-direction. For collision to occur, the relative motion in x-direction should be zero. In other words, the component velocities in x-direction should be equal to the horizontal distance that the two projectiles cover equally at any given time. Hence,



$$u_{Ox} = u_{Ax}$$

$\Rightarrow$ $$u_0 \cos\theta = u_A$$

$\Rightarrow$ $$\cos\theta = \frac{u_A}{u_O} = \frac{5}{10}$$

$$= \frac{1}{2} = \cos 60°$$

$\Rightarrow$ $$\theta = 60°$$

We should ensure that collision does occur at the point of return. It means that by the time projectiles travel horizontal distances required, they should also cover vertical distances so that both the projectiles are at C at the same time. In a nutshell, their times of flight should be equal.

For projectile from O,

$$T = \frac{2u_o \sin\theta}{g}$$

For projectile from A,

$$T = \sqrt{\left(\frac{2H}{g}\right)}$$

For projectile from A,

$$T = \frac{2u_o \sin\theta}{g} = \sqrt{\left(\frac{2H}{g}\right)}$$

Squaring both sides and inserting the values,

$\Rightarrow$ $$H = \frac{4u_o^2 \sin^2\theta}{2g}$$

$\Rightarrow$ $$H = \frac{4 \times 10^2 \sin^2 60°}{2 \times 10}$$

$$H = 20\left(\frac{\sqrt{3}}{2}\right)^2 = 15\,\text{m}$$

We have deliberately worked out this problem taking advantage of the fact that projectiles are colliding at the end of their flights and hence their times of flight should be equal. We can, however, proceed to analyse in typical manner, using the concept of relative velocity. The initial separation between the two projectiles in the vertical direction is H. This separation is covered with the component of relative velocity in the vertical direction.

$$\Rightarrow \quad v_{OAy} = u_{Oy} - u_{Ay} = u_0 \sin 60° - 0$$

$$= 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ m/s}.$$

Now, time of flight of the projectile from the ground is

$$T = \frac{2u_0 \sin\theta}{g} = \frac{2 \times 10 \times \sin 60°}{10}$$

$$= \sqrt{3}$$

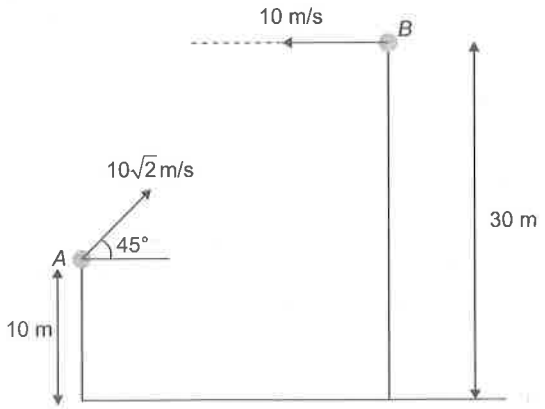
Hence, the vertical displacement of the projectile from A before collision is

$$\Rightarrow \quad H = v_{OAy} \, X \, T = 5\sqrt{3} \times \sqrt{3}$$

$$= 15 \text{ m/s}. \quad \blacksquare$$

EXAMPLE 59

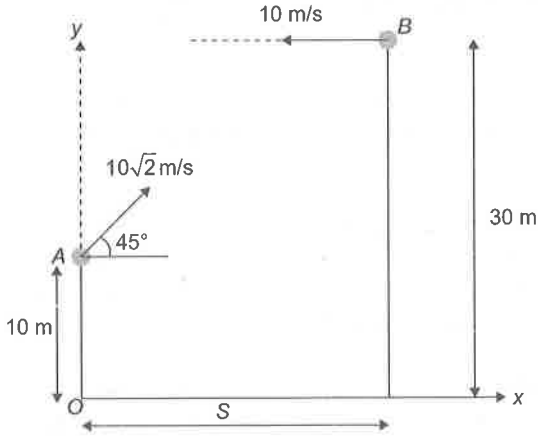
Two projectiles are projected simultaneously from two towers as shown in the figure. If the projectiles collide in air, then find the distance s between the towers.



SOLUTION

We see here that the projectiles are approaching both horizontally and vertically. Their movement in two component directions should be synchronized so that they are at the same position at a particular given time. For collision, the necessary requirement is that relative velocity and displacement should be in the same direction.

It is given that collision does occur. It means that two projectiles should cover the displacement with relative velocity in each of the component directions.



In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2} \cos 45° - (-10)$$

$$= 10\sqrt{2} \frac{1}{\sqrt{2}} + 10 = 20 \text{ m/s},$$

If t is the time after which collision occurs, then

$$\Rightarrow \quad s = v_{Ay} - u_{By}$$

$$\Rightarrow \quad v_{ABy} = u \cos 45° - 0$$

$$= 10\sqrt{2} \times \frac{1}{\sqrt{2}} = 10 \text{ m/s}.$$

The initial vertical distance between the points of projection is 30 − 10 = 20 m. This vertical distance is covered with the component of relative velocity in vertical direction. Hence, the time taken to collide t is

$$\Rightarrow \quad t = \frac{20}{10} = 2$$

Putting this value in the earlier equation for "s", we have :

$$\Rightarrow \quad s = 20t = 20 \times 2 = 40 \text{ m} \quad \blacksquare$$

Note

Now you can try all the questions related to relative motion.

Questions for Short Answer

1. A vector $\vec{a}$ is turned without a change in its length through a small angle $d\theta$. What are $|\vec{a}|$ and Δa?

2. Does the speedometer of a car measure speed or velocity? Explain.

3. When a particle moves with constant velocity, its average velocity and its instantaneous velocity and speed are equal. Comment on this statement.

4. In a given time interval, is the total displacement of a particle equal to the product of the average velocity and the time interval, even when the velocity is not constant? Explain.

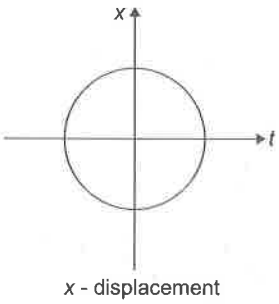
5. Can you have zero displacement and a non-zero average velocity? Can you have a zero displacement and a non-zero velocity? Illustrate your answer on an x–t graph.

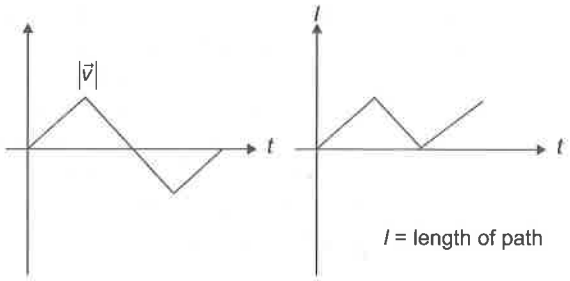
6. At which point on its path a projectile has the smallest speed?

7. A person standing on the edge of a cliff at some height above the ground below throws one ball straight up with initial speed u and then throws another ball straight down with the same initial speed. Which ball, if either, has the larger speed when it hits the ground? Neglect air resistance.

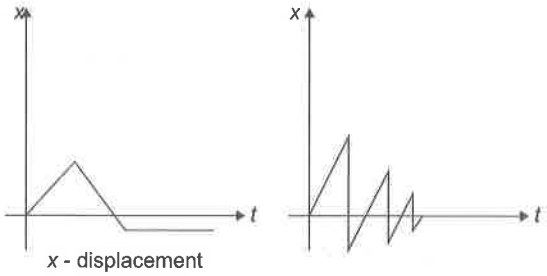
8. An airplane on floor relief mission has to drop a sack of rice exactly in the centre of a circle on the ground while flying at a predetermined height and speed. What is so difficult about that? Why does not it just drop the sack when it is directly above the circle?

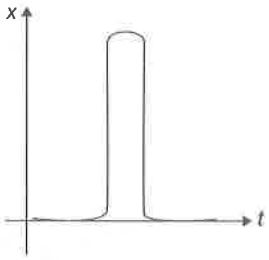
9. Which of the following graphs cannot possibly represent one-dimensional motion of a particle?





10. Can you suggest a suitable situation from observation around you for each of the following?





11. One of the following statements is incorrect.

    (a) The car travelled around the track at a constant velocity.

    (b) The car travelled around the track at a constant speed.

    Which statement is incorrect and why?

12. Give an example from your own experience in which the velocity of an object is zero for just an instant of time, but its acceleration is not zero.

13. A ball is dropped from rest from the top of a building and strikes the ground with a speed vf. From the ground level, a second ball is thrown straight upward at the same instant that the first ball is dropped. The initial speed of the second ball is $v_0 = vf$, the same speed with which the first ball will eventually strike the ground. Ignoring air resistance, decide whether the balls cross paths at half the height of the building

above the halfway point or below the halfway point. Give your reasoning.

14. The muzzle velocity of a gun is the velocity of the bullet when it leaves the barrel. The muzzle velocity of one rifle with a short barrel is greater than the muzzle velocity of another rifle that has a longer barrel. In which rifle is the acceleration of the bullet larger? Explain your reasoning.

15. On a riverboat cruise, a plastic bottle is accidentally dropped overboard. A passenger on the boat estimates that the boat pulls ahead of the bottle by 5 m each second. Is it possible to conclude that the boat is moving at 5 m/s with respect to the shore? Account for your answer.

16. A wrench is accidentally dropped from the top of the mast on a sailboat. Will the wrench hit at the same place on the deck whether the sailboat is at rest or moving with a constant velocity? Justify your answer.

17. Is the acceleration of a projectile equal to zero when it reaches the top of its trajectory? If not, why not?

18. A child is playing on the floor of a recreational vehicle (*RV*) as it moves along the highway at a constant velocity. He has a toy cannon, which shoots a marble at a fixed angle and speed with respect to the floor. The cannon can be aimed toward the front or the rear of the *RV*. Is the range towards the front the same as, less than, or greater than the range towards the rear? Answer this question (a) from the child's point of view and (b) from the point of view of an observer standing still on the ground. Justify your answers.

19. Three swimmers can swim equally fast relative to the water. They have a race to see who can swim across a river in the least time. Swimmer *A* swims perpendicular to the current and lands on the far shore downstream, because the current has swept him in that direction. Swimmer *B* swims upstream at an angle to the current and lands on the far shore directly opposite the starting point. Swimmer *C* swims downstream at an angle to the current in an attempt to take advantage of the current. Who crosses the river in the least time? Account for your answer.

EXERCISES

JEE Main

1. A particle covers half of the circle of radius r. Then the displacement and distance of the particle are, respectively,
   (A) $2\pi r, 0$ (B) $2r, \pi r$
   (C) $\frac{\pi r}{2}, 2r$ (D) $\pi r, r$

2. A hall has the dimensions 10 m × 10 m × 10 m. A fly starting at one corner ends up at a diagonally opposite corner. The magnitude of its displacement is nearly
   (A) $5\sqrt{3}$ m (B) $10\sqrt{3}$ m
   (C) $20\sqrt{3}$ m (D) $30\sqrt{3}$ m

3. A car travels from *A* to *B* at a speed of 20 km h^{-1} and returns at a speed of 30 km h^{-1}. The average speed of the car for the whole journey is
   (A) 5 km h^{-1} (B) 24 km h^{-1}
   (C) 25 km h^{-1} (D) 50 km h^{-1}

4. A car travels a distance of 2000 m. If the first half distance is covered at 40 km/h and the second half at velocity v and if the average velocity is 48 km/h, then the value of v is
   (A) 56 km/h (B) 60 km/h
   (C) 50 km/h (D) 8 km/h

5. At an instant t, the co-ordinates of a particle are $x = at^2$, $y = bt^2$ and $z = 0$, then its velocity at the instant t will be
   (A) $t\sqrt{a^2+b^2}$ (B) $2t\sqrt{a^2+b^2}$
   (C) $\sqrt{a^2+b^2}$ (D) $2t^2\sqrt{a^2+b^2}$

6. A car runs at constant speed on a circular track of radius 100 m taking 62.8 s for each lap. What is the average speed and average velocity on each complete lap?
   (A) Velocity 10 m/s speed 10 m/s
   (B) Velocity zero, speed 10 m/s
   (C) Velocity zero, speed zero
   (D) Velocity 10 m/s, speed zero

7. The displacement of a body is given by $2s = gt^2$, where g is a constant. The velocity of the body at any time t is
   (A) gt (B) $gt/2$
   (C) $gt^2/2$ (D) $gt^3/3$

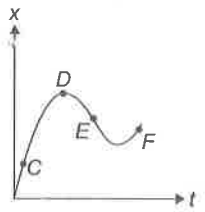
8. A particle is moving so that its displacement s is given as $s = t^3 - 6t^2 + 3t + 4$ m. Its velocity at the instant when its acceleration is zero will be

(A) 3 m/s (B) −12 m/s
(C) 42 m/s (D) −9 m/s

9. A body starts from rest and is uniformly accelerated for 30 s. The distance travelled in the first 10 s is x_1, next 10 s is x_2 and the last 10 s is x_3. Then $x_1:x_2:x_3$ is the same as

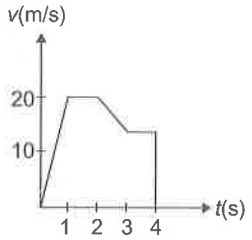
(A) 1:2:4 (B) 1:2:5
(C) 1:3:5 (D) 1:3:9

10. The displacement–time graph of a moving particle is shown in the figure. The instantaneous velocity of the particle is negative at the point



(A) C (B) D
(C) E (D) F

11. The variation of velocity of a particle moving along a straight line is shown in the figure. The distance travelled by the particle in 4 s is

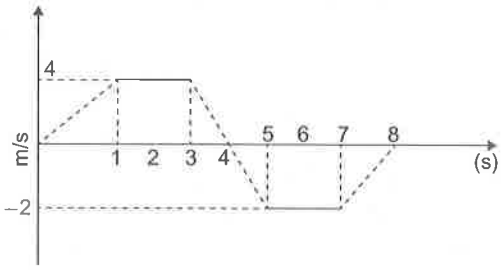


(A) 25 m (B) 30 m
(C) 55 m (D) 60 m

12. The displacement time graphs of two particles A and B are straight lines making angles of, respectively, 30° and 60° with the time axis. If the velocity of A is v_A and that of B is v_B, then the value of $\frac{v_A}{v_B}$ is

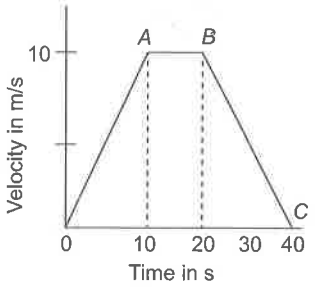
(A) 1/2 (B) $1/\sqrt{3}$
(C) $\sqrt{3}$ (D) 1/3

13. The v–t graph of a linear motion is shown in the adjoining figure. The distance from the origin after 8 s is



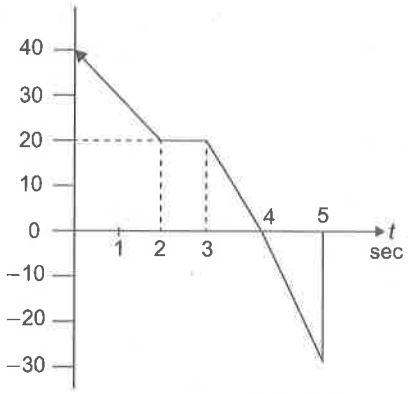
(A) 18 m (B) 16 m
(C) 8 m (D) 6 m

14. The adjoining curve represents the velocity–time graph of a particle, its acceleration values along OA, AB and BC in m/s² are, respectively,



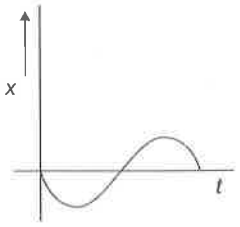
(A) 1, 0, −0.5 (B) 1, 0, 0.5
(C) 1, 1, 0.5 (D) 1, 0.5, 0

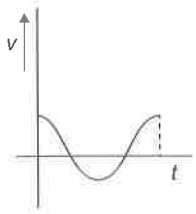
15. In the following velocity-time graph of a body, the distance and displacement travelled by the body in 5 s in metres will be

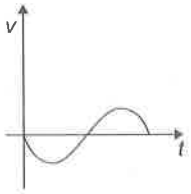


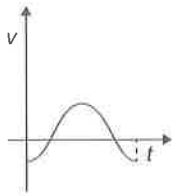
(A) 75, 115 (B) 105, 75
(C) 45, 75 (D) 95, 55

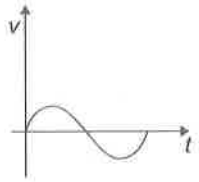
16. If position–time graph of a particle is a sine curve as shown, what will be its velocity–time graph?



(A)


(B)


(C)


(D)


17. A particle, after starting from rest, experiences constant acceleration for 20 s. If it covers a distance of S_1 in the first 10 s and distance S_2 in the next 10 s, then

(A) $S_2 = S_1/2$ (B) $S_2 = S_1$

(C) $S_2 = 2S_1$ (D) $S_2 = 3S_1$

18. A body sliding on a smooth inclined plane requires 4 s to reach the bottom after starting from rest at the top. How much time does it take to cover one fourth the distance starting from the top?

(A) 1 s (B) 2 s

(C) 0.4 s (D) 1.6 s

19. The initial velocity of a particle is 10 m/s and its retardation is 2 m/s^2. The distance covered in the fifth second of the motion will be

(A) 1 m (B) 19 m

(C) 50 m (D) 75 m

20. A body starts from rest, the ratio of distances travelled by the body during the 3rd and 4th seconds is

(A) 7/5 (B) 5/7

(C) 7/3 (D) 3/7

21. A body is dropped from a height h under acceleration due to gravity g. If t_1 and t_2 are time intervals for its fall for the first half and the second half distance, the relation between them is

(A) $t_1 = t_2$ (B) $t_1 = 2t_2$

(C) $t_1 = 2.414t_2$ (D) $t_1 = 4t_2$

22. Two bodies of different masses m_a and m_b are dropped from two different heights, viz a and b. The ratio of times taken by the two to drop through these distances is

(A) $a:b$ (B) $\frac{m_a}{m_b} : \frac{b}{a}$

(C) $\sqrt{a} : \sqrt{b}$ (D) $a^2:b^2$

23. A body is thrown upward and reaches its maximum height. At that position

(A) its velocity is zero and its acceleration is also zero

(B) its velocity is zero but its acceleration is maximum

(C) its acceleration is minimum

(D) its velocity is zero and its acceleration is the acceleration due to gravity

24. Two trains each of length 50 m are approaching each other on parallel rails. Their velocities are 10 m/s and 15 m/s. They will cross each other in

(A) 2 s (B) 4 s

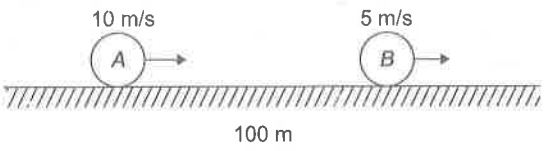
(C) 10 s (D) 6 s

25. A car A is going north-east at 80 km/h and another car B is going south-east at 60 km/h. Then the direction of the velocity of A relative to B makes with the north an angle α such that tan α is

(A) 1/7 (B) 3/4

(C) 4/3 (D) 3/5

26. An object A is moving at 10 m/s and B is moving at 5 m/s in the same direction of positive x-axis. A is 100 m behind B as shown. Find the time taken by A to meet B.



(A) 18 s (B) 16 s

(C) 20 s (D) 17 s

27. A ball is thrown upwards. It returns to the ground describing a parabolic path. Which of the following remains constant?
(A) Speed of the ball
(B) Kinetic energy of the ball
(C) Vertical component of velocity
(D) Horizontal component of velocity

28. The angle of projection of a body is 15°. The other angle for which the range is the same as the first one is equal to
(A) 30° (B) 45°
(C) 60° (D) 75°

29. A ball is thrown at an angle of 45° with the horizontal with kinetic energy E. The kinetic energy at the highest point during the flight is
(A) Zero (B) $E/2$
(C) E (D) $(2)^{1/2}E$

30. The maximum range of a projectile is 22 m. When it is thrown at an angle of 15° with the horizontal, its range will be
(A) 22 m (B) 6 m
(C) 15 m (D) 11 m

31. The equation of a projectile is $y = 16x - \frac{5x^2}{4}$. The horizontal range is
(A) 16 m (B) 8 m
(C) 3.2 m (D) 12.8 m

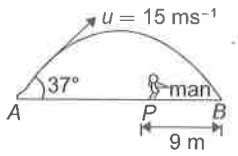
32. If four balls A, B, C, D are projected with the same speed at angles of 15°, 30°, 45° and 60° with the horizontal, respectively, the two balls which will fall at the same place will be
(A) A and B (B) A and D
(C) B and D (D) A and C

33. The velocity at the maximum height of a projectile is half of its initial velocity u. Its range on the horizontal plane is
(A) $\frac{2u^2}{3g}$ (B) $\frac{\sqrt{3}u^2}{2g}$
(C) $\frac{u^2}{3g}$ (D) $\frac{u^2}{2g}$

34. A marble A is dropped vertically, another identical marble B is projected horizontally from the same point at the same instant
(A) A will reach the ground earlier than B
(B) B will reach the ground earlier than A
(C) both A and B will reach the ground at the same instant
(D) none of the above

35. An aeroplane is flying at a height of 1960 m in horizontal direction with a velocity of 360 km/h. When it is vertically above the point. A on the ground, it drops a bomb. The bomb strikes a point B on the ground, then the time taken by the bomb to reach the ground is
(A) $20\sqrt{2}$ s (B) 20 s
(C) $10\sqrt{2}$ s (D) 10 s

36. A ball is hit by a batsman at an angle of 37° as shown in figure. The man standing at P should run at what minimum velocity so that he catches the ball before it strikes the ground. Assume that height of man is negligible in comparison to maximum height of projectile.



(A) 3 ms^{-1} (B) 5 ms^{-1}
(C) 9 ms^{-1} (D) 12 ms^{-1}

37. Two projectiles A and B are thrown with the same speed such that A makes angle θ with the horizontal and B makes angle θ with the vertical, then
(A) both must have the same time of flight
(B) both must achieve the same maximum height
(C) A must have more horizontal range than B
(D) both may have the same time of flight

38. Suppose a player hits several baseballs. Which baseball will be in the air for the longest time?
(A) The one with the farthest range
(B) The one which reaches maximum height
(C) The one with the greatest initial velocity
(D) The one leaving the bat at 45° with respect to the ground

39. A ball is projected from the top of a tower with a velocity of 5 m/s at an angle of 53° to the horizontal. Its speed when it is at a height of 0.45 m from the point of projection is
(A) 2 m/s (B) 3 m/s
(C) 4 m/s (D) data insufficient

40. From the top of a tower of height h a body of mass m is projected in the horizontal direction with a velocity v, It falls on the ground at a distance x from the tower. If a body of mass 2 m is projected from the top of another tower of height $2h$ in the horizontal direction so that it falls on the ground at a distance $2x$ from the tower, the horizontal velocity of the second body is
(A) $2v$ (B) $\sqrt{2}\,v$
(C) $\frac{v}{2}$ (D) $\frac{v}{\sqrt{2}}$

41. One stone is projected horizontally from a 20-m-high cliff with an initial speed of 10 ms^{-1}. A second stone is simultaneously dropped from that cliff. Which of the following is true?
(A) Both strike the ground with the same velocity
(B) The ball with initial speed 10 ms^{-1} reaches the ground first
(C) Both the balls hit the ground at the same time
(D) One cannot say without knowing the height of the building

42. An aeroplane flying at a constant velocity releases a bomb. As the bomb drops down from the aeroplane,
(A) it will always be vertically below the aeroplane
(B) it will always be vertically below the aeroplane only if the aeroplane is flying horizontally
(C) it will always be vertically below the aeroplane only if the aeroplane is flying at an angle of 45° to the horizontal
(D) it will gradually fall behind the aeroplane if the aeroplane is flying horizontally

43. A helicopter is flying south with a speed of 50 kmh^{-1}. A train is moving with the same speed towards east. The relative velocity of the helicopter as seen by the passengers in the train will be towards.
(A) north-east (B) south-east
(C) north-west (D) south-west

44. Two particles are moving with velocities v_1 and v_2. Their relative velocity is the maximum, when the angle between their velocities is
(A) zero (B) $\pi/4$
(C) $\pi/2$ (D) π

45. A swimmer's speed in the direction of flow of river is 16 km h^{-1}. Against the direction of flow of the river, the swimmer's speed is 8 km h^{-1}. Calculate the swimmer's speed in still water and the velocity of flow of the river.
(A) 12 km/h, 4 km/h (B) 10 km/h, 3 km/h
(C) 10 km/h, 4 km/h (D) 12 km/h, 2 km/h

46. A man is walking on a road with a velocity 3 km/h. Suddenly rain starts falling. The velocity of rain is 10 km/h in vertically downward direction. The relative velocity of the rain is
(A) $\sqrt{13}$ km/h (B) $\sqrt{7}$ km/h
(C) $\sqrt{109}$ km/h (D) 13 km/h

JEE Advanced

Single Correct

1. A body covers the first 1/3 part of its journey with a velocity of 2 m/s, the next 1/3 part with a velocity of 3 m/s and rest of the journey with a velocity of 6m/s. The average velocity of the body will be
(A) 3 m/s (B) $\frac{11}{3}$ m/s
(C) $\frac{8}{3}$ m/s (D) $\frac{4}{3}$ m/s

2. A particle has a velocity u towards east at $t = 0$. Its acceleration is towards west and is constant, Let x_A and x_B be the magnitude of displacements in the first 10 s and the next 10 s.
(A) $x_A < x_B$
(B) $x_A = x_B$
(C) $x_A > x_B$
(D) The information is insufficient to decide the relation of x_A with x_B.

3. A stone is dropped into a well in which the level of water is h below the top of the well. If v is velocity of sound, the time T after which the splash is heard is given by

(A) $T = 2h/v$ (B) $T = \sqrt{\frac{2h}{g}} + \frac{h}{v}$

(C) $T = \sqrt{\frac{2h}{g}} + \frac{h}{2v}$ (D) $T = \sqrt{\frac{h}{2g}} + \frac{2h}{v}$

4. The co-ordinates of a moving particle at a time t are given by, $x = 5 \sin 10t$, $y = 5 \cos 10t$. The speed of the particle is

(A) 25 (B) 50

(C) 10 (D) none of these

5. A body moves with velocity $v = \ln x$ m/s, where x is its position. The net force acting on the body is zero at

(A) 0 m (B) $x = e^2$ m

(C) $x = e$ m (D) $x = 1$ m

6. A body of mass 1 kg is acted upon by a force $\vec{F} = 2\sin 3\pi t\,i + 3\cos 3\pi t\,j$. Find its position at $t = 1$ s if at $t = 0$ it is at rest at origin.

(A) $\left(\frac{3}{3\pi^2}, \frac{3}{9\pi^2}\right)$ (B) $\left(\frac{2}{3\pi^2}, \frac{2}{3\pi^2}\right)$

(C) $\left(\frac{2}{3\pi}, \frac{2}{3\pi^2}\right)$ (D) none of these

7. A force $F = Be^{-Ct}$ acts on a particle whose mass is m and whose velocity is 0 at $t = 0$. Its terminal velocity (velocity after a long time) is

(A) $\frac{C}{mB}$ (B) $\frac{B}{mC}$

(C) $\frac{BC}{m}$ (D) $-\frac{B}{mC}$

8. A particle starts moving rectilinearly at time $t = 0$ such that its velocity v changes with time t according to the equation $v = t^2 - t$, where t is in seconds and v is in m/s. The time interval for which the particle retards is

(A) $t < 1/2$ (B) $1/2 < t < 1$

(C) $t > 1$ (D) $t < 1/2$ and $t > 1$

9. A ball is thrown vertically down with a velocity of 5 m/s. With what velocity should another ball be thrown down after 2 s so that it can hit the first ball in 2 s?

(A) 40 m/s (B) 55 m/s

(C) 15 m/s (D) 25 m/s

10. A particle is projected vertically upwards from a point A on the ground. It takes t_1 time to reach a point B but it still continues to move up. If it takes further t_2 time to reach the ground from point B, then height of point B from the ground is

(A) $\frac{1}{2}g(t_1 + t_2)^2$ (B) gt_1t_2

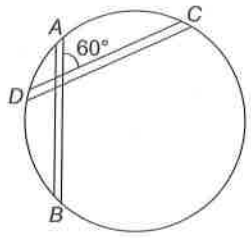
(C) $\frac{1}{8}g(t_1 + t_2)^2$ (D) $\frac{1}{2}gt_1t_2$

11. Balls are thrown vertically upward in such a way that the next ball is thrown when the previous one is at the maximum height. If the maximum height is 5 m, the number of balls thrown per minute will be

(A) 40 (B) 50

(C) 60 (D) 120

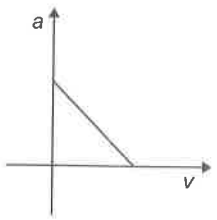
12. A disc arranged in a vertical plane has two groves of the same length directed along the vertical chord AB and CD as shown in the figure. The same particles slide down along AB and CD. The ratio of the time t_{AB}/t_{CD} is

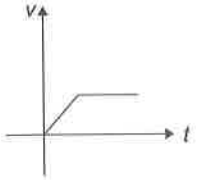


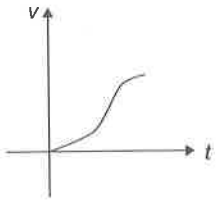
(A) 1:2 (B) $1:\sqrt{2}$

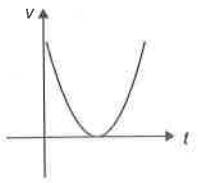
(C) 2:1 (D) $\sqrt{2}:1$

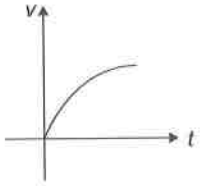
13. Acceleration versus velocity graph of a particle moving in a straight line starting form rest is as shown in the figure. The corresponding velocity–time graph would be



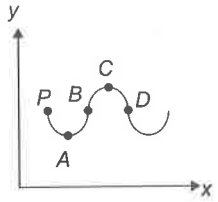
(A) 

(B) 

(C) 

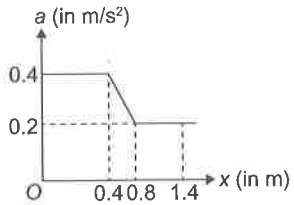
(D) 

14. A man moves in the x–y plane along the path shown. At what point is his average velocity vector in the same direction as his instantaneous velocity vector. The man starts from point P.



(A) A (B) B
(C) C (D) D

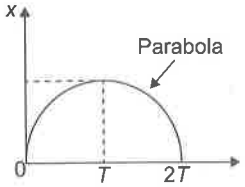
15. The acceleration of a particle which moves along the positive x-axis varies with its position as shown. If the velocity of the particle is 0.8 m/s at $x = 0$, the velocity of the particle at $x = 1.4$ is (in m/s)



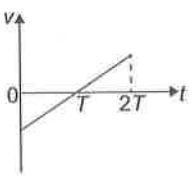
(A) 1.6 (B) 1.2
(C) 1.4 (D) none of these

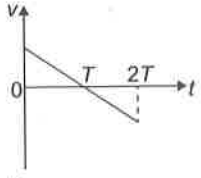
Question No. 16 to 18 (3 Questions)

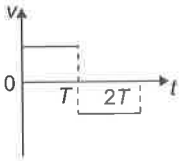
The x–t graph of a particle moving along a straight line is shown in the figure.

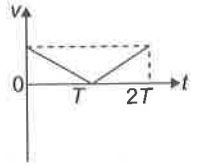


16. The v–t graph of the particle is correctly shown by

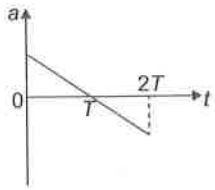
(A) 

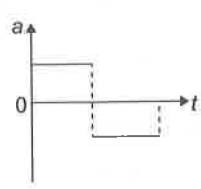
(B) 

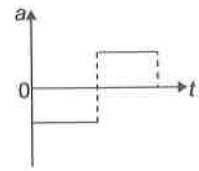
(C) 

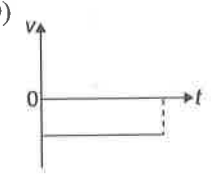
(D) 

17. The a–t graph of the particle is correctly shown by

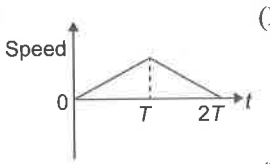
(A) 

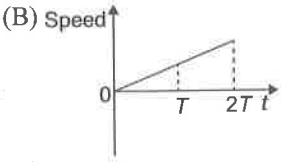
(B) 

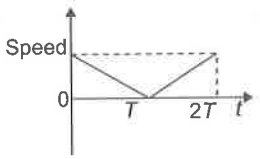
(C) 

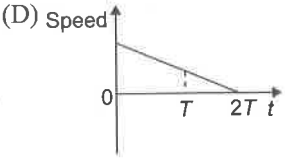
(D) 

18. The speed–time graph of the particle is correctly shown by

(A) 

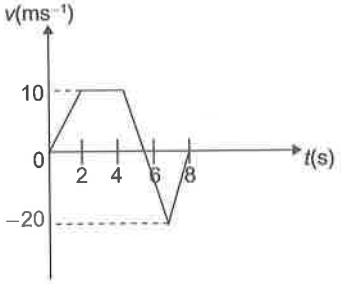
(B) 

(C) 

(D) 

Question No. 19 to 24 (6 Questions)

The figure shows a velocity–time graph of a particle moving along a straight line



19. Choose the incorrect statement. The particle comes to rest at

(A) $t = 0$ s (B) $t = 5$ s

(C) $t = 8$ s (D) none of these

20. Identify the region in which the rate of change of velocity $\left|\frac{\Delta \vec{v}}{\Delta t}\right|$ of the particle is maximum.

(A) 0 – 2 s (B) 2 – 4 s

(C) 4 – 6 s (D) 6 – 8 s

21. If the particle starts from the position $x_0 = -15$ m, then its position at $t = 2$ s will be

(A) −5 m (B) 5 m

(C) 10 m (D) 15 m

22. The maximum displacement of the particle is

(A) 33.3 m (B) 23.3 m

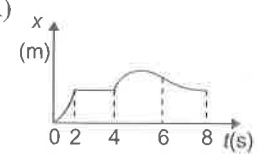
(C) 18.3 (D) zero

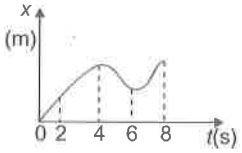
23. The total distance travelled by the particle is

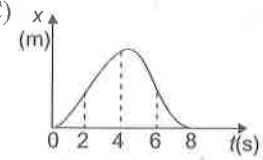
(A) 66.6 m (B) 51.6 m

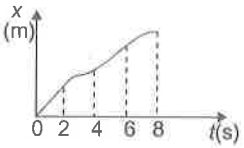
(C) zero (D) 36.6 m

24. The correct displacement-time graph of the particle is shown as

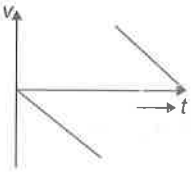
(A) 

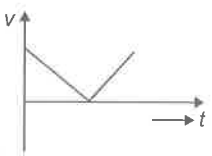
(B) 

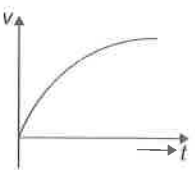
(C) 

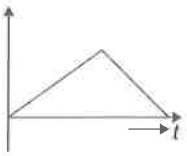
(D) 

25. The velocity-time graph of a body falling from rest under gravity and rebounding from a solid surface is represented by which of the following graphs?

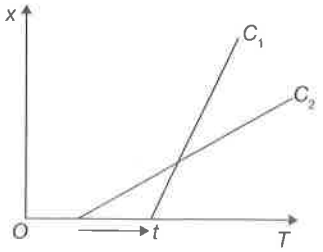
(A) 

(B) 

(C) 

(D) 

26. Shown in the figure are the displacement–time graph for two children going home from the school. Which of the following statements about their relative motion is true after both of them started moving? Their relative velocity



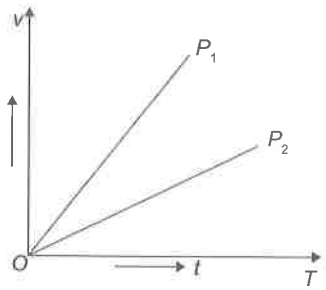
(A) first increases and then decreases

(B) first decreases and then increases

(C) is zero

(D) is non-zero constant

27. Shown in the figure are the velocity–time graphs of the two particles P_1 and P_2. Which of the following statements about their relative motion is true? Their relative velocity



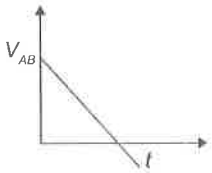
(A) is zero

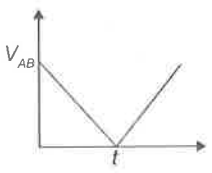
(B) is non-zero but constant

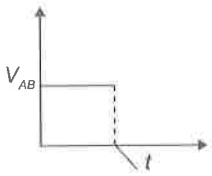
(C) continuously decreases

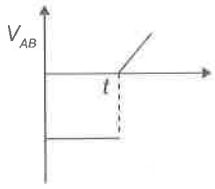
(D) continuously increases

28. A body A is thrown vertically upwards with such a velocity that it reaches a maximum height of h. Simultaneously, another body B is dropped from height h. It strikes the ground and does not rebound. The velocity of A relative to B versus time graph is best represented by (upward direction is positive)

(A) 

(B) 

(C) 

(D) 

29. It takes 1 min for a passenger standing on an escalator to reach the top. If the escalator does not move, it takes him 3 min to walk up. How long will it take for the passenger to arrive at the top if he walks up the moving escalator?

(A) 30 s (B) 45 s
(C) 40 s (D) 35 s

30. A body is thrown up in a lift with a velocity u relative to the lift and the time of flight is found to be t. The acceleration with which the lift is moving up is

(A) $\frac{u-gt}{t}$ (B) $\frac{2u-gt}{t}$
(C) $\frac{u+gt}{t}$ (D) $\frac{2u+gt}{t}$

31. A point mass is projected, making an acute angle with the horizontal. If the angle between velocity $\vec{v}$ and acceleration $\vec{g}$ is θ, then θ is given by

(A) $0° < \theta < 90°$ (B) $\theta = 90°$
(C) $\theta = 90°$ (D) $0° < \theta < 180°$

Question No. 32 to 34 (3 Questions)

A projectile is thrown with a velocity of 50 ms^{-1} at an angle of 53° with the horizontal.

32. Choose the incorrect statement:

(A) It travels vertically with a velocity of 40 ms^{-1}
(B) It travels horizontally with a velocity of 30 ms^{-1}
(C) The minimum velocity of the projectile is 30 ms^{-1}
(D) None of these

33 Determine the instants at which the projectile is at the same height.

(A) $t = 1$ s and $t = 7$ s (B) $t = 3$ s and $t = 5$ s
(C) $t = 2$ s and $t = 6$ s (D) All of these

34. The equation of the trajectory is given by

(A) $180y = 240x - x^2$ (B) $180y = x^2 - 240x$
(C) $180y = 135x - x^2$ (D) $180y = x^2 - 135x$

35. A particle is projected from a horizontal plane (x–z plane) such that its velocity vector at time t is given by $\vec{V} = a\hat{i} + (b - tc)\hat{j}$. Its range on the horizontal plane is given by

(A) $\frac{ba}{c}$ (B) $\frac{2ba}{c}$
(C) $\frac{3ba}{c}$ (D) None of these

36. A ball is thrown from a point on the ground at some angle of projection. At the same time, a bird starts from a point directly above this point of projection at a height h horizontally with speed u. Given that in its flight the ball just touches the bird at one point. Find the distance on ground where the ball strikes

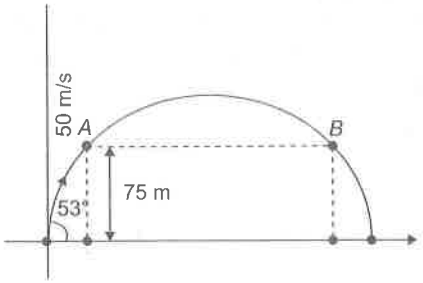
(A) $2u\sqrt{\frac{h}{g}}$ (B) $u\sqrt{\frac{2h}{g}}$
(C) $2u\sqrt{\frac{2h}{g}}$ (D) $u\sqrt{\frac{h}{g}}$

37. A projectile is fired with a speed u at an angle θ with the horizontal. Its speed when its direction of motion makes an angle α with the horizontal is

(A) $u \sec\theta \cos\alpha$ (B) $u \sec\theta \sin\alpha$
(C) $u \cos\theta \sec\alpha$ (D) $u \sin\theta \sec\alpha$

Question No. 38 and 39 (2 Questions)

At $t = 0$, a projectile is fired from a point O (taken as origin) on the ground with a speed of 50 m/s at an angle of 53° with the horizontal. It just passes two points A and B each at height 75 m above the horizontal as shown.



38. The horizontal separation between the points A and B is
(A) 30 m (B) 60 m
(C) 90 m (D) None of these

39. The distance (in metres) of the particle from the origin at $t = 2$ s is
(A) $60\sqrt{2}$ (B) 100
(C) 60 (D) 120

40. Particle is dropped from the height of 20 m from horizontal ground. There is wind blowing due to which the horizontal acceleration of the particle becomes 6 ms^{-2}. Find the horizontal displacement of the particle till it reaches ground.
(A) 6 m (B) 10 m
(C) 12 m (D) 24 m

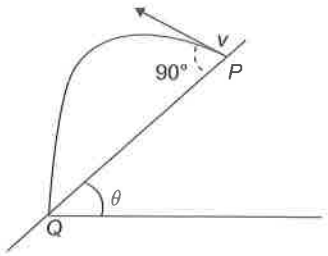
41. Find the time of flight of the projectile thrown horizontally with speed 10 ms^{-1} from a long inclined plane which makes an angle of $\theta = 45°$ from the horizontal.
(A) $\sqrt{2}$ s (B) $2\sqrt{2}$ s
(C) 2 s (D) none of these

42. A particle is projected at angle 37° with the incline plane in the upward direction with speed 10 m/s. The angle of the incline plane is given as 53°. Then the maximum height above the incline plane attained by the particle will be
(A) 3 m (B) 4 m
(C) 5 m (D) zero

43. On an inclined plane of inclination 30°, a ball is thrown at an angle of 60° with the horizontal from the foot of the incline with a velocity of $10\sqrt{3}$ ms^{-1}. If $g = 10$ ms^{-2}, then the time in which ball with hit the inclined plane is
(A) 1.15 s (B) 6 s
(C) 2 s (D) 0.92 s

44. A projectile is fired with a velocity at right angles to the slope which is inclined at an angle θ with the horizontal. The expression for the range R along the incline is
(A) $\frac{2v^2}{g}\sec\theta$ (B) $\frac{2v^2}{g}\tan\theta$
(C) $\frac{2v^2}{g}\tan\theta\sec\theta$ (D) $\frac{v^2}{g}\tan^2\theta$

45. If the time taken by the projectile to reach Q is T, then $PQ =$



(A) $Tv\sin\theta$ (B) $Tv\cos\theta$
(C) $Tv\sec\theta$ (D) $Tv\tan\theta$

Question No. 46 to 49 (4 Questions)

Two projectiles are thrown simultaneously in the same plane from the same point. If their velocities are v_1 and v_2 at angles θ_1 and θ_2, respectively, from the horizontal, then answer the following questions.

46. The trajectory of particle 1 with respect to particle 2 will be
(A) a parabola
(B) a straight line
(C) a vertical straight line
(D) a horizontal straight line

47. If $v_1\cos\theta_1 = v_2\cos\theta_2$, then choose the incorrect statement
(A) one particle will remain exactly below or above the other particle
(B) the trajectory of one with respect to the other will be a vertical straight line
(C) both will have the same range
(D) none of these

48. If $v_1\sin\theta_1 = v_2\sin\theta_2$, then choose the incorrect statement
(A) the time of flight of both the particles will be the same
(B) the maximum height attained by the particles will be the same
(C) the trajectory of one with respect to the other will be a horizontal straight line
(D) none of these

49. If $v_1 = v_2$ and $\theta_1 > \theta_2$, then choose the incorrect statement.
(A) Particle 2 moves under particle .
(B) The slope of the trajectory of particle 2 with respect to particle 1 is always positive.

$\because$ The body is in equilibrium so equate the force in the x and y directions

In the x-direction,

$$\frac{3N_1}{5}=\frac{4N_2}{5} \quad (1)$$

In the y-direction,

$$\frac{3N_2}{5}+\frac{4N_1}{5}=100 \quad (2)$$

After solving Eqs. (1) and (2), we get

$$N_1 = 80 \text{ N}, N_2 = 60 \text{ N}$$ ■

Tension

Tension in a string is an electromagnetic force. It arises when a string is pulled. If a massless string is not pulled, tension in it is zero. A string suspended by rigid support is pulled by a force 'F' as shown in Fig. 4.12, for calculating the tension at point 'A' we draw F.B.D. of marked portion of the string. Here, string is massless.

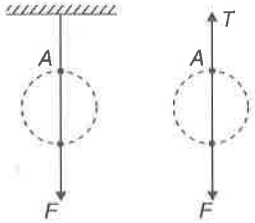


Figure 4.12

F.B.D. of marked portion

$$\Rightarrow \quad T = F$$

String is considered to be made of a number of small segments which attracts each other due to electromagnetic nature. The attraction force between two segments is equal and opposite due to Newton's third law.

Conclusion

$$T = mg$$

1. Tension always acts along the string and in such a direction that it tries to reduce the length of string.
2. If the string is massless, then the tension will be the same along the string but if the string has some mass, then the tension will continuously change along the string.

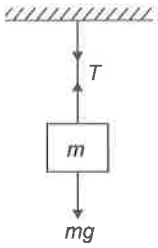
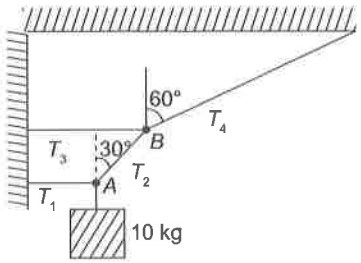


Figure 4.13

SOLVED EXAMPLES

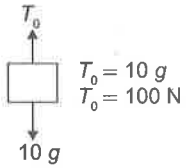
EXAMPLE 5

The system shown in the figure is in equilibrium. Find the magnitude of tension in each string; T_1, T_2, T_3 and T_4 (g = 10 m/s^{-2}).

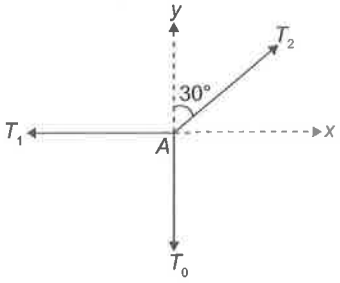


SOLUTION

F.B.D. of block 10 kg



F.B.D. of point 'A'



$$\sum F_y = 0$$

$$T_2 \cos 30° = T_0 = 100 \text{ N}$$

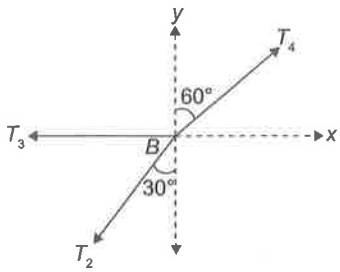
$$T_2 = \frac{200}{\sqrt{3}} \text{ N}$$

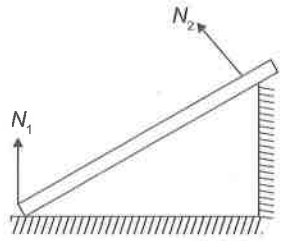
$$\sum F_x = 0$$

$$T_1 = T_2 \sin 30° = \frac{200}{\sqrt{3}} \cdot \frac{1}{2}$$

$$= \frac{100}{\sqrt{3}} \text{ N}$$

F.B.D of point 'B'

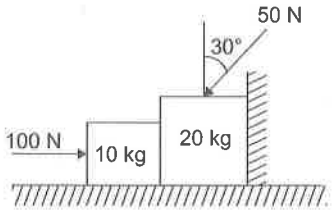




■

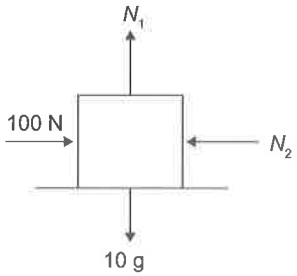
EXAMPLE 3

Two blocks are kept in contact as shown in the figure. Find:
(a) forces exerted by surfaces (floor and wall) on blocks,
(b) contact force between two blocks.



SOLUTION

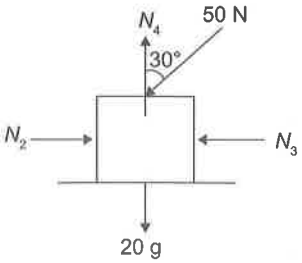
F.B.D. of 10 kg block



$$N_1 = 10\text{ g} = 100\text{ N} \quad (1)$$

$$N_2 = 100\text{ N} \quad (2)$$

F.B.D. of 20 kg block



$$N_2 = 50\sin 30° + N_3$$

$$N_3 = 100 - 25 = 75\text{ N} \quad (3)$$

and

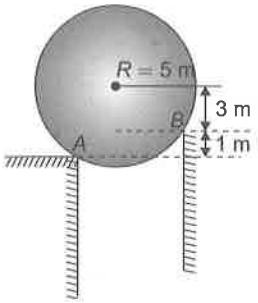
$$N_4 = 50\cos 30° + 20\text{ g}$$

$$N_4 = 243.30\text{ N}$$

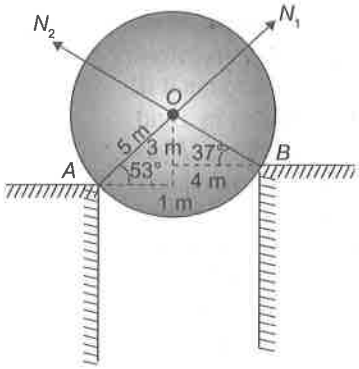
■

EXAMPLE 4

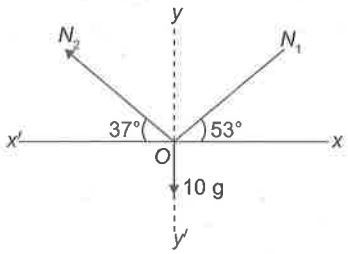
Find out the normal reaction at points A and B if the mass of sphere is 10 kg.



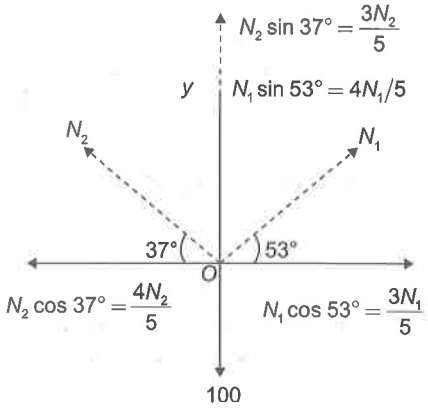
SOLUTION



Now F.B.D.



Now resolve the forces along the x and y directions



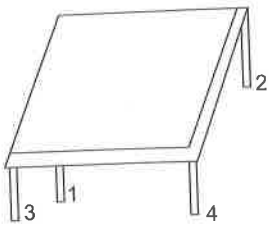


Figure 4.6

- Here, table presses the Earth. So, the normal force exerted by four legs of the table on the Earth is as shown in Fig. 4.7.

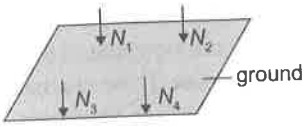


Figure 4.7

- Now, a boy pushes a block kept on a frictionless surface.

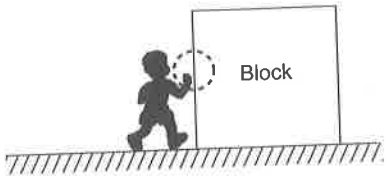


Figure 4.8

Here, force exerted by the boy on the block is electromagnetic interaction which arises due to similar charges appearing on finger and contact surface of the block; it is normal force.

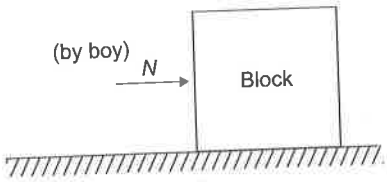


Figure 4.9

- A block is kept on the inclined surface. Component of its weight presses the surface perpendicularly due to which contact force acts between surface and block.

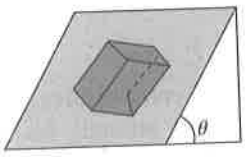


Figure 4.10

Normal force exerted by the block on the surface of inclined plane is shown in Fig. 4.11.

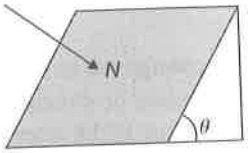


Figure 4.11

Force acts perpendicular to the surface.

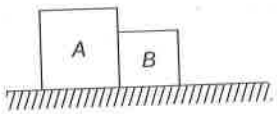
Notes

- Normal force acts in such a fashion that it tries to compress the body.
- Normal is a dependent force; it comes in role when one surface presses the other.

SOLVED EXAMPLES

EXAMPLE 1

Two blocks are kept in contact on a smooth surface as shown in the figure. Draw normal force exerted by A on B.

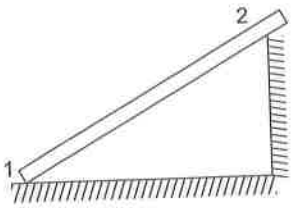


SOLUTION

In the above figure, Block A does not push Block B. So there is no molecular interaction between A and B. Hence, normal force exerted by A on B is zero. ■

EXAMPLE 2

Draw normal forces on the massive rod at points 1 and 2 as shown in the figure.



SOLUTION

Normal force acts perpendicular to extended surface at point of contact.

NEWTON'S LAW OF MOTION

FORCE

A pull or push which changes or tends to change the state of rest or of uniform motion or direction of motion of any object is called force. Force is the interaction between the object and the source (providing the pull or push). It is a vector quantity.

Effect of resultant force

- may change only speed
- may change only direction of motion
- may change both the speed and direction of motion
- may change size and shape of a body

Unit of Force

$$\text{Newton and } \frac{\text{kg}\cdot\text{m}}{\text{s}^2} \text{ (MKS System)}$$

$$\text{dyne and } \frac{\text{g}\cdot\text{cm}}{\text{s}^2} \text{ (CGS System)}$$

$$1 \text{ Newton} = 10^5 \text{ dyne}$$

Kilogram Force (kgf)

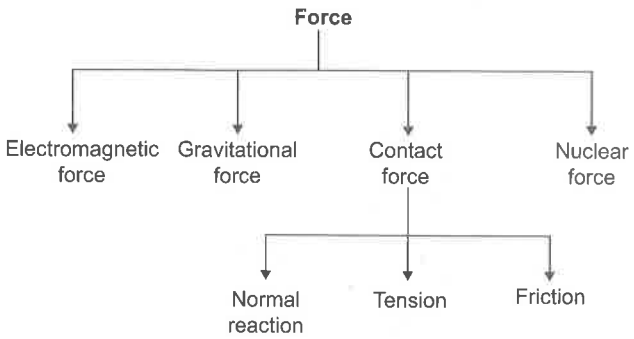
The force with which earth attracts a 1 kg body towards its centre is called kilogram force. Thus

$$\text{kgf} = \frac{\text{Force in Newton}}{\text{g}}$$

Dimensional Formula of Force [MLT^{-2}]

For full information of force, we require

- Magnitude of force
- Direction of force
- Point of application of the force



Electromagnetic Force

Force exerted by one particle on the other because of the electric charge on the particles is called electromagnetic force.

The following are the main characteristics of electromagnetic force:

- These can be attractive or repulsive.
- These are long range forces.
- These depend on the nature of medium between the charged particles.
- All macroscopic forces (except gravitational) which we experience as push or pull or by contact are electromagnetic, i.e., tension in a rope, the force of friction, normal reaction, muscular force and force experienced by a deformed spring are electromagnetic forces. These are manifestations of the electromagnetic attractions or repulsions between atoms/molecules.

Gravitational Force

It acts between any two masses kept anywhere in the universe. It follows inverse square rule $\left(F \propto \frac{1}{\text{distance}^2}\right)$ and is attractive in nature.

$$F = \frac{GM_1M_2}{R^2}$$

The force, mg, which the Earth applies on the bodies, is gravitational force.

Nuclear Force

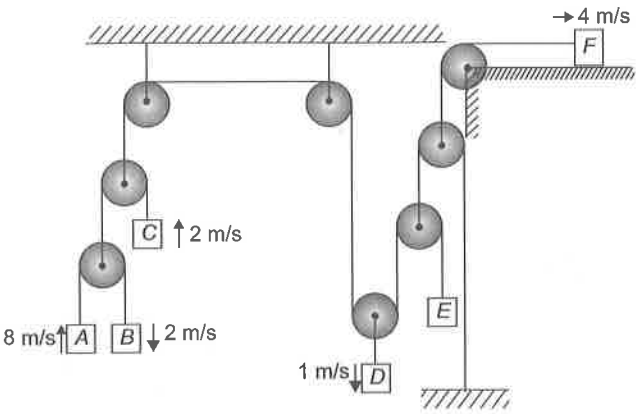
It is the strongest force. It keeps nucleons (neutrons and protons) together inside the nucleus in spite of large electric repulsion between protons. Radioactivity, fission and fusion result because of unbalancing of nuclear forces. It acts within the nucleus that too up to a very small distance.

Contact Force

Forces which are transmitted between bodies by short range atomic molecular interactions are called contact forces. When two objects come in contact, they exert contact forces on each other.

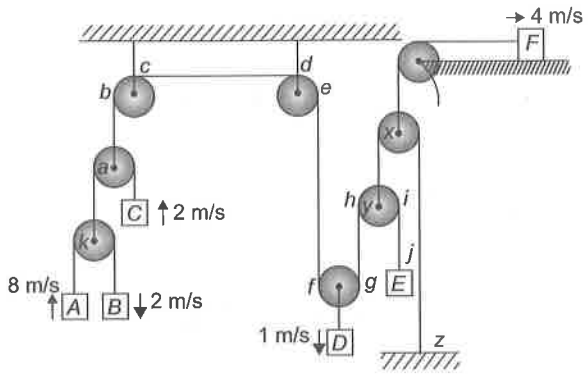
Normal Force (N)

It is the component of contact force perpendicular to the surface. It measures how strongly the surfaces in contact are pressed against each other. It is the electromagnetic force. A table is placed on the Earth as shown in Fig. 4.6.



SOLUTION

Step 1: We first choose the longest string in which point j (Block E) lies (*abcdefghij*).



Step 2: Now write the equation according to the velocity of each point (either increase or decrease the length)

$$v_a + v_b + v_c + v_d + v_e + v_f + v_g + v_h + v_i + v_j = 0 \quad (1)$$

Now find the values of v_a, v_b ... in the following way:

$$v_k = \frac{v_A + v_B}{2}$$

(from the second format)

$$= \frac{8-2}{2} = 3 \text{ m/s}$$

(upward)

$$v_a = \frac{v_K + v_C}{2}$$

(from the second format)

$$= \frac{3+2}{2} = 5/2 \text{ m/s}$$

(upward)

$$v_x = 4 \text{ m/s}$$

(from the first format of constrain)

From the second format of constrain,

$$v_x = \frac{v_y + v_z}{2}$$

$\therefore$ $v_z = 0$

(fixed)

$\Rightarrow$ $v_y = 2v_x = 8$ m/s

(upward)

$\Rightarrow$ Now $v_a = -5/2$ m/s

(decreases the length)

$$v_b = v_c = v_d = v_e = 0$$

(attached to fixed object)

$$v_f = v_g = 1 \text{ m/s}$$

(increases the length)

$$v_h = v_i = v_y = 8 \text{ m/s}$$

(increases the length)

Let us assume that Block E moves upwards, then $v_j = -v_E$ (decreases the length)

Putting the above values in Eq. (1)

$$-5/2 + 1 + 1 + 8 + 8 - v_E = 0$$

$$v_E = 31/2 \text{ m/s}$$

(upward) ■

Note

In Fig. 4.4, pulley is moving with the velocity v at an angle θ with the horizontal.

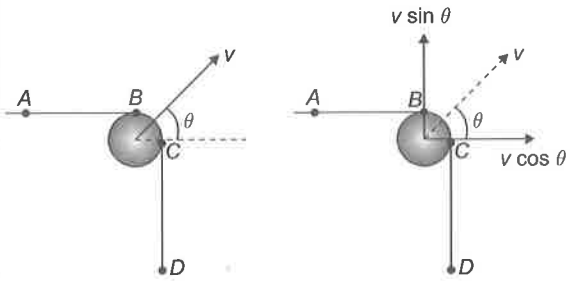


Figure 4.4

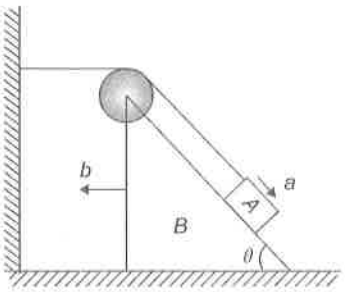
Note (contd)

Only $v \cos \theta$ is responsible to increase or decrease the length AB and $v \sin \theta$ is responsible to either decrease or increase the length CD.

- Further solving strategy is the same as the third format.

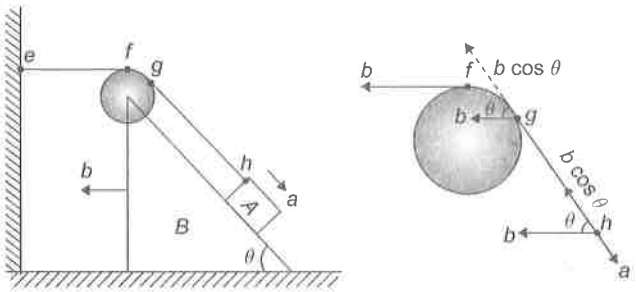
EXAMPLE 11

Find out the relation between acceleration a and b as shown in the following figure.



SOLUTION

Step 1: Mark the points on the string which is attached to the real object (e, f, g, h)



Step 2: Acceleration of each point which are responsible to affect the length of the string

$a_e = 0$ (because it is attached to fixed object)

$a_f = -b$ (attach to the pulley which is moving with wedge's acceleration and $-$ve because it decreases the length)

$a_g = b \cos \theta$ (only this component is responsible to affect the length of the string)

$a_h = (a - b \cos \theta)$ (resultant velocity at point h along the string)

Now from the third format

$$a_e + a_f + a_g + a_h = 0$$

$$\Rightarrow \quad 0 + (-b) + b \cos \theta + (a - b \cos \theta) = 0$$

$$a - b = 0$$

$$\Rightarrow \quad a = b \quad \blacksquare$$

WEDGE CONSTRAINT

Conditions

1. Contact must not be lost between two bodies.
2. Bodies are rigid.

The relative velocity/acceleration perpendicular to the contact surface of the two rigid objects is always zero. Wedge constraint is applicable for each contact.

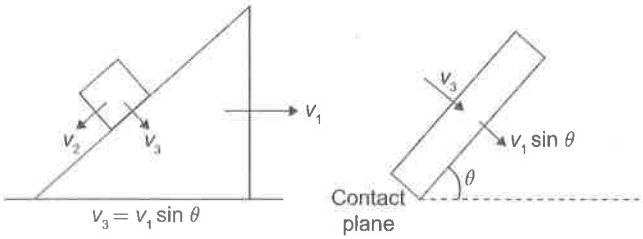


Figure 4.5

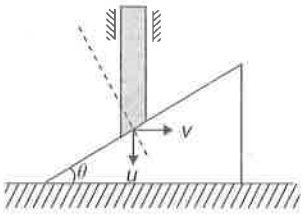
In other words,

Components of velocity and acceleration perpendicular to the contact surface of the two objects are always equal if there is no deformation and they remain in contact.

SOLVED EXAMPLE

EXAMPLE 12

Find the relation between the velocity of rod and that of the wedge at any instant in the figure shown.



SOLUTION

Using wedge constraint,

Component of the velocity of rod along perpendicular to the inclined surface is equal to the velocity of wedge along that direction.

$$u \cos \theta = v \sin \theta$$

$$\frac{u}{v} = \tan \theta$$

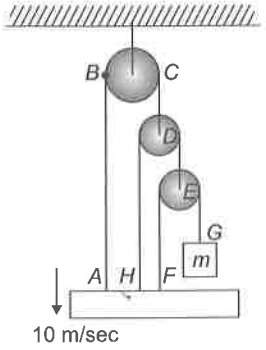
$$u = v \tan \theta \quad \blacksquare$$

$$v_D = 23 \text{ m/s}$$

(upward direction) ■

EXAMPLE 8

Find the velocity of point G.



SOLUTION

In the string $ABCD$ from the first format of constraint

$$V_D = 10 \text{ m/s}\uparrow$$

Now $$v_D = \frac{v_H + v_E}{2}$$

$v_H = 10$ m/s $\downarrow$ if upward direction is taken to be positive

$$+10 = \frac{-10 + v_E}{2}$$

$$\Rightarrow \quad v_E = 30 \text{ m/s}\uparrow$$

Now $$\frac{v_F + v_G}{2} = v_E$$

$$\Rightarrow \quad 30 = \frac{-10 + v_G}{2}$$

$$60 + 10 = v_G$$

$$v_G = 70 \text{ m/s}\uparrow$$ ■

Note

In the second format, three or four points of the string are attached to the moving bodies.

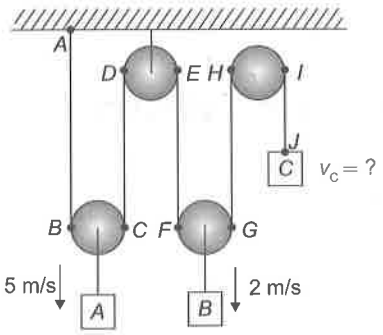
Third Format

Solving Strategy

1. First choose the longest string in the given problem which contains the point of which velocity/acceleration to be found out.
2. Now mark a point on the string wherever it comes in contact or leaves the contact of real bodies.
3. If due to motion of a point, length of the part of a string with point is related, increases then its speed will be taken +ve otherwise −ve.

SOLVED EXAMPLES

EXAMPLE 9



SOLUTION

Step 1: We choose a longest string $ABCDEFGHIJ$ in which we have to find out velocity of point J (v_c)

Step 2: Mark all the point A, B, ...

Step 3: Write the equation

$$v_A + v_B + v_C + v_D + v_E + v_F + v_G + v_H + v_I + v_J = 0$$

$$v_A = v_D = v_E = v_H = v_I = 0$$

(No movement of that point because attached to fixed objects)

$$\Rightarrow \quad v_B + v_c + v_F + v_G + v_J = 0 \qquad (1)$$

$$v_B = v_C = 5 \text{ m/s}$$

(increases the length)

$$v_F = v_G = 2 \text{ m/s}$$

(It also increases the length)

Let us assume C is moving upwards with velocity v_C, so v_C is negative because it decreases the length.

$$\Rightarrow \quad 5 + 5 + 2 + 2 - v_C = 0$$

$$v_C = 14 \text{ m/s}$$

(upward) ■

EXAMPLE 10

Find out the velocity of Block E as shown in the figure.

(−ve sign indicates that the direction of each block is opposite with respect to the pulley.)

$$v_A - v_P = -v_B + v_P$$

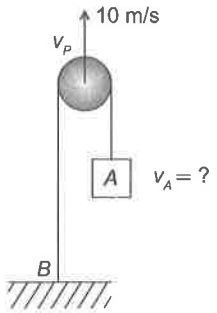
$$\Rightarrow \quad v_P = \frac{v_A + v_B}{2}$$

Note

To solve the problem, put the values of v_A, v_B and v_P with sign.

SOLVED EXAMPLES

EXAMPLE 5



SOLUTION

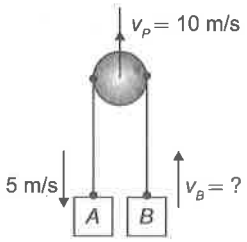
$$v_P = \frac{v_A + v_B}{2}$$

Putting $v_p = 10\ \text{ms}^{-1}$, $v_B = 0$, we get

$$v_A = 20\ \text{ms}^{-1}$$

(upward direction) ■

EXAMPLE 6



SOLUTION

If we take upward direction as positive, then

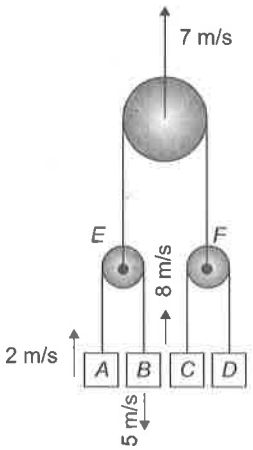
$$10 = \frac{-5 + v_B}{2}$$

$$v_B = 25\ \text{m/s}$$

(in upward direction) ■

EXAMPLE 7

Find out the velocity of Block D.



SOLUTION

From the second format of constrained motion

$$v_E = \frac{v_A + v_B}{2}$$

$$v_E = \frac{2-5}{2} = -3/2$$

(If upward direction is taken to be +ve)

$$v_E = -3/2\ \text{m/s}$$

Now $\quad \frac{v_E + v_F}{2} = 7\ \text{m/s}$

$$\Rightarrow \quad 7 = \frac{-3/2 + v_F}{2}$$

$$\Rightarrow \quad 14 + 3/2 = v_F$$

$$\Rightarrow \quad v_F = \frac{31}{2}$$

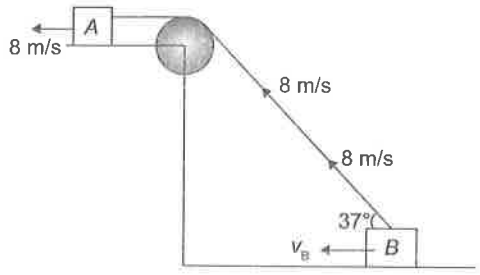
Now $\quad \frac{v_C + v_D}{2} = v_F$

$$\Rightarrow \quad \frac{8 + v_D}{2} = \frac{31}{2}$$

$$\Rightarrow \quad v_D = 31 - 8$$

SOLUTION

Since Block A is moving with velocity 8 ms^{-1}.



$\therefore$ Velocity of each point on the string must be 8 m/s along the string.

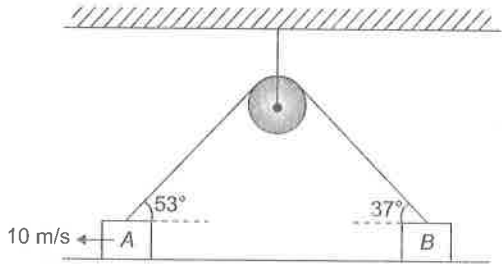
The real velocity of B is v_B. Then the string will not break only when the component of v_B along string is 8 m/s.

$$\Rightarrow \quad v_B \cos 37° = 8$$

$$\Rightarrow \quad v_B = \frac{8}{\cos 37°} = 10 \text{ m/s}$$ ■

EXAMPLE 3

Find out the velocity of Block B in a pulley block system as shown in the figure.

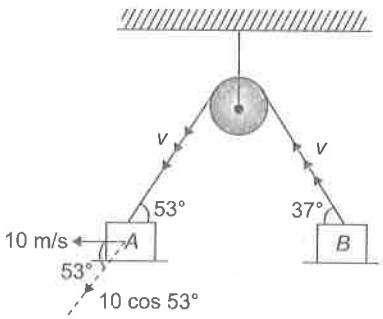


SOLUTION

In a given pulley block system, the velocity of all the particle of string is let us assume v, then

10 m/s is the real velocity of Block A. Then its component along string is v.

$$\Rightarrow \quad 10 \cos 53° = v \qquad (1)$$



If v_B is the real velocity of Block B, then its component along string is v.

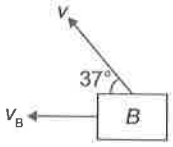
Then, $\quad v_B \cos 37° = v \qquad (2)$

From (1) and (2), we get

$$v_B \cos 37° = 10 \cos 53°$$

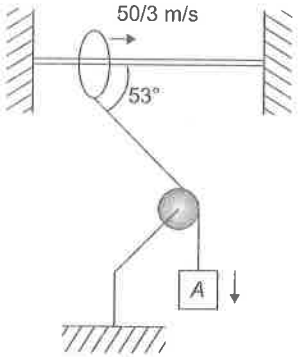
$$\Rightarrow \quad v_B = \frac{10 \times 3/5}{4/5}$$

$$= \frac{30}{4} = \frac{15}{2} \text{ m/s}$$ ■



EXAMPLE 4

What is the velocity of Block A in the figure as shown below?



SOLUTION

The component of velocity of ring along the string = the velocity of A

$$= \frac{50}{3} \cos 53° = v_A$$

$$\Rightarrow \quad v_A = 10 \text{ m/s}$$ ■

Note

In the first format, only two points of string are attached or touched to moving bodies.

Second Format (When Pulley is also Moving)

To understand this format, we consider the following example in which pulley is moving with velocity v_p and both the blocks have velocity v_A and v_B, respectively, as shown in Fig. 4.3.

If we observe the motion of A and B with respect to pulley, then the pulley is at rest. Then from the first format

$$v_{AP} = -v_{BP}$$

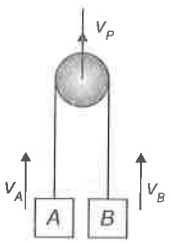


Figure 4.3

CHAPTER 4

Constrained, NLM, Friction

CONSTRAINED MOTION

String Constraint

When the two objects are connected through a string and if the string has the following properties:

- The length of the string remains constant, i.e., it is inextensible string.
- Always remains taut, i.e., it does not slacks.

Then the parameters of the motion of the objects along the length of the string have a definite relation between them.

First Format (When String is Fixed)

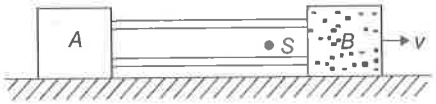


Figure 4.1

Block B moves with the velocity v, i.e., each particle of Block B moves with the velocity v.

If string remains attached to Block B it is necessary that the velocity of each particle of string is the same $= v$ ($v_s = v$)

Now we can say that Block A also moves with velocity v.

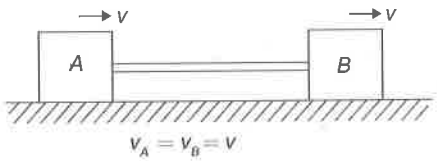


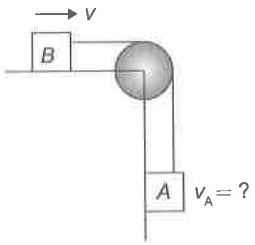
Figure 4.2

Note

If pulley is fixed then the velocity of all the particles of string is same along the string.

SOLVED EXAMPLES

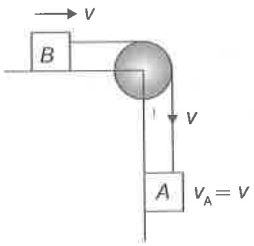
EXAMPLE 1



SOLUTION

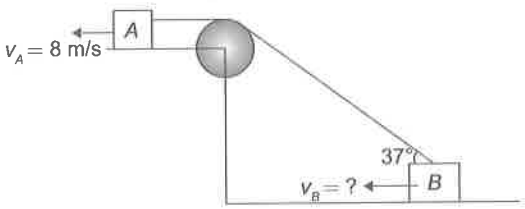
In the above situation, Block B is moving with velocity v. Then the speed of each point of the string is v along the string.

$\therefore$ Speed of Block A is also v.



■

EXAMPLE 2



9.

v_0	a
−	+

Vel

10. $\left(\frac{\pi\sqrt{2}}{3}\right)$ cm/min **11.** 25 m **12.** (a) 2.7 km, (b) 60 m/s, (c) 225 m and 2.25 km **13.** 240 m

14. $\sqrt{\left(\frac{h}{8g}\right)}$ **15.** 50 m **16.** 100 m, zero **17.** 5 s **18.** 36.2 s. **19.** $20\sqrt{5}$

20. 20 s **21.** 60, 2 m/s. **22.** $u = 50\left(\sqrt{3}-1\right)$ m/s, $H = 125\left(-\sqrt{3}+2\right)$ m

23. (i) 1503.2 m (ii) 35.54 s (iii) 3970.56 m **24.** 100/3 m/s **25.** 10 m/s

26. 75 m **27.** $R + 2H$ **28.** 6 m/s **29.** $20 \times \sqrt{2/3}$

30. 10 m **31.** $50\sqrt{3}$ cm **32.** $\tan^{-1}(1/2)$ **33.** $\tan^{-1}(3)$

34. $\theta = \tan^{-1}\left(\frac{1}{2}\right), \frac{4}{3}$ **35.** 200 m, 20 m/min, 12 m/min

Level II

1. 122.7 km/h **2.** $\frac{\pi v^4}{g^2}$ **3.** 60°, **4.** $u = 3$m/s **5.** (a) 10 ms^{-1}, (b) 2 s, (c) 5 m, (d) 16.25 m, (e) 20 m

6. $\frac{1}{\sqrt{3}}$ s **7.** 5 m/s **8.** 1 s **9.** $2\tan^{-1}(1/3)$ **10.** $\theta = 37°$, $v = 6$ m/s

Previous Year Questions

JEE Main

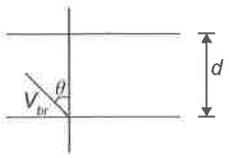
1. A	**2.** B	**3.** C	**4.** D	**5.** C	**6.** B	**7.** D	**8.** C	**9.** A	**10.** C
11. D	**12.** C	**13.** C	**14.** D	**15.** A	**16.** C	**17.** D	**18.** A	**19.** C	**20.** B
21. B	**22.** C	**23.** B	**24.** D	**25.** A	**26.** A	**27.** D	**28.** C	**29.** A	

JEE Advanced

1. B	**2.** A	**3.** $u = 7.29$ m/s, $t = 1$ s		**4.** (a) 45°,	(b) 2 m/s
5. B	**6.** C	**7.** B	**8.** B	**9.** 5 m/s^2	**10.** 0005

For rear range, $V_{cannon} = VC\cos\theta - VT$
Range will be less.

19.



$t = \dfrac{d}{V_{br}\cos\theta}$ for t_{min} $\cos\theta = 1$ maximum. Hence, A will reach the opposite end in the least time.

Exercises

JEE Main

1. B	**2.** B	**3.** B	**4.** B	**5.** B	**6.** B	**7.** A	**8.** D	**9.** C	**10.** C
11. C	**12.** D	**13.** A	**14.** A	**15.** B	**16.** C	**17.** D	**18.** B	**19.** A	**20.** B
21. C	**22.** C	**23.** D	**24.** B	**25.** A	**26.** C	**27.** D	**28.** D	**29.** B	**30.** D
31. D	**32.** C	**33.** B	**34.** C	**35.** B	**36.** B	**37.** D	**38.** B	**39.** C	**40.** B
41. C	**42.** A	**43.** D	**44.** D	**45.** A	**46.** C				

JEE Advanced

1. A	**2.** D	**3.** B	**4.** B	**5.** D	**6.** C	**7.** B	**8.** B	**9.** A	**10.** D
11. C	**12.** B	**13.** D	**14.** C	**15.** B	**16.** B	**17.** D	**18.** C	**19.** B	**20.** C
21. A	**22.** A	**23.** A	**24.** C	**25.** A	**26.** D	**27.** D	**28.** C	**29.** B	**30.** B
31. D	**32.** A	**33.** D	**34.** A	**35.** B	**36.** C	**37.** C	**38.** B	**39.** A	**40.** C
41. C	**42.** A	**43.** C	**44.** C	**45.** D	**46.** B	**47.** C	**48.** D	**49.** B	**50.** C
51. A	**52.** A	**53.** A	**54.** B	**55.** B	**56.** C	**57.** B	**58.** B	**59.** A, B, C, D	
60. A, C	**61.** B, C, D		**62.** B, D	**63.** C, D	**64.** A, C	**65.** A, C	**66.** A, D	**67.** C	
68. A, B, C, D		**69.** B, C, D		**70.** A, C, D		**71.** A, B, C		**72.** C, D	
73. D, C	**74.** A, B, C, D		**75.** A, B	**76.** A, B, C, D		**77.** A, C, D		**78.** B, C	

JEE Advanced

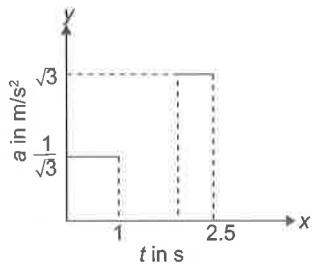
level I

1. (a) $y^2 + 8y + 12 = x$; (b) crosses x-axis when $t = 4$ s, crosses y-axis when $t = \pm 2$ s.

2. (a) $5i + 3j + 8k$, (b) $2i + 16k$, (c) $7\sqrt{2}$, (d) $2\sqrt{65}$ **3.** $-18t\hat{j} - 3\cos t\hat{k}$; 3π; $3\sqrt{4+\pi^4}$

4. (a) $\dfrac{1}{\sqrt{3}}$ m/s, (b) $\dfrac{\sqrt{3}}{2}$ m/s^2, **5.** $\vec{v} = -30\hat{i} - 40\hat{j}$, $\vec{a} = -16\hat{i} - 8\hat{j}$

(c)



6. $a/2b$, $a^2/4b$ **7.** $\dfrac{2}{3}$ m **8.** 42 km/h

9. I curve: at particular time x has more than one value. Hence, it is not a 1-D motion.

   II curve: $|\vec{V}|$ cannot be negative.

   III curve: Length of a moving body cannot decrease with time.

10. I curve: A ball moving forward collides with the surface and rebounds and stops after the II collision.

    II curve: A ball repeatedly making inelastic collisions with the floor.

    III curve: Collision of a ball with a surface. {Surface has a large velocity for a short time.}

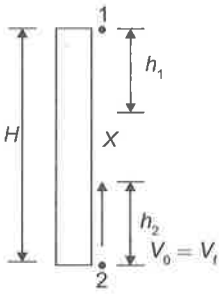
11. (a) is incorrect. Car cannot travel around the track with constant velocity as the direction is continuously changing.

    (b) is correct.

12. Ball is at maximum height $V = 0$ for just an instant but acceleration due to gravity.

13. $$V_f = \sqrt{2gH}$$

    Let balls meet after t seconds.



$$h_1 = \frac{1}{2}gt^2 \text{ and } h_2 = V_f t = \frac{1}{2}gt^2$$

$$H = \sqrt{2gHt} \qquad t = \sqrt{\frac{H}{2g}}$$

$$h_1 + h_2 = H = Vft$$

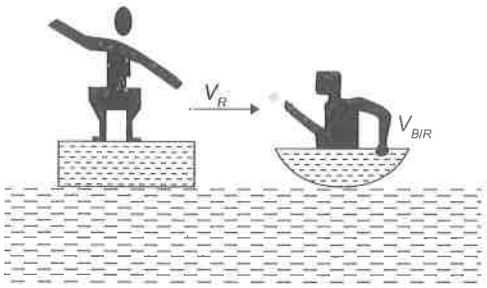
$$\therefore \qquad h_1 = \frac{1}{2}g\frac{H}{2g} = \frac{H}{4}.$$

    Hence, they will meet above half the height of building.

14. Initially, the bullet is at rest $u = 0$, $V^2 = 0 + 2as$.

    $\therefore \quad a = \dfrac{V^2}{2S}$ muzzle velocity is more for short barrel and S is also less. Hence, acceleration will be more in that case.

15.



    Hence, we cannot conclude that the velocity of the boat is 5 m/sec with respect to the shore:

$$V_{\text{Bottle}} = V_{\text{river}}; \quad V_B - V_R = 5.$$

16. Yes, the wrench will hit at the same place on the deck regardless of whether the boat is at rest or moving because when the boat is at rest the wrench will have zero horizontally velocity and when the boat is moving both will have the same horizontal velocity.

17. Acceleration of the projectile remains constant throughout the journey $= g$.

18. (a) From the child's point of view, the range will be the same in both the cases.

    (b) In ground frame of reference,

$$VCT = VC - VT$$

$$VC = VCT + VT$$

    For front range, $V_{\text{cannon}} = VC\cos\theta + VT$

    Range will be more.

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1.
(B) STATEMENT-1 is True, STATEMENT-2 is True. STATEMENT-2 is NOT a correct explanation for STATEMENT-1
(C) STATEMENT-1 is True, STATEMENT-2 is False.
(D) STATEMENT-1 is False, STATEMENT-2 is True.

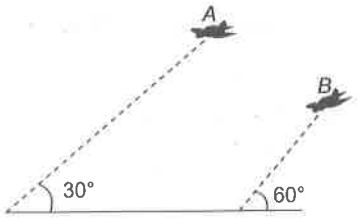
[JEE-2008]

9. A train is moving along a straight line with a constant acceleration a. A boy standing in the train throws a ball forward with a speed of 10 m/s, at an angle of 60° to the horizontal. The boy has to move forward by 1.15 m inside the train to catch the ball back at the initial height. The acceleration of the train in m/s^2 is

[JEE-2011]

10. Airplanes A and B are flying at a constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal, respectively as shown in the figure. The speed of A is $100\sqrt{3}$ ms^{-1}. At time $t = 0$ s, an observer in A finds B at a distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at $t = t_0$, A just escapes being hit by B, t_0 in seconds is

[JEE Advanced-2014]



ANSWER KEYS

Questions for Short Answer

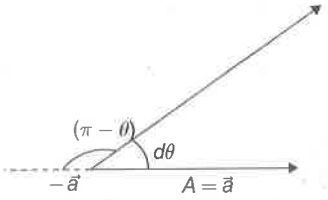
1. Magnitude $|\vec{a}|$ will remain unchanged.

$$B = \vec{a}$$

$$\Delta a = \vec{B} - \vec{A}$$

$$= \sqrt{a^2 + a^2 + 2a^2 \cos(\pi - d\theta)}$$

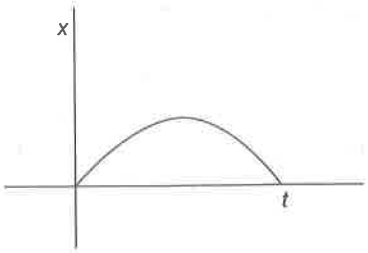
$$= \sqrt{2a^2(1 - \cos\theta)}$$



$$\Rightarrow \quad \sqrt{2a^2(1 - 1 + 2\sin 2d\theta/2)}$$

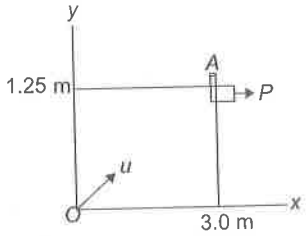
$$= 2a \sin d\theta/2$$

2. Speedometer measures the speed of car as it only gives the magnitude.
3. When a particle is moving with constant velocity, its average velocity and instantaneous velocity will be the same and the magnitude of instantaneous velocity will also be the same.
4. $\vec{V}_{Avg} = \dfrac{\Delta\vec{S}}{\Delta t}$, $\Delta\vec{S} = V_{Avg} \times \Delta t$
5. $\vec{V}_{Avg} = \dfrac{\Delta\vec{S}}{\Delta t}$ $\Delta\vec{S} = 0$. With zero displacement, non-zero $\vec{V}_{Avg}$ is not possible. Zero displacement and non-zero $\vec{V}$ is possible if the particle is reversing and coming to the starting point. Show on $x-t$ graph by an example.

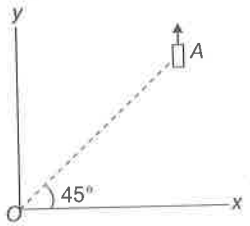


6. Speed of a projectile is smallest at the highest point.
7. Both the balls will hit the ground with same speed.
8. If the sack of rice is dropped when it is just above the centre, it will fall ahead of the circle because the sack will have the same velocity as the plane in the horizontal direction.

with an acceleration 1.5 m/s². At the same instant, a stone is projected from the origin with a velocity u as shown. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in the x-y plane. Find u and the time after which the stone hits the object. Take $g = 10$ m/s². **[JEE-2000]**



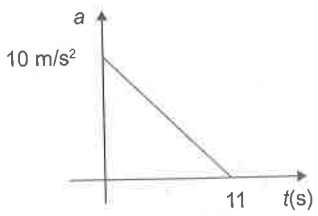
4. On a frictionless horizontal surface, assumed to be the x–y plane, a small trolley A is moving along a straight line parallel to the y-axis (see figure) with a constant velocity of $\left(\sqrt{3}-1\right)$ m/s. At a particular instant, when the line OA makes an angle of 45° with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley.



(a) The motion of the ball is observed from the frame of the trolley. Calculate the angle θ made by the velocity vector of the ball with the x-axis in this frame.

(b) Find the speed of the ball with respect to the surface, if $\phi = \frac{4\theta}{3}$. **[JEE-2002]**

5. A particle starts from rest. Its acceleration (a) versus time (t) is as shown in the figure. The maximum speed of the particle will be **[JEE-2004 (Scr)]**



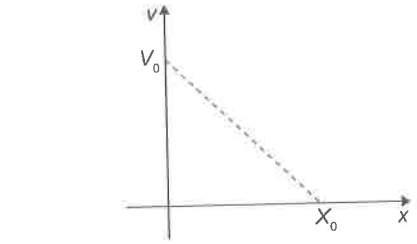
(A) 110 m/s (B) 55 m/s
(C) 550 m/s (D) 660 m/s

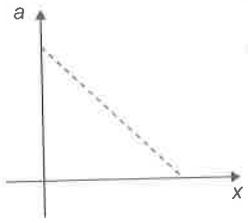
6. A small block slides without friction down an inclined plane starting from rest. Let S_n be the distance travelled from time $t = n - 1$ to $t = n$. Then $\frac{S_n}{S_{n+1}}$ is **[JEE-2004 (Scr)]**

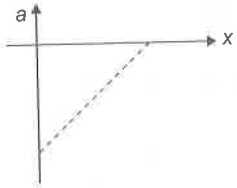
(A) $\frac{2n-1}{2n}$ (B) $\frac{2n+1}{2n-1}$

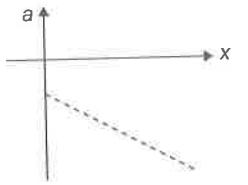
(C) $\frac{2n-1}{2n+1}$ (D) $\frac{2n}{2n+1}$

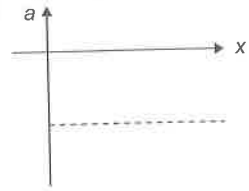
7. The velocity–displacement graph of a particle moving along a straight line is shown. The most suitable acceleration-displacement graph will be **[JEE-2005 (Scr)]**



(A)


(B)


(C)


(D)


8. **STATEMENT-1**
For an observer looking out through the window of a fast moving train, the nearby objects appear to move in the opposite direction to the train, while the distant objects appear to be stationary.
STATEMENT-2
If the observer and the object are moving at velocities $\vec{V}_1$ and $\vec{V}_2$, respectively, with reference to a laboratory frame, the velocity of the object with respect to the observer is $\vec{V}_2 - \vec{V}_1$.

25. A water fountain on the ground sprinkles water all around it. If the speed of water coming out of the fountain is v, the total area around the fountain that gets wet is **[AIEEE-2011]**

(A) $\pi\frac{v^4}{g^2}$ (B) $\frac{\pi}{2}\frac{v^4}{g^2}$

(C) $\pi\frac{v^2}{g^2}$ (D) $\pi\frac{v^2}{g}$

26. An object, moving with a speed of 6.25 m/s, is decelerated at a rate given by $\frac{dv}{dt} = -2.5\sqrt{v}$, where v is the instantaneous speed. The time taken by the object to come to rest would be **[AIEEE-2011]**

(A) 2 s (B) 4 s

(C) 8 s (D) 1 s

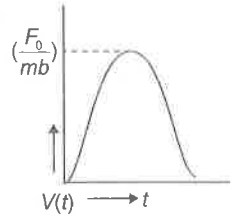
27. A boy can throw a stone up to a maximum height of 10 m. The maximum horizontal distance that the boy can throw the same stone up to will be **[AIEEE-2012]**

(A) $20\sqrt{2}$ m (B) 10 m

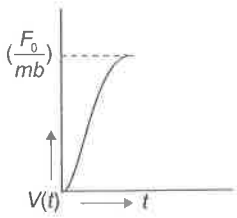
(C) $10\sqrt{2}$ m (D) 20 m

28. A particle of mass m is at rest at the origin at time $t = 0$. It is subjected to a force $F(t) = F_0e^{-bt}$ in the x direction. Its speed $v(t)$ is depicted by which of the following curves? **[AIEEE-2012]**

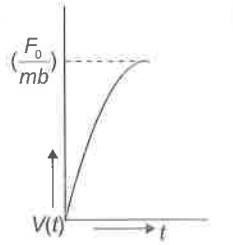
(A)



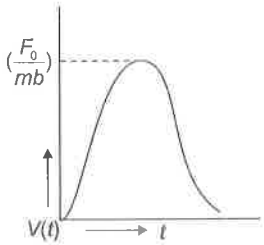
(B)



(C)



(D)



29. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground is n times that taken by it to reach the highest point of its path.
The relation between H, u and n is

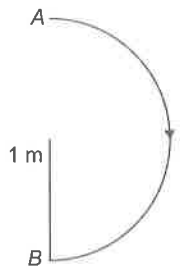
(A) $2gH = nu^2(n-2)$ (B) $gH = (n-2)u^2$

(C) $2gH = n^2u^2$ (D) $gH = (n-2)^2u^2$

[JEE Main-2014]

JEE Advanced

1. In 1.0 s a particle goes from point A to point B moving in a semicircle of radius 1.0 m. The magnitude of average velocity is **[JEE-1999]**

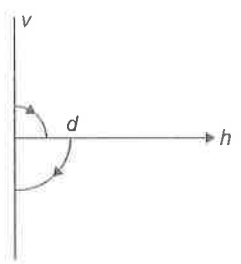


(A) 3.14 m/s (B) 2.0 m/s

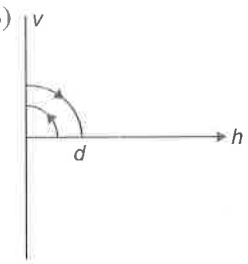
(C) 1.0 m/s (D) zero

2. A ball is dropped vertically from a height d above the ground. It hits the ground and bounces up vertically to a height $d/2$. Neglecting subsequent motion and air resistances, its velocity v varies with the height h above the ground as **[JEE-2000 (Scr)]**

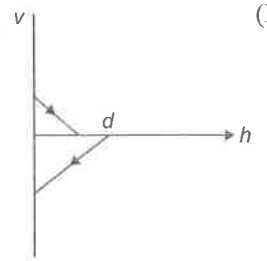
(A)



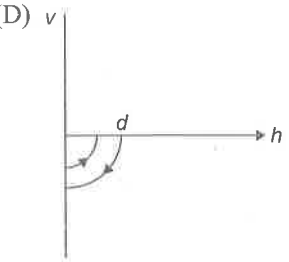
(B)



(C)



(D)



3. An object A is kept fixed at the point $x = 3$ m and $y = 1.25$ m on a plank P raised above the ground. At time $t = 0$, the plank starts moving along the $+x$ direction

varies as $v = \alpha\sqrt{x}$. The displacement of the particle varies with time as **[AIEEE-2006]**

(A) t^2 (B) t

(C) $t^{1/2}$ (D) t^3

19. A particle is projected at 60° to the horizontal with a kinetic energy K. The kinetic energy at the highest point is **[AIEEE-2007]**

(A) K (B) zero

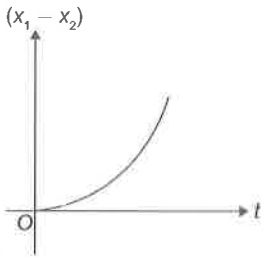
(C) $\frac{K}{4}$ (D) $\frac{K}{2}$

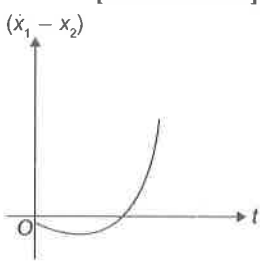
20. The velocity of a particle is $v = v_0 + gt + ft^2$. If its position is $x = 0$ at $t = 0$, then its displacement after unit time ($t = 1$) is **[AIEEE-2007]**

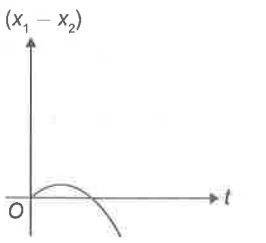
(A) $v_0 + 2g + 3f$ (B) $v_0 + \frac{g}{2} + \frac{f}{3}$

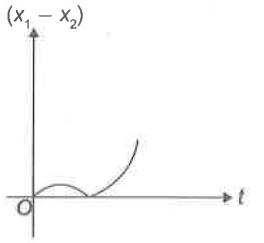
(C) $v_0 + g + f$ (D) $v_0 + \frac{g}{2} + f$

21. A body is at rest at $x = 0$. At $t = 0$, it starts moving in the positive x-direction with a constant acceleration. At the same instant, another body passes through $x = 0$ moving in the positive x-direction with a constant speed. The position of the first body is given by $x_1(t)$ after time t and that of the second body by $x_2(t)$ after the same time interval. Which of the following graphs correctly describes $(x_1 - x_2)$ as a function of time? **[AIEEE-2008]**

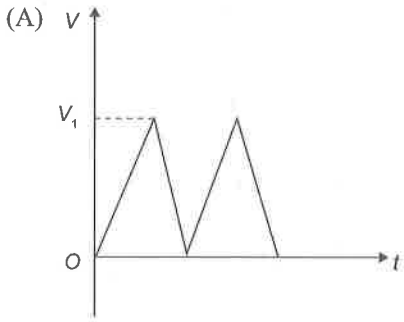
(A) 

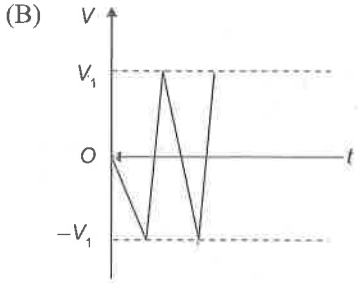
(B) 

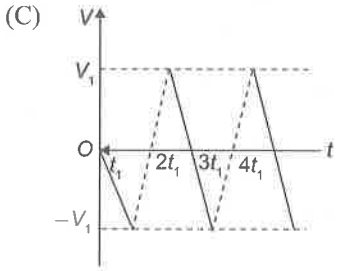
(C) 

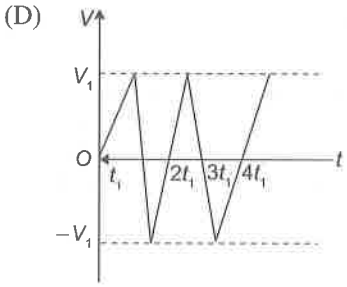
(D) 

22. Consider a rubber ball freely falling from a height $h = 4.9$ m onto a horizontal elastic plate. Assume that the duration of collision is negligible and the collision with the plate is totally elastic. Then the velocity as a function of time the height as function of time will be **[AIEEE-2009]**

(A) 

(B) 

(C) 

(D) 

23. A particle has an initial velocity $3\hat{i} + 4\hat{j}$ and an acceleration of $0.4i + 0.3j$. Its after 10 s is **[AIEEE-2009]**

(A) 10 unit (B) $7\sqrt{2}$ unit

(C) 7 unit (D) 8.5 unit

24. A particle is moving with velocity $\vec{v} = k\left(y\hat{i} + x\hat{j}\right)$ where k is a constant. The general equation for its path is **[AIEEE-2010]**

(A) $y = x^2$+ constant (B) $y^2 = x$ + constant

(C) xy = constant (D) $y^2 = x^2$+ constant

ground? [$g = 10$ m/s^2, $\sin 30° = 1/2$, $\cos 30° = \sqrt{3}/2$] **[AIEEE-2003]**

(A) 5.20 m (B) 4.33 m
(C) 2.60 m (D) 8.66 m

8. A car moving with a speed of 50 km/h can be stopped by brakes after at least 6 m. If the same car is moving at a speed of 100 km/h, the minimum stopping distance is **[AIEEE-2003]**

(A) 12 m (B) 18 m
(C) 24 m (D) 6 m

9. A ball is thrown from a point with a speed v_0 at an angle of projection θ. From the same point and at the same instant, a person starts running with a constant speed $\frac{v_0}{2}$ to catch the ball. Will the person be able to catch the ball? If yes, what should be the angle of projection? **[AIEEE-2004]**

(A) Yes, 60° (B) Yes, 30°
(C) No (D) Yes, 45°

10. A particle is acted upon by a force of constant magnitude which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane, it follows that **[AIEEE-2004]**

(A) its velocity is constant
(B) its acceleration is constant
(C) its kinetic energy is constant
(D) it moves in straight line

11. An automobile travelling with a speed of 60 km/h can brake to stop within a distance of 20 m. If the car is going twice as fast, i.e. 120 km/h, the stopping distance will **[AIEEE-2004]**

(A) 20 m
(B) 40 m
(C) 60 m
(D) 80 m

12. A ball is released from the top of a tower of height h metre. It takes T seconds to reach the ground. What is the position of the ball at $\frac{T}{3}$ s ? **[AIEEE-2004]**

(A) h/9 m from the ground
(B) 7 h/9 m from the ground
(C) 8 h/9 m from the ground
(D) 17 h/18 m from the ground

13. A parachutist after bailing out falls 50 m without friction. When the parachute opens, it decelerates at 2 ms^{-2}. He reaches the ground with a speed of 3 ms^{-1}. At what height, did he bail out? **[AIEEE-2005]**

(A) 91 m
(B) 182 m
(C) 293 m
(D) 111 m

14. A particle is moving eastwards with a velocity of 5 ms^{-1}. In 10 s the velocity changes to 5 ms^{-1} northwards. The average acceleration in this time is **[AIEEE-2005]**

(A) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-east
(B) $\frac{1}{2}$ ms^{-2} towards north
(C) zero
(D) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-west

15. A car, starting from rest, accelerates at the rate f through a distance s, then continues at constant speed for time t and then decelerates at the rate $f/2$ to come to rest. If the total distance travelled is 15 s, then **[AIEEE-2005]**

(A) $s = \frac{1}{72} ft^2$
(B) $s = \frac{1}{36} ft^2$
(C) $s = \frac{1}{2} ft^2$
(D) $s = \frac{1}{4} ft^2$

16. The relation between time t and distance x is $t = ax^2 + bx$, where a and b are constants. The acceleration is **[AIEEE-2005]**

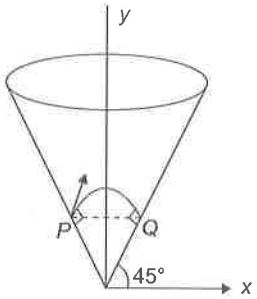
(A) $-2abv^2$ (B) $2bv^3$
(C) $-2av^3$ (D) $2av^2$

17. A projectile can have the same range R for two angles of projection. If t_1 and t_2 be the times of flights in the two cases, then the product of the two times of flights is proportional to **[AIEEE-2005]**

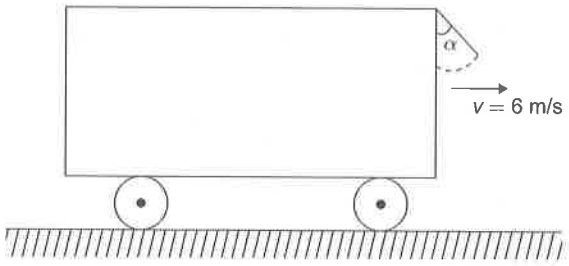
(A) R^2 (B) $\frac{1}{R^2}$
(C) $\frac{1}{R}$ (D) R

18. A particle located at $x = 0$ at time $t = 0$ starts moving along the positive x-direction with a velocity v that

8. A particle is projected from point P with velocity $5\sqrt{2}$ m/s perpendicular to the surface of a hollow right angle cone whose axis is vertical. It collides at Q normally. Find the time of the flight of the particle.



9. A glass wind screen whose inclination with the vertical can be changed is mounted on a cart as shown in the figure. The cart moves uniformly along the horizontal path with a speed of 6 m/s. At what maximum angle α to the vertical can the wind screen be placed so that the rain drops falling vertically downwards with velocity 2 m/s do not enter the cart?



10. A hunter is riding an elephant of height 4 m moving in straight line with uniform speed of 2 m/s. A deer running with a speed V in front it is at a distance of $4\sqrt{5}$ m moving perpendicular to the direction of motion of the elephant. If the hunter can throw his spear with a speed of 10 m/s relative to the elephant, then at what angle θ to its direction of motion must he throw his spear horizontally for a successful hit. Find also the speed V of the deer.

Previous Year Questions

JEE Main

1. If a body losses half of its velocity on penetrating 3 cm in a wooden block, then how much will it penetrate more before coming to rest? **[AIEEE-2002]**
   (A) 1 cm (B) 2 cm
   (C) 3 cm (D) 4 cm

2. From a building, two balls A and B are thrown such that A is thrown upwards and B downwards (both vertically with the same velocity). If v_A and v_B are their respective velocities on reaching the ground, then: **[AIEEE-2002]**
   (A) $v_B > v_A$ (B) $v_A = v_B$
   (C) $v_A > v_B$ (D) their velocities depend on their masses

3. A ball whose kinetic energy is E is projected at an angle of 45° to the horizontal. The kinetic energy of the ball at the highest point of its flight will be **[AIEEE-2002]**
   (A) E (B) $\frac{E}{\sqrt{2}}$
   (C) $\frac{E}{2}$ (D) zero

4. Speeds of two identical cars u and $4u$ at a specific instant. The ratio of the respective distances at which the two cars are stopped from that instant is **[AIEEE-2002]**
   (A) 1:1 (B) 1:4
   (C) 1:8 (D) 1:16

5. A lift is moving down with acceleration a. A man in the lift drops a ball inside the lift. The acceleration of the ball as observed by the man in the lift and a man standing stationary on the ground are, respectively **[AIEEE-2002]**
   (A) g, g (B) $g-a, g-a$
   (C) $g-a, g$ (D) a, g

6. The coordinates of a moving particle at any time t are given by $x = \alpha t^3$ and $y = \beta t^3$.The speed of the particle at time t is given by **[AIEEE-2003]**
   (A) $3t\sqrt{\alpha^2+\beta^2}$ (B) $3t^2\sqrt{\alpha^2+\beta^2}$
   (C) $t^2\sqrt{\alpha^2+\beta^2}$ (D) $\sqrt{\alpha^2+\beta^2}$

7. A boy playing on the roof of a 10-m-high building throws a ball with a speed of 10 m/s at an angle 30° with the horizontal. How far from the throwing point will the ball be at the height of 10 m from the

with the vertical at which the person should hold his umbrella so that he may not get drenched.

33. A glass wind screen whose inclination with the vertical can be changed is mounted on a car. The car moves horizontally with a speed of 2 m/s. At what angle α with the vertical should the wind screen be placed so that the rain drops falling vertically downwards with velocity 6 m/s strike the wind screen perpendicularly?

34. A man with some passengers in his boat, starts rowing the boat perpendicular to the flow of river 200 m wide and flowing at a speed of 2 m/s. The boat speed in still water is 4 m/s. When he reaches half the width of the river, the passengers told him that they wanted to reach the end from where they have started.

(i) Find the direction to which he must row to reach the required end.

(ii) How many times more total time, it would take to that if he would have denied the passengers?

35. A man crosses a river in a boat. If he crosses the river in minimum time, he takes 10 min with a drift of 120 m. If he crosses the river taking the shortest path, he takes 12.5 min, find

(i) the width of the river

(ii) the velocity of the boat with respect to water

(iii) the speed of the current.

Assume $v_{b/r} > v_r$.

Level II

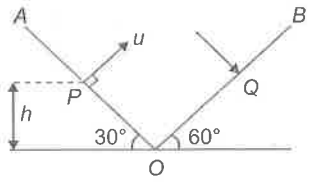
1. A speeder in an automobile passes a stationary policeman who is hiding behind a bill board with a motorcycle. After a 2-s delay (reaction time), the policeman accelerates to his maximum speed of 150 km/h in 12 s and catches the speeder 1.5 km beyond the billboard. Find the speed of speeder in km/h.

2. A large number of bullets are fired in all directions with the same speed v. What is the maximum area on ground on which these bullets can spread?

3. The speed of a particle when it is at its greatest height is $\sqrt{2/5}$ times of its speed when it is at its half the maximum height. The angle of projection is _____ and the velocity vector angle at half the maximum height is _____.

4. A projectile is to be thrown horizontally from the top of a wall of height 1.7 m. Calculate the initial velocity of projection if it hits perpendicularly an incline of angle 37°, which starts from the ground at the bottom of the wall. The line of greatest slope of incline lies in the plane of motion of the projectile.

5. Two inclined planes OA and OB having inclination (with horizontal) 30° and 60°, respectively, intersect each other at O as shown in the figure. A particle is projected from point P with velocity $u = 10\sqrt{3}\ \text{ms}^{-1}$ along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicularly at Q, calculate



(a) velocity with which the particle strikes the plane OB,

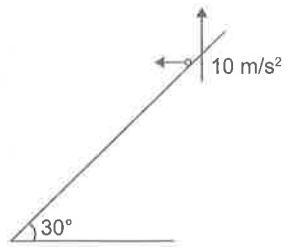
(b) time of flight,

(c) vertical height h of P from O,

(d) maximum height from O attained by the particle and

(e) distance PQ.

6. A particle is thrown horizontally with a relative velocity 10 m/s from an inclined plane, which is also moving with acceleration 10 m/s² vertically upward. Find the time after which it lands on the plane ($g = 10$ m/s²).

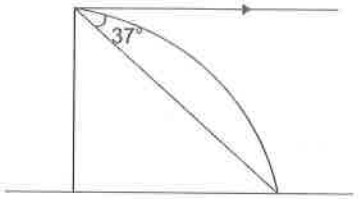


7. A, B and C are three objects each moving with constant velocity. A's speed is 10 m/s in a direction $\overrightarrow{PQ}$. The velocity of B relative to A is 6 m/s at an angle of $\cos^{-1}(15/24)$ to PQ. The velocity of C relative to B is 12 m/s in a direction $\overrightarrow{QP}$. Then find the magnitude of the velocity of C.

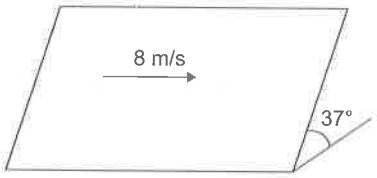
ground in 5 s at an angle of 45° with the horizontal. Find the height of the tower and the speed with which it was projected (use $g = 10$ m/s^2).

23. A rocket is launched at an angle 53° to the horizontal with an initial speed of 100 ms^{-1}. It moves along its initial line of motion with an acceleration of 30 ms^{-2} for 3 s. At this time, its engine falls and the rocket proceeds like a free body. Find
(i) the maximum altitude reached by the rocket
(ii) the total time of flight
(iii) the horizontal range [sin 53° = 4/5].

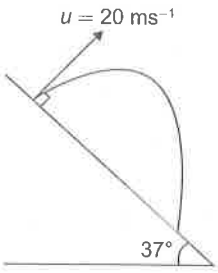
24. A ball is thrown horizontally from a cliff such that it strikes the ground after 5 s. The line of sight from the point of projection to the point of hitting makes an angle of 37° with the horizontal. What is the initial velocity of projection?



25. A ball is projected on a smooth inclined plane in a direction perpendicular to the line of greatest slope with velocity of 8 m/s. Find its speed after 1 s.

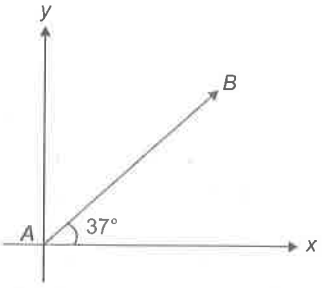


26. Find the range of a projectile on the inclined plane, which is projected perpendicular to the inclined plane with velocity 20 m/s as shown in the figure.

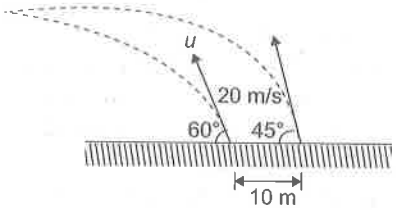


27. The horizontal range of a projectiles is R and the maximum height attained by it is H. A strong wind now begins to blow in the direction of motion of the projectile, giving it a constant horizontal acceleration $= g/2$. Under the same conditions of projection, find the horizontal range of the projectile.

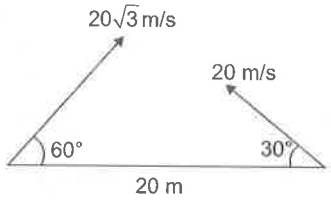
28. A butterfly is flying with velocity $10\hat{i} + 12\hat{j}$ m/s and wind is blowing along the x-axis with velocity u. If the butterfly starts motion from A and after some time reaches point B, find the value of u.



29. In the below figure, two projectiles are fired simultaneously. What should be the initial speed of the left-side projectile for the two projectiles to hit in mid-air?



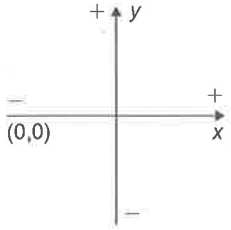
30. In the figure shown, the two projectiles are fired simultaneously. Find the minimum distance between them during their flight?



31. Two particles are moving along two long straight lines, in the same plane, with the same speed = 20 cm/s. The angle between the two lines is 60°, and their intersection point is O. At a certain moment, the two particles are located at a distance 3 m and 4 m from O and are moving towards O. Find the shortest distance between them subsequently.

32. Rain is falling vertically with a speed of 20 ms^{-1} relative to air. A person is running in the rain with a velocity of 5 ms^{-1} and wind is also blowing with a speed of 15 ms^{-1} (both towards east). Find the angle

9. A particle is moving along the x-axis. Initially, it is located 5 m left of the origin and it is moving away from the origin and slowing down. In this coordinate system, the signs of the initial velocity and acceleration are



V_0	a

10. Find the change in velocity of the tip of the minute hand (radius = 10 cm) of a clock in 45 min.

11. At a distance $L = 400$ m from the traffic light, brakes are applied to a locomotive moving at a velocity $v = 54$ km/h. Determine the position of the locomotive relative to the traffic light 1 min after the application of the breaks if its acceleration is -0.3 m/s^2.

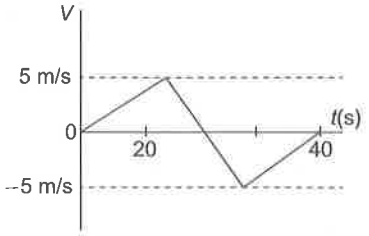
12. A train starts from rest and moves with a constant acceleration of 2.0 m/s^2 for half a minute. The brakes are then applied and the train comes to rest in 1 minute. Find
(a) the total distance moved by the train
(b) the maximum speed attained by the train and
(c) the position(s) of the train at half the maximum speed.

13. A car is moving along a straight line. It is taken from rest to a velocity of 20 ms^{-1} by a constant acceleration of 5 ms^{-2}. It maintains a constant velocity of 20 ms^{-1} for 5 s and then is brought to rest again by a constant acceleration of -2 ms^{-2}. Draw a velocity-time graph and find the distance covered by the car.

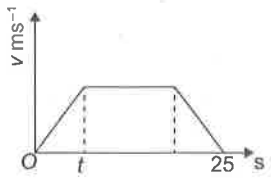
14. A stone is dropped from a height h. Simultaneously, another stone is thrown up from the ground with such a velocity that it can reach a height of 4 h. Find the time when two stones cross each other.

15. A balloon is ascending vertically with an acceleration of 0.2 m/s^2. Two stones are dropped from it at an interval of 2 s. Find the distance between them 1.5 s after the second stone is released (use $g = 9.8$ m/s^2).

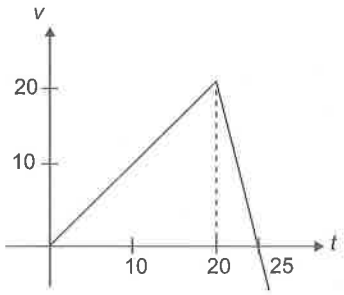
16. From the velocity-time plot shown in the figure, find the distance travelled by the particle during the first 40 s. Also, find the average velocity during this period.



17. The velocity–time graph of the particle moving along a straight line is shown. The rate of acceleration and deceleration is constant and it is equal to 5 ms^{-2}. If the average velocity during the motion is 20 ms^{-1}, then find the value of t.



18. The figure shows the v-t graph of a particle moving in a straight line. Find the time when the particle returns to the starting point.



19. A particle is projected in the x–y plane, and 2 s after projection, the velocity of the particle makes an angle 45° with the x-axis; 4 s after projection, it moves horizontally. Find the velocity of projection (use $g = 10$ ms^{-2}).

20. A particle is projected upwards with a velocity of 100 m/s at an angle of 60° with the vertical. Find the time when the particle will move perpendicular to its initial direction, taking $g = 10$ m/s^2.

21. The equation of a projectile is $y = \sqrt{3}x - \frac{gx^2}{2}$. The angle of projectile is _____ and initial velocity is _____.

22. A ball is projected at an angle of 30° above with the horizontal from the top of a tower and strikes the

(C) At the origin, its velocity makes an angle $\tan^{-1}(a)$ with the x-axis.
(D) The particle moves exactly like a projectile.

77. A ball is rolled off along the edge of a horizontal table with a velocity 4 m/s. It hits the ground after time 0.4 s. Which of the following are correct?
(A) The height of the table is 0.8 m.
(B) It hits the ground at an angle of 60° with the vertical.
(C) It covers a horizontal distance 1.6 m from the table.
(D) It hits the ground with vertical velocity 4 m/s.

78. A particle is projected from the ground with velocity u at angle θ with horizontal. The horizontal range, maximum height and time of flight are R, H and T, respectively. They are given by

$$R = \frac{u^2 \sin 2\theta}{g}, H = \frac{u^2 \sin^2 \theta}{2g}$$

and $T = \frac{2u \sin \theta}{g}$

Now keeping u as fixed, θ is varied from 30° to 60°. Then,
(A) R will first increase then decrease, H will increase and T will decrease
(B) R will first increase then decrease while H and T both will increase
(C) R will decrease while H and T will increase
(D) R will increase while H and T will increase

JEE Advanced

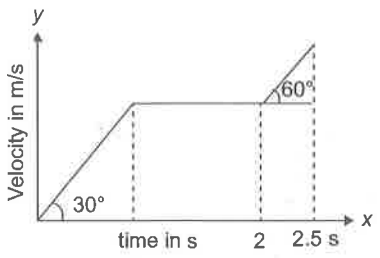
Level I

1. The position vector of a particle moving in the x–y plane is given by $r = (t^2 - 4)i + (t - 4)j$. Find the
(a) equation of trajectory of the particle
(b) time when it crosses x-axis and y-axis

2. A particle moves along the space curve $\vec{r} = (t^2 + t)i + (3t - 2)j + (2t^3 - 4t^2)k$ (t in s, r in m). Find at time $t = 2$, the (a) velocity, (b) acceleration, (c) speed or magnitude of velocity and (d) magnitude of acceleration.

3. At time t, the position vector of a particle of mass $m = 3$ kg is given by $\vec{r} = 6t\hat{i} - t^3\hat{j} + \cos t\hat{k}$. Find the resultant force $\vec{F}(t)$, magnitude of its acceleration when $t = \frac{\pi}{2}$, and speed when $t = \pi$.

4. The velocity–time graph of a body moving in a straight line is shown. Find its



(a) instantaneous velocity at $t = 1.5$ s
(b) average acceleration from $t = 1.5$ s to $t = 2.5$ s
(c) draw its acceleration–time graph from $t = 0$ to $t = 2.5$ s

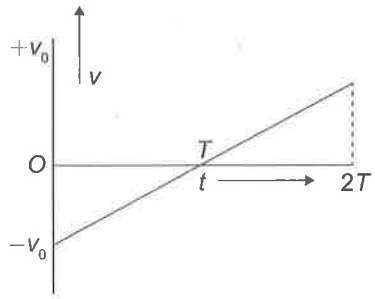
5. The curvilinear motion of a particle is defined by $vx = 50 - 16t$ and $y = 100 - 4t^2$, where vx is in metres per second, y is in metres and t is in seconds. It is also known that $x = 0$ at $t = 0$. Determine the velocity (v) and acceleration (a) when the position $y = 0$ is reached.

6. Velocity v of a car is given by $v = at - bt^2$, where a and b are positive constants and t is time elapsed. Find the value of time for which velocity is maximum and also the corresponding values of velocity.

7. The force acting on a body moving in a straight line is given by $F = (3t^2 - 4t + 1)$ Newton, where t is in seconds. If the mass of the body is 1 kg and initially it was at rest at the origin, find the displacement between time $t = 0$ and $t = 2$ s.

8. A particle goes from A to B with a speed of 40 km/h and B to C with a speed of 60 km/h. If $AB = 6BC$, the average speed in km/h between A and C is _____

[**Hint:** Average speed $= \frac{\text{total distance travelled}}{\text{time taken}}$]



(A) The particle changes its direction of motion at some point.
(B) The acceleration of the particle remains constant.
(C) The displacement of the particle is zero.
(D) The initial and final speeds of the particle are the same.

69. A block is thrown with a velocity of 2 ms^{-1} (relative to ground) on a belt, which is moving with velocity 4 ms^{-1} in the opposite direction of the initial velocity of block. If the block stops slipping on the belt after 4 s after it is thrown, then choose the correct statements(*s*).
(A) Displacement with respect to ground is zero after 2.66 s and magnitude of displacement with respect to ground is 12 m after 4 s.
(B) Magnitude of displacement with respect to ground in 4 s is 4 m.
(C) Magnitude of displacement with respect to belt in 4 s is 12 m.
(D) Displacement with respect to ground is zero in 8/3 s.

70. A particle moves with constant speed v along a regular hexagon *ABCDEF* in the same order. Then the magnitude of the average velocity for its motion from *A* to
(A) *F* is $v/5$ (B) *D* is $v/3$
(C) *C* is $v\sqrt{3/2}$ (D) *B* is v

71. An observer moves with a constant speed along the line joining two stationary objects. He will observe that the two objects
(A) have the same speed
(B) have the same velocity
(C) move in the same direction
(D) move in opposite directions

72. A man on a rectilinearly moving cart, facing the direction of motion, throws a ball straight up with respect to himself
(A) The ball will always return to him
(B) The ball will never return to him
(C) The ball will return to him if the cart moves with constant velocity
(D) The ball will fall behind him if the cart moves with some acceleration

73. A projectile of mass 1 kg is projected with a velocity of $\sqrt{20}$ m/s such that it strikes on the same level as the point of projection at a distance of $\sqrt{3}$ m. Which of the following options are incorrect?
(A) The maximum height reached by the projectile can be 0.25 m.
(B) The minimum velocity during its motion can be $\sqrt{15}$ m/s.
(C) The time taken for the flight can be $\sqrt{\frac{3}{5}}$ s.
(D) The maximum potential energy during its motion can be 6 J.

74. Choose the correct alternative (s):
(A) If the greatest height to which a man can throw a stone is h, then the greatest horizontal distance up to which he can throw the stone is $2h$.
(B) The angle of projection for a projectile motion whose range R is n times the maximum height is $\tan^{-1}(4/n)$.
(C) The time of flight T and the horizontal range R of a projectile are connected by the equation $gT^2 = 2R \tan\theta$, where θ is the angle of projection.
(D) A ball is thrown vertically up. Another ball is thrown at an angle θ with the vertical. Both of them remain in air for the same period of time. Then the ratio of heights attained by the two ball 1:1.

75. If T is the total time of flight, h is the maximum height and R is the range for horizontal motion, then x and y co-ordinates of projectile motion and time t are related as
(A) $y = 4h\left(\frac{t}{T}\right)\left(1-\frac{t}{T}\right)$ (B) $y = 4h\left(\frac{X}{R}\right)\left(1-\frac{X}{R}\right)$
(C) $y = 4h\left(\frac{T}{t}\right)\left(1-\frac{T}{t}\right)$ (D) $y = 4h\left(\frac{R}{X}\right)\left(1-\frac{R}{X}\right)$

76. A particle moves in the xy plane with a constant acceleration g in the negative y-direction. Its equation of motion is $y = ax - bx^2$, where a and B constants. Which of the following are correct?
(A) The x-component of its velocity is constant.
(B) At the origin, the y-component of its velocity is $a\sqrt{\frac{g}{2b}}$.

60. A particle has initial velocity 10 m/s. It moves due to a constant retarding force along the line of velocity which produces a retardation of 5 m/s^2. Then

(A) the maximum displacement in the direction of initial velocity is 10 m

(B) the distance travelled in the first 3 s is 7.5 m

(C) the distance travelled in the first 3 s is 12.5 m

(D) the distance travelled in the first 3 s is 17.5 m

61. Mark the correct statements for a particle going on a straight line

(A) if the velocity is zero at any instant, the acceleration should also be zero at that instant

(B) if the velocity is zero for a time interval, the acceleration is zero at any instant within the time interval

(C) if the velocity and acceleration have opposite sign, the object is slowing down

(D) if the position and velocity have opposite sign, the particle is moving towards the origin.

62. A particle initially at rest is subjected to two forces. One is constant, and the other is a retarding force proportional to the particle velocity. In the subsequent motion of the particle,

(A) the acceleration will increase from zero to a constant value

(B) the acceleration will decrease from its initial value to zero

(C) the velocity will increase from zero to maximum and then decrease

(D) the velocity will increase from zero to a constant value.

63. Let $\vec{v}$ and $\vec{a}$ denote the velocity and acceleration, respectively, of a body in one-dimensional motion

(A) $|\vec{v}|$ must decrease when $\vec{a} < 0$

(B) Speed must increase when $\vec{a} > 0$

(C) Speed will increase when both $\vec{v}$ and $\vec{a}$ are < 0

(D) Speed will decrease when $\vec{v} < 0$ and $\vec{a} > 0$

64. Which of the following statements are true for a moving body?

(A) If its speed changes, its velocity must change and it must have some acceleration

(B) If its velocity changes, its speed must change and it must have some acceleration

(C) If its velocity changes, its speed may or may not change, and it must have some acceleration

(D) If its speed changes but direction of motion does not changes, its velocity may remain constant

65. Let v and a denote the velocity and acceleration, respectively, of a body

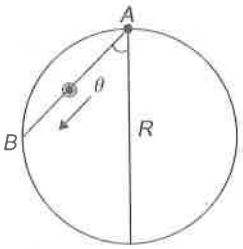
(A) a can be non-zero when $v = 0$

(B) a must be zero when $v = 0$

(C) a may be zero when $v \neq 0$

(D) The direction of a must have some correlation with the direction of v

66. A bead is free to slide down a smooth wire tightly stretched between points A and B on a vertical circle. If the bead starts from rest at A, the highest point on the circle,



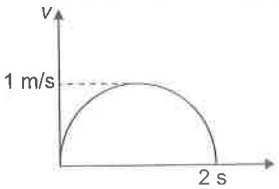
(A) its velocity v on arriving at B is proportional to $\cos\theta$

(B) its velocity v on arriving B is proportional to $\tan\theta$

(C) time to arrive at B is proportional to $\cos\theta$

(D) time to arrive at B is independent of θ

67. Velocity–time graph for a car is a semi-circle as shown below. Which of the following is correct?



(A) Car must move in circular path.

(B) Acceleration of the car is never zero.

(C) Mean speed of the particle is $\pi/4$ m/s.

(D) The car makes a turn once during its motion.

68. The figure shows the velocity (v) of a particle plotted against time (t).

(C) Both the particles will have the same range if $\theta_1 > 45°$ and $\theta_2 < 45°$ and $\theta_1 + \theta_2 = 90°$
(D) none of these

50. A ship is travelling due east at 10 km/h. A ship heading 30° east of north is always due north from the first ship. The speed of the second ship in km/h is

(A) $20\sqrt{2}$ (B) $20\sqrt{3/2}$
(C) 20 (D) $20/\sqrt{2}$

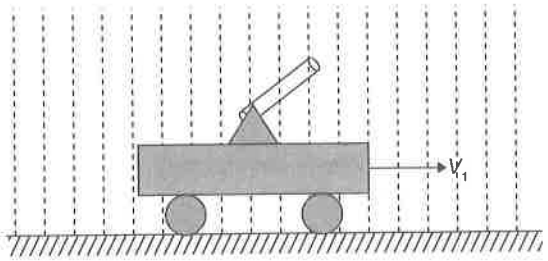
51. A particle is kept at rest at origin. Another particle starts from (5, 0) with a velocity of $-4\hat{i} + 3\hat{j}$. Find their closest distance of approach.

(A) 3 m (B) 4 m
(C) 5 m (D) 2 m

52. Four particles situated at the corners of a square of side a move at a constant speed v. Each particle maintains a direction towards the next particle in succession. Calculate the time the particles will take to meet each other.

(A) $\frac{a}{v}$ (B) $\frac{a}{2v}$
(C) $\frac{a}{3v}$ (D) $\frac{2a}{3v}$

53. A pipe which can rotate in a vertical plane is mounted on a cart. The cart moves uniformly along a horizontal path with a speed $v_1 = 2$ m/s. At what angle α to the horizontal should the pipe be placed so that drops of rain falling with a velocity v_2 6 m/s move parallel to the walls of the pipe without touching them? Consider the velocity of the drops as constant due to the resistance of the air.



(A) $\tan^{-1}(3)$ (B) $\tan^{-1}\left(\frac{1}{3}\right)$
(C) $\tan^{-1}\left(\frac{1}{2}\right)$ (D) none of these

54. A swimmer swims in still water at a speed = 5 km/ h. He enters a 200-m-wide river, having river flow speed = 4 km/h at point A and proceeds to swim at an angle of 127° (sin 37° = 0.6) with the river flow direction. Another point B is located directly across A on the other side. The swimmer lands on the other bank at a point C, from which he walks the distance CB with a speed = 3 km/h. The total time in which he reaches from A to B is

(A) 5 min (B) 4 min
(C) 3 min (D) None of these

55. A boat having a speed of 5 km/h in still water crosses a river of width 1 km along the shortest possible path in 15 min. The speed of the river in km/h is

(A) 1 (B) 3
(C) 4 (D) $\sqrt{41}$

56. A flag is mounted on a car moving due north with a velocity of 20 km/h. Strong winds are blowing due east with a velocity of 20 km/h. The flag will point in the direction

(A) east (B) north-east
(C) south-east (D) south-west

57. A man is crossing a river flowing with a velocity of 5 m/s. He reaches a point directly across at a distance of 60 m in 5 s. His velocity in still water should be

(A) 12 m/s (B) 13 m/s
(C) 5 m/s (D) 10 m/s

58. Wind is blowing in the north direction at a speed of 2 m/s which causes the rain to fall at some angle with the vertical. With what velocity should a cyclist drive so that the rain appears vertical to him

(A) 2 m/s south (B) 2 m/s north
(C) 4 m/s west (D) 4 m/s south

Multiple Correct

59. The displacement x of a particle depends on time t as $x = \alpha t^2 - \beta t^3$

(A) particle will return to its starting point after time α/β.
(B) the particle will come to rest after time $\frac{2\alpha}{3\beta}$
(C) the initial velocity of the particle was zero but its initial acceleration was not zero.
(D) no net force acts on the particle at time $\frac{\alpha}{3\beta}$

$$\sum F_y = 0$$

$$\Rightarrow \quad T_4 \cos 60° = T_2 \cos 30°$$

and $$\sum F_x = 0$$

$$\Rightarrow \quad T_3 + T_2 \sin 30° = T_4 \sin 60°$$

$$\therefore \quad T_3 = \frac{200}{\sqrt{3}} \text{N},$$

and $$T_4 = 200 \text{ N}$$ ■

Frictional Force

It is the component of contact force tangential to the surface. It opposes the relative motion (or attempted relative motion) of the two surfaces in contact (which is explained later).

NEWTON'S FIRST LAW OF MOTION

According to this law, 'A system will remain in its state of rest or of uniform motion unless a net external force acts on it.'

First law can also be stated as 'If the net external force acting on a body is zero, only then the body remains at rest.'

- The word external means external to the system (object under observation); interactions within the system should not be considered.
- The word net means the resultant of all the forces acting on the system.
- Newton's first law is nothing but Galileo's law of inertia.
- Inertia means inability of a body to change its state of motion or rest by itself.
- The property of a body that determines its resistance to a change in its motion is its mass (inertia). Greater the mass, greater the inertia.
- An external force is needed to set the system into motion, but no external force is needed to keep a body moving with constant velocity in its uniform motion.
- Newton's laws of motion are valid only in a set of frame of references; these frames of reference are known as inertial frames of reference.
- Generally, we take earth as an inertial frame of reference, but strictly speaking it is not an inertial frame.
- All frames moving uniformly with respect to an inertial frame are themselves inertial.
- We take all frames at rest or moving uniformly with respect to earth, as inertial frames.

NEWTON'S SECOND LAW OF MOTION

Newton's second law states that 'The rate of change of a momentum of a body is directly proportional to the applied force and takes place in the direction in which the force acts.'

i.e., $$\vec{F} \propto \frac{d\vec{p}}{dt}$$

or $$\vec{F} = k\frac{d\vec{p}}{dt}$$

where k is a constant of proportionality.

$$\vec{p} = m\vec{v}$$

So $$\vec{F} = k\frac{(dm\vec{v})}{dt}$$

For a body having constant mass,

$$\Rightarrow \quad \vec{F} = km\frac{d\vec{v}}{dt} = km\,\vec{a}$$

From experiments, the value of k is found to be 1.

So, $$\vec{F}_{net} = m\vec{a}$$

- Force cannot change the momentum along a direction normal to it, i.e., the component of velocity normal to the force doesn't change.
- Newton's second law is strictly applicable to a single point particle. In case of rigid bodies or system of particles or system of rigid bodies, $\vec{F}$ refers to total external force acting on system and $\vec{a}$ refers to acceleration of centre of mass of the system. The internal forces, if any, in the system are not to be included in $\vec{F}$.
- Acceleration of a particle at any instant and at a particular location is determined by the force (net) acting on the particle at the same instant and at same location and is not in any way depending on the history of the motion of the particle.

Problem Solving Strategy

Newton's laws refer to a particle and relate the forces acting on the particle to its mass and to its acceleration. But before writing any equation from Newton's law, you should be careful about which particle you are considering. The laws are applicable to an extended body too which is nothing but collection of a large number of particles.

Follow the steps given below in writing the equations:

Step 1: Select the body

The first step is to decide the body on which the laws of motion are to be applied. The body may be a single particle, an extended body like a block, a combination of two blocks-one kept over another or connected by a string. The only condition is that *all the parts of the body or system must have the same acceleration.*

Step 2: Identify the forces

Once the system is decided, list down all the force acting on the system due to all the objects in the environment such as inclined planes, strings, springs, etc. However, any force applied by the system should not be included in the list. You should also be clear about the nature and direction of these forces.

Step 3: Make a free-body diagram (F.B.D.)

Make a separate diagram representing the body by a point and draw vectors representing the forces acting on the body with this point as the common origin. This is called a free-body diagram of the body.

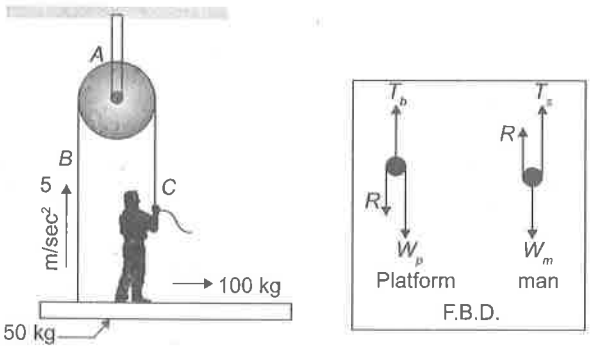


Figure 4.14

Look at the adjoining free-body diagrams for the platform and the man. Note that the force applied by the man on the rope has not been included in the F.B.D.

Once you get enough practice, you would be able to identify and draw forces in the main diagram itself instead of making a separate one.

Step 4: Select axes and write equations

When the body is in equilibrium then choose the axis in such a fashion that maximum number of force lie along the axis.

If the body is moving with some acceleration, then first find out the direction of real acceleration and choose the axis one is along the real acceleration direction and the other perpendicular to it.

Write the equations according to the Newton's second law ($F_{net} = ma$) in the corresponding axis.

APPLICATIONS

Motion of a Block on a Horizontal Smooth Surface

Case (i): When Subjected to a Horizontal Pull

The distribution of forces on the body is shown. As there is no motion along the vertical direction, hence,

$$R = mg$$

For horizontal motion,

$$F = ma$$

or $$a = F/m$$

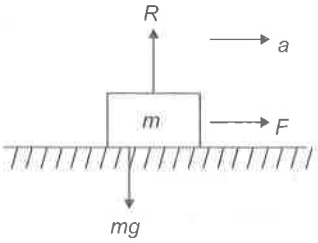


Figure 4.15

Case (ii): When Subjected to a Pull Acting at an Angle (θ) to the Horizontal

Now F has to be resolved into two components, $F\cos\theta$ along the horizontal and $F\sin\theta$ along the vertical direction.

For no motion along the vertical direction, we have

$$R + F\sin\theta = mg$$

or $$R = mg - F\sin\theta$$

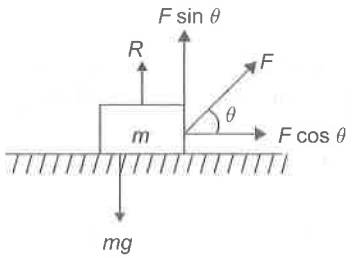


Figure 4.16

Note

Hence,

$$R \neq mg \cdot R < mg$$

For horizontal motion,

$$F\cos\theta = ma$$

$$\Rightarrow \quad a = \frac{F\cos\theta}{m}$$

Case (iii): When the Block is Subjected to a Push Acting at an Angle θ to the Horizontal (Downward)

The force equation in this case

$$R = mg + F\sin\theta$$

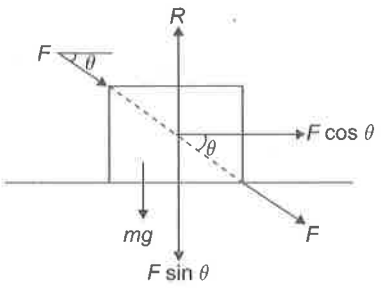


Figure 4.17

Note

$$R \neq mg, R > mg$$

For horizontal motion,

$$F\cos\theta = ma$$

$$\Rightarrow \quad a = \frac{F\cos\theta}{m}$$

Motion of Bodies in Contact

Case (i): Two-body System

Let a force F be applied on mass m_1.

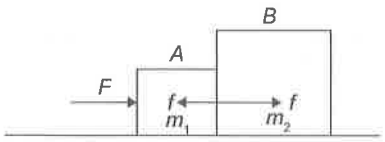


Figure 4.18

Free-body Diagrams

(Vertical force do not cause motion; hence, they have not been shown in the diagram.)

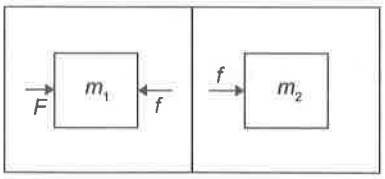


Figure 4.19

$$\Rightarrow \quad a = \frac{F}{m_1 + m_2}$$

and

$$f = \frac{m_2 F}{m_1 + m_2}$$

1. Here f is known as force of contact.
2. Acceleration of system can be found simply by

$$a = \frac{\text{force}}{\text{total mass}}$$

Note

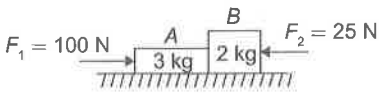
If force F be applied on m_2, the acceleration will remain the same, but the force of contact will be different.

i.e., $$f' = \frac{m_1 F}{m_1 + m_2}$$

SOLVED EXAMPLES

EXAMPLE 6

Find the contact force between the 3 kg and 2 kg blocks as shown in the figure.



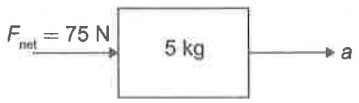
SOLUTION

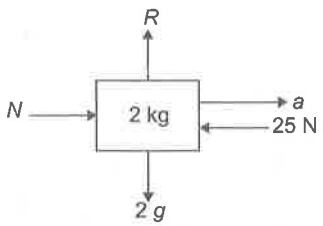
Considering both blocks as a system to find the common acceleration

$$F_{net} = F_1 - F_2 = 100 - 25$$
$$= 75 \text{ N}$$

Common acceleration

$$a = \frac{F_{net}}{5 \text{ kg}} = \frac{75}{5} = 15 \text{ m/s}^2$$





To find the contact force between A and B, we draw F.B.D of 2 kg block from $(\sum F_{net})_x = ma_x$

$\Rightarrow \quad N - 25 = (2)(15)$

$\Rightarrow \quad N = 55 \text{ N}$ ■

Case (ii): Three-body System

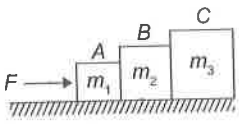
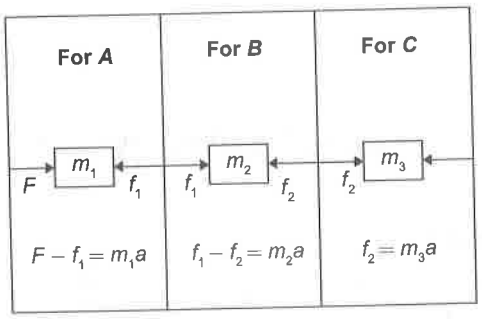


Figure 4.20

Free-body Diagrams



$$\Rightarrow \quad a = \frac{F}{m_1 + m_2 + m_3}$$

and $$f_1 = \frac{(m_2 + m_3)F}{(m_1 + m_2 + m_3)}$$

$$f_2 = \frac{m_3 F}{(m_1 + m_2 + m_3)}$$

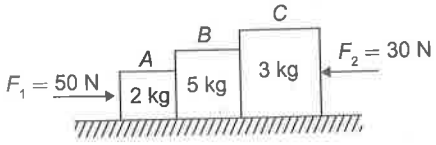
f_1 = contact force between masses m_1 and m_2
f_2 = contact force between masses m_2 and m_3

Remember

Contact forces will be different if force F will be applied on mass C.

EXAMPLE 7

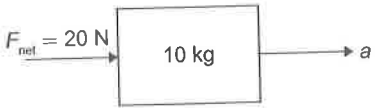
Find the contact force between the block and the acceleration of the blocks as shown in the figure.

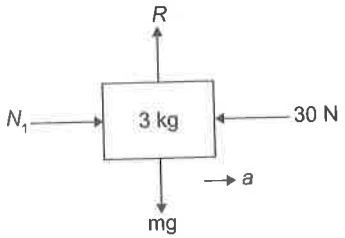


SOLUTION

Considering all the three blocks as a system to find the common acceleration

$$F_{net} = 50 - 30 = 20 \text{ N}$$





$$a = \frac{20}{10} = 2 \text{ m/s}^2$$

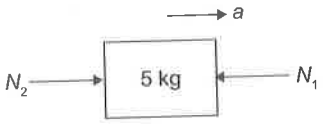
To find the contact force between B and C, we draw F.B.D. of 3 kg block.

$$(\Sigma F_{net})_x = ma$$

$$\Rightarrow \quad N_1 - 30 = 3(2)$$

$$\Rightarrow \quad N_1 = 36 \text{ N}$$

To find contact force between A and B, we draw F.B.D. of 5 kg block.



$$\Rightarrow \quad N_2 - N_1 = 5a$$

$$N_2 = 5 \times 2 + 36$$

$$\Rightarrow \quad N_2 = 46 \text{ N}$$ ■

Motion of Connected Bodies

Case (i): For Two Bodies

F is the pull on body A of mass m_1. The pull of A on B is exercised as tension through the string connecting A and B. The value of tension throughout the string is T only.

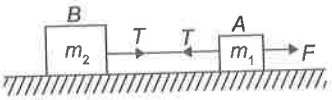
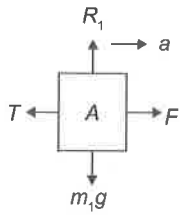
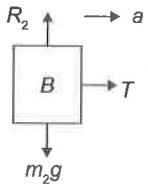


Figure 4.21

Free-body Diagrams

For body A	For body B
	
$R_1 = m_1 g$ $F - T = m_1 a$	$R_2 = m_2 g$ $T = m_2 a$

$$\Rightarrow \quad a = \frac{F}{m_1 + m_2}$$

Case (ii): For Three Bodies

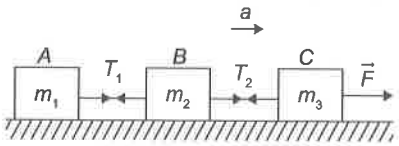
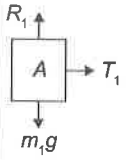
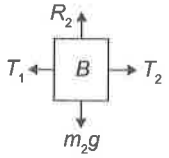
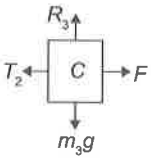


Figure 4.22

Free-body Diagrams

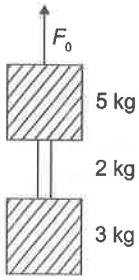
For A	For B	For C
		
$R_1 = m_1 g$ $T_1 = m_1 a$	$R_2 = m_2 g$ $T_2 - T_1 = m_2 a$ $\Rightarrow \; T_2 = m_2 a + T_1$ $T_2 = (m_2 + m_1)a$	$R_3 = m_3 g$ $F - T_2 = m_3 a$ $\Rightarrow \; F = m_3 a + T_2$ $= m_3 a + (m_1 + m_2)a$ $F = (m_1 + m_2 + m_3)a$

$$\Rightarrow \quad a = \frac{F}{m_1 + m_2 + m_3}$$

SOLVED EXAMPLES

EXAMPLE 8

A 5 kg block has a rope of mass 2 kg attached to its underside and a 3 kg block is suspended from the other end of the rope. The whole system is accelerated upwards at 2 m/s² by an external force F_0.



(a) What is F_0?
(b) What is the force on rope?
(c) What is the tension at middle point of the rope?

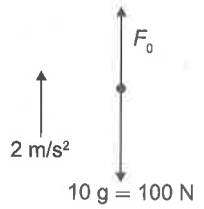
($g = 10$ m/s²)

SOLUTION

For calculating the value of F_0, consider two blocks with the rope as a system.

F.B.D. of whole system

(a)

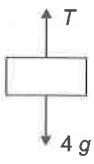


$$F_0 - 100 = 10 \times 2$$
$$F = 120 \text{ N} \qquad (1)$$

(b) According to Newton's second law, net force on rope

$$F = ma = (2)\,(2) = 4 \text{ N} \qquad (2)$$

(c) For calculating tension at the middle point, we draw F.B.D. of 3 kg block with half of the rope (mass 1 kg) as shown.



$$T - 4g = 4(2) = 48 \text{ N}$$ ■

Motion of a Body on a Smooth Inclined Plane

Natural acceleration down the plane $= g \sin\theta$

Driving force for acceleration a up the plane,

$$F = m(a + g \sin\theta)$$

and for an acceleration a down the plane,

$$F = m(a - g \sin\theta)$$

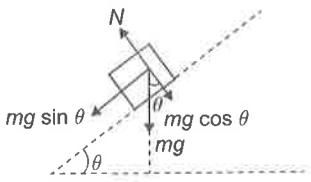
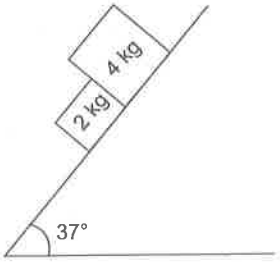


Figure 4.23

EXAMPLE 9

Find out the contact force between the 2 kg and 4 kg blocks as shown in the figure.

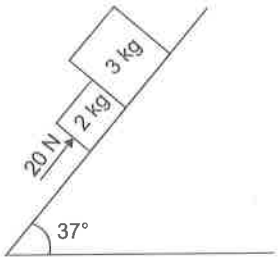


SOLUTION

On an incline plane, acceleration of the block is independent of mass. So both the blocks will move with the same acceleration (g sin 37°) so the contact force between them is zero. ■

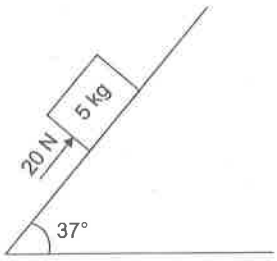
EXAMPLE 10

Find out the contact force between 2 kg and 3 kg blocks placed on the incline plane as shown in the figure.

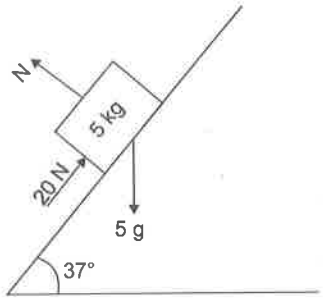


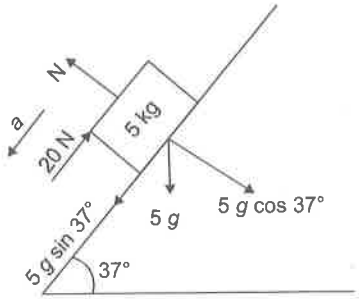
SOLUTION

Considering both the blocks as a 5-kg system because both will move the same acceleration.



Now show forces on the 5 kg block





∵ Acceleration of 5 kg block is down the incline.

So choose one axis down the incline and the other perpendicular to it.

From Newton's second law,

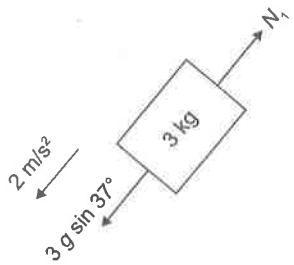
$$N = 5g \cos 37° \quad (1)$$

$$5g \sin 37° - 20 = 5a \quad (2)$$

$$30 - 20 = 5a$$

$$a = 2 \text{ m/s}^2 \text{ (down the incline)}$$

For contact force (N_1) between 2 kg and 3 kg blocks, we draw F.B.D. of 3 kg block.



From $F_{net} = ma$

$$\Rightarrow \quad 3g \sin 37° - N_1 = 3 \times 2$$

$$18 - N_1 = 6$$

$$N_1 = 12 \text{ N}$$ ■

Pulley Block System

SOLVED EXAMPLES

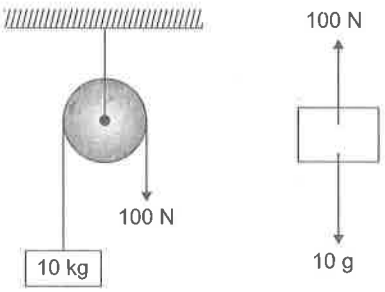
EXAMPLE 11

One end of the string which passes through the pulley and connected to 10 kg mass at the other end is pulled by 100 N force. Find out the acceleration of 10 kg mass ($g = 9.8$ m/s^2).

SOLUTION

Since string is pulled by 100 N force, tension in the string is 100 N.

F.B.D. of 10 kg block



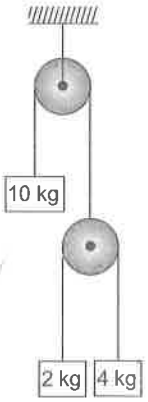
$$100 - 10g = 10a$$

$$100 - 10 \times 9.8 = 10a$$

$$a = 0.2 \text{ m/s}^2$$ ■

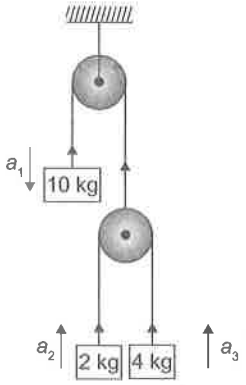
EXAMPLE 12

In the figure shown, find out acceleration of each block.

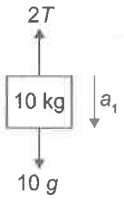


SOLUTION

Now F.B.D. of each block and apply Newton's second law on each F.B.D.

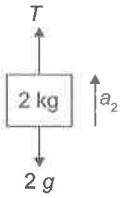


(a)



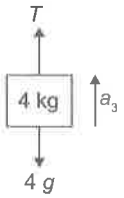
$$\Rightarrow \quad 10g - 2T = 10a_1 \qquad (1)$$

(b)



$$\Rightarrow \quad T - 2g = 2a_2 \qquad (2)$$

(c)



$$\Rightarrow \quad T - 4g = 4a_3 \qquad (3)$$

From constrain relation,

$$2a_1 = a_2 + a_3 \qquad (4)$$

Solving Eqs. (1), (2), (3) and (4), we get

$$T = \frac{800}{23} \text{ N}$$

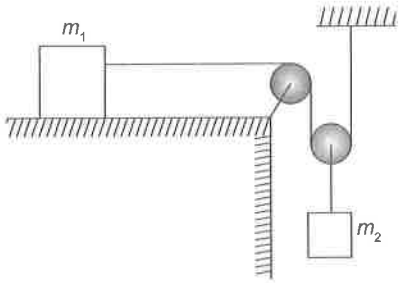
$a_1 = 70/23$ m/s^2 (downward),

$a_2 = 170/23$ m/s^2 (upward),

$a_3 = 30/23$ m/s^2 (downward) ■

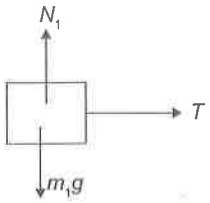
EXAMPLE 13

Find the acceleration of each block in the figure shown below, in terms of their masses m_1, m_2 and g. Neglect any friction.



SOLUTION

Let T be the tension in the string that is assumed to be massless.

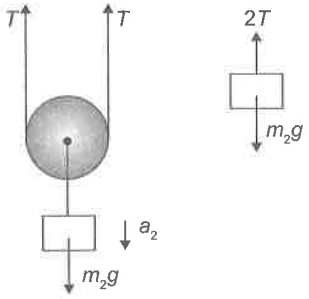


For mass m_1, the F.B.D. shows that

$$N_1 = m_1 g$$

where N_1 is the force applied upwards by the plane on the mass m_1. If acceleration of m_1 along horizontal is a_1, then

$$T = m_1 a_1 \tag{1}$$



For mass m_2, the F.B.D. shows that

$$m_2 g - 2T = m_2 a_2 \tag{2}$$

where a_2 is the vertical acceleration of mass m_2.

Note that upward tension on m_2 is $2T$ applied by both sides of the string.

From constrain relation,

$$a_2 = \frac{a_1}{2}$$

Thus, the acceleration of m_1 is twice that of m_2.

With this input, solving (1) and (2) we find

$$a_1 = \frac{2m_2 g}{4m_1 + m_2}$$

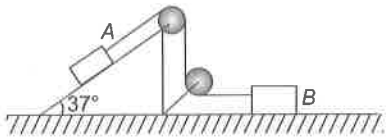
$$a_2 = \frac{m_2 g}{4m_1 + m_2}$$

EXAMPLE 14

Two blocks A and B each having a mass of 20 kg, rest on frictionless surfaces as shown in the figure below. Assuming the pulleys to be light and frictionless, compute:

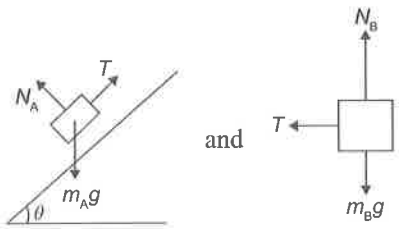
(a) the time required for block A, to move down by 2 m on the plane, starting from rest,

(b) tension in the string, connecting the blocks.

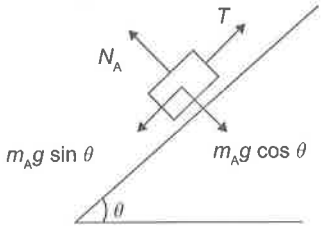


SOLUTION

Step 1: Draw the F.B.D.s for both the blocks. If tension in the string is T, then we have



Note that m_Ag should better be resolved along and perpendicular to the plane, as Block A is moving along the plane.



Step 2: From F.B.D.s, we write the force equations for Block A where

$$N_A = m_A g \cos\theta = 20 \times 10 \times 0.8$$

$$= 160 \text{ N}$$

and $\quad m_A g \sin\theta - T = m_A a \tag{1}$

where 'a' is the acceleration of masses of Blocks A and B.

Similarly, force equations for Block B are

$$N_B = m_B g = 20 \times 10$$
$$= 200 \text{ N}$$

and $$T = m_B a \quad (2)$$

From (1) and (2), we obtain

$$a = \frac{m_A g \sin\theta}{m_A + m_B} = \frac{20 \times 10 \times 0.6}{40}$$
$$= 3 \text{ ms}^{-2}$$
$$T = m_B a = 20 \times 3$$
$$= 60 \text{ N}$$

Step 3: With constant acceleration $a = 3$ ms^{-2}, Block A moves down the inclined plane a distance $S = 2$ m in time t given by

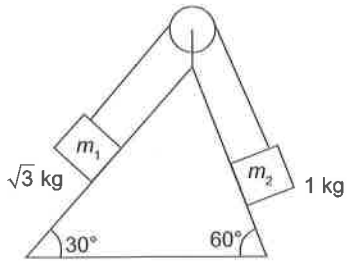
$$S = \frac{1}{2}at^2$$

or $$t = \sqrt{\frac{2S}{a}} = \frac{2}{\sqrt{3}} \text{s}$$ ■

EXAMPLE 15

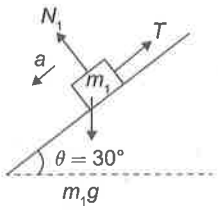
Two blocks m_1 and m_2 are placed on a smooth inclined plane as shown in the figure. If they are released from rest, then find:

(a) acceleration of mass m_1 and m_2
(b) tension in the string
(c) net force on pulley exerted by string



SOLUTION

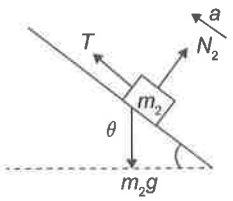
F.B.D of m_1:



$$m_1 g \sin\theta - T = m_1 a$$

$$\frac{\sqrt{3}}{2}g - T = \sqrt{3}a \quad (1)$$

F.B.D. of m_2:



$$T - m_2 g \sin\theta = m_2 a$$

$$T - 1 \cdot \frac{\sqrt{3}}{2}g = 1a \quad (2)$$

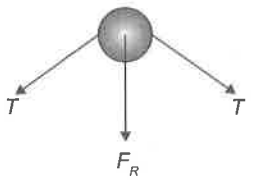
Adding Eqs. (1) and (2), we get

$$a = 0$$

Putting this value in Eq. (1), we get

$$T = \frac{\sqrt{3}g}{2},$$

F.B.D. of pulley:



$$F_R = \sqrt{2}T$$

$$F_R = \sqrt{\frac{3}{2}}g$$ ■

NEWTONS' THIRD LAW OF MOTION

Statement: *To every action there is equal and opposite reaction.*

But what is the meaning of action and reaction and which force is action and which force is reaction?

Every force that acts on body is due to the other bodies in environment. Suppose that a body A experiences a force $\vec{F}_{AB}$ due to other body B. Also body B will experience

a force $\vec{F}_{BA}$ due to A. According to Newton's third law, two forces are equal in magnitude and opposite in direction. Mathematically, we write it as

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

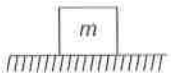
Here, we can take either $\vec{F}_{AB}$ or $\vec{F}_{BA}$ as action force and other will be the reaction force.

Notes

1. Action-Reaction pair acts on two different bodies.
2. Magnitude of force is same.
3. Direction of forces is in opposite direction.
4. For action-reaction pair, there is no need of contact.

EXAMPLE 16

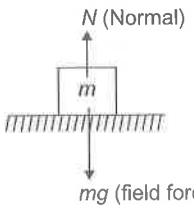
A block of mass 'm' is kept on the ground as shown in the figure.



(a) Draw F.B.D. of block.
(b) Are forces acting on block action-reaction pair?
(c) If answer is no, draw action-reaction pair.

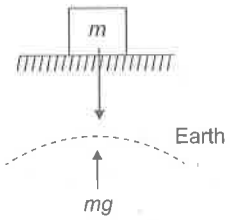
SOLUTION

(a) F.B.D. of block



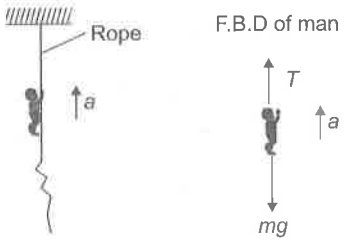
(b) 'N' and mg are not action-reaction pair, since pair acts on different bodies and they are of same nature.

(c) Pair of 'mg' of block acts on the Earth in opposite direction.



and pair of 'N' acts on surface as shown in the figure.

Climbing on the Rope



Now three conditions arise.
If

$T > mg \Rightarrow$ man accelerates in upward direction
$T < mg \Rightarrow$ man accelerates in downward direction
$T = mg \Rightarrow$ man's acceleration is zero ■

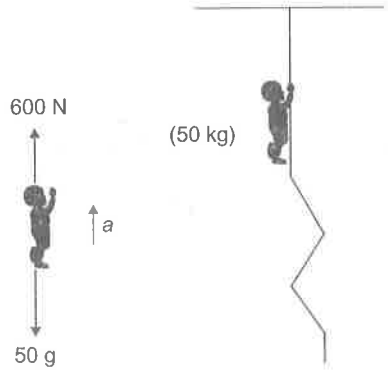
Note

Either climbing or descending on the rope man exerts force downward.

EXAMPLE 17

If the breaking strength of rope is 600 N, then find out the maximum acceleration of the man with which he can climb up the rope.

SOLUTION



Maximum force that can be exerted on the man by the rope is 600 N.
F.B.D of man

$$\Rightarrow \quad 600 - 50g = 50a$$
$$a_{max} = 2 \text{ m/s}^2$$ ■

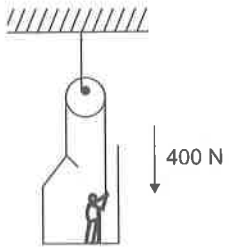
EXAMPLE 18

A 60 kg painter stands on a 15 kg platform. A rope attached to the platform and passing over an overhead pulley allows the painter to raise himself along with the platform.

(a) To get started, he pulls the rope down with a force of 400 N. Find the acceleration of the platform as well as that of the painter.

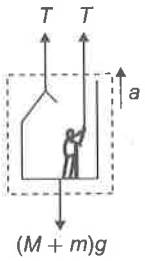
(b) What force must he exert on the rope so as to attain an upward speed of 1 m/s in 1 s?

(c) What force should apply now to maintain the constant speed of 1 m/s?



SOLUTION

The free-body diagram of the painter and the platform as a system can be drawn as shown in the figure. Note that the tension in the string is equal to the force by which he pulls the rope.



(a) Applying Newton's second law

$$2T - (M+m)g = (M+m)a$$

or
$$a = \frac{2T-(M+m)g}{M+m}$$

Here, $M = 60$ kg; $m = 15$ kg;

$T = 400$ N

$g = 10$ m/s^2

$$a = \frac{2(400)-(60+15)(10)}{60+15}$$

$$= 0.67 \text{ m/s}^2$$

(b) To attain a speed of 1 m/s in one second, the acceleration a must be 1 m/s^2.
Thus, the applied force is

$$F = \frac{1}{2}(M+m)\,(g+a)$$

$$= (60+15)\,(10+1)$$

$$= 412.5 \text{ N}$$

(c) When the painter and the platform move (upward) together with a constant speed, it is in a state of dynamic equilibrium.

Thus, $2F - (M+m)g = 0$

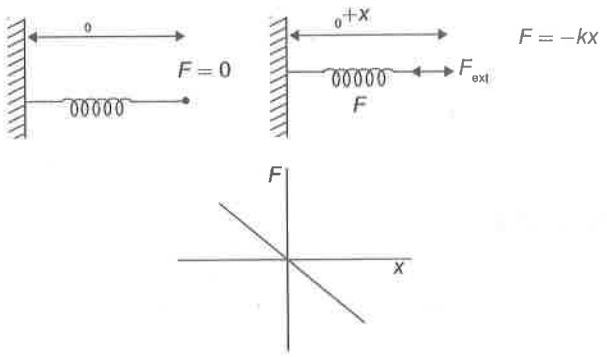
or
$$F = \frac{(M+m)g}{2} = \frac{(60+15)(10)}{2}$$

$$= 375 \text{ N}$$ ■

SPRING FORCE

Every spring resists any attempt to change its length; when it is compressed or extended, it exerts force at its ends. The force exerted by a spring is given by $F = -kx$, where x is the change in length and k is the stiffness constant or spring constant (unit Nm^{-1}).

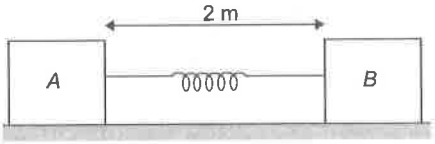
When spring is in its natural length, spring force is zero.



Graph between spring force v/s x

EXAMPLE 19

Two blocks are connected by a spring of natural length 2 m. The force constant of spring is 200 N/m. Find spring force in the following situations.



(a) If both Blocks A and B are displaced by 0.5 m in the same direction.

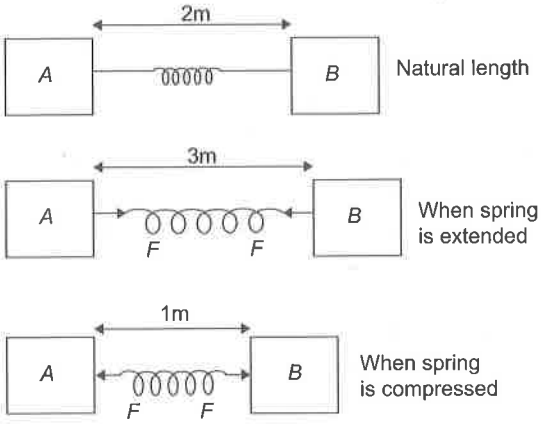
(b) If both Blocks A and B both are displaced by 0.5 m in the opposite direction.

SOLUTION

(a) Since both blocks are displaced by 0.5 m in the same direction, so change in length of spring is zero. Hence, spring force is zero.

(b) In this case, change in length of spring is 1 m. So spring force is

$$F = -Kx$$
$$= -(200)\cdot(1)$$
$$F = -200 \text{ N}$$



■

EXAMPLE 20

Force constant of a spring is 100 N/m. If a 10 kg block attached with the spring is at rest, then find extension in the spring ($g = 10$ m s^2).

SOLUTION

In this situation, spring is in extended state, so spring force acts in upward direction. Let x be the extension in the spring.

F.B.D. of 10 kg block:

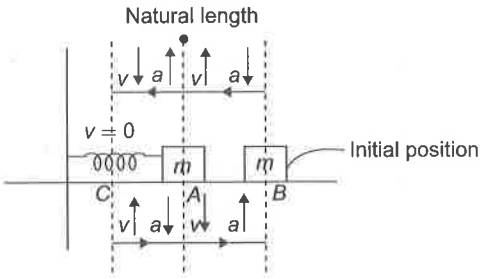
$$F_s = 10 \text{ g}$$
$$\Rightarrow \quad Kx = 100$$
$$\Rightarrow \quad (100)x = (100)$$
$$\Rightarrow \quad x = 1 \text{ m}$$

■

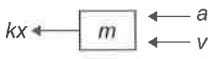
Spring Force System

Initially the spring is in natural length at A with block m. But when the block displaced towards right then the spring is elongated and now block is released at B then the block move towards left due to spring force (kx).

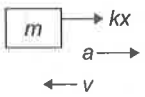
Analysis of Motion of Block



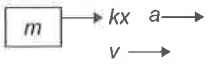
1. From B to A, speed of block increases and acceleration decreases (due to decrease in spring force kx)



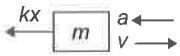
2. Due to inertia block crosses natural length at A. From A to C, speed of the block decreases and acceleration increases (due to increase in spring force kx)



3. At C, the block stops momentarily at this instant and since the spring is compressed spring force is towards right and the block starts to move towards right. From C to A, speed of the block increases and acceleration decreases (due to decrease in spring force kx).



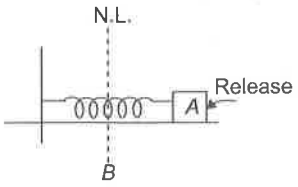
4. Again block crosses point A due to inertia then from A to B speed decreases and acceleration increases.



In this way, block does SHM (to be explained later) if no resistive force is acting on the block.

Notes

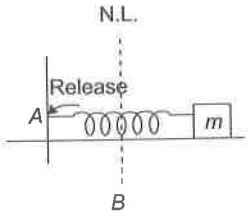
1.



When Block A is released, then it take some finite time to reach at B, i.e., spring force does not change instantaneously.

Notes (contd)

2.

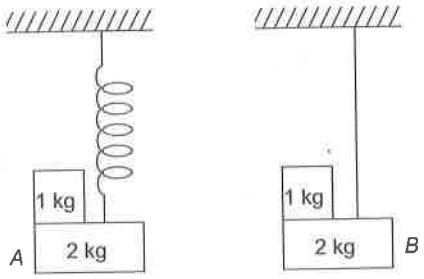


When point A of the spring is released in the above situation, then the spring forces changes instantaneously and becomes zero because one end of the spring is free.

3. In string, tension may change instantaneously.

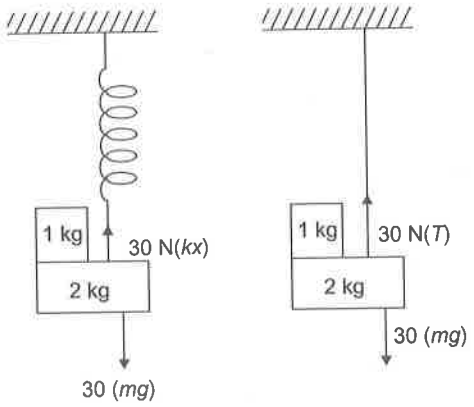
EXAMPLE 21

Find out the acceleration of 2 kg block in the figures shown at the instant 1 kg block falls from 2 kg block (at $t = 0$).



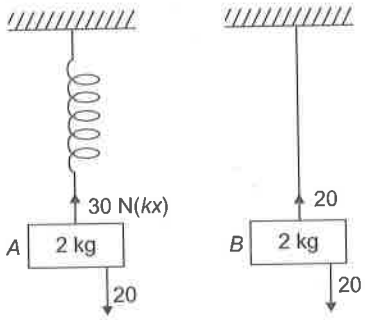
SOLUTION

F.B.D.s before fall of 1 kg block:



After the fall of the 1 kg block, tension will change instantaneously but spring force (kx) does not change instantaneously.

F.B.D.s just after the fall of 1 kg block:



$$a_A = \frac{30-20}{2} = 5 \text{ m/s}^2 \text{ (upward)}$$

$$a_B = 0 \text{ m/s}^2$$

SOLVED EXAMPLES

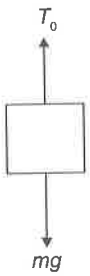
EXAMPLE 22

Two Blocks A and B of same mass 'm' attached with a light spring are suspended by a string as shown in the figure. Find the acceleration of Blocks A and B just after the string is cut.

SOLUTION

When Blocks A and B are in equilibrium position

F.B.D of B:



$$T_0 = mg \quad (1)$$

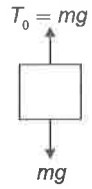
F.B.D of A:

$$T = mg + T_0 \quad (2)$$

$$T = 2mg$$

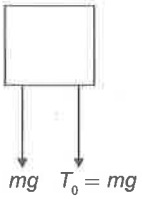
When string is cut, tension T becomes zero. But spring does not change its shape just after cutting. So spring force acts on mass B, again draw F.B.D. of Blocks A and B as shown in the figure.

F.B.D of B:



$$T_0 - mg = m \cdot a_B$$
$$a_B = 0$$

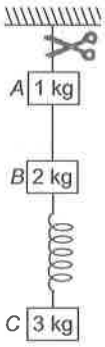
F.B.D. of A:



$$mg + T_0 = m \cdot a_A$$
$$2\,mg = m \cdot a_A$$
$$a_A = 2g \text{ (downwards)}$$

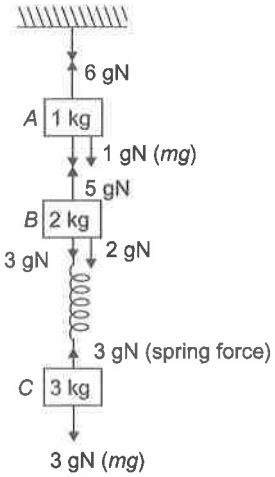
EXAMPLE 23

Find out the acceleration of 1 kg, 2 kg and 3 kg blocks and tension in the string between 1 kg and 2 kg blocks just after cutting the string as shown in the figure.

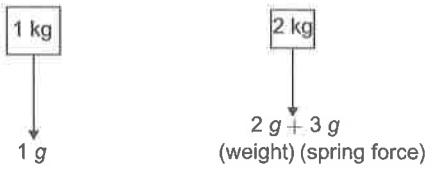


SOLUTION

F.B.D before cutting of string



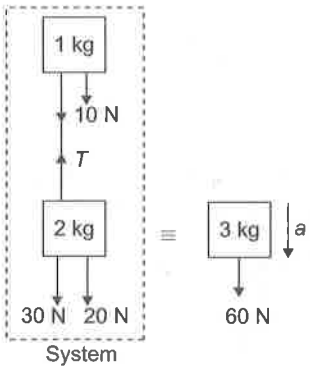
Let us assume the tension in the string connecting Blocks A and B becomes zero just after cutting the string then.



$$a_1 = \frac{1g}{1} = g \text{ ms}^{-2} \qquad a_2 = \frac{5g}{2} = 2.5\,g \text{ ms}^{-2}$$

$\because \quad a_2 > a_1 \qquad \therefore \quad T \neq 0$

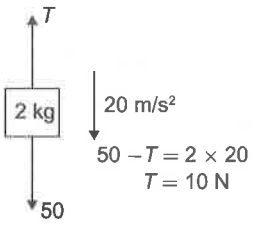
If $T \neq 0$ that means string is tight and both Blocks A and B will have same acceleration. So it will take as a system of 3 kg mass.



Total force downward $= 10 + 30 + 20 = 60$ N

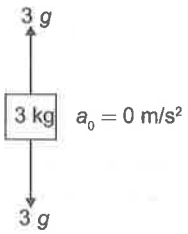
Total mass $= 3$ kg

$$\Rightarrow \qquad a = \frac{60}{3} = 20 \text{ m/s}^2$$



Now apply $F_{net} = ma$ at Block B.

$\because$ The spring force does not change instantaneously the F.B.D of 'C' is



Reference Frame

A frame of reference is basically a coordinate system in which motion of object is analysed. There are two types of reference frames.

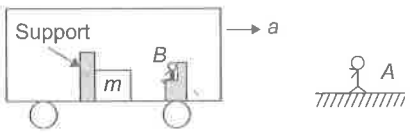
1. **Inertial reference frame:** Frame of reference moving with constant velocity or stationary.
2. **Non-inertial reference frame:** A frame of reference moving with non-zero acceleration.

Notes

1. Although the Earth is a non-inertial frame (due to rotation) but we always consider it as an inertial frame.
2. A body moving in circular path with constant speed is a non-inertial frame (direction change cause acceleration).

PSEUDO FORCE

Consider the following example to understand the pseudo force concept:



The block m in the bus is moving with constant acceleration a with respect to man A at ground. Force required for this acceleration is the normal reaction exerted by the support

So, $$N = ma \quad (1)$$

This block m is at rest with respect to man B who is in the bus (a non-inertial frame). So the acceleration of the block with respect to man B is zero.

$$N = m(0) = 0 \quad (2)$$

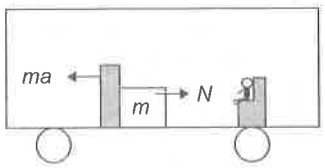
But the normal force is exerted in a non-inertial frame also. So Eq. (2) is wrong; therefore, we conclude that Newton's law is not valid in non-inertial frame.

If we want to apply Newton's law in non-inertial frame, then we can do so by using of the concept pseudo force.

Pseudo force is an imaginary force, which in actual is not acting on the body. But after applying it on the body we can use Newton's laws in non-inertial frames.

This imaginary force is acting on the body only when we are solving the problem in a non-inertial frame of reference.

In the above example, the net force on the block m is zero with respect to man B after applying the pseudo force.



$N = ma$

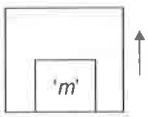
Notes

1. Direction of pseudo force is opposite to the acceleration of frame.
2. Magnitude of pseudo force is equal to mass of the body which we are analysing multiplied by acceleration of frame.
3. Point of application of pseudo force is the centre of mass of the body which we are analysing.

SOLVED EXAMPLES

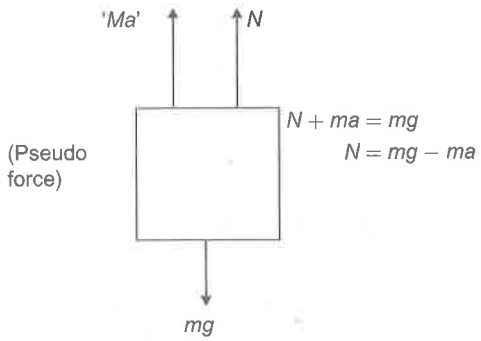
EXAMPLE 24

A box is moving upwards with retardation 'a' $< g$, find the direction and magnitude of 'pseudo force' acting on block of mass 'm' placed inside the box. Also calculate normal force exerted by surface on the block.



SOLUTION

Pseudo force acts opposite to the direction of acceleration of reference frame.



pseudo force = ma in upward direction

F.B.D of 'm' w.r.t box (non-inertial)

On adding Eqs. (1) and (2), we get

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)(g + a_0) \qquad (3)$$

Substituting a in Eq. (1), we get

$$T = \frac{2m_1 m_2 (g + a_0)}{m_1 + m_2}$$

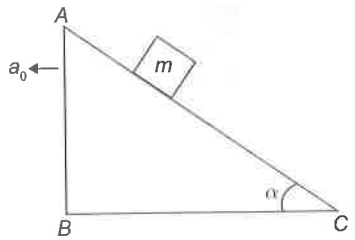
$$\therefore \quad x = \frac{F}{k} = \frac{2T}{k}$$

$$= \frac{4m_1 m_2 (g + a_0)}{(m_1 + m_2)k}$$

■

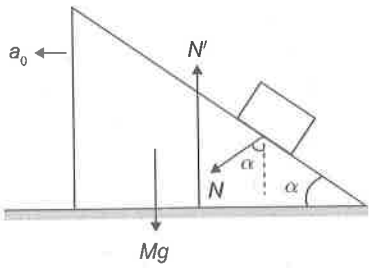
EXAMPLE 27

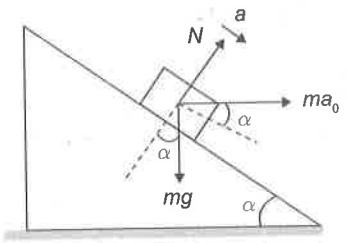
All the surfaces shown in the figure are assumed to be frictionless. The block of mass m slides on the prism which in turn slides backwards on the horizontal surface. Find the acceleration of the smaller block with respect to the prism.



SOLUTION

Let the acceleration of the prism be a_0 in the backward direction. Consider the motion of the smaller block from the frame of the prism. The forces on the block are (figure)





(a) N normal force
(b) mg downward (gravity),
(c) ma_0 forward (pseudo force)

The block slides down the plane. Components of the forces parallel to the incline give

$$ma_0 \cos\alpha + mg \sin\alpha = ma$$

or,
$$a = a_0 \cos\alpha + g \sin\alpha \qquad (1)$$

Components of the forces perpendicular to the incline give

$$N + ma_0 \sin\alpha = mg \cos\alpha \qquad (2)$$

Now consider the motion of the prism from the ground frame. No pseudo force is needed as the frame used is inertial. The forces are (figure)

(a) Mg downwards
(b) N normal to the incline (by the block)
(c) N' upwards (by the horizontal surface)

Horizontal components give

$$N \sin\alpha = Ma_0$$

or
$$N = Ma_0 / \sin\alpha \qquad (3)$$

Putting in Eq. (2)

$$\frac{Ma_0}{\sin\alpha} + ma_0 \sin\alpha = mg \cos\alpha$$

or,
$$a_0 = \frac{mg \sin\alpha \cos\alpha}{M + m \sin^2\alpha}$$

From Eq. (1)

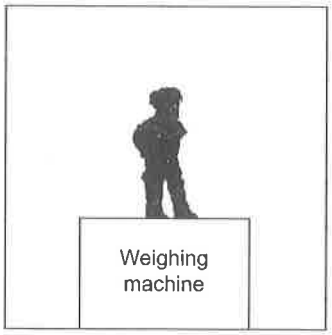
$$a = \frac{mg \sin\alpha \cos^2\alpha}{M + m \sin^2\alpha} + g \sin\alpha$$

$$= \frac{(M + m)\, g \sin\alpha}{M + m \sin^2\alpha}$$

■

WEIGHING MACHINE

A weighing machine does not measure the weight but measures the force exerted by object on its upper surface or we can say weighing machine measure normal force on the man.



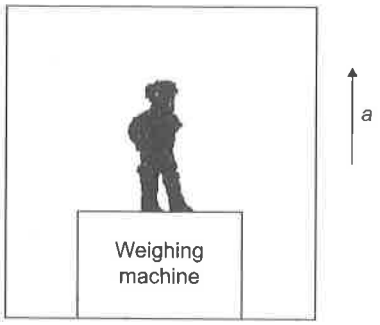
Motion in a Lift

1. If the lift is unaccelerated ($v = 0$ or constant)
   In this case, no pseudo force acts on the man.
   In this case, the F.B.D. of the man

   $$N = mg$$

   In this case, machine reads the actual weight.

2. If the lift is accelerated upwards (where a = constant).



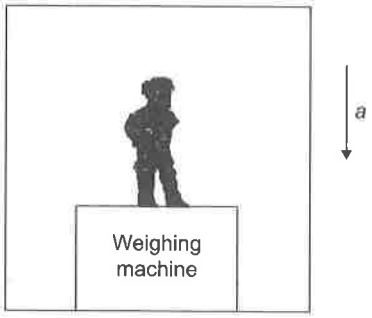
F.B.D of man with respect to lift
So weighing machine reads

$$N = m(g + a)$$

Apparent weight

$N >$ Actual weight (mg)

3. If the lift is accelerated downwards.



F.B.D of man with respect to lift
So weighing machine reads

$$N = m(g - a)$$

Apparent weight

$N <$ Actual weight (mg)

Notes

1. If $a = g \Rightarrow N = 0$
   Thus, in a freely falling lift, the man will experience a state of weightlessness.
2. If the lift is accelerated downwards such that $a > g$, then the man will be accelerated upwards and will stay at the ceiling of the lift.
3. Apparent weight is greater than or less than actual weight only depends on the direction and magnitude of acceleration. Magnitude and direction of velocity does not play any role in apparent weight.

FRICTION

Friction is a contact force that opposes the relative motion or tendency of relative motion of two bodies.

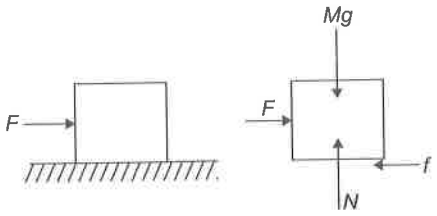


Figure 4.24

Consider a block on a horizontal table as shown in Fig. 4.23. If we apply a force, acting to the right, the block remains stationary if F is not too large. The force that counteracts F and keeps the block in rest from moving is called frictional force. If we keep on increasing the force, the block will remain at rest and for a particular value of applied force, the body comes to state of about to move. Now if we slightly increase the force from this value, block starts its motion with a jerk and we observe that to keep the block moving we need less effort than to start its motion.

So from this observation, we see that we have three states of block: first, block does not move; second, block is about to move; and third, block starts moving. The friction forces acting in three states are called static frictional force, limiting frictional force and kinetic frictional force, respectively. If we draw the graph between applied force and frictional force for this observation, its nature is as shown in the figure.

Static Frictional Force

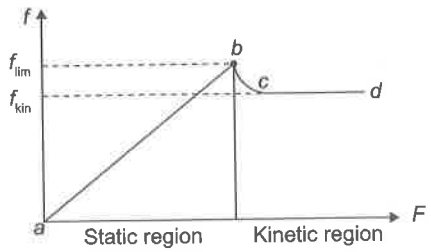


Figure 4.25

When there is no relative motion between the contact surfaces, frictional force is called static frictional force. It is a self-adjusting force; it adjusts its value according to requirement (of no relative motion). In the taken example, static frictional force is equal to applied force. Hence, one can say that the portion of the graph *ab* will have a slope of 45°.

Direction of Static Friction

The direction of static friction on a body is such that the total force acting on it keeps it at rest with respect to the body in contact.

The direction of static friction is as follows. For a moment, consider the surfaces to be frictionless. In the absence of friction, the bodies will start slipping against each other. One should then find the direction of friction as opposite to the velocity with respect to the body applying the friction.

Limiting Frictional Force

This frictional force acts when body is about to move. This is the maximum frictional force that can exist at the contact surface.

The magnitude of limiting frictional force is proportional to the normal force at the contact surface.

$$f_{lim} \propto N$$

$$\Rightarrow \quad f_{lim} = \mu_s N$$

Here, μ_s is a constant, the value of which depends on nature of surfaces in contact, and is called 'coefficient of static friction'.

Kinetic Frictional Force

Once relative motion starts between the surfaces in contact, the frictional force is called kinetic frictional force. The magnitude of kinetic frictional force is also proportional to normal force.

$$f_k = \mu_k N$$

From the previous observation, we can say that $\mu_k < \mu_s$.

Although the coefficient of kinetic friction varies with speed, we shall neglect any variation, i.e., when relative motion starts a constant frictional force starts opposing its motion.

Direction of Kinetic Friction

The kinetic friction on a body A slipping against another body B is opposite to the velocity of A with respect to B.

It should be carefully noted that the velocity coming into picture is with respect to the body applying the force of friction.

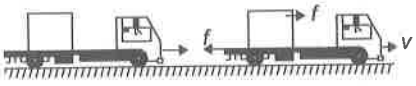


Figure 4.26

Suppose we have a long truck moving on a horizontal road. A small block is placed on the truck which slips on the truck to fall from the rear end. As seen from the road, both the truck and the block are moving towards right; of course, the velocity of the block is smaller than that of the truck. What is the direction of the kinetic friction acting on the block due to the truck? The velocity of the block as seen from the truck is towards left. Thus, the friction on the block is towards right. The friction acting on the truck due to the block is towards left.

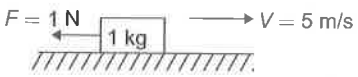
SOLVED EXAMPLES

EXAMPLE 1

Find the direction of kinetic friction force:

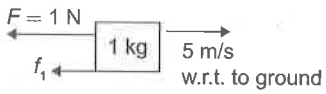
(a) on the block, exerted by the ground.

(b) on the ground, exerted by the block.

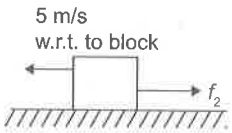


SOLUTION

(a)



(b)



where f_1 and f_2 are the friction forces on the block and ground, respectively. ■

EXAMPLE 2

The correct relation between magnitude of f_1 and f_2 in the above problem is:

(A) $f_1 > f_2$, **(B)** $f_2 > f_1$,

(C) $f_1 = f_2$, **(D)** not possible to decide due to insufficient data.

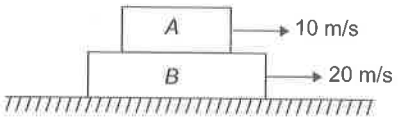
SOLUTION

By Newton's third law, the above frictional forces are action-reaction pair and equal but opposite to each other in direction. Hence, (C).

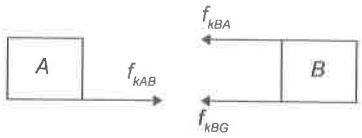
Also note that the direction of kinetic friction has nothing to do with applied force F. ■

EXAMPLE 3

All surfaces are rough. Draw the friction force on A and B.



SOLUTION



Kinetic friction acts to reduce relative motion. ■

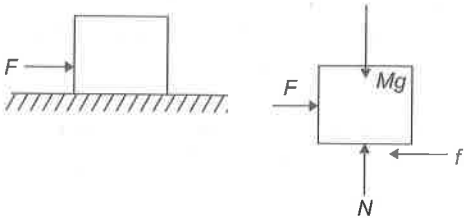
Summary

We can summarize the laws of friction between two bodies in contact as follows:

1. If the bodies slip over each other, the force of friction is given by
$$f_k = \mu_k N,$$
where N is the normal contact force and μ_k is the coefficient of kinetic friction between the surfaces.
2. The direction of kinetic friction on a body is opposite to the velocity of this body with respect to the body applying the force of friction.
3. If the bodies do not slip over each other, the force of friction is given by
$$f_s \leq \mu_s N,$$
where μ_s is the coefficient of static friction between the bodies and N is the normal force between them. The direction and magnitude of static friction are such that the condition of no slipping between the bodies is ensured.
4. The frictional force f_k or f_s does not depend on the area of contact as long as the normal force N is same.

EXAMPLE 4

A block of mass 5 kg is resting on a rough surface as shown in the figure. It is acted upon by a force of F towards right. Find frictional force acting on the block when (a) $F = 5$ N, (b) 25 N, (c) 50 N ($\mu_s = 0.6$, $\mu_k = 0.5$) [$g = 10\ \text{ms}^{-2}$].



SOLUTION

The maximum value of frictional force that the surface can offer is

$$f_{max} = f_{lim} = \mu_s N$$

$$= 0.6 \times 5 \times 10 = 30 \text{ N}$$

Therefore, if $F \leq f_{max}$ body will be at rest and $f = F$ or $F > f_{max}$ body will move and $f = f_k$

(a) $F = 5 \text{ N} < F_{max}$

So body will not move; hence, static frictional force will act and $f_s = f = 5$ N.

(b) $F = 25 \text{ N} < F_{max}$

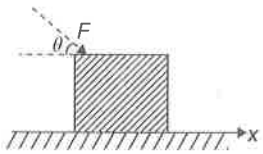
$f_s = 25$ N

(c) $F = 50 \text{ N} > F_{max}$

So body will move and kinetic frictional force will act; its value will be $f_k = \mu_k N = 0.5 \times 5 \times 10 = 25$ N. ■

EXAMPLE 5

A block having a mass 3 kg is initially at rest on a horizontal surface. The coefficient of static friction $\mu_s = 0.3$ between the block and the surface and μ_k is 0.25. A constant force F of 50 N, acts on the body at the angle $\theta = 37°$. What is the acceleration of the block?



SOLUTION

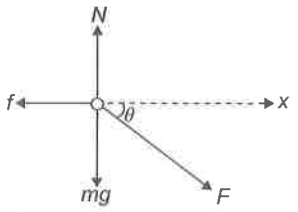
We have two possibilities here; the block may remain at rest, or it may accelerate towards the right. The decision hinges on whether or not the x-component of the force F has magnitude, less than or greater than the maximum static friction force.

The x-component of F is

$$F_x = F \cos\theta$$

$$= (50 \text{ N})\,(0.8) = 40 \text{ N}$$

To find $f_{s,\,max}$, we first calculate the normal force N, whether or not the block accelerates horizontally, the sum of the y-component of all the forces on the block is zero.



$$N - F \sin\theta - mg = 0$$

or $$N = F \sin\theta + mg$$

$$= (50 \text{ N})\,(0.6) + (3 \text{ kg})(9.8 \text{ ms}^{-2})$$

$$= 59.4 \text{ N}$$

The maximum static frictional force

$$f_{s,\,max} = \mu_s N$$

$$= (0.3)\,(59.4 \text{ N}) = 17.8 \text{ N}$$

This value is smaller than the x-component of F; hence, the block moves. We now interpret the force f in the figure as a kinetic frictional force. This value is obtained as

$$f_k = \mu_k N$$

$$= (0.25)\,(59.4 \text{ N}) = 14.8 \text{ N}$$

Therefore, the resultant force in the x-direction is

$$\Sigma F_x = F\cos\theta - f$$

$$= 40 \text{ N} - 14.8 \text{ N} = 25.2 \text{ N}$$

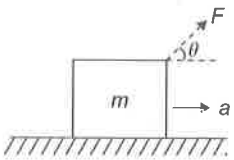
Then the acceleration 'a' of the block is

$$a = \frac{25.2 \text{ N}}{3 \text{ kg}} = 8.4 \text{ ms}^{-2}$$ ■

Think: What would happen if the magnitude of F_x happened to be less than $f_{s,\,max}$ but larger than f_k?

EXAMPLE 6

In the previous example, suppose we move the block by pulling it with the help of a massless string tied to the block as shown here. What is the force F required to produce the same acceleration in the block as obtained in the last example?



SOLUTION

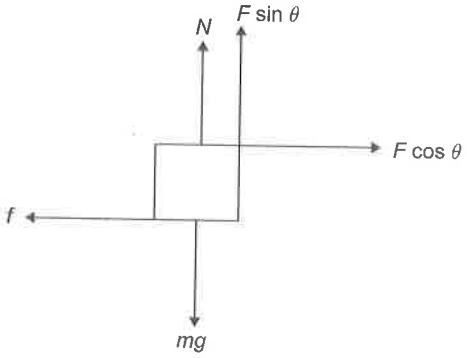
We are given that

$$m = 3 \text{ kg}, \mu_s = 0.3,$$

$$\mu_k = 0.25, \theta = 37°$$

and $$a = 8.4 \text{ ms}^{-2}$$

In order to determine the force F, we first draw the F.B.D. as shown below:



The equations of motion, therefore, are

$$N + F \sin\theta = mg$$

$$N = mg - F \sin\theta$$

$$F\cos\theta - f = ma$$

and where $f = \mu_s$ N before the start of the motion, once motion is set, $f = \mu_k$N.

Hence, force F which produces $a = 8.4$ m/s² is given by

$$F\cos\theta - \mu_k(mg - F\sin\theta) = ma$$

or

$$F = \frac{ma + \mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

$$= \frac{3(0.4 + 0.25 \times 9.8)}{0.8 + 0.25 \times 0.6}$$

$$= 34.26 \text{ N}$$ ■

Note

$F \sin\theta$ works out to be less than mg. Otherwise, we would lift the block up in the above analysis.

Comment

It is easier to pull than to push. Only about 34 N force is required to pull than 50 N required during pushing, why?

Because, when we pull at an angle, the effective normal force N by which block is pressing down on surface is reduced and consequently friction is reduced. Just the contrary happens when you are pushing.

MINIMUM FORCE REQUIRED TO MOVE THE PARTICLE

A body of mass m rests on a horizontal floor with which it has a coefficient of static friction μ. It is desired to make the body slide by applying the minimum possible force F.

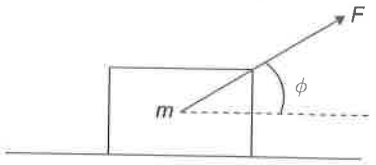


Figure 4.27

Let the applied force F be at angle ϕ with the horizontal.

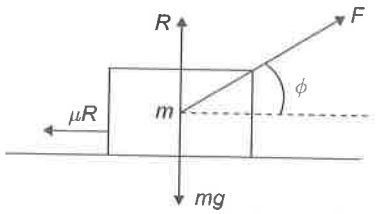


Figure 4.28

R = Normal force

For vertical equilibrium,

$$R + F\sin\phi = mg$$

or, $$R = (mg - F\sin\phi) \quad (1)$$

For horizontal equilibrium, i.e., when the block is just about to slide,

$$F\cos\phi = \mu R \quad (2)$$

Substituting for R,

$$F\cos\phi = \mu(mg - F\sin\phi)$$

or $$F = \mu\, mg/(\cos\phi + \mu\sin\phi)$$

For minimum F $(\cos\phi + \mu\sin\phi)$ is maximum.

⇒ Let $$x = \cos\phi + \mu\sin\phi$$

$$\frac{dx}{d\phi} = -\sin\phi + \mu\cos\theta$$

For maximum of x,

$$\frac{dx}{d\phi} = 0$$

$\tan\phi = \mu$ and at this value of ϕ

$$F_{min} = \frac{\mu mg}{\sqrt{1+\mu^2}}$$

FRICTION AS THE COMPONENT OF CONTACT FORCE

When two bodies are kept in contact, electromagnetic forces act between the charged particles at the surfaces of the bodies. As a result, each body exerts a contact force on other. The magnitudes of the contact forces acting on the two bodies are equal but their directions are opposite and hence the contact forces obey Newton's third law.

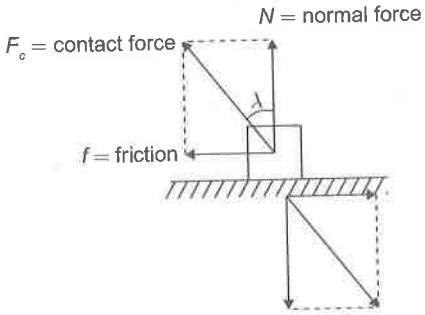


Figure 4.29

The direction of the contact force acting on a particular body is not necessarily perpendicular to the contact surface. We can resolve this contact force into two components: one perpendicular to the contact surface and the other parallel to it. The perpendicular component is called the normal contact force or normal force and the parallel component is called friction.

$$\text{Contact force} = \sqrt{f^2 + N^2}$$

$$F_{c\,min} = N$$

$$(\text{when } f_{min} = 0)$$

$$F_{c\,max} = \sqrt{\mu^2 N^2 + N^2}$$

$$(\text{when } f_{max} = \mu N)$$

$$N \le F_c \le \sqrt{(\mu^2+1)}N$$

$$0 \le \lambda \le \tan^{-1}\mu$$

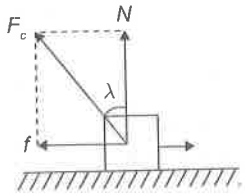
SOLVED EXAMPLE

EXAMPLE 7

A body of mass 400 g slides on a rough horizontal surface. If the frictional force is 3.0 N, find (a) the angle made by the contact force on the body with the vertical and (b) the magnitude of the contact force. Take $g = 10\ m/s^2$.

SOLUTION

Let the contact force on the block by the surface be F_c which makes an angle λ with the vertical (shown in the figure).



The component of F_c perpendicular to the contact surface is the normal force N and the component of F parallel to the surface is the friction f. As the surface is horizontal, N is vertically upward. For vertical equilibrium,

$$N = mg$$

$$= (0.400\ kg)\ (10\ m/s^2)$$

$$= 4.0\ N$$

The frictional force is

$$f = 3.0\ N$$

(a)
$$\tan\lambda = \frac{f}{N} = \frac{3}{4}$$

or,
$$\lambda = \tan^{-1}(3/4) = 37°$$

(b) The magnitude of the contact force is

$$F = \sqrt{N^2 + f^2}$$

$$= \sqrt{(4.0\,N)^2 + (3.0\,N)^2}$$

$$= 5.0\,N$$

■

MOTION ON A ROUGH INCLINED PLANE

Suppose a motion up the plane takes place under the action of pull P acting parallel to the plane

$$N = mg\cos\alpha$$

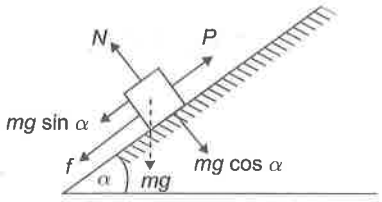


Figure 4.30

Frictional force acting down the plane

$$F = \mu N$$

$$= \mu\, mg \cos \alpha$$

Appling Newton's second law for motion up the plane

$$P - (mg \sin \alpha + f) = ma$$

$$P - mg \sin \alpha - \mu\, mg \cos \alpha = ma$$

If $P = 0$, the block may slide downwards with an acceleration a. The frictional force would then act up the plane

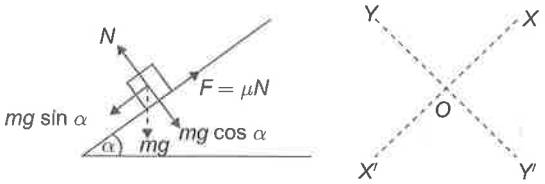
$$mg \sin \alpha - F = ma$$

or,
$$mg \sin \alpha - \mu\, mg \cos \alpha = ma$$

SOLVED EXAMPLES

EXAMPLE 8

A 20 kg box is gently placed on a rough inclined plane of inclination 30° with horizontal. The coefficient of sliding friction between the box and the plane is 0.4. Find the acceleration of the box down the incline.



SOLUTION

In solving inclined plane problems, the X and Y directions along which the forces are to be considered, may be taken as shown. The components of weight of the box are

(a) $mg \sin \alpha$ acting down the plane and
(b) $mg \cos \alpha$ acting perpendicular to the plane.

$$N = mg \cos \alpha$$

$$mg \sin \alpha - \mu N = ma$$

$$\Rightarrow \quad mg \sin \alpha - \mu\, mg \cos \alpha = ma$$

$$a = g \sin \alpha - \mu g \cos \alpha$$

$$= g\,(\sin \alpha - \mu \cos \alpha)$$

$$= 9.8\left(\frac{1}{2} - 0.4 \times \frac{\sqrt{3}}{2}\right)$$

$$= 4.9 \times 0.3072$$

$$= 1.505 \text{ m/s}^2$$

The box accelerates down the plane at 1.505 m/s². ■

EXAMPLE 9

A force of 400 N acting horizontal pushes up a 20 kg block placed on a rough inclined plane which makes an angle of 45° with the horizontal. The acceleration experienced by the block is 0.6 m/s². Find the coefficient of sliding friction between the box and the incline.

SOLUTION

The horizontally directed force 400 N and weight 20 kg of the block are resolved into two mutually perpendicular components, parallel and perpendicular to the plane as shown.

$$N = 20g \cos 45° + 400 \sin 45°$$

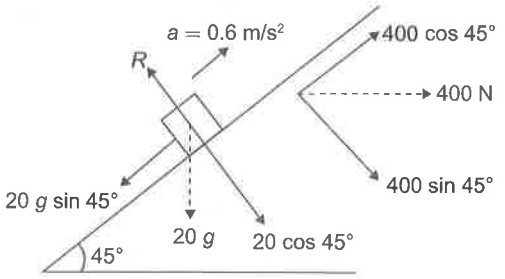
$$= 421.4 \text{ N}$$

The frictional force experienced by the block

$$F = \mu N$$

$$= \mu \times 421.4$$

$$= 421.4\ \mu N$$



As the accelerated motion is taking place up the plane,

$$400 \cos 45° - 20g \sin 45° - f = 20a$$

$$\frac{400}{\sqrt{2}} - \frac{20 \times 9.8}{\sqrt{2}} - 421.4\mu = 20a$$

$$= 20 \times 0.6 = 12$$

$$\mu = \left(\frac{400}{\sqrt{2}} = -\frac{196}{\sqrt{2}} - 12\right) \times \frac{1}{421.4}$$

$$= \frac{282.8 - 138.6 - 12}{421.4} = 0.3137$$

The coefficient of sliding friction between the block and the incline = 0.3137 ■

ANGLE OF REPOSE

Consider a rough inclined plane whose angle of inclination θ with ground can be changed. A block of mass m is resting on the plane. Coefficient of (static) friction between the block and the plane is μ.

For a given angle θ, the F.B.D. (free-body diagram) of the block is

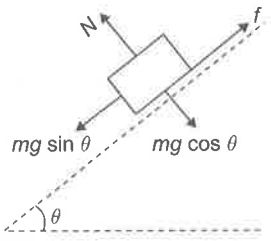


Figure 4.31

where f is the force of static friction on the block. For normal direction to the plane, we have $N = mg \cos \theta$.

As θ increases, the force of gravity down the plane, $mg \sin \theta$, increases. Friction force resists the slide till it attains its maximum value.

$$f_{max} = \mu N = \mu mg \cos \theta$$

which decreases with θ (because $\cos \theta$ decreases as θ increases)

Hence, beyond a critical value $\theta = \theta_c$, the block starts to slide down the plane. The critical angle is the one when $mg \sin \theta$ is just equal of f_{max}, i.e., when

$$mg \sin \theta_c = \mu mg \cos \theta_c$$

or $$\tan \theta_c = \mu$$

where θ_c is called the angle of repose.

If $\theta > \theta_c$, the block will slide down. For $\theta < \theta_c$, the block stays at rest on the incline.

TWO BLOCKS ON AN INCLINED PLANE

Consider two blocks having masses m_1 and m_2 placed on a rough inclined plane. μ_1 and μ_2 are the friction coefficients for m_1 and m_2, respectively. If N is the normal force between the contact surfaces of m_1 and m_2,

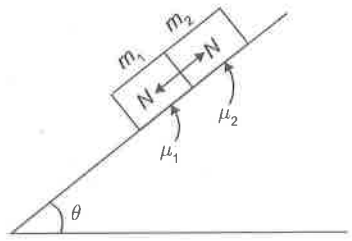


Figure 4.32

Now three conditions arise.

(a) If $\mu_1 = \mu_2 = \mu$, then
$N = 0$, because both the blocks are in contact but does not press each other.

$$a_1 = a_2 = g \sin \theta - \mu mg \cos \theta$$

(a_1, a_2 are the acceleration of blocks μ_1 and μ_2, respectively)

(b) If $\mu_1 < \mu_2$, then
$N = 0$, because there is no contact between the blocks.

$$a_1 = g \sin \theta - \mu_1 g \cos \theta$$

$$a_2 = g \sin \theta - \mu_2 g \cos \theta$$

$$\Rightarrow \quad a_1 > a_2$$

(c) If $\mu_1 > \mu_2$, then

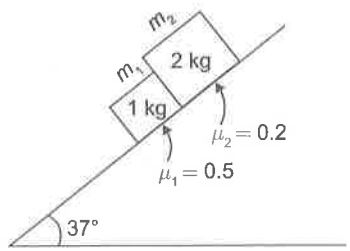
$$N \neq 0$$

$$a_1 = a_2$$

SOLVED EXAMPLES

EXAMPLE 10

Masses m_1 and m_2 are placed on a rough inclined plane as shown in the figure. Find out the acceleration of the blocks and contact force in between these surfaces.

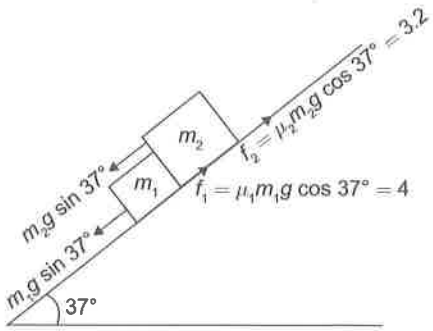


SOLUTION

As we know if $\mu_1 > \mu_2$ both will travel together so

$$a_1 = a_2 = a$$

F.B.D

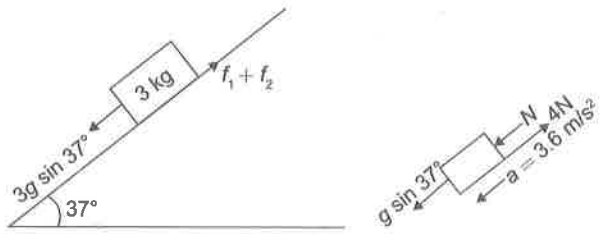


which is equivalent to

$$a = \frac{3g \sin 37° - (f_1 + f_2)}{3}$$

$$a = \frac{18 - 7.2}{2} = 3.6 \text{ m/s}^2$$

Now F.B.D of 1 kg block is



$$g \sin 37° + N - 4 = (1)\, a$$

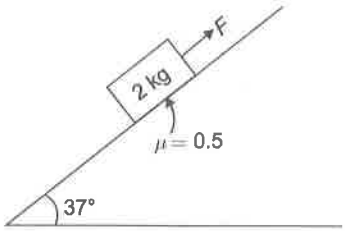
$$N = 3.6 + 4 - 6 = 1.6 \text{ N}$$

RANGE OF FORCE *F* FOR WHICH ACCELERATION OF BODY IS ZERO

SOLVED EXAMPLES

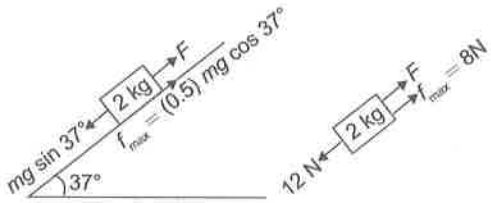
EXAMPLE 11

Find out the range of force in the above situation for which 2 kg block does not move on the incline.

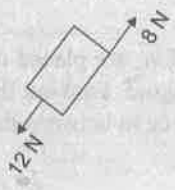
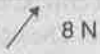
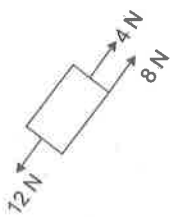


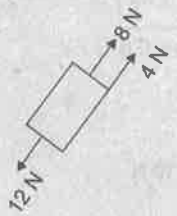
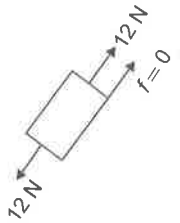
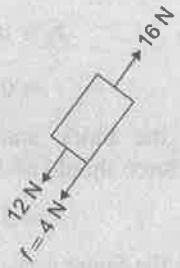
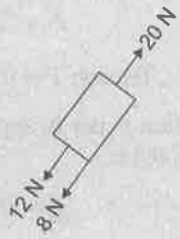
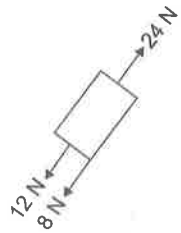
SOLUTION

F.B.D of 2 kg block



Now take different value of F

Force (F)	F.B.D.	Acceleration $a = \frac{F_{net}}{m}$	Direction and magnitude	Friction type
(i) $F = 0$ N		2 m/s²	 8 N	Kinetic
(ii) $F = 4$ N		0 m/s²	8 N	Static

Force (F)	F.B.D.	Acceleration $a = \frac{F_{net}}{m}$	Direction and magnitude	Friction type
(iii) $F = 8$ N		0 m/s²	4 N	Static
(iv) $F = 12$ N		0 m/s²	0	Static
(v) $F = 16$ N		0 m/s²	4 N	Static
(in this condition friction changes its direction to stop relative slipping)				
(vi) $F = 20$ N		0 m/s²	8 N	Static
(vii) $F = 24$ N		2 m/s²	8 N	Kinetic

From the above table, block does not move from $F = 4$ N ($mg \sin\theta - \mu mg \cos\theta$) to $F = 20$ N ($mg \sin\theta + \mu mg \cos\theta$). So friction develops a range of force for which block does not move. ■

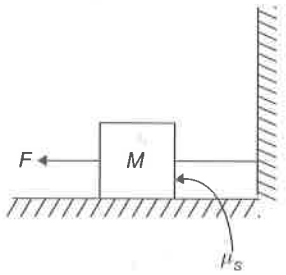
Note

If friction is not present, then only for $F = 12$ N the block will not move but friction develop a range of force 4 N to 20 N to prevent slipping. So we can write the range of force F for which acceleration of the body is zero.

$mg \sin\theta - \mu mg \cos\theta \leq F \leq mg \sin\theta + \mu mg \cos\theta$

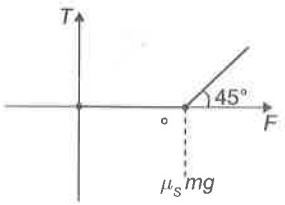
EXAMPLE 12

In the following figure, force F is gradually increased from zero. Draw the graph between applied force F and tension T in the string. The coefficient of static friction between the block and the ground is μ_s.



SOLUTION

As the external force F is gradually increased from zero, it is compensated by the friction and the string bears no tension. When limiting friction is achieved by increasing force F to a value till $\mu_s mg$, the further increase in F is transferred to the string.

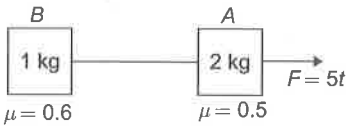


■

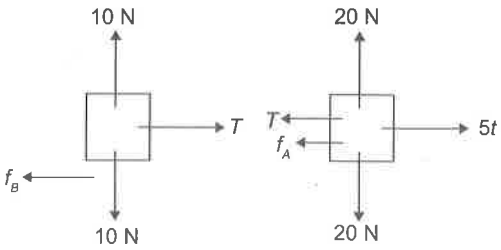
EXAMPLE 13

Figure shows two blocks tied by a string. A variable force $F = 5t$ is applied on the block. The coefficients of friction for the blocks are 0.6 and 0.5, respectively. Find the frictional force between blocks and ground as well as tension in the string at

(a) $t = 1$ s, **(b)** $t = 2$ s, **(c)** $t = 3$ s



SOLUTION



(a) At $t = 1$ s, $F = 5 \times 1 = 5$ N
Maximum value of friction force

$$f_A = \mu N$$

$$= 0.5 \times 20 = 10 \text{ N}$$

To keep the block stationary, the magnitude of frictional force should be 5 N. So

$$f_A = 5 \text{ N}$$

Now from the figure it is clear that if

$$f_A = 5 \text{ N}$$

and $F = 5$ N,

then Tension $T = 0$

Since tension is not in application, frictional force on Block B is 0, i.e.,

$$f_B = 0$$

(b) At $t = 2$ s, $F = 5 \times 2 = 10$ N
Maximum value of friction force

$$f = \mu N$$

$$= 0.5 \times 20 = 10 \text{ N}$$

To keep the block stationary, the magnitude of friction force should be 10 N. So

$$f_A = 10 \text{ N}$$

From the figure, it is clear that if

$$f_A = 10 \text{ N}$$

and $F = 10$ N,

then Tension $T = 0$

Hence friction force on block B is $f_B = 0$

(c) At $t = 3$ s, $F = 5 \times 3 = 15$ N
Maximum value of friction force

$$f = \mu N$$

$$= 0.5 \times 20 = 10 \text{ N}$$

Again applying the same analogy

$$f_A = 10 \text{ N}$$

From the figure, it is clear that if

$$f_A = 10 \text{ N}$$

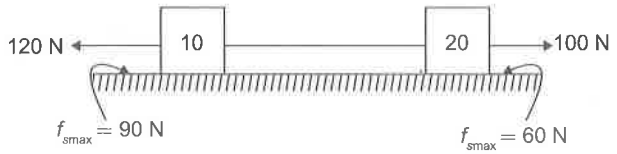
and $F = 15$ N

then Tension $T = 5$ N

So frictional force on Block B is $f_B = 5$ N ■

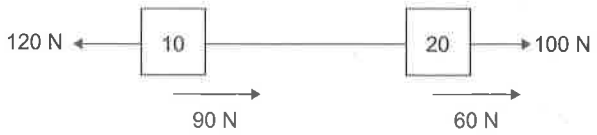
EXAMPLE 14

Find the tension in the string in situation as shown in the figure below. Forces 120 N and 100 N start acting when the system is at rest.



SOLUTION

(a) Let us assume that system moves towards left then as it is clear from F.B.D., net force in horizontal direction is towards right. Therefore, the assumption is not valid.

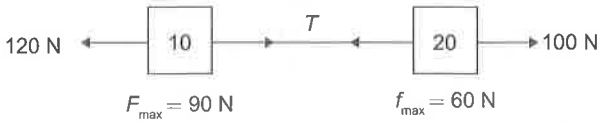


Above assumption is not possible as net force on the system comes towards right. Hence, system is not moving towards left.

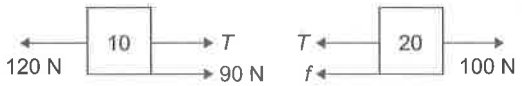
(b) Similarly, let us assume that system moves towards right.

Above assumption is also not possible as net force on the system is towards left in this situation. Hence, the assumption is again not valid.

Therefore, it can be concluded that the system is stationary.



Assuming that the 10 kg block reaches limiting friction first then using F.B.D.



$$120 = T + 90$$

$$\Rightarrow \quad T = 30 \text{ N}$$

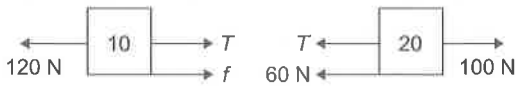
Also $T + f = 100$

$$\therefore \quad 30 + f = 100$$

$$\Rightarrow \quad f = 70 \text{ N}$$

which is not possible as the limiting value is 60 N for this surface of block.

$\therefore$ Our assumption is wrong and now taking the 20 kg surface to be limiting, we have



$$T + 60 = 100 \text{ N}$$

$$\Rightarrow \quad T = 40 \text{ N}$$

Also $f + T = 120$ N

$$\Rightarrow \quad f = 80 \text{ N}$$

This is acceptable as static friction at this surface should be less than 90 N.

Hence, the tension in the string is $T = 40$ N. ■

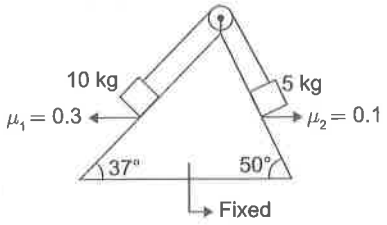
PULLEY BLOCK SYSTEM INVOLVING FRICTION

If friction force is acting and value of acceleration of a particle is negative, then it means direction of friction force is opposite to that what we assumed and acceleration would be having a different numerical value.

SOLVED EXAMPLES

EXAMPLE 15

Two blocks of masses 5 kg and 10 kg are attached with the help of light string and placed on a rough incline as shown in the figure. Coefficients of friction are as marked in the figure. The system is released from rest. Determine the acceleration of the two blocks.



SOLUTION

Let 10 kg block is sliding down, then acceleration of both the blocks are given by

$$a = \frac{10g\sin 37° - \mu_1 \times 10g\cos 37° - 5g\sin 53° - \mu_2 \times 5g\cos 53°}{15}$$

$$= -\text{ve}$$

It means our assumed direction of motion is wrong and 5 kg block is going to slide down, if this would be the case, the direction of friction force will reverse and acceleration of blocks would be given by

$$a_1 = \frac{5g\sin 53° - \mu_2 \times 5g\cos 53° - \mu_1 \times 10g\cos 37° - 10g\sin 37°}{15}$$

$$= -\text{ve}$$

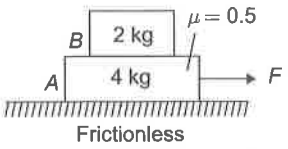
It means in this direction also there is no motion. So we can conclude that the system remains at rest and friction force is static in nature. ■

TWO-BLOCK SYSTEM

SOLVED EXAMPLES

EXAMPLE 16

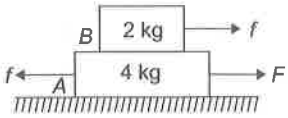
Find out the maximum value of F for which both the blocks will move together.



SOLUTION

In the given situation, 2 kg block will move only due to friction force exerted by the 4 kg block

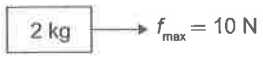
F.B.D.



The maximum friction force exerted on the Block B is

$$f_{max} = \mu N$$

$$f_{max} = (0.5)(20) = 10 \text{ N}$$

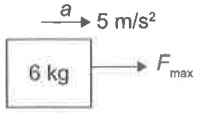


So the maximum acceleration of 2 kg block is

$$a_{2_{max}} = \frac{10}{2} = 5 \text{ m/s}^2$$

a_{max} is the maximum acceleration for which both the block will move together, i.e., for $a \leq 5 \text{ ms}^{-2}$ acceleration of both blocks will be same and we can take both the blocks as a system.

F.B.D.



$$F_{max} = 6 \times 5 = 30 \text{ N}$$

for $0 < F < 30$

Both the block move together. ■

EXAMPLE 17

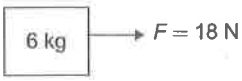
In the above question, find the acceleration of both the block when

(a) $F = 18$ N, **(b)** $F = 36$ N

SOLUTION

(a) Since $F < 30$ both the blocks will move together

F.B.D.

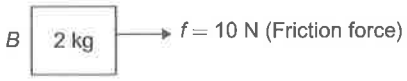


$$a = \frac{18}{6} = 3 \text{ m/s}^2$$

(b) When $F = 36$ N

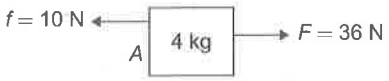
When $F > 30$ both the blocks will move separately so we treat each block independently.

F.B.D of 2 kg block



$$a_B = 5 \text{ m/s}^2$$

F.B.D of 4 kg block



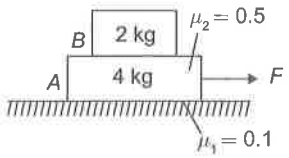
$$a_A = \frac{36-10}{4}$$

$$= \frac{26}{4} \text{ m/s}^2$$

■

EXAMPLE 18

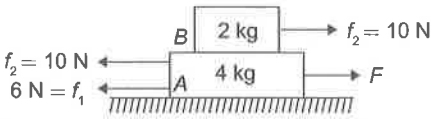
Find out the range of force in which both the blocks move together.



SOLUTION

If f_1 is friction force between Block A and lower surface and f_2 is friction force between both the blocks' surface.

F.B.D.



$$f_{1\,max} = \mu_1 N_1$$

$$= (0.1)\,(60) = 6 \text{ N}$$

$$f_{2\,max} = \mu_2 N_2$$

$$= (0.5)\,(20) = 10 \text{ N}$$

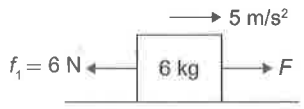
Upper 2 kg block moves only due to friction force; so the maximum acceleration of that block is

$$a_{max} = \frac{10}{2} = 5 \text{ m/s}^2$$

This is the maximum acceleration for which both the blocks will move together.

Therefore, for $a \leq 5 \text{ ms}^{-2}$ we can take both the blocks as one system.

F.B.D.



For $F < 6$ N, blocks will not move at all.

Now the value of F_{max} for which both the blocks will move together

$$F_{max} - 6 = 6 \times 5$$

$$F_{max} = 36 \text{ N}$$

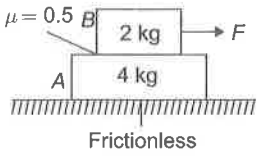
Conclusion:

$0 \text{ N} < F < 6 \text{ N}$ = No blocks will move

$6 \text{ N} < F < 36 \text{ N}$ = Both blocks will move together

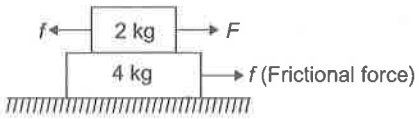
$F > 36$ N = Both move separately ■

EXAMPLE 19



The lower block A will move only due to friction force

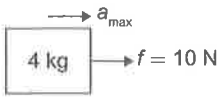
F.B.D.



$$f_{max} = \mu N$$

$$= (0.5)\,(20) = 10 \text{ N}$$

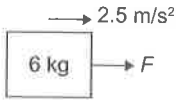
F.B.D. of 4 kg block



The maximum acceleration of 4 kg block is

$$\Rightarrow \quad a_{max} = \frac{10}{4} = 2.5 \text{ m/s}^2$$

This is the maximum acceleration for which both the blocks move together

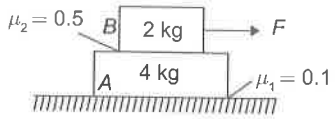


F_{max} for which both the blocks will move together

$$F_{max} = 2.5 \times 6 = 15 \text{ N}$$

■

EXAMPLE 20



If f_2 is the friction force between A and B and f_1 is the friction force between A and floor

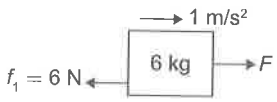
$$f_{1\,max} = 6 \text{ N}$$

$$f_{2\,max} = 10 \text{ N}$$

Lower Block A will move only due to friction force. So a_{max} for 4 kg block

$$a_{max} = \frac{10-6}{4} = 1 \text{ m/s}^2$$

This is the maximum acceleration for which both the blocks will move together



$$F - 6 = 6 \times 1$$

$$F = 12 \text{ N}$$

If F is less than 6 N both the blocks will be stationary

Conclusion:

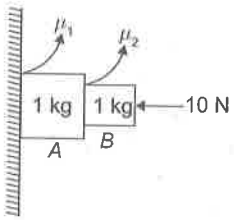
$0 < F < 6$ N = Both blocks are stationary

6 N $< F <$ 12 N = Both move together

$F >$ 12 N = Both move separately ■

EXAMPLE 21

Find the accelerations of Blocks A and B for the following cases.



(A) $\mu_1 = 0$ and $\mu_2 = 0.1$

(B) $\mu_2 = 0$ and $\mu_1 = 0.1$

(C) $\mu_1 = 0.1$ and $\mu_2 = 1.0$

(D) $\mu_1 = 1.0$ and $\mu_2 = 0.1$

(P) $a_A = a_B = 9.5$ m/s²

(Q) $a_A = 9$ m/s², $a_B = 10$ m/s²

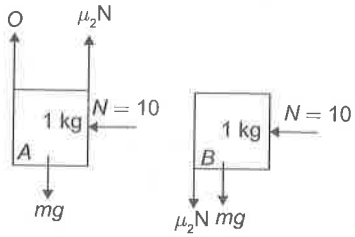
(R) $a_A = a_B = g = 10$ m/s²

(S) $a_A = 1$, $a_B = 9$ m/s²

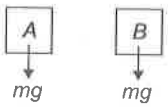
SOLUTION

(A) R, (B) Q, (C) P, (D) S

(i) F.B.D. in (case (i))
($\mu_1 = 0$, $\mu_2 = 0.1$)

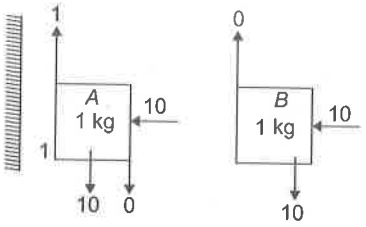


While friction's work is to oppose the relative motion and here if relative motion will start then friction comes and without relative motion there is no friction so both the blocks move together with same acceleration and friction will not come.



$\Rightarrow$ $a_A = a_B = 10$ m/s²

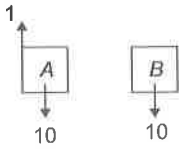
(ii)



Friction between wall and Block A opposes relative motion since wall is stationary so friction wants to stop Block A also and maximum friction will act between wall and block while there is no friction between blocks.

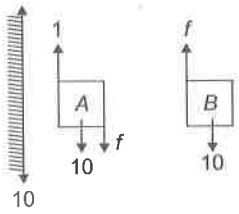
Note

Friction between wall and block will oppose relative motion between wall and block only it will not do anything for two-block motion.

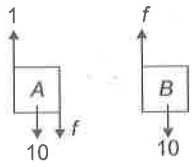


$a_A = 9\ \text{m/s}^2;\ a_B = 10\ \text{m/s}^2$

(iii)



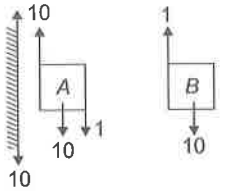
Friction between wall and block will be applied maximum equal to 1 N but maximum friction available between Blocks A and B is 10 N but if this will be there then relative motion will increase while friction is to oppose relative motion. So friction will come less than 10 so friction will be f that will be static.



by system $(20 - 1) = 2 \times a$

$\Rightarrow \quad a = \dfrac{19}{2} = 9.5\ \text{m/s}^2$

(iv) $a_A = \dfrac{11-10}{1} = 1\ \text{m/s}^2$



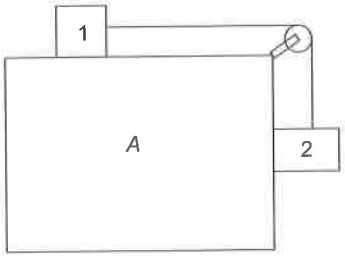
$a_B = \dfrac{10-1}{1} = 9\ \text{m/s}^2$ ■

FRICTION INVOLVING PSEUDO CONCEPT

SOLVED EXAMPLES

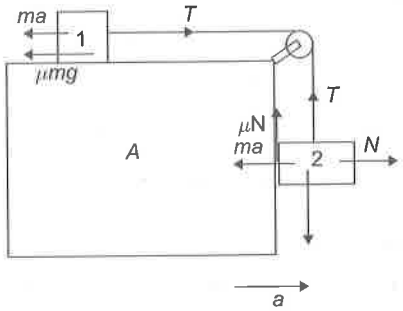
EXAMPLE 22

What is the minimum acceleration with which bar A should be shifted horizontally to keep the bodies 1 and 2 stationary relative to the bar? The masses of the bodies are equal and the coefficient of friction between the bar and the bodies is equal to μ. The masses of the pulley and the threads are negligible whereas the friction in the pulley is absent (see the figure).



SOLUTION

Let us place the observer on A.



Since we have non-inertial frame, we have pseudo forces.

For body '1', we have

$$T = ma + \mu mg \qquad (1)$$

For body '2', we have

$$N = ma$$

$$mg - T - \mu ma = 0$$

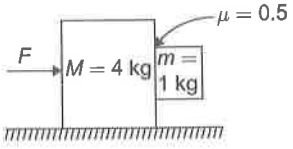
$$\therefore \quad mg = T + \mu mg \qquad (2)$$

From (1) and (2)

$$a_{\min} = g\left(\frac{1-\mu}{1+\mu}\right)$$ ■

EXAMPLE 23

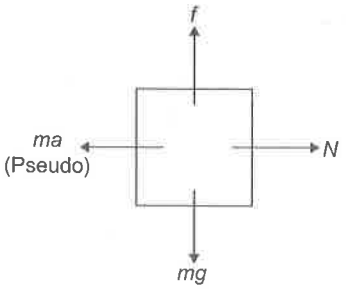
Find out the range of force for which smaller block is at rest with respect to bigger block.



SOLUTION

Smaller block is at rest w.r.t. the bigger block. Let both the blocks travel together with acceleration a.

F.B.D. of smaller block w.r.t. the bigger block



$$f_{max} = \mu \times N$$

$$N = ma$$

$$f = \mu ma \quad (1)$$

$$\Rightarrow \quad f = mg \quad (2)$$

From (1) and (2),

$$a = g/\mu = 20 \text{ m/s}^2$$

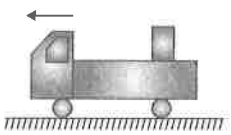
So $$F = 20\,(M + m)$$

$$= 20(5) = 100 \text{ N}$$

If $F \geq 100$ N, both will travel together. ■

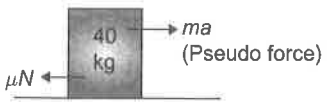
EXAMPLE 24

The rear side of a truck is open and a box of 40 kg mass is placed 5 m away from the open end as shown. The coefficient of friction between the box and the surface below it is 0.15. On a straight road, the truck starts from rest and accelerates with 2 ms^{-2}. At what distance from the starting point does the box fall off the truck (i.e., distance travelled by the truck)? [Ignore the size of the box]



SOLUTION

In the reference frame of the truck, F.B.D. of 40 kg block



Net force

$$\Rightarrow \quad ma - \mu N$$

$$\Rightarrow \quad 40 \times 2 - \frac{15}{100} \times 40 \times 10$$

$$ma_{block} \Rightarrow 80 - 60$$

$$\Rightarrow \quad a_{block} = \frac{20}{40} = \frac{1}{2} \text{ m/s}^2$$

This is the acceleration of the block in reference frame of truck.

So the time taken by the box to fall down from the truck

$$S_{rel} = u_{rel}t + \frac{1}{2}a_{rel}t^2$$

$$\Rightarrow \quad 5 = 0 + \frac{1}{2} \times \frac{1}{2} \times t^2$$

$$\Rightarrow \quad t^2 = 20$$

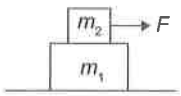
So distance moved by the truck

$$\Rightarrow \quad \frac{1}{2} \times a_{truck} \times t^2$$

$$\Rightarrow \quad \frac{1}{2} \times 2 \times (20) = 20 \text{ m}$$ ■

EXAMPLE 25

Mass m_2 placed on a plank of mass m_1 lying on a smooth horizontal plane. A horizontal force $F = \alpha_0 t$ (α_0 is a constant) is applied to a bar. If accelerations of the plank and the bar are a_1 and a_2, respectively, and the coefficient of friction between m_1 and m_2 is μ. Then find acceleration a with time t.



SOLUTION

If $F < \mu m_2 g$, then both blocks move with common acceleration, i.e., $a_1 = a_2$

When $F > \mu m_2 g$, then

Equation for block of mass m

$$F - \mu m_2 g = m_2 a_2 \quad (1)$$

and $$\mu m_2 g = m_1 a_1 \quad (2)$$

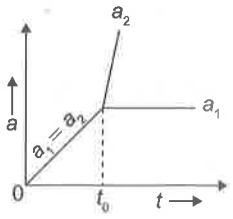
From Eq. (1),

$$\alpha_0 t - \mu m_2 g = m_2 a_2$$

i.e., acceleration a_2 varies with time linearly, its slope positive and intercept negative.

From Eq. (2), a_1 is independent of time.

So, the graph between a and t is as follows.



■

EXERCISES

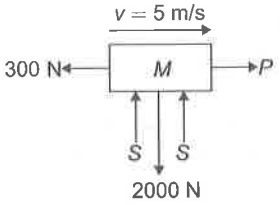
JEE Main

(A) Newton's Law of Motion

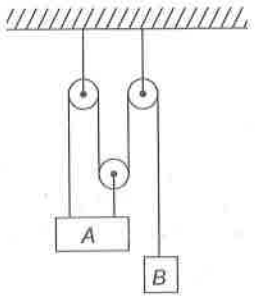
1. A man getting down a running bus, falls forward because
   (A) due to inertia of rest, road is left behind and man reaches forward
   (B) due to inertia of motion, upper part of body continues to be in motion in forward direction while feet comes to rest as soon as they touch the road
   (C) he leans forward as a matter of habit
   (D) of the combined effect of all the three factors stated in (A), (B) and (C)

2. You are on a friction less horizontal plane. How can you get off if no horizontal force is exerted by pushing against the surface?
   (A) by jumping
   (B) by spitting or sneezing
   (C) by rolling your body on the surface
   (D) by running on the plane

3. The forces acting on an object are shown in the figure. If the body moves horizontally at a constant speed of 5 m/s, then the values of the forces P and S are, respectively,
   (A) 0 N, 0 N (B) 300 N, 200 N
   (C) 300 N, 1000 N (D) 2000 N, 300 N

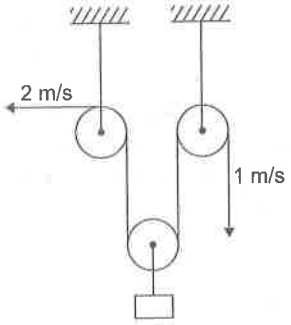


4. At a given instant, A is moving with the velocity of 5 m/s upwards. What is the velocity of B at that time?



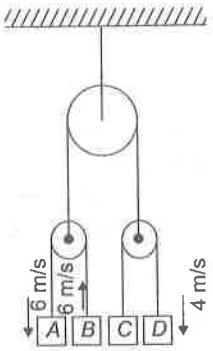
(A) 15 m/s ↓ (B) 15 m/s ↑
(C) 5 m/s ↓ (D) 5 m/s ↑

5. Find the velocity of the hanging block if the velocities of the free ends of the rope are as indicated in the figure.



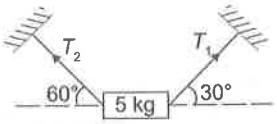
(A) 3/2 m/s ↑ (B) 3/2 m/s ↓
(C) 1/2 m/s ↑ (D) 1/2 m/s ↓

6. In the figure, the velocity of different blocks is shown. The velocity of C is



(A) 6 m/s (B) 4 m/s
(C) 0 m/s (D) none of these

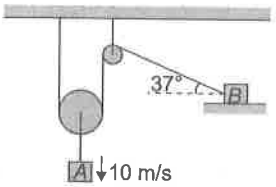
7. A body of mass 5 kg is suspended by the strings making angles 60° and 30° with the horizontal.



(A) $T_1 = 25$ N (B) $T_2 = 25$ N
(C) $T_1 = 25\sqrt{3}$ N (D) $T_2 = 25\sqrt{3}$ N

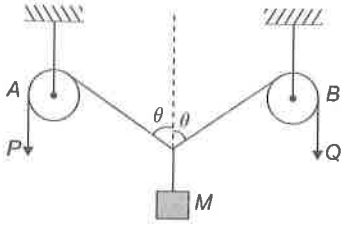
(A) a, b
(B) a, d
(C) c, d
(D) b, c

8. Find the velocity of Block B at the instant shown in the figure.



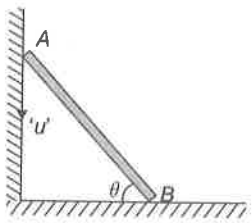
(A) 25 m/s
(B) 20 m/s
(C) 22 m/s
(D) 30 m/s

9. In the arrangement shown in the figure, the ends P and Q of an unstretchable string move downwards with uniform speed U. Pulleys A and B are fixed. Mass M moves upwards with a speed.



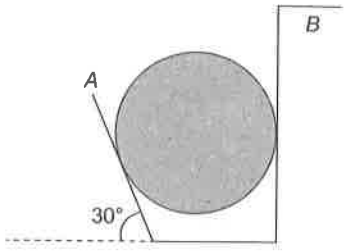
(A) $2U\cos\theta$
(B) $U\cos\theta$
(C) $\dfrac{2U}{\cos\theta}$
(D) $\dfrac{U}{\cos\theta}$

10. The velocity of end 'A' of rigid rod placed between two smooth vertical walls moves with velocity 'u' along the vertical direction. Find out the velocity of end 'B' of that rod, rod always remains in constant with the vertical walls.



(A) $u\tan 2\theta$
(B) $u\cot\theta$
(C) $u\tan\theta$
(D) $2u\tan\theta$

11. The 50 kg homogeneous smooth sphere rests on the 30° incline A and bears against the smooth vertical wall B. Calculate the contact forces at A and B.



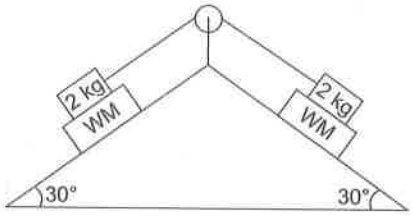
(A) $N_B = \dfrac{1000}{\sqrt{3}}\text{N},\ N_A = \dfrac{500}{\sqrt{3}}\text{N}$

(B) $N_A = \dfrac{1000}{\sqrt{3}}\text{N},\ N_B = \dfrac{500}{\sqrt{3}}\text{N}$

(C) $N_A = \dfrac{1000}{\sqrt{3}}\text{N},\ N_B = \dfrac{500}{\sqrt{3}}\text{N}$

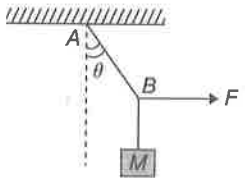
(D) $N_A = \dfrac{1000}{\sqrt{3}}\text{N},\ N_B = \dfrac{50}{\sqrt{3}}\text{N}$

12. Find out the reading of the weighing machine in the following cases.



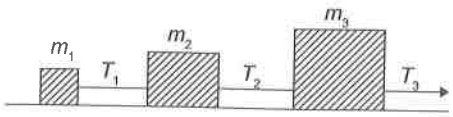
(A) $10\sqrt{3}$
(B) $10\sqrt{2}$
(C) $20\sqrt{3}$
(D) $30\sqrt{3}$

13. A mass M is suspended by a rope from a rigid support at A as shown in the figure. Another rope is tied at the end B, and it is pulled horizontally with a force F. If the rope AB makes an angle θ with the vertical in equilibrium, then the tension in the string AB is



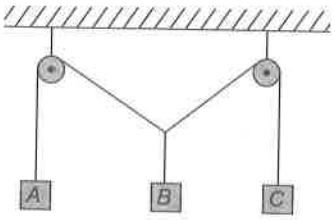
(A) $F\sin\theta$
(B) $F/\sin\theta$
(C) $F\cos\theta$
(D) $F/\cos\theta$

14. Three blocks are connected as shown, on a horizontal frictionless table and pulled to the right with a force $T_3 = 60$ N. If $m_1 = 10$ kg, $m_2 = 20$ kg and $m_3 = 30$ kg, the tension T_2 is



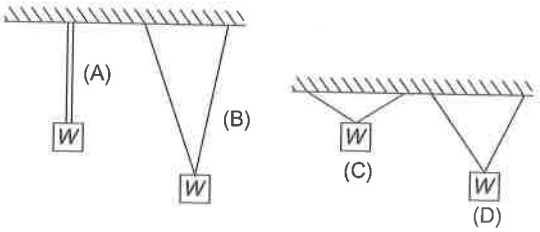
(A) 10 N
(B) 20 N
(C) 30 N
(D) 60 N

15. Three blocks A, B and C are suspended as shown in the figure. Mass of each blocks A and C is m. If system is in equilibrium and mass of B is M, then



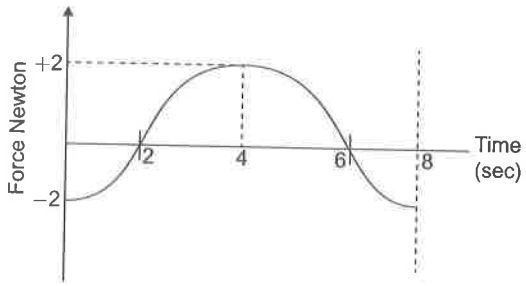
(A) $M = 2$ m
(B) $M < 2$ m
(C) $M > 2$ m
(D) $M =$ m

16. A weight can be hung in any of the following four ways by string of the same type. In which case is the string most likely to break?



(A) A
(B) B
(C) C
(D) D

17. A force–time graph for a linear motion is shown in the figure where the segments are circular. The linear momentum gained between zero and 8 s in



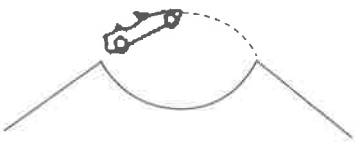
(A) -2π N·s
(B) 0 N.s
(C) 4π N·s
(D) -6π N·s

18. A particle moves in the xy-plane under the action of a force F such that the value of its linear momentum (P) at any time t is, $P_x = 2\cos t$, $P_y = 2\sin t$. The angle θ between P and F at that time t will be

(A) 0°
(B) 30°
(C) 90°
(D) 180°

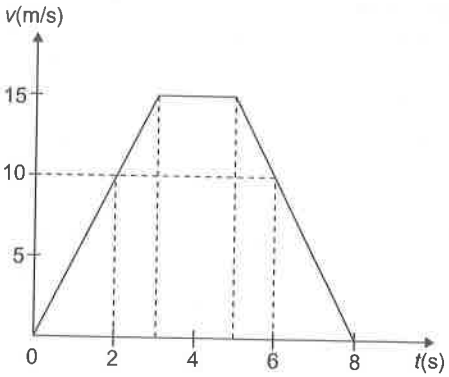
19. A stunt man jumps his car over a crater as shown (neglect air resistance)

(A) during the whole flight, the driver experiences weightlessness
(B) during the whole flight, the driver never experiences weightlessness
(C) during the whole flight, the driver experiences weightlessness only at the highest point



(D) the apparent weight increases during upward journey

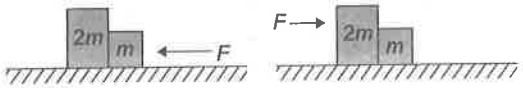
20. A particle of mass 50 g moves on a straight line. The variation of speed with time is shown in the figure. Find the force acting on the particle at $t = 2$, 4 and 6 s.



(A) 0.25 N along motion, zero, 0.25 opposite to motion
(B) 0.25 N along motion, zero, 0.25 along to motion
(C) 0.25 N opposite motion, zero, 0.25 along to motion
(D) 0.25 N opposite motion, zero, 0.25 opposite to motion

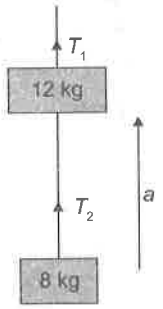
21. Two blocks are in contact on a frictionless table. One has mass m and the other $2m$. A force F is applied on $2m$ as shown in the figure. Now the same force F is applied

from the right on m. In the two cases, respectively, the ratio force of contact between the two blocks will be



(A) same (B) 1:2
(C) 2:1 (D) 1:3

22. A body of mass 8 kg is hanging another body of mass 12 kg. The combination is being pulled by a string with an acceleration of 2.2 ms^{-2}. The tensions T_1 and T_2 will be, respectively (use $g = 9.8$ m/s^2)



(A) 200 N, 80 N (B) 220 N, 90 N
(C) 240 N, 96 N (D) 260 N, 96 N

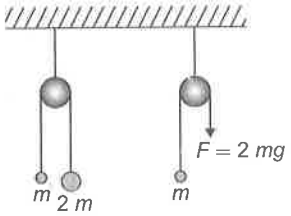
23. A rope of mass 5 kg is moving vertically in the vertical position with an upward force of 100 N acting at the upper end and a downward force of 70 N acting at the lower end. The tension at mid-point of the rope is

(A) 100 N (B) 85 N
(C) 75 N (D) 105 N

24. A particle of small mass m is joined to a very heavy body by a light string passing over a light pulley. Both bodies are free to move. The total downward force in the pulley is

(A) mg (B) $2mg$
(C) $4mg$ (D) cannot be determined

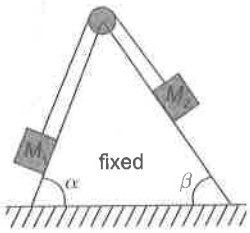
25. The pulley arrangements shown in the figure are identical; the mass of the rope being negligible. In case I, the mass m is lifted by attaching a mass $2m$ to the other end of the rope. In case II, the mass m is lifted by pulling the other end of the rope with constant downward force $F = 2mg$, where g is the acceleration due to gravity. The acceleration of mass in case I is



(I) (II)

(A) zero
(B) more than that in case II
(C) less than that in case II
(D) equal to that in case II

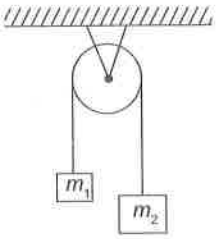
26. Two masses M_1 and M_2 are attached to the ends of a light string which passes over a massless pulley attached to the top of a double inclined smooth plane of angles of inclination α and β. The tension in the string is



(A) $\frac{M_2(\sin\beta)g}{M_1+M_2}$ (B) $\frac{M_1(\sin\alpha)g}{M_1+M_2}$

(C) $\frac{M_1M_2(\sin\beta+\sin\alpha)g}{M_1+M_2}$ (D) zero

27. Two masses are hanging vertically over frictionless pulley. The acceleration of the two masses is

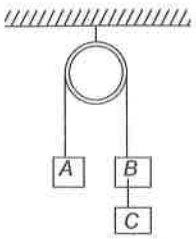


(A) $\frac{m_1}{m_2}g$ (B) $\frac{m_2}{m_1}g$

(C) $\left(\frac{m_2-m_1}{m_1+m_2}\right)g$ (D) $\left(\frac{m_1+m_2}{m_2-m_1}\right)g$

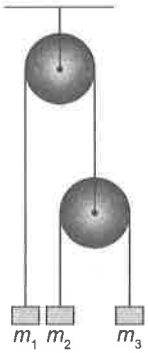
28. Three equal weights A, B, C of mass 2 kg each are hanging on a string passing over a fixed frictionless

pulley as shown in the figure. The tension in the string connecting weights B and C is



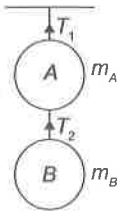
(A) zero
(B) 13 Newton
(C) 3.3 Newton
(D) 19.6 Newton

29. In the arrangement shown in the figure, pulleys are massless and frictionless and threads are inextensible. The block of mass m_1 will remain at rest, if



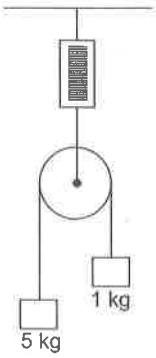
(A) $\frac{1}{m_1}=\frac{1}{m_2}+\frac{1}{m_3}$
(B) $m_1 = m_2 + m_3$
(C) $\frac{4}{m_1}=\frac{1}{m_2}+\frac{1}{m_3}$
(D) $\frac{1}{m_3}=\frac{2}{m_2}+\frac{3}{m_1}$

30. Two objects A and B of masses m_A and m_B are attached by strings as shown in the figure. If they are given upward acceleration, then the ratio of tension $T_1:T_2$ is



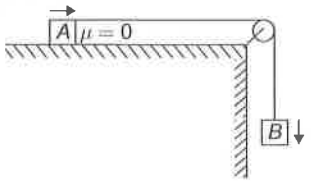
(A) $(m_A + m_B)/m_B$
(B) $(m_A + m_B)/m_A$
(C) $\frac{m_A+m_B}{m_A-m_B}$
(D) $\frac{m_A-m_B}{m_A+m_B}$

31. In the figure, a smooth pulley of negligible weight is suspended by a spring balance. Weights of 1 kg and 5 kg are attached to the opposite ends of a string passing over the pulley and move with acceleration because of gravity. During the motion, the spring balance reads a weight of



(A) 6 kg
(B) less than 6 kg
(C) more than 6 kg
(D) may be more or less than 6 kg

32. Both the blocks shown here are of mass m and are moving with constant velocity in direction shown in a resistive medium which exerts equal constant force on both blocks in direction opposite to the velocity. The tension in the string connecting both of them will be (Neglect friction)

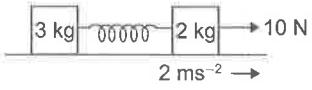


(A) mg
(B) $mg/2$
(C) $mg/3$
(D) $mg/4$

33. A monkey of mass 20 kg is holding a vertical rope. The rope can break when a mass of 25 kg is suspended from it. What is the maximum acceleration with which the monkey can climb up along the rope?

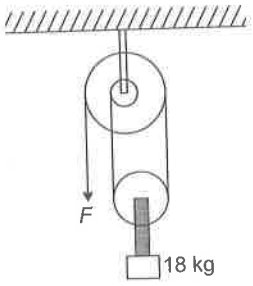
(A) 7 ms^{-2}
(B) 10 ms^{-2}
(C) 5 ms^{-2}
(D) 2.5 ms^{-2}

34. Find the acceleration of 3 kg mass when the accele ration of 2 kg mass is 2 ms^{-2} as shown in the figure.



(A) 3 ms^{-2} (B) 2 ms^{-2}
(C) 0.5 ms^{-2} (D) zero

35. In the figure at the free end, a force F is applied to keep the suspended mass of 18 kg at rest. The value of F is

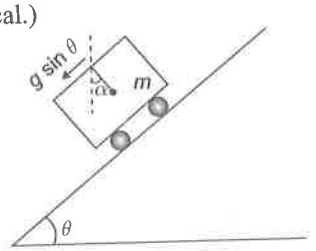


(A) 180 N (B) 90 N
(C) 60 N (D) 30 N

36. If the tension in the cable supporting an elevator is equal to the weight of the elevator, the elevator may be
(a) going up with increasing speed
(b) going down with increasing speed
(c) going up with uniform speed
(d) going down with uniform speed
(A) a, d (B) a, b, c
(C) c, d (D) a, b

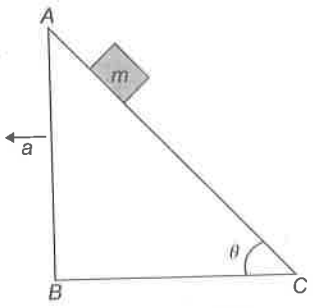
37. As an inclined plane is made slowly horizontal by reducing the value of angle θ with horizontal, the component of weight parallel to the plane of a block resting on the inclined plane
(A) decreases
(B) remains same
(C) increases
(D) increases if the plane is smooth

38. A trolley is accelerating down an incline of angle θ with acceleration $g \sin \theta$. Which of the following is correct. (α is the constant angle made by the string with vertical.)



(A) $\alpha = \theta$ (B) $\alpha = 0°$
(C) Tension in the string, $T = mg$
(D) Tension in the string, $T = mg \sec \theta$

39. A block of mass m resting on a wedge of angle θ as shown in the figure. The wedge is given an acceleration a. What is the minimum value of a so that the mass m falls freely?



(A) g (B) $g \cos \theta$
(C) $g \cot \theta$ (D) $g \tan \theta$

Friction

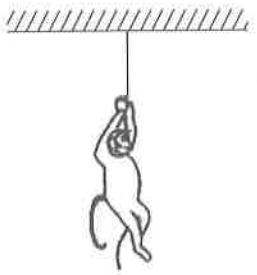
40. A block is placed on a rough floor and a horizontal force F is applied on it. The force of friction f by the floor on the block is measured for different values of F and a graph is plotted between them.
(a) The graph is a straight line of slope 45°.
(b) The graph is a straight line parallel to the F-axis
(c) The graph is a straight line of slope 45° for small F and a straight line parallel to the F-axis for large F.
(d) There is small kink on the graph.
(A) c, d (B) a, d
(C) a, b (D) a, c

41. Mark the correct statements about the friction between two bodies:
(a) Static friction is always greater than the kinetic friction.
(b) Coefficient of static friction is always greater than the coefficient of kinetic friction.
(c) Limiting friction is always greater than the kinetic friction.
(d) Limiting friction is never less than the static friction
(A) b, c, d (B) a, b, c
(C) a, c, d (D) a, b, d

42. A block A kept on an inclined surface just begins to slide if the inclination is 30°. The block is replaced by another block B and it is found that it just begins to slide if the inclination is 40°.
(A) Mass of $A >$ mass of B
(B) Mass of $A <$ mass of B
(C) Mass of $A =$ mass of B
(D) All the three are possible

43. Two cars of unequal masses use similar tyres. If they are moving at the same initial speed, the minimum stopping distance
(A) is smaller for the heavier car
(B) is smaller for the lighter car
(C) is same for both cars
(D) depends on the volume of the car

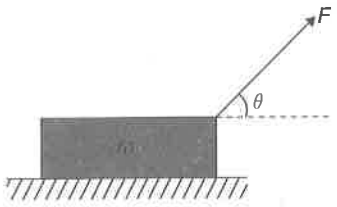
44. A monkey of mass m is climbing a rope hanging from the roof with acceleration a. The coefficient of static friction between the body of the monkey and the rope is μ. Find the direction and the value of friction force on the monkey.



(A) Upward, $F = m(g + a)$
(B) Downward, $F = m(g + a)$
(C) Upward, $F = mg$
(D) Downward, $F = mg$

45. A body is placed on a rough inclined plane of inclination θ. As the angle θ is increased from 0° to 90°, the contact force between the block and the plane
(A) remains constant
(B) first remains constant then decreases
(C) first decreases then increases
(D) first increases then decreases

46. A wooden block of mass m resting on a rough horizontal table (coefficient of friction $= \mu$) is pulled by a force F as shown in the figure. The acceleration of the block moving horizontally is



(A) $\frac{F\cos\theta}{m}$ (B) $\frac{\mu F\sin\theta}{M}$
(C) $\frac{F}{m}(\cos\theta + \mu\sin\theta) - \mu g$ (D) none of these

47. A chain is lying on a rough table with a fraction $1/n$ of its length hanging down from the edge of the table. If it is just on the point of sliding down from the table, then the coefficient of friction between the table and the chain is

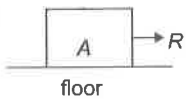
(A) $\frac{1}{n}$ (B) $\frac{1}{(n-1)}$
(C) $\frac{1}{(n+1)}$ (D) $\frac{(n-1)}{(n+1)}$

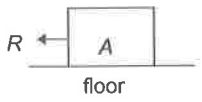
48. A body of mass m moves with a velocity v on a surface whose friction coefficient is μ. If the body covers a distance s, then v will be

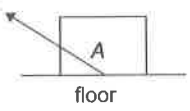
(A) $\sqrt{2\mu gs}$ (B) $\sqrt{\mu gs}$
(C) $\sqrt{\mu gs/2}$ (D) $\sqrt{3\mu gs}$

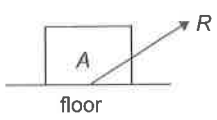
49. A block of mass 2 kg rests on a rough inclined plane making an angle of 30° with the horizontal. The coefficient of static friction between the block and the plane is 0.7. The frictional force on the block is
(A) 0.7×9.8 Newton
(B) 9.8 Newton
(C) $0.7 \times 9.8\sqrt{3}$ Newton
(D) $9.8 \times \sqrt{3}$ Newton

50. A box 'A' is lying on the horizontal floor of the compartment of a train running along horizontal rails from left to right. At time 't', it decelerates. Then the reaction R by the floor on the box is given best by

(A) 

(B) 

(C) 

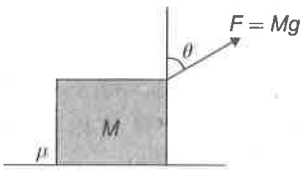
(D) 

51. For the equilibrium of a body on an inclined plane of inclination 45°, the coefficient of static friction will be
(A) greater than one (B) less than one
(C) zero (D) less than zero

52. Starting from rest, a body slides down a 45° inclined plane in twice the time it takes to slide down the same distance in the absence of friction. The coefficient of friction between the body and the inclined plane is

(A) 0.75 (B) 0.33
(C) 0.25 (D) 0.80

53. A block of mass M rests on a rough horizontal surface as shown in the figure. Coefficient of friction between the block and the surface is μ. A force $F = Mg$ acting at an angle θ with the vertical side of the block pulls it in which of the following cases the block can be pulled along the surface?



(A) $\tan\theta \geq \mu$ (B) $\tan(\theta/2) \geq \mu$
(C) $\cot\theta \geq \mu$ (D) $\cot(\theta/2) \geq \mu$

54. A block moves down a smooth inclined plane of inclination θ. Its velocity on reaching the bottom is v. If it slides down a rough inclined plane of some inclination, its velocity on reaching the bottom is v/n, where n is a number greater than 0. The coefficient of friction is given by

(A) $\mu = \tan\theta\left(1-\frac{1}{n^2}\right)$

(B) $\mu = \cot\theta\left(1-\frac{1}{n^2}\right)$

(C) $\mu = \tan\theta\left(1-\frac{1}{n^2}\right)^{\frac{1}{2}}$

(D) $\mu = \cot\theta\left(1-\frac{1}{n^2}\right)^{\frac{1}{2}}$

55. A block of mass 5 kg and surface area 2 m^2 just begins to slide down an inclined plane when the angle of inclination is 30°. Keeping mass same, the surface area of the block is doubled. The angle at which this starts sliding down is

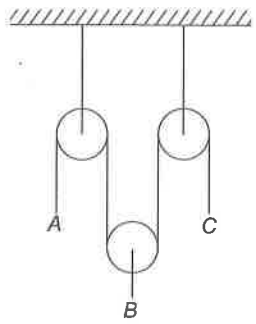
(A) 30° (B) 60°
(C) 15° (D) none of these

JEE Avanced

(A) Newton's Law of Motion

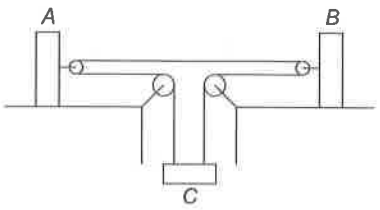
Single Correct

1. The pulleys in the diagram are all smooth and light. The acceleration of A is a upwards and the acceleration of C is f downwards. The acceleration of B is



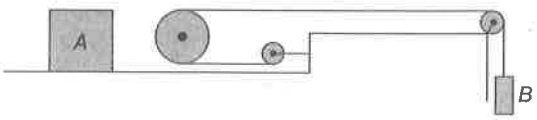
(A) 1/2 $(f-a)$ up (B) 1/2 $(a+f)$ down
(C) 1/2 $(a+f)$ up (D) 1/2 $(a-f)$ up

2. If the acceleration of A is 2 m/s² to left and the acceleration of B is 1 m/s² to left, then the acceleration of C is



(A) 1 m/s² upwards (B) 1 m/s² downwards
(C) 2 m/s² downwards (D) 2 m/s² upwards

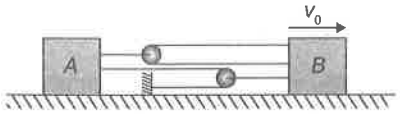
3. If block A has a velocity of 0.6 m/s to the right, determine the velocity of block B.



(A) 1.8 m/s in downward direction
(B) 1.8 m/s in upward direction

(C) 0.6 m/s in downward direction
(D) 0.6 m/s in upward direction

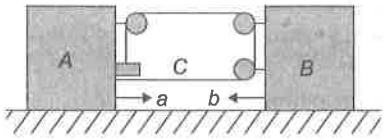
4. Block B moves to the right with a constant velocity v_0. The velocity of body A relative to B is



(A) $\frac{v_0}{2}$, towards left (B) $\frac{v_0}{2}$, towards right

(C) $\frac{3v_0}{2}$, towards left (D) $\frac{3v_0}{2}$, towards right

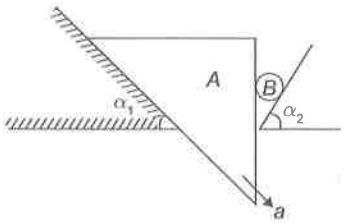
5. Find the acceleration of C w.r.t. ground.



(A) $a\hat{i}-(2a+2b)\hat{j}$ (B) $a\hat{i}-(2a+b)\hat{j}$

(C) $a\hat{i}-(a+2b)\hat{j}$ (D) $b\hat{i}-(2a+2b)\hat{j}$

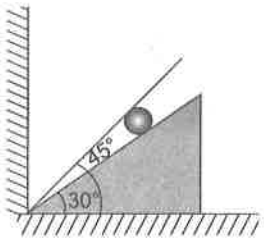
6. Find the acceleration of B.



(A) $\frac{a\cos\alpha_1}{\cos\alpha_2}$ (B) $\frac{a\sin\alpha_1}{\cos\alpha_2}$

(C) $\frac{a\cos\alpha_2}{\cos\alpha_1}$ (D) $\frac{\cos\alpha_1}{\cos\alpha_2}$

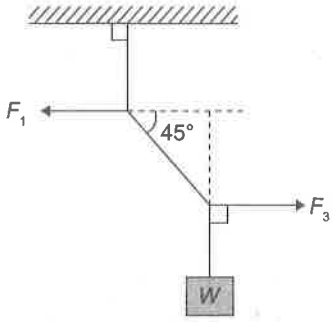
7. A spherical ball of mass $m = 5$ kg rests between two planes which make angles of 30° and 45°, respectively, with the horizontal. The system is in equilibrium. Find the normal forces exerted on the ball by each of the planes. The planes are smooth.



(A) $N_{45} = 96.59$ N, $N_{30} = 136.6$ N
(B) $N_{30} = 96.59$ N, $N_{45} = 136.6$ N
(C) $N_{45} = 136.6$ N, $N_{30} = 96.56$ N
(D) none of these

Question No. 8 to 9

In the figure, the tension in the diagonal string is 60 N.



8. Find the magnitude of the horizontal force $\vec{F}_1$ and $\vec{F}_2$ that must be applied to hold the system in the position shown.

(A) $\frac{60}{\sqrt{3}}$ N (B) $\frac{20}{\sqrt{2}}$ N

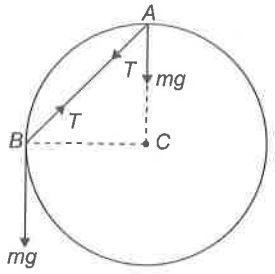
(C) $\frac{40}{\sqrt{2}}$ N (D) $\frac{60}{\sqrt{2}}$ N

9. In the above questions, what is the weight of the suspended block?

(A) $\frac{60}{\sqrt{2}}$ N (B) $\frac{40}{\sqrt{2}}$ N

(C) $\frac{60}{\sqrt{3}}$ N (D) $\frac{50}{\sqrt{2}}$ N

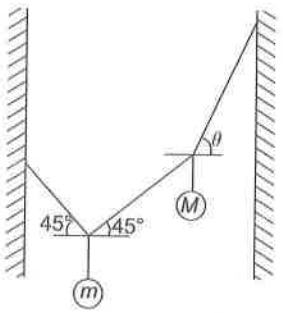
10. Objects A and B each of mass m are connected by light inextensible cord. They are constrained to move on a frictionless ring in a vertical plane as shown in the figure. The objects are released from rest at the positions shown. The tension in the cord just after release will be



(A) $mg\sqrt{2}$ (B) $\frac{mg}{\sqrt{2}}$

(C) $\frac{mg}{2}$ (D) $\frac{mg}{4}$

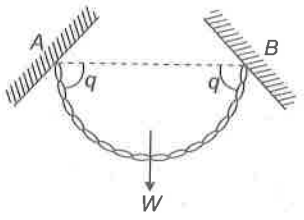
11. Two masses m and M are attached to the strings as shown in the figure. If the system is in equilibrium, then



(A) $\tan\theta = 1+\frac{2M}{m}$ (B) $\tan\theta = 1+\frac{2m}{M}$

(C) $\cot\theta = 1+\frac{2M}{m}$ (D) $\cot\theta = 1+\frac{2m}{M}$

12. A flexible chain of weight W hangs between two fixed points A and B which are at the same horizontal level. The inclination of the chain with the horizontal at both the points of support is θ. What is the tension of the chain at the mid-point?

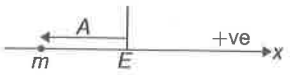


(A) $\frac{W}{2}\cdot\operatorname{cosec}\theta$ (B) $\frac{W}{2}\cdot\tan\theta$

(C) $\frac{W}{2}\cdot\cot\theta$ (D) none of these

Question No. 13 to 15

A particle of mass m is constrained to move on the x-axis. A force F acts on the particle. F always points towards the position labelled E. For example, when the particle is to the left of E, F points to the right. The magnitude of F is constant except at point E where it is zero.



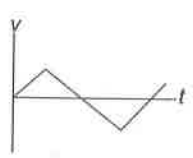
The system is horizontal. F is the net force acting on the particle. The particle is displaced a distance A towards left from the equilibrium position E and released from rest at $t = 0$.

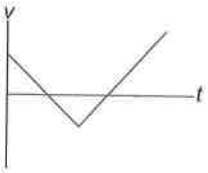
13. What is the period of the motion?

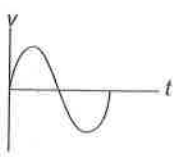
(A) $4\left(\sqrt{\frac{2Am}{F}}\right)$ (B) $2\left(\sqrt{\frac{2Am}{F}}\right)$

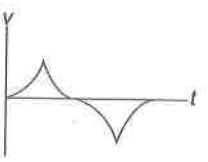
(C) $\left(\sqrt{\frac{2Am}{F}}\right)$ (D) None of these

14. Velocity–time graph of the particle is

(A) 

(B) 

(C) 

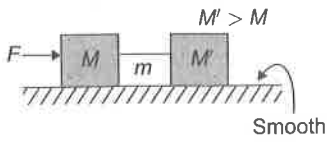
(D) 

15. Find the minimum time it will take to reach from $x = -\frac{A}{2}$ to 0.

(A) $\frac{3}{2}\sqrt{\frac{mA}{F}}(\sqrt{2}-1)$ (B) $\sqrt{\frac{mA}{F}}(\sqrt{2}-1)$

(C) $2\sqrt{\frac{mA}{F}}(\sqrt{2}-1)$ (D) None of these

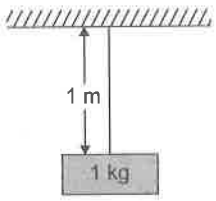
16. A constant force F is applied in horizontal direction as shown in the figure. Contact force between M and m is N and between m and M' is N', then



(A) N or N' equal
(B) $N > N'$
(C) $N' > N$
(D) cannot be determined

Question No. 17 to 19

A block of mass 1 kg is suspended by a string of mass 1 kg, length 1 m as shown in the figure ($g = 10$ m/s^2). Calculate:

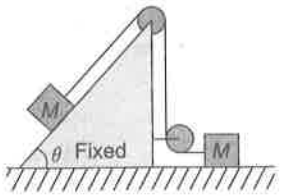


17. The tension in the string at its lowest point.
(A) 10 N
(B) 15 N
(C) 20 N
(D) 25 N

18. The tension in the string at its mid-point.
(A) 10 N
(B) 15 N
(C) 20 N
(D) 25 N

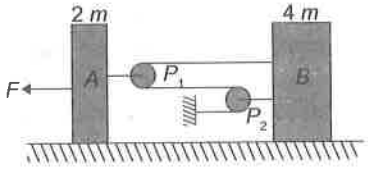
19. Force exerted by support on the string.
(A) 10 N
(B) 15 N
(C) 20 N
(D) 25 N

20. Two blocks, each having mass M, rest on frictionless surfaces as shown in the figure. If the pulleys are light and frictionless, and M on the incline is allowed to move down, then the tension in the string will be



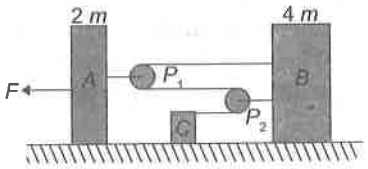
(A) $\frac{2}{3}Mg\sin\theta$
(B) $\frac{3}{2}Mg\sin\theta$
(C) $\frac{Mg\sin\theta}{2}$
(D) $2Mg\sin\theta$

21. Calculate the acceleration of block B in the below figure, assuming the surfaces and the pulleys P_1 and P_2 are all smooth and pulleys and string are light.



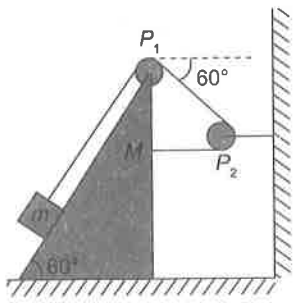
(A) $a = \frac{3F}{17m}$ m/s^2
(B) $a = \frac{2F}{17m}$ m/s^2
(C) $a = \frac{3F}{15m}$ m/s^2
(D) $a = \frac{3F}{12m}$ m/s^2

22. In the previous question, surface is replaced by block C of mass m as shown in the figure. Find the acceleration of block B.



(A) $a = \frac{3F}{20m}$ m/s^2
(B) $a = \frac{3F}{21m}$ m/s^2
(C) $a = \frac{2F}{12m}$ m/s^2
(D) $a = \frac{3F}{18m}$ m/s^2

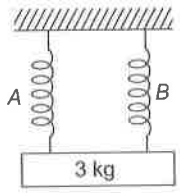
23. In the arrangement shown in the figure, the block of mass $m = 2$ kg lies on the wedge on mass $M = 8$ kg. Find the initial acceleration of the wedge if the surfaces are smooth and pulley and strings are massless.



(A) $a = \frac{30\sqrt{3}}{23}$ m/s^2 (B) $a = \frac{20\sqrt{3}}{23}$ m/s^2

(C) $a = \frac{30\sqrt{2}}{23}$ m/s^2 (D) None of these

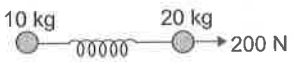
24. The block of 3 kg is initially in equilibrium and is hanging by two identical springs A and B as shown in figure. If spring A is cut from lower point at $t = 0$, then find the acceleration of block in ms^{-2} at $t = 0$.



(A) 5 (B) 10

(C) 15 (D) 0

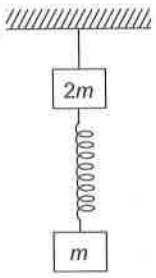
25. Two masses of 10 kg and 20 kg, respectively, are connected by a massless spring as shown in the figure. A force of 200 N acts on the 20 kg mass at the instant when the 10 kg mass has an acceleration of 12 ms^{-2} towards right, the acceleration of the 20 kg mass is :



(A) 2 ms^{-2} (B) 4 ms^{-2}

(C) 10 ms^{-2} (D) 20 ms^{-2}

26. Two blocks are connected by a spring. The combination is suspended, at rest, from a string attached to the ceiling, as shown in the figure. The string breaks suddenly. Immediately after the string breaks, what is the initial downward acceleration of the upper block of mass $2m$?



(A) 0 (B) 3g/2

(C) g (D) $2g$

27. A man of mass 60 kg is standing on a weighing machine placed in a lift moving with velocity 'v' and acceleration 'a' as shown in the figure. Calculate the reading of weighing machine in following situation:

($g = 10$ m/s^2)

(i) $a = 0$, $v = 0$

(A) 600 N (B) 500 N

(C) 450 N (D) 700 N

(ii) $a = 0$, $v = 2$ m/s

(A) 600 N (B) 500 N

(C) 450 N (D) 700 N

(iii) $a = 0$, $v = -2$ m/s

(A) 450 N (B) 500 N

(C) 600 N (D) 700 N

(iv) $a = 2$ m/s^2, $v = 0$

(A) 600 N (B) 500N

(C) 450 N (D) 720 N

(v) $a = -2$ m/s^2 $v = 0$

(A) 600 N (B) 480 N

(C) 450 N (D) 700 N

(vi) $a = 2$ m/s^2, $v = 2$ m/s

(A) 600 N (B) 480 N

(C) 450 N (D) 720 N

(vii) $a = 2$ m/s^2, $v = -2$ m/s

(A) 600 N (B) 720 N

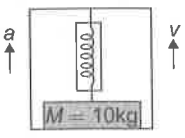
(C) 450 N (D) 700 N

(viii) $a = -2$ m/s^2 $v = -2$ m/s

(A) 600 N (B) 480 N

(C) 450 N (D) 700 N

28. What will be the reading of spring balance in the figure shown in following situations. ($g = 10$ m/s^2)



(i) $a = 0$, $v = 0$

(A) 100 N (B) 80N

(C) 120 N (D) 150 N

(ii) $a = 0$, $v = 2$ m/s

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(iii) $a = 0$, $v = -2$ m/s

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(iv) $a = 2$ m/s^2, $v = 0$

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(v) $a = -2$ m/s^2, $v = 0$

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(vi) $a = 2$ m/s^2, $v = 2$ m/s

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(vii) $a = 2$ m/s^2, $v = -2$ m/s

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

(viii) $a = -2$ m/s^2, $v = -2$ m/s

(A) 100 N (B) 80 N

(C) 120 N (D) 150 N

Question No. 29 to 31

An object of mass 2 kg is placed at rest in a frame (S_1) moving with velocity $10\hat{i} + 5\hat{j}$ m/s and having acceleration $5\hat{i} + 10\hat{j}$ m/s^2. The object is also seen by an observer standing in a frame (S_2) moving with velocity $5\hat{i} + 10\hat{j}$ m/s.

29. Calculate 'Pseudo force' acting on object. Which frame is responsible for this force?

(A) $F = -10\hat{i} - 20\hat{j}$ due to acceleration of frame S_1

(B) $F = -20\hat{i} - 20\hat{j}$ due to acceleration of frame S_1

(C) $F = -10\hat{i} - 30\hat{j}$ due to acceleration of frame S_1

(D) None of these

30. Calculate net force acting on object with respect to S_2 frame.

(A) $F = 20\hat{i} + 20\hat{j}$ (B) $F = 10\hat{i} + 20\hat{j}$

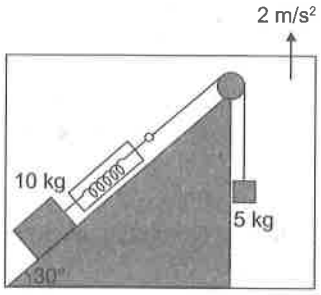
(C) $F = 5\hat{i} + 20\hat{j}$ (D) $F = 10\hat{i} + 5\hat{j}$

31. Calculate net force acting on object with respect of S_1 frame.

(A) 0 (B) 1

(C) 2 (D) None of these

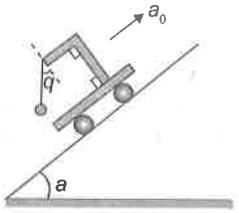
32. In the following figure, the reading of the spring balanced will be ($g = 10$ m/s^2)



(A) 50 N (B) 40 N

(C) 60 N (D) 70 N

33. A pendulum of mass m hangs from a support fixed to a trolley. The direction of the string when the trolley rolls up of plane of inclination α with acceleration a_0 is (String and bob remain fixed with respect to trolley.)

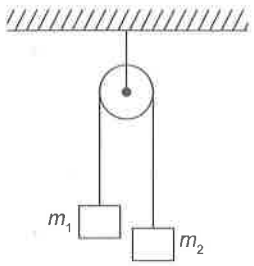


(A) $\theta = \tan^{-1} \alpha$ (B) $\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$

(C) $\theta = \tan^{-1}\left(\frac{g}{a_0}\right)$ (D) $\theta = \tan^{-1}\left(\frac{a_0 + g \sin\alpha}{g \cos\alpha}\right)$

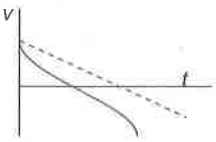
Multiple Correct

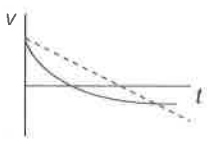
34. A student calculates the acceleration of m_1 in the figure shown as $a_1 = \frac{(m_1 - m_2)g}{m_1 + m_2}$. Which assumption is not required to do this calculation?

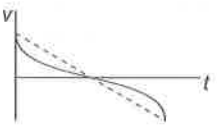


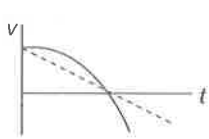
(A) pulley is frictionless (B) string is massless
(C) pulley is massless (D) string is inextensible

35. Which graph shows best the velocity–time graph for an object launched vertically into the air when air resistance is given by $|D| = bv$? The dashed line shows the velocity graph if there were no air resistance.

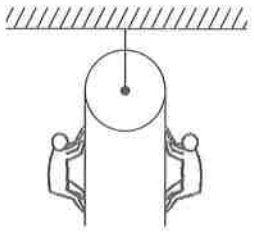
(A) 

(B) 

(C) 

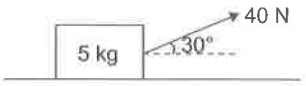
(D) 

36. Two men of unequal masses hold on to the two sections of a light rope passing over a smooth light pulley. Which of the following are possible?



(A) The lighter man is stationary while the heavier man slides with some acceleration.
(B) The heavier man is stationary while the lighter man climbs with some acceleration.
(C) The two men slide with the same acceleration in the same direction.
(D) The two men move with accelerations of the same magnitude in opposite directions.

37. The following figure shows a force of 40 N acting at 30° to the horizontal on a body of mass 5 kg resting on a smooth horizontal surface. Assuming that the acceleration of free-fall is 10 ms^{-2}, which of the following statements *A*, *B*, *C*, *D*, *E* is (are) correct?



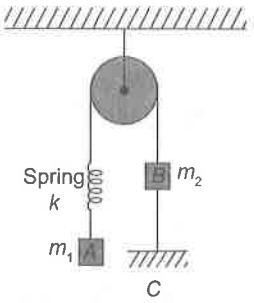
1. The horizontal force acting on the body is 20 N.
2. The weight of the 5 kg mass acts vertically downwards.
3. The net vertical force acting on the body is 30 N.

(A) 1, 2, 3 (B) 1, 2
(C) 2 only (D) 1 only

38. For ordinary terrestrial experiments, which of the following observers below are inertial?

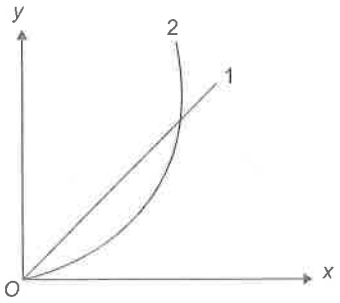
(A) A child revolving in a 'giant wheel'.
(B) A driver in a sports car moving with a constant high speed of 200 km/h on a straight road.
(C) The pilot of an aeroplane which is taking off.
(D) A cyclist negotiating a sharp turn.

39. In the system shown in the figure, $m_1 > m_2$. System is held at rest by thread *BC*. Just after the thread *BC* is burnt,



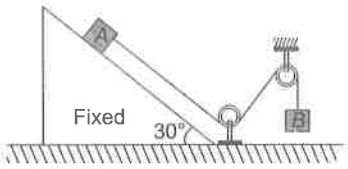
(A) acceleration of m_2 will be upwards
(B) magnitude of acceleration of both blocks will be equal to $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$
(C) acceleration of m_1 will be equal to zero
(D) magnitude of acceleration of two blocks will be non-zero and unequal

40. A particle is resting on a smooth horizontal floor. At $t = 0$, a horizontal force starts acting on it. Magnitude of the force increases with time according to law $F = \alpha t$, where α is a constant. For the following figure, which of the following statements is/are correct?



(A) Curve 1 shows acceleration against time
(B) Curve 2 shows velocity against time
(C) Curve 2 shows velocity against acceleration
(D) None of these

41. Two blocks A and B of equal mass m are connected through a massless string and arranged as shown in the figure. Friction is absent everywhere. When the system is released from rest,



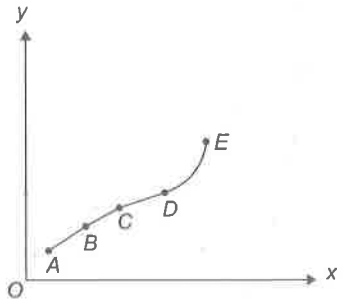
(A) tension in string is $\frac{mg}{2}$

(B) tension in string is $\frac{mg}{4}$

(C) acceleration of A is $g/2$

(D) acceleration of A is $\frac{3}{4}g$

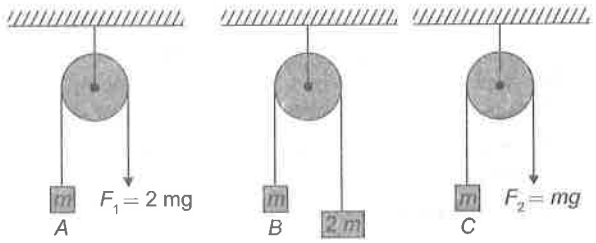
42. Figure shows the displacement of a particle going along the x-axis as a function of time:



(A) The force acting on the particle is zero in the region AB.
(B) The force acting on the particle is zero in the region BC.
(C) The force acting on the particle is zero in the region CD.
(D) The force is zero nowhere.

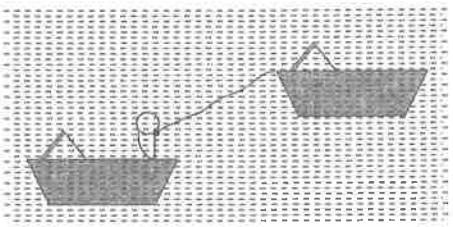
43. A force of magnitude F_1 acts on a particle so as to accelerate it from rest to a velocity v. The force F_1 is then replaced by another force of magnitude F_2 which decelerates it to rest.
(A) F_1 must be equal to F_2
(B) F_1 may be equal to F_2
(C) F_1 must be unequal to F_2
(D) None of these

44. In the figure, blocks A, B and C of mass m each have acceleration a_1. a_2 and a_3, respectively. F_1 and F_2 are external forces of magnitudes $2mg$ and mg, respectively.



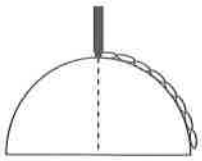
(A) $a_1 = a_2 = a_3$ (B) $a_1 > a_2 > a_3$
(C) $a_1 = a_2, a_2 > a_3$ (D) $a_1 > a_2, a_2 = a_3$

45. A rope is stretched between two boats at rest. A sailor in the first boat pulls the rope with a constant force of 100 N. First boat with the sailor has a mass of 250 kg whereas mass of the second boat is double of that mass. If the initial distance between the boats was 100 m, the time taken for two boats to meet each other is



(A) 13.8 s (B) 18.3 s
(C) 3.18 s (D) 31.8 s

46. A chain of length ℓ is placed on a smooth spherical surface of radius r with one of its ends fixed at the top of the surface. Length of chain is assumed to be $\ell < \frac{\pi r}{2}$. Acceleration of each element of chain when upper end is released is



(A) $\frac{\ell g}{r}\left(1-\cos\frac{r}{\ell}\right)$ (B) $\frac{rg}{\ell}\left(1-\cos\frac{\ell}{r}\right)$

(C) $\frac{\ell g}{r}\left(1-\sin\frac{\ell}{r}\right)$ (D) $\frac{rg}{\ell}\left(1-\sin\frac{r}{\ell}\right)$

47. Five persons A, B, C, D and E are pulling a cart of mass 100 kg on a smooth surface and cart is moving with acceleration 3 m/s² in the east direction. When person 'A' stops pulling, it moves with acceleration 1 m/s² in the west direction. When person 'B' stops pulling, it moves with acceleration 24 m/s² in the north direction. The magnitude of acceleration of the cart when only A and B pull the cart keeping their directions same as the old directions, is

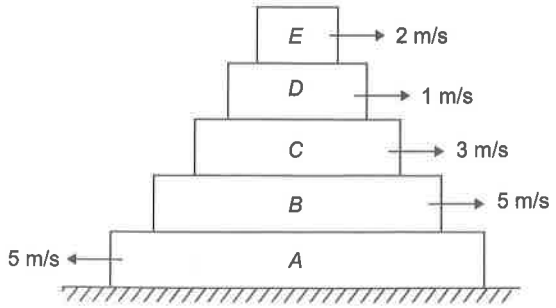
(A) 26 m/s² (B) $3\sqrt{71}$ m/s²

(C) 25 m/s² (D) 30 m/s²

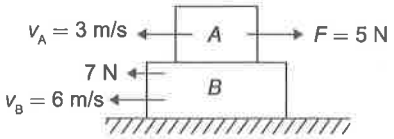
(B) Friction

Single Correct

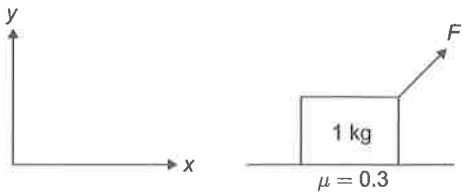
48. Find the direction of frictional forces on each block and the ground. (Assume all surfaces are rough and all velocities are with respect to ground.)



49. In the following figure, find the direction of friction on the blocks and ground.



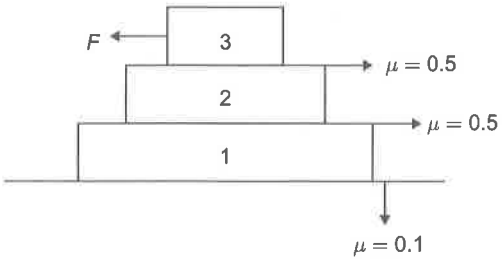
50. A force $\vec{F} = \hat{i} + 4\hat{j}$ acts on the block shown. The force of friction acting on the block is



(A) $-\hat{i}$ (B) $-1.8\hat{i}$

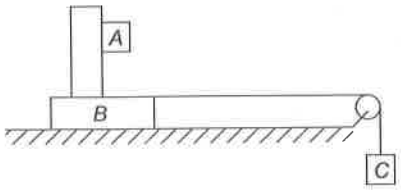
(C) $-2.4\hat{i}$ (D) $-3\hat{i}$

51. If force F is increasing with time and at $t = 0$, $F = 0$ where will slipping first start?



(A) Between 3 kg and 2 kg
(B) Between 2 kg and 1 kg
(C) Between 1 kg and ground
(D) Both (A) and (B)

52. In the arrangement shown in the figure, masses of the blocks B and A is $2m$ and m, respectively. Surface between block B and the floor is smooth. Block B is connected to block C by means of a string pulley system. If the whole system is released, then find the minimum value of mass of block C so that block A remains stationary w.r.t. B. The coefficient of friction between A and B is μ.



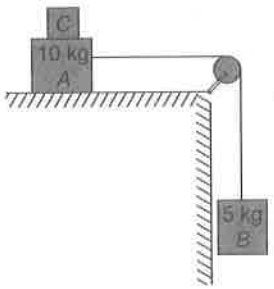
(A) $\frac{m}{\mu}$ (B) $\frac{2m+1}{\mu+1}$

(C) $\frac{3m}{\mu-1}$ (D) $\frac{6m}{\mu+1}$

53. Two masses A and B of 10 kg and 5 kg, respectively, are connected with a string passing over a frictionless pulley fixed at the corner of a table as shown in the figure. The coefficient of friction of A with the table is 0.2. The minimum mass of C that may be placed on A to prevent it from moving is equal to

(A) 15 kg (B) 10 kg

(C) 5 kg (D) zero

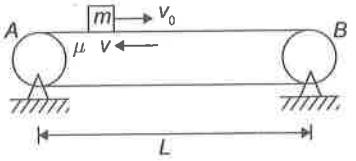


54. If the coefficient of friction between an insect and bowl is μ and the radius of the bowl is r, the maximum height to which the insect can crawl in the bowl is

(A) $\frac{r}{\sqrt{1+\mu^2}}$ (B) $r\left(1-\frac{1}{\sqrt{1+\mu^2}}\right)$

(C) $r\sqrt{1+\mu^2}$ (D) $r\sqrt{1+\mu^2}-1$

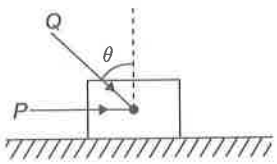
55. With what minimum velocity should block be projected from left end A towards end B such that it reaches the other end B of conveyer belt moving with constant velocity v? The friction coefficient between the block and the belt is μ.



(A) $\sqrt{\mu gL}$ (B) $\sqrt{2\mu gL}$

(C) $\sqrt{3\mu gL}$ (D) $2\sqrt{\mu gL}$

56. A block of mass m lying on a rough horizontal plane is acted upon by a horizontal force P and another force Q inclined at an angle θ to the vertical. The minimum value of coefficient of friction between the block and the surface for which the block will remain in equilibrium is



(A) $\frac{P+Q\sin\theta}{mg+Q\cos\theta}$ (B) $\frac{P\cos\theta+Q}{mg-Q\sin\theta}$

(C) $\frac{P+Q\cos\theta}{mg+Q\sin\theta}$ (D) $\frac{P\sin\theta-Q}{mg-Q\cos\theta}$

57. A small mass slides down an inclined plane of inclination θ with the horizontal. The coefficient of friction is $\mu = m_0 x$, where x is the distance through which the mass slides down and m_0, a constant. Then the distance covered by the mass before it stops is

(A) $\frac{2}{\mu_0}\tan\theta$ (B) $\frac{4}{\mu_0}\tan\theta$

(C) $\frac{1}{2\mu_0}\tan\theta$ (D) $\frac{1}{\mu_0}\tan\theta$

58. In the above question, the speed of the mass when travelled half the maximum distance is

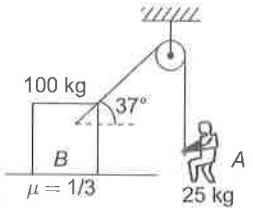
(A) $\sqrt{\frac{g\tan\theta\sin\theta}{\mu_0}}$ (B) $\sqrt{\frac{g\tan\theta\sin\theta}{2\mu_0}}$

(C) $\sqrt{\frac{g\tan\theta\sin\theta}{8\mu_0}}$ (D) none of these

59. A body is moving down a long inclined plane of slope 37°. The coefficient of friction between the body and the plane varies as $\mu = 0.3x$, where x is the distance travelled down the plane. The body will have maximum speed ($\sin 37° = \frac{3}{5}$ and $g = 10\ m/s^2$)

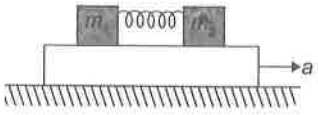
(A) at $x = 1.16$ m (B) at $x = 2$ m
(C) at the bottom of plane (D) at $x = 2.5$ m

60. Block B of mass 100 kg rests on a rough surface of friction coefficient $\mu = 1/3$. A rope is tied to block B as shown in the following figure. The maximum acceleration with which boy A of 25 kg can climb on the rope without making the block to move is



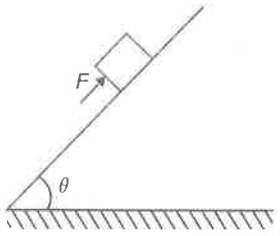
(A) $\frac{4g}{3}$ (B) $\frac{g}{3}$
(C) $\frac{g}{2}$ (D) $\frac{3g}{4}$

61. Two blocks of masses m_1 and m_2 are connected with a massless unstretched spring and placed over a plank moving with an acceleration 'a' as shown in the figure. The coefficient of friction between the blocks and the platform is μ.



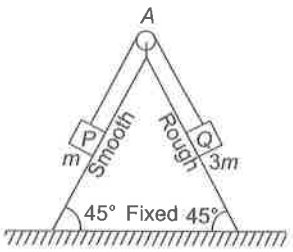
(A) Spring will be stretched if $a > mg$.
(B) Spring will be compressed if $a \leq mg$.
(C) Spring will neither be compressed nor be stretched for $a \leq mg$.
(D) Spring will be in its natural length under all conditions.

62. A block placed on a rough inclined plane of inclination ($\theta = 30°$) can just be pushed upwards by applying a force 'F' as shown in the figure. If the angle of inclination of the inclined plane is increased to ($\theta = 60°$), the same block can just be prevented from sliding down by application of a force of same magnitude. The coefficient of friction between the block and the inclined plane is



(A) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (B) $\frac{2\sqrt{3}-1}{\sqrt{3}+1}$
(C) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (D) none of these

63. A fixed wedge with both surface is inclined at 45° to the horizontal as shown in the figure. A particle P of mass m is held on the smooth plane by a light string which passes over a smooth pulley A and attached to a particle Q of mass $3m$ which rests on the rough plane. The system is released from rest. Given that the acceleration of each particle is of magnitude $\frac{g}{5\sqrt{2}}$ then



(a) the tension in the string is:

(A) mg (B) $\frac{6mg}{5\sqrt{2}}$
(C) $\frac{mg}{2}$ (D) $\frac{mg}{4}$

(b) In the above question, the coefficient of friction between Q and the rough plane is:

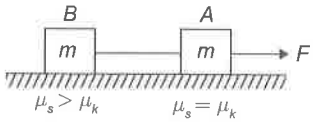
(A) $\frac{4}{5}$ (B) $\frac{1}{5}$
(C) $\frac{3}{5}$ (D) $\frac{2}{5}$

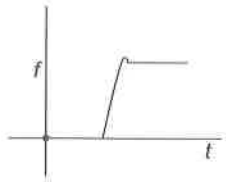
(c) In the above question, the magnitude and direction of the force exerted by the string on the pulley is:

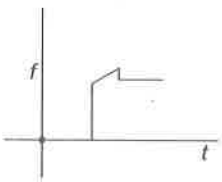
(A) $\frac{6mg}{5}$ downward (B) $\frac{6mg}{5}$ upward

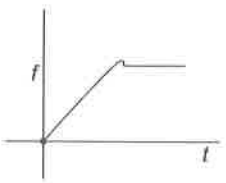
(C) $\frac{mg}{5}$ downward (D) $\frac{mg}{4}$ downward

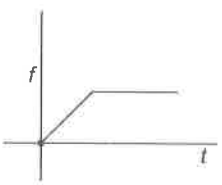
64. A force $F = t$ is applied to block A as shown in the figure. The force is applied at $t = 0$ s when the system is at rest and the string is just straight without tension. Which of the following graphs give the friction force between B and the horizontal surface as a function of time 't'?



(A)


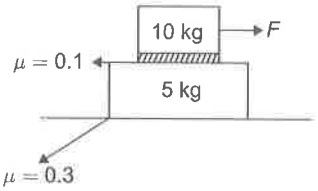
(B)


(C)


(D)


For Question No. 65. to 69, Refer Given Figure

65. When $F = 2$N, the frictional force between 5 kg block and ground is



(A) 2 N (B) 0
(C) 8 N (D) 10 N

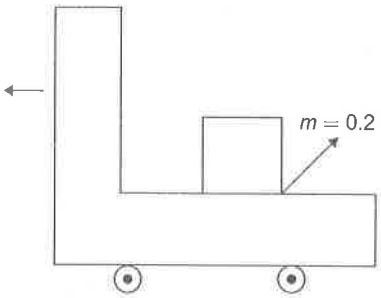
66. When $F = 2$ N, the frictional force between 10 kg block and 5 kg block is
(A) 2 N (B) 15 N
(C) 10 N (D) none of these

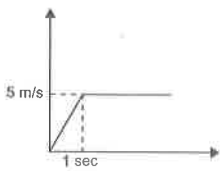
67. The maximum 'F' which will not cause motion of any of the blocks is
(A) 10 N (B) 15 N
(C) data insufficient (D) none of these

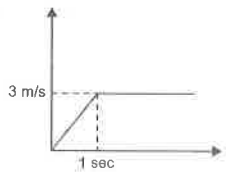
68. The maximum acceleration of 5 kg block is
(A) 1 m/s^2 (B) 3 m/s^2
(C) 0 (D) none of these

69. The acceleration of 10 kg block when $F = 30$ N is
(A) 2 m/s^2 (B) 3 m/s^2
(C) 1 m/s^2 (D) none of these

70. A truck starting from rest moves with an acceleration of 5 m/s^2 for 1 s and then moves with constant velocity. The velocity w.r.t. ground versus time graph for block in truck is (Assume that block does not fall off the truck.)



(A)


(B)


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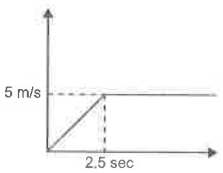
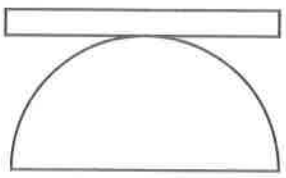


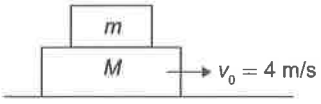
(D) None of these

71. A board is balanced on a rough horizontal semi-circular log. Equilibrium is obtained with the help of addition of a weight to one of the ends of the board when the board makes an angle θ with the horizontal. Coefficient of friction between the log and the board is



(A) $\tan\theta$ (B) $\cos\theta$
(C) $\cot\theta$ (D) $\sin\theta$

72. A stationary body of mass m is slowly lowered onto a massive platform of mass M ($M >> m$) moving at a speed $V_0 = 4$ m/s as shown in the figure. How far will the body slide along the platform ($\mu = 0.2$ and $g = 10$ m/s^2)?



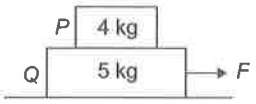
(A) 4 m (B) 6 m
(C) 12 m (D) 8 m

Multiple Correct

73. A block of mass 2.5 kg is kept on a rough horizontal surface. It is found that the block does not slide if a horizontal force less than 15 N is applied to it. Also it is found that it takes 5 s to slide throughout the first 10 m if a horizontal force of 15 N is applied and the block is gently pushed to start the motion. Taking $g = 10$ m/s^2, then

(A) $\mu_s = 0.60$ (B) $\mu_k = 0.52$
(C) $\mu_k = 0.60$ (D) $\mu_s = 0.52$

74. The coefficient of friction between 4 kg and 5 kg blocks is 0.2 and between 5 kg block and ground is 0.1, respectively. Choose the correct statements.

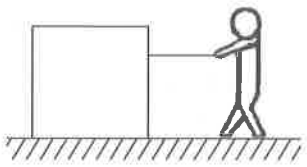


(A) Minimum force needed to cause system to move is 17 N.
(B) When force is 4 N static friction at all surfaces is 4 N to keep system at rest.
(C) Maximum acceleration of 4 kg block is 2 m/s^2.
(D) Slipping between 4 kg and 5 kg blocks start when F is 17 N.

75. In a tug-of-war contest, two men pull on a horizontal rope from opposite sides. The winner will be the man who

(A) exerts greater force on the rope
(B) exerts greater force on the ground
(C) exerts a force on the rope which is greater than the tension in the rope
(D) makes a smaller angle with the vertical

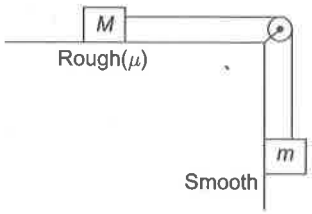
76. A man pulls a block heavier than himself with a light horizontal rope. The coefficient of friction is the same between the man and the ground, and between the block and the ground.



(A) The block will not move unless the man also moves.
(B) The man can move even when the block is stationary.
(C) If both move, the acceleration of the man is greater than the acceleration of the block.
(D) None of the above assertions is correct.

Question No. 77 to 78

In the figure, two blocks M and m are tied together with an inextensible and light string. The mass M is placed on a rough horizontal surface with coefficient of friction μ and the mass m is hanging vertically against a smooth vertical wall. The pulley is frictionless.



77. Choose the correct statement(s).

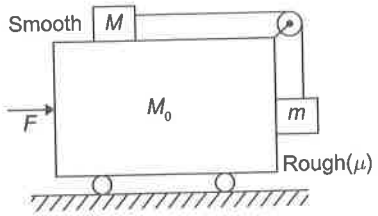
(A) The system will accelerate for any value of m.
(B) The system will accelerate only when $m > M$.
(C) The system will accelerate only when $m > \mu M$.
(D) Nothing can be said.

78. Choose the correct statement(s) related to the tension T in the string.

(A) When $m < \mu M$, $T = mg$
(B) When $m < \mu M$, $T = Mg$
(C) When $m > \mu M$, $\mu Mg < T < mg$
(D) When $m > \mu M$, $mg < T < \mu Mg$

Question No. 79 to 81

Imagine a situation in which the horizontal surface of block M_0 is smooth and its vertical surface is rough with a coefficient of friction μ.



79. Identify the correct statement(s).

(A) If $F = 0$, the blocks cannot remain stationary.

(B) For one unique value of F, the blocks M and m remain stationary with respect to M_0.

(C) The limiting friction between m and M_0 is independent of F.

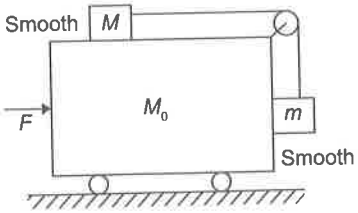
(D) There exists a value of F at which friction force is equal to zero.

80. In the above problem, choose the correct value(s) of F which the blocks M and m remain stationary with respect to M_0.

(A) $(M_0 + M + m)\dfrac{g}{\mu}$ (B) $\dfrac{m(M_0 + M + m)g}{M - \mu m}$

(C) $(M_0 + M + m)\dfrac{mg}{M}$ (D) None of these

81. Consider a special situation in which both the faces of the block M_0 are smooth, as shown in the following figure. Mark out the correct statement(s).



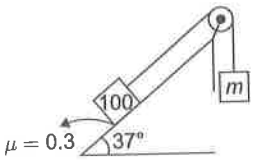
(A) If $F = 0$, the blocks cannot remain stationary.

(B) For one unique value of F, the blocks M and m remain stationary with respect to block M_0

(C) There exists a range of F for which blocks M and m remain stationary with respect to block M_0.

(D) Since there is no friction, therefore, blocks M and m cannot be in equilibrium with respect to M_0.

82. The value(s) of mass m for which the 100 kg block remains in static equilibrium is



(A) 35 kg (B) 37 kg

(C) 83 kg (D) 85 kg

83. The contact force exerted by one body on another body is equal to the normal force between the bodies. It can be said that

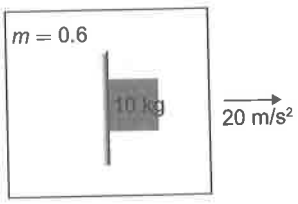
(A) the surface must be frictionless

(B) the force of friction between the bodies is zero

(C) the magnitude of normal force equals that of friction

(D) it is possible that the bodies are rough and they do not slip on each other

84. Car is accelerating with acceleration = 20 m/s^2. A box that is placed inside the car, of mass $m = 10$ kg, is put in contact with the vertical wall as shown. The friction coefficient between the box and the wall is $\mu = 0.6$.



(A) The acceleration of the box will be 20 m/s^2.

(B) The frictional force acting on the box will be 100 N

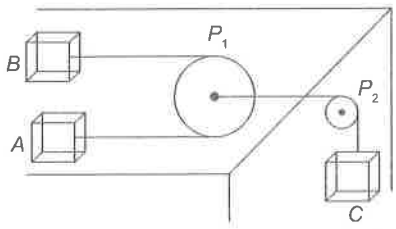
(C) The contact force between the vertical wall and the box will be $100\sqrt{5}$ N.

(D) The net contact force between the vertical wall and the box is only of electromagnetic in nature.

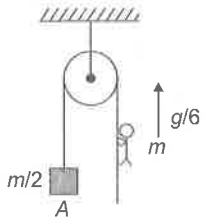
JEE Advanced

Level I

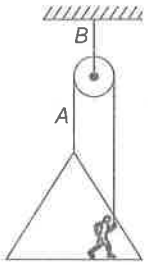
1. Two masses A and B, lie on a frictionless table. They are attached to either end of a light rope which passes around a horizontal movable pulley of negligible mass. Find the acceleration of each mass $M_A = 1$ kg, $M_B = 2$ kg, $M_C = 4$ kg. The pulley P_2 is vertical.



2. Block A of mass $m/2$ is connected to one end of light rope which passes over a pulley as shown in the figure. Man of mass m climbs the other end of rope with a relative acceleration of $g/6$ with respect to rope find acceleration of block A and tension in the rope.

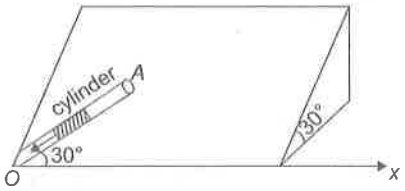


3. To paint the side of a building, painter normally hoists himself up by pulling on the rope A as in the figure. The painter and platform together weigh 200 N. Rope B can withstand 300 N. Find

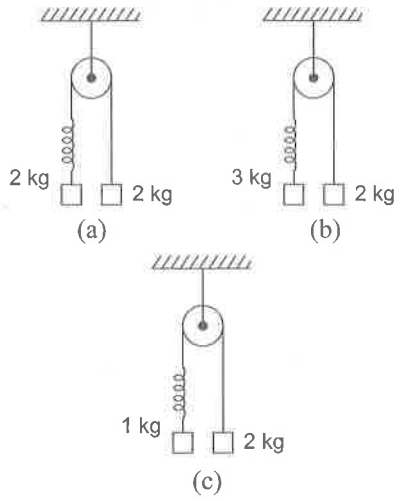


(A) the maximum acceleration of the painter
(B) tension in rope A
(i) when painter is at rest
(ii) when painter moves up with an acceleration 2 m/s^2.

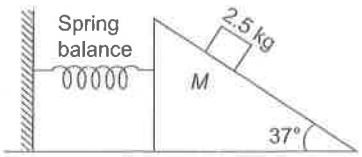
4. An inclined plane makes an angle 30° with the horizontal. A groove $OA = 5$ m cut in the plane makes an angle 30° with OX. A short smooth cylinder is free to slide down the influence of gravity. Find the time taken by the cylinder due to reach from A to O.
($g = 10$ m/s^2)



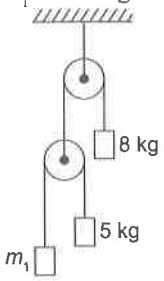
5. Same spring is attached with 2 kg, 3 kg and 1 kg blocks in three different cases as shown in the figure. If x_1, x_2 and x_3 be the extensions in the spring in these three cases, then find the ratio of their extensions.



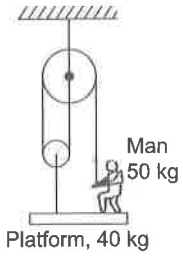
6. Find the reading of spring balance as shown in the figure. Assume that mass M is in equilibrium. (All surfaces are smooth.)



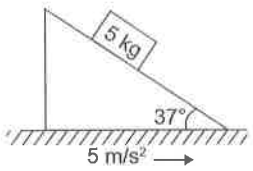
7. At what value of m_1 will 8 kg mass be at rest.



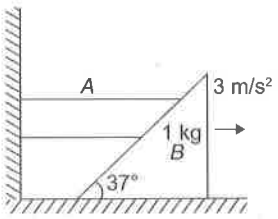
8. What force must man exert on rope to keep platform in equilibrium?



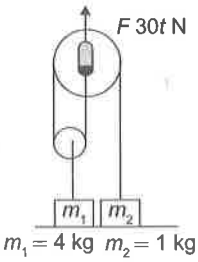
9. Inclined plane is moved towards right with an acceleration of 5 ms^{-2} as shown in the figure. Find force in Newton in which block of mass 5 kg exerts on the incline plane. (All surfaces are smooth.)



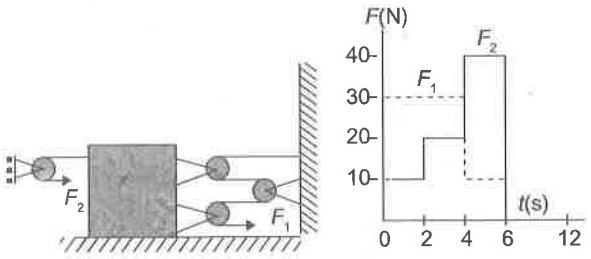
10. Find force in Newton in which mass A exerts on mass B if B is moving towards right with 3 ms^{-2}. Also find mass of A. (All surfaces are smooth.)



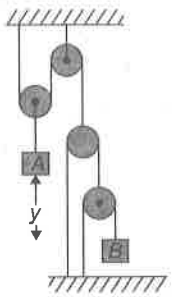
11. Force F is applied on upper pulley. If $F = 30t$ where t is time in seconds. Find the time when m_1 loses contact with floor.



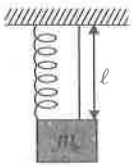
12. The 40 kg block is moving to the right with a speed of 1.5 m/s when it is acted upon by forces F_1 and F_2. These forces vary in the manner shown in the graph. Find the velocity of the block after $t = 12$ s. Neglect friction and masses of the pulleys and cords.



13. The vertical displacement of block A in metre is given by $y = t^2/4$ where t is in seconds. Calculate the downward acceleration aB of block B.



14. An object of mass m is suspended in equilibrium using a string of length l and a spring having spring constant K ($< 2\ mg/l$) and unstretched length $l/2$.



(A) Find the tension in the string.

(B) What happens if $K > 2\ mg/l$?

Q13

15. Three monkeys A, B and C with masses of 10, 15 and 8 kg, respectively, are climbing up and down the rope suspended from D. At the instant represented, A is descending the rope with an acceleration of 2 m/s² and C is pulling himself up with an acceleration of 1.5 m/s². Monkey B is climbing up with a constant speed of 0.8 m/s. Treat the rope and monkeys as a complete system and calculate the tension T in the rope at D. ($g = 10$ m/s^{-2})

16. A person of mass m is standing on a platform of mass M and wants to raise this platform. Massless pulleys are configured in two different ways as shown in the figure. We would like to know which configuration makes it easier to raise the platform? Answer the following questions in terms of m, M, a and constant as appropriate. (Assume the rope is also massless and does not stretch.)

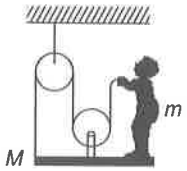


Fig. 1

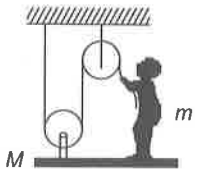


Fig. 2

(A) For configuration (1), find the force, F, the person must exert straight up in order to accelerate the platform + person system with an acceleration a. Include a free-body diagram in your solution.

(B) What force does the platform exert on the person when the acceleration of the system is a? Include a free-body diagram in your solution.

(C) If platform is massless, $M = 0$, and he wants to raise it with a constant velocity, find F. Does this configuration offer a mechanical advantage? (i.e., Is $F < mg$?)

(D) Now repeat the above for configuration (2). First, find the force, F, the person must exert straight down in order to accelerate the platform + person system with an upward acceleration a. Include a free-body diagram in your solution.

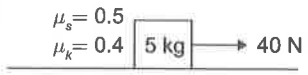
(E) Now, what force does the platform exert on the person when the acceleration of the system is a? Include a free-body diagram in your solution.

(F) Again, if the platform is massless, $M = 0$, and he wants to raise it with a constant velocity, find F. Does this configuration offer a mechanical advantage? (i.e., Is $F < mg$?)

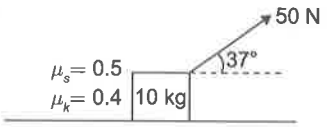
(B) Friction

17. Give the acceleration of blocks:

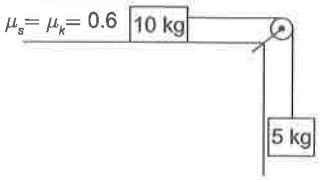
(A)



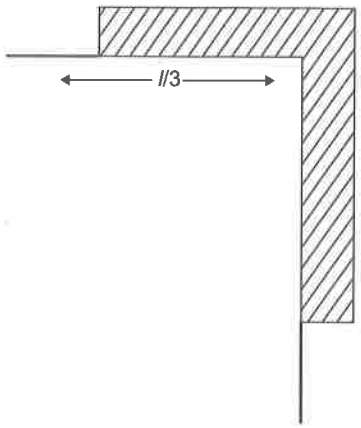
(B)



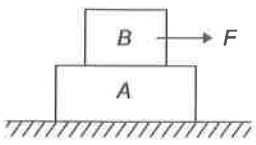
(C)



18. Determine the coefficient of friction (μ), so that rope of mass m and length l does not slide down.



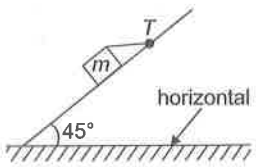
19. A wooden block A of mass M is placed on a frictionless horizontal surface. On top of A, another lead block B also of mass M is placed. A horizontal force of magnitude F is applied to B. Force F is increased continuously from zero. Then draw the graph between A and F. [$\mu_k < \mu_s$]



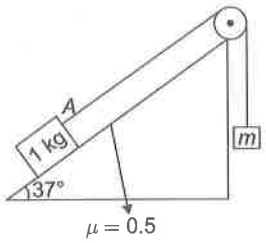
20. A rope so lies on a table that part of it lays over. The rope begins to slide when the length of hanging part is 25% of entire length. The coefficient of friction between the rope and the table is

21. A worker wishes to pile a cone of sand into a circular area in his yard. The radius of the circle is r, and no sand is to spill onto the surrounding area. If μ is the static coefficient of friction between each layer of sand along the slope and the sand, the greatest volume of sand that can be stored in this manner is:

22. A block of mass 15 kg is resting on a rough inclined plane as shown in the figure. The block is tied up by a horizontal string which has a tension of 50 N. The coefficient of friction between the surfaces of contact is ($g = 10$ m/s^2):

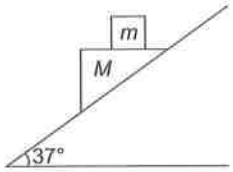


23. In the figure, what should be mass m so that block A slide up with a constant velocity?

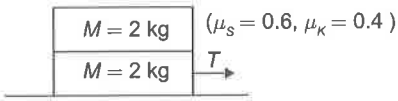


24. A block of mass 1 kg is horizontally thrown with a velocity of 10 m/s on a stationary long plank of mass 2 kg whose surface has $\mu = 0.5$. Plank rests on frictionless surface. Find the time when m_1 comes to rest w.r.t. plank.

25. Block M slides down on frictionless incline as shown. Find the minimum friction coefficient so that m does not slide with respect to M.



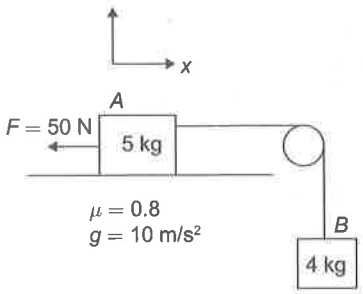
26. The coefficients of static and kinetic friction between the two blocks and also between the lower block and the ground are $\mu_s = 0.6$ and $\mu_k = 0.4$. Find the value of tension T applied on the lower block at which the upper block begins to slip relative to the lower block.



27. A thin rod of length 1 m is fixed in a vertical position inside a train, which is moving horizontally with constant acceleration 4 m/s^2. A bead can slide on the rod, and friction coefficient between them is 1/2. If the bead is released from rest at the top of the rod, find the time when it will reach at the bottom.

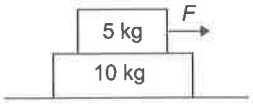
28 A body of mass 2 kg rests on a horizontal plane having coefficient of friction $\mu = 0.5$. At $t = 0$, a horizontal force is applied that varies with time $F = 2t$. The time constant t_0 at which motion starts and distance moved in $t = 2t_0$ second will be _____ and _____ respectively.

29. Find the acceleration of the blocks and magnitude and direction of frictional force between block A and table, if block A is pulled towards left with a force of 50 N.



30. A block A of mass 2 kg rests on another block B of mass 8 kg which rests on a horizontal floor. The coefficient of friction between A and B is 0.2 whereas that between B and floor is 0.5. When a horizontal force F of 25 N is applied on block B, the force of friction between A and B is _____.

31. The coefficient of friction between 5 kg and 10 kg block is 0.5. If friction between them is 20 N. What is the value of force being applied on 5 kg? The floor is frictionless.



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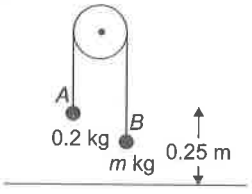
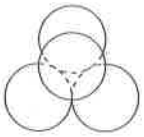
Level II

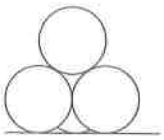
1. The diagram shows particles A and B, of masses 0.2 kg and m kg, respectively, connected by a light inextensible string which passes over a fixed smooth peg. The system is released from rest, with B at a height of 0.25 m above the floor. B descends, hitting the floor 0.5 s later. All resistances to motion may be ignored.



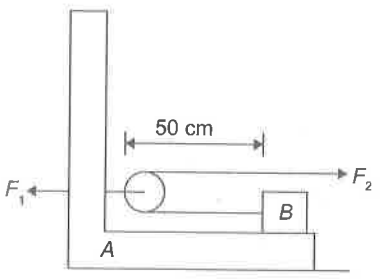
(A) Find the acceleration of B as it descends.
(B) Find the tension in the string while B is descending and find also the value of m.
(C) When B hits the floor it comes to rest immediately, and the string becomes slack. Find the length of time for which B remains at rest on the ground before being jerked into motion again.

2. An ornament for a courtyard at a word's fair is to be made up of four identical, frictionless metal sphere, each weighing $2\sqrt{6}$ Newton. The spheres are to be arranged as shown in the figure, with three resting on a horizontal surface and touching each other; the fourth is to rest freely on the other three. The bottom three are kept from separating by spot welds at the points of contact with each other. Allowing for a factor of safety of 3 N, how much tension must the spot welds with stand?

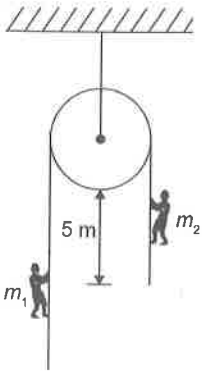

Top view


Horizontal view

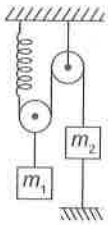
3. A 1 kg block 'B' rests as shown on a bracket 'A' of same mass. Constant forces $F_1 = 20$ N and $F_2 = 8$ N start to act at time $t = 0$ when the distance of block B from pulley is 50 cm. Time when block B reaches the pulley is _____.



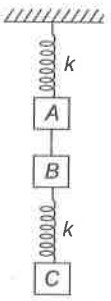
4. Two men of masses m_1 and m_2 hold on the opposite ends of a rope passing over a frictionless pulley. The man m_1 climbs up the rope with an acceleration of 1.2 m/s² relative to the rope. The man m_2 climbs up the rope with an acceleration of 2.0 m/s² relative to the rope. Find the tension in the rope if $m_1 = 40$ kg and $m_2 = 60$ kg. Also find the time after which they will be at same horizontal level if they start from rest and are initially separated by 5 m.



5. In the figure shown, pulleys are ideal $m_1 > 2m_2$. Initially, the system is in equilibrium and string connecting m_2 to rigid support below is cut. Find the initial acceleration of m_2.

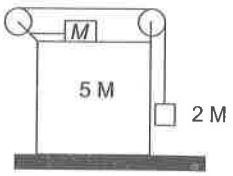


6. The system shown adjacent is in equilibrium. Find the acceleration of the blocks A, B and C all of equal masses m at the instant when
(Assume springs to be ideal)
(A) The spring between ceiling and A is cut.
(B) The string (inextensible) between A and B is cut.
(C) The spring between B and C is cut.



Also find the tension in the string when the system is at rest and in the above three cases.

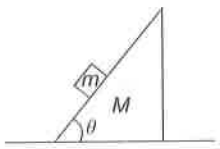
7. In the system shown, find the initial acceleration of the wedge of mass $5M$. The pulleys are ideal and the cords are inextensible (there is no friction anywhere).



8. A smooth right circular cone of semi-vertical angle $\alpha = \tan^{-1}(5/12)$ is at rest on a horizontal plane. A rubber ring of mass 2.5 kg which requires a force of 15 N for an extension of 10 cm is placed on the cone. Find the increase in the radius of the ring in equilibrium.

9. A block of mass m lies on wedge of mass M as shown in the figure. Answer the following parts separately.

(A) With what minimum acceleration must the wedge be moved towards right horizontally so that block m falls freely.



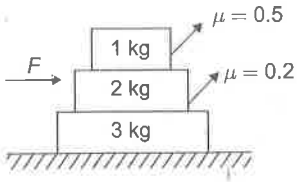
(B) Find the minimum friction coefficient required between wedge M and ground so that it does not move while block m slips down on it.

10. A car begins to move at time $t = 0$ and then accelerates along a straight track with a speed given by $V(t) = 2t^2$ ms^{-1} for $0 \leq t \leq 2$. After the end of acceleration, the car continues to move at a constant speed. A small block initially at rest on the floor of the car begins to slip at $t = 1$ s and stops slipping at $t = 3$ s. Find the coefficient of static and kinetic friction between the block and the floor.

11. In the figure shown,

(A) For what maximum value of force F can all these blocks move together?

(B) Find the value of force F at which sliding starts at other rough surfaces

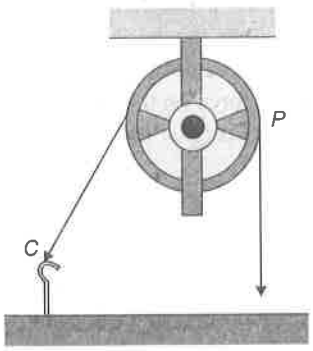


(C) Find acceleration of all blocks, nature and value of friction force of force $F = 18$ N.

Previous Year Questions

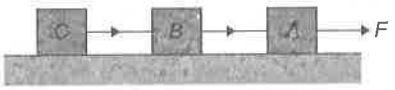
JEE Main

1. One end of massless rope, which passes over a massless and frictionless pulley P is tied to a hook C while the other end is free. The maximum tension that the rope can bear is 360 N. With what value of maximum safe acceleration (in ms^{-2}) can a man of 60 kg climb down the rope? **[AIEEE 2002]**



(A) 16 (B) 6
(C) 4 (D) 80

2. Three identical blocks of masses $m = 2$ kg are drawn by a force $F = 10.2$ N with an acceleration of 0.6 ms^{-2} on a frictionless surface, then what is the tension (in N) in the string between that blocks B and C? **[AIEEE 2002]**



(A) 9.2 (B) 7.8
(C) 4 (D) 9.8

3. A light string passing over a smooth light pulley connects two blocks of masses m_1 and m_2 (vertically). If the acceleration of the system is $g/8$, then the ratio of the masses is **[AIEEE 2002]**

(A) 8:1 (B) 9:7
(C) 4:3 (D) 5:3

4. When forces F_1, F_2, F_3 are acting on a particle of mass m such that F_2 and F_3 are mutually perpendicular, then the particle remains stationary. If the force F_1 is now removed, then the acceleration of the particle is **[AIEEE 2002]**

(A) $\frac{F_1}{m}$ (B) $\frac{F_1F_3}{mF_1}$

(C) $\frac{(F_2 - F_3)}{m}$ (D) $\frac{F_2}{m}$

5. A rocket with a lift-off mass 3.5×10^4 kg is blasted upwards with an initial acceleration of 10 ms^{-2}. Then the initial thrust of the blast is **[AIEEE 2003]**

(A) 3.5×10^5 N (B) 7.0×10^5 N
(C) 14.0×10^5 N (D) 1.75×10^5 N

6. A light spring balance hangs from the hook of the other light spring balance and a block of mass M kg hangs from the former one. Then the true statement about the scale reading is **[AIEEE 2003]**

(A) Both the scales read M kg each
(B) The scale of the lower one reads M kg and of the upper one zero
(C) The reading of the two scales can be anything but the sum of the readings will be M kg
(D) Both the scales read $M/2$ kg

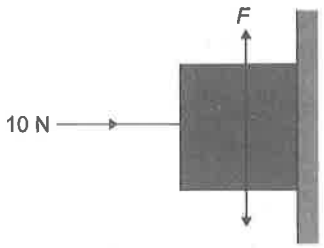
7. A block of mass M is pulled along a horizontal frictionless surface by a rope of mass m. If a force P is applied at the free end of the rope, the force exerted by the rope on the block is **[AIEEE 2003]**

(A) $\frac{Pm}{M+m}$ (B) $\frac{Pm}{M-m}$
(C) P (D) $\frac{PM}{M+m}$

8. A marble block of mass 2 kg lying on ice when given a velocity of 6 ms^{-1} is stopped by friction in 10 s. Then the coefficient of friction is **[AIEEE 2003]**

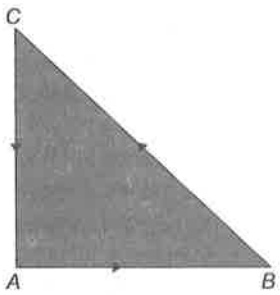
(A) 0.02 (B) 0.03
(C) 0.06 (D) 0.01

9. A horizontal force of 10 N is necessary to just hold a block stationary against a wall. The coefficient of friction between the block and the wall is 0.2. The weight of the block is **[AIEEE 2003]**



(A) 20 N (B) 50 N
(C) 100 N (D) 2 N

10. Three forces start acting simultaneously on a particle moving with velocity $\vec{v}$. These forces are represented in magnitude and direction by the three sides of a triangle ABC (as shown). The particle will now move with velocity **[AIEEE 2003]**



(A) less than $\vec{v}$
(B) greater than $\vec{v}$
(C) $|v|$ in the direction of largest force BC
(D) $\vec{v}$, remaining unchanged

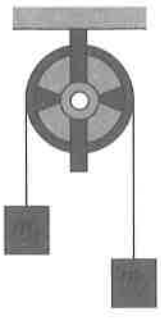
11. A spring balance is attached to the ceiling of a lift. A man hangs his bag on the spring and the spring reads 49 N, when the lift is stationary. If the lift moves downwards with an acceleration of 5 ms^{-2}, the reading of the spring balance will be **[AIEEE 2003]**

(A) 24 N (B) 74 N
(C) 15 N (D) 49 N

12. A block rests on a rough inclined plane making an angle of 30° with the horizontal. The coefficient of static friction between the block and the plane is 0.8. If the frictional force on the block is 10 N, the mass of the block (in kg) is ($g = 10$ m/s^2) **[AIEEE 2004]**

(A) 2.0 (B) 4.0
(C) 1.6 (D) 2.5

13. Two masses $m_1 = 5$ kg and $m_2 = 4.8$ kg tied to a string are hanging over a light frictionless pulley. What is the acceleration of the masses when lift is free to move? ($g = 9.8$ ms^{-2}) **[AIEEE 2004]**



(A) 0.2 ms^{-2} (B) 9.8 ms^{-2}
(C) 5 ms^{-2} (D) 4.8 ms^{-2}

14. A machine gun fires a bullet of mass 40 g with a velocity 1200 ms^{-1}. The man holding it, can exert a maximum force of 144 N on the gun. How many bullets can he fire per second at the most? **[AIEEE 2004]**

(A) 1 (B) 4
(C) 2 (D) 3

15. The upper half of an inclined plane with inclination ϕ is perfectly smooth, while the lower half is rough. A body starting from rest at the top will again come to rest at the bottom, if the coefficient of friction for the lower half is given by **[AIEEE 2005]**

(A) $2\sin\phi$ (B) $2\cos\phi$
(C) $2\tan\phi$ (D) $\tan\phi$

16. A smooth block is released at rest on a 45° incline and then slides a distances d. The time taken to slide is n times as much to slide on rough incline than on a smooth incline. The coefficient of friction is **[AIEEE 2005]**

(A) $\mu_k = 1-\frac{1}{n^2}$ (B) $\mu_k = \sqrt{1-\frac{1}{n^2}}$

(C) $\mu_s = 1-\frac{1}{n^2}$ (D) $\mu_s = \sqrt{1-\frac{1}{n^2}}$

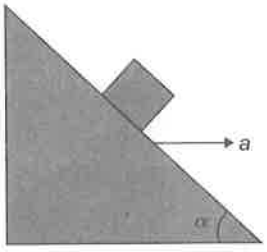
17. A particle of mass 0.3 kg is subjected to a force $F = -kx$ with $k = 15$ Nm^{-1}. What will be its initial acceleration, if it is released from a point 20 cm away from the origin? **[AIEEE 2005]**

(A) 3 ms^{-2} (B) 15 ms^{-2}
(C) 5 ms^{-2} (D) 10 ms^{-2}

18. Consider a car moving on a straight road with a speed of 100 ms^{-1}. The distance at which car can be stopped, is $[\mu_k = 0.5]$ **[AIEEE 2005]**

(A) 800 m (B) 1000 m
(C) 100 m (D) 400 m

19. A block is kept on a frictionless inclined surface with angle of inclination α. The incline is given an acceleration a to keep the block stationary. Then a is equal to **[AIEEE 2005]**



(A) $\frac{g}{\tan\alpha}$ (B) $g\,\text{cosec}\,\alpha$
(C) g (D) $g\tan\alpha$

20. A ball of mass 0.2 kg is thrown vertically upwards by applying a force by hand. If the hand moves 0.2 m while applying the force and the ball goes upto 2 m height further, find the magnitude of the force. Consider $g =$ 10 m/s^2. **[AIEEE 2006]**

(A) 4 N (B) 16 N
(C) 20 N (D) 22 N

21. A mass of M kg is suspended by a weightless string. The horizontal force that is required to displace it until the string makes an angle of 45° with the initial vertical direction is **[AIEEE 2006]**

(A) $Mg(\sqrt{2}+1)$ (B) $Mg\sqrt{2}$

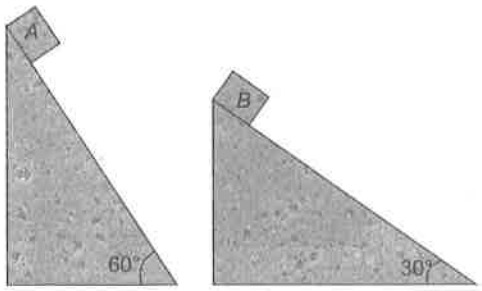
(C) $\frac{Mg}{\sqrt{2}}$ (D) $Mg(\sqrt{2}-1)$

22. A block of mass m is connected to another block of mass M by a spring (massless) of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstreched. Then a constant force F starts acting on the block of mass M to pull it. Find the force on the block of mass m. **[AIEEE 2007]**

(A) $\frac{mF}{M}$ (B) $\frac{(M+m)F}{m}$

(C) $\frac{mF}{(m+M)}$ (D) $\frac{MF}{(m+M)}$

23. Two fixed frictionless inclined planes making an angle 30° and 60° with the vertical are shown in the figure. Two blocks A and B are placed on the two planes. What is the relative vertical acceleration of A with respect to B? **[AIEEE 2010]**



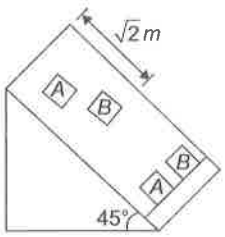
(A) 4.9 ms^{-2} in horizontal direction
(B) 9.8 ms^{-2} in vertical direction
(C) Zero
(D) 4.9 ms^{-2} in vertical direction

24. The minimum force required to start pushing a body up a rough (frictional coefficient μ) inclined plane is F_1 while the minimum force needed to prevent it from sliding down is F_2. If the inclined plane makes an angle θ from the horizontal such $\tan\theta = 2\mu$, then the ratio $\frac{F_1}{F_2}$ is **[AIEEE 2011]**

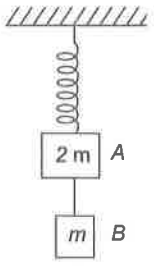
(A) 4 (B) 1
(C) 2 (D) 3

JEE Advanced

1. Two blocks A and B of equal masses are released from an inclined plane of inclination 45° at $t = 0$. Both the blocks are initially at rest. The coefficient of kientic friction between block A and the inclined plane is 0.2 while it is 0.3 for block B. Initially, block A is $\sqrt{2}$ m behind the block B. When and where their front faces will come in line. (Take $g = 10$ m/s²)

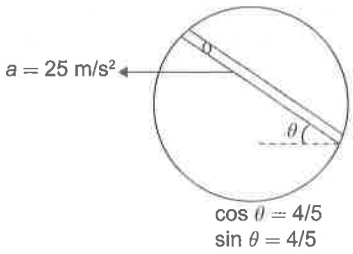


2. Two blocks A and B masses $2m$ and m, respectively, are connected by a massless and inextensible string. The whole system is suspended by a massless spring as shown in the figure. The magnitudes of acceleration of A and B, immediately after the string is cut, are respectively. **[JEE 2006]**

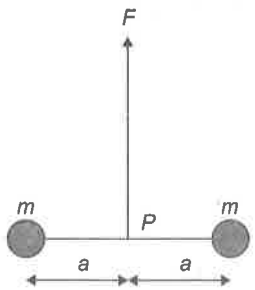


(A) g, g (B) $g, g/2$
(C) $g/2, g$ (D) $g/2, g/2$

3. A circular disc with a groove along its diameter is placed horizontally. A block of mass 1 kg is placed as shown. The coefficient of friction between the block and all surfaces of groove in contact is $\mu = 2/5$. The disc has an acceleration of 25 m/s². Find the acceleration of the block with respect to disc. **[JEE 2006]**



4. Two particles of mass m each are tied at the ends of a light string of length $2a$. The whole system is kept on a frictionless horizontal surface with the string held tight so that each mass is at a distance 'a' from the centre P (as shown in the figure). Now, the mid-point of the string is pulled vertically upwards with a small but constant force F. As a result, the particles move towards each other on the surfaces. The magnitude of acceleration, when the separation between them becomes $2x$, is **[JEE 2007]**



(A) $\frac{F}{2m}\frac{x}{\sqrt{a^2 - x^2}}$ (B) $\frac{F}{2m}\frac{x}{\sqrt{a^2 - x^2}}$

(C) $\frac{F}{2m}\frac{x}{a}$ (D) $\frac{F}{2m}\frac{\sqrt{a^2-x^2}}{x}$

5. **Statement 1**
A cloth covers a table. Some dishes are kept on it. The cloth can be pulled out without dislodging the dishes from the table
because
Statement 2
For every action, there is an equal and opposite reaction
(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1
(C) Statement 1 is True, Statement 2 is False
(D) Statement 1 is False, Statement 2 is True

[JEE 2007]

6. **Statement 1**
It is easier to pull a heavy object than to push it on a level ground.
and
Statement 2
The magnitude of frictional force depends on the nature of the two surfaces in contact.
(A) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is true; Statement 2 is NOT a correct explanation for Statement 1
(C) Statement 1 is true, Statement 2 is false
(D) Statement 1 is false, Statement 2 is true

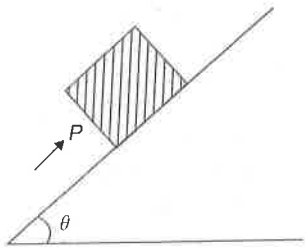
[JEE 2008]

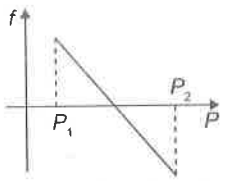
7. A block of base 10 cm × 10 cm and height 15 cm is kept on an inclined plane. The coefficient of friction between them is $\sqrt{3}$. The inclination θ of this inclined plane from the horizontal plane is gradually increased from 0°. Then **[JEE 2009]**
(A) at $\theta = 30°$, the block will start sliding down the plane
(B) the block will remain at rest on the plane up to certain θ and then it will topple
(C) at $\theta = 60°$, the block will start sliding down the plane and continue to do so at higher angles
(D) at $\theta = 60°$, the block will start sliding down the plane and on further increasing θ, it will topple at certain θ

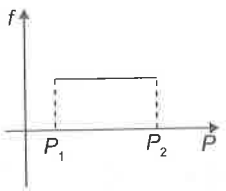
8. A piece of wire is bent in the shape of a parabola $y = kx^2$ (y-axis vertical) with a bead of mass m on it. The bead can slide on the wire without friction. It stays at the lowest point of the parabola when the wire is at rest. The wire is now accelerated parallel to the x-axis with a constant acceleration a. The distance of the new equilibrium position of the bead, where the bead can stay at rest with respect to the wire, from the y-axis is **[JEE 2009]**

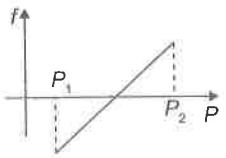
(A) $\frac{a}{gk}$ (B) $\frac{a}{2gk}$
(C) $\frac{2a}{gk}$ (D) $\frac{a}{4gk}$

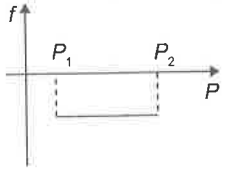
9. A block of mass m is on an inclined plane of angle θ. The coefficient of friction between the block and the plane is μ and $\tan\theta > \mu$. The block is held stationary by applying a force P parallel to the plane. The direction of force pointing up the plane is taken to the positive. As P is varied from $P = mg(\sin\theta - \mu\cos\theta)$ to $P = mg(\sin\theta + \mu\cos\theta)$, the frictional force f versus P graph will look like **[JEE 2010]**



(A) 

(B) 

(C) 

(D) 

10. A block is moving on an inclined plane making an angle 45° with horizontal and the coefficient of friction is μ. The force required to just push it up the inclined plane is 3 times the force required to just prevent it from sliding down. If we define $N = 10\mu$, then N is **[JEE 2011]**

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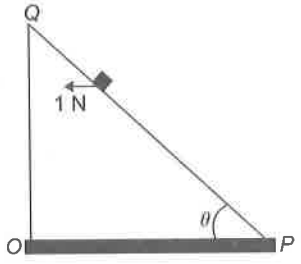
11. A small block of mass of 0.1 kg lies on a fixed inclined plane PQ which makes an angle θ with the horizontal. A horizontal force of 1 N acts on the block through its centre of mass as shown in the figure. The block remains stationary if (take $g = 10$ m/s^2)



(A) $\theta = 45°$

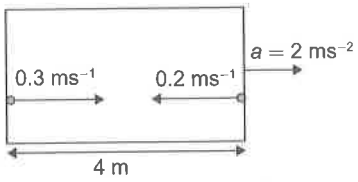
(B) $\theta > 45°$ and a frictional force acts on the block towards P

(C) $\theta > 45°$ and a frictional force acts on the block towards Q

(D) $\theta < 45°$ and a frictional force acts on the block towards Q

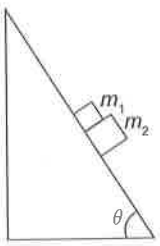
12. A rocket is moving in a gravity-free space with a constant acceleration of 2 ms^{-2} along the $+x$ direction (see the figure). The length of a chamber inside the rocket is 4 m. A ball is thrown from the left end of the chamber is $+x$ direction with a speed of 0.3 ms^{-1} relative to the rocket. At the same time, another ball is thrown in $-x$ direction with a speed of 0.2 ms^{-1} from its right end relative to the rocket. The time in seconds when the two balls hit each other is

[JEE Advanced 2014]



13. A block of mass $m_1 = 1$ kg and another mass $m_2 = 2$ kg are placed together (see the figure) on an inclined plane with angle of inclination θ. Various values of θ are given in List I. The coefficients of static and dynamic friction between the block m_2 and the plane are equal to $\mu = 0.3$. In List II, expressions for the friction on block m_2 are given. Match the correct expression of the friction in List II with the angles given in List I, and choose the correct option. The acceleration due to gravity is denoted by g.

[JEE Advanced 2014]



List I	List II
P. $\theta = 5°$	1. $m_2 g \sin\theta$
Q. $\theta = 10°$	2. $(m_1 + m_2) g \sin\theta$
R. $\theta = 15°$	3 $\mu m_2 g \cos\theta$
S. $\theta = 20°$	4. $\mu(m_1+m_2) g \cos\theta$

(A) P-1, Q-1, R-1, S-3

(B) P-2, Q-2, R-2, S-3

(C) P-2, Q-2, R-2, S-4

(D) P-2, Q-2, R-3, S-3

ANSWER KEYS

Exercises

JEE Main

(A) Newtons's Law of Motion

1. B	**2.** B	**3.** C	**4.** A	**5.** A	**6.** B	**7.** B	**8.** A	**9.** D	**10.** C
11. B	**12.** A	**13.** B	**14.** C	**15.** B	**16.** C	**17.** B	**18.** C	**19.** A	**20.** A
21. B	**22.** C	**23.** B	**24.** C	**25.** C	**26.** C	**27.** C	**28.** B	**29.** C	**30.** A
31. B	**32.** B	**33.** D	**34.** B	**35.** B	**36.** C	**37.** A	**38.** A	**39.** C	**40.** A
41. A	**42.** D	**43.** C	**44.** A	**45.** B	**46.** C	**47.** B	**48.** A	**49.** B	**50.** C
51. A	**52.** A	**53.** D	**54.** A	**55.** A					

JEE Advanced

(A) Newtons's Law of Motion

Single Correct

1. A **2.** A **3.** A **4.** B **5.** A **6.** A **7.** A **8.** D **9.** A **10.** B
11. A **12.** C **13.** A **14.** A **15.** B **16.** B **17.** A **18.** B **19.** C **20.** C
21. A **22.** B **23.** A **24.** A **25.** B **26.** B
27. (i) A (ii) A (iii) C (iv) D (v) B (vi) D (vii) B (viii) B
28. (i) A (ii) A (iii) A (iv) C (v) B (vi) C (vii) C (viii) B
29. A **30.** B **31.** A **32.** C **33.** D

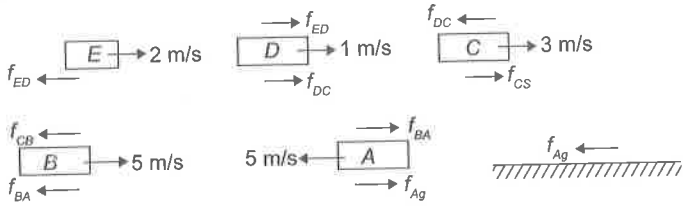
Multiple Correct

34. C **35.** B **36.** A, B, D **37.** C **38.** B **39.** A, C **40.** A, B, C **41.** B, D **42.** A, B, C **43.** B
44. B **45.** B **46.** B **47.** C

(B) Friction

Single Correct

48.



49.

50. A **51.** C **52.** C **53.** A **54.** B **55.** B
56. A **57.** A **58.** A **59.** D **60.** B **61.** D **62.** C **63.** (a) B (b) D (c) A
64. A **65.** A **66.** A **67.** A **68.** C **69.** A **70.** C **71.** A **72.** A **73.** A, B
74. C **75.** B **76.** A, B, C **77.** C **78.** A, C **79.** A, D **80.** B, C **81.** A, B **82.** B, C **83.** B,D
84. A, B, C, D

JEE Advanced

Level I

(A) Newtons's Law of Motion

1. $\frac{4g}{5}, \frac{2g}{5}, \frac{3g}{5}$ **2.** $a = \frac{4g}{9}, T = \frac{13mg}{18}$ **3.** (A) 5 m/s², (B) (i) 100 N, (ii) 120 N **4.** 2 s

5. $x_2 > x_1 > x_3\, x_1 : x_2 : x_3$: 15 : 18 : 10 **6.** 12 N **7.** 10/3 kg **8.** 300 N **9.** 55 **10.** 5 N, 16/31 kg
11. 2 s **12.** 12 m/s **13.** $aB = 4$ m/s² (↑)

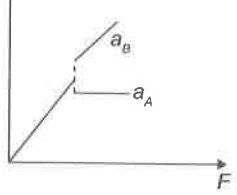
14. (A) $T = mg - \frac{kl}{2}$, (B) length of spring will less than 'l' and $T = 0$ in the string **15.** 322 N

16. (A) $T = \frac{(m+M)(a+g)}{2}$, (B) $N = m(a+g) + T$, (C) $T = \frac{mg}{2}$, (D) $T = \frac{(m+M)(a+g)}{3}$
(E) $N = m(a+g) - T$, (F) $T = \frac{mg}{2}$

(B) Friction

17. (A) 4 m/s², (B) 1.2 m/s², (C) 0 **18.** $\mu = 2$

19. a



20. 0.33 **21.** $\frac{1}{3}\mu\pi r^3$ **22.** 1/2 **23.** 1 kg **24.** 4/3 s **25.** 3/4 **26.** 40 N **27.** 1/2 s **28.** 5 s, 125/6 m

29. $10\hat{i}$ **30.** 0 **31.** 30 N

Level II

1. (A) 2 ms^{-2}, (B) 2.4 N 0.3 (C) 0.2 s **2.** 2 N **3.** 0.5 s **4.** 556.8 N , 1.47 s **5.** $\left(\frac{m_1 - 2m_2}{2m_2}\right)g$

6. (A) $a_A = \frac{3g\downarrow}{2} = a_B;\ a_C = 0; T = mg/2$ (B) $a_A = 2g\uparrow;\ a_B = 2g\downarrow;\ a_C = 0,\ T = 0$

(C) $a_A = a_B = g/2\uparrow; T = \frac{3mg}{2}; T = 2mg$ **7.** 2g/23 **8.** $\Delta r = \frac{mg\cot\alpha}{4\pi^2 k}$, 1 cm

9. (A) $a = g\cot\theta$, (B) $\mu_{min} = \frac{m\sin\theta\cos\theta}{m\cos^2\theta + M}$ **10.** $\mu_s = 0.4$, $\mu_k = 0.3$ **11.** 12 N, 21 N, 4 m/s², 2 m/s², 4 N, 6 N

Previous Year Questions

JEE Main

1. C	**2.** B	**3.** C	**4.** A	**5.** A	**6.** A	**7.** D	**8.** C	**9.** D	**10.** D
11. A	**12.** A	**13.** A	**14.** D	**15.** C	**16.** A	**17.** D	**18.** B	**19.** D	**20.** D
21. D	**22.** C	**23.** D	**24.** D						

JEE Advanced

1. 11.313 m	**2.** B	**3.** 10 m/s²	**4.** B	**5.** B	**6.** B	**7.** B	**8.** B	**9.** A
10. 5 N	**11.** A, C	**12.** 0002 and 0008		**13.** D				

CHAPTER 5

Circular Motion, Work, Power and Energy

CIRCULAR MOTION

When a particle moves in a plane such that its distance from a fixed (or moving) point remains constant, then its motion is called as the circular motion with respect to that fixed (or moving) point. That fixed point is called the centre and the distance between the fixed point and the particle is called the radius.

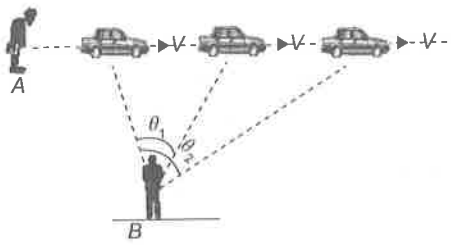


Figure 5.1

The car is moving in a straight line with respect to the man A (Fig. 5.1). But the man B continuously rotates his face to see the car. So with respect to man A, $\frac{d\theta}{dt} = 0$.

But with respect to man B, $\frac{d\theta}{dt} \neq 0$.

Therefore, we conclude that with respect to A, the motion of the car is in a straight line, but for man B, it has some angular velocity.

KINEMATICS OF CIRCULAR MOTION

Variables of Motion

(a) Angular Position

The angle made by the position vector with a given line (reference line) is called angular position Circular motion is a two-dimensional motion or motion in a plane. Suppose a particle P is moving in a circle of radius r and centre O (Fig. 5.2). The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the angular position of the particle. As the particle moves on the circle, its angular position θ changes. Suppose the point rotates an angle $\Delta\theta$ in time Δt.

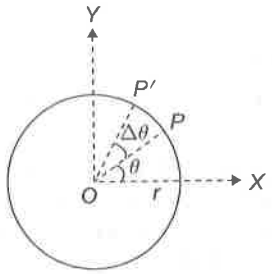


Figure 5.2

(b) Angular Displacement

Definition

The angle rotated by a position vector of the moving particle in a given time interval with some reference line is called its angular displacement.

Important Points

- It is dimensionless and has proper unit SI unit radian while other units are degree or revolution 2π rad $=$ $360° = 1$ rev.
- Infinitely small angular displacement is a vector quantity but finite angular displacement is not because the addition of the small angular displacement is commutative, while for large angular displacement, it is not. $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ but $\theta_1 + \theta_2 \neq \theta_2 + \theta_1$
- Direction of small angular displacement is decided by the right-hand thumb rule. When the fingers are directed along the motion of the point, then the thumb will represent the direction of angular displacement.

- Angular displacement can be different for different observers.

(c) *Angular Velocity*, ω

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Total angle of rotation}}{\text{Total time taken}};$$

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t},$$

where θ_1 and θ_2 are the angular position of the particle at time t_1 and t_2, respectively.

(ii) Instantaneous Angular Velocity

The rate at which the position vector of a particle with respect to the centre rotates is called as the instantaneous angular velocity with respect to the centre.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

Important Points

- It is an axial vector with dimensions $[T^{-1}]$ and SI unit rad/s.
- For a rigid body, as all the points will rotate through the same angle in the same time, angular velocity is a characteristic of the body as a whole, e.g., angular velocity of all points of earth about its own axis is $(2\pi/24)$ rad/h.
- If a body makes n rotations in t seconds, then angular velocity in radians per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

If T is the period and f the frequency of uniform circular motion, then

$$\omega_{av} = \frac{2\pi \times 1}{T} = 2\pi f.$$

- If $\theta = a - bt + ct^2$, then $\omega = \dfrac{d\theta}{dt} = -b + 2ct.$

Relation between speed and angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The rate of change of angular velocity is called the angular acceleration (α). Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

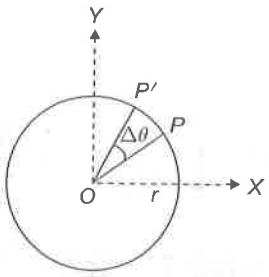


Figure 5.3

The linear distance PP' travelled by the particle (Fig. 5.3) in time Δt is

$$\Delta s = r\Delta\theta$$

or
$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

$$= r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

or
$$\frac{\Delta s}{\Delta t} = r\frac{d\theta}{dt}$$

or
$$v = r\omega$$

Here, v is the linear speed of the particle.
It is only valid for circular motion.

$v = r\omega$ is a scalar quantity $\left(\vec{\omega} \neq \dfrac{\vec{v}}{r}\right)$.

SOLVED EXAMPLE

EXAMPLE 1

If θ depends on time t in the following way, $\theta = 2t^2 + 3$, then
(a) find out ω average up to 3 s,
(b) ω at 3 s.

SOLUTION

$$\omega_{avg} = \frac{\text{Total angular displacement}}{\text{total time}}$$

$$= \frac{\theta_f - \theta_i}{t_2 - t_1}$$

and
$$\theta_f = 2\,(3)^2 + 3 = 21 \text{ rad}$$

$$\theta_i = 2\,(0) + 3 = 3 \text{ rad}.$$

So,
$$\omega_{avg} = \frac{21 - 3}{3} = 6 \text{ rad/s},$$

$$\omega_{\text{instantaneous}} = \frac{d\theta}{dt} = 4t,$$

$$\omega_{at\ t} = 3\text{ s} = 4 \times 3 = 12\text{ rad/s}.$$ ■

(d) Relative Angular Velocity

Angular velocity is defined with respect to the point from which the position vector of the moving particle is drawn. Here angular velocity of the particle w.r.t. 'O' and 'A' will be different (Fig. 5.4).

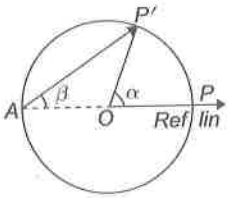


Figure 5.4

$$\omega_{PO} = \frac{d\alpha}{dt};\ \omega_{PA} = \frac{d\beta}{dt}.$$

Definition

Relative angular velocity of a particle A with respect to the other moving particle B is the angular velocity of the position vector of A with respect to B (Fig. 5.5). It is the rate at which the position vector of A with respect to B rotates at that instant.

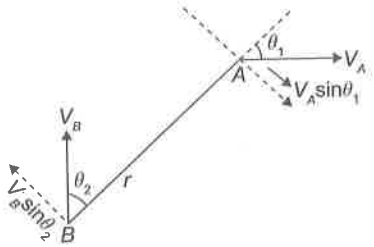


Figure 5.5

$$\omega_{AB} = \frac{(V_{AB})_{\perp}}{r_{AB}}.$$

Here,

$V_{AB\perp}$ = Relative velocity $\perp$ to position vector AB

$$= \frac{\text{Relative velocity of } A \text{ w.r.t. } B \text{ perpendicular to line } AB}{\text{Seperation between } A \text{ and } B}$$

$$(V_{AB})_{\perp} = V_A \sin\theta_1 + V_B \sin\theta_2$$

$$r_{AB} = r$$

$$\omega_{AB} = \frac{V_A \sin\theta_1 + V_B \sin\theta_2}{r}.$$

Important Points

- If two particles are moving on the same circle or different coplanar concentric circles in the same direction with different uniform angular speed ω_A and ω_B (Fig. 5.6), respectively, then the rate of change of angle between $\vec{OA}$ and $\vec{OB}$ is given as

$$\frac{d\theta}{dt} = \omega_B - \omega_A.$$

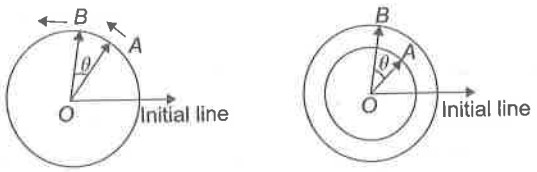


Figure 5.6

Thus, the time taken by one to complete one revolution around O w.r.t. the other is given as

$$T = \frac{2\pi}{\omega_{\text{rel}}} = \frac{2\pi}{\omega_2 - \omega_1}$$

$$= \frac{T_1 T_2}{T_1 - T_2}.$$

- If two particles are moving on two different concentric circles with different velocities, then the angular velocity of B relative to A as observed by A will depend on their positions and velocities. Consider the case when A and B are closest to each other and moving in the same direction as shown in Fig. 5.7. In this condition,

$$V_{\text{rel}} = |\vec{V}_B - \vec{V}_A| = V_B - V_A$$

$$r_{\text{rel}} = |\vec{r}_B - \vec{r}_A| = r_B - r_A.$$

So,
$$\omega_{BA} = \frac{(V_{\text{rel}})_{\perp}}{r_{\text{rel}}} = \frac{V_B - V_A}{r_B - r_A}$$

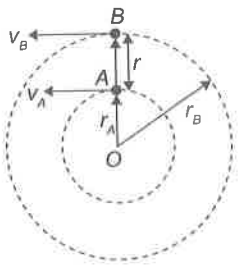


Figure 5.7

$(V_{rel})_{\perp}$ = Relative velocity perpendicular to the position vector.

SOLVED EXAMPLES

EXAMPLE 2

Two particles move on a circular path (one just inside and the other just outside) with angular velocities ω and 5ω starting from the same point. Then, which is incorrect:

(a) They cross each other at regular intervals of time $\frac{2\pi}{4\omega}$ when their angular velocities are oppositely directed.

(b) They cross each other at points on the path subtending an angle of 60° at the centre if their angular velocities are oppositely directed.

(c) They cross at intervals of time $\frac{\pi}{3\omega}$ if their angular velocities are oppositely directed.

(d) They cross each other at points on the path subtending 90° at the centre if their angular velocities are in the same sense.

SOLUTION

If the angular velocities are oppositely directed, they meet at intervals of

$$\text{time } t = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{6\omega}$$

$$= \frac{\pi}{3\omega}.$$

Angle subtended at the centre by the crossing points

$$\theta = \omega t = \frac{\pi}{3}$$

$$= 60°.$$

When their angular velocities are in the same direction,

$$t' = \frac{2\pi}{\omega_{rel}} = \frac{2\pi}{4\omega}$$

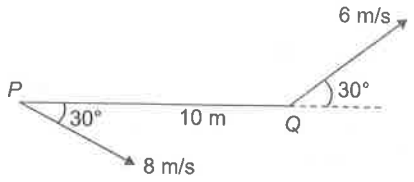
$$= \frac{\pi}{2\omega}$$

and $$\theta' = \frac{\pi}{2\omega} \times \omega = \frac{\pi}{2}.$$

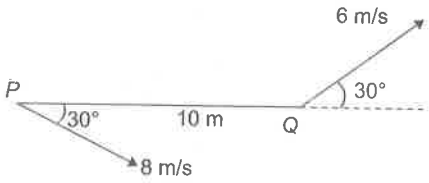
Ans. (a)

EXAMPLE 3

Two moving particles P and Q are 10 m apart at a certain instant. The velocity of P is 8 m/s making 30° with the line joining P and Q and that of Q is 6 m/s making 30° with PQ in the figure. Find the angular velocity of Q with respect to P in rad/s at that instant.



(A) 0 (B) 0.1
(C) 0.4 (D) 0.7



SOLUTION

Angular velocity of Q relative to

$$P = \frac{\text{Projection of } V_{QP} \text{ perpendicular to the line } PQ}{\text{Separation between } P \text{ and } Q}$$

$$\frac{V_Q \sin\theta_2 - V_p \sin\theta_1}{PQ} = \frac{6\sin 30° - (-8\sin 30°)}{10}$$

$$= 0.7 \text{ rad/s}$$

$\therefore$ (D).

(e) Angular Acceleration, α

(i) Average Angular Acceleration

Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2, respectively, then the average angular acceleration α_{av} is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

(ii) Instantaneous Angular Acceleration

It is the limit of average angular acceleration as Δt approaches zero, i.e.,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

$$= \omega \frac{d\omega}{d\theta}$$

Important Points

- It is also an axial vector with dimension $[T^{-2}]$ and unit rad/s².

- If $\alpha = 0$, circular motion is said to be uniform.

  As $\omega = \frac{d\theta}{dt}$, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$,

  i.e., second derivative of angular displacement w.r.t time gives angular acceleration.
- α is an axial vector, and the direction of α is along ω. If ω increases and opposite to ω, if ω decreases.

(f) Radial and Tangential Acceleration

Acceleration of a particle moving in a circle has two components: one is along $\hat{e}_t$ (along tangent) and the other is along $-\hat{e}_r$ (or towards centre). Of these, the first one is called the tangential acceleration (a_t) and the other is called radial or centripetal acceleration (a_r). Thus,

$$a_t = \frac{dv}{dt} = \text{rate of change of speed}$$

and

$$a_r = \omega^2 r = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$$

Here, a_t is the component which is responsible for changing the magnitude of speed of the particle in circular motion. a_r is the component which is responsible for changing the direction of the particle in circular motion.

The two components are mutually perpendicular. Therefore, net acceleration of the particle will be

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(r\omega^2)^2 + \left(\frac{dv}{dt}\right)^2}$$

$$= \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}.$$

Following three points are important regarding the above discussion.

1. In uniform circular motion, speed (v) of the particle is constant, i.e., $\frac{dv}{dt} = 0$. Thus, $a_t = 0$ and $a = a_r = r\omega^2$.
2. In accelerated circular motion, $\frac{dv}{dt}$ = positive, i.e., a_t is along $\hat{e}_t$ or tangential acceleration of particle is parallel to velocity $\vec{v}$ because $\vec{v} = r\omega\hat{e}_t$ and $\vec{a}_t = \frac{dv}{dt}\hat{e}_t$.
3. In decelerated circular motion, $\frac{dv}{dt}$ = negative and hence, tangential acceleration is anti-parallel to velocity $\vec{v}$.

(g) Relation Between Angular Acceleration and Tangential Acceleration

We know that

$$v = r\omega.$$

Here, v is the linear speed of the particle.

Differentiating again with respect to time, we have

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

or

$$a_t = r\alpha.$$

Here, $a_t = \frac{dv}{dt}$ is the rate of change of speed (not the rate of change of velocity).

SOLVED EXAMPLES

EXAMPLE 4

A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes from 5.0 m/s to 6.0 m/s in 2.0 s, find the angular acceleration.

SOLUTION

The tangential acceleration is given by

$$a_t = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

($\because$ Here speed increases uniformly, $a_t = \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$.)

$$= \frac{6.0 - 5.0}{2.0} \text{ m/s}^2 = 0.5 \text{ m/s}^2.$$

The angular acceleration is $\alpha = a_t/r$,

$$= \frac{0.5 \text{ m/s}^2}{20 \text{ cm}} = 2.5 \text{ rad/s}^2. \quad \blacksquare$$

EXAMPLE 5

A particle moves in a circle of radius 20 cm. Its linear speed at any time is given by $v = 2t$, where v is in m/s and t is in seconds. Find the radial and tangential acceleration at $t = 3$ s and hence calculate the total acceleration at this time.

SOLUTION

The linear speed at 3 s is

$$v = 2 \times 3 = 6 \text{ m/s}.$$

The radial acceleration at 3 s

$$= \frac{v^2}{r} = \frac{6\times6}{0.2}$$
$$= 180 \text{ m/s}^2.$$

The tangential acceleration is given by

$$\frac{dv}{dt} = 2 \text{ , because } v = 2t.$$

$\therefore$ Tangential acceleration is 2 m/s^2.

$$\text{Net acceleration} = \sqrt{a_r^2 + a_t^2}$$
$$= \sqrt{(180)^2 + (2)^2}$$
$$= 180.01 \text{ m/s}^2. \quad \blacksquare$$

T-1 Is it possible for a car to move in a circular path in such a way that it has a tangential acceleration but no centripetal acceleration?

EXAMPLE 6

A particle moves in a circle of radius 2.0 cm at a speed given by $v = 4t$, where v is in cm/s and t is in seconds.
(a) Find the tangential acceleration at $t = 1$ s.
(b) Find the total acceleration at $t = 1$ s.

SOLUTION

Tangential acceleration

$$a_t = \frac{dv}{dt}$$

or
$$a_t = \frac{dv}{dt}(4t) = 4 \text{ cm/s}^2.$$

$$a_c = \frac{v^2}{R} = \frac{(4)^2}{2}$$
$$= 8$$

$\Rightarrow$
$$a = \sqrt{a_t^2 + a_c^2}$$
$$= \sqrt{(4)^2 + (8)^2}$$
$$= 4\sqrt{5} \text{ m/s}^2. \quad \blacksquare$$

EXAMPLE 7

A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above the level ground. The string breaks, and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion?

SOLUTION

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times2}{9.8}}$$
$$= 0.64 \text{ s},$$
$$v = \frac{10}{t} = 15.63 \text{ m/s}.$$
$$a = \frac{v_B^2}{R} = 0.45 \text{ m/s}^2. \quad \blacksquare$$

EXAMPLE 8

Find the magnitude of the acceleration of a particle moving in a circle of radius 10 cm with uniform speed completing the circle in 4 s.

SOLUTION

The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$$v = 2\pi r/t = \frac{2\pi \times 10 \text{ cm}}{4 \text{ s}}$$
$$= 5\pi \text{ cm/s}.$$

The acceleration is

$$a_r = \frac{v^2}{r} = \frac{(5\pi \text{ cm/s})^2}{10 \text{ cm}}$$
$$= 2.5\ \pi^2 \text{ cm/s}^2. \quad \blacksquare$$

EXAMPLE 9

A particle moves in a circle of radius 20 cm. Its linear speed is given by $v = 2t$, where t is in seconds and v is in m/s. Find the radial and tangential acceleration at $t = 3$ s.

SOLUTION

The linear speed at $t = 3$ s is

$$v = 2t = 6 \text{ m/s}.$$

The radial acceleration at $t = 3$ s is

$$a_r = v^2/r = \frac{36\ \text{m}^2/\text{s}^2}{0.20\ \text{m}}$$

$$= 180\ \text{m/s}^2.$$

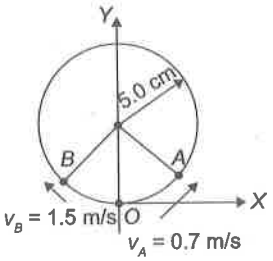
The tangential acceleration is

$$a_t = \frac{dv}{dt} = \frac{d(2t)}{dt}$$

$$= 2\ \text{m/s}^2.$$ ■

EXAMPLE 10

Two particles A and B start at the origin O and travel in opposite directions along the circular path at constant speeds $v_A = 0.7$ m/s and $v_B = 1.5$ m/s, respectively. Determine the time when they collide and the magnitude of the acceleration of B just before this happens.



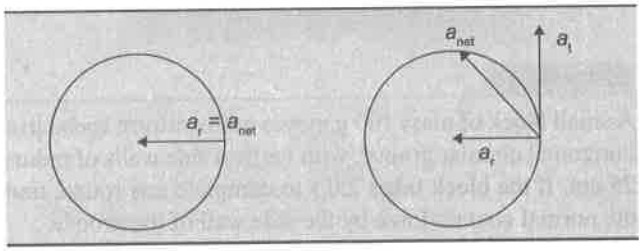
SOLUTION

$$1.5t + 0.7t = 2\pi R = 10\pi$$

$$\therefore \quad t = \frac{10\pi}{2.2} = 14.3\ \text{s}$$

$$a = \frac{v_B^2}{R} = 0.45\ \text{m/s}^2.$$ ■

Uniform circular motion	Non-uniform circular motion
1. Speed of the particle is constant, i.e., ω is constant	1. Speed of the particle is not constant, i.e., ω is not constant
2. $a_t = \frac{d\lvert\vec{v}\rvert}{dt} = 0$	2. $a_t = \frac{d\lvert\vec{v}\rvert}{dt} \neq 0$
$a_r = \frac{v^2}{r} \neq 0$	$a_r \neq 0$
$\therefore a_{net} = a_r$	$\therefore \vec{a}_{net} = \vec{a}_r + \vec{a}_t$

Uniform circular motion	Non-uniform circular motion
	

(h) Relations Among Angular Variables

These relations are also referred as equations of rotational motion (Fig. 5.8) and are

$$\omega = \omega_0 + \alpha t \qquad (1)$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad (2)$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \qquad (3)$$

These are valid only if angular acceleration is constant and are analogous to equations of translatory motion, i.e.,

$$v = u + at;$$

$$s = ut + \frac{1}{2}at^2$$

and $$v^2 = u^2 + 2as.$$

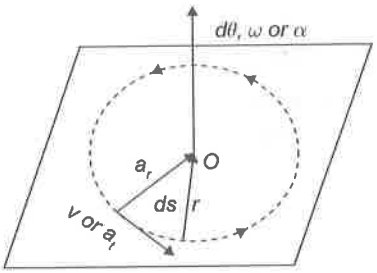


Figure 5.8

Dynamics of Circular Motion

In circular motion or motion along any curved path, Newton's law is applied in two perpendicular directions: one along the tangent and the other perpendicular to it, i.e., towards the centre. The component of net force towards the centre is called the **centripetal force.** The component of net force along the tangent is called the **tangential force.**

Tangential force $(F_t) = Ma_t = M\frac{dv}{dt} = M\alpha r$, where α is the angular acceleration.

Centripetal force $(F_c) = m\omega^2 r = \frac{mv^2}{r}$.

SOLVED EXAMPLE

EXAMPLE 11

A small block of mass 100 g moves with uniform speed in a horizontal circular groove, with vertical side walls of radius 25 cm. If the block takes 2.0 s to complete one round, find the normal contact force by the side wall of the groove.

SOLUTION

The speed of the block is

$$v = \frac{2\pi \times (25\text{ cm})}{2.0\text{ s}} = 0.785\text{ m/s}.$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0.785\text{ m/s})^2}{0.25}$$

$$= 2.46\text{ m/s}^2$$

towards the centre. The only force in this direction is the normal contact force due to the side walls. Thus, from Newton's second law, this force is

$$N = ma = (0.100\text{ kg})\,(2.46\text{ m/s}^2)$$

$$= 0.246\text{ N}.$$ ■

Centripetal Force

Concepts: The necessary resultant force towards the centre is called the centripetal force:

$$F = \frac{mv^2}{r} = m\omega^2 r.$$

1. A body moving with constant speed in a circle is not in equilibrium.
2. It should be remembered that in the absence of the centripetal force, the body will move in a straight line with constant speed.
3. It is not a new kind of force which acts on bodies. In fact, any force which is directed towards the centre may provide the necessary centripetal force.

Centrifugal Force

When a body is rotating in a circular path and the centripetal force vanishes, the body would leave the circular path. To an observer A who is not sharing the motion along the circular path, the body appears to fly off tangentially at the point of release. To another observer B who is sharing the motion along the circular path (i.e., the observer B is also rotating with the body which is released), it appears as if it has been thrown off along the radius away from the centre by some force. This inertial force is called centrifugal force.

Its magnitude is equal to that of the centripetal force $= \frac{mv^2}{r}$. Centrifugal force is a fictitious force which has to be applied as a concept only in a rotating frame of reference to apply Newton's law of motion in that frame.

FBD of the ball w.r.t non-inertial frame rotating with the ball (Fig. 5.9).

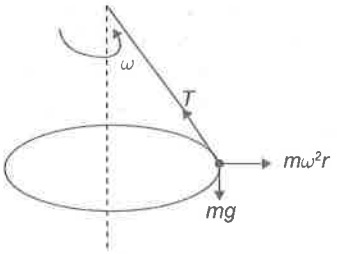


Figure 5.9

Suppose we are working from a frame of reference that is rotating at a constant, angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass m kept at a distance r from the axis of rotation, we have to assume that a force $mr\omega^2$ acts radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

T-2 A particle of mass m rotates in a circle of radius r with a uniform angular speed ω. It is viewed from a frame rotating about the same axis with a uniform angular speed ω_0. The centrifugal force on the particle is

(A) $m\omega^2 r$ (B) $m\omega_0^2 r$

(C) $m\left(\frac{\omega+\omega_0}{2}\right)^2 r$ (D) $m\omega_0\omega r$

Note

A rod moves with angular velocity ω (Fig. 5.10). Then we conclude the following for the points A and B in a rod.

$\alpha_A = \alpha_B$ $\quad s_B > s_A$

$\theta_A = \theta_B$ $\quad v_B > v_A$

$\omega_A = \omega_B$ $\quad a_{tB} > a_{tA}$

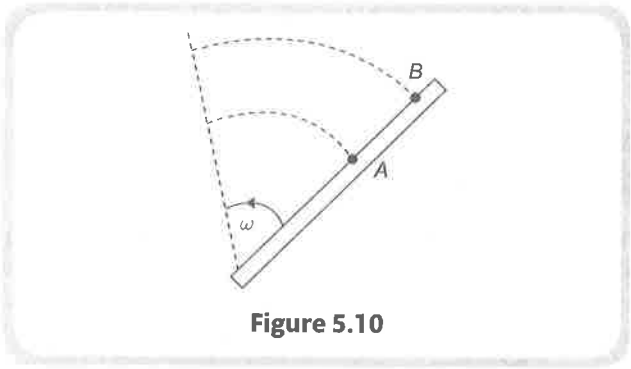
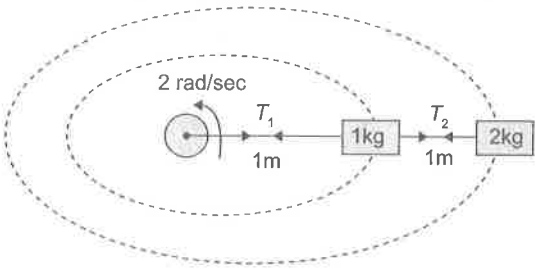


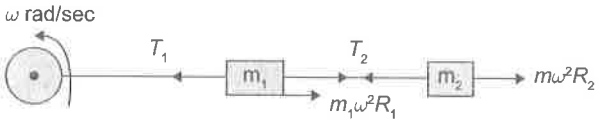
Figure 5.10

SOLVED EXAMPLES

EXAMPLE 12

Find out the tension of the strings T_1 and T_2 shown in the figure.





We know that $\omega_{m_1} = \omega_{m_2}$

SOLUTION

$$\Rightarrow \quad T_1 = m_1\omega^2 R_1 + T_2$$

$$T_2 = m_2\omega^2 R_2$$

$$T_2 = 2 \times 4 \times 2 = 16 \text{ N}$$

So $$T_1 = (1)(2)^2(1) + 16 \text{ N} = 4 + 16 \text{ N}$$

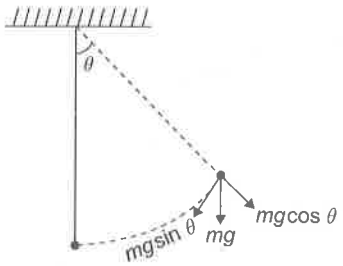
$$T_1 = 20 \text{ N}$$ ■

SIMPLE PENDULUM

EXAMPLE 13

A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle θ with the vertical. Find the tension in the string at this instant.

SOLUTION



As shown in the figure, the forces acting on the bob are the following: (a) the tension T, (b) the weight mg.

As the bob moves in a vertical circle with centre at O, the radial acceleration is v^2/L towards O. Taking the components along this radius and applying Newton's second law, we get

$$T - mg\cos\theta = mv^2/L$$

or, $$T = m(g\cos\theta + v^2/L)$$

$$|\vec{F}_{net}| = \sqrt{(mg\sin\theta)^2 + \left(\frac{mv^2}{L}\right)^2} = m\sqrt{g^2\sin^2\theta + \frac{v^4}{L^2}}.$$ ■

CIRCULAR MOTION IN HORIZONTAL PLANE

A ball of mass m attached to a light and inextensible string rotates in a horizontal circle of radius r with an angular speed ω about the vertical (Fig. 5.11). If we draw the force diagram of the ball, we can easily see that the component of tension force along the centre gives the centripetal force and the component of tension along the vertical balances the gravitational force. Such a system is called a conical pendulum.

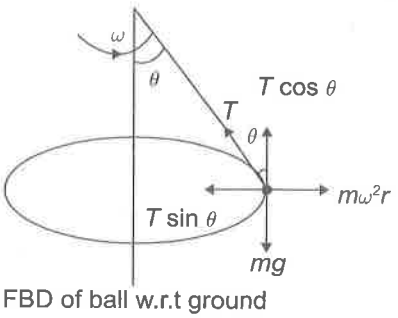


Figure 5.11

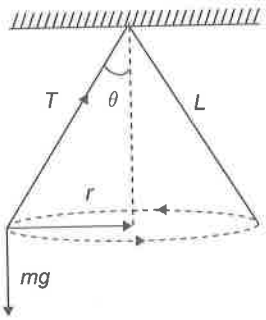
SOLVED EXAMPLE

EXAMPLE 14

A particle of mass m is suspended from a ceiling through a string of length L. The particle moves in a horizontal circle of radius r. Find (a) the speed of the particle and (b) the tension in the string.

SOLUTION

The situation is shown in the figure.



The angle θ made by the string with the vertical is given by

$$\sin\theta = r/L \qquad (1)$$

The forces on the particle are the following:

(i) the tension T along the string and

(ii) the weight mg vertically downward.

The particle is moving in a circle with a constant speed v. Thus, the radial acceleration towards the centre has magnitude v^2/r. Resolving the forces along the radial direction and applying. Newton's second law,

$$T\sin\theta = m(v^2/r). \qquad (2)$$

As there is no acceleration in the vertical direction, we have from Newton's law,

$$T\cos\theta = mg. \qquad (3)$$

Dividing (2) by (3),

$$\tan\theta = \frac{v^2}{rg}$$

or $$v = \sqrt{rg\tan\theta}\,.$$

And from Eq. (3), $$T = \frac{mg}{\cos\theta}\,.$$

Using Eq. (1), $$v = \frac{r\sqrt{g}}{\left(L^2 - r^2\right)^{1/4}}$$

and $$T = \frac{mgL}{\left(L^2 - r^2\right)^{1/2}}\,.$$

MOTION OF A MOTORCYCLIST ON A CURVED PATH

A cyclist having mass m moves with a constant speed v on a curved path as shown in Fig. 5.12.

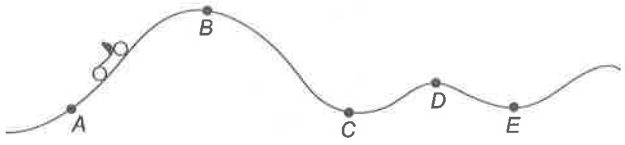


Figure 5.12

We divide the motion of the cyclist into four parts:

(1) from A to B **(2)** from B to C

(3) from C to D **(4)** from D to E

(1 and 3 are the same type of motion).

(1) Motion of the cyclist from A to B (Fig. 5.13)

$$N + \frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow \quad N = mg\cos\theta - \frac{mv^2}{R} \qquad (1)$$

$$f = mg\sin\theta. \qquad (2)$$

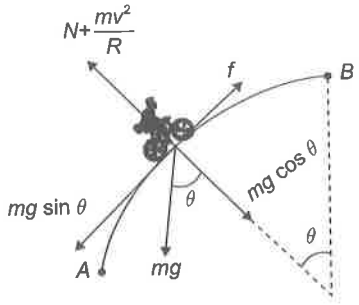


Figure 5.13

As the cyclist moves upward,

θ decreases and $\cos\theta$ increases.

$\therefore N$ increases

and

θ decreases $\sin\theta$ decreases.

$\therefore$ Frictional force required to balance $mg\sin\theta$ (as the cyclist is moving with constant speed) also decreases.

(2) Motion of the cyclist from B to C (Fig. 5.14)

$$N + \frac{mv^2}{R} = mg\cos\theta$$

$$\Rightarrow \quad N = mg\cos\theta - \frac{mv^2}{R} \qquad (1)$$

$$f = mg\sin\theta. \qquad (2)$$

Therefore, from B to C the normal force decrease but frictional force increases because θ increases.

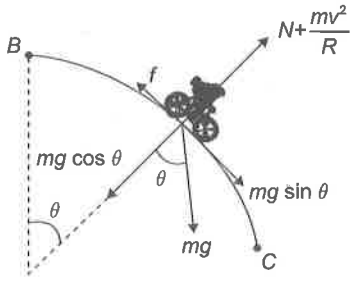


Figure 5.14

(3) Motion of the cyclist from D to E (Fig. 5.15)

$$N = \frac{mv^2}{R} + mg\cos\theta$$

$$f = mg\sin\theta.$$

From D to E, θ decreases; therefore, $mg\cos\theta$ increases. So N increase but f decreases.

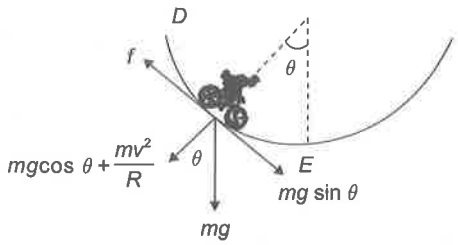


Figure 5.15

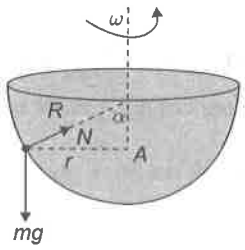
SOLVED EXAMPLES

EXAMPLE 15

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping from its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α, find the angular speed at which the bowl is rotating.

SOLUTION

Let ω be the angular speed of rotation of the bowl.
Two forces are acting on the ball:
(a) normal reaction N, (b) weight mg.



The ball is rotating in a circle of radius r $(= R\sin\alpha)$ with centre at A at an angular speed ω. Thus,

$$N\sin\alpha = mr\omega^2$$

$$= mR\omega^2\sin\alpha$$

$$N = mR\omega^2 \qquad (1)$$

and $$N\cos\alpha = mg \qquad (2)$$

Dividing Eqs. (1) by (2), we get

$$\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$$

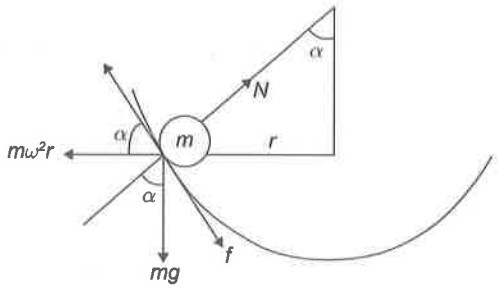
$$\therefore \quad \omega = \sqrt{\frac{g}{R\cos\alpha}}. \qquad \blacksquare$$

EXAMPLE 16

If friction is present between the surface of the ball and bowl, then find out the range of ω for which the ball does not slip (μ is the friction coefficient). Friction develops a range of ω for which the particle will be at rest.

SOLUTION

(a) When $\omega > \omega_0$



In this situation, the ball has a tendency to slip upwards. So the frictional force will act downwards. So F.B.D. of the ball is given as

$$N = m\omega^2 r \sin\alpha + mg\cos\alpha \quad (1)$$

$$f + mg\sin\alpha = m\omega^2 r\cos\alpha \quad (2)$$

$$\therefore \quad f_{max} = \mu N = \mu(m\omega^2 r\sin\alpha + mg\cos\alpha)$$

$$r = R\sin\alpha.$$

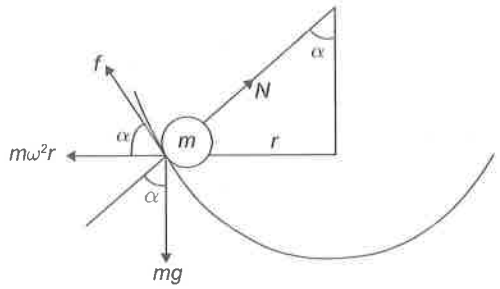
Substituting the values of f_{max} and r in Eq. (2), we get

$$\Rightarrow \mu(m\omega^2 r\sin\alpha + mg\cos\alpha) \geq m\omega^2 r\cos\alpha - mg\sin\alpha$$

$$\therefore \quad \mu\,(m\omega^2 R\sin 2\alpha + mg\cos\alpha) \geq m\omega^2 R\sin\alpha\cos\alpha - mg\sin\alpha$$

$$\omega \leq \sqrt{\frac{\mu g\cos\alpha + g\sin\alpha}{R\sin\alpha(\cos\alpha - \mu\sin\alpha)}}$$

(b) When $\omega < \omega_0$



In this situation, the ball has a tendency to slip downwards. So the frictional force will act upwards. So F.B.D. of the ball is given as

$$\Rightarrow \quad f + m\omega^2 r\cos\alpha = mg\sin\alpha$$

$$f = m\,(g\sin\alpha - \omega^2 r\cos\alpha) \quad (1)$$

$$N = mg\cos\alpha + m\omega^2 r\sin\alpha \quad (2)$$

$$f_{max} = \mu N$$

$$= \mu\,(mg\cos\alpha + m\omega^2 r\sin\alpha).$$

For equilibrium,

$$\Rightarrow \quad \mu\,(g\cos\alpha + \omega^2 r\sin\alpha) \geq (g\sin\alpha - \omega^2 r\cos\alpha).$$

$$\Rightarrow \quad \text{Substituting} \quad r = R\sin\alpha,$$

$$\omega \geq \sqrt{\frac{g(\sin\alpha - mg\cos\alpha)}{R\sin\alpha(\mu\sin\alpha + \cos\alpha)}}.$$

CIRCULAR TURNING ON ROADS

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways:

1. by friction only
2. by banking of roads only
3. by both friction and banking of roads.

In real life, the necessary centripetal force is provided by both friction and banking of roads. Now let us write the equations of motion in each of the three cases separately and see what are the constants in each case.

By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by the force of friction f acting towards the centre.

Thus, $$f = \frac{mv^2}{r}.$$

Further, the limiting value of f is μN

or $$f_L = \mu N = \mu mg \text{ (as } N = mg).$$

Therefore, for a safe turn without sliding, $\frac{mv^2}{r} \leq f_L$

or $$\frac{mv^2}{r} \leq \mu mg$$

or $$\mu \geq \frac{v^2}{rg}$$

or $$v \leq \sqrt{\mu rg}.$$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence

on friction, the roads are banked at the turn so that the outer part of the road is somewhat lifted compared to the inner part (Fig. 5.16).

Applying Newton's second law along the radius and the first law in the vertical direction,

$$N \sin \theta = \frac{mv^2}{r}$$

or $N \cos \theta = mg.$

From these two equations, we get

$$\tan \theta = \frac{v^2}{rg}$$

or $v = \sqrt{rg \tan \theta}$.

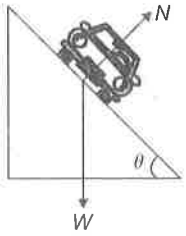


Figure 5.16

By Both Friction and Banking of Roads

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle. Of these forces, the first force, i.e., weight (mg) is fixed both in magnitude and direction (Fig. 5.17).

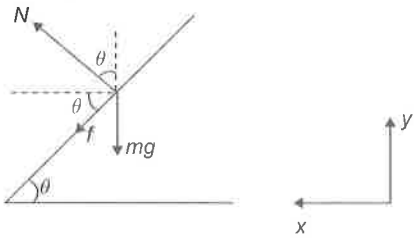


Figure 5.17

The direction of the second force, i.e., normal reaction N is also fixed (perpendicular or road) while the direction of the third, i.e., friction f can be either inwards or outwards while its magnitude can be varied up to a maximum limit ($f_L = \mu N$). So the magnitude of normal reaction N and directions plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre. Of these, m and r are also constant. Therefore, magnitude of N and directions plus magnitude of friction mainly depend on the speed of the vehicle v. Thus, this situation varies from problem to problem. Even though we can see the following:

1. Friction f will be outwards if the vehicle is at rest $v = 0$. Because in that case the component weight $mg \sin \theta$ is balanced by f.
2. Friction f will be inwards if

$$v > \sqrt{rg \tan \theta}\ .$$

3. Friction f will be outwards if

$$v < \sqrt{rg \tan \theta}\ .$$

4. Friction f will be zero if

$$v = \sqrt{rg \tan \theta}\ .$$

5. For maximum safe speed (Fig. 5.17),

$$N \sin \theta + f \cos \theta = \frac{mv^2}{r} \quad (1)$$

$$N \cos \theta - f \sin \theta = mg. \quad (2)$$

We know that maximum value of friction is

$$f = \mu N.$$

$$\therefore \quad \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} = \frac{v^2}{rg}$$

$$\therefore \quad v_{max} = \sqrt{\frac{rg(\mu + \tan \theta)}{(1 - \mu \tan \theta)}}\ .$$

Similarly, $$v_{min} = \sqrt{\frac{rg(\mu - \tan \theta)}{(1 + \mu \tan \theta)}}\ .$$

Note

- The expression $\tan \theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.
- The expression $\tan \theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical to negotiate a safe turn.

DEATH WELL

A motorcyclist is driving in a horizontal circle on the inner surface of a vertical cylinder of radius R. Friction coefficient between the tyres of the motorcyclist and the surface

of the cylinder is μ. Find out the minimum velocity for which the motorcyclist can do this. v is the speed of the motorcyclist and m is his mass (Fig. 5.18).

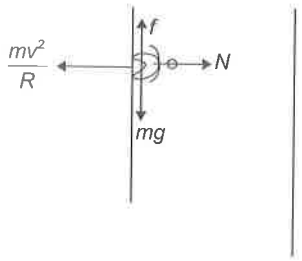


Figure 5.18

$$N = \frac{mv^2}{R}$$

$$f = mg$$

$$f_{max} = \frac{\mu mv^2}{R}.$$

Cyclist does not drop down when

$$f_{max} \geq mg$$

$$\Rightarrow \quad \frac{\mu mv^2}{R} \geq mg$$

$$v \geq \sqrt{\frac{gR}{\mu}}.$$

MOTION OF A CYCLIST ON A CIRCULAR PATH

Suppose a cyclist is going at a speed v on a circular horizontal road of radius r which is not banked. Consider the cycle and the rider together as the system. The centre of mass C (Fig. 5.19) of the system is going in a circle with the centre at O and radius r.

Let us choose O as the origin, OC as the x-axis and vertically upward as the Z-axis. This frame is rotating at an angular speed $\omega = \frac{v}{r}$ about the z-axis. In this frame, the system is at rest. Since we are working from a rotating frame of reference, we will have to apply a centrifugal force on each particle. The net centrifugal force on the system will be $M\omega^2 r = Mv^2/r$, where M is the total mass of the system. This force will act through the centre of mass. Since the system is at rest in this frame, no other pseudo force is needed.

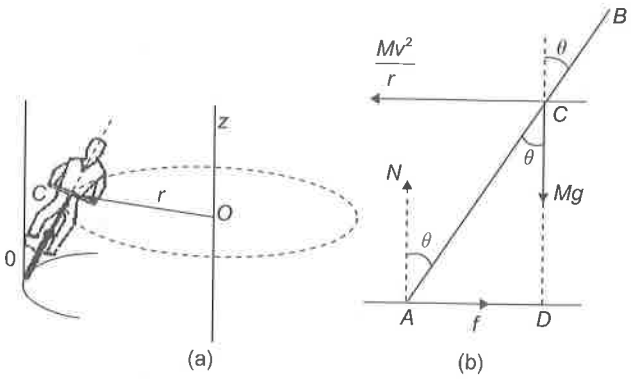


Figure 5.19

Figure 5.19 shows the forces. The cycle is bent at an angle θ with the vertical. The forces acting on the cycle are the following:

(i) weight mg,
(ii) normal force N
(iii) friction f and
(iv) centrifugal force $\frac{Mv^2}{r}$.

In the frame considered, the system is at rest. Thus, the total external force and the total external torque must be zero. Let us consider the torques of all the forces about the point A. The torques of N and f about A are zero because these forces pass through A. The torque of Mg about A is $Mg(AD)$ in the clockwise direction and that of $\frac{Mv^2}{r}$ is $\frac{Mv^2}{r}(CD)$ in the anticlockwise direction. For rotational equilibrium,

$$Mg(AD) = \frac{Mv^2}{r}(CD)$$

or,
$$\frac{AD}{CD} = \frac{v^2}{rg}$$

or,
$$\tan\theta = \frac{v^2}{rg}. \qquad (1)$$

Thus, the cyclist bends at an angle $\tan^{-1}\left[\frac{v^2}{rg}\right]$ with the vertical.

T-3 A car driver going at a speed of v suddenly finds a wide wall at a distance r. Should he apply breaks or turn the car in a circle of radius r to avoid hitting the wall?

EFFECT OF EARTH'S ROTATION ON APPARENT WEIGHT

The earth rotates about its axis at an angular speed of 1 revolution per 24 h. The line joining the north and the south poles is the axis of rotation.

Every point on the earth moves in a circle. A point at the equator moves in a circle of radius equal to the radius of the earth and the centre of the circle is the same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place point on the earth (Fig. 5.20).

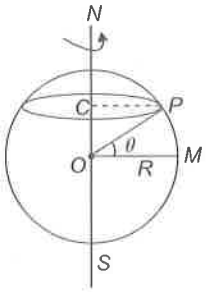


Figure 5.20

Drop a perpendicular PC from P to the axis SN. The place P rotates in a circle with the centre at C. The radius of this circle is CP. The angle between the line OM and the radius OP through P is called the latitude of the place point. We have

$$CP = OP \cos\theta$$

or,
$$r = R\cos\theta,$$

where R is the radius of the earth.

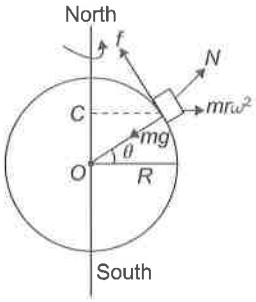


Figure 5.21

If we calculate work from the frame of reference of the earth, we shall have to assume the existence of a pseudo force. In particular, a centrifugal force $m\omega^2 r$ has to be assumed on any particle of mass m placed at P. If we consider a block of mass m at point P (Fig. 5.21), then this block is at rest with respect to the earth. If we resolve the forces along and perpendicular to the line joining the centre of earth, then

$$N + mr\omega^2\cos\theta = mg$$

$$\Rightarrow \quad N = mg - mr\omega^2\cos\theta$$

$$\Rightarrow \quad N = mg - mR\omega^2\cos^2\theta.$$

SOLVED EXAMPLE

EXAMPLE 17

A body weighs 98 N on a spring balance at the north pole. What will be its weight recorded on the same scale if it is shifted to the equator? Use $g = GM/R^2 = 9.8$ m/s^2 and the radius of the earth $R = 6400$ km.

SOLUTION

At the poles, the apparent weight is the same as the true weight.

Thus, $$98\text{ N} = mg = m\,(9.8\text{ m/s}^2).$$

At the equator, the apparent weight is

$$mg' = mg - m\omega^2 R$$

The radius of the earth is 6400 km and the angular speed is

$$\omega = \frac{2\pi\text{ rad}}{24\times60\times60\text{ s}}$$

$$= 7.27\times10^{-6}\text{ rad/s}$$

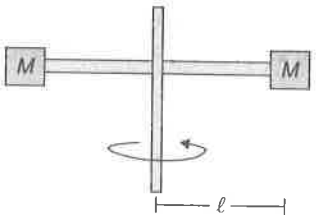
$$mg' = 98\text{ N} - (10\text{ kg})\,(7.27\times10^{-5}\text{ s}^{-1})^2\,(6400\text{ km})$$

$$= 97.66\text{ N}.$$ ■

SOME SOLVED EXAMPLES

EXAMPLE 1

Two blocks each of mass M are connected to the ends of a light frame as shown in the figure. The frame is rotated about the vertical line of symmetry. The rod breaks if the tension in it exceeds T_0. Find the maximum frequency with which the frame may be rotated without breaking the rod.



SOLUTION

Consider one of the blocks. If the frequency of revolution is f, the angular velocity is $\omega = 2\pi f$. The acceleration towards the centre is $\omega^2\ell = 4\pi^2 f^2\ell$. The only horizontal force on the block is the tension of the rod. At the point of breaking, this force is T_0. So from Newton's law,

$$T_0 = M \times 4\pi^2 f^2 \ell$$

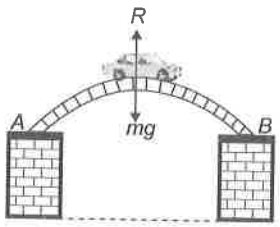
or,
$$f = \frac{1}{2\pi}\left[\frac{T_0}{M\ell}\right]^{1/2}.$$ ■

EXAMPLE 2

Prove that a motor car moving over a convex bridge is lighter than the same car resting on the same bridge.

SOLUTION

The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (figure).



The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.

$$\therefore \quad mg - R = \frac{mv^2}{r}$$

or
$$R = mg - \frac{mv^2}{r}.$$

Clearly, $R < mg$, i.e., the weight of the moving car is less than the weight of the stationary car. ■

EXAMPLE 3

A body weighing 0.4 kg is whirled in a vertical circle with a string making two revolutions per second. If the radius of the circle is 1.2 m. Find the tension (a) at the top of the circle and (b) at the bottom of the circle. Consider $g = 10\ \text{ms}^{-2}$ and $\pi = 3.14$.

SOLUTION

Mass, $m = 0.4$ kg;

time period $= \frac{1}{2}$ s; radius, $r = 1.2$ m.

Angular velocity,

$$\omega = \frac{2\pi}{1/2} = 4\pi \text{ rad s}^{-1}$$
$$= 12.56 \text{ rad s}^{-1}.$$

(a) At the top of the circle,

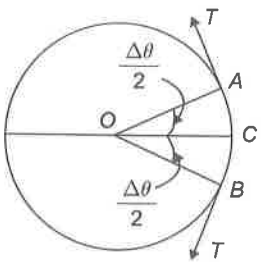
$$T = \frac{mv^2}{r} - mg$$
$$= mr\omega^2 - mg = m(r\omega^2 - g)$$
$$= 0.4\,(1.2 \times 12.56 \times 12.56 - 9.8)\ \text{N}$$
$$= 71.2\ \text{N}.$$

(b) At the lowest point,

$$T = m(r\omega^2 + g) = 80\ \text{N}.$$ ■

EXAMPLE 4

A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of ring moves with velocity v. Find the tension in the ring.



SOLUTION

Consider a small part ACB of the ring that subtends an angle $\Delta\theta$ at the centre as shown in the figure.

Let the tension in the ring be T.

The forces on this elementary portion ACB are the following:

(i) tension T by the part of the ring left to A
(ii) tension T by the part of the ring to B
(iii) weight $(\Delta m)g$
(iv) normal force N by the table.

As the elementary portion ACB moves in a circle of radius R at constant speed v, its acceleration towards the centre is $\frac{(\Delta m)v^2}{R}$.

Resolving the force along the radius CO,

$$T\cos\left(90° - \frac{\Delta\theta}{2}\right) + T\cos\left(90° - \frac{\Delta\theta}{2}\right) = \Delta m\frac{v^2}{R} \quad (1)$$

$$2T\sin\frac{\Delta\theta}{2} = \Delta m\frac{v^2}{R} \quad (2)$$

Length of the part $ACB = R\Delta\theta$. The mass per unit length of the ring is $\frac{m}{2\pi R}$.

$\therefore$ Mass of this portion ACB,

$$\Delta m = \frac{R\Delta\theta m}{2\pi R} = \frac{m\Delta\theta}{2\pi}$$

Using this value of Δm in Eq. (2),

$$2T\sin\frac{\Delta\theta}{2} = \frac{m\Delta\theta v^2}{2\pi R}$$

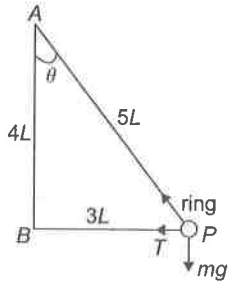
$$\therefore \quad T = \frac{mv^2}{2\pi R}\left(\frac{\frac{\Delta\theta}{2}}{\sin\frac{\Delta\theta}{2}}\right)$$

Since $\Delta\theta$ is small, $\left(\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)$ is equal to 1,

$$T = \frac{mv^2}{2\pi R}. \quad \blacksquare$$

EXAMPLE 5

A small smooth ring of mass m is threaded on a light inextensible string of length $8L$ which has its ends fixed at points in the same vertical line at a distance $4L$ apart. The ring describes horizontal circles at a constant speed with both parts of the string taut and with the lower portion of the string horizontal. Find the speed of the ring and the tension in the string. The ring is then tied at the mid-point of the string and made to perform horizontal circles at constant speed of $3\sqrt{Lg}$. Find the tension in each part of the string.



SOLUTION

When the string passes through the ring, the tension in the string is the same in both parts. Also from geometry

$BP = 3L$ and $AP = 5L$.

$$T\cos\theta = \frac{4}{5}T = mg \quad (1)$$

$$T + T\sin\theta = T\left(1 + \frac{3}{5}\right) = \frac{8}{5}T$$

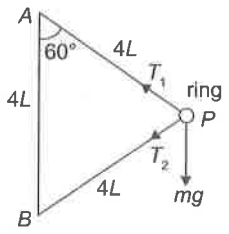
$$= \frac{mv^2}{BP} = \frac{mv^2}{3L} \quad (2)$$

$$\frac{v^2}{3Lg} = 2$$

$$v = \sqrt{6Lg}.$$

From Eq. (1),

$$T = \frac{mg}{4/5} = \frac{5}{4}mg.$$



In the second case, ABP is an equilateral triangle.

$$T_1\cos 60° = mg + T_2\cos 60°$$

$$T_1 - T_2 = \frac{mg}{\cos 60°} = 2mg \quad (3)$$

$$T_1 \sin 60° + T_2 \sin 60° = \frac{mv^2}{r}$$

$$= \frac{9mgL}{4L \sin 60°}$$

$$T_1 + T_2 = \frac{9mg}{4 \sin^2 60°} = 3mg \qquad (4)$$

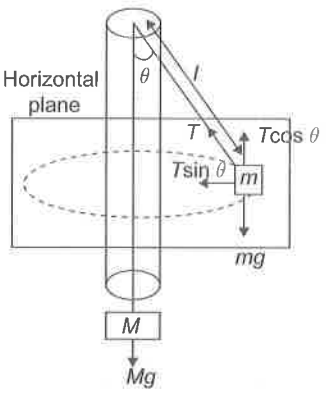
Solving Eqs. (3) and (4),

$$T_1 = \frac{5}{2}mg;$$

$$T_2 = \frac{1}{2}mg.$$ ■

EXAMPLE 6

A large mass M and a small mass m hang at the two ends of the string that passes through a smooth tube as shown in the figure. The mass m moves around in a circular path, which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is ℓ and θ is the angle this length makes with vertical. What should be the frequency of rotation of mass m so that M remains stationary?



SOLUTION

The forces acting on mass m and M are shown in the figure. When mass M is stationary,

$$T = Mg, \qquad (1)$$

where T is the tension in the string.

For the smaller mass, the vertical component of tension $T \cos \theta$ balances mg and the horizontal component $T \sin \theta$ supplies the necessary centripetal force.

$$T \cos \theta = mg \qquad (2)$$

$$T \sin \theta = mr\omega^2 \qquad (3)$$

ω is the angular velocity and r is the radius of horizontal circular path.

From Eqs. (1) and (3), $Mg \sin \theta = mr\omega^2$

$$\omega = \sqrt{\frac{Mg \sin \theta}{mr}}$$

$$= \sqrt{\frac{Mg \sin \theta}{m\ell \sin \theta}} = \sqrt{\frac{Mg}{m\ell}}.$$

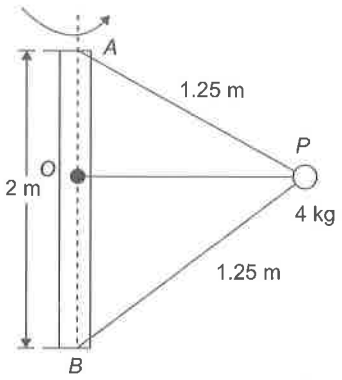
$$\text{Frequency of rotation} = \frac{1}{T}$$

$$= \frac{1}{2\pi / \omega} = \frac{\omega}{2\pi}$$

$$\therefore \quad \text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}.$$ ■

EXAMPLE 7

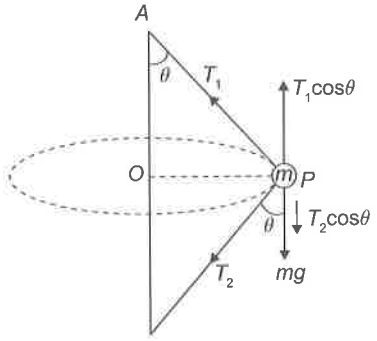
The 4-kg block in the figure is attached to the vertical rod by means of two strings. When the system rotates about the axis of the rod, the two strings are extended as indicated in the figure. How many revolutions per minute must the system make in order that the tension in the upper string is 60 N. What is tension in the lower string?



SOLUTION

The forces acting on block P of mass 4 kg are shown in the figure. If θ is the angle made by the strings with the vertical, T_1 and T_2 are the tensions in strings for equilibrium in the vertical direction.

$$T_1 \cos \theta = T_2 \cos \theta + mg$$



$$(T_1 - T_2)\cos\theta = mg$$

$$\cos\theta = \frac{1}{1.25} = \frac{4}{5}$$

$$\left[\because \cos\theta = \frac{OA}{AP} = \frac{1}{1.25}\right]$$

$$\therefore \quad T_1 - T_2 = \frac{mg}{\cos\theta} = \frac{5mg}{4}$$

$$= \frac{5}{4} \times 4 \times 9.8 = 49 \text{ N}.$$

Given $T_1 = 60$ N

$$T_2 = T_1 - 49 = 60 \text{ N} - 49 \text{ N}$$

$$= 11 \text{ N}.$$

The net horizontal force ($T_1 \sin\theta + T_2 \sin\theta$) provides the necessary centripetal force $m\omega^2 r$.

$$\therefore \quad (T_1 + T_2)\sin\theta = m\omega^2 r$$

$$\Rightarrow \quad \omega^2 = \frac{(T_1 + T_2)\sin\theta}{mr}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{1 - (4/5)^2} = \frac{3}{5}.$$

$$r = OP = \sqrt{1.25^2 - 1^2}$$

$$= 0.75$$

$$\therefore \quad \omega^2 = \frac{(60 + 11)\frac{3}{5}}{4 \times 0.75} = 0.75$$

$$\Rightarrow \quad \omega = \sqrt{14.2} = 3.768 \text{ rad/s}.$$

$$\text{Frequency of revolution} = \frac{\omega}{2\pi}$$

$$= \frac{3.768}{2 \times 3.14} = 0.6 \text{ rev/s}$$

or 36 rev/min. ■

WORK

Work is said to be done by a force when the force produces a displacement in the body on which it acts in any direction except perpendicular to the direction of the force.

Work Done by a Constant Force

Consider an object that undergoes a displacement S along a straight line while being acted upon by a force F that makes an angle θ with S as shown (Fig. 5.22).

The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement,

i.e., $$W = FS\cos\theta \qquad (1)$$

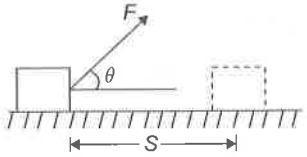


Figure 5.22

Work done is a scalar quantity and its SI unit is N-m or joule (J). We can also write work done as a scalar product of force and displacement,

$$W = \vec{F} \cdot \vec{S}, \qquad (2)$$

where S is the displacement of the point of application of the force.

From this definition, we conclude the following points:

(A) *Work Done by a Force is Zero if the Displacement is Perpendicular to the Force ($\theta = 90°$).*

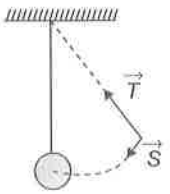


Figure 5.23

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Example

The tension in the string of a simple pendulum is always perpendicular to displacement (Fig. 5.23).

So, work done by the tension is zero.

(B) *If The Angle Between Force and Displacement is Acute ($\theta < 90°$), We Say that the Work Done by the Force is Positive.*

Examples

1. When a load is lifted, the lifting force and the displacement act in the same direction. So, work done by the lifting force is positive.
2. When a spring is stretched, both the stretching force and the displacement act in the same direction. So work done by the stretching force is positive.

(C) *If The Angle Between Force and Displacement is Obtuse ($\theta > 90°$), We Say that the Work Done by the Force is Negative.*

Example

When a body is lifted, the work done by the gravitational force is negative. This is because the gravitational force acts vertically downwards, while the displacement is in the vertically upwards direction.

Important Points about Work

1. Work is said to be done by a force when its point of application moves by some distance. Force does no work if the point of application of force does not move ($S = 0$).

   Example

   A person carrying a load on his head and standing at a given place does no work.

2. Work is defined for an interval or displacement. There is no term like instantaneous work similar to instantaneous velocity.

   Work done by 10 N force in both the cases are same = 20 N (Figs. 5.24 and 5.25).

3. For a particular displacement, work done by a force is independent of the type of motion, i.e., whether it moves with constant velocity, constant acceleration or retardation, etc.
4. If a body is in dynamic equilibrium under the action of certain forces, then total work done on the body is zero but work done by individual forces may not be zero.
5. When several forces act, work done by a force for a particular displacement is independent of other forces.

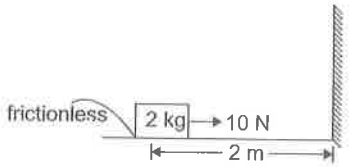

Figure 5.24

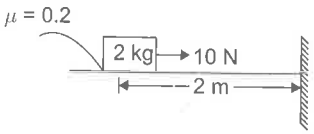

Figure 5.25

6. A force is independent of the reference frame. Its displacement depends on the frame, so the work done by a force is frame dependent. Therefore, work done by a force can be different in different reference frames.

UNITS OF WORK

In CGS system, the unit of work is erg.

One erg of work is said to be done when a force of 1 dyne displaces a body through 1 cm in its own direction.

$$\therefore \quad 1 \text{ erg} = 1 \text{ dyne} \times 1 \text{ cm}$$
$$= 1 \text{ g cm s}^{-2} \times 1 \text{ cm}$$
$$= 1 \text{ g cm}^2 \text{ s}^{-2}.$$

Note

Another name for joule is newton metre.

Relation Between Joule and erg

$$1 \text{ joule} = 1 \text{ newton} \times 1 \text{ metre}$$
$$1 \text{ joule} = 10^5 \text{ dyne} \times 10^2 \text{ cm}$$
$$= 10^7 \text{ dyne cm}$$
$$1 \text{ joule} = 10^7 \text{ erg}$$
$$1 \text{ erg} = 10^{-7} \text{ joule}.$$

Dimensions of Work

$$[\text{Work}] = [\text{Force}]\,[\text{Distance}]$$
$$= [MLT^{-2}]\,[L] = [ML^2T^{-2}]$$

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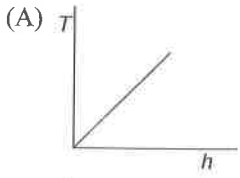
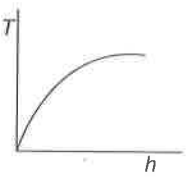
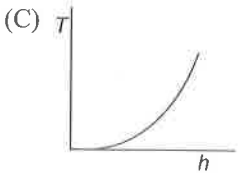
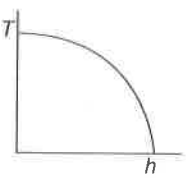
52. A bob attached to a string is held horizontal and released. The tension and vertical distance from the point of suspension can be represented by

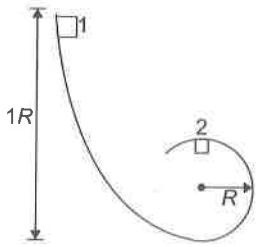
(A) 

(B) 

(C) 

(D) 

53. A small cube with mass M starts from at rest point 1 at a height $4R$, where R is the radius of the circular part of the track. The cube slides down the frictionless track and around the loop. The force that the track exerts on the cube at point 2 is nearly _____ times the cube's weight Mg.



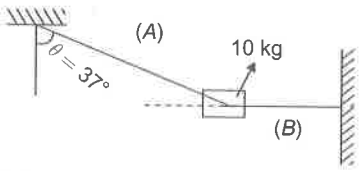
(A) 1
(B) 2
(C) 3
(D) 4

JEE Advanced

Level I

(A) Circular Motion

1. The 10-kg block is in equilibrium.



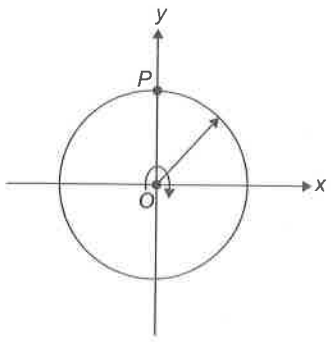
(i) Find the tension in string A.
(ii) Find the tension in string A just after the string B is cut.

2. A particle moves in the x–y plane with the velocity $\vec{v} = a\hat{i} + bt\hat{j}$. At the instant, $t = a\sqrt{3}/b$ the magnitudes of tangential, normal and total acceleration are _____, _____ and _____.

3. A particle moves clockwise in a circle of radius 1 m with centre at $(x, y) = (1\text{ m}, 0)$. It starts at rest at the origin at time $t = 0$. Its speed increases at the constant rate of $\left(\frac{\pi}{2}\right)$ m/s². (A) How long does it take to travel halfway around the circle? (B) What is the speed at that time?

4. A point moves along a circle having a radius 20 cm with a constant tangential acceleration 5 cm/s². How much time is needed after motion begins for the normal acceleration of the point to be equal to the tangential acceleration?

5. A ring rotates about the z axis as shown in the figure. The plane of rotation is xy. At a certain instant, the acceleration of a particle P (shown in figure) on the ring is $\left(6\hat{i} - 8\hat{j}\right)$ m/s². Find the angular acceleration of the ring and the angular velocity at that instant. Radius of the ring is 2 m.



41. To the observer A, the work done by the spring force is
(A) negative but nothing can be said about its magnitude
(B) $-\frac{1}{2}mv_0^2$
(C) positive but nothing can be said about its magnitude
(D) $+\frac{1}{2}mv_0^2$

42. To the observer A, the work done by the normal reaction N between the block and the spring on the block is
(A) zero (B) $-\frac{1}{2}mv_0^2$
(C) $+\frac{1}{2}mv_0^2$ (D) none of these

43. To the observer A, the net work done on the block is
(A) $-mv_0^2$ (B) $+mv_0^2$
(C) $-\frac{1}{2}mv_0^2$ (D) zero

44. According to the observer A,
(A) the kinetic energy of the block is converted into the potential energy of the spring
(B) the mechanical energy of the spring–mass system is conserved
(C) the block loses its kinetic energy because of the negative work done be the conservative force of the spring
(D) all the above

45. To an observer B, when the block is compressing the spring,
(A) velocity of the block is decreasing
(B) retardation of the block is increasing
(C) kinetic energy of the block is zero
(D) all the above

46. According to observer B, the potential energy of the spring increases
(A) due to the positive work done by pseudo force
(B) due to the positive work done by a normal reaction between the spring and the wall
(C) due to the decrease in the kinetic energy of the block
(D) all the above

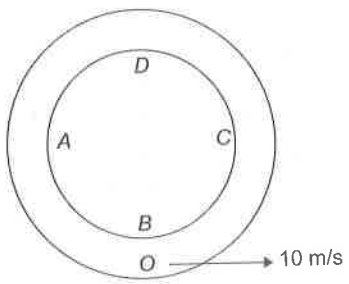
47. Assume the aerodynamic drag force on a car is proportional to its speed. If the power output from the engine is doubled, then the maximum speed of the car
(A) is unchanged
(B) increases by a factor of $\sqrt{2}$
(C) is also doubled
(D) increases by a factor of 4

48. A body of mass 1 kg starts moving from rest at $t = 0$ in a circular path of radius 8 m. Its kinetic energy varies as a function of time as $KE = 2t^2$ Joules, where t is in seconds. Then
(A) tangential acceleration $= 4$ m/s^2
(B) power of all forces at $t = 2$ s is 8 watt
(C) first round is completed in 2 s
(D) tangential force at $t = 2$ s is 4 N

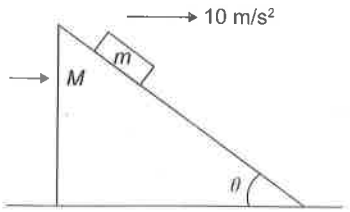
49. The potential energy for a force field $\vec{F}$ is given by $U(x, y) = \sin(x + y)$. The force acting on the particle of mass m at $\left(0, \frac{\pi}{4}\right)$ is
(A) 1 (B) $\sqrt{2}$
(C) $\frac{1}{\sqrt{2}}$ (D) 0

50. A particle originally at rest at the highest point of a smooth vertical circle is slightly displaced. It will leave the circle at a vertical distance h below the highest point, such that
(A) $h = R$ (B) $h = R/3$
(C) $h = R/2$ (D) $h = 2R$

51. A ball whose size is slightly smaller than the width of the tube of radius 2.5 m is projected from the bottom-most point of a smooth tube fixed in a vertical plane with velocity of 10 m/s. If N_1 and N_2 are the normal reactions exerted by the inner side and the outer side of the tube on the ball,



(A) $N_1 > 0$ for motion in ABC, $N_2 > 0$ for motion in CDA
(B) $N_1 > 0$ for motion in CDA, $N_2 > 0$ for motion in ABC
(C) $N_2 > 0$ for motion in ABC and part of CDA
(D) N_1 is always zero



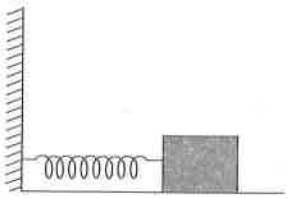
(A) 30 J (B) 60 J

(C) 150 J (D) $100\sqrt{3}$ J

Question No. 36 to 37

A spring block system is placed on a rough horizontal floor. The block is pulled towards right to give the spring an elongation less than $\frac{2\mu mg}{k}$ but more than $\frac{\mu mg}{k}$ and released.

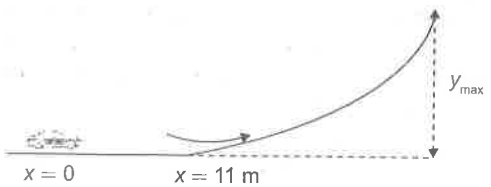
36. Which of the following laws/principles of physics can be applied on the spring block system?

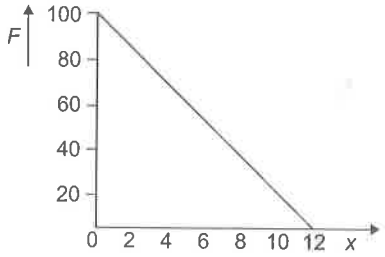


(A) Conservation of mechanical energy
(B) Conservation of momentum
(C) Work–energy principle
(D) None

37. Which of the following is a correct statement?
(A) The block will cross the mean position.
(B) The block comes to rest when the forces acting on it are exactly balanced.
(C) The block will come to rest when the work done by the friction becomes equal to the change in the energy stored in the spring.
(D) None of these

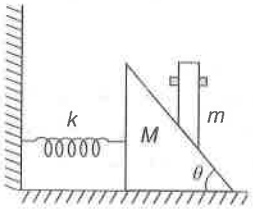
38. A toy car of mass 5 kg moves up a ramp under the influence of force F plotted against displacement x. The maximum height attained is given by





(A) $y_{max} = 20$ m (B) $y_{max} = 15$ m
(C) $y_{max} = 11$ m (D) $y_{max} = 5$ m

39. A wedge of mass M fitted with a spring of stiffness k is kept on a smooth horizontal surface. A rod of mass m is kept on the wedge as shown in the figure. The system is in equilibrium. Assuming that all surfaces are smooth, the potential energy stored in the spring is



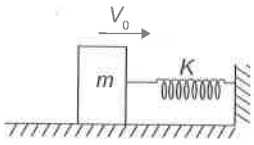
(A) $\frac{mg^2 \tan^2\theta}{2K}$ (B) $\frac{m^2 g \tan^2\theta}{2K}$

(C) $\frac{m^2 g^2 \tan^2\theta}{2K}$ (D) $\frac{m^2 g^2 \tan^2\theta}{K}$

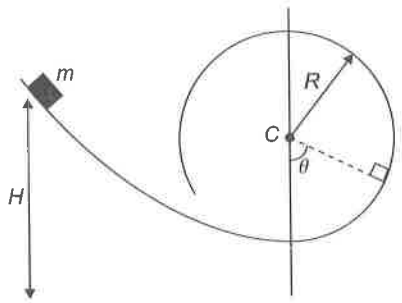
40. A block of mass m is hung vertically from an elastic thread of force constant mg/a. Initially, the thread was at its natural length and the block is allowed to fall freely. The kinetic energy of the block when it passes through the equilibrium position will be
(A) mga (B) $mga/2$
(C) zero (D) $2mga$

Question No. 41 to 46

A block of mass m moving with a velocity v_0 on a smooth horizontal surface strikes and compresses a spring of stiffness k till mass comes to rest as shown in the figure. This phenomenon is observed by two observers:



A: standing on the horizontal surface
B: standing on the block

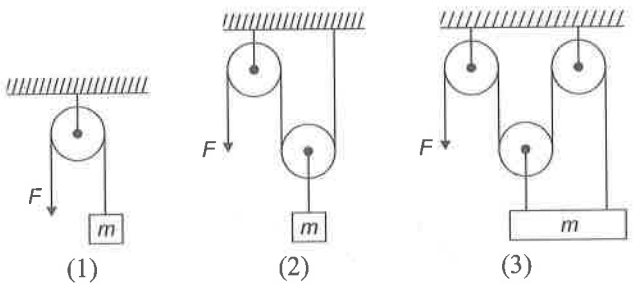


28. Choose the correct alternative(s) if $H = 2R$.
(A) The particle reaches the top of the loop with zero velocity.
(B) The particle cannot reach the top of the loop.
(C) The particle breaks off at a height $H = R$ from the base of the loop.
(D) The particle breaks off at a height $R < H < 2R$.

29. If θ is instantaneous angle which the line joining the particle and the centre of the loop makes with the vertical, then identify the correct statement(s) related to the normal reaction N between the block and the surface.
(A) The maximum value N occurs at $\theta = 0$.
(B) The minimum value of N occurs at $N = \pi$ for $H > 5R/2$.
(C) The value of N becomes negative for $\pi/2 < \theta < 3\pi/2$.
(D) The value of N becomes zero only when $\theta \geq \pi/2$.

30. The minimum value of H required so that the particle makes a complete vertical circle is given by
(A) $5R$ (B) $4R$
(C) $2.5R$ (D) $2R$

31. Equal force $F(>mg)$ is applied to a string in all the three cases. Starting from rest, the point of application of force moves a distance of 2 m down in all cases. In which case does the block have maximum kinetic energy?



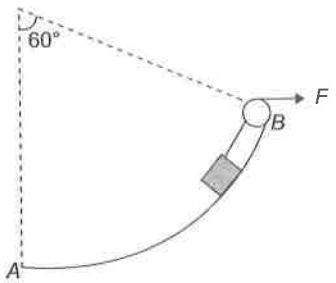
(1) (2) (3)

(A) 1 (B) 2
(C) 3 (D) Equal in all three cases

32. Two springs have their force constant as k_1 and $k_2 (k_1 > k_2)$. When they are stretched by the same force,
(A) no work is done by this force in case of both the springs
(B) equal work is done by this force in the case of both the springs
(C) more work is done by this force in the case of the second spring
(D) more work is done by this force in the case of the first spring

33. A force $\vec{F} = k\left[y\hat{i} + x\hat{j}\right]$, where K is a positive constant, acts on a particle moving in the x–y plane. Starting from the point (3, 5), the particle is taken along a straight line to (5, 7). The work done by the force is
(A) zero (B) $35K$
(C) $20K$ (D) $15K$

34. A 10-kg block is pulled in the vertical plane along a frictionless surface in the form of an arc of a circle of radius 10 m. The applied force is 200 N as shown in the figure. If the block started from rest at A, the velocity at B would be



(A) 1.732 m/s (B) 17.32 m/s
(C) 173.2 m/s (D) none of these

35. In the figure, all the surfaces are frictionless, and mass of the block, $m = 1$ kg. The block and wedge are held initially at rest. Now the wedge is given a horizontal acceleration of 10 m/s^2 by applying a force on the wedge, so that the block does not slip on the wedge. Then work done by the normal force in ground frame on the block in $\sqrt{3}$ s is

(B) Work, Power and Energy

18. No work is done by a force on an object if
(A) the force is always perpendicular to its velocity
(B) the force is always perpendicular to its acceleration
(C) the object is stationary but the point of application of the force moves on the object
(D) the object moves in such a way that the point of application of the force remains fixed

19. One end of a light spring of spring constant k is fixed to a wall and the other end is tied to a block placed on a smooth horizontal surface. In a displacement, the work done by the spring is $\frac{1}{2}kx^2$. The possible cases are
(A) the spring was initially compressed by a distance x and was finally in its natural length
(B) it was initially stretched by a distance x and finally was in its natural length
(C) it was initially in its natural length and finally in a compressed position.
(D) it was initially in its natural length and finally in a stretched position.

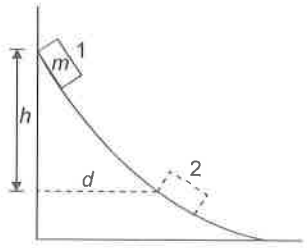
20. Work done by force of friction
(A) can be zero
(B) can be positive
(C) can be negative
(D) information insufficient

21. When total work done on a particle is positive,
(A) KE remains constant
(B) momentum increases
(C) KE decreases
(D) KE increases

22. A particle with constant total energy E moves in one dimension in a region where the potential energy is $U(x)$. The speed of the particle is zero where
(A) $U(x) = E$ (B) $U(x) = 0$
(C) $\frac{dU(x)}{dx} = 0$ (D) $\frac{d^2U(x)}{dx^2} = 0$

23. A block of mass m slides down a plane inclined at an angle θ. Which of the following will NOT increase the energy lost by the block due to friction?
(A) Increasing the angle of inclination
(B) Increasing the distance that the block travels
(C) Increasing the acceleration due to gravity
(D) Increasing the mass of the block

24. A box of mass m is released from rest at position on the frictionless curved track shown. It slides a distance d along the track in time t to reach position 2, dropping a vertical distance h. Let v and a be the instantaneous speed and instantaneous acceleration, respectively, of the box at position 2. Which of the following equations is valid for this situation?



(A) $h = vt$ (B) $h = (1/2)gt^2$
(C) $d = (1/2)at^2$ (D) $mgh = (1/2)mv^2$

25. A ball of mass m is attached to the lower end of a light vertical spring of force constant k. The upper end of the spring is fixed. The ball is released from rest with the spring at its normal (unstretched) length and comes to rest again after descending through a distance x.
(A) $x = mg/k$
(B) $x = 2\,mg/k$
(C) The ball will have no acceleration at the position where it has descended through $x/2$.
(D) The ball will have an upward acceleration equal to g at its lowermost position.

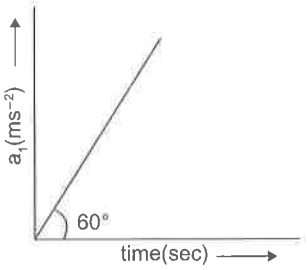
26. A cart moves with a constant speed along a horizontal circular path. From the cart, a particle is thrown up vertically with respect to the cart
(A) The particle will land somewhere on the circular path
(B) The particle will land outside the circular path
(C) The particle will follow an elliptical path
(D) The particle will follow a parabolic path

27. The potential energy in joules of a particle of mass 1 kg moving in a plane is given by $U = 3x + 4y$, the position coordinates of the point being x and y, measured in metres. If the particle is initially at rest at (6, 4), then
(A) its acceleration is of magnitude 5 m/s^2
(B) its speed when it crosses the y-axis is 10 m/s
(C) it crosses the y-axis ($x = 0$) at $y = -4$
(D) it moves in a straight line passing through the origin (0, 0)

Question No. 28 to 30

A particle of mass m is released from a height H on a smooth curved surface which ends into a vertical loop of radius R, as shown.

the particle is zero). Time after which total acceleration of the particle makes an angle of 30° with the radial acceleration is



(A) 4 s (B) 4/3 s

(C) $2^{2/3}$ s (D) $\sqrt{2}$ s

10. A stone of mass of 16 kg is attached to a string 144 m long and is whirled in a horizontal smooth surface. The maximum tension the string can withstand is 16 N. The maximum speed of revolution of the stone without breaking it will be

(A) 20 m/s (B) 16 m/s

(C) 14 m/s (D) 12 m/s

11. Three identical particles are joined together by a thread as shown in the figure. All the three particles are moving on a smooth horizontal plane about point O. If the speed of the outermost particle is v_0, then the ratio of the tensions in the three sections of the string is (assume that the string remains straight)

(A) 3:5:7 (B) 3:4:5

(C) 7:11:6 (D) 3:5:6

12. A particle is kept fixed on a turntable rotating uniformly. As seen from the ground, the particle goes in a circle, its speed is 20 cm/s and acceleration is 20 cm/s^2. The particle is now shifted to a new position to make the radius half of the original value. The new values of the speed and acceleration will be

(A) 10 cm/s, 10 cm/s^2 (B) 10 cm/s, 80 cm/s^2

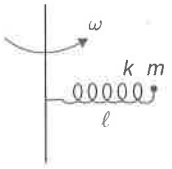
(C) 40 cm/s, 10 cm/s^2 (D) 40 cm/s, 40 cm/s^2

13. The kinetic energy k of a particle moving along a circle of radius R depends on the distance covered s as $k = as^2$, where a is a positive constant. The total force acting on the particle is

(A) $2a\dfrac{S^2}{R}$ (B) $2as\left(1+\dfrac{S^2}{R^2}\right)^{1/2}$

(C) $2as$ (D) $2a\dfrac{R^2}{S}$

14. A particle of mass m is fixed *to one end* of a light spring of force constant k and *unstretched* length ℓ. The system is rotated about the other end of the spring with an angular velocity ω, in gravity-free space. The increase in length of the spring will be



(A) $\dfrac{m\omega^2\ell}{k}$ (B) $\dfrac{m\omega^2\ell}{k-m\omega^2}$

(C) $\dfrac{m\omega^2\ell}{k+m\omega^2}$ (D) none of these

15. A uniform circular ring of mass per unit length λ and radius R is rotating with angular velocity ω about its own axis in a gravity-free space. Tension in the ring is

(A) zero (B) $\dfrac{1}{2}lR^2w^2$

(C) lR^2w^2 (D) lRw^2

16. A uniform rod of mass m and length ℓ rotates in a horizontal plane with an angular velocity ω about a vertical axis passing through one end. The tension in the rod at distance x from the axis is

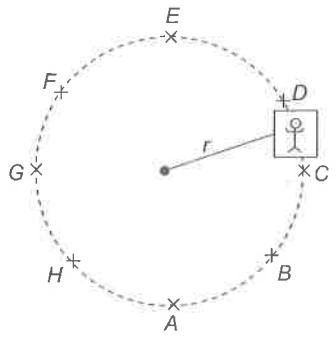
(A) $\dfrac{1}{2}m\omega^2x$ (B) $\dfrac{1}{2}m\omega^2\dfrac{x^2}{\ell}$

(C) $\dfrac{1}{2}m\omega^2\ell\left(1-\dfrac{x}{\ell}\right)$ (D) $\dfrac{1}{2}\dfrac{m\omega^2}{\ell}[\ell^2-x^2]$

17. The ratio of period of oscillation of the conical pendulum to that of the simple pendulum is (assume the strings are of the same length in the two cases and θ is the angle made by the string with the vertical in case of a conical pendulum)

(A) $\cos\theta$ (B) $\sqrt{\cos\theta}$

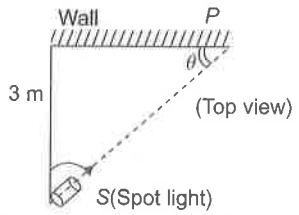
(C) 1 (D) none of these



(A) The reading of his weight on the machine is the same at all positions.
(B) The weight reading at A is greater than the weight reading at E by $2w$.
(C) The weight reading at $G = w$.
(D) The ratio of the weight reading at E to that at $A = 0$.
(E) The ratio of the weight reading at A to that at $C = 2$.

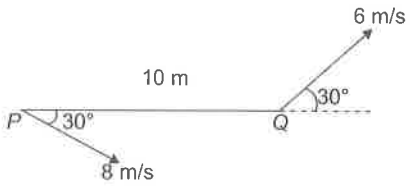
3. A simple pendulum of length l and mass (bob) M is oscillating in a plane about a vertical line between angular limits $-\phi$ and ϕ. For an angular displacement θ, $[|\theta| < \phi]$ the tension in the string and velocity of the bob are T and v, respectively. Which of the following relations hold good under the above conditions?
(A) $T \cos\theta = Mg$
(B) $T - Mg\cos\theta = \dfrac{Mv^2}{L}$
(C) Tangential acceleration $= g\sin\theta$
(D) $T = Mg\cos\theta$

4. A spotlight S rotates in a horizontal plane with a constant angular velocity of 0.1 rad/s. The spot of light p moves along the wall at a distance 3 m. What is the velocity of the spot P when $\theta = 45°$?



(A) 0.6 m/s (B) 0.5 m/s
(C) 0.4 m/s (D) 0.3 m/s

5. Two moving particles P and Q are 10 m apart at a certain instant. The velocity of P is 8 m/s making an angle 30° with the line joining P and Q and that of Q is 6 m/s making an angle 30° with PQ as shown in the figure. Then, the angular velocity of P with respect to Q is

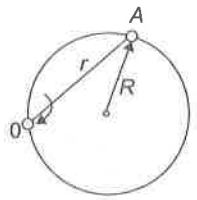


(A) zero (B) 0.1 rad/s
(C) 0.4 rad/s (D) 0.7 rad/s

6. The magnitude of displacement of a particle moving in a circle of radius a with constant angular speed ω varies with time t as
(A) $2a\sin wt$ (B) $2a\sin\dfrac{\omega t}{2}$
(C) $2a\cos wt$ (D) $2a\cos\dfrac{\omega t}{2}$

7. A particle moves along an arc of a circle of radius R. Its velocity depends on the distance covered as $v = a\sqrt{s}$, where a is a constant. Then the angle α between the vector of the total acceleration and the vector of velocity as a function of s will be
(A) $\tan\alpha = \dfrac{R}{2s}$ (B) $\tan\alpha = 2s/R$
(C) $\tan\alpha = \dfrac{2R}{s}$ (D) $\tan\alpha = \dfrac{s}{2R}$

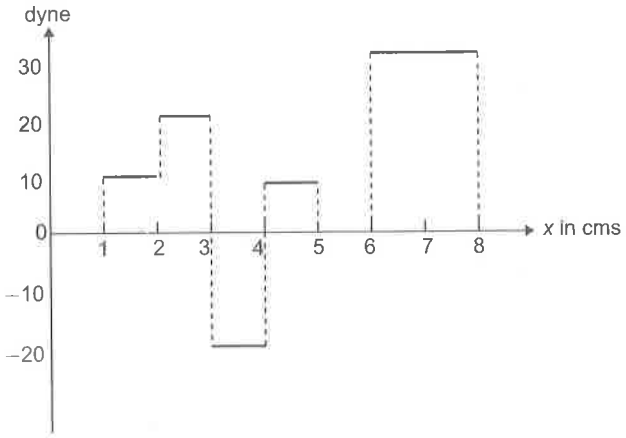
8. A particle A moves along a circle of radius $R = 50$ cm so that its radius vector r relative to the point O (figure) rotates with the constant angular velocity $\omega = 0.40$ rad/s. Then modulus of the velocity of the particle and the modulus of its total acceleration will be



(A) $v = 0.4$ m/s, $a = 0.4$ m/s^2
(B) $v = 0.32$ m/s, $a = 0.32$ m/s^2
(C) $v = 0.32$ m/s, $a = 0.4$ m/s^2
(D) $v = 0.4$ m/s, $a = 0.32$ m/s^2

9. Tangential acceleration of a particle moving in a circle of radius 1 m varies with time t as (initial velocity of

57. The relationship between force and position is shown in the figure (in one-dimensional case). The work done in displacing a body from $x = 1$ cm to $x = 5$ cm is

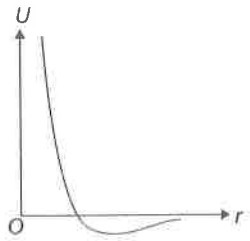


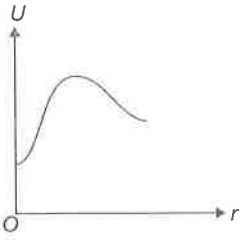
(A) 20 erg (B) 60 erg
(C) 70 erg (D) 700 erg

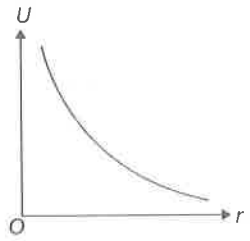
58. A truck of mass 30,000 kg moves up an inclined plane of slope 1 in 100 at a speed of 30 km/h. The power of the truck is (given $g = 10$ ms^{-2})
(A) 25 kW (B) 10 kW
(C) 5 kW (D) 2.5 kW

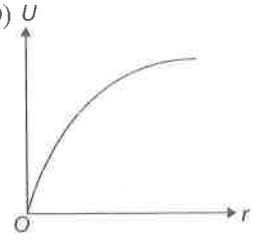
59. A particle moves with a velocity $\vec{v} = (5\hat{i} - 3\hat{j} + 6\hat{k})$ m/s under the influence of a constant force $\vec{F} = (10\hat{i} + 10\hat{j} + 20\hat{k})$ N. The instantaneous power applied to the particle is
(A) 200 J/s (B) 40 J/s
(C) 140 J/s (D) 170 J/s

60. The diagrams represent the potential energy U of a function of inter-atomic distance r. Which diagram corresponds to stable molecules found in nature?

(A) 

(B) 

(C) 

(D) 

61. $F = 2x^2 - 3x - 2$. Choose the correct option.
(A) $x = -1/2$ is the position of stable equilibrium
(B) $x = 2$ is the position of stable equilibrium
(C) $x = -1/2$ is the position of unstable equilibrium
(D) $x = 2$ is the position of neutral equilibrium

62. A particle is rotated in a vertical circle by connecting it to a light rod of length l and keeping the other end of the rod fixed. The minimum speed of the particle when the light rod is horizontal for which the particle will complete the circle is

(A) $\sqrt{gl}$ (B) $\sqrt{2gl}$
(C) $\sqrt{3gl}$ (D) none of these

JEE Advanced

(A) Circular Motion

1. A person applies a constant force $\vec{F}$ on a particle of mass m and finds that the particle moves in a circle of radius r with a uniform speed v as seen (in the plane of motion) from an inertial frame of reference.
(A) This is not possible.
(B) There are other forces on the particle.
(C) The resultant of the other forces is $\frac{mv^2}{r}$ towards the centre.
(D) The resultant of the other forces varies in magnitude as well as in direction.

2. A machine, in an amusement park, consists of a cage at the end of one arm, hinged at O. The cage revolves along a vertical circle of radius r ($ABCDEFGH$) about its hinge O, at constant linear speed $v = \sqrt{gr}$. The cage is attached such that the man of weight w standing on a weighing machine, inside the cage, is always vertical. Then which of the following is correct?

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46. A body of mass m accelerates uniformly from rest to a speed v_0 in time t_0. The work done on the body till any time t is

(A) $\frac{1}{2}mv_0^2\left(\frac{t^2}{t_0^2}\right)$ (B) $\frac{1}{2}mv_0^2\left(\frac{t_0}{t}\right)$

(C) $mv_0^2\left(\frac{t}{t_0}\right)$ (D) $mv_0^2\left(\frac{t}{t_0}\right)^3$

47. A light spring of length 20 cm and force constant 2 kg/cm is placed vertically on a table. A small block of mass 1 kg falls on it. The length h from the surface of the table at which the ball will have the maximum velocity is

(A) 20 cm (B) 15 cm
(C) 10 cm (D) 5 cm

48. The work done is joules in increasing the extension of a spring of stiffness 10 N/cm from 4 cm to 6 cm is

(A) 1 (B) 10
(C) 50 (D) 100

49. When a conservative force does positive work on a body,

(A) the potential energy increases
(B) the potential energy decreases
(C) total energy increases
(D) total energy decreases

50. The PE of a certain spring when stretched from natural length through a distance 0.3 m is 10 J. The amount of work in joule that must be done on this spring to stretch it through an additional distance 0.15 m will be

(A) 10 J (B) 20 J
(C) 7.5 J (D) 12.5 J

51. A man who is running has half the kinetic energy of a boy of half his mass. The man speeds up by 1 m/s and then has the same kinetic energy as the boy. The original speed of the man was

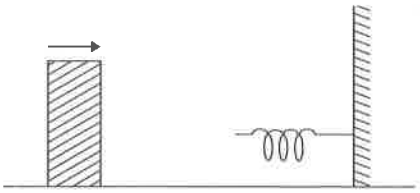
(A) $\sqrt{2}$ m/s (B) $(\sqrt{2}-1)$ m/s

(C) 2 m/s (D) $(\sqrt{2}+1)$ m/s

52. A particle is released from rest at origin. It moves under the influence of potential field $U = x^2 - 3x$. The kinetic energy at $x = 2$ is

(A) 2 J (B) 1 J
(C) 1.5 J (D) 0 J

53. A 1.0-kg block collides with a horizontal weightless spring of force constant 2.75 Nm^{-1} as shown in the figure. The block compresses the spring 4.0 m from the rest position. If the coefficient of kinetic friction between the block and the horizontal surface is 0.25, the speed of the block at the instant of collision is



(A) 0.4 ms^{-1} (B) 4 ms^{-1}
(C) 0.8 ms^{-1} (D) 8 ms^{-1}

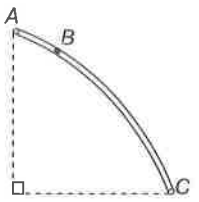
54. A car of mass m is driven with acceleration a along a straight level road against a constant external resistive force R. When the velocity of the car is V, the rate at which the engine of the car is doing work will be

(A) RV (B) maV
(C) $(R + ma)V$ (D) $(ma - R)V$

55. A pump ejects 12,000 kg of water at a speed of 4 m/s in 40 s. Find the average rate at which the pump is working.

(A) 0.24 kW (B) 2.4 W
(C) 2.4 kW (D) 24 W

56. The tube AC forms a quarter circle in a vertical plane. The ball B has an area of cross-section slightly smaller than that of the tube and can move without friction through it. B is placed at A and displaced slightly. It will



(A) always be in contact with the inner wall of the tube
(B) always be in contact with the outer wall of the tube
(C) initially be in contact with the inner wall and later with the outer wall
(D) initially be in contact with the outer wall and later with the inner wall

(A) a car is heavier than a cycle
(B) a car has four wheels while a cycle has only two
(C) the difference in the speed of the two
(D) the cyclist has to counteract the centrifugal force while in the case of a car only the passenger is thrown by this force

35. A cyclist is moving on a circular track of radius 80 m with a velocity of 72 km/h. He has to lean from the vertical approximately through an angle
(A) $\tan^{-1}(1/4)$ (B) $\tan^{-1}(1)$
(C) $\tan^{-1}(1/2)$ (D) $\tan^{-1}(2)$

36. Keeping the banking angle same to increase the maximum speed with which a vehicle can travel on a curved road by 10%, the radius of curvature of the road has to be changed from 20 m to
(A) 16 m (B) 18 m
(C) 24.25 m (D) 30.5 m

37. A motorcyclist moving with a velocity of 72 km/h on a flat road takes a turn on the road at a point where the radius of curvature of the road is 20 m. The acceleration due to gravity is 10 m/s^2. In order to avoid skidding, he must not bend with respect to the vertical plane by an angle greater than
(A) $\theta = \tan^{-1} 6$ (B) $\theta = \tan^{-1} 2$
(C) $\theta = \tan^{-1} 25.92$ (D) $\theta = \tan^{-1} 4$

(B) Work, Power and Energy

38. A rigid body of mass m is moving in a circle of radius r with a constant speed v. The force on the body is $\frac{mv^2}{r}$ and is directed towards the centre. What is the work done by this force in moving the body over half the circumference of the circle?
(A) $\frac{mv^2}{\pi r^2}$ (B) Zero
(C) $\frac{mv^2}{r^2}$ (D) $\frac{\pi r^2}{mv^2}$

39. A body travels through a distance of 10 m on a straight line, under the influence of 5 N. If the work done by the force is 25 J, the angle between the force and displacement is
(A) 0° (B) 30°
(C) 60° (D) 90°

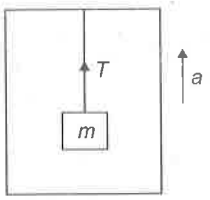
40. A block is moved from rest through a distance of 4 m along a straight line path. The mass of the blocks is 5 kg and the force acting on it is 20 N. If the kinetic energy acquired by the block be 40 J, at what angle to the path the force is acting?
(A) 30° (B) 60°
(C) 45° (D) None of these

41. The work done in pushing a block of mass 10 kg from bottom to the top of a frictionless inclined plane 5 m long and 3 m high is (considering $g = 9.8$ m/s^2)
(A) 392 J (B) 294 J
(C) 98 J (D) 0.98 J

42. A force $F = (3x + 4)$ N (where x is in metres) acts on a particle which moves from a position (2 m, 3 m) to (3 m, 0 m). Then the work done is
(A) 7.5 J (B) −12 J
(C) −4.5 J (D) +4.5 J

43. If a man increases his speed by 2 m/s, his KE is doubled. The original speed of the man is
(A) $(2 + \sqrt{2})$ m/s (B) $(2 + 2\sqrt{2})$ m/s
(C) 4 m/s (D) $(1 + \sqrt{2})$ m/s

44. A block of mass m is suspended by a light thread from an elevator. The elevator is accelerating upward with uniform acceleration a. The work done by tension on the block during t seconds is



(A) $\frac{m}{2}(g+a)at^2$ (B) $\frac{m}{2}(g-a)at^2$
(C) $\frac{m}{2}gat^2$ (D) 0

45. The work done by the frictional force on a surface in drawing a circle of radius r on the surface by a pencil of negligible mass with a normal pressing force N (coefficient of friction m_k) is
(A) $4\pi r^2 \mu_k$ N (B) $-2\pi r^2 \mu_k$ N
(C) $-3\pi r^2 \mu_k$ N (D) $-2\pi r \mu_k$ N

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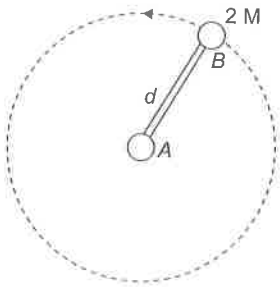
26. A road is banked at an angle of 30° to the horizontal for negotiating a curve of radius $10\sqrt{3}$ m. At what velocity will a car experience no friction while negotiating the curve?

(A) 54 km/h (B) 72 km/h
(C) 36 km/h (D) 18 km/h

27. A particle is moving in a circle:

(A) The resultant force on the particle must be towards the centre.
(B) The cross product of the tangential acceleration and the angular velocity will be zero.
(C) The direction of the angular acceleration and the angular velocity must be the same.
(D) The resultant force may be towards the centre.

28. The dumbbell is placed on a frictionless horizontal table. Sphere A is attached to a frictionless pivot so that B can be made to rotate about A with constant angular velocity. If B makes one revolution in period P, the tension in the rod is

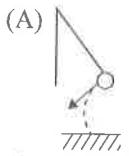
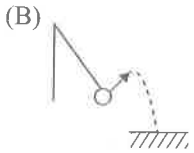
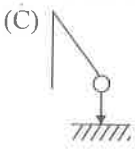
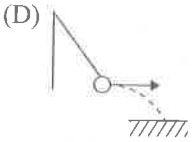


(A) $\frac{4\pi^2 Md}{P^2}$ (B) $\frac{8\pi^2 Md}{P^2}$
(C) $\frac{4\pi^2 Md}{P}$ (D) $\frac{2Md}{P}$

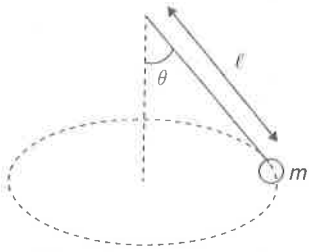
29. A string can bear a maximum tension of 100 N without breaking. A body of mass 1 kg is attached to one end of 1 m length of thin string and it is revolved in a horizontal plane. The maximum linear velocity which can be imparted to the body without breaking the string, will be

(A) 10 m/s (B) 1 m/s
(C) 100 m/s (D) 1000 m/s

30. A pendulum bob is swinging in a vertical plane such that its angular amplitude is less than 90°. At its highest point, the string is cut. Which trajectory is possible for the bob afterwards?

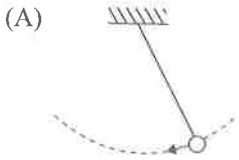
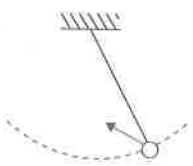
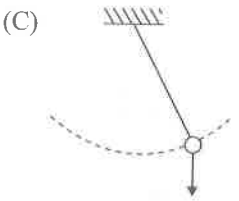
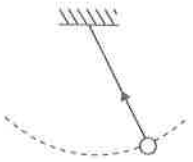
(A) 
(B) 
(C) 
(D) 

31. A conical pendulum is moving in a circle with angular velocity ω as shown. If tension in the string is T, which of following equations are correct?



(A) $T = mw^2 l$ (B) $T \sin\theta = mw^2 l$
(C) $T = mg\cos\theta$ (D) $T = mw^2 l \sin\theta$

32. Which vector in the figures best represents the acceleration of a pendulum mass at an intermediate point in its swing?

(A) 
(B) 
(C) 
(D) 

33. Two racing cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2, respectively. Their speeds are such that each makes a complete circle in the same time t. The ratio of the angular speeds of the first to the second car is

(A) 1:1 (B) $m_1:m_2$
(C) $r_1:r_2$ (D) $m_1m_2:r_1r_2$

34. A cyclist taking turn bends inwards while a car passenger taking the same turn is thrown outwards. The reason is

(A) 160π m/s^2 (B) 40π m/s^2
(C) 40 m/s^2 (D) 640π m/s^2

17. A particle moves with deceleration along the circle of radius R so that at any moment of time its tangential and normal accelerations are equal in moduli. At the initial moment $t = 0$, the speed of the particle equals v_0, then

(i) the speed of the particle as a function of the distance covered s will be

(A) $v = v_0 e^{-s/R}$ (B) $v = v_0 e^{s/R}$
(C) $v = v_0 e^{-R/s}$ (D) $v = v_0 e^{R/s}$

(ii) the total acceleration of the particle as function of velocity and distance covered is given as

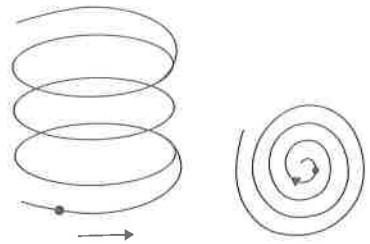
(A) $a = \sqrt{2}\dfrac{v^2}{R}$ (B) $a = \sqrt{2}\dfrac{v}{R}$

(C) $a = \sqrt{2}\dfrac{R}{v}$ (D) $a = \dfrac{2R}{v}$

18. If angular velocity of a disc depends an angle rotated θ as $\omega = q^2 + 2\theta$, then its angular acceleration α at $\theta =$ 1 rad is

(A) 8 rad/s^2 (B) 10 rad/s^2
(C) 12 rad/s^2 (D) none of these

19. A particle is going in a uniform helical and spiral path separately as shown in the figure with a constant speed.



(i) (ii)

(A) The velocity of the particle is constant in both cases.
(B) The acceleration of the particle is constant in both cases.
(C) The magnitude of acceleration is constant in (i) and decreasing in (ii).
(D) The magnitude of acceleration is decreasing continuously in both the cases.

20. If the radii of circular paths of two particles of the same mass are in the ratio of 1:2, then in order to have the same centripetal force, their speeds should be in the ratio of

(A) 1:4 (B) 4:1
(C) $1:\sqrt{2}$ (D) $\sqrt{2}:1$

21. A particle moving along a circular path due to a centripetal force having constant magnitude is an example of motion with

(A) constant speed and velocity
(B) variable speed and velocity
(C) variable speed and constant velocity
(D) constant speed and variable velocity

22. A curved section of a road is banked for a speed v. If there is no friction between the road and the tires of a car, then

(A) the car is more likely to slip at speeds higher than v than speeds lower than v
(B) the car cannot remain in static equilibrium on the curved section
(C) the car will not slip when moving with speed v
(D) none of the these

23. Water in a bucket is whirled in a vertical circle with a string attached to it. The water does not fall down even when the bucket is inverted at the top of its path. We conclude that

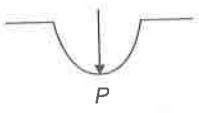
(A) $mg = \dfrac{mv^2}{R}$ (B) $mg > \dfrac{mv^2}{R}$

(C) $mg < \dfrac{mv^2}{R}$ (D) none of these

24. A man is standing on a rough ($\mu = 0.5$) horizontal disc rotating with a constant angular velocity of 5 rad/s. At what distance from the centre should he stand so that he does not slip on the disc?

(A) $R \leq 0.2$ m (B) $R > 0.2$ m
(C) $R > 0.5$ m (D) $R > 0.3$ m

25. A car travelling on a smooth road passes through a curved portion of the road in the form of an arc of circle of radius 10 m. If the mass of the car is 500 kg, the reaction on the car at the lowest point P where its speed is 20 m/s is



(A) 35 kN (B) 30 kN
(C) 25 kN (D) 20 kN

4. A wheel having a diameter of 3 m starts from rest and accelerates uniformly to an angular velocity of 210 r.p.m. in 5 s. Angular acceleration of the wheel is
(A) 1.4π rad/s^2 (B) 3.3π rad/s^2
(C) 2.2π rad/s^2 (D) 1.1π rad/s^2

5. The second's hand of a watch has length 6 cm. Speed of the end point and the magnitude of difference of the velocities at two perpendicular positions will be
(A) 2π and 0 mm/s
(B) $2\sqrt{2}\pi$ and 4.44 mm/s
(C) $2\sqrt{2}\pi$ and 2π mm/s
(D) 2π and $2\sqrt{2}\pi$ mm/s

6. The ratio of angular speed of the hours hand and seconds hand of a clock is
(A) 1:1 (B) 1:60
(C) 1:720 (D) 3600:1

7. The angular velocity of the earth about its axis of rotation is
(A) $2\pi/(60 \times 60 \times 24)$ rad/s
(B) $2\pi/(60 \times 60)$ rad/s
(C) $2\pi/60$ rad/s
(D) $2\pi/(365 \times 24 \times 60 \times 60)$ rad/s

8. A stone of mass 0.5 kg tied with a string of length 1 m is moving in a circular path with a speed of 4 m/s. The tension acting on the string in Newton is
(A) 2 (B) 8
(C) 0.2 (D) 0.8

9. Two bodies A and B rotate about an axis such that angle θ_A (in radians) covered by the first body is proportional to the square of time and θ_B (in radians) covered by the second body varies linearly. At $t = 0$, $\theta_A = \theta_B = 0$. If A completes its first revolution in $\sqrt{\pi}$ s and B needs 4π s to complete half revolution, then angular velocity $\omega_A:\omega_B$ at $t = 5$ s is in the ratio
(A) 4:1 (B) 20:1
(C) 80:1 (D) 20:4

10. A car of mass 1000 kg moves on a circular track of radius 20 m. If the coefficient of friction is 0.64, what is the maximum velocity with which the car can be moved?
(A) 1.12 m/s (B) 11.2 m/s
(C) $\frac{0.64 \times 20}{1000}$ m/s (D) $\frac{1000}{0.64 \times 20}$ m/s

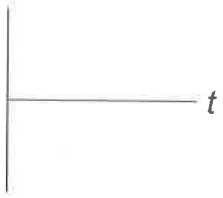
11. The earth (radius 6400 km) makes one revolution about its own axis in 24 hours. The centripetal acceleration of a point on its equator is nearly
(A) 340 cm/s^2 (B) 3.4 cm/s^2
(C) 34 cm/s^2 (D) 0.34 cm/s^2

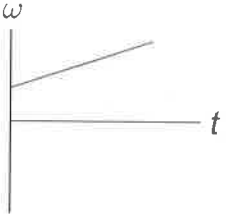
12. A stone of mass 0.1 kg tied to one end of a string 1.0 m long is revolved in a horizontal circle at a rate of $10/\pi$ revolution per second. Calculate the tension of the string.
(A) 30 N (B) 40 N
(C) 50 N (D) 60 N

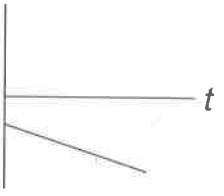
13. A coin placed on a rotating turntable just slips if it is at a distance of 40 cm from the centre. If the angular velocity of the turntable is doubled, it will just slip at a distance of
(A) 10 cm (B) 20 cm
(C) 40 cm (D) 80 cm

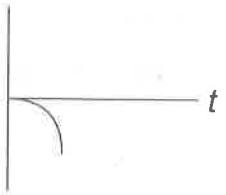
14. A particle moves in a circle of radius 25 cm at two revolutions per second. The acceleration of particle in m/s^2 is
(A) p^2 (B) $8p^2$
(C) $4p^2$ (D) $2p^2$

15. In the graphs, angular velocity is shown as a function of time. In which of these graphs is the magnitude of the angular acceleration constantly decreasing?

(A) 

(B) 

(C) 

(D) 

16. A particle moves along a circle of radius $\left(\frac{20}{\pi}\right)$ m with constant tangential acceleration. If the velocity of the particle is 80 m/s at the end of the second revolution after motion has begun, the tangential acceleration is

$$\cos\phi = \frac{u^2 - 2gR}{3gR}.$$

(d) Body Moving on a Spherical Surface

The small body of mass m is placed on the top of a smooth sphere of radius R and the body slides down the surface.

At any instant, i.e., at point C, the forces are the normal reaction N and the weight mg. The radial component of the weight is $mg\cos\phi$ acting towards the centre. The centripetal force is

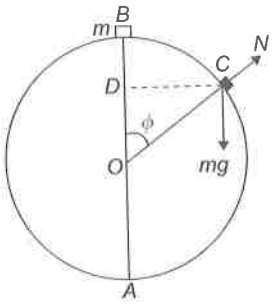


Figure 5.24

$$mg\cos\phi - N = \frac{mv^2}{R},$$

where v is the velocity of the body at O.

$$N = m\left(g\cos\phi - \frac{v^2}{R}\right). \quad (1)$$

The body flies off the surface at the point where N becomes zero.

i.e., $$g\cos\phi = \frac{v^2}{R};$$

$$\cos\phi = \frac{v^2}{gR}. \quad (2)$$

To find v, we use conservation of energy,

i.e., $$\frac{1}{2}mv^2 = mg\,(BD)$$

$$= mg\,(OB - OD)$$

$$= mgR\,(1 - \cos\phi)$$

$$v^2 = 2Rg\,(1 - \cos\phi)$$

$$2(1 - \cos\phi) = \frac{v^2}{gR}. \quad (3)$$

From Eqs. (2) and (3), we get

$$\cos\phi = 2 - 2\cos\phi;\ 3\cos\phi = 2$$

$$\cos\phi = \frac{2}{3};\ \phi = \cos^{-1}\left(\frac{2}{3}\right). \quad (4)$$

This gives the angle at which the body goes off the surface. The height from the ground ay that point

$$= AD = R(1 + \cos\phi)$$

$$= R\left(1 + \frac{2}{3}\right) = \frac{5}{3}R.$$

EXERCISES

JEE Main

(A) Circular Motion

1. A wheel is at rest. Its angular velocity increases uniformly and becomes 80 rad/s after 5 s. The total angular displacement is
   (A) 800 rad (B) 400 rad
   (C) 200 rad (D) 100 rad

2. A wheel is subjected to uniform angular acceleration about its axis. Initially, its angular velocity is zero. In the first 2 s, it rotates through an angle θ_1. In the next 2 s, it rotates through an additional angle θ_2. The ratio of θ_2/θ_1 is
   (A) 1 (B) 2
   (C) 3 (D) 5

3. The linear and angular acceleration of a particle are 10 m/s² and 5 rad/s², respectively. It will be at a distance of _____ from the axis of rotation.
   (A) 50 m (B) 1/2 m
   (C) 1 m (D) 2 m

Equation (1) shows that tension in the string decreases as θ increases, since the term '$g \cos \theta$' decreases as θ increases.

When $u = \sqrt{5gR}$,

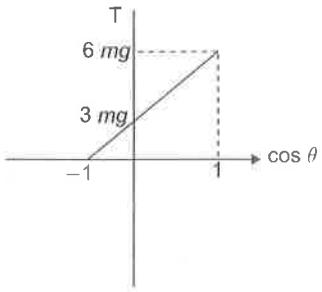


Figure 5.20

$$\Rightarrow \quad T = 3mg(1 + \cos\theta)$$

Now $\theta = 0$

$$\Rightarrow \quad \cos\theta = 1$$

$$\Rightarrow \quad T_A = 6\,mg$$

If $\theta = 90°$

$$\Rightarrow \quad T_B = 3\,mg$$

$\theta = 180°, \cos\theta = -1$

$$T_C = 0$$

Different Situations

(a) A Body Moving Inside a Hollow Tube or Sphere

The previous discussion holds good for this case, but instead of tension in the string, we have the normal reaction of the surface. If N is the normal reaction at the lowest point, then the condition $u \geq \sqrt{5gR}$ for the body to complete the circle holds for this case also (Fig. 5.21). All other equations (can be) similarly obtained by replacing tension T by normal reaction N.

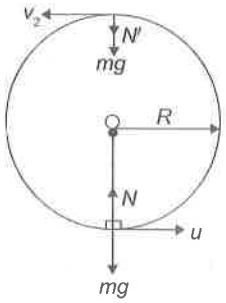


Figure 5.21

(b) When a Body is Attached to a Rod of Length R

In this case since the body is attached to a rigid rod, the body cannot leave the circular path.

Therefore, if the speed of the body becomes zero before the highest point C, it's motion will be oscillatory about the centre of the rod.

(i) Condition for completing the circle

If the body just reaches the highest point, then it will complete the vertical circle (Fig. 5.22).

Applying energy conservation between the lowest and highest point of circle, we get

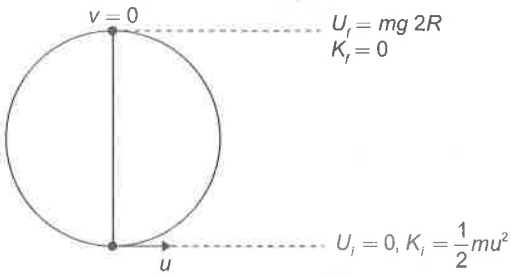


Figure 5.22

$$2mgR = \frac{1}{2}mu^2$$

$$\Rightarrow \quad u = \sqrt{4gR}.$$

So, if the velocity at point A is greater than or equal to $\sqrt{4gR}$, then the body will complete the vertical circle.

(c) Vertical Motion in a Dual Ring

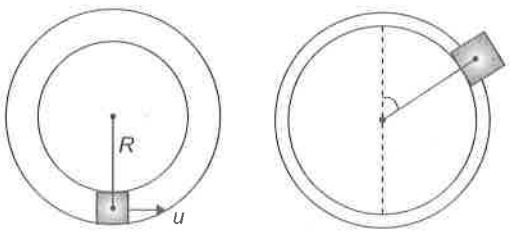


Figure 5.23

This system will behave as the previous system. So u_{min} to complete vertical circle $u_{min} = \sqrt{4gR}$ (Fig. 5.23)

Angle at which the normal reaction on the body will change its direction from inward to outward the ring is given by

Substituting in Eq. (1),

$$T = m\left[\frac{u^2}{R} - 2g(1+\cos\phi) - g\cos\phi\right]$$

$$T = m\left[\frac{u^2}{R} - 3g\left(\cos\phi + \frac{2}{3}\right)\right]$$

This equation shows that the tension becomes zero if

$$\frac{u^2}{R} = 3g\left(\cos\phi + \frac{2}{3}\right) \qquad (2)$$

If the tension is not to become zero,

$$u^2 > 3Rg\left(\cos\phi + \frac{2}{3}\right).$$

Equation (2) gives the values of ϕ at which the string becomes slack.

$$\cos\phi + \frac{2}{3} = \frac{u^2}{3gR}$$

$$\cos\phi = \frac{u^2}{3gR} - \frac{2}{3}$$

$$\cos\phi = \frac{u^2 - 2gR}{3gR}.$$

It is the angle from the vertical at which the tension in the string vanishes to zero. And after that its motion is projectile.

Tension in the String Versus θ

We may find an expression for the tension in the string when it makes an angle θ with the vertical (Fig. 5.19). At C, the weight of the body acts vertically downwards, and the tension in the string is towards the centre O.

The weight mg is resolved radially and tangentially.

The radial component is $mg\cos\theta$ and the tangential component is $mg\sin\theta$.

$$T - mg\cos\theta = \frac{mv^2}{R}, \text{ where } v \text{ is the velocity at } C.$$

i.e., $$T = m\left(\frac{v^2}{R} + g\cos\theta\right). \qquad (1)$$

The velocity v can be expressed in terms of velocity u at A.

The total energy at $A = \frac{1}{2}mu^2$.

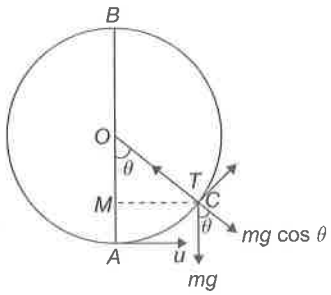


Figure 5.19

The kinetic energy at $C = \frac{1}{2}mv^2$.

The potential energy at

$$C = mg\,(AM)$$

$$= mg\,(AO - MO)$$

$$= mg\,(R - R\cos\theta)$$

$$= mgR\,(1 - \cos\theta).$$

The total energy at

$$C = \frac{1}{2}mv^2 + mgR(1-\cos\theta)$$

$\therefore$ From conservation of energy,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgR\,(1 - \cos\theta)$$

$$u^2 = v^2 + 2gR\,(1 - \cos\theta)$$

or $$v^2 = u^2 - 2gR(1 - \cos\theta).$$

Substituting in Eq. (5),

$$T = m\left[g\cos\theta + \frac{u^2}{R} - 2g(1-\cos\theta)\right]$$

$$= \frac{mu^2}{R} + 3mg\left(\cos\theta - \frac{2}{3}\right) \qquad (2)$$

This expression gives the value of the tension in the string in terms of the velocity at the lowest point and the angle θ.

then $$T_A = mg + \frac{5mgR}{R}$$

$\Rightarrow$ $$T_A = 6mg.$$

Tension at *B* (Fig. 5.16)

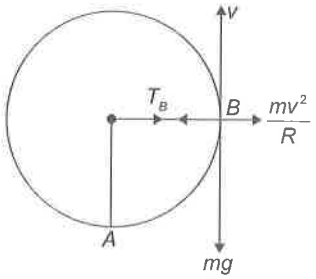


Figure 5.16

$$T_B = \frac{mv^2}{R}$$

Energy conservation from A to B

$$\frac{1}{2}mu_{min}^2 = mgR + \frac{1}{2}mv^2$$

$\Rightarrow$ $$v^2 = 3gR$$

$\Rightarrow$ $$T_B = 3mg.$$

Condition for the Body to Reach *B*

Let us calculate u_{min} such that the body just reaches B (Fig. 5.17).

Work done by tension = 0.

Only gravitational force is working on the body, which is a conservative force. Therefore, applying conservation of energy, we get

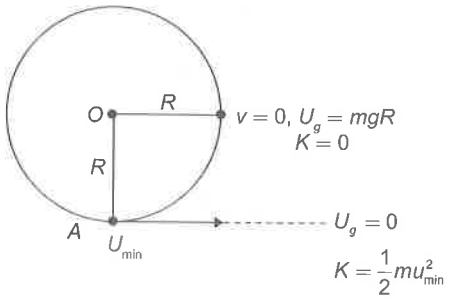


Figure 5.17

$$mgR = \frac{1}{2}mu_{min}^2$$

$\Rightarrow$ $$u_{min} = \sqrt{2gR}$$

$\therefore$ If $u \leq \sqrt{2gR}$, then the body will oscillate about A.

When $\sqrt{2gR} < u < \sqrt{5gR}$

If the velocity of projection is greater than $\sqrt{2gR}$ but less than $\sqrt{5gR}$, the particle rises above the horizontal diameter and the tension vanishes before reaching the highest point (Fig. 5.18).

We have seen that the tension in the string at the highest point is lower than the tension at the lowest point.

At the point D, the string OD makes an angle ϕ with the vertical. The radial component of the weight is $mg\cos\phi$ towards the centre O.

$$T + mg\cos\phi = \frac{mv^2}{R}$$

$\Rightarrow$ $$T = m\left(\frac{v^2}{R} - g\cos\phi\right) \quad (1)$$

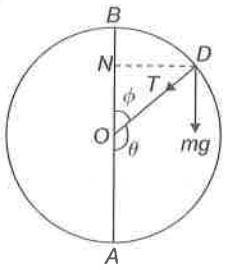


Figure 5.18

Kinetic energy at $D = \frac{1}{2}mv^2$

Potential energy at $D = mg(AN)$

$$= mg(AO + ON)$$

$\Rightarrow$ $$mg(R + R\cos\phi) = mgR(1 + \cos\phi).$$

From conservation of energy,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgR(1 + \cos\phi)$$

$$v^2 = u^2 - 2gR(1 + \cos\phi)$$

Because of this $mg\cos\theta$ will decrease. From A to B, speed of the body also decreases due to which $\frac{mv^2}{R}$ decreases. Therefore, tension in the string decreases from A to B.

But due to $mg\cos\theta$ tension can never be zero.

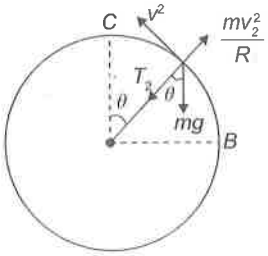


Figure 5.13

(b) Motion of a Body from B to C (Fig. 5.13)

$$T_2 = \frac{mv_2^2}{R} - mg\cos\theta \qquad (2)$$

From $B \rightarrow C$, speed decreases, due to which $\frac{mv_2^2}{R}$ decreases.

θ decreases due to which $mg\cos\theta$ increases.

Therefore, from $B \rightarrow C$, tension in the string decreases.

String slacks at a point where $\frac{mv_2^2}{R} = mg\cos\theta$, i.e., $T = 0$.

Minimum Velocity at Point A for Which Body Can Complete the Vertical Circle

The condition for the body to complete the vertical circle is that the string should be taut all the time, i.e., the tension is greater than zero.

So the body can complete the vertical circle if the tension is not zero in the region B to C.

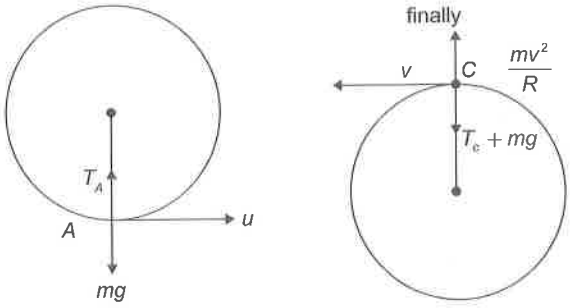
Initially,

from Fig. 5.14(b), $\quad \frac{mv^2}{R} = T_C + mg \qquad (1)$

Apply energy conservation from A to C, then

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv^2 + 2mgR = \frac{1}{2}mu^2 + 0 \qquad (2)$$



(a) (b)

Figure 5.14

The body can complete a vertical circle, when

$$T_C \geq 0$$

$$\frac{mv^2}{R} - mg \geq 0$$

$$\Rightarrow \quad v^2 \geq gR. \qquad (3)$$

Put the value from Eq. (3) to Eq. (2) and

$$u = u_{min}$$

$$\Rightarrow \quad \frac{1}{2}m(Rg) + 2mgR = \frac{1}{2}mu_{min}^2$$

$$\Rightarrow \quad u_{min}^2 = 5gR$$

$$\Rightarrow \quad u_{min} = \sqrt{5gR}\,.$$

If the velocity is greater than or equal to $\sqrt{5gR}$, then the body will complete the vertical circle.

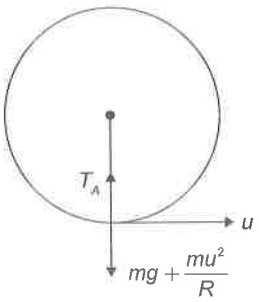


Figure 5.15

Tension at A (Fig. 5.15)

$$\text{If } T_A = mg + \frac{mu^2}{R}$$

$$u = u_{min} = \sqrt{5gR}\,,$$

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EXAMPLE 25

The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through in time t? (b) What is the kinetic energy of the air? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy and that $A = 30\ m^2$, $v = 36\ kmh^{-1}$ and the density of air is $1.2\ kg\ m^{-3}$. What is the electrical power produced?

SOLUTION

(a) Volume of wind flowing per second $= Av$.
Mass of wind flowing per second $= Av\rho$.
Mass of air passing in t seconds $= Av\rho t$.

(b) Kinetic energy of air

$$= \frac{1}{2}mv^2 = \frac{1}{2}(Av\rho t)v^2$$

$$= \frac{1}{2}Av^3\rho t.$$

(c) Electrical energy produced

$$= \frac{25}{100} \times \frac{1}{2}Av^3\rho t = \frac{Av^3\rho t}{8}$$

$$\text{Electrical power} = \frac{Av^3\rho t}{8t} = \frac{Av^3\rho}{8}.$$

Now, $A = 30\ m^2$, $v = 36\ kmh^{-1} = 36 \times \frac{5}{18}\ ms^{-1} = 10\ ms^{-1}$, $\rho = 1.2\ kg\ ms^{-1}$.

$$\therefore \quad \text{Electrical power} = \frac{30 \times 10 \times 10 \times 1.2}{8}\ W$$

$$= 4500\ W = 4.5\ kW.$$ ■

EXAMPLE 26

One coolie takes 1 min to raise a box through a height of 2 m. Another one takes 30 s for the same job and does the same amount of work. Which one of the two has greater power and which one uses greater energy?

SOLUTION

$$\text{Power of first coolie} = \frac{\text{Work}}{\text{Time}}$$

$$= \frac{M \times g \times S}{t}$$

$$= \frac{M \times 9.8 \times 2}{60}\ Js^{-1}$$

$$\text{Power of second coolie} = \frac{M \times 9.8 \times 2}{30}\ Js^{-1}$$

$$= 2\left(\frac{M \times 9.8 \times 2}{60}\right) Js^{-1}$$

$$= 2 \times \text{Power of first coolie.}$$

So, the power of the second coolie is double that of the first. Both the coolies spend the same amount of energy.

We know that $W = Pt$.

For the same work,

$$W = P_1t_1 = P_2t_2$$

or $$\frac{P_2}{P_1} = \frac{t_1}{t_2}$$

$$= \frac{1\ \text{min}}{30\ \text{s}} = 2$$

or $$P_2 = 2P_1.$$ ■

VERTICAL CIRCULAR MOTION

To understand this, consider the motion of a small body (say a stone) tied to a string and whirled in a vertical circle.

Now we study the circular motion of the body in two parts.

(a) Motion of a body from A to B

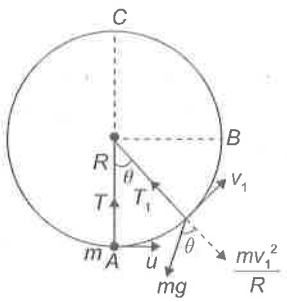


Figure 5.12

(b) Motion of a body from B to C

(a) Motion of a Body from A to B (Fig. 5.12)

$$T_1 = mg\cos\theta + \frac{mv_1^2}{R} \qquad (1)$$

During the motion of the body from A to B, θ will increase and $\cos\theta$ will decrease.

$\Rightarrow \quad W_N = 0$

(as normal reaction is perpendicular to displacement at all points).

W_f can be calculated as shown below:

$$f = \mu mg \cos\theta$$

$$\therefore \quad (dW_{AB})_f = -fds$$

$$= -(\mu mg \cos\theta)ds = -\mu mg(dl)$$

(as $ds \cos\theta = dl$)

$$\therefore \quad f = -\mu mg \Sigma dl$$

$$= -\mu mgl.$$

Substituting these values, we get

$$W_F = mgh + \mu mgl$$ ■

Note

Here again, if we want to solve this problem without using work–energy theorem, we will first find magnitude of the applied force $\vec{F}$ at different locations and then integrate $dW(= \vec{F} \cdot d\vec{r})$ with proper limits.

POWER

Power is defined as the time rate of doing work.
When the time taken to complete a given amount of work is important, we measure the power of the agent doing work.

The average power $\left(\bar{P} \text{ or } P_{avg}\right)$ delivered by an agent is given by

$$\bar{P} \text{ or } P_{avg} = \frac{\Delta W}{\Delta t} = \frac{\text{Total work done}}{\text{Total time}},$$

where ΔW is the amount of work done in time Δt.

Power is the ratio of two scalars—work and time. So, power is a scalar quantity. If time taken to complete a given amount of work is more, then power is less.

- The instantaneous power is $P = \frac{dW}{dt}$, where dW is the work done by a force $\vec{F}$ in a small time dt.
- $P = \frac{dW}{dt}$

$$= \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$, where $\vec{v}$ is the velocity of the body.

By definition of dot product,

$$P = Fv \cos\theta,$$

where θ is the smaller angle between $\vec{F}$ and $\vec{v}$.

This P is called as instantaneous power if dt is very small.

Unit of Power

A unit power is the power of an agent which does unit work in unit time.

The power of an agent is said to be 1 watt if it does 1 joule of work in 1 second.

$$1 \text{ watt} = 1 \text{ joule/s} = 10^7 \text{ erg/s}.$$

Also,
$$1 \text{ watt} = \frac{1 \text{ newton} \times 1 \text{ metre}}{1 \text{ second}}$$

$$= 1 \text{ Nms}^{-1}.$$

(a) Dimensional Formula of Power

$$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]}$$

$$= \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$$

SOLVED EXAMPLES

EXAMPLE 24

A 1 kW motor pumps out water from a well 10 m deep. Calculate the quantity of water pumped out per second.

SOLUTION

Power, $P = 1 \text{ kW} = 10^3 \text{ W}$
$S = 10$ m; time, $t = 1$ s; mass of water, $m = ?$

$$\text{Power} = \frac{mg \times S}{t}$$

$$\therefore \quad 10^3 = \frac{m \times 9.8 \times 10}{1}$$

or
$$m = \frac{10^3}{9.8 \times 10} \text{kg} = 10.204 \text{ kg}.$$ ■

$$\Rightarrow \quad maR\sin\theta = (k_f + u_f) - (k_i + u_i)$$

$$maR\sin\theta = \frac{1}{2}mv^2 - mgR(1-\cos\theta)$$

$$\Rightarrow \quad \frac{1}{2}mv^2 = maR\sin\theta + mgR(1-\cos\theta)$$

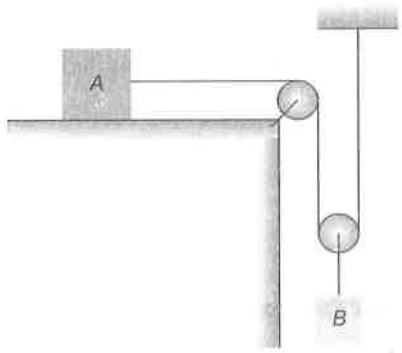
$$\Rightarrow \quad v^2 = 2R(a\sin\theta + g - g\cos\theta)$$

$$\Rightarrow \quad v = [2R\,(a\sin\theta + g - g\cos\theta)]^{1/2}\ \text{m/s}.$$

■

EXAMPLE 22

In the arrangement shown in the figure, $m_A = 4.0$ kg and $m_B = 4.0$ kg. The system is released from rest and the block B is found to have a speed 0.3 m/s after it has descended through a distance of 1 m. Find the coefficient of friction between the block and the table. Neglect friction elsewhere (take $g = 10\ \text{m/s}^2$).



SOLUTION

From constraint relations, we can see that

$$v_A = 2v_B$$

Therefore, $v_A = 2(0.3) = 0.6$ m/s

as $v_B = 0.3$ m/s (given).

Applying $W_{nc} = \Delta U + \Delta K$,

we get $-\mu m_A g S_A = -m_B g S_B + \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$.

Here, $S_A = 2S_B = 2$ m

as $S_B = 1$ m (given).

$$\therefore \quad -\mu(4.0)\,(10)\,(2) = -(1)(10)(1) + \frac{1}{2}(4)(0.6)^2 + \frac{1}{2}(1)\,(0.3)^2$$

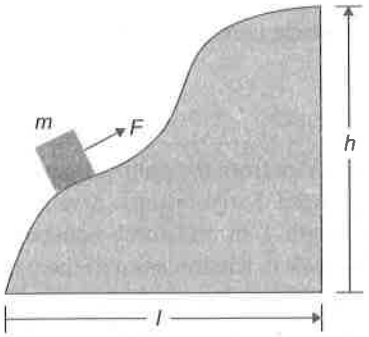
or $$-80\mu = -10 + 0.72 + 0.045$$

or $$80\mu = 9.235$$

or $$\mu = 0.115.$$ ■

EXAMPLE 23

A body of mass m was slowly hauled up the hill as shown in the figure by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.

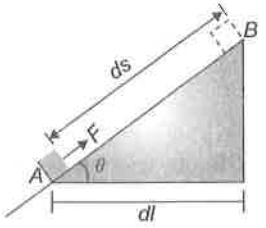


SOLUTION

Four forces are acting on the body:

(1) weight (mg) (2) normal reaction (N)

(3) friction (f) and (4) the applied force (F).



Using the work–energy theorem,

$$W_{\text{net}} = \Delta KE$$

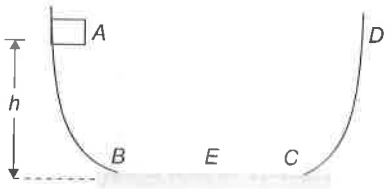
or $$W_{mg} + W_N + W_f + W_F = 0.$$

Here, $\Delta KE = 0$, because $K_i = 0 = K_f$

$$W_{mg} = -mgh$$

After starting from point B, the particle will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend and at C will have the same value as it had when ascending, but now it will move from C to B. The same will be repeated and finally the particle will come to rest at E such that,

$$BC + CB + BE = 7.5$$



or $$3 + 3 + BE = 7.5$$

i.e., $$BE = 1.5.$$

So, the particle comes to rest at the centre of the flat part. ■

EXAMPLE 20

A 0.5 kg block slides from the point A on a horizontal track with an initial speed 3 m/s towards a weightless horizontal spring of length 1 m and force constant 2 N/m. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as 0.22 and 0.20, respectively. If the distance AB and BD are 2 m and 2.14 m, respectively, find the total distance through which the block moves before it comes to rest completely (g= 10 m/s^2).

SOLUTION

As the track AB is frictionless, the block moves this distance without loss in its initial KE:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 3^2$$

$$= 2.25 \text{ J}.$$

In the path BD, as friction is present, work done against friction

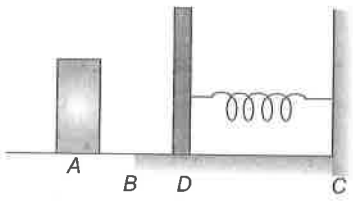
$$= \mu_k mgs$$

$$= 0.2 \times 0.5 \times 10 \times 2.14$$

$$= 2.14 \text{ J}.$$

So, at D, the KE of the block $= 2.25 - 2.14 = 0.11$ J.
Now, if the spring is compressed by x,

$$0.11 = \frac{1}{2} \times k \times x^2 + \mu_k mgx$$



i.e., $$0.11 = \frac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x$$

or $$x^2 + x - 0.11 = 0,$$

which on solving gives positive value of $x = 0.1$ m.

After moving the distance $x = 0.1$ m, the block comes to rest. Now the compressed spring exerts a force:

$$F = kx = 2 \times 0.1$$

$$= 0.2 \text{ N}$$

On the block, while the limiting frictional force between the block and track is

$$f_L = \mu_s mg = 0.22 \times 0.5 \times 10$$

$$= 1.1 \text{ N}.$$

Since $F < f_L$, the block will not move back. So, the total distance moved by the block

$$= AB + BD + 0.1$$

$$= 2 + 2.14 + 0.1$$

$$= 4.24 \text{ m}$$ ■

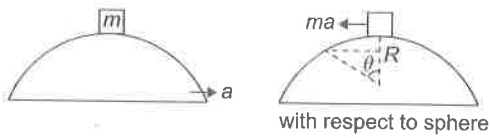
(e) Important Examples

EXAMPLE 21

A smooth sphere of radius R is made to translate in a straight line with a constant acceleration a. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of the angle θ it slides.

SOLUTION

We solve the above problem with respect to the sphere. So apply a pseudo force on the particle.



Now from work energy theorem,

work done by ma = change in mechanical energy

or $$mgh+\frac{1}{2}ky^2=\frac{1}{2}mv^2$$

(as for, B, $h=0$ and $y=0$)

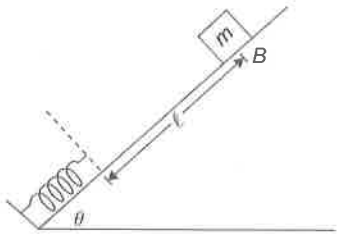
or $$\frac{3}{4}mgd+\frac{1}{2}k\left(\frac{d}{4}\right)^2=\frac{1}{2}mv^2$$

(as for A, $h=\frac{3}{4}d$ and $y=\frac{1}{4}d$)

or $$v=d\sqrt{\frac{3g}{2d}+\frac{k}{16m}}$$ ■

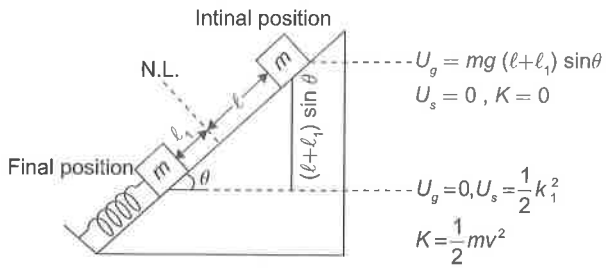
EXAMPLE 17

The block shown in the figure is released from rest and initially the spring is at its natural length. Write down the energy conservation equation when the spring is compressed by ℓ_1?



SOLUTION

Here two conservative forces are included in the problem: (i) gravitational force and (ii) spring force.



We assume zero gravitational potential energy at A as shown in the figure.

From mechanical energy conservation,

$$k_f+u_f=k_i+u_i \qquad (1)$$

$$\frac{1}{2}mv^2+\frac{1}{2}k\ell_1^2=mg(\ell_1+\ell)\sin\theta\,.$$ ■

(C) When Only One Non-Conservative Force is Included in the Problem

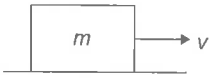
EXAMPLE 18

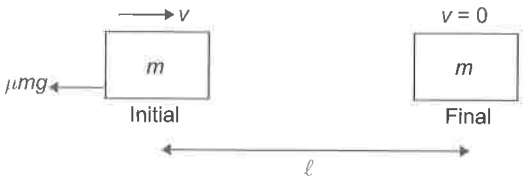
Find out the distance travelled by the block as shown in the figure, if the initial speed of the block is v and μ is the friction coefficient between the surface of block and ground.

SOLUTION

Applying work energy theorem, we get

$$\Rightarrow (-\mu mg\ell)=(0+0)-\left(\frac{1}{2}mv^2+0\right)$$





$$\Rightarrow \quad \mu g\ell=\frac{1}{2}v^2$$

$$\Rightarrow \quad \ell=\frac{v^2}{2\mu g}\,.$$

(D) When Both Conservative and Non-conservative Forces are Included in the Problem

EXAMPLE 19

A particle slides along a track with elevated ends and a flat central part as shown in the figure. The flat portion BC has a length $l = 3.0$ m. The curved portions of the track are frictionless. For the flat part, the coefficient of kinetic friction is $\mu_k = 0.20$. The particle is released at point A, which is at height $h = 1.5$ m above the flat part of the track. Where does the particle finally come to rest?

SOLUTION

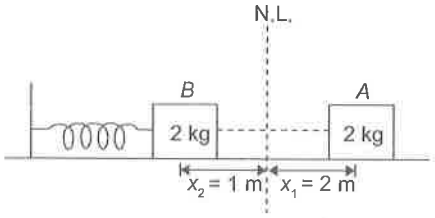
As initial mechanical energy of the particle is mgh and final is zero, loss in mechanical energy $= mgh$. This mechanical energy is lost in doing work against friction in the flat part.

So, loss in mechanical energy = work done against friction

or $$mgh=\mu mgs,$$

i.e., $$s=\frac{h}{\mu}=\frac{1.5}{0.2}$$

$$=7.5\,\text{m}$$



$$u_f = \frac{1}{2}kx_2^2 = \frac{1}{2}\times k\times 1$$
$$= \frac{k}{2}$$

Substituting the above values in Eq. (1), we get

$$\Rightarrow \quad v^2 + \frac{k}{2} = 2k$$

$$\Rightarrow \quad v^2 = \frac{3k}{2}$$

$$\Rightarrow \quad v = \sqrt{\frac{3k}{2}} \text{ m/s}.$$ ■

EXAMPLE 15

A block of mass m is dropped from height h above the ground. Find out the speed of the block when it reaches the ground.

SOLUTION

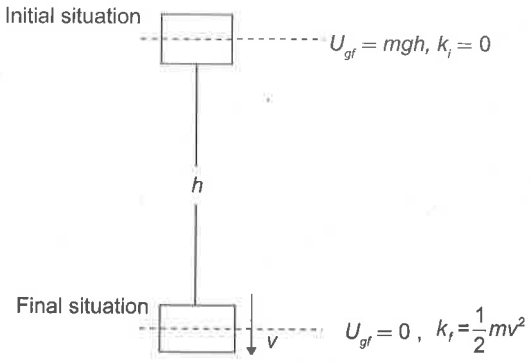


Figure shows the complete description of the problem. Only one conservative force is working on the block. So from mechanical energy conservation,

$$k_f + u_f = k_i + u_i$$

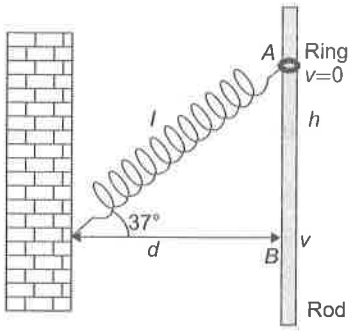
$$\Rightarrow \quad \frac{1}{2}mv^2 + 0 = 0 + mgh$$

$$v = \sqrt{2gh} \text{ m/s}.$$ ■

(b) When Two Conservative Forces are Included in the Problem

EXAMPLE 16

One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other end is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially, the spring makes an angle of 37° with the horizontal as shown in the figure. When the system is released from rest, find the speed of the ring when the spring becomes horizontal ($\sin 37° = 3/5$).



SOLUTION

. If l is the stretched length of the spring, then from the figure,

$$\frac{d}{l} = \cos 37° = \frac{4}{5},$$

i.e., $$l = \frac{5}{4}d.$$

So, the stretch

$$y = l - d = \frac{5}{4}d - d$$
$$= \frac{d}{4}$$

and $$h = l\sin 37°$$

$$= \frac{5}{4}d\times\frac{3}{5} = \frac{3}{4}d.$$

Now, taking point B as reference level and applying the law of conservation of mechanical energy between A and B,

$$E_A = E_B$$

WORK-ENERGY THEOREM

If the resultant or net force acting on a body is F_{net} (Fig. 5.11), then Newton's second law states that

$$F_{net} = ma. \quad (1)$$

If the resultant force varies with x, the acceleration and speed also depend on x.

Then, $$a = v\frac{dv}{dx}. \quad (2)$$

From Eq. (1),

$$F_{net} = mv\frac{dv}{dx}$$

$$\Rightarrow \quad F_{net}dx = mvdv.$$

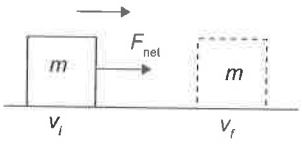


Figure 5.11

$$\int F_{net}dx = \int_{v_i}^{v_f} mvdv$$

$$W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$W_{net} = k_f - k_i$$

$$W_{net} = \Delta K. \quad (3)$$

Work done by net force F_{net} in displacing a particle is equal to the change in kinetic energy of the particle, i.e., we can write Eq. (3) in the following way:

$$(W \cdot D)_c + (W \cdot D)_{NC} + (W \cdot D)_{ext} + (W \cdot D)_{pseudo} = \Delta K, \quad (4)$$

where $(W \cdot D)_c$ = work done by the conservative force
$(W \cdot D)_{NC}$ = work done by non-conservative force
$(W \cdot D)_{ext}$ = work done by the external force
$(W \cdot D)_{pseudo}$ = work done by the pseudo force.
We know that

$$(W \cdot D)_c = -\Delta U$$

$$\Rightarrow \quad -\Delta U + (W \cdot D)_{NC} + (W \cdot D)_{ext} + (W \cdot D)_{pseudo} = \Delta K$$

$$\Rightarrow \quad (W \cdot D)_{NC} + (W \cdot D)_{ext} + (W \cdot D)_{pseudo} = (k_f + u_f) - (k_i + u_i)$$

$$\because \quad k + u = \text{Mechanical energy}$$

$$\Rightarrow$$ Work done by forces (except conservative forces) = change in mechanical energy

If $(W \cdot D)_{NC} = (W \cdot D)_{ext} = (W \cdot D)_{pseudo} = 0$,

$$K_f + U_f = K_i + U_i$$

Initial mechanical energy = Final mechanical energy
This is called mechanical energy conservation law.

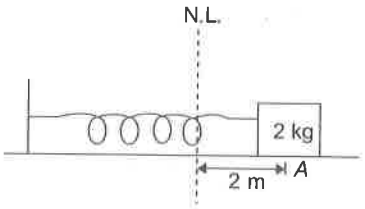
SOLVED EXAMPLES

Questions Based on Work–Energy Theorem

(a) When Only One Conservative Force is Acting

EXAMPLE 14

The block shown in the figure is released from rest. Find out the speed of the block when the spring is compressed by 1 m.



SOLUTION

In the above problem, only one conservative force (spring force) is working on the block. So from mechanical energy conservation,

$$k_f + u_f = k_i + u_i. \quad (1)$$

At position A, block is at rest.
So $k_i = 0$.

$$u_i = \frac{1}{2}kx_1^2 = \frac{1}{2}k(2)^2$$

$$= 2k \text{ Joule.}$$

At position B, if the speed of the block is v, then

$$k_f = \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times v^2 = v^2$$

Case II On increasing x, if U decreases, force is in (+) ve x-direction, i.e., repulsion force.

Different Positions of a Particle

(a) Position of Equilibrium

If net force acting on a body is zero, it is said to be in equilibrium. For equilibrium, $\frac{dU}{dx} = 0$. Points P, Q, R and S are the states of equilibrium positions.

(b) Types of Equilibrium

(i) Stable Equilibrium

When a particle is displaced slightly from a position and a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position.

Necessary conditions: $-\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} = +\text{ve}$.

In Fig. 5.10, points P and R are stable equilibrium points.

(ii) Unstable Equilibrium

When a particle is displaced slightly from a position and force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium.

Condition: $\frac{dU}{dx} = 0$; potential energy is maximum, i.e., $\frac{d^2U}{dx^2} = -\text{ve}$.

Point Q in Fig. 5.10 shows unstable equilibrium.

(iii) Neutral Equilibrium

In neutral equilibrium, potential energy is constant. When a particle is displaced from its position, it does not experience any force acting on it and continues to be in equilibrium in the displaced position. This is said to be neutral equilibrium.

In Fig. 5.10, S is the neutral point.

Condition: $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} = 0$.

SOLVED EXAMPLES

EXAMPLE 12

The potential energy between two atoms in a molecule is given by $U_{(x)} = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants and x is the distance between the atoms. The system is in stable equilibrium when

(A) $x = 0$ (B) $x = \frac{a}{2b}$

(C) $x = \left(\frac{2a}{b}\right)^{1/6}$ (D) $x = \left(\frac{11a}{5b}\right)$

SOLUTION (C)

Given that, $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$

We, know $F = -\frac{du}{dx}$

$= (-12)ax^{-13} - (-6b)x^{-7} = 0$

or $\frac{6b}{x^7} = \frac{12a}{4x^{13}}$

or $x^6 = 12a/6b = 2a/b$

or $x = \left(\frac{2a}{b}\right)^{1/6}$. ■

EXAMPLE 13

The potential energy of a conservative system is given by $U = ax^2 - bx$, where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

SOLUTION

In a conservative field,

$$F = -\frac{dU}{dx}$$

$\therefore$ $$F = -\frac{d}{dx}(ax^2 - bx)$$

$$= b - 2ax.$$

For equilibrium, $F = 0$ or $b - 2ax = 0$

$\therefore$ $$x = \frac{b}{2a}.$$

From the given equation, we can see that $\frac{d^2U}{dx^2} = 2a$ (positive), i.e., U is minimum.

Therefore, $x = \frac{b}{2a}$ is the stable equilibrium position. ■

equivalent to the amount of work necessary to produce the extension.

$$\text{Elastic potential energy} = \frac{1}{2}kx^2. \qquad (3)$$

It is never negative whether the spring is extended or compressed.

Proof

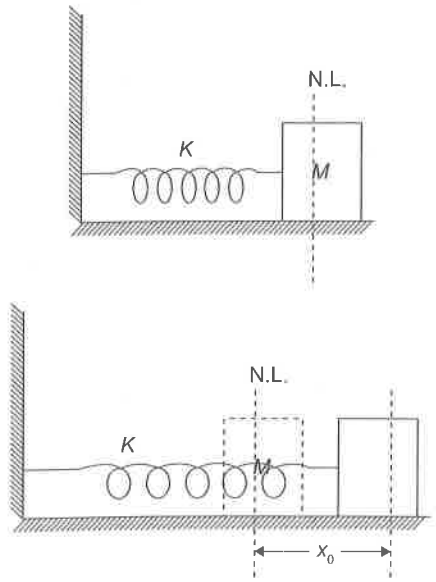


Figure 5.9

Consider a spring block system as shown in Fig. 5.9. Let us calculate work done by the spring when the block is displaced by x_0 from the natural length.

At any moment, if the elongation in spring is x, then the force on the block by the spring is kx towards left. Therefore, the work done by the spring when the block further displaces by dx is given as

$$dW = -kx\,dx.$$

$\therefore$ Total work done by the spring,

$$W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2.$$

Similarly, work done by the spring when it is given a compression x_0 is $-\frac{1}{2}kx_0^2$.

Note

We assume zero potential energy at natural length of the spring.

CONSERVATIVE FORCE AND POTENTIAL ENERGY

$$F_s = -\frac{\partial U}{\partial s},$$

i.e., the projection of the force field , the vector F, at a given point in the direction of the displacement r equals the derivative of the potential energy U with respect to a given direction, taken with the opposite sign. The designation of a partial derivative $\partial/\partial s$ emphasizes the fact of deriving with respect to a definite direction.

So, having reversed the sign of the partial derivatives of the function U with respect to x, y, z, we obtain the projection F_x, F_y and F_z of the vector F on the unit vectors i, j and k. Hence, one can readily find the vector itself:

$$F = F_x i + F_y j + F_z k,$$

or

$$F = -\left(\frac{\partial U}{\partial x}i + \frac{\partial U}{\partial y}j + \frac{\partial U}{\partial z}k\right).$$

The quantity in parentheses is referred to as the scalar gradient of the function U and is denoted by grad U or ∇U. We shall use the second, more convenient, designation where ∇ (nabla) signifies the symbolic vector or operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Potential Energy Curve

- A graph plotted between the PE of a particle and its displacement from the centre of the force field is called PE curve (Fig. 5.10).
- Using this graph, we can predict the rate of motion of a particle at various positions.
- Force on the particle is $F_{(x)} = -\frac{dU}{dx}$.

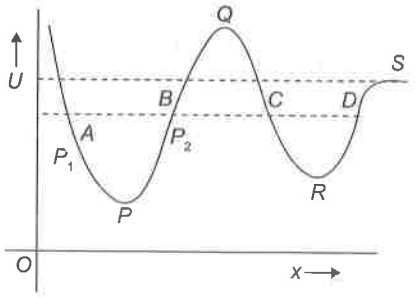


Figure 5.10

Case I On increasing x, if U increases, force is in $(-)$ ve x direction, i.e., attraction force.

with mechanical energy. This type of energy is a property of movement or position.

Kinetic Energy

Kinetic energy (KE) is the capacity of a body to do work by virtue of its motion.

If a body of mass m has velocity v, its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest up to its velocity v.

The numerical value of the kinetic energy can be calculated from the formula

$$KE = \frac{1}{2}mv^2 \qquad (1)$$

- Since both m and v^2 are always positive, KE is always positive and does not depend upon the direction of motion of the body.

Potential Energy

Potential energy is the energy of the body by virtue of its position. A body is capable to do work by virtue of its position, configuration or state of strain.

Now, the relation between potential energy and work done is given as

$$WD = -\Delta U,$$

where ΔU is the change in potential energy.

There are two common forms of potential energy: gravitational and elastic.

Important Points Related to Potential Energy

1. Potential energy is a straight function (defined only for position).
2. Potential energy of a point depends on a reference point.
3. Potential energy difference between two positions does not depend on the frame of reference.
4. Potential energy is defined only for conservative force because work done by conservative force is path independent.
5. If we define potential energy for non-conservative force, then we have to define PE of a single point through different paths, which gives different values of PE at a single point that does not make any sense.

(a) Gravitational Potential Energy

It is possessed by virtue of height.

When an object is allowed to fall from one level to a lower level, it gains speed due to gravitational pull, i.e., it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its gravitational potential energy into kinetic energy.

The gravitational potential energy is equivalent to the negative of the amount of work done by the weight of the body in causing the descent.

If a mass m is at a height h above a lower level (Fig. 5.8), the PE possessed by the mass is $(mg)(h)$.

Since h is the height of an object above a specified level, an object below the specified level has negative potential energy.

Therefore, $\text{GPE} = \pm mgh$ (2)

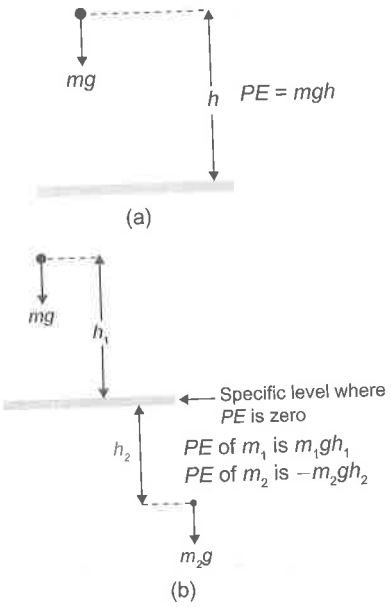


Figure 5.8

- The chosen level from which height is measured has no absolute position. It is important, therefore, to indicate clearly the zero PE level in any problem in which PE is to be calculated.
- $\text{GPE} = \pm mgh$ is applicable only when h is very small in comparison to the radius of the earth. We have discussed GPE in detail in GRAVITATION.

(b) Elastic Potential Energy

It is a property of stretched or compressed springs.

The end of a stretched elastic spring will begin to move if it is released. The spring, therefore, possesses potential energy due to its elasticity (i.e., due to change in its configuration).

The amount of elastic potential energy stored in a spring of natural length a and spring constant k when it is extended by a length x **(from the natural length)** is

$\Rightarrow$ $dw = d(xy)$.

For total work done, we integrate both side

$$\int dw = \int d(xy).$$

Put $xy = k$.

Then at (2, 3), $k_i = 2 \times 3 = 6$

at (5, 6), $k_f = 5 \times 6 = 30$.

Then, $w = \int_6^{30} dk = [k]_6^{30}$

$\Rightarrow$ $w = (30 - 6) = 24$ J. ■

NON-CONSERVATIVE FORCES

A force is said to be non-conservative if work done by or against the force in moving a body depends upon the path between the initial and final positions.

The frictional forces are non-conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does not depend only on the initial and final positions. Note that the work done by frictional force in a round trip is not zero.

The velocity-dependent forces such as air resistance, viscous force, magnetic force, etc., are non-conservative forces.

SOLVED EXAMPLE

EXAMPLE 11

Calculate the work done by the force $\vec{F} = y\hat{i}$ to move the particle from (0, 0) to (1, 1) in the following conditions: (a) $y = x$ and (b) $y = x^2$.

SOLUTION

We know that,

$$dw = \vec{F} \cdot d\vec{s}$$

$\Rightarrow$ $dw = (y\hat{i}) \cdot (dx\hat{i})$

$$dw = ydx. \quad (1)$$

In Eq. (1), we can calculate work done only when we know the path taken by the particle.

Either

$$y = x \text{ or } y = x^2$$

So now,

(a) when $y = x$

$$\int dw = \int_0^1 xdx$$

$\Rightarrow$ $w = \frac{1}{2}$ J.

(b) when $y = x^2$

$$\int dw = \int_0^1 x^2 dx$$

$\Rightarrow$ $w = \frac{1}{3}$ J. ■

Difference Between Conservative and Non-conservative Forces

S.No.	Conservative forces	Non-conservative forces
1	Work done does not depend on path	Work done depends on path
2	Work done in a round trip is zero	Work done in a round trip is not zero
3	Central in nature	Forces are velocity-dependent and retarding in nature
4	When only a conservative force acts within a system, the kinetic energy and potential energy can change. However, their sum, the mechanical energy of the system, does not change	Work done against a non-conservative force may be dissipated as heat energy
5	Work done is completely recoverable.	Work done is not completely recoverable.

ENERGY

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up. Conversely, if some work is done upon an object, the object will be given some energy. Energy and work are mutually convertible.

There are various forms of energy. Heat, electricity, light, sound and chemical energy are all familiar forms. In studying mechanics, we are, however, concerned chiefly

4. Magnetic force between two magnetic poles is a conservative force.

Forces acting along the line joining the centres of two bodies are called central forces. Gravitational force and Electrostatic force are two important examples of central forces. Central forces are conservative forces.

Properties of Conservative Forces

- Work done by or against a conservative force depends only on the initial and final position of the body.
- Work done by or against a conservative force does not depend upon the nature of the path between initial and final positions of the body.
- Work done by or against a conservative force in a round trip is zero.

  If a body moves under the action of a force that does no total work during any round trip, then the force is conservative; otherwise, it is non-conservative.

  The concept of potential energy exists only in the case of conservative forces.
- The work done by a conservative force is completely recoverable.

  Complete recoverability is an important aspect of the work done by a conservative force.

Work Done by Conservative Forces

SOLVED EXAMPLES

(a) I Format: When Constant Force is Given

EXAMPLE 8

Calculate the work done to displace the particle from (1, 2) to (4, 5) if $\vec{F} = 4\hat{i} + 3\hat{j}$.

SOLUTION

$$dw = \vec{F} \quad d\vec{r}\,(d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$dw = (4\hat{i} + 3\hat{j})(dx\hat{i} + dy\hat{j})$$

$$\Rightarrow \qquad dw = 4dx + 3dy$$

$$\int_0^w dw = \int_1^4 4dx + \int_2^5 3dy$$

$$\Rightarrow \qquad w = [4x]_1^4 + [3y]_2^5$$

$$w = (16 - 4) + (15 - 6)$$

$$\Rightarrow \qquad w = 12 + 9 = 21 \text{ Joule.}$$ ■

(b) II Format: When F is Given as a Function of x, y, z

If $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, then

$$dw = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\Rightarrow \qquad dw = F_x dx + F_y dy + F_z dz.$$

EXAMPLE 9

An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})$ m to $\vec{r}_2 = (4\hat{i} + 6\hat{j})$ m under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})$ N. Find the work done by this force.

SOLUTION

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot \vec{dr}$$

$$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2 dx + 2ydy) = [x^3 + y^2]_{(2,3)}^{(4,6)}$$

$$= 83 \text{ J.}$$ ■

(c) III Format: Perfect Differential Format

EXAMPLE 10

If $\vec{F} = y\hat{i} + x\hat{j}$, then find out the work done in moving the particle from position (2, 3) to (5, 6).

SOLUTION

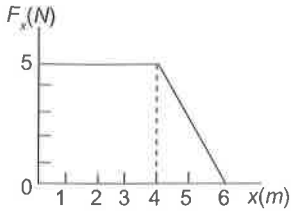
$$dw = \vec{F}.\vec{ds}$$

$$dw = (y\hat{i} + x\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$

$$dw = ydx + xdy.$$

Now $ydx + xdy = d(xy)$

(perfect differential equation)



SOLUTION

The work done by the force is equal to the area under the curve from $x = 0$ to $x = 6.0$ m.

This area is equal to the area of the rectangular section from $x = 0$ to $x = 4.0$ m plus the area of the triangular section from $x = 4.0$ m to $x = 6.0$ m. The area of the rectangle is (4.0)(5.0) N.m = 20 J, and the area of the triangle is $\frac{1}{2}(2.0)\times(5.0)$ N.m = 5.0 J. Therefore, the total work done is 25 J. ■

INTERNAL WORK

Suppose that a man sets himself in motion backward by pushing against a wall (Fig. 5.6). The forces acting on the man are his weight W, the upward force N exerted by the ground and the horizontal force N exerted by the wall. The works of W and of N are zero because they are perpendicular to the motion.

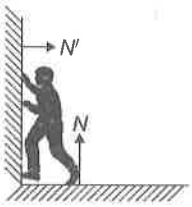


Figure 5.6

The force N' is the unbalanced horizontal force that imparts to the system a horizontal acceleration. The work of N', however, is zero because there is no motion at its point of application. We are, therefore, confronted with a curious situation in which a force is responsible for acceleration, but its work, being zero, is not equal to the increase in kinetic energy of the system.

The new feature in this situation is that the man is a composite system with several parts that can move in relation to each other and thus can do work on each other, even in the absence of any interaction with externally applied forces. Such work is called internal work. Although internal forces play no role in the acceleration of the composite system, their points of application can move so that work is done; thus, the man's kinetic energy can change even though the external forces do no work.

Basic Concept of Work Lies in Following Lines

Draw the force at a proper point where it acts that give proper importance of the point of application of force.

Think independently for displacement of point of application of force. Instead of relation, the displacement of applicant point with force relates it with the observer or reference frame in which work is calculated.

$$W = (\text{Force vector}) \times \begin{pmatrix} \text{displacement vector of point of} \\ \text{application of force as seen by} \\ \text{observer} \end{pmatrix}.$$

CONSERVATIVE FORCE

A force is said to be conservative if the work done by or against the force in moving a body depends only on the initial and final positions of the body and does not depend on the nature of the path followed between the initial and final positions.

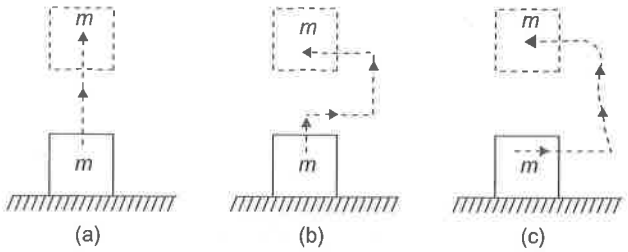
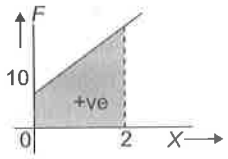


Figure 5.7

Consider a body of mass m being raised to a height h vertically upwards as shown in Fig. 5.7. The work done is mgh. Suppose we take the body along the path as in Figure 5.7(b). The work done during horizontal motion is zero. Adding up the works done in the two vertical parts of the paths, we get the result mgh once again. Any arbitrary path like the one shown in Fig. 5.7(c) can be broken into elementary horizontal and vertical portions. Work done along the horizontal path is zero. The work done along the vertical parts add up to mgh. Thus, we conclude that the work done in raising a body against gravity is independent of the path taken. It only depends upon the initial and final positions of the body. We conclude from this discussion that the force of gravity is a conservative force.

Examples of Conservative Forces

1. Gravitational force, not only due to Earth but in its general form as given by the universal law of gravitation, is a conservative force.
2. Elastic force in a stretched or compressed spring is a conservative force.
3. Electrostatic force between two electric charges is a conservative force.



or $dW = Fdx$ $[\because \cos 0° = 1]$

Total work done,

$$W = \int_{x=0}^{x=2} Fdx = \int_{x=0}^{x=2} (0.5x + 10)\,dx$$

$$= \int_{x=0}^{x=2} 0.5x\,dx + \int_{x=0}^{x=2} 10\,dx$$

$$= 0.5\left|\frac{x^2}{2}\right|_{x=0}^{x=2} + 10|x|_{x=0}^{x=2}$$

$$= \frac{0.5}{2}[2^2 - 0^2] + 10[2 - 0]$$

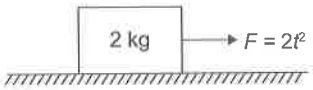
$$= (1 + 20) = 21 \text{ J.}$$ ■

SOLVED EXAMPLE

When F is given as a function of time (t)

EXAMPLE 6

The force $F = 2t^2$ is applied on the 2 kg block. Then find out the work done by this force in 2 s. Initially at time $t = 0$, the block is at rest.



SOLUTION

$$F = ma$$

$$\Rightarrow \quad 2t^2 = 2a$$

$$\Rightarrow \quad a = t^2$$

$$\Rightarrow \quad \frac{dv}{dt} = t^2$$

$$\Rightarrow \quad \int_0^v dv = \int_0^t t^2 dt$$

(At $t = 0$, it is at rest).

$$\Rightarrow \quad v = \frac{t^3}{3}.$$

Let the displacement of the block be dx from $t = t$ to $t = t + dt$. Then, work done by the force F in this time interval dt is

$$dw = F \cdot dx = 2t^2 \cdot dx$$

$$dw = 2t^2 \cdot \frac{dx}{dt} \cdot dt$$

$$\Rightarrow \quad dw = 2t^2(v)dt$$

$$\int_0^w dw = \int_0^2 2t^2 \cdot \frac{t^3}{3} dt$$

$$\Rightarrow \quad W = \frac{2}{3}\int_0^2 t^5 dt$$

$$\Rightarrow \quad W = \frac{2}{3}\left|\frac{t^6}{6}\right|_0^2 = \frac{64}{9} \text{ J.}$$ ■

AREA UNDER FORCE–DISPLACEMENT CURVE

Graphically, area under the force–displacement curve is the work done (Fig. 5.5).

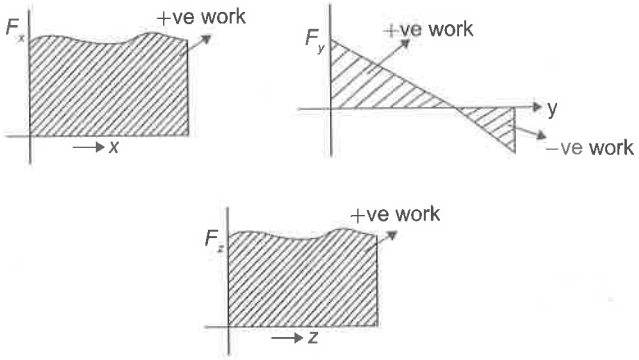


Figure 5.5

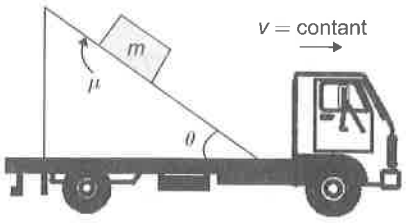
The work done can be positive or negative as per the area above the x-axis or below the x-axis, respectively.

SOLVED EXAMPLE

EXAMPLE 7

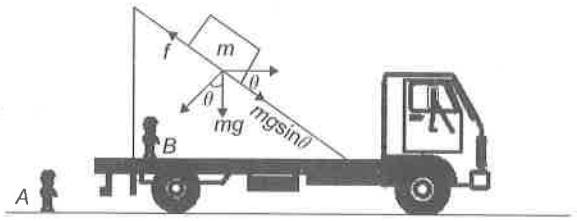
Force acting on a particle varies with x as shown in the figure. Calculate the work done by the force as the particle moves from $x = 0$ to $x = 6.0$ m.

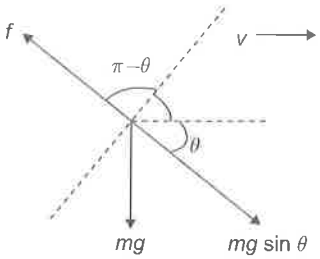
friction in time t if the block is at rest with respect to the incline plane.



SOLUTION

F.B.D of the block with respect to the ground.





The block is at rest with respect to the wedge

$$\Rightarrow \qquad f = mg \sin\theta$$

In time t, the displacement of the block with respect to the ground, $d = vt$.

Work done by friction for man A,

W_f = (component of frictional force along displacement) × displacement

$W_f = mg \sin\theta \cdot vt \cos(180° - \theta)$

$W_f = -mg\, vt \cos\theta \sin\theta$

W_f for man $B = 0$ (displacement is zero with respect to man B). ■

WORK DONE BY A VARIABLE FORCE

When *F* is a function of *x*, *y*, *z*

When the magnitude and direction of a force vary in three dimensions, it can be expressed as a function of the position. For a variable force, work is calculated for infinitely small displacement, and for this displacement, force is assumed to be constant.

$$dW = \vec{F} \cdot d\vec{s}\,.$$

The total work done will be the sum of infinitely small work

$$W_{A\to B} = \int_A^B \vec{F} \cdot d\vec{s} = \int_A^B (\vec{F}\cos\theta)\, d\vec{s}\,.$$

In terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$W_{A\to B} = \int_{x_A}^{x_B} F_x\,dx + \int_{y_A}^{y_B} F_y\,dy + \int_{z_A}^{z_B} F_z\,dz\,.$$

SOLVED EXAMPLES

EXAMPLE 4

A force $F = (4.0x\,\hat{i} + 3.0y\,\hat{j})$ N acts on a particle which moves in the x-direction from the origin to $x = 5.0$ m. Find the work done on the object by the force.

SOLUTION

Here the work done is only due to x component of force because displacement is along the x-axis.

i.e., $$W = \int_{x_1}^{x_2} F_x\,dx = \int_0^5 4x\,dx$$

$$= [2x^2]_0^5 = 50\,\text{J}.$$ ■

EXAMPLE 5

A force $F = 0.5x + 10$ acts on a particle. Here F is in newton and x is in metre. Calculate the work done by the force during the displacement of the particle from $x = 0$ to $x = 2$ m.

SOLUTION

Small amount of work done dW in giving a small displacement $\overrightarrow{dx}$ is given by

$$dW = \vec{F} \cdot \overrightarrow{dx}$$

or $$dW = F\,dx \cos 0°$$

Work has one dimension in mass, two dimensions in length and '−2' dimensions in time. On the basis of dimensional formula, the unit of work is kg $m^2 s^{-2}$.

Note that 1 kg $m^2 s^{-2}$ = (1 kg m s^{-2}) m = 1 Nm = 1 J.

WORK DONE BY MULTIPLE FORCES

If several forces act on a particle, then we can replace $\vec{F}$ in equation $W = \vec{F} \cdot \vec{S}$ by the net force $\sum \vec{F}$, where

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$$

$$\therefore \quad W = \left[\sum \vec{F}\right] \cdot \vec{S} \qquad (1)$$

This gives the work done by the net force during a displacement $\vec{S}$ of the particle.

We can rewrite Eq. (1) as

$$W = \vec{F}_1 \cdot \vec{S} + \vec{F}_2 \cdot \vec{S} + \vec{F}_3 \cdot \vec{S} + \cdots$$

or $$W = W_1 + W_2 + W_3 + \cdots$$

So, the work done on the particle is the sum of the individual work done by all the forces acting on the particle.

SOLVED EXAMPLES

EXAMPLE 1

A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with the horizontal. Coefficient of friction between the block and the surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block.

SOLUTION

The forces acting on the block are shown in the figure. As the block moves with uniform velocity, the resultant force on it is zero.

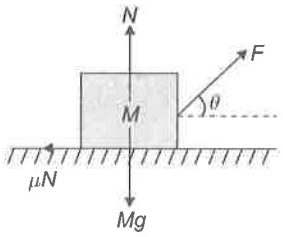
$$\therefore \quad F\cos\theta = \mu N \qquad (1)$$

$$F\sin\theta + N = Mg \qquad (2)$$

Eliminating N from Eqs. (1) and (2),

$$F\cos\theta = \mu(Mg - F\sin\theta)$$

$$F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta}.$$



Work done by this force during a displacement d

$$W = F \cdot d\cos\theta$$

$$= \frac{\mu Mgd\cos\theta}{\cos\theta + \mu\sin\theta}. \quad \blacksquare$$

EXAMPLE 2

A particle moving in the xy-plane undergoes a displacement $\vec{s} = (2.0\hat{i} + 3.0\hat{j})m$, while a constant force $\vec{F} = (5.0\hat{i} + 2.0\hat{j})$N acts on the particle.

(a) Calculate the magnitude of the displacement and that of the force.

(b) Calculate the work done by the force.

SOLUTION

(a) $\vec{s} = (2.0\hat{i} + 3.0\hat{j})$

$$\vec{F} = (5.0\hat{i} + 2.0\hat{j})$$

$$|\vec{s}| = \sqrt{x^2 + y^2} = \sqrt{(2.0)^2 + (3.0)^2}$$
$$= \sqrt{13}\ \text{m}$$

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2}$$
$$= 5.4\ \text{N}$$

(b) Work done by the force, $W = \vec{F} \cdot \vec{s}$

$$= (5.0\hat{i} + 2.0\hat{j}) \cdot (2.0\hat{i} + 3.0\hat{j})\ N.m$$

$$= 10 + 0 + 0 + 6$$

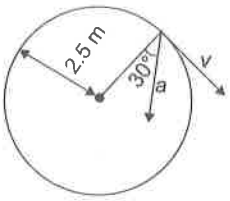
$$= 16\ \text{N} \cdot \text{m} = 16\ \text{J}. \quad \blacksquare$$

EXAMPLE 3

A block of mass m is placed on an inclined plane which is moving with constant velocity v in the horizontal direction as shown in the figure. Then find out the work done by the

6. A particle is revolving in a circle of radius 1 m with an angular speed of 12 rad/s. At $t = 0$, it was subjected to a constant angular acceleration α and its angular speed increased to $(480/\pi)$ r.p.m. in 2 s. The particle then continues to move with attained speed. Calculate:
(A) angular acceleration of the particle
(B) tangential velocity of the particle as a function of time
(C) acceleration of the particle at $t = 0.5$ s and at $t = 3$ s
(D) angular displacement at $t = 3$ s

7. The figure shows the total acceleration and velocity of a particle moving clockwise in a circle of radius 2.5 m at a given instant of time. At this instant, find

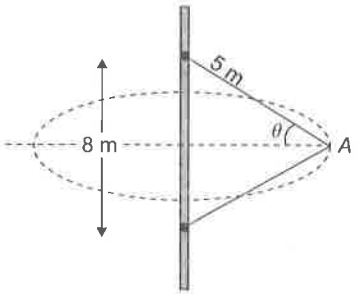


(A) the radial acceleration
(B) the speed of the particle and
(C) its tangential acceleration

8. A stone is launched upward at 45° with speed v_0. A bee follows the trajectory of the stone at a constant speed equal to the initial speed of the stone.
(A) Find the radius of curvature at the top point of the trajectory.
(B) What is the acceleration of the bee at the top point of the trajectory? For the stone, neglect the air resistance.

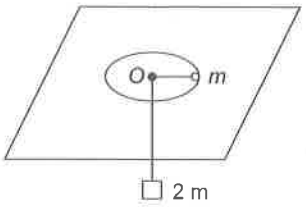
9. A particle moves in a circle of radius R with a constant speed v. Then, find the magnitude of average acceleration during a time interval $\frac{\pi R}{2v}$.

10. A 4-kg block is attached to a vertical rod by means of two strings of equal length. When the system rotates about the axis of the rod, the strings are extended as shown in the figure.
(A) How many revolutions per minute must the system make in order for the tension in the upper chord to be 20 kgf?
(B) What is the tension in the lower chord?

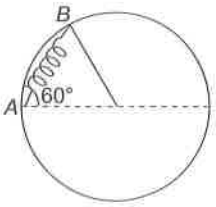


11. A rod of length 1 m is being rotated about its end in a gravity-free space with a constant angular acceleration of 5 rad/s² starting from rest. A sleeve is fitted on the rod at a distance of 0.5 m from the centre. The coefficient of friction between the rod and the sleeve is 0.05. Find the time after which the sleeve will start slipping on the rod.

12. A mass m rotating freely in a horizontal circle of radius 1 m on a frictionless smooth table supports a stationary mass $2m$, attached to the other end of the string passing through a smooth hole O in the table, hanging vertically. Find the angular velocity of rotation.

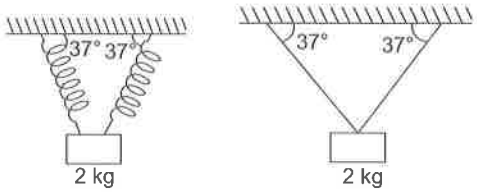


13. A beam of mass m is attached to one end of a spring of natural length $\sqrt{3}R$ and spring constant $k = \frac{(\sqrt{3}+1)mg}{R}$. The other end of the spring is fixed at point A on a smooth fixed vertical ring of radius R as shown in the figure. What is the normal reaction at B just after the bead is released?



14. The blocks of mass 2 kg are shown to be in equilibrium. At $t = 0$, right spring in Figure (i) and right string in

Figure (ii) breaks. Find the ratio of instantaneous acceleration of blocks.



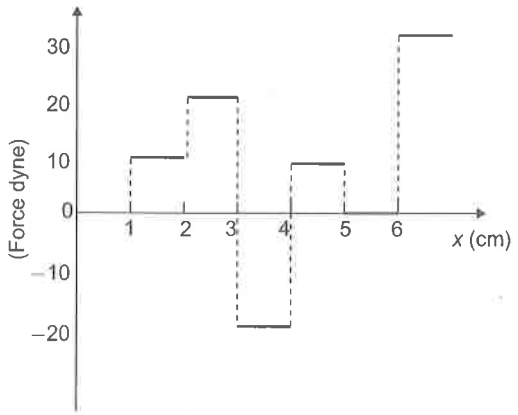
(B) *Work, Power and Energy*

15. A block of mass m is pulled on a rough horizontal surface, which has a coefficient of friction μ. A force F is applied, which is capable of moving the body uniformly with speed v. Find the work done on the block in time t by (A) weight of the block, (B) normal reaction by the surface on the block, (C) friction and (D) F.

16. Calculate the work done against gravity by a coolie in carrying a load of mass 10 kg on his head when he walks uniformly a distance of 5 m in the (A) horizontal direction (B) vertical direction (take $g = 10\ m/s^2$).

17. A body is constrained to move in the y-direction. It is subjected to a force $(-2i + 15j + 6k)$ newton. What is the work done by this force in moving the body through a distance of 10 m?

18. The relationship between force and position is shown in the figure given (in one-dimensional case). What will be the work done by the force in displacing a body from $x = 1$ cm to $x = 5$ cm?



19. It is well known that a raindrop or a small pebble falls under the influence of the downward gravitational force and the opposing resistive force. The latter is known to be proportional to the speed of the drop but is otherwise undetermined. Consider a drop or small pebble of 1 g falling from a cliff of height 1.00 km. It hits the ground with a speed of 50.0 ms^{-1}. What is the work done by the unknown resistive force?

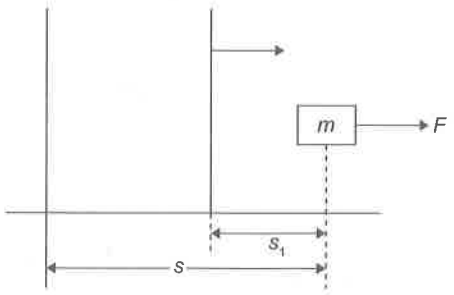
20. A rigid body of mass 2 kg initially at rest moves under the action of an applied horizontal force 7 N on a table with coefficient of kinetic friction = 0.1. Calculate the

(A) work done by the applied force on the body in 10 s
(B) work done by friction on the body in 10 s
(C) work done by the net force on the body in 10 s
(D) change in kinetic energy of the body is 10 s

21. A rigid body of mass 0.3 kg is taken slowly up an inclined plane of length 10 m and height 5 m and then allowed to slide down to the bottom again. The co-efficient of friction between the body and the plane is 0.15. Using $g = 9.8\ m/s^2$, find the

(A) work done by the gravitational force over the round trip
(B) work done by the applied force (assuming it to be parallel to the inclined plane) over the upward journey
(C) work done by frictional force over the round trip
(D) kinetic energy of the body at the end of the trip

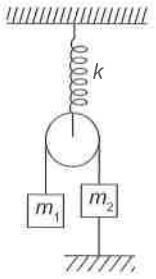
22. A block of mass m sits at rest on a frictionless table in a rail car that is moving with speed v_c along a straight horizontal track (as shown in the figure). A person riding in the car pushes on the block with a net horizontal force F for a time t in the direction of the car's motion.



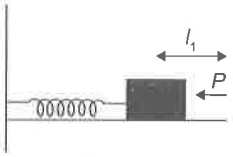
(A) What is the final speed of the block according to (i) a person in the car?

(B) According to a person standing on the ground outside the train.
(C) How much did K of the block change (i) according to the person in the car?
(D) According to the person on the ground.
(E) In terms of F, m, and t, how far did the force displace the object according to the person in car?
(F) According to the person on the ground.
(G) How much work does each say the force did?
(H) Compare the work done to the K gain according to each person.
(I) What can your conclude from this computation?

23. In the figure shown, pulley and spring are ideal. Find the potential energy stored in the spring ($m_1 > m_2$).



24. A block of mass m placed on a smooth horizontal surface is attached to a spring and is held at rest by a force P as shown. Suddenly, the force P changes its direction opposite to the previous one. How many times is the maximum extension l_2 of the spring longer compared to its initial compression l_2?



25. (A) Power applied to a particle varies with time as $P = (3t^2 - 2t + 1)$ watt, where t is in second. Find the change in its kinetic energy between time $t =$ 2 s and $t = 4$ s.
(B) The potential function for a conservative force is given by $U = k(x + y)$. Find the work done by the conservative force in moving a particle from the point $A(1, 1)$ to point B (2, 3).

26. A labourer lifts 100 stones to a height of 6 m in 2 min. If mass of each stone be 1 kg, calculate the average power. Given $g = 10\ \text{ms}^{-2}$.

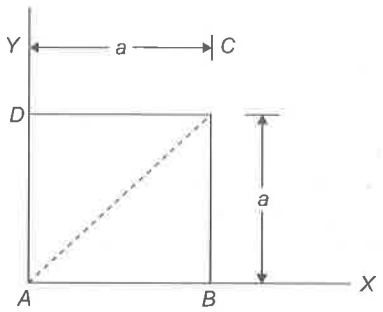
27. An engine develops 10 kW of power. How much time will it take to lift a mass of 200 kg through a height of 40 m? Given $g = 10\ \text{ms}^{-2}$.

28. Two trains of equal masses are drawn along smooth level lines by engines: one of them X exerts a constant force while the other Y works at a constant rate. Both start from rest and after a time t both again have the same velocity v.. Find the ratio of travelled distance during the interval.

29. Water is pumped from a depth of 10 m and delivered through a pipe of cross section 10^{-2}m^2 up to a height of 10 m. If it is needed to deliver a volume 0.2 m^3/s, find the power required. [Use $g = 10\ \text{m/s}^2$.]

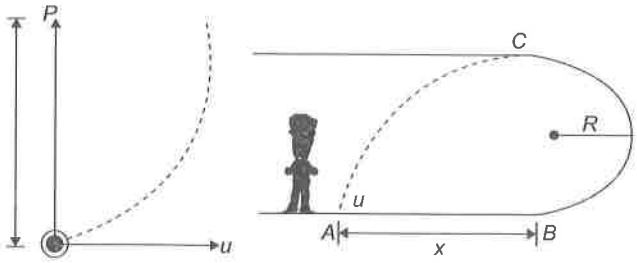
30. A particle moves along a straight line. A force acts on the particle which produces a constant power. It starts with initial velocity 3 m/s and after moving a distance 252 m, its velocity is 6 m/s. Find the time taken.

31. A force $F = x^2y^2i + x^2y^2j$ (N) acts on a particle which moves in the x–y plane. Find the work done by $F\frac{\vec{u}}{F}$ as it moves the particle from A to C (figure) along each of the paths ABC, ADC and AC.



32. Calculate the forces $F(y)$ associated with the following one-dimensional potential energies:
(A) $U = -wy$ (B) $U = ay^3 - by^2$
(C) $U = U_0 \sin by$

33. Consider the shown arrangement when a is a bob of mass m is suspended by means of a string connected to peg P. If the bob is given a horizontal velocity $\sqrt{3gl}$ having magnitude, find the minimum speed of the bob in subsequent motion.

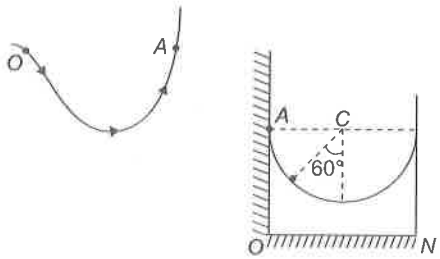


34. A person rolls a small ball with speed u along the floor from point A. If $x = 3R$, determine the required speed u so that the ball returns to A after rolling on the circular surface in the vertical plane from B to C and becoming a projectile at C. What is the minimum value of x for which the game could be played if the contact must be maintained to point C? Neglect friction.

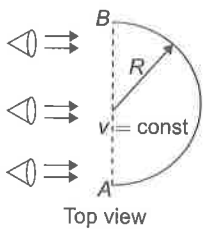
35. A toy rocket of mass 1 kg has a small fuel of mass 0.02 kg which it burns out in 3 s. Starting from rest on a horizontal smooth track, it gets a speed of 20 ms^{-1} after the fuel is burnt out. What is the average thrust of the rocket? What is the energy content per unit mass of the fuel? (Ignore the small mass variation of the rocket during fuel burning.)

Level II

1. A particle which moves along the curved path shown passes point O with a speed of 12 m/s and slows down to 5 m/s at point A in a distance of 18 m measured along the curve from O. The deceleration measured along the curve is proportional to the distance from O. If the total acceleration of the particle is 10 m/s^2 as it passes A, find the radius of curvature of A.

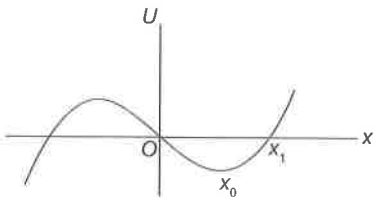


2. A ball of mass 1 kg is released from position A inside a wedge with a hemispherical cut of radius 0.5 m as shown in the figure. Find the force exerted by the vertical wall OM on the wedge, when the ball is in position B (neglect friction everywhere and take $g = 10$ m/s^2).

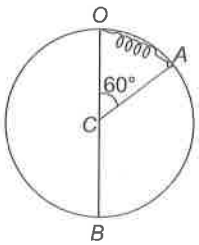


Top view

3. A small block can move in a straight horizontal line a along AB. Flash lights from one side projects its shadow on a vertical wall, which has horizontal cross-section as a circle. Find the tangential and normal acceleration of shadow of the block on the wall as a function of time if the velocity of the block is constant (v).



4. A particle is confined to move along the $+x$ axis under the action of a force $F(x)$ that is derivable from the potential $U(x) = ax^3 - bx$.



(A) Find the expression for $F(x)$.
(B) When the total energy of the particle is zero, the particle can be trapped within the interval $x = 0$ to $x = x_1$. For this case, find the values of x_1.
(C) Determine the maximum kinetic energy that the trapped particle has in its motion. Express all answers in terms of a and b.

5. A particle of mass 5 kg is free to slide on a smooth ring of radius $r = 20$ cm fixed in a vertical plane. The particle is attached to one end of a spring whose other end is fixed to the top point O of the ring. Initially, the particle is at rest at a point A of the ring such that $\angle OCA = 60°$, C being the centre of the ring. The natural length of the spring is also equal to $r = 20$ cm. After the particle is released and slides down the ring,

the contact force between the particle and the ring becomes zero when it reaches the lowest position B. Determine the force constant of the spring.

$$k = \frac{4mg}{R}.$$

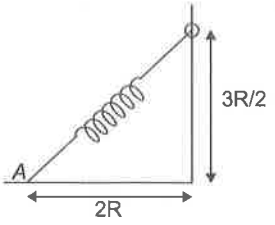
6. Two blocks of mass $m_1 = 10$ kg and $m_2 = 5$ kg connected to each other by a massless inextensible string of length 0.3 m are placed along a diameter of a turntable. The coefficient of friction between the table and m_1 is 0.5, while there is no friction between m_2 and the table. The table is rotating with an angular velocity of 10 rad/s about a vertical axis passing through its centre. The masses are placed along the diameter of the table on either side of the centre O such that m_1 is at a distance of 0.124 m from O. The masses are observed to be at rest with respect to an observer on the turntable.

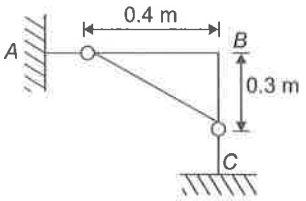
(A) Calculate the frictional force on m_1.

(B) What should be the minimum angular speed of the turntable so that the masses will slip from this position?

(C) How should the masses be placed with the string remaining taut, so that there is no frictional force acting on the mass m_1?

7. A ring of mass m can slide over a smooth vertical rod. The ring is connected to a spring of force constant, where $2R$ is the natural length of the spring. The other end of the spring is fixed to the ground at a horizontal distance $2R$ from the base of the rod. The mass is released at a height of $1.5R$ from the ground.

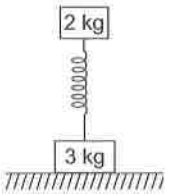




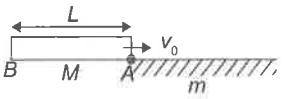
(A) Calculate the work done by the spring

(B) Calculate the velocity of the ring as it reaches the ground.

8. Two identical beads of mass 1 kg each are connected by an inextensible massless string and they can slide along the two arms AB and BC of a rigid smooth wire frame in vertical plane. If the system is released from rest, find the speeds of the particles when they have moved by a distance of 0.1 m. Also find tension in the string.



9. The ends of spring are attached to blocks of mass 2 kg and 3 kg. The 3-kg block rests on a horizontal surface and the 2-kg block which is vertically above it is in equilibrium producing a compression of 1 cm of the spring. The 2-kg mass must be compressed further by at least _____, so that when it is released, the 3-kg block may be lifted off the ground.

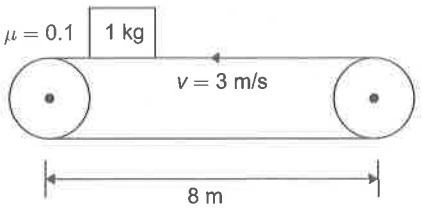


10. A uniform rod of mass m length L is sliding along its length on a horizontal table whose top is partly smooth and rest rough with friction coefficient μ. If the rod after moving through smooth part enters the rough part with velocity v_0,

(A) what will be the magnitude of the friction force when its x length ($<L$) lies in the rough part during sliding?

(B) Determine the minimum velocity v_0 with which it must enter so that it lies completely in the rough region before coming to rest.

(C) If the velocity is double the minimum velocity as calculated in part (A), then what distance would its front end A have travelled in the rough region before the rod comes to rest.



11. Find the velocity with which a block of mass 1 kg must be horizontally projected on a conveyer belt moving uniformly at a velocity of 3 m/s so that maximum heat is liberated. Take coefficient of friction of 0.1. Also find the corresponding amount of heat liberated. What happens when the belt velocity is 5 m/s?

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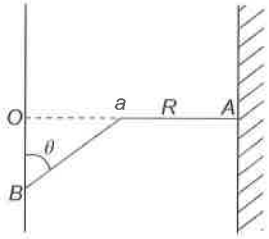
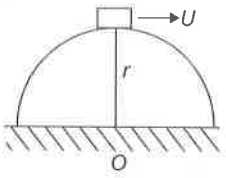


12. A small bead of mass m is free to slide on a fixed smooth vertical wire, as indicated in the diagram. One end of a light elastic string of unstretched length a and force constant $2mg/a$ is attached to B. The string passes through a smooth fixed ring R and the other end of the string is attached to the fixed point A, AR being horizontal. The point O on the wire is at the same horizontal level as R, and $AR = RO = a$.

(A) In the equilibrium, find OB.

(B) The bead B is raised to a point C of the wire above O, where $OC = a$ and is released from rest. Find the speed of the bead as it passes O and find the greatest depth below O of the bead in the subsequent motion.



13. A small block of mass m is projected horizontally from the top of the smooth hemisphere of radius r with speed u as shown. For values of $u \geq u_0$, it does not slide on the hemisphere (i.e., leaves the surface at the top itself)

(A) For $u = 2u_0$ it lands at point P on ground, find OP.

(B) For $u = u_0/3$, find the height from the ground at which it leaves the hemisphere.

(C) Find its net acceleration at the instant it leaves the hemisphere.

Previous Year Questions

JEE Main

(A) Circular Motion

1. The minimum velocity (in ms^{-1}) with which a car driver must traverse a flat curve of radius 150 m and coefficient of friction 0.6 to avoid skidding is **[AIEEE, 2002]**

(A) 60 (B) 30

(C) 15 (D) 25

2. Which of the following statements is false for a particle moving in a circle with a constant angular speed ? **[AIEEE, 2004]**

(A) The velocity vector is tangent to the circle.

(B) The acceleration vector is tangent to the circle.

(C) The acceleration vector points to the centre of the circle.

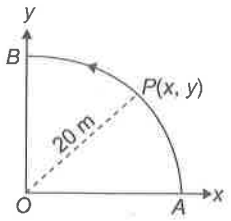
(D) The velocity and acceleration vectors are perpendicular to each other.

3. An angular ring with inner and outer radii R_1 and R_2 is rolling without slipping with a uniform angular speed. The ratio of the forces experienced by the two particles situated on the inner and outer parts of the ring $\frac{F_1}{F_2}$ is **[AIEEE, 2005]**

(A) $\frac{R_2}{R_1}$ (B) $\left(\frac{R_1}{R_2}\right)^2$

(C) 1 (D) $\frac{R_1}{R_2}$

4. A point P moves in the counter-clockwise direction on a circular path as shown in the figure. The movement of P is such that it sweeps out a length $s = t^3 + 5$, where s is in metre and t is in seconds. The radius of the path is 20 m. The acceleration of P when $t = 2s$ is nearly **[AIEEE, 2010]**



(A) 13 ms^{-2} (B) 12 ms^{-2}

(C) 7.2 ms^{-2} (D) 14 ms^{-2}

5. For a particle in uniform circular motion, the acceleration $\vec{a}$ at a point $P(R,\theta)$ on the circle of radius R is (here θ is measured from the x-axis) **[AIEEE, 2010]**

(A) $-\frac{v^2}{R}\cos\theta\hat{i}+\frac{v^2}{R}\sin\theta\hat{j}$

(B) $-\frac{v^2}{R}\sin\theta\hat{i}+\frac{v^2}{R}\cos\theta\hat{j}$

(C) $-\frac{v^2}{R}\cos\theta\hat{i}-\frac{v^2}{R}\sin\theta\hat{j}$

(D) $\frac{v^2}{R}\hat{i}+\frac{v^2}{R}\hat{j}$

6. Two cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2, respectively. Their speeds are such that they make complete circles in the same time t. The ratio of their centripetal acceleration is **[AIEEE, 2012]**

(A) $m_1r_1:m_2r_2$ (B) $m_1:m_2$
(C) $r_1:r_2$ (D) 1:1

(B) Work Power, Energy

7. A spring of force constant 800 N/m has an extension of 5 cm. The work done in extending it from 5 cm to 15 cm is **[AIEEE, 2002]**

(A) 16 J (B) 8 J
(C) 32 J (D) 24 J

8. A body is moved along a straight line by a machine delivering a constant power. The distance moved by the body in time t is proportional to **[AIEEE, 2003]**

(A) $t^{3/4}$ (B) $t^{3/2}$
(C) $t^{1/4}$ (D) $t^{1/2}$

9. A spring of spring constant 5×10^3 N/m is stretched initially by 5 cm from the unstretched position. Then the work required to stretch it further by another 5 cm is **[AIEEE, 2003]**

(A) 12.50 N-m (B) 18.75 N-m
(C) 25.00 N-m (D) 6.25 N-m

10. A force $\vec{F}=\left(5\hat{i}+3\hat{j}+2\hat{k}\right)$N is applied over a particle which displaces it from its origin to the point $\vec{r}=\left(2\hat{i}-\hat{j}\right)$m. The work done on the particle in joules is **[AIEEE, 2004]**

(A) −7 (B) +7
(C) +10 (D) +13

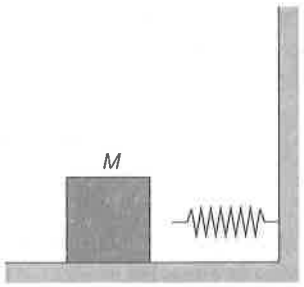
11. A uniform chain of length 2 m is kept on a table such that a length of 60 cm hangs freely from the edge of the table. The total mass of the chain is 4 kg. What is the work done in pulling the entire chain on the table? **[AIEEE, 2004]**

(A) 7.2 J (B) 3.6 J
(C) 120 J (D) 1200 J

12. A particle moves in a straight line with retardation proportional to its displacement. Its loss of kinetic energy for any displacement x is proportional to **[AIEEE, 2004]**

(A) x^2 (B) e^x
(C) x (D) $\log_e x$

13. The block of mass M moving on the frictionless horizontal surface collides with the spring of spring constant k and compresses it by length L. The maximum momentum of the block after collision **[AIEEE, 2005]**



(A) $\sqrt{MK}L$ (B) $\frac{kL^2}{2M}$

(C) zero (D) $\frac{ML^2}{k}$

14. A spherical ball of mass 20 kg is stationary at the top of a hill of height 100 m. It rolls down a smooth surface to the ground, then climbs up another hill of height 30 m and finally rolls down to a horizontal base at a height of 20 m above the ground. The velocity attained by the ball is **[AIEEE, 2005]**

(A) 40 m/s (B) 20 m/s
(C) 10 m/s (D) $10\sqrt{30}$ m/s

15. A body of mass m is accelerated uniformly from rest to a speed V in time T. The instantaneous power delivered to the body as function of time is given by **[AIEEE, 2005]**

(A) $\frac{mv^2}{T^2}t$ (B) $\frac{mv^2}{T^2}t^2$

(C) $\frac{1}{2}\frac{mv^2}{T^2}t$ (D) $\frac{1}{2}\frac{mv^2}{T^2}t^2$

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16. A bullet fired into a fixed target loses half of its velocity after penetrating 3 cm. How much further it will penetrate before coming to rest assuming that it faces constant resistance to motion? **[AIEEE, 2005]**
(A) 3.0 cm (B) 2.0 cm
(C) 1.5 cm (D) 1.0 cm

17. A mass of M kg is suspended by a weightless string. The horizontal force that is required to displace it until the string makes an angle of 45° with the initial vertical direction is **[AIEEE, 2006]**
(A) $Mg\left(\sqrt{2}+1\right)$ (B) $Mg\sqrt{2}$
(C) $\frac{Mg}{\sqrt{2}}$ (D) $Mg\left(\sqrt{2}-1\right)$

18. A ball of mass 0.2 kg is thrown vertically upwards by applying a force by hand. If the hand moves 0.2 m while applying the force and the ball goes up to 2 m height further, find the magnitude of the force. Consider $g = 10\ m/s^2$. **[AIEEE, 2006]**
(A) 4 N (B) 16 N
(C) 20 N (D) 22 N

19. The potential energy of a 1-kg particle free to move along the x-axis is given by **[AIEEE, 2006]**

$$V(x)=\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\text{J}$$

The total mechanical energy of the particle is 2 J. Then, the maximum speed (in ms^{-1}) is
(A) $\frac{3}{\sqrt{2}}$ (B) $\sqrt{2}$
(C) $\frac{1}{\sqrt{2}}$ (D) 2

20. A 2-kg block slides on a horizontal floor with a speed of 4 m/s. It strikes a uncompressed spring and compresses it till the block is motionless. The kinetic friction force is 15 N and spring constant is 10,000 N/m. The spring compresses by **[AIEEE, 2007]**
(A) 5.5 cm (B) 2.5 cm
(C) 11.0 cm (D) 8.5 cm

21. An athlete in the Olympic Games covers a distance of 100 m in 10 s. His kinetic energy can be estimated to be in the range **[AIEEE, 2008]**
(A) 200 J–500 J (B) 2×10^5 J–3×10^5 J
(C) 20,000 J–50,000 J (D) 2000 J–5000 J

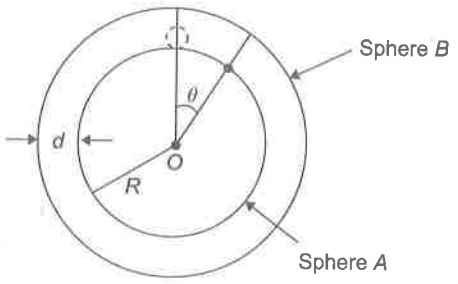
22. At time $t = 0$ s, a particle starts moving along the x-axis. If its kinetic energy increases uniformly with time t, the net force acting on it must be proportional **[AIEEE, 2011]**
(A) $\sqrt{t}$ (B) constant
(C) t (D) $\frac{1}{\sqrt{t}}$

23. When a rubber-band is stretched by a distance x, it exerts a restoring force of magnitude $F = ax + bx^2$, where a and b are constants. The work done in stretching the unstretched rubber-band by L is
(A) $\frac{aL^2}{2}+\frac{bL^3}{3}$ (C) $\frac{1}{2}\left(\frac{aL^2}{2}+\frac{bL^3}{3}\right)$
(C) aL^2+bL^3 (D) $\frac{1}{2}(aL^2+bL^3)$

JEE Advanced

1. An ideal spring with spring constant k is hung from the ceiling and a block of mass M is attached to its lower end. The mass is released with the spring initially unstretched. Then the maximum extension in the spring is **[JEE(Scr.), 2002]**
(A) $4\ Mg/k$ (B) $2\ Mg/k$
(C) Mg/k (D) $Mg/2k$

2. A spherical ball of mass m is kept at the highest point in the space between two fixed, concentric spheres A and B (see figure). The smaller sphere A has a radius R and the space between the two spheres has a width d. The ball has a diameter very slightly less than d. All surfaces are frictionless. The ball is given a gentle push (towards the right in the figure). The angle made by the radius vector of the ball with the upward vertical is denoted by θ (shown in the figure). **[JEE, 2002]**



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6. **Statement I** A block of mass m starts moving on a rough horizontal surface with a velocity v. It stops due to friction between the block and the surface after moving through a certain distance. The surface is now tilted to an angle of 30° with the horizontal and the same block is made to go up on the surface with the same initial velocity v. The decrease in the mechanical energy in the second situation is smaller than that in the first situation.

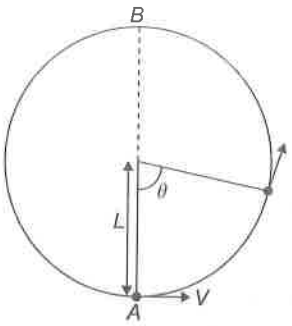
Statement II The coefficient of friction between the block and the surface decreases with the increase in the angle of inclination. **[JEE, 2007]**

(A) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is true; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is true, Statement 2 is false.
(D) Statement 1 is false, Statement 2 is true.

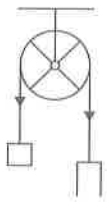
7. A bob of mass M is suspended by a massless string of length L. The horizontal velocity V at the position A is just sufficient to make it reach the point B. The angle θ at which the speed of the bob is half of that at A satisfies **[JEE, 2008]**

(A) $\theta = \frac{\pi}{4}$ (B) $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

(C) $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ (D) $\frac{3\pi}{4} < \theta < \pi$

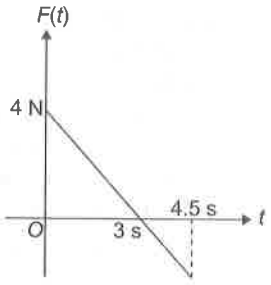


8. A light inextensible string that goes over a smooth fixed pulley as shown in the figure connects two blocks of masses 0.36 kg and 0.72 kg. Taking $g = 10$ m/s², find the work done (in joules) by the string on the block of mass 0.36 kg during the first second after the system is released from rest.



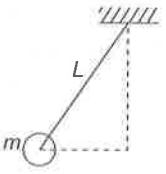
[JEE, 2009]

9. A block of mass 2 kg is free to move along the x-axis. It is at rest and from $t = 0$ onwards, it is subjected to a time-dependent force $F(t)$ in the x-direction. The force $F(t)$ varies with t as shown in the figure. The kinetic energy of the block after 4.5 s is **[JEE, 2010]**



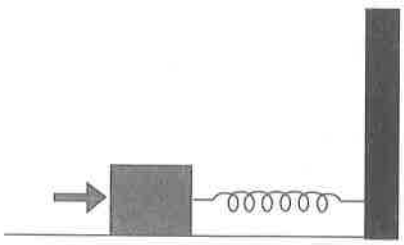
(A) 4.50 J (B) 7.50 J
(C) 5.06 J (D) 14.06 J

10. A ball of mass (m) 0.5 kg is attached to the end of a string having length (L) 0.5 m. The ball is rotated on a horizontal circular path about the vertical axis. The maximum tension that the string can bear is 324 N. The maximum possible value of angular velocity of ball (in radian/s) is **[JEE 2011]**



(A) 9 (B) 18
(C) 27 (D) 36

11. A block of mass 0.18 kg is attached to a spring of force-constant 2 N/m. The coefficient of friction between the block and the floor is 0.1. Initially, the block is at rest and the spring is unstretched. An impulse is given to the block as shown in the figure. The block slides a distance of 0.06 m and comes to rest for the first time. The initial velocity of the block in m/s is $V = N/10$. Then, N is **[JEE, 2011]**

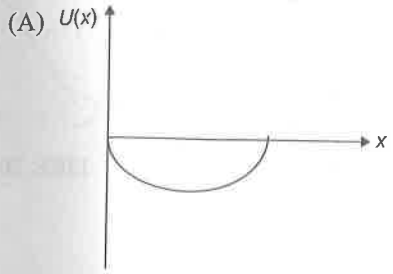


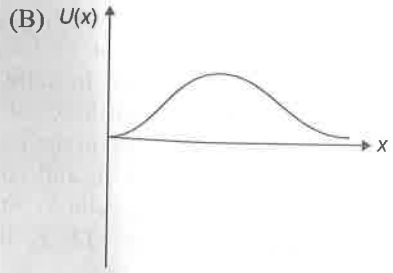
12. Two identical discs of the same radius R are rotating about their axes in opposite directions with the same constant angular speed ω. The discs are in the same horizontal plane. At time $t = 0$, the points P and Q are facing each other as shown in the figure. The relative speed between the two points P and Q is v_r. In one time period (T) of rotation of the discs, v_r as a function of time is best represented by **[JEE, 2012]**

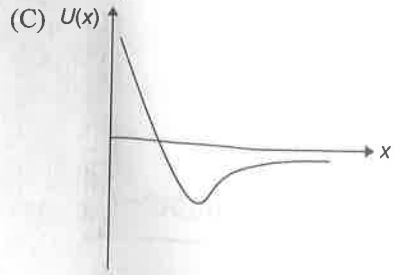
(A) Express the total normal reaction force exerted by the spheres on the ball as a function of angle θ.

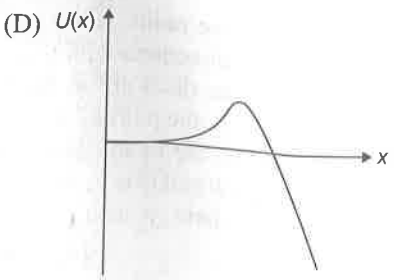
(B) Let N_A and N_B denote the magnitudes of the normal reaction force on the ball exerted by the spheres A and B, respectively. Sketch the variations of N_A and N_B as functions of $\cos\theta$ in the range $0 \leq \theta \leq \pi$ by drawing two separate graphs in your answer book, taking $\cos\theta$ on the horizontal axes.

3. A particle, which is constrained to move along the x-axis, is subjected to a force in the same direction which varies with the distance x of the particle from the origin as $F(x) = -kx + ax^3$. Here, k and a are positive constants. For $x \geq 0$, the functional form of the potential energy $U(x)$ of the particle is **[JEE, 2002]**

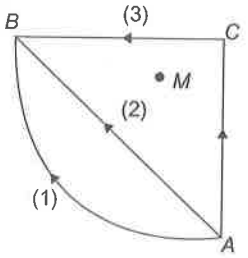
(A) 

(B) 

(C) 

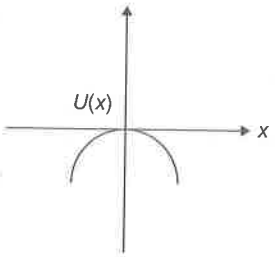
(D) 

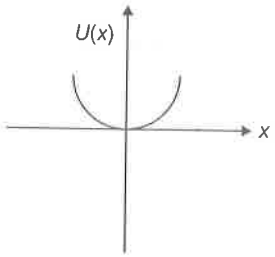
4. A particle is shifted from A to B via three different paths in the figure. The work done in different paths are W_1, W_2, W_3, respectively. Then, **[JEE(Scr.), 2003]**

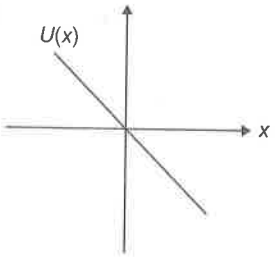


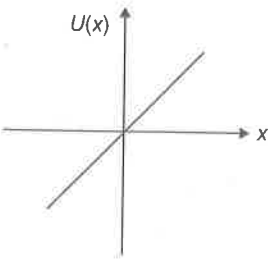
(A) $W_1 = W_2 = W_3$ (B) $W_1 = W_2 > W_3$
(C) $W_1 > W_2 > W_3$ (D) $W_1 < W_2 < W_3$

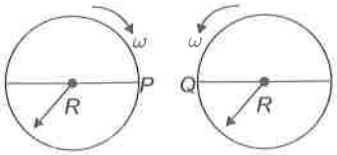
5. A particle is placed at the origin and a force $F = kx$ is acting on it (where k is a positive constant). If $U(0) = 0$, the graph of $U(x)$ versus x will be (where U is the potential energy function) **[JEE(Scr.), 2004]**

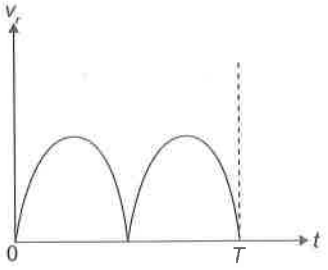
(A) 

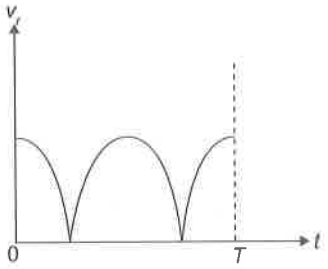
(B) 

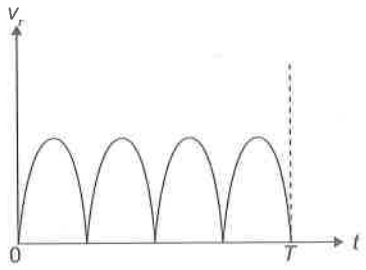
(C) 

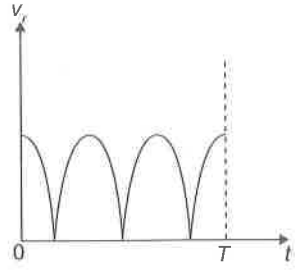
(D) 



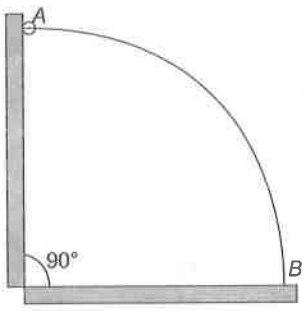
(A) 

(B) 

(C) 

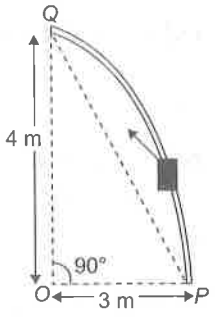
(D) 

13. A wire, which passes through the hole in a small bead, is bent in the form of quarter of a circle. The wire is fixed vertically on the ground as shown in the figure. The bead is released from near the top of the wire and it slides along the wire without friction. As the bead moves from A to B, the force it applies on the wire is **[JEE Advanced, 2014]**



(A) always radially outwards
(B) always radially inwards
(C) radially outwards initially and radially inwards later
(D) radially inwards initially and radially outwards later

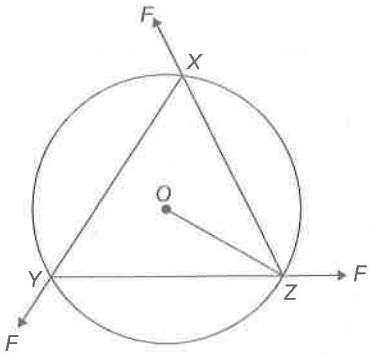
14. Consider an elliptically shaped rail PQ in the vertical plane with $OP = 3$ m and $OQ = 4$ m. A block of mass 1 kg is pulled along the rail from P to Q with a force of 18 N, which is always parallel to line PQ (see the figure given). Assuming no frictional losses, the kinetic energy of the block when it reaches Q is ($n \times 10$) Joules. The value of n is (take acceleration due to gravity $= 10\ \text{ms}^{-2}$)



[JEE Advanced, 2014]

15. A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude $F = 0.5$ N are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see the figure). One second after applying the forces, the angular speed of the disc in radians per second is

[JEE Advanced, 2014]



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ANSWER KEYS

Exercises

JEE Main

(A) Circular Motion

1. C	**2.** C	**3.** D	**4.** A	**5.** D	**6.** C	**7.** A	**8.** B	**9.** C	**10.** B
11. B	**12.** B	**13.** A	**14.** C	**15.** A	**16.** C	**17.** (i) A, (ii) A		**18.** C	**19.** C
20. C	**21.** D	**22.** C	**23.** C	**24.** A	**25.** C	**26.** C	**27.** D	**28.** B	**29.** A
30. C	**31.** A	**32.** B	**33.** A	**34.** D	**35.** C	**36.** C	**37.** B		

(B) Work, Power and Energy

38. B	**39.** C	**40.** B	**41.** B	**42.** C	**43.** B	**44.** A	**45.** D	**46.** A	**47.** B
48. A	**49.** B	**50.** D	**51.** D	**52.** A	**53.** D	**54.** C	**55.** C	**56.** C	**57.** A
58. A	**59.** C	**60.** A	**61.** A	**62.** B					

JEE Advanced

(A) Circular Motion

1. BD	**2.** BCDE	**3.** BC	**4.** A	**5.** D	**6.** B	**7.** B	**8.** D	**9.** C	**10.** D
11. D	**12.** A	**13.** B	**14.** B	**15.** C	**16.** D	**17.** B			

(B) Work, Power and Energy

18. ACD	**19.** AB	**20.** ABC	**21.** BD	**22.** A	**23.** A	**24.** D	**25.** BCD	**26.** BD	**27.** ABC
28. BD	**29.** ABD	**30.** C	**31.** C	**32.** C	**33.** C	**34.** B	**35.** C	**36.** C	**37.** C
38. C	**39.** C	**40.** B	**41.** B	**42.** B	**43.** C	**44.** D	**45.** C	**46.** B	**47.** B
48. B	**49.** A	**50.** B	**51.** C	**52.** A	**53.** C				

JEE Advanced

Level I

(A) Circular Motion

1. (A) 125 N, (B) 80 N **2.** $\sqrt{3}b/2, b/2, b$ **3.** (A) $t = 2$s, (B) 3.14 m/s **4.** 2 s **5.** $-3\hat{k}$ rad/s^2, $-2\hat{k}$ rad/s

6. (A) 2 rad/s^2, (B) $12 + 2t$ for $t \le 2$ s, 16 for $t \ge 2$ s, (C) $\sqrt{28,565} \approx 169$, 256 m/s^2 (D) 44 rad

7. (A) $25\frac{\sqrt{3}}{2}$ m/s^2, (B) $\left(125\frac{\sqrt{3}}{4}\right)^{1/2}$ m/s, (C) 25/2 m/s^2 **8.** (A) $v_0^2/2g$, (B) $2g$ **9.** $2\sqrt{2}\frac{v^2}{\pi R}$

10. (A) $\frac{30}{\pi}\sqrt{\frac{35}{2}}$ per min, (B) 150 N **11.** 0.1 s **12.** $\sqrt{2}$g rad/s **13.** $\left(1-\sqrt{3}/2\right)mg$ **14.** $\frac{25}{24}$

(B) Work, Power and Energy

15. (A) Zero, (B) Zero, (C) $-mmgvt$, (D) $mmgvt$ **16.** (A) Zero, (B) 500 J **17.** 150 J **18.** 20 Ergs **19.** −8.75 J

20. (A) 875 J (B) −250 J (C) 625 J.
(D) Change in kinetic energy of the body is equal to the work done by the net force in 10 s. This is in accordance with work–energy theorem.

21. (A) Since the gravitational force is a conservative force, the work done in round trip is zero, (B) 18.5 J
(C) −7.6 J, (D) 10.9 J

22. (A) $a_1 = F/m$, so $v_1 = a_1 t = Ft/m$, (B) Since velocities and $v = v_c + v_1 = v_c + Ft/m$

(C) $\Delta K_1 = \frac{m(v_1)^2}{2} = \frac{F^2t^2}{2m}$, (D) $\Delta K = \frac{m(v_c + v_1)^2}{2} - \frac{mv_c^2}{2}$, (E) s_1 is $\frac{a_1t^2}{2} = \frac{Ft^2}{2m}$ (F) $s_1 + v_ct$

(H) Comparing W and W_1 and ΔK and ΔK_1, they are, respectively, equal.

(I) The work–energy theorem holds for moving observers.

23. $\frac{2m_1^2g^2}{k}$ **24.** 3 **25.** (A) 46 J, (B) −3k **26.** 50 W **27.** 8 s **28.** 3/4 **29.** 80 kW **30.** 54 s

31. $W_{ABC} = W_{ADC} = \frac{a^5}{3}$ J, $W_{AC} = \frac{2a^5}{5}$ J **32.** (A) $F = -\frac{dU}{dy} = \omega$, (B) $F = -\frac{dU}{dy} = -3ay^2 + 2by$

(C) $F = -\frac{dU}{dy} = -bU_0 \cos by$ **33.** $\frac{1}{3}\sqrt{\frac{gl}{3}}$ **34.** $\frac{5}{2}\sqrt{gR}, x_{min} = 2R$ **35.** 20/3 N, 10,000 J/kg

Level II

1. 3.3 m **2.** $\frac{15\sqrt{3}}{2}$ N **3.** $a_N = \frac{vR}{(2Rt - vt^2)}, a_t = \frac{R(vt - R)v^{1/2}}{(2Rt - vt^2)^{3/2}}$ **4.** $F = -3ax^2 + b$, $x = \sqrt{\frac{b}{a}}$, $KE_{max} = \frac{2b}{3\sqrt{3}}\sqrt{\frac{b}{a}}$

5. 500 N/m **6.** (A) 36 N, (B) 11.66 rad/s, (C) 0.1 m, 0.2 m **7.** $mgR/2$, $2\sqrt{gR}$

8. $\frac{4\sqrt{2}}{5}, \frac{3\sqrt{2}}{5}$ $3a_1 = 4a_2 + 20$, $T = 12$ N **9.** 2.5 cm **10.** (A) $f = -\frac{\mu m}{\ell} \times g$; (B) $\sqrt{\mu g\ell}$; (C) $\frac{5\ell}{2}$

11. 4 m/s, 24.5 J, 40 J **12.** (A) $\frac{a}{2}$, (B) $2\sqrt{ag}, 2a$ **13.** (A) $2\sqrt{2r}$, (B) $h = \frac{19r}{27}$, (C) g

Previous Year Questions

JEE Main

(A) Circular Motion

1. B **2.** B **3.** D **4.** D **5.** C **6.** C

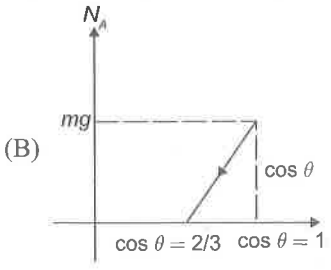
(B) Work, Power and Energy

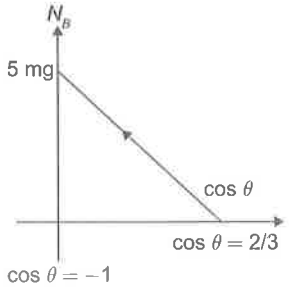
7. B **8.** B **9.** B **10.** B **11.** B **12.** A **13.** A **14.** A **15.** A **16.** D
17. D **18.** D **19.** A **20.** A **21.** D **22.** D **23.** A

JEE Advanced

1. B **2.** (A) $N = 3mg\cos\theta - 2mg$,

(B)





3. D **4.** A **5.** A **6.** C **7.** D **8.** Work = T.S. = 4.8 × 10/6 = 8 N
9. C **10.** D **11.** 4 **12.** A **13.** D **14.** 5 **15.** 2

CHAPTER 6

Centre of Mass

CENTRE OF MASS

Every physical system has associated with it a certain point whose motion characterizes the motion of the whole system. When the system moves under some external forces, then this point moves as if the entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is called the centre of mass of the system.

Centre of Mass of a System of *N* Discrete Particles

Consider a system of N point masses m_1, m_2, m_3,..., m_n whose position vectors from the origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$, respectively (Fig. 6.1). Then the position vector of the centre of mass C of the system is given by

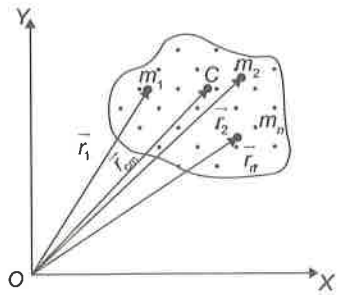


Figure 6.1

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n}$$

$$\vec{r}_{COM} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$$\vec{r}_{COM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

where $m_i\vec{r}_i$ is called the **moment of mass** of particle with respect to the origin.

$M = \left(\sum_{i=1}^{n} m_i\right)$ is the total mass of the system.

Further, $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$

and $\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j} + z_{COM}\hat{k}$.

So, the Cartesian co-ordinates of the COM will be

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$$

or $$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}.$$

Similarly, $$y_{COM} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

and $$z_{COM} = \frac{\sum_{i=1}^{n} m_i z_i}{M}.$$

Notes

- If the origin is taken at the centre of mass, then $\sum_{i=1}^{n} m_i \vec{r}_i = 0$. Hence, the COM is the point about which the sum of 'mass moments' of the system is zero.
- If we change the origin, then $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ changes. So $\vec{r}_{COM}$ also changes, but the exact location of the centre of mass does not change.

Position of COM of Two Particles

Consider two particles of masses m_1 and m_2 separated by a distance l as shown in Fig. 6.2.

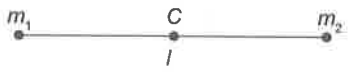


Figure 6.2

Let us assume that m_1 is placed at the origin and m_2 is placed at the position $(l, 0)$ and the distance of centre of mass from m_1 and m_2 is r_1 and r_2, respectively.

So, $$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$r_1 = \frac{0 + m_2 l}{m_1 + m_2} = \frac{m_2 l}{m_1 + m_2} \quad (1)$$

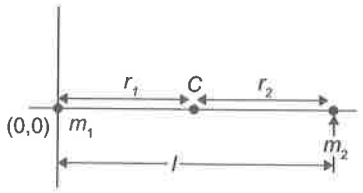


Figure 6.3

$$r_2 = l - \frac{m_2 l}{m_1 + m_2} = \frac{m_1 l}{m_1 + m_2} \quad (2)$$

From the above discussion, we see that $r_1 = r_2 = \frac{l}{2}$ if $m_1 = m_2$, i.e., COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_2 < m_1$, i.e., COM is nearer to the particle having larger mass.

From Eqs. (1) and (2),

$$m_1 r_1 = m_2 r_2 \quad (3)$$

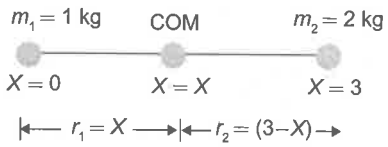
Centre of mass of a two-particle system lies on the line joining the centre of mass of the two-particle system.

SOLVED EXAMPLES

EXAMPLE 1

Two particles of masses 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.

SOLUTION



Since, both the particles lie on the x-axis, the COM will also lie on the x-axis. Let the COM be located at $x = x$, then r_1 = distance of COM from the particle of mass 1 kg = x and r_2 = distance of COM from the particle of mass 2 kg = $(3 - x)$.

Using $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

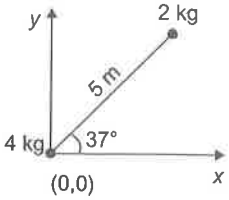
or $$\frac{x}{3-x} = \frac{2}{1}$$

or $$x = 2 \text{ m}.$$

Thus, the COM of the two particles is located at $x = 2$ m. ■

EXAMPLE 2

Two particles of masses 4 kg and 2 kg are located as shown in the figure. Find out the position of centre of mass.



SOLUTION

First find out the position of the 2-kg mass.

$$x_{2\,kg} = 5\cos 37° = 4\text{ m}$$

$$y_{2\,kg} = 5\sin 37° = 3\text{ m}$$

So this system is like a two-particle system of masses 4 kg and 2 kg that are located at (0, 0) and (4, 3), respectively. Then,

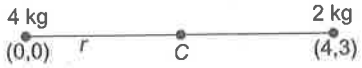
$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$= \frac{0 + 2\times 4}{4+2} = \frac{8}{6}$$

$$= \frac{4}{3}$$

$$y_{COM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

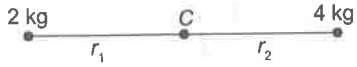
$$= \frac{0 + 3\times 2}{4+2} = 1\text{ m}.$$



So position of COM is $\left(\frac{4}{3}, 1\right)$. ■

EXAMPLE 3

Two particles of masses 2 kg and 4 kg lie on the same line. If 4 kg is displaced rightwards by 5 m, then by what distance 2 kg should be moved for which the centre of mass will remain at the same position.



SOLUTION

Let us assume that the COM lies at point C and the distance of C from the 2-kg and 4-kg particles is r_1 and r_2, respectively. Then from the relation

$$m_1r_1 = m_2r_2$$

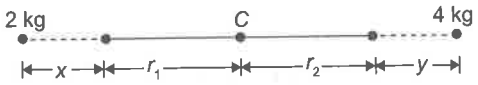
$$2r_1 = 4r_2 \qquad (1)$$

Now the 4-kg mass is displaced rightwards by 5 m. Assume that the 2-kg mass is displaced leftwards by x distance to keep the COM at rest.

From the relation

$$m_1r_1' = m_2r_2',$$

$$m_1(r_1 + x) = m_2(r_2 + y)$$



$$2(r_1 + x) = 4(r_2 + 5) \qquad (2)$$

$$2x = 20$$

$$x = 10\text{ m}.$$

To keep the COM at rest, the 2-kg mass has to be displaced 10 m leftwards.

Aliter: If the centre of mass is at rest, then we can write

$$m_1x = m_2y$$

$$2 \times x = 4 \times 5$$

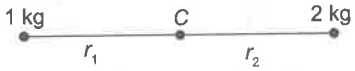
$$x = 10\text{ m}$$ ■

EXAMPLE 4

Two particles of mass 1 kg and 2 kg lie on the same line. If 2 kg is displaced 10 m rightwards, then by what distance 1 kg should be displaced so that the centre of mass will be displaced 2 m rightwards.

SOLUTION

Initially, let us assume that COM is at point C, which is r_1 and r_2 distance apart from masses m_1 and m_2, respectively, as shown in the figure.



From the relation

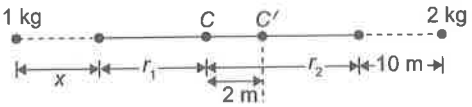
$$m_1r_1 = m_2r_2,$$

$$r_1 = 2r_2 \qquad (1)$$

Now 2 kg is displaced 10 m rightwards, then we assume that 1 kg is displaced x m leftwards to move the COM 2 m rightwards.

Thus, from the relation

$$m_1r_1' = m_2r_2',$$



$$\Rightarrow \quad 1(x + r_1 + 2) = 2(10 + r_2 - 2)$$

$$\Rightarrow \quad x + r_1 + 2 = 20 + 2r_2 - 4 \qquad (2)$$

From Eqs. (1) and (2),

$$x = 14 \text{ m (leftwards)}.$$

EXAMPLE 5

Three particles of masses 1 kg, 2 kg and 3 kg are placed at the corners A, B and C, respectively, of an equilateral triangle ABC of edge 1 m. Find the distance of their centre of mass from A.

SOLUTION

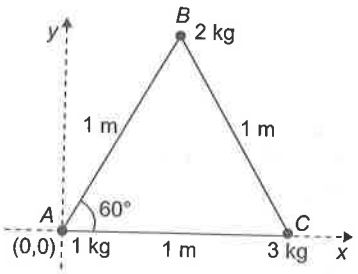
Assume that the 1-kg mass is placed at the origin as shown in the figure.

Co-ordinate of $A = (0, 0)$.

Co-ordinate of $B = (1 \cos 60°, 1 \sin 60°)$

$$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Co-ordinate of $C = (1, 0)$.



Let us assume that the position of COM is given by

$$\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j}$$

Now

$$x_{COM} = \frac{m_A x_A + m_B x_B + m_C x_C}{m_A + m_B + m_C}$$

$$= \frac{1(0) + 2\left(\frac{1}{2}\right) + 3(1)}{1+2+3}$$

$$= \frac{4}{6} = \frac{2}{3}$$

$$y_{COM} = \frac{1(0) + 2\left(\frac{\sqrt{3}}{2}\right) + 3(0)}{1+2+3}$$

$$= \frac{\sqrt{3}}{6}$$

Position of centre of mass $= \left(\frac{2}{3}, \frac{\sqrt{3}}{6}\right)$.

Distance of COM from the point

$$A = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2}$$

$$= \frac{\sqrt{19}}{6} \text{ m}.$$

Centre of Mass of a Continuous Mass Distribution

For continuous mass distribution, the centre of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation,

$$x_{COM} = \frac{\int x\,dm}{\int dm}, \quad y_{COM} = \frac{\int y\,dm}{\int dm}, \quad z_{COM} = \frac{\int z\,dm}{\int dm} \tag{1}$$

$$\int dm = M \text{ (mass of the body)}.$$

Here x, y, z in the numerator of Eq. (1) is the coordinate of the centre of mass of the dm mass.

$$\vec{r}_{COM} = \frac{1}{M}\int \vec{r}\,dm.$$

Note

If an object has symmetric mass distribution about x-axis, then the y-coordinate of COM is zero and vice-versa.

(a) Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$ (Fig. 6.4). Mass per unit length of the rod $\lambda = \frac{M}{L}$.

Hence, dm (the mass of the element dx situated at $x = x$ is) $= \lambda dx$.

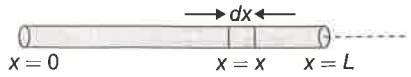


Figure 6.4

The coordinates of the element dx are $(x, 0, 0)$. Therefore, the x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm}$$

$$= \frac{\int_0^L (x)(\lambda dx)}{\int_0^L \lambda dx}$$

$$= \frac{1}{L}\int_0^L x dx = \frac{L}{2}.$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0$$

Similarly, $z_{COM} = 0$,

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$, i.e., it lies at the centre of the rod.

SOLVED EXAMPLE

EXAMPLE 6

A rod of length L is placed along the x-axis between $x = 0$ and $x = L$. The linear density (mass/length) λ of the rod varies with the distance x from the origin as $\lambda = Rx$. Here, R is a positive constant. Find the position of centre of mass of this rod.

SOLUTION

Mass of element dx situated at $x = x$ is

$$dm = \lambda dx = Rx\,dx.$$

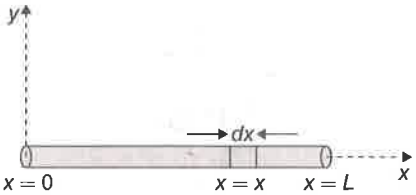
The COM of the element has coordinates $(x, 0, 0)$. Therefore, the x-coordinates of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x dm}{\int dm}$$

$$= \frac{\int_0^L (x)(Rx)dx}{\int_0^L (Rx)dx}$$

$$= \frac{R\int_0^L x^2 dx}{R\int_0^L x dx}$$

$$= \frac{\left[\frac{x^3}{3}\right]_0^L}{\left[\frac{x^2}{2}\right]_0^L} = \frac{2L}{3}$$



The y-coordinates of COM of the rod is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \text{ (as } y = 0).$$

Similarly, $z_{COM} = 0$.

Hence, the centre of mass of the rod lies at $\left[\frac{2L}{3}, 0, 0\right]$. ■

(b) Centre of Mass of a Semi-circular Ring

Figure 6.5 shows the object (semi-circular ring). By observation, we can say that the x-coordinate of the centre of mass of the ring is zero as the half ring is symmetrical about the y-axis on both sides of the origin. We are required to find only the y-coordinate of the centre of mass.

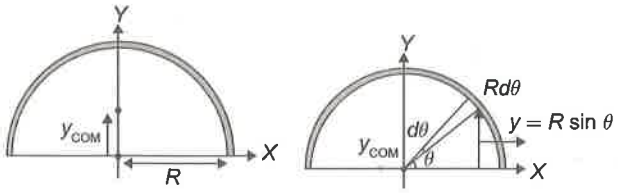


Figure 6.5

To find y_{COM}, we use

$$y_{COM} = \frac{\int (dm)y}{\int dm} \quad (1)$$

Here, y is the position of COM of dm mass.

Here for dm, we consider an elemental arc of the ring at an angle θ from the x-direction of angular width $d\theta$. If radius of the ring is R, then its y–coordinate will be $R \sin\theta$. Here, dm is given as

$$dm = \lambda R d\theta,$$

where λ = mass density of the semi-circular ring.

So from Eq. (1), we have

$$y_{COM} = \frac{\int_0^{\pi} \lambda R d\theta (R\sin\theta)}{\int_0^{\pi} \lambda R d\theta}$$

$$= \frac{R}{\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$y_{COM} = \frac{2R}{\pi} \quad (2)$$

(c) Centre of Mass of a Semi-circular Disc

Figure 6.6 shows the half disc of mass M and radius R. Here, we are only required to find the y-coordinate of the centre of mass of this disc as centre of mass will be located on its half vertical diameter. Here to find y_{COM}, we consider a small elemental ring of mass dm of radius r on the disc (disc can be considered to be made up of such thin rings of increasing radii), which will be integrated from 0 to R. Here dm is given as

$$dm = \sigma\pi r dr,$$

where σ is the mass density of the semi-circular disc.

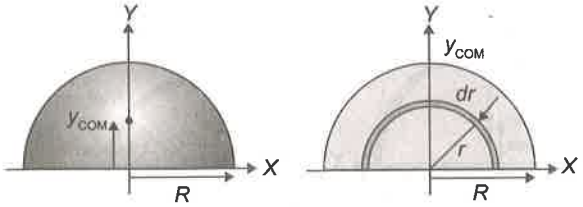


Figure 6.6

$$\sigma = \frac{M}{\pi R^2/2}$$

$$= \frac{2M}{\pi R^2}.$$

Now the y-coordinate of the element is taken as $\frac{2r}{\pi}$ (as in the previous section, we have derived that the centre of mass of a semi-circular ring is concentrated at $\frac{2R}{\pi}$),

$$y_{COM} = \frac{\int_0^R dm \cdot y}{\int_0^R dm}.$$

Here, y is the position COM of dm mass.

Here, y_{COM} is given as

$$y_{COM} = \frac{\int_0^R dm \frac{2r}{\pi}}{\int_0^R \sigma\pi r dr}$$

$$= \int_0^R \frac{4}{\pi R^2} r^2 d$$

$$\Rightarrow y_{COM} = \frac{4R}{3\pi}.$$

(d) Centre of Mass of a Hollow Hemisphere

Consider a hollow hemisphere of mass M and radius R (Fig. 6.7). Now we consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere. This strip will have an area

$$dS = 2\pi R\cos\theta \, R d\theta$$

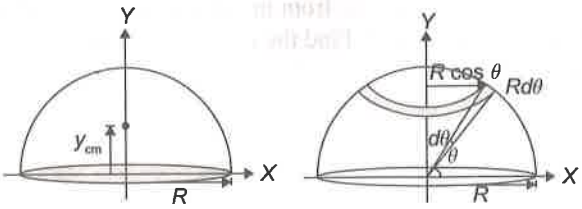


Figure 6.7

Its mass dm is given as

$$dm = \sigma 2\pi R\cos\theta \, R d\theta$$

Here, σ is the mass density of a hollow hemisphere,

$$\sigma = \frac{M}{2\pi R^2}.$$

Here, the y-coordinate of this strip of mass dm can be taken as $R \sin\theta$. Now we can obtain the centre of mass of the system as

$$y_{COM} = \frac{\int_0^{\pi/2} dm\, R\sin\theta}{\int_0^{\pi/2} dm}$$

$$= \frac{\int_0^{\pi/2} (\sigma 2\pi R^2 \cos\theta\, d\theta) R\sin\theta}{\int_0^{\pi/2} (\sigma 2\pi R^2 \cos\theta\, d\theta)}$$

$$= R\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$y_{COM} = \frac{R}{2}.$$

(e) Centre of Mass of a Solid Cone

A solid cone has mass M, height H and base radius R (Fig. 6.8). Obviously, the centre of mass of this cone will lie somewhere on its axis, at a height less than $H/2$. To locate the centre of mass, we consider an elemental disc of width dy and radius r, at a distance y from the apex of the cone. Let the mass of this disc be dm, which can be given as

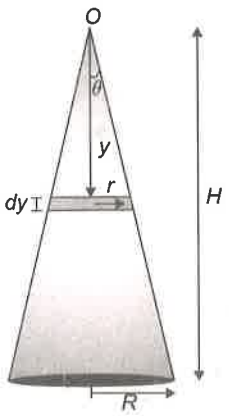


Figure 6.8

$$dm = \rho \times \pi r^2 dy.$$

Here, ρ is the mass density of the solid cone.

Here, y_{COM} can be given as

$$y_{COM} = \frac{1}{M}\int_0^H y\, dm$$

$$= \frac{1}{M}\int_0^H \left(\frac{3M}{\pi R^2 H}\pi\left(\frac{Ry}{H}\right)^2 dy\right) y$$

$$= \frac{3}{H^3}\int_0^H y^3\, dy = \frac{H}{4}.$$

(f) Centre of Mass of a Solid Hemisphere

A hemisphere is of mass density ρ and radius R. To find its centre of mass (only y co-ordinate), we consider an elemental hollow hemisphere of radius r on the solid hemisphere (solid hemisphere can be considered to be made up of such hollow hemispheres of increasing radii; Fig. 6.9), which will be integrated from O to R.

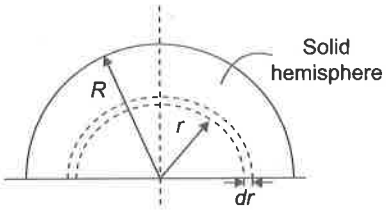


Figure 6.9

Here, the y-co-ordinate of centre of mass of the elemental hollow hemisphere is $(0, r/2, 0)$.

$$dm = \rho\, 2\pi r^2 dr$$

$$y_{COM} = \frac{\int_0^R dm \cdot y}{\int_0^R dm};$$

$$y_{COM} = \frac{\int_0^R \rho(2\pi r^2) dr(r/2)}{\int_0^R \rho \cdot 2\pi r^2 \cdot dr};$$

$$y_{COM} = \frac{3R}{8}.$$

(g) Centre of Mass of a Triangular Plate

A triangular plate has mass density σ, height H and base $2R$ (Fig. 6.10). Obviously, the centre of mass of this plate will lie somewhere on its axis at a height less than $H/2$. To locate the centre of mass, we consider an elemental rod of width dy and length $2r$ at a distance y from the apex of the plate. Let the mass of this rod be dm, which can be gives as

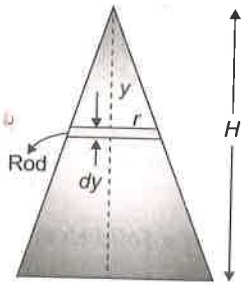


Figure 6.10

$$dm = \sigma(2r)dy.$$

From the theorem of triangle,

$$\frac{H}{R} = \frac{y}{r}$$

$$\Rightarrow \qquad r = \frac{Ry}{H}$$

Here, y_{COM} can be given as

$$y_{COM} = \frac{\int_0^H y\,dm}{\int_0^H dm};$$

$$y_{COM} = \frac{\int_0^H \sigma(2r)dy \cdot y}{\int_0^H \sigma(2r)dy}$$

$$y_{COM} = \frac{\int_0^H \sigma\left(\frac{2Ry}{H}\right)ydy}{\int_0^H \sigma\left(\frac{2Ry}{H}\right)dy};$$

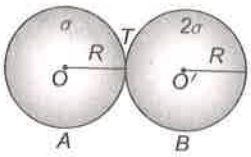
$$y_{COM} = \frac{2H}{3}.$$

COMBINATION OF STRUCTURE

SOLVED EXAMPLES

EXAMPLE 7

Two circular discs having radius R and mass density σ and 2σ, respectively, are placed as shown in the figure. Then find out the position of COM of the system.



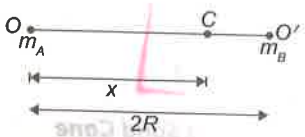
SOLUTION

Mass of disc A

$$m_A = \sigma\pi R^2$$

Mass of disc B

$$m_B = 2\sigma\pi R^2$$



Due to symmetry, the COM of disc A lies at point O and COM of disc B lies at point O'. So we realize the above problem as follows:

Centres of mass due to both the discs lie at point C (assume), at a distance x from m_A

$$\Rightarrow \qquad x = \frac{m_B(2R)}{m_A + m_B};$$

$$x = \frac{2\sigma\pi R^2(2R)}{\sigma(\pi R^2 + 2\pi R^2)};$$

$$x = \frac{4R}{3}.$$

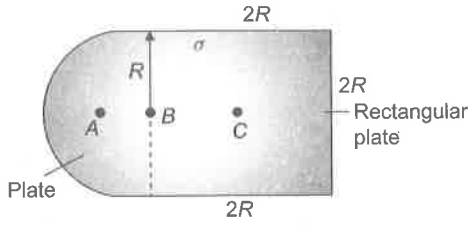
So the centre of mass lies in the disc B having distance $\frac{4R}{3}$ from O.

■

EXAMPLE 8

Find out the position of centre of mass of the figure shown below.

SOLUTION



We divide the above problem in to two parts:

(i) First find out the position of centre of mass of both the semi-circular plate and the rectangular plate separately.

(ii) Then find the position of centre of mass of the given structure.

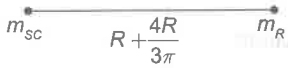
Centre of mass of the semi-circular disc lies at $\frac{4R}{3\pi}$.

$$\Rightarrow \qquad AB = \frac{4R}{3\pi}.$$

Centre of mass of the rectangular plate lies at the centre of the plate at point C.

$$\Rightarrow \qquad BC = R$$

$$\Rightarrow$$



$$m_{sc} = \frac{\sigma\pi R^2}{2};$$

$$m_R = \sigma 4R^2$$

$$\Rightarrow$$

$\Rightarrow$ Let us assume COM is at r_1 distance from m_R,

$$\Rightarrow \qquad r_1 = \frac{\sigma\frac{\pi R^2}{2}\left(R+\frac{4R}{3\pi}\right)}{\sigma\frac{\pi R^2}{2}+\sigma 4R^2}$$

$$\Rightarrow \qquad r_1 = \frac{\pi R(3\pi+4)}{3(\pi+8)}$$

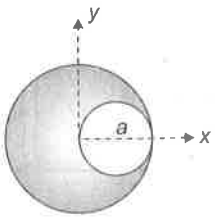
CAVITY PROBLEMS

If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained by assuming that in the remaining part, $+m$ and $-m$ mass is present. Further steps are explained by the following example.

SOLVED EXAMPLES

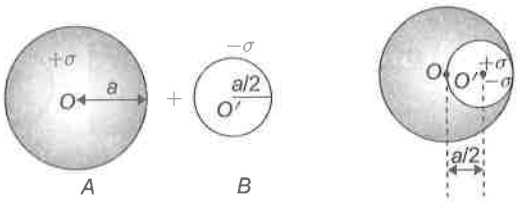
EXAMPLE 9

Find the position of centre of mass of the uniform lamina shown in the figure if the mass density of the lamina is σ.



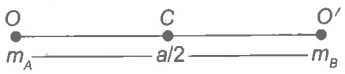
SOLUTION

We assume that in the remaining portion a disc of radius $a/2$ having mass density $+\sigma$ is present. We also include one disc of $a/2$ radius having $-\sigma$ mass density. So now the problem changes as follows:



So the centre of mass of both discs A and B lies in their respective centre, i.e., O and O'.

Now,



$\Rightarrow$ COM of the lamina

$$\Rightarrow \qquad \frac{m_A\, a/2}{m_A + m_B}$$

$$m_A = \sigma(\pi a^2)$$

$$m_B = -\sigma(\pi)(a/2)^2$$

$$= -\sigma\pi\frac{a^2}{4}$$

$$\Rightarrow \qquad c = \frac{\sigma\pi a^2 \cdot a/2}{\sigma\pi a^2 - \sigma\pi\frac{a^2}{4}}$$

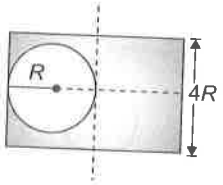
$$= \frac{a^3/2}{3a^2/4}$$

$$\frac{a^3}{2} \times \frac{4}{3a^2} = \frac{2a}{3},$$

i.e., COM lies towards the left side from point O. ■

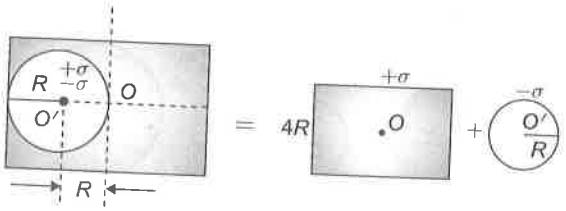
EXAMPLE 10

Find out the position of centre of mass of the uniform lamina as shown in the figure.



SOLUTION

We assume that a disc of radius R having mass density $\pm\sigma$ is in the removed section. Now, the problem changes in the following form:



When the disc of mass density $+\sigma$ and radius R is included, then a complete rectangular plate is formed having centre of mass at point O. When we consider only the disc having mass density $-\sigma$ and radius R, then COM of this disc lies at point O'.

Then the position of COM

$$= \frac{\sigma(4R)^2 \cdot R}{-\sigma\pi R^2 + \sigma(4R^2)}$$

$$= \frac{\sigma 16R^3}{\sigma R^2(16-\pi)}$$

$$= \frac{16R}{16-\pi}$$

i.e., the centre of mass lies towards the right side from of cavity. ■

EXAMPLE 11

The centre of mass of a rigid body always lies inside the body. Is this statement true or false?

SOLUTION

False. ■

EXAMPLE 12

The centre of mass always lies on the axis of symmetry if it exists. Is this statement true or false?

SOLUTION

True ■

EXAMPLE 13

If all the particles of a system lie in the y–z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or not?

SOLUTION

True ■

MOTION OF CENTRE OF MASS AND CONSERVATION OF MOMENTUM

The position of centre of mass is given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (1)$$

Here, m_1, m_2, m_3,... are the masses in the system and $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$,... are the corresponding position vectors of m_1, m_2, m_3,..., respectively.

Velocity of COM of System

To find the velocity of centre of mass, we differentiate equation (1) with respect to time

$$\frac{d\vec{r}_{COM}}{dt} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$\Rightarrow \quad \vec{V}_{COM} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$\vec{V}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (2)$$

Acceleration of Centre of Mass of the System

To find the acceleration of COM, we differentiate Eq. (2),

$$\Rightarrow \quad \frac{d\vec{V}_{COM}}{dt} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \qquad (3)$$

Now

$$(m_1 + m_2 + m_3)\,\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots$$

$$\vec{F}_{net(system)} = \vec{F}_{1net} + \vec{F}_{2net} + \vec{F}_{3net} + \cdots$$

The internal forces which the particles exert on one another play absolutely no role in the motion of the centre of mass.

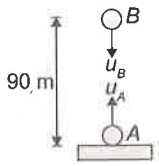
SOLVED EXAMPLES

EXAMPLE 14

Two particles A and B of mass 1 kg and 2 kg, respectively, are projected in the directions shown in the figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially, they were 90 m apart. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant ($g = 10$ m/s^2).

SOLUTION

Using $\quad m_A r_A = m_B r_B$



or $\quad (1)\,(r_A) = (2)\,(r_B)$

or $\quad r_A = 2r_B \qquad (1)$

and $\quad r_A + r_B = 90$ m $\qquad (2)$

Solving these two equations, we get

$$r_A = 60 \text{ m}$$

and $\quad r_B = 30$ m

i.e., COM is at height 60 m from the ground at time $t = 0$.

Further, $\quad \vec{a}_{COM} = \dfrac{m_A\vec{a}_A + m_B\vec{a}_B}{m_A + m_B}$

$$= g = 10 \text{ m/s}^2 \text{ (downwards)}$$

As $\quad \vec{a}_A = \vec{a}_B = g$ (downwards),

$$\vec{u}_{COM} = \frac{m_A\vec{u}_A + m_B\vec{u}_B}{m_A + m_B}$$

$$= \frac{(1)(200) - (2)(50)}{1+2}$$

$$= \frac{100}{3} \text{ m/s (upwards)}$$

Let h be the height attained by COM beyond 60 m. Using

$$v_{COM}^2 = u_{COM}^2 + 2a_{COM}h$$

or $\quad 0 = \left(\frac{100}{3}\right)^2 - (2)(10)h$

or $\quad h = \frac{(100)^2}{180} = 55.55$ m

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 55.55 = 115.55 \text{ m}$$ ■

Case I: If $F_{net} = 0$, then we conclude the following:

1. The acceleration of centre of mass is zero ($\vec{a}_{COM} = 0$). If $a_1, a_2, a_3, \ldots$ is acceleration of $m_1, m_2, m_3, \ldots$ mass in the system, then $a_1, a_2, a_3, \ldots$ may or may not be zero.
2. KE of the system is not constant. It may change due to internal force.
3. Velocity of centre of mass is constant ($\vec{v}_{COM} = $ constant), but v_1, v_2, v_3 may or may not be constant. It may change due to internal force.
   From Eq. (2),
   $$m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \text{constant}$$
   This is called momentum conservation.

'If the resultant external force is zero on the system, then the net momentum of the system must remain constant.'

Case II: When the centre of mass is at rest

$\vec{v}_{COM} = 0$, then

$$\frac{d\vec{r}_{COM}}{dt} = 0$$

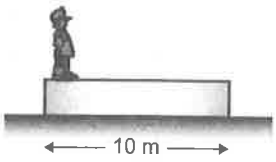
$$\Rightarrow \quad \vec{r}_{COM} = \text{constant},$$

i.e., $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots$ may or may not change

SOLVED EXAMPLES

EXAMPLE 15

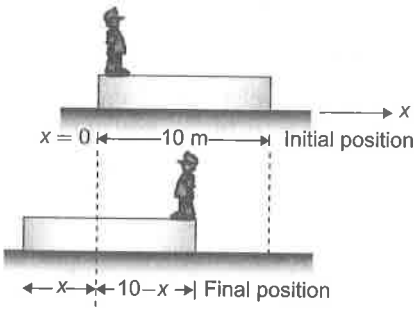
A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.



SOLUTION

Here, the system is man + plank. Net force on this system is horizontal direction and is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in the horizontal direction.

Let x be the displacement of the plank. Assuming the origin, i.e., $x = 0$ at the position shown in the figure,



As we said earlier also, the centre of mass will not move in horizontal direction (x-axis). Therefore, for centre of mass to remain stationary,

$$x_i = x_f$$

$$20x = 60 \times (10 - x)$$

or $$x = \frac{30}{4}\text{m}$$

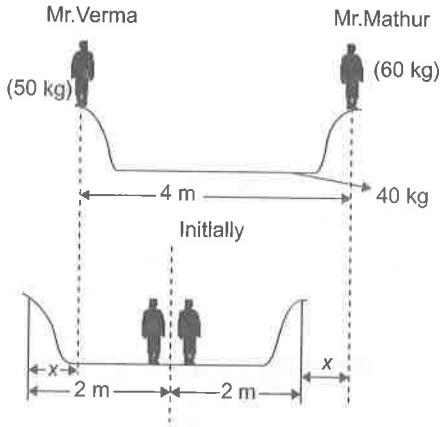
or $$x = 7.5 \text{ m}$$ ■

EXAMPLE 16

Mr. Verma (weighing 50 kg) and Mr. Mathur (weighing 60 kg) are sitting at the two extremes of a 4m-long boat (40 kg) standing still in water. To discuss a mechanics problem, they come to the middle of the boat. Neglecting friction with water, how far does the boat move on the water during the process?

SOLUTION

Here, the system is Mr. Verma + Mr. Mathur + boat. Net force on this system is in the horizontal direction and is zero. Initially, the centre of mass of the system is at rest. Therefore, the COM does not move in the horizontal direction. Let x be the displacement of the boat. Then we can use the concept



$$m_1 x_1 = m_v x_v + m_M x_M$$

$$40 \times x = 50 \times (2 - x) - 60\,(2 + x)$$

$$40x = 100 - 50x - 120 - 60x$$

$$150x = -20$$

$$x = -\frac{2}{15}\text{m}$$

$x \approx 13$ cm (right wards). ■

Case III: When the net force is zero only in one direction

Let us assume that F_{net} in the x-direction is zero, then we conclude,

(a) acceleration of the system in the x-direction is zero ($a_x = 0$)

(b) $v_{(COM)x}$ = constant

$$\Rightarrow \quad m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} = \text{constant},$$

i.e., momentum is conserved only in the x-direction.

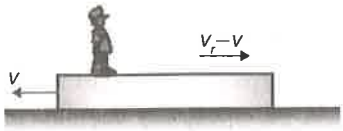
SOLVED EXAMPLES

EXAMPLE 17

A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of the platform.

SOLUTION

Absolute velocity of man $= v_r - v$, where v = recoil velocity of the platform. Taking the platform and the man as a system, net external force on the system in the horizontal direction is zero. The linear momentum of the system remains constant. Initially, both the man and the platform were at rest.



Hence, $$0 = m_1(v_r - v) - m_2 v$$

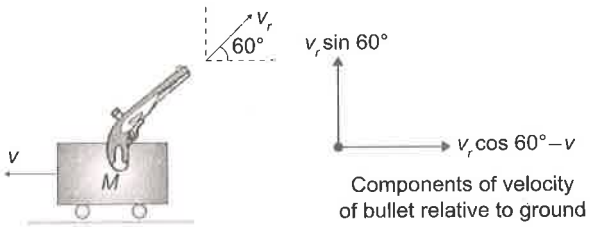
$$\therefore \quad v = \frac{m_1 v_r}{m_1 + m_2}$$

EXAMPLE 18

A gun (mass $= M$) fires a bullet (mass $= m$) with speed v_r relative to barrel of the gun, which is inclined at an angle of 60° with the horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of the gun.

SOLUTION

Let the recoil speed of the gun is v. Taking gun + bullet as the system. Net external force on the system in the horizontal direction is zero. Initially, the system was at rest. Therefore, applying the principle of conservation of linear momentum in the horizontal direction, we get



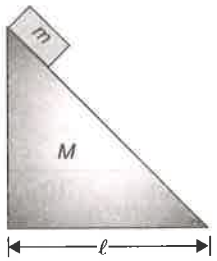
$$Mv - m(v_r \cos 60° - v) = 0$$

$$\therefore \quad v = \frac{mv_r \cos 60°}{M + m}$$

or $$v = \frac{mv_r}{2(M + m)}$$

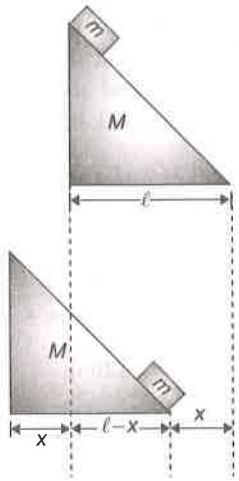
EXAMPLE 19

A particle of mass m is placed at rest on the top of a smooth wedge of mass M, which in turn is placed at rest on a smooth horizontal surface as shown in the figure. Then estimate the distance moved by the wedge as the particle reaches the foot of the wedge.



SOLUTION

There is no external force in the horizontal direction on the wedge block system. So the x-coordinate of the COM of the wedge block system is at rest.



Let us assume that the wedge moves distance x when the block reaches the ground. We can use the following relation when the x-coordinate of COM is at rest.

$$m_1 x_1 = m_2 x_2$$

$$Mx = m(\ell - x)$$

$$x = \frac{m\ell}{m + M}.$$

EXAMPLE 20

A projectile is fired at a speed of 100 m/s at an angle of 37° above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio 1:3, the lighter piece coming to rest. Find the distance from the launching point to the point where the heavier piece lands.

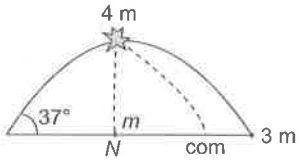
SOLUTION

Internal force does not affect the motion of the centre of mass. The centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is,

$$x_{COM} = \frac{2u^2 \sin\theta\cos\theta}{g}$$

$$= \frac{2\times10^4 \times \frac{3}{5}\times\frac{4}{5}}{10}\ \text{m}$$

$$= 960\ \text{m}$$



The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range, i.e., at $x = 480$ m. If the heavier block hits the ground at x_2, then

$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$\Rightarrow \qquad 960 = \frac{(m)(480)+(3m)(x_2)}{(m+3m)}$$

$$x_2 = 1120\ \text{m}.$$ ■

EXAMPLE 21

A shell is fired from a cannon with a speed of 100 m/s at an angle 60° with the horizontal (positive x-direction). At the highest point of its trajectory, the shell explodes into two equal fragments. One of the fragments moves along the negative x-direction with a speed of 50 m/s. What is the speed of the other fragment at the time of explosion?

SOLUTION

As we know, in the absence of external force, the motion of centre of mass of a body remains unaffected. Thus, here the centre of mass of the two fragments will continue to follow the original projectile path. The velocity of the shell at the highest point of trajectory is $v_M = u\cos\theta = 100 \times \cos 60° = 50$ m/s. Let v_1 be the speed of the fragment which moves along the negative x-direction and the other fragment has speed v_2, which must be along the positive x-direction. Now from momentum conservation, we have

$$mv = \frac{-m}{2}v_1 + \frac{m}{2}v_2$$

or $$2v = v_2 - v_1$$

or $$v_2 = 2v + v_1$$

$$= (2 \times 50) + 50 = 150\ \text{m/s}.$$ ■

EXAMPLE 22

A particle of mass 2 m is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s After 1 s, explosion takes place and the particle is broken into two equal pieces. As a result of the explosion, one part comes to rest. Find the maximum height attained by the other part. (Take $g = 10\ \text{m/s}^2$)

SOLUTION

Applying conservation of linear momentum at the time of collision, or at $t = 1$ s,

$$m\vec{v} + m(0) = 2m(20\hat{i} + 10\hat{j})$$

$$\therefore \qquad \vec{v} = 40\hat{i} + 20\hat{j}$$

At 1 s, masses will be at height

$$h_1 = u_y t + \frac{1}{2}v_y t^2$$

$$= (20)(1) + \frac{1}{2}(-10)(1)^2 = 15\,\text{m}$$

After explosion, the other mass will further rise to a height

$$h_2 = \frac{u_y^2}{2g}$$

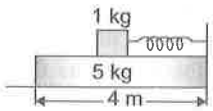
$$= \frac{(20)^2}{2\times10} = 20\ \text{m}$$

$u_y = 20$ m/s just after collision.

$\therefore$ Total height $h = h_1 + h_2 = 35$ m. ■

EXAMPLE 23

A plank of mass 5 kg is placed on a frictionless horizontal plane. Further, a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of the spring is compressed by the block by half of the spring's natural length. The system is now released from rest. What is the velocity of the plank when the block leaves the plank? (The stiffness constant of the spring is 100 N/m.)



SOLUTION

Let the velocity of the block and the plank when the block leaves the spring be u and v, respectively.

By conservation of energy,

$$\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$$

[M = mass of the plank, m = mass of the block]

$$\Rightarrow \quad 100 = u^2 + 5v^2 \qquad (1)$$

By conservation of momentum,

$$mu + Mv = 0$$

$$\Rightarrow \quad u = -5v \qquad (2)$$

Solving Eqs. (1) and (2),

$$30v^2 = 100$$

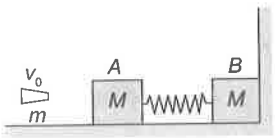
$$\Rightarrow \quad v = \sqrt{\frac{10}{3}}\ \text{m/s}.$$

From this moment until the block falls, both the plank and the block keep their velocity constant.

Thus, when the block falls, velocity of the plank = $\sqrt{\frac{10}{3}}$ m/s. ■

EXAMPLE 24

Two identical blocks each of mass $M = 9$ kg are placed on a rough horizontal surface of frictional coefficient $\mu = 0.1$. The two blocks are joined by a light spring and block B is in contact with a vertical fixed wall as shown in the figure. A bullet of mass $m = 1$ kg and $v_0 = 10$ m/s hits block A and gets embedded in it. Find the maximum compression of the spring. (Spring constant = 240 N/m, $g = 10$ m/s^2.)



SOLUTION

For the collision,

$$1 \times 10 = 10 \times v$$

$$\Rightarrow \quad v = 1\ \text{m/s}$$

If x be the maximum compression,

$$\frac{1}{2} \times 10 \times 1^2 = \mu(m + M)gx + \frac{1}{2}kx^2$$

$$5 = 10x + 120x^2$$

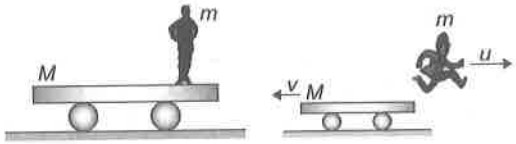
$$\Rightarrow \quad x = \frac{1}{6}\text{m}.$$ ■

EXAMPLE 25

A flat car of mass M is at rest on a frictionless floor with a child of mass m standing at its edge. If the child jumps off from the car towards right with an initial velocity u, with respect to the car, find the velocity of the car after its jump.

SOLUTION

Let the car attain a velocity v, and the net velocity of the child with respect to the earth will be $u - v$, as u is its velocity with respect to the car.



Initially, the system was at rest. Thus, according to the momentum conservation, momentum after jump must be zero, as

$$m(u - v) = Mv$$

$$v = \frac{mu}{m + M}.$$ ■

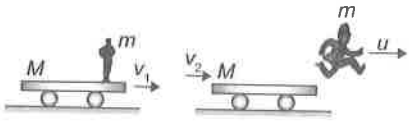
EXAMPLE 26

A flat car of mass M with a child of mass m is moving with a velocity v_1 on a frictionless surface. The child jumps in the direction of motion of the car with a velocity u with respect to the car. Find the final velocities of the child and that of the car after jump.

SOLUTION

This case is similar to the previous example, except now the car is moving before jump. Here also no external force is acting on the system in the horizontal direction; hence, momentum remains conserved in this direction. After the jump, the car attains a velocity v_2 in the same direction, which is less than v_1, due to backward push of the child for jumping. After jump, the child attains a velocity $u + v_2$ in the direction of motion of the car, with respect to the ground.

807



According to momentum conservation,

$$(M+m)v_1 = Mv_2 + m(u+v_2).$$

Velocity of the car after the jump is

$$v_2 = \frac{(M+m)v_1 - mu}{M+m}.$$

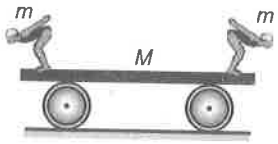
Velocity of the child after the jump is

$$u+v_2 = \frac{(M+m)v_1 + (M)u}{M+m}.$$

■

EXAMPLE 27

Two persons A and B, each of mass m are standing at the two ends of a rail-road car of mass M. The person A jumps to the left with a horizontal speed u with respect to the car. Thereafter, the person B jumps to the right, again with the same horizontal speed u with respect to the car. Find the velocity of the car after both the persons have jumped off.



SOLUTION

Let the car attain the velocity v in the rightward direction and velocity of man A with respect to ground is v', then

$$v' = v - u.$$

From momentum conservation,

$$0 = mv' + (M+m)v$$

$$\Rightarrow \quad m(v-u) + (M+m)v = 0$$

$$\Rightarrow \quad v = \frac{mu}{(M+2m)}.$$

Following this, mass B jumps to the right with the same horizontal speed u with respect to the car, then the car attains v'' velocity from linear momentum conservation.

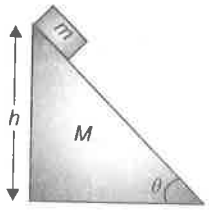
$$(M+m)v = m(u+v'') + Mv''$$

$$(M+m)\left[\frac{mu}{M+2m}\right] = mu + (m+M)v''$$

Now $$v'' = \frac{m^2u}{(M+2m)(M+m)}.$$ ■

EXAMPLE 28

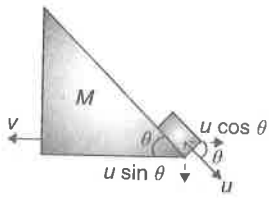
A block of mass m is placed on a triangular block of mass M, which in turn is placed on a horizontal surface as shown in the figure. Assuming frictionless surfaces, find the velocity of the triangular block when the smaller block reaches the bottom end.



SOLUTION

Let us assume that the wedge moves leftward with velocity v and block moves downward with velocity u with respect to the wedge.

∴ Net force in the horizontal direction is zero. So momentum is conserved in the x-direction.



Now velocity of the block with respect to the ground is

$$\vec{v}_m = \vec{v}_{mw} + \vec{v}_w$$

$$\vec{v}_m = \vec{u} + \vec{v}$$

$$\vec{v}_m = (u\cos\theta - v)\hat{i} - u\sin\theta\,\hat{j}.$$

Now from momentum conservation in the x-direction,

$$0 = -Mv + mv_{mx}$$

$$\Rightarrow \quad Mv = m(u\cos\theta - v) \qquad (1)$$

From energy conservation,

$$mgh = \frac{1}{2}Mv^2 + \frac{1}{2}mv_m^2$$

$$\Rightarrow \quad mgh = \frac{1}{2}mv^2 + \frac{1}{2}m(u^2+v^2-2uv\cos\theta) \qquad (2)$$

From Eqs. (1) and (2),

$$v = \left[\frac{2m^2 gh\cos^2\theta}{(M+m)(M+m\sin^2\theta)}\right]^{1/2}$$

SPRING BLOCK SYSTEM

A light spring of spring constant k and natural length l_0 attached in a compressed condition between two blocks of masses m_1 and m_2 on a smooth horizontal surface is shown in Fig. 6.11. The spring is initially compressed by a distance x_0.

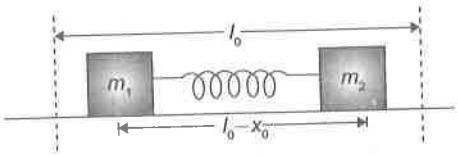


Figure 6.11

When the system is released, the blocks acquire velocities in the opposite directions. Let us assume that the velocities of block m_1 and m_2 are v_1 and v_2, respectively, at natural length of the spring and since no external force acts on this system in the horizontal direction. Hence, the linear momentum remains constant. Then from momentum conservation,

$$0 = m_2 v_2 - m_1 v_1$$

$$m_2 v_2 = m_1 v_1 \quad (1)$$

From mechanical energy conservation,

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow \quad 0 + \frac{1}{2}kx_0^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + 0$$

$$\Rightarrow \quad \frac{1}{2}kx_0^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad (2)$$

In the initial condition, there is no external force on the system and both the blocks are at stationary condition. Therefore, centre of mass of the system is at rest. So we can write,

$$m_1 x_1 = m_2 x_2 \quad (3)$$

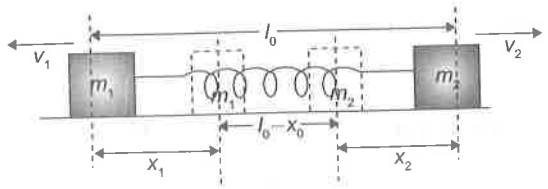


Figure 6.12

From Fig. 6.12, we can conclude

$$l_0 - x_0 + x_1 + x_2 = l_0$$

$$\Rightarrow \quad x_0 = x_1 + x_2 \quad (4)$$

Due to inertia, both the blocks move further from the position of the natural length of the spring.

Maximum extension occurs when both the blocks come to rest. Let us assume that x_1' and x_2' are the extensions in the spring from the initial position due to blocks m_1 and m_2 from natural length (Fig. 6.13).

So at maximum extension, $v_1 = v_2 = 0$.

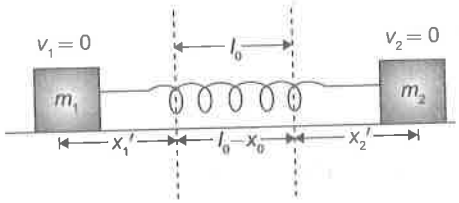


Figure 6.13

$\therefore$ Centre of mass is at rest. Therefore, we can write

$$m_1 x_1' = m_2 x_2'$$

$$x_1' + x_2' + l_0 - x_0 = l_0 + x_0$$

$$x_1' + x_2' = 2x_0.$$

SOLVED EXAMPLE

EXAMPLE 29

A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface. When released, the blocks acquire velocities in the opposite directions. The spring loses contact with the blocks when it acquires natural length. If the spring was initially compressed through a distance x, find the final speeds of the two blocks.

SOLUTION

Consider the two blocks plus the spring to be the system. No external force acts on this system in the horizontal direction. Hence, the linear momentum will remain constant. Suppose, the block of mass M moves with a speed v_1 and the other block with a speed v_2 after losing contact with the spring. From conservation of linear momentum in horizontal direction, we have

$$Mv_1 - mv_2 = 0$$

or $$v_1 = \frac{m}{M} v_2 \quad (1)$$

Initially, the energy of the system

$$= \frac{1}{2} kx^2.$$

Finally, the energy of the system

$$= \frac{1}{2} m v_2^2 + \frac{1}{2} M v_1^2.$$

As there is no friction, mechanical energy will remain conserved.

Therefore,

$$\frac{1}{2} m v_2^2 + \frac{1}{2} M v_1^2 = \frac{1}{2} kx^2 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$v_2 = \left[\frac{kM}{m(M+m)}\right]^{1/2} x$$

and $$v_1 = \left[\frac{kM}{m(M+m)}\right]^{1/2} x$$ ■

II format: Figure 6.14 shows two blocks of masses $2m$ and m that are placed on a frictionless surface and connected with a spring. An external kick gives a velocity v_0 m/s to the m mass towards right

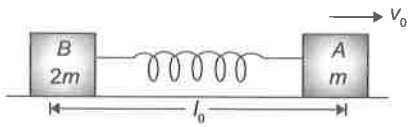


Figure 6.14

Now the velocity of centre of mass is

$$\vec{V}_{COM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\Rightarrow \quad V_{COM} = \frac{mv_0 + 0}{2m + m}$$

$$= \frac{v_0}{3} \text{ m/s}.$$

Due to the kick on m mass block, it starts moving with a velocity v_0 towards right immediately but due to inertia the $2m$ block remains at rest at that moment. Thus, the velocity of blocks A and B with respect to the centre of mass is

$$v_A = v_0 - \frac{v_0}{3}$$

$$= \frac{2v_0}{3} \text{ m/s}$$

(towards right)

$$v_B = 0 - \frac{v_0}{3}$$

$$= -\frac{v_0}{3} = \frac{v_0}{3}$$

(towards left)

Figure 6.15 shows the condition when the centre of mass is at rest.

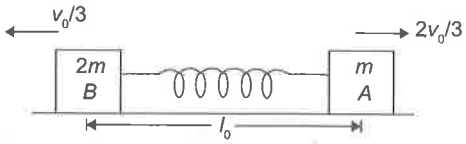


Figure 6.15

If the maximum extension of the spring is x_0 then at this position both the blocks come to rest with respect to COM, so from mechanical energy conservation,

$$\Rightarrow \quad K_i + U_i = K_f + U_f \quad (1)$$

$$K_i = \frac{1}{2} m \left(\frac{2v_0}{3}\right)^2 + \frac{1}{2} 2m \left(\frac{v_0}{3}\right)^2$$

$$U_i = 0$$

(spring is in natural length)

$$K_f = 0$$

$$(V_A = V_B = 0)$$

$$U_f = \frac{1}{2} kx_0^2.$$

Substituting the above values in Eq. (1),

$$\Rightarrow \quad \frac{1}{2} m \left(\frac{2v_0}{3}\right)^2 + \frac{1}{2} 2m \left(\frac{v_0}{3}\right)^2 + 0 = \frac{1}{2} kx_0^2$$

$$\Rightarrow \quad kx_0^2 = \frac{2}{3} mv_0^2.$$

Maximum extension

$$x_0 = v_0 \sqrt{\frac{2}{3k} m}$$

III format: Example

A block of mass m is connected to another block of mass M by a massless spring of spring constant k (Fig. 6.16). The blocks are kept on a smooth horizontal plane and are at rest. The spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.

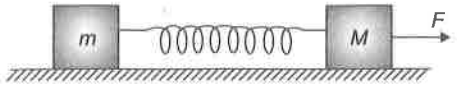


Figure 6.16

We solve the situation in the reference frame of centre of mass. As only F is the external force acting on the system, due to this force, the acceleration of the centre of mass is $F/(M+m)$. Thus, with respect to the centre of mass, there is a pseudo force on the two masses in the opposite direction. The free body diagram of m and M with respect to the centre of mass (taking centre of mass at rest) is shown in Fig. 6.17.

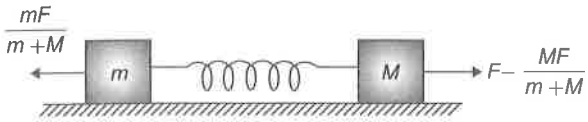


Figure 6.17

Taking the centre of mass at rest, if m moves maximum by a distance x_1 and M moves maximum by a distance x_2 (Fig. 6.18), then the work done by the external forces (including pseudo force) will be

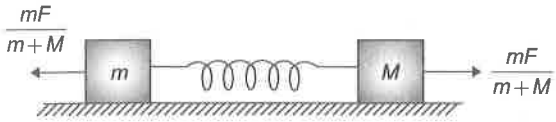


Figure 6.18

$$W = \frac{mF}{m+M}\cdot x_1 + \left(F - \frac{MF}{m+M}\right)\cdot x_2$$

$$= \frac{mF}{m+M}\cdot(x_1+x_2).$$

This work is stored in the form of potential energy of the spring as

$$U = \frac{1}{2}k(x_1+x_2)^2.$$

Thus, on equating we get the maximum extension in the spring, as after this instant the spring starts contracting

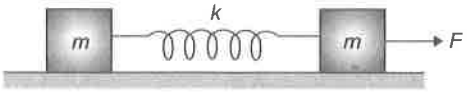
$$\frac{1}{2}k(x_1+x_2)^2 = \frac{mF}{m+M}\cdot(x_1+x_2)$$

$$x_{max} = x_1+x_2 = \frac{2mF}{k(m+M)}.$$

SOLVED EXAMPLE

EXAMPLE 30

Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in the figure.



(a) Find the displacement of the centre of mass at time t.

(b) If the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.

SOLUTION

(a) The acceleration of the centre of mass is

$$a_{COM} = \frac{F}{2m}.$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2}a_{COM}t^2 = \frac{Ft^2}{4m}$$

(b) Suppose the displacement of the first block is x_1 and that of the second block is x_2. Then,

$$x = \frac{mx_1+mx_2}{2m}$$

or $$\frac{Ft^2}{4m} = \frac{x_1+x_2}{2}$$

or $$x_1+x_2 = \frac{Ft^2}{2m} \quad (1)$$

Further, the extension of the spring is x_1-x_2. Therefore,

$$x_1-x_2 = x_0 \quad (2)$$

From Eqs. (1) and (2),

$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m}+x_0\right)$$

and $$x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right)$$ ■

IMPULSE

Impulse of a force $\vec{F}$ acting on a body for the time interval $t = t_1$ to $t = t_2$ is defined as

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}dt$$

$$\vec{I} = \int \vec{F}dt$$

$$= \int m\frac{d\vec{v}}{dt}dt = \int m\,d\vec{v}$$

$$\vec{I} = m(\vec{v}_2 - \vec{v}_1)$$

$$= \Delta\vec{P}$$

= change in momentum due to force $\vec{F}$.

Also $$\vec{I}_{Re} = \int_{t_1}^{t_2} \vec{F}_{Res}dt = \Delta\vec{P}$$

(impulse–momentum theorem) ■

Note

Impulse applied to an object in a given time interval can also be calculated from the area under force–time (F–t) graph in the same time interval.

Instantaneous Impulse

There are many cases when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown, the value of their product (i.e., impulse) can be known by measuring the initial and final momentum. Thus, we can write

$$\vec{I} = \int \vec{F}\,dt$$

$$= \Delta\vec{P} = \vec{P}_f - \vec{P}_i.$$

Important Points

1. It is a vector quantity.
2. Dimensions = $[MLT^{-1}]$
3. SI unit = kg m/s
4. Direction is along the change in momentum.
5. Magnitude is equal to the area under the F–t graph.
6. $\vec{I} = \int \vec{F}\,dt = \vec{F}_{av}\int dt = \vec{F}_{av}\Delta t$.
7. It is not a property of a particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

SOLVED EXAMPLES

EXAMPLE 31

The hero of a stunt film fires 50-g bullets from a machine gun, each at a speed of 1.0 km/s. If he fires 20 bullets in 4 s, what average force does he exert against the machine gun during this period.

SOLUTION

The momentum of each bullet

= (0.050 kg) (1000 m/s)

= 50 kg m/s.

The gun has been imparted this much amount of momentum by each bullet fired. Thus, the rate of change of momentum of the gun

$$= \frac{(50\text{ kg m/s}) \times 20}{4\text{ s}}$$

= 250 N.

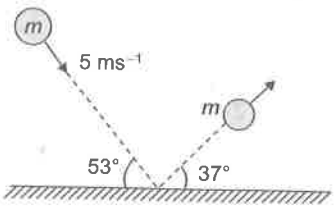
In order to hold the gun, the hero must exert a force of 250 N against the gun. ■

EXAMPLE 32

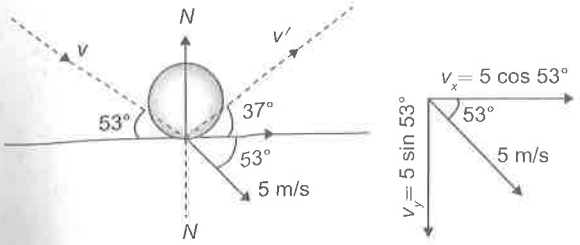
A ball of mass m = 1 kg strikes a smooth horizontal floor shown in the figure. Find out the impulse exerted on the floor.

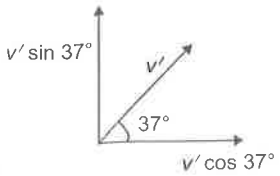
SOLUTION

As the ball strikes on the surface, an impulsive normal force is exerted on the ball as shown in the figure.



The normal force can change only the component v_y. So in the x-direction, momentum is conserved ($F_{net}x = 0$)





$$\Rightarrow \quad v' \cos 37° = 5 \cos 53°$$

$$v' = \frac{5\times3\times5}{5\times4} = \frac{15}{4} \text{ m/s}$$

So, $$v'_y = v' \sin 37°$$

$$= \frac{15}{4}\times\frac{3}{5} = \frac{9}{4} \text{ m/s}$$

Impulse = change in linear momentum in the y-direction,

$$I = \int N \cdot dt$$

$$= m\left(v_y - (-v'_y)\right)$$

$$= 1\left(4 + \frac{9}{4}\right)$$

$$= 6.25 \text{ Ns.}$$

Impulsive Force

A force, of relatively higher magnitude and acting for relatively shorter time, is called impulsive force. An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval. Impulsive force is a relative term. There is no clear boundary between an impulsive and non-impulsive force.

Note

Usually, colliding forces are impulsive in nature.

Since, the application time is very small, very little motion of the particle takes place.

Important points

1. Gravitational force and spring force are always non-impulsive.
2. Normal, tension and friction are case dependent.
3. An impulsive force can only be balanced by another impulsive force.

- **Impulsive normal:** In case of collision, normal forces at the surface of collision are always impulsive (Fig. 6.19).

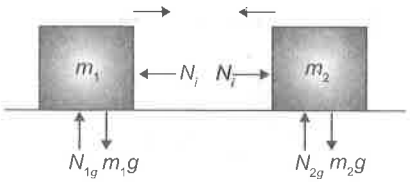


Figure 6.19

For example, N_i = impulsive; N_g = non-impulsive

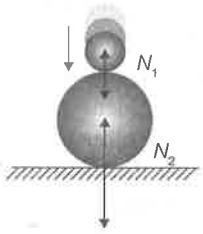


Figure 6.20

Both normals are impulsive.

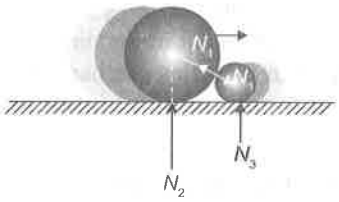


Figure 6.21

N_1, N_3 = impulsive; N_2 = non-impulsive.

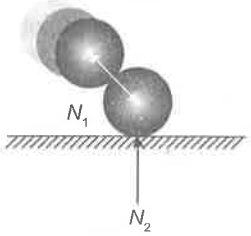
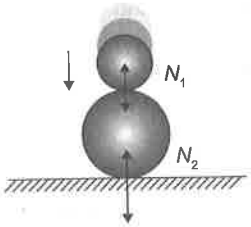


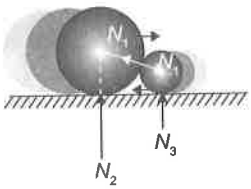
Figure 6.22

Both normals are impulsive.

- **Impulsive friction:** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



Friction at both the surfaces is impulsive.



Friction due to N_2 is non-impulsive and due to N_3 and N_1 are impulsive.

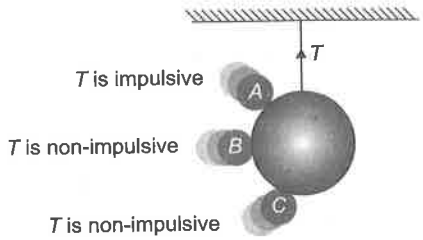
- **Impulsive tensions:** When a string jerks, equal and opposite tensions act suddenly at each end. Consequently, equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

  - **One end of the string is fixed**
    The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object. The object attached to the free end, however, will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

  - **Both ends of the string are attached to movable objects**
    In this case, equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string, however, no impulse acts and the momentum of each particle in this direction is unchanged.

Example:

In case of a rod, tension is always impulsive, and in case of a spring, tension is always non-impulsive.



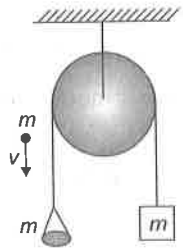
All normal are impulsive but tension T is impulsive only for the ball A

Figure 6.23

SOLVED EXAMPLES

EXAMPLE 33

A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley. Initially, the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v, find the speed with which the system moves just after the collision.



SOLUTION

Let the required speed be V.

Further, let J_1 = impulse between the particle and the pan and J_2 = impulse imparted to the block and the pan by the string.

Using impulse = change in momentum,

for particle $\quad J_1 = mv - mV \quad (1)$

for pan $\quad J_1 - J_2 = mV \quad (2)$

for block $\quad J_2 = mV \quad (3)$

Solving, these three equations, we get

$$V = \frac{v}{3}.$$

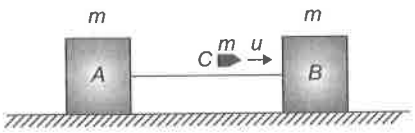
Alternative solution:
Applying conservation of linear momentum along the string,

$$mv = 3mV$$

we get, $$V = \frac{v}{3}$$ ■

EXAMPLE 34

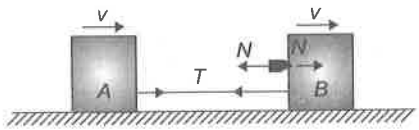
Two identical blocks A and B, connected by a massless string are placed on a frictionless horizontal plane. A bullet having the same mass, moving with speed u strikes block B from behind as shown. If the bullet gets embedded into the block B, then find the following:



(a) The velocity of A, B, C after collision.
(b) Impulse on A due to tension in the string.
(c) Impulse on C due to normal force of collision.
(d) Impulse on B due to normal force of collision.

SOLUTION

Let us assume that all the three are moving with velocity v.



Consider rightward direction as +ve. Then first we write impulse equation on bullet as

$$-\int N \cdot dt = mv - mu \qquad (1)$$

Now impulse equation on block B *is*

$$\int (N - T)\,dt = mv \qquad (2)$$

Impulse equation on block A *is*

$$\int T \cdot dt = mv \qquad (3)$$

(a) Add Eqs. (1), (2) and (3),

$$0 = 3mv - mu$$

$$\Rightarrow \quad v = \frac{u}{3}.$$

(b) Impulse on A due to tension in the string from Eq. (3) is $\int T \cdot dt = \frac{mu}{3}$.

(c) Impulse on C due to normal force of collision is obtained as follows:
From Eq. (1),

$$\int N \cdot dt = m\left(\frac{u}{3} - u\right)$$

$$= -\frac{2mu}{3}.$$

(d) Impulse on B due to normal force of collision is obtained from Eq. (2) as

$$\int (N - T)\,dt = \frac{mu}{3}$$

$$\int N \cdot dt = \frac{mu}{3} + \int T \cdot dt$$

$$= \frac{mu}{3} + \frac{mu}{3} = \frac{2mu}{3}$$ ■

COEFFICIENT OF RESTITUTION (e)

The coefficient of restitution is defined as the ratio of the impulses of reformation and deformation of either body.

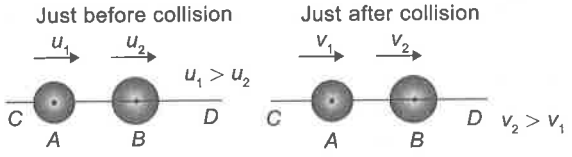
$$e = \frac{\text{impulse of reformation}}{\text{impulse of deformation}}$$

$$= \frac{\int F_r\,dt}{\int F_d\,dt}$$

$$e = \frac{\text{Velocity of separation of point of contact}}{\text{Velocity of approach of point of contact}}$$

Example for calculation of e

Two smooth balls A and B are approaching each other such that their centres are moving along the line CD in the absence of an external impulsive force. Let the velocities of A and B just before collision be u_1 and u_2, respectively. Let the velocities of A and B just after collision be v_1 and v_2, respectively.



$$e = \frac{v_2 - v_1}{u_1 - u_2}.$$

Notes

Coefficient of restitution is a factor between two colliding bodies which depends on the material of the body but is independent of the shape.

We can say e is a factor which relates deformation and reformation of the body:

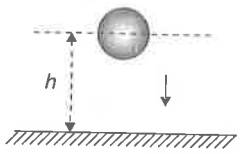
$$0 \le e \le 1$$

SOLVED EXAMPLES

EXAMPLE 35

If a body falls normally on a surface from height h, what will be the height regained after collision if the coefficient of restitution is e?

SOLUTION



If a body falls from height h, from equations of motion we know that it will hit the ground with a velocity say $u = \sqrt{2gh}$, which is also the velocity of approach here.

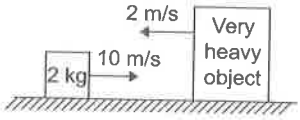
Now if after collision it regains a height h_1, then again by equations of motion $v = \sqrt{2gh_1}$, which is also the velocity of separation. So, by definition of e,

$$e = \sqrt{\frac{2gh_1}{2gh}}$$

or $$h_1 = e^2 h.$$ ■

EXAMPLE 36

A block of mass 2 kg is pushed towards a very heavy object moving with 2 m/s closer to the block (as shown in the figure). Assuming elastic collision and frictionless surface, find the final velocities of the blocks.



SOLUTION

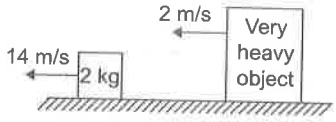
Let v_1 and v_2 be the final velocities of the 2-kg block and the heavy object, respectively. Then,

$$v_1 = u_1 + 1(u_1 - u_2)$$

$$= 2u_1 - u_2$$

$$= -14 \text{ m/s}$$

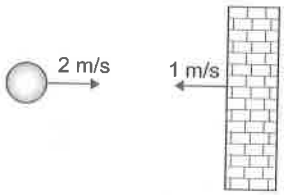
$$v_2 = -2 \text{ m/s}.$$



■

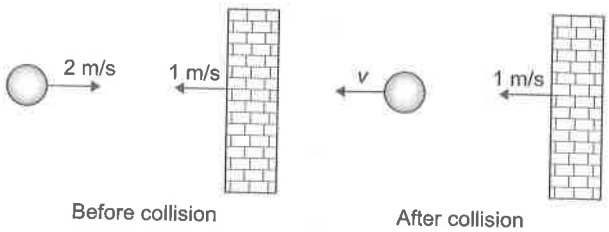
EXAMPLE 37

A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in the figure. Assuming collision to be elastic, find the velocity of the ball immediately after the collision.



SOLUTION

The speed of the wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in the figure. Since collision is elastic ($e = 1$).



separation speed = approach speed

or $$v - 1 = 2 + 1$$

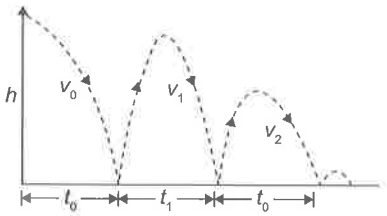
or $$v = 4 \text{ m/s}.$$ ■

EXAMPLE 38

A ball is dropped from a height h on to a floor. If in each collision its speed becomes e times its striking value, (a) find the time taken by the ball to stop rebounding; (b) find the total change in momentum in this time; (c) find the average force exerted by the ball on the floor using results of parts (a) and (b).

SOLUTION

(a) When the ball is dropped from a height h, the time taken by it to reach the ground will be $t_0 = \sqrt{\frac{2h}{g}}$ and its speed $v_0 = \sqrt{2gh}$.



Now after collision, its speed will become e times, i.e., $v_1 = ev_0 = e\sqrt{2gh}$ and so, it will take time to go up till its speed becomes zero $= (v_1/g)$. It will take the same time to come down. So the total time between I and II collision will be $t_1 = 2v_1/g$. Similarly, total time between II and III collision is $t_2 = 2v_2/g$.

So the total time of motion

$$\Delta T = t_0 + t_1 + t_2 + \cdots$$

or $$\Delta T = t_0 + \frac{2v_1}{g} + \frac{2v_2}{g} + \cdots$$

or $$\Delta T = t_0 + \frac{2ev_0}{g} + \frac{2e^2v_0}{g} + \cdots$$

$$[\text{as } v_2 = ev_1 = e^2v_0]$$

or $$\Delta T = \sqrt{\frac{2h}{g}}\left[1 + 2e(1 + e + e^2 + \cdots)\right]$$

$$= \sqrt{\frac{2h}{g}}\left[1 + 2e\left(\frac{1}{1-e}\right)\right]$$

$$= \sqrt{\frac{2h}{g}}\left[\frac{1+e}{1-e}\right]$$

(b) Change in momentum in the I collision

$$= mv_1 - (-mv_0)$$

$$= m(v_1 + v_0).$$

Change in momentum in the II collision

$$= m(v_2 + v_1).$$

Change in momentum in the nth collision

$$= m(vn + v_{n-1}).$$

Adding these, total change in momentum

$$\Delta p = m[v_0 + 2v_1 + \cdots + 2v_{n-1} + v_n]$$

or $$\Delta p = mv_0[1 + 2e + e^2 + \cdots]$$

or $$\Delta p = mv_0\left[1 + 2e\left(\frac{1}{1-e}\right)\right]$$

$$= m\sqrt{2gh}\left[\frac{1+e}{1-e}\right] \quad (2)$$

(c) Now as $\vec{F} = \frac{d\vec{p}}{dt}$,

so $$F_{avg} = \frac{\Delta p}{\Delta T}.$$

Substituting the values of ΔT and Δp from Eqs. (1) and (2)

$$F_{avg} = m\sqrt{2gh}\left[\frac{1+e}{1-e}\right] \times \sqrt{\frac{g}{2h}}\left[\frac{1-e}{1+e}\right]$$

$$= mg \quad (3)$$

■

Line of Motion

The line passing through the centre of the body along the direction of resultant velocity is called the line of motion.

Line of Impact

The line passing through the common normal to the surfaces in contact during impact is called the line of impact. The force during collision acts along this line on both the bodies.

Direction of line of impact can be determined by

(a) Geometry of colliding objects like spheres, discs, wedges, etc.

(b) Direction of change of momentum.

If one particle is stationary before the collision, then the line of impact will be along its motion after collision.

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Examples of line of impact

1. Two balls A and B are approaching each other such that their centres are moving along line CD (Fig. 6.24).

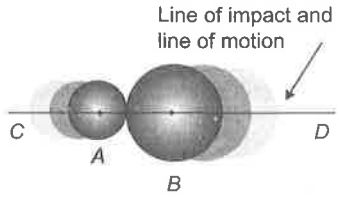


Figure 6.24

2. Two balls A and B are approaching each other such that their centres are moving along dotted the lines as shown in Fig. 6.25.

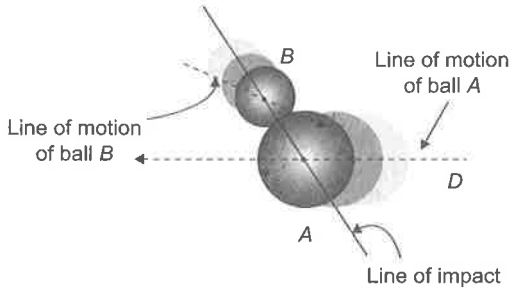
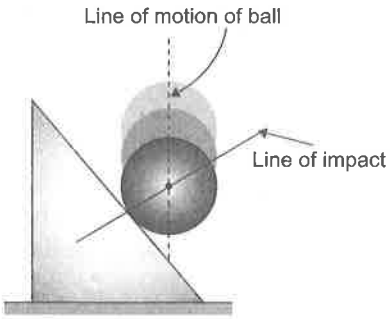


Figure 6.25

3. A ball is falling on a stationary wedge.



Note

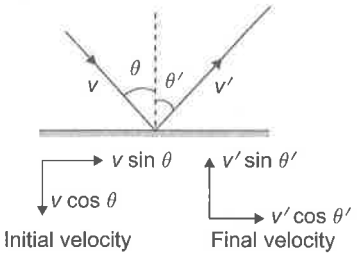
In the examples discussed earlier, line of motion is the same as the line of impact. But in problems in which line of impact and line of motion is different, then e will be

$$e = \frac{\text{Velocity of separation along line of impact}}{\text{Velocity of approach along line of impact}}$$

SOLVED EXAMPLES

EXAMPLE 39

A ball of mass m hits a floor with a speed v making an angle of incident θ with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.



SOLUTION

Suppose the angle of reflection is θ' and the speed after the collision is v' (shown the figure), the floor exerts a force on the ball along the normal during the collision. There is no force parallel to the surface. Thus, the parallel component of the velocity of the ball remains unchanged. This gives

$$v' \sin\theta' = v \sin\theta \qquad (1)$$

For the components normal to the floor, the velocity of separation is $v' \cos\theta'$ and the velocity of approach is $v \cos\theta$.

Hence, $$v' \cos\theta' = ev \cos\theta \qquad (2)$$

From (1) and (2),

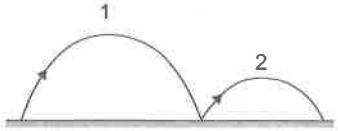
$$v' = v\sqrt{\sin^2\theta + e^2\cos^2\theta}\,.$$

Hence, $$\tan\theta' = \frac{\tan\theta}{e}\,.$$

For elastic collisions, $e = 1$, so that $\theta' = \theta$ and $v' = v$. ■

EXAMPLE 40

A ball is projected from the ground at some angle with the horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e.



SOLUTION

Let us assume that ball is projected with speed u at an angle θ with the horizontal. Then, before first collision with the ground,

Time of flight

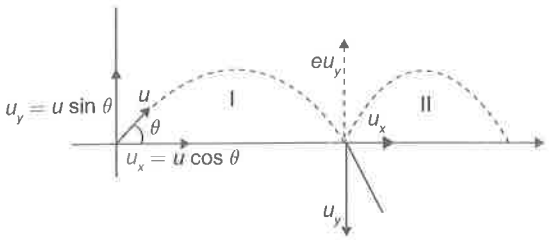
$$T = \frac{2u_y}{g},$$

Horizontal range

$$R = \frac{2u_x u_y}{g},$$

Maximum height

$$H_{max} = \frac{u_y^2}{2g} \quad (1)$$



After striking the ground, the component uy changes to euy. So

Time of flight,

$$T' = \frac{2eu_y}{g},$$

$$R' = \frac{2u_x(eu_y)}{g}$$

$$H'_{max} = \frac{(eu_y)^2}{2g} \quad (2)$$

From Eqs. (1) and (2)

Now, $$\frac{T}{T'} = a = \frac{1}{e}$$

$$\frac{R}{R'} = b = \frac{1}{e};$$

$$\frac{H_{max}}{H'_{max}} = \frac{1}{e^2} = c. \quad \blacksquare$$

EXAMPLE 41

A ball is projected from the ground with speed u at an angle α with the horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.

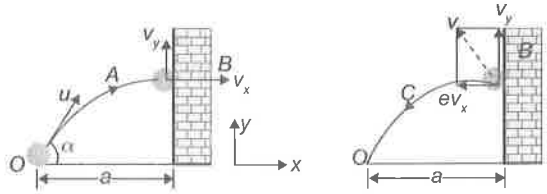
SOLUTION

A ball is a projected with a speed u at an angle α with the horizontal. It collides at a distance a with a wall parallel to the y-axis as shown in the figure.

Let v_x and v_y be the components of its velocity along the x- and y-directions at the time of impact with the wall. Coefficient of restitution between the ball and the wall is e.

Component of its velocity along the y-direction (common tangent) v_y will remain unchanged while component of its velocity along the x-direction (common normal) v_x will become ev_x in the opposite direction.

- Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and the maximum height attained by the ball remain the same as it would have been in the absence of collision with the wall. Thus,



From OAB,

$$R = a = u \cos\alpha \, t_{OAB}$$

From BCO,

$$R = a = eu \cos\alpha \, t_{BCO}$$

$$T = t_{OAB} + t_{BCO}$$

or $$\frac{2u\sin\alpha}{g} = \frac{a}{u\cos\alpha} + \frac{a}{eu\cos\alpha}$$

or $$\frac{a}{eu\cos\alpha} = \frac{2u\sin\alpha}{g} - \frac{a}{u\cos\alpha}$$

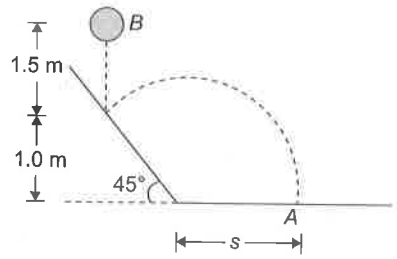
or $$\frac{a}{eu\cos\alpha} = \frac{2u^2\sin\alpha\cos\alpha - ag}{gu\cos\alpha}$$

$\therefore$ $$e = \frac{ag}{2u^2\sin\alpha\cos\alpha - ag}$$

or $$e = \frac{1}{\left(\frac{u^2\sin 2\alpha}{ag} - 1\right)}. \quad \blacksquare$$

EXAMPLE 42

To test the manufactured properties of 10 N steel balls, each ball is released from rest as shown and strikes a 45° inclined surface. If the coefficient of restitution is $e = 0.8$, determine the distance s where the ball must strike the horizontal plane at A. At what speed does the ball strike at A? ($g = 9.8$ m/s^2)



SOLUTION

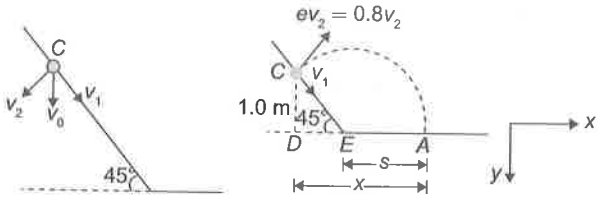
$$v_0 = \sqrt{2gh}$$

$$= \sqrt{2 \times 9.8 \times 1.5} = 5.42 \text{ m/s}.$$

The component of velocity parallel and perpendicular to the plane at the time of collision is given as

$$v_1 = v_2 = \frac{v_0}{\sqrt{2}}$$

$$= 3.83 \text{ m/s}.$$



Component parallel to the plane (v_1) remains unchanged, while the component perpendicular to plane becomes ev_2, where

$$ev_2 = 0.8 \times 3.83$$

$$= 3.0 \text{ m/s}.$$

$\therefore$ Component of velocity in the horizontal direction after collision

$$v_x = \frac{(v_1 + ev_2)}{\sqrt{2}}$$

$$= \frac{(3.83 + 3.0)}{\sqrt{2}} = 4.83 \text{ m/s}.$$

While the component of velocity in the vertical direction after collision,

$$v_y = \frac{v_1 - ev_2}{\sqrt{2}}$$

$$= \frac{3.83 - 3.0}{\sqrt{2}} = 0.59 \text{ m/s}.$$

Let t be the time the particle takes from point C to A, then

$$1.0 = 0.59t + \frac{1}{2} \times 9.8 \times t^2;$$

$$t = 0.4 \text{ s}.$$

Solving this, we get,

$\therefore$ $DA = v_x t$

$= (4.83)(0.4) = 1.93$ m

$\therefore$ $S = DA - DE$

$= 1.93 - 1.0$

$S = 0.93$ m

$v_{yA} = v_{yc} + gt$

$= (0.59) + (9.8)\,(0.4) = 4.51$ m/s

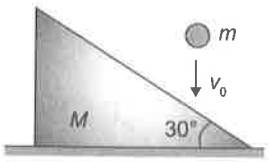
$v_{xA} = v_{xC} = 4.83$ m/s

$\therefore$ $v_A = \sqrt{(v_{xA})^2 + (v_{yA})^2}$

$= 6.6$ m/s. ■

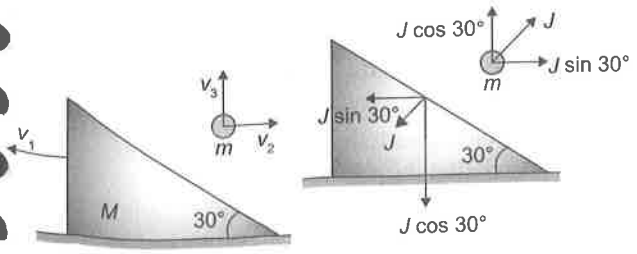
EXAMPLE 43

A ball of mass $m = 1$ kg falling vertically with a velocity $v_0 = 2$ m/s strikes a wedge of mass $M = 2$ kg kept on a smooth, horizontal surface as shown in the figure. The coefficient of restitution between the ball and the wedge is $e = \frac{1}{2}$. Find the velocity of the wedge and the ball immediately after collision.



SOLUTION

Given $M = 2$ kg and $m = 1$ kg.



Let J be the impulse between the ball and the wedge during collision and v_1, v_2 and v_3 be the components of velocity of the wedge and the ball in the horizontal and vertical directions, respectively.

Applying impulse = change in momentum, we get

$$J \sin 30° = Mv_1 = mv_2$$

or
$$\frac{J}{2} = 2v_1 = v_2 \quad (1)$$

$$J \cos 30° = m(v_3 + v_0)$$

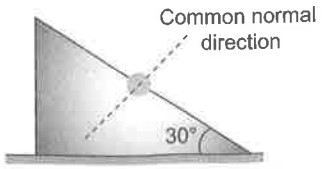
or
$$\frac{\sqrt{3}}{2} J = (v_3 + 2) \quad (2)$$

Applying relative speed of separation = e (relative speed of approach) in common normal direction, we get

$$(v_1 + v_2) \sin 30° + v_3 \cos 30° = \frac{1}{2}(v_0 \cos 30°)$$

or
$$v_1 + v_2 + \sqrt{3}v_3 = \frac{\sqrt{3}}{2} \quad (3)$$

Solving Eqs. (1), (2) and (3), we get

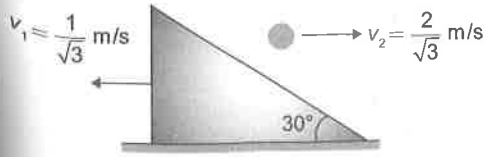


$$v_1 = \frac{1}{\sqrt{3}} \text{ m/s}$$

$$v_2 = \frac{2}{\sqrt{3}} \text{ m/s}$$

and
$$v_3 = 0.$$

Thus, velocities of the wedge and ball are $v_1 = \frac{1}{\sqrt{3}}$ m/s and $v_2 = \frac{2}{\sqrt{3}}$ m/s in the horizontal direction as shown in the figure. ■



COLLISION OR IMPACT

Collision is an event in which an impulsive force acts between two or more bodies for a short time, which results in change of their velocities.

Notes

- In a collision, particles may or may not come in physical contact.
- The duration of collision Δt is negligible as compared to the usual time intervals of observation of motion.
- In a collision the effect of external non-impulsive forces such as gravity are not taken into account as due to small duration of collision (Δt) average impulsive force responsible for collision is much larger than the external forces acting on the system.

The collision is in fact a redistribution of total momentum of the particle:

Thus, the law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles. Consider a situation shown in Fig. 6.26.

Two balls of masses m_1 and m_2 are moving with velocities v_1 and v_2 ($< v_1$) along the same straight line in a smooth horizontal surface. Now let us see what happens during the collision between two particles.

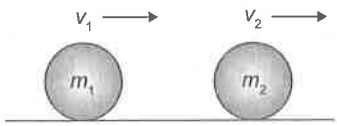


Figure (a)

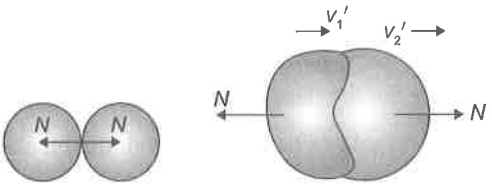


Figure (b) Figure (c)

Figure 6.26

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Figure 6.26 (a): Ball of mass m_1 is behind m_2. Since $v_1 > v_2$, the balls will collide after some time.
Figure 6.26 (b): During collision, both the balls are a little bit deformed. Due to deformation, two equal and opposite normal forces act on both the balls. These forces decreases the velocity of m_1 and increase the velocity of m_2.
Figure 6.26 (c): Now velocity of ball m_1 decreases from v_1 to v_1' and velocity of ball m_2 increases from v_2 to v_2'. But still $v_1' > v_2'$, so both the balls are continuously deformed.
Figure 6.26 (d): Contact surface of both the balls are deformed till the velocity of both the balls becomes equal. So at maximum deformation, velocities of both the blocks are equal.

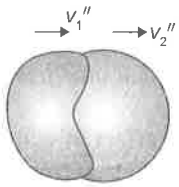


Figure (d)

At maximum deformation,

$$v_1'' = v_2''$$

Figure 6.26 (e): Normal force is still in the direction shown in Fig. 6.26 (e), i.e., velocity of m_1 is further decreased and that of m_2 is increased. Now both the balls start to regain their original shape and size.

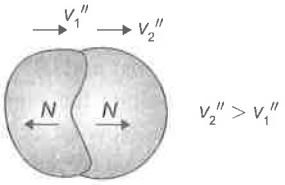


Figure (e)

Figure 6.26 (f): These two forces redistribute their linear momentum in such a manner that both the blocks are separated from one another. Velocity of ball m_2 becomes more than the velocity of block m_1, i.e., $v_2 > v_1$.

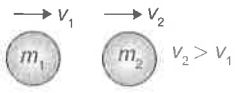


Figure (f)

The collision is said to be elastic if both the blocks regain their original form. The collision is said to be inelastic if the deformation is permanent. If the blocks move together with the same velocity after the collision, the collision is said to be perfectly inelastic.

Classification of Collisions

(1) On the basis of the line of impact

(a) Head-on collision: If the velocities of the colliding particles are along the same line before and after the collision.

(b) Oblique collision: If the velocities of the colliding particles are along different lines before and after the collision.

(2) On the basis of energy

(a) Elastic collision:

(i) In an elastic collision, the colliding particles regain their shape and size completely after collision, i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies.

(ii) Thus, kinetic energy of the system after collision is equal to the kinetic energy of the system before collision.

(iii) $e = 1$.

(iv) Due to F_{net} on the system, zero linear momentum remains conserved.

(b) Inelastic collision:

(i) In an inelastic collision, the colliding particles do not regain their shape and size completely after collision.

(ii) Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved.

(iii) However, in the absence of external forces, law of conservation of linear momentum still holds good.

(iv) $(\text{Energy loss})_{\substack{\text{perfectly}\\\text{inelastic}}} > (\text{Energy loss})_{\substack{\text{partial}\\\text{inelastic}}}$

(v) $0 < e < 1$

(c) Perfectly inelastic collision:

(i) In this, the colliding bodies do not return to their original shape and size after collision, i.e., both the particles stick together after the collision and move with the same velocity.

(ii) But due to F_{net} of the system, zero linear momentum remains conserved.

(iii) Total energy is conserved.

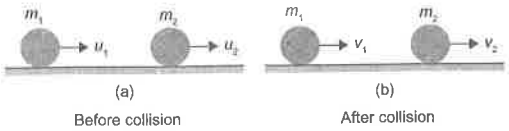
(iv) Initial kinetic energy > final kinetic energy.

(iv) Loss in kinetic energy goes to the deformation potential energy.

(v) $e = 0$.

Value of Velocities after Collision

Let us now find the velocities of two particles after collision if they collide directly. The coefficient of restitution between them is given as e.



$u_1 > u_2$ $\qquad$ $v_2 > v_1$

Figure 6.27

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

$$\Rightarrow \qquad (u_1 - u_2)e = (v_2 - v_1) \qquad (1)$$

By momentum conservation,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad (2)$$

$$v_2 = v_1 + e(u_1 - u_2) \qquad (3)$$

From the above equation,

$$v_1 = \frac{m_1u_1 + m_2u_2 + m_2e(u_2 - u_1)}{m_1 + m_2} \qquad (4)$$

$$v_2 = \frac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2} \qquad (5)$$

Special cases:

1. If $m_1 >> m_2$ and $u_2 = 0$ and $u_1 = u$

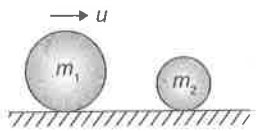


Figure 6.28

and $\qquad e = 1$

$$m_1 = m_2$$

From Eqs. (3) and (4)

$$v_1 = \frac{m_1u - m_2u}{m_1 + m_2}$$

$$= \frac{u(m_1 - m_2)}{m_1 + m_2}$$

$$v_1 \simeq u$$

$$v_2 = \frac{m_1u + m_2u}{m_1 + m_2}$$

$$= \frac{2m_1u}{m_1 + m_2};$$

$$v_2 = 2u$$

2. If $m_1 = m_2 = m$ and $e = 1$, then

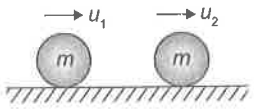


Figure 6.29

From Eqs. (3) and (4),

$$v_1 = \frac{m(u_1 + u_2) + m(u_2 - u_1)}{2m}$$

$$v_1 = u_2.$$

In this way,

$$v_2 = u_1,$$

i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities.

Collision in Two Dimension (Oblique)

1. A pair of equal and opposite impulses act along a common normal direction. Hence, linear momentum of individual particles changes along the common normal direction. If the mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles changes during collision in this direction.
2. No component of impulse acts along the common tangent direction. Hence, linear momentum or linear velocity of the individual particles (if mass is constant) remains unchanged along this direction.
3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remains conserved before and after collision in any direction.
4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction, we can apply relative speed of separation = e (relative speed of approach).

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SOLVED EXAMPLES

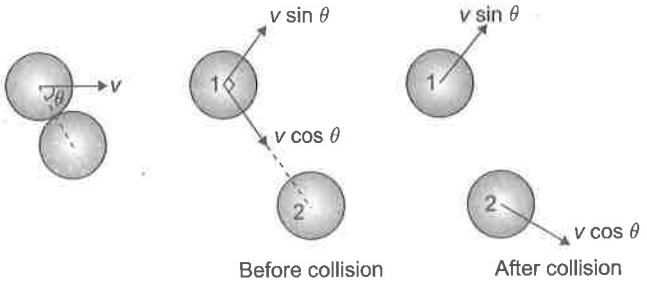
EXAMPLE 44

A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.

SOLUTION

In head-on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along the common normal direction $v \cos \theta$ becomes zero after collision, while that of ball 2 becomes $v \cos \theta$.

The components along the common tangent direction of both the particles remain unchanged. Thus, the components along the common tangent and common normal direction of both the balls in tabular form are given a head:



Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin \theta$	$v \sin \theta$	$v \cos \theta$	0
2	0	0	0	$v \cos \theta$

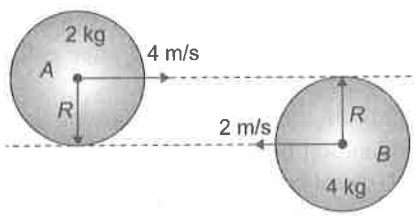
From the above table and figure, we see that both the balls move at right angles after collision with velocities $v \sin \theta$ and $v \cos \theta$. ■

Note

When two identical bodies have an oblique elastic collision, with one body at rest before collision, then the two bodies will go in perpendicular directions.

EXAMPLE 45

Two spheres are moving towards each other. Both have same the radius but their masses are 2 kg and 4 kg . If the velocities are 4 m/s and 2 m/s, respectively, and the coefficient of restitution is $e = 1/3$, find



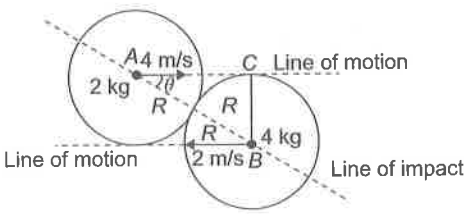
(a) the common velocity along the line of impact
(b) final velocities along the line of impact.
(c) impulse of deformation
(d) impulse of reformation
(e) maximum potential energy of deformation
(f) loss in kinetic energy due to collision.

SOLUTION

In ΔABC,

$$\sin \theta = \frac{BC}{AB} = \frac{R}{2R} = \frac{1}{2}$$

or $$\theta = 30°.$$

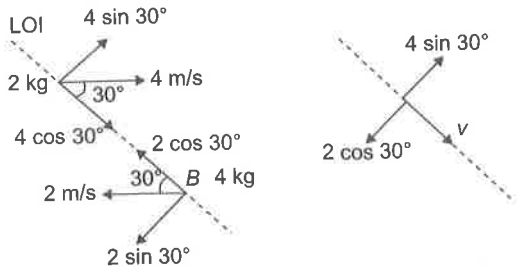


(a) By conservation of momentum along the line of impact,

$$2(4 \cos 30°) - 4(2 \cos 30°) = (2 + 4)v$$

or $$v = 0$$

(common velocity along LOI).

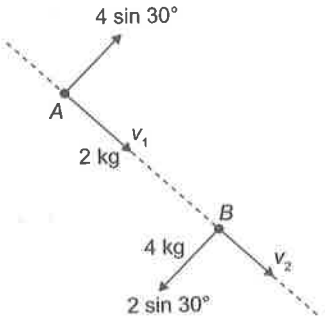


Just before collision along LOI

Maximum deformed state

(b) Let v_1 and v_2 be the final velocity of A and B, respectively. Then, by conservation of momentum along the line of impact,

$$2(4\cos 30°) - 4(2\cos 30°) = 2(v_1) + 4(v_2)$$



Just after collision along LOI

or $$0 = v_1 + 2v_2 \quad (1)$$

By coefficient of restitution,

$$e = \frac{\text{Velocity of separation along LOI}}{\text{Velocity of approach along LOI}}$$

or $$\frac{1}{3} = \frac{v_2 - v_1}{4\cos 30° + 2\cos 30°}$$

or $$v_2 - v_1 = \sqrt{3} \quad (2)$$

From the above two equations,

$$v_1 = \frac{-2}{\sqrt{3}} \text{ m/s}$$

and $$v_2 = \frac{1}{\sqrt{3}} \text{ m/s}$$

(c) $J_0 = m_1(v - u_1)$

$$= 2\,(0 - 4\cos 30°)$$

$$= -4\sqrt{3} \text{ Ns}$$

(d) $J_R = eJ_0$

$$= \frac{1}{3}(-4\sqrt{3}) = -\frac{4}{\sqrt{3}} \text{ Ns.}$$

(e) Maximum potential energy of deformation is equal to the loss in kinetic energy during deformation up to maximum deformed state,

$$U = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \frac{1}{2}2(4\cos 30°)^2 + \frac{1}{2}4(-2\cos 30°)^2 - \frac{1}{2}(2+4)(0)$$

or $U = 18$ Joule.

(f) Loss in kinetic energy

$$\Delta KE = \frac{1}{2}m_1(u_1\cos\theta)^2 + \frac{1}{2}m_2(u_2\cos\theta)^2 - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$= \frac{1}{2}2(4\cos 30°) + \frac{1}{2}4(-2\cos 30°) - \left(\frac{1}{2}2\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}4\left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$\Delta KE = 16$ Joule. ■

VARIABLE MASS

In our discussion of the conservation linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those mass as variable, i.e., those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping thereby continuously decreasing its mass.

In such a problem you have nothing to do but apply a thrust force $\left(\vec{F}_t\right)$ to the main mass in addition to all the other forces acting on it. This thrust force is given by,

$$\vec{F}_t = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

Here $\vec{v}_{rel}$ is the velocity of the mass gained or mass ejected relative to the main mass. In case of a rocket, this is

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sometimes called the exhaust velocity of the gases. $\left(\frac{dm}{dt}\right)$ is the rate at which mass is increasing or decreasing.

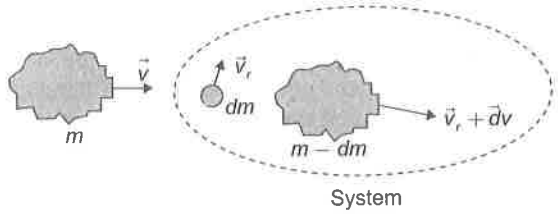


Figure 6.30

The expression for the thrust force can be derived from the conservation of linear momentum in the absence of any external forces on a system as follows:

Suppose at some moment $t = t$, mass of a body is m and its velocity is $\vec{v}$. After some time at $t = t + dt$, its mass becomes $(m - dm)$ and velocity becomes $\vec{v} + \vec{dv}$. The mass dm is ejected with a relative velocity $\vec{v}_r$. Absolute velocity of mass dm is therefore $\vec{v}_r + \vec{v} + \vec{dv}$. If no external forces are acting on the system, the linear momentum of the system will remain conserved,

or $$\vec{P}_i = \vec{P}_f$$

or $$m\vec{v} = (m - dm)(\vec{v} + \vec{dv}) + dm(\vec{v}_r + \vec{v} + \vec{dv})$$

or $$m\vec{v} = m\vec{v} + md\vec{v} - dm\vec{v} - (dm)(\vec{dv}) + dm\vec{v} + \vec{v}_r dm + (dm)(\vec{dv})$$

$\therefore$ $$m\vec{dv} = -\vec{v}_r dm$$

or $$m\left(\frac{\vec{dv}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$$

Here, $m\left(\frac{\vec{dv}}{dt}\right)$ = thrust force $\left(\vec{F}_1\right)$

and $-\frac{dm}{dt}$ = rate at which mass is ejecting.

Problems related to variable mass can be solved in the following three steps:

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find the net force on the mass and apply $\vec{F}_{net} = m\frac{\vec{dv}}{dt}$ (m = mass at that particular instant).

Rocket Propulsion

Let m_0 be the mass of the rocket at time $t = 0$, m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.

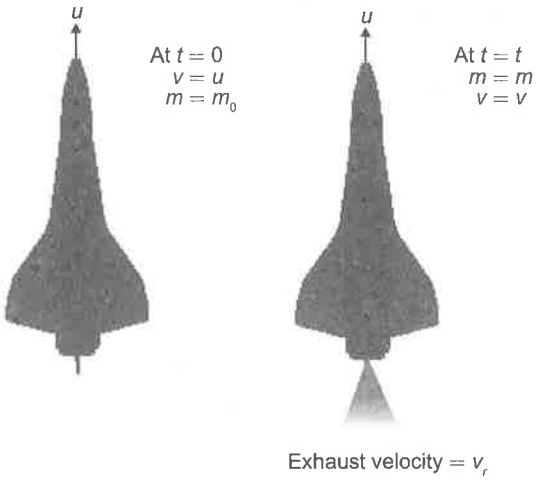


Figure 6.31

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases. Usually, $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$

1. Thrust force on the rocket
$$F_t = v_r\left(-\frac{dm}{dt}\right) \text{ (upwards)}$$
2. Weight of the rocket
$$W = mg \text{ (downwards)}$$
3. Net force on the rocket
$$F_{net} = Ft - W \text{ (upwards)}$$

or $$F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg\,.$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or $$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

or $$\int_u^v dv = v_r\int_{m_0}^{m}\frac{-dm}{m} - g\int_0^t dt$$

or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt\,.$$

Thus, $$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right) \qquad (1)$$

Notes

1. $F_t = v_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.
2. If gravity is ignored and initial velocity of the rocket $u = 0$, Eq. (1) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$.

SOLVED EXAMPLE

EXAMPLE 46

A uniform chain of mass per unit length ℓ begins to fall with a velocity v on the table. Find the thrust force exerted by the chain on the table.

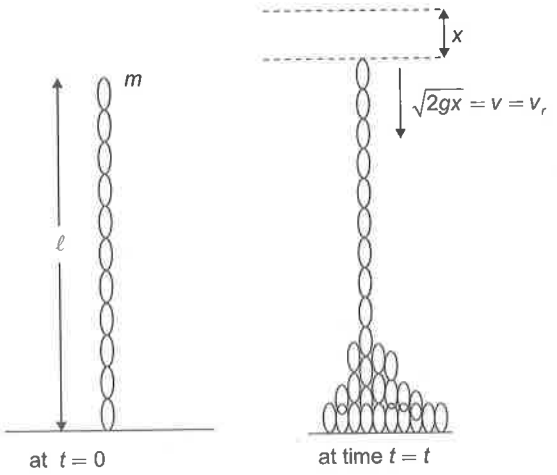
SOLUTION

Let us assume that the mass of the chain is m and length ℓ.

We assume that after time t, x length of the chain has fallen on the table. Then the speed of the upper part of the chain is $\sqrt{2gx}$ as shown in the figure.

Now in time $t + dt$, length of chain that has fallen on the table is vdt. Then the mass of chain that has fallen on the table is

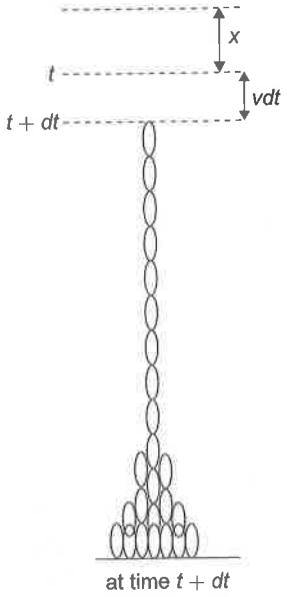
$$dm = \frac{m}{\ell}\cdot vdt$$



Now the rate of increase of mass

$$\frac{dm}{dt} = \frac{m}{\ell}v$$

$$= \frac{m}{\ell}\sqrt{2gx}$$



Here v is downward and mass is increasing so thrust force acts in the downward direction and is given by

$$f_t = v_r\frac{dm}{dt}$$

$$= \sqrt{2gx}\,\frac{m}{\ell}\left(\sqrt{2gx}\right)$$

$$f_t = \ell v^2.$$

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EXERCISES

JEE Main

1. A man of mass M stands at one end of a plank of length L which lies at rest on a frictionless surface. The man walks to other end of the plank. If the mass of the plank is $\frac{M}{3}$, then the distance that the man moves relative to ground is

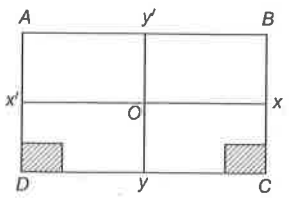
(A) $\frac{3L}{4}$ (B) $\frac{L}{4}$

(C) $\frac{4L}{5}$ (D) $\frac{L}{3}$

2. The centre of mass of two particles lies
(A) on the line perpendicular to the line joining the particles
(B) on a point outside the line joining the particles
(C) on the line joining the particles
(D) none of these

3. Two balls A and B of masses 100 g and 250 g, respectively, are connected by a stretched spring of negligible mass and placed on a smooth table. When the balls are released simultaneously the initial acceleration of B is 10 cm/s^2 westward. What is the magnitude and direction of initial acceleration of the ball A?
(A) 25 cm/s^2 Eastward
(B) 25 cm/s^2 Northward
(C) 25 cm/s^2 Westward
(D) 25 cm/s^2 Southward

4. A uniform square plate $ABCD$ has a mass of 10 kg. If two point masses of 3 kg each are placed at the corners C and D as shown in the adjoining figure, then the centre of mass shifts to the point which lies on



(A) OC (B) OD
(C) OY (D) OX

5. A particle of mass $3m$ is projected from the ground at some angle with horizontal. The horizontal range is R. At the highest point of its path, it breaks into two pieces m and $2m$. The smaller mass comes to rest and larger mass finally falls at a distance x from the point of projection where x is equal to

(A) $\frac{3R}{4}$ (B) $\frac{3R}{2}$

(C) $\frac{5R}{4}$ (D) $3R$

6. A man weighing 80 kg is standing at the centre of a flat boat and he is 20 m from the shore. He walks 8 m on the boat towards the shore and then halts. The boat weight 200 kg. How far is he from the shore at the end of this time?
(A) 11.2 m (B) 13.8 m
(C) 14.3 m (D) 15.4 m

7. Two particles having mass ratio $n : 1$ are interconnected by a light inextensible string that passes over a smooth pulley. If the system is released, then the acceleration of the centre of mass of the system is

(A) $(n-1)^2 g$ (B) $\left(\frac{n+1}{n-1}\right)^2 g$

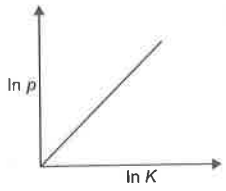
(C) $\left(\frac{n-1}{n+1}\right)^2 g$ (D) $\left(\frac{n+1}{n-1}\right) g$

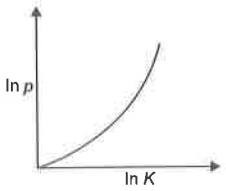
8. Internal forces can change
(A) the linear momentum but not the kinetic energy of the system.
(B) the kinetic energy but not the linear momentum of the system.
(C) linear momentum as well as kinetic energy of the system.
(D) neither the linear momentum nor the kinetic energy of the system.

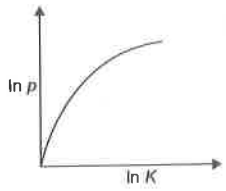
9. A small sphere is moving at a constant speed in a vertical circle. Below is a list of quantities that could be used to describe some aspects of the motion of the sphere.

I kinetic energy
II gravitational potential energy
III momentum
Which of these quantities will change as this sphere moves around the circle?
(A) I and II only (B) I and III only
(C) III only (D) II and III only

10. Which of the following graphs represent the graphical relation between momentum (p) and kinetic energy (K) for a body in motion?

(A) 

(B) 

(C) 

(D) None of these

11. Two balls are thrown in air. The acceleration of the centre of mass of the two balls while in air (neglect air resistance)
(A) depends on the direction of the motion of the balls
(B) depends on the masses of the two balls
(C) depends on the speeds of the two balls
(D) is equal to g

12. A bomb at rest explodes into two parts of masses m_1 and m_2. If the momentums of the two parts be p_1 and p_2, then their kinetic energies will be in the ratio of
(A) m_1/m_2 (B) m_2/m_1
(C) p_1/p_2 (D) p_2/p_1

13. Conservation of linear momentum is equivalent to
(A) Newton's second law of motion
(B) Newton's first law of motion
(C) Newton's third law of motion
(D) Conservation of angular momentum

14. A body of mass m collides against a wall with the velocity v and rebounds with the same speed. Its change of momentum is
(A) $2mv$ (B) mv
(C) $-mv$ (D) 0

15. A bomb initially at rest explodes by itself into three equal mass fragments. The velocities of two fragments are $\left(3\hat{i}+2\hat{j}\right)$ m/s and $\left(-\hat{i}-4\hat{j}\right)$m/s. The velocity of the third fragment is (in m/s)
(A) $2\hat{i}+2\hat{j}$ (B) $2\hat{i}-2\hat{j}$
(C) $-2\hat{i}+2\hat{j}$ (D) $-2\hat{i}-2\hat{j}$

16. A stone of mass m_1 moving with a uniform speed v suddenly explodes on its own into two fragments. If the fragment of mass m_2 is at rest, the speed of the other fragment is
(A) $\frac{m_1 v}{(m_1-m_2)}$ (B) $\frac{m_2 v}{(m_1-m_2)}$
(C) $\frac{m_1 v}{(m_1+m_2)}$ (D) $\frac{m_1 v}{m_2}$

17. A nucleus of mass number A originally at rest emits α-particle with speed v. The recoil speed of daughter nucleus is
(A) $\frac{4v}{A-4}$ (B) $\frac{4v}{A+4}$
(C) $\frac{v}{A-4}$ (D) $\frac{v}{A+4}$

18. There are some passengers inside a stationary railway compartment. The track is frictionless. The centre of mass of the compartment itself (without the passengers) is C_1, whereas the centre of mass of the compartment plus passengers system is C_2. If the passengers move about inside the compartment along the track,
(A) both C_1 and C_2 will move with respect to the ground
(B) neither C_1 nor C_2 will move with respect to the ground
(C) C_1 will move but C_2 will be stationary with respect to the ground
(D) C_2 will move but C_1 will be stationary with respect to the ground

19. A system of N particles is free from any external forces.
(a) Which of the following is true for the magnitude of the total momentum of the system?
(A) It must be zero.
(B) It could be non-zero, but it must be constant

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(C) It could be non-zero, and it might not be constant
(D) It could be zero, even if the magnitude of the total momentum is not zero.

(b) Which of the following must be true for the sum of the magnitudes of the momenta of the individual particles in the system?

(A) It must be zero.
(B) It could be non-zero, but it must be constant.
(C) It could be non-zero, and it might not be constant.
(D) The answer depends on the nature of the internal forces in the system.

20. An isolated rail car of mass M is moving along a straight, frictionless track at an initial speed v_0. The car is passing under a bridge when a crate filled with N bowling balls, each of mass m, is dropped from the bridge into the bed of the rail car. The crate splits open and the bowling balls bounce around inside the rail car, but none of them fall out.

(a) Is the momentum of the rail car + bowling balls system conserved in this collision?

(A) Yes, the momentum is completely conserved.
(B) Only the momentum component in the vertical direction is conserved.
(C) Only the momentum component parallel to the track is conserved.
(D) No components are conserved.

(b) What is the average speed of the rail car + bowling balls system sometime after the collision?

(A) $(M + Nm)v_0/M$
(B) $Mv_0/(Nm + M)$
(C) Nmv_0/M
(D) The speed cannot be determined because there is not enough information.

21. A super-ball is to bounce elastically back and forth between two rigid walls at a distance d from each other. Neglecting gravity and assuming the velocity of super-ball to be v_0 horizontally, the average force being exerted by the super-ball on each wall is

(A) $\frac{1}{2}\frac{mv_0^2}{d}$ (B) $\frac{mv_0^2}{d}$

(C) $\frac{2mv_0^2}{d}$ (D) $\frac{4mv_0^2}{d}$

22. A force exerts an impulse I on a particle changing its speed from u to $2u$. The applied force and the initial velocity are oppositely directed along the same line. The work done by the force is

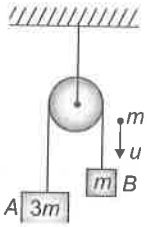
(A) $\frac{3}{2}Iu$ (B) $\frac{1}{2}Iu$

(C) Iu (D) $2Iu$

23. A boy hits a baseball with a bat and imparts an impulse J to the ball. The boy hits the ball again with the same force, except that the ball and the bat are in contact for twice the amount of time as in the first hit. The new impulse equals

(A) half the original impulse
(B) the original impulse
(C) twice the original impulse
(D) four times the original impulse

24. A system of two blocks A and B are connected by an inextensible massless strings as shown in the figure. The pulley is massless and frictionless. Initially, the system is at rest, when a bullet of mass 'm' moving with a velocity 'u' as shown hits the block 'B' and gets embedded into it. The impulse imparted by tension force to the block of mass $3m$ is



(A) $\frac{5mu}{4}$ (B) $\frac{4mu}{5}$

(C) $\frac{2mu}{5}$ (D) $\frac{3mu}{5}$

25. A ball strikes a smooth horizontal ground at an angle of 45° with the vertical. What cannot be the possible angle of its velocity with the vertical after the collision? (Assume $e \leq 1$.)

(A) 45° (B) 30°
(C) 53° (D) 60°

26. In an inelastic collision,

(A) momentum is conserved but kinetic energy is not
(B) momentum is not conserved but kinetic energy is conserved
(C) neither momentum nor kinetic energy is conserved
(D) both the momentum and the kinetic energy are conserved

3 90

27. Two perfectly elastic balls of same mass m are moving with velocities u_1 and u_2. They collide elastically n times. The kinetic energy of the system finally is

(A) $\frac{1}{2}\frac{m}{u}u_1^2$ (B) $\frac{1}{2}\frac{m}{u}(u_1^2+u_2^2)$

(C) $\frac{1}{2}m(u_1^2+u_2^2)$ (D) $\frac{1}{2}mn(u_1^2+u_2^2)$

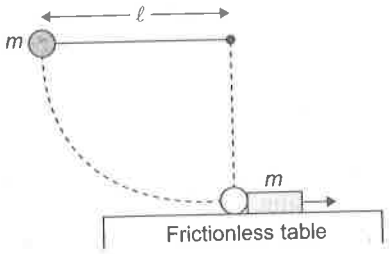
28. When two bodies collide elastically, then
(A) kinetic energy of the system alone is conserved
(B) only momentum is conserved
(C) both energy and momentum are conserved
(D) neither energy nor momentum is conserved

29. A ball hits the floor and rebounds after an inelastic collision. In this case-
(A) the momentum of the ball just after the collision is the same as that just before the collision
(B) the mechanical energy of the ball remains the same in the collision
(C) the total momentum of the ball and the earth is conserved.
(D) the total energy of the ball and the earth is conserved

30. Six steel balls of identical size are lined up along a straight frictionless groove. Two similar balls moving with a speed V along the groove collide with this row on the extreme left hand then

(A) all the balls will start moving to the right with speed 1/8 each
(B) all the six balls initially at rest will move on with speed $V/6$ each and two identical balls will come to rest
(C) two balls from the extreme right end will move on with speed V each and the remaining balls will remain at rest
(D) one ball from the right end will move on with speed $2V$, the remaining balls will be at rest.

31. The bob of a simple pendulum of length ℓ dropped from a horizontal position strikes a block of the same mass, placed on a horizontal table (frictionless) as shown in the figure, the block shall have kinetic energy



(A) Zero (B) $mg\ell$
(C) $1/2\ mg\ell$ (D) $2mg\ell$

32. Two balls A and B having masses 1 kg and 2 kg, moving with speeds 21 m/s and 4 m/s, respectively, in opposite direction, collide head on. After collision, A moves with a speed of 1 m/s in the same direction, then the coefficient of restitution is
(A) 0.1 (B) 0.2
(C) 0.4 (D) None of these

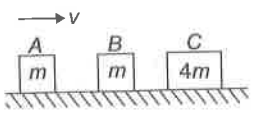
33. A truck moving on horizontal road east with velocity 20 ms^{-1} collides elastically with a light ball moving with velocity 25 ms^{-1} along west. The velocity of the ball just after collision is
(A) 65 ms^{-1} towards east
(B) 25 ms^{-1} towards west
(C) 65 ms^{-1} towards west
(D) 20 ms^{-1} towards east

34. A sphere of mass m moving with a constant velocity hits another stationary sphere of the same mass. If e is the coefficient of restitution, then the ratio of the speed of the first sphere to the speed of the second sphere after collision will be

(A) $\left(\frac{1-e}{1+e}\right)$ (B) $\left(\frac{1+e}{1-e}\right)$

(C) $\left(\frac{e+1}{e-1}\right)$ (D) $\left(\frac{e-1}{e+1}\right)$

35. Three blocks are initially placed as shown in the figure. Block A has mass m and initial velocity v to the right. Block B with mass m and block C with mass $4m$ are both initially at rest. Neglect friction. All collisions are elastic. The final velocity of block A is



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(A) $0.60v$ to the left (B) $1.4v$ to the left
(C) v to the left (D) $0.4v$ to the left

36. Two billiard balls undergo a head-on collision. Ball 1 is twice as heavy as ball 2. Initially, ball 1 moves with a speed v towards ball 2 which is at rest. Immediately after the collision, ball 1 travels at a speed of $v/3$ in the same direction. What type of collision has occurred?
(A) Inelastic
(B) Elastic
(C) Completely inelastic
(D) Cannot be determined from the information given

37. A ball is dropped from a height h. As is bounces off the floor, its speed is 80 percent of what it was just before it hit the floor. The ball will then rise to a height of most nearly
(A) $0.80h$ (B) $0.75h$
(C) $0.64h$ (D) $0.50h$

38. A ball is thrown vertically downwards with velocity $\sqrt{2gh}$ from a height h. After colliding with the ground, it just reaches the starting point. The coefficient of restitution is
(A) $1/\sqrt{2}$ (B) $1/2$
(C) 1 (D) $\sqrt{2}$

39. A ball is dropped from height 5 m. The time after which ball stops rebounding if coefficient of restitution between ball and ground $e = 1/2$ is
(A) 1 s (B) 2 s
(C) 3 s (D) infinite

40. A neutron travelling with a velocity v and K.E. E collides perfectly elastically head on with the nucleus of an atom of mass number A at rest. The fraction of total energy retained by neutron is
(A) $\left(\frac{A-1}{A+1}\right)^2$ (B) $\left(\frac{A+1}{A-1}\right)^2$
(C) $\left(\frac{A-1}{A}\right)^2$ (D) $\left(\frac{A+1}{A}\right)^2$

41. An object of mass 5 kg and speed 10 ms^{-1} explodes into two pieces of equal mass. One piece comes to rest. The kinetic energy added to the system during the explosion is
(A) zero. (B) 50 J
(C) 250 J (D) 500 J

42. Two particles of same mass m moving with velocities u_1 and u_2 collide perfectly inelastically. The loss of energy would be
(A) $1/2\ m(u_1 - u_2)^2$ (B) $1/4\ m(u_1 - u_2)^2$
(C) $m(u_1 - u_2)^2$ (D) $2\ m(u_1 - u_1)^2$

43. A ball of mass m moving with a speed u undergoes a head-on elastic collision with a ball of mass nm initially at rest. The fraction of the incident energy transferred to the heavier ball is
(A) $\frac{n}{1+n}$ (B) $\frac{n}{(1+n)^2}$
(C) $\frac{2n}{(1+n)^2}$ (D) $\frac{4n}{(1+n)^2}$

44. Which of the following does not hold when two particles of masses m_1 and m_2 undergo elastic collision?
(A) When $m_1 = m_2$ and m_2 is stationary, there is maximum transfer of kinetic energy in head on collision.
(B) When $m_1 = m_2$ and m_2 is stationary, there is maximum transfer of momentum in head on collision.
(C) When $m_1 >> m_2$ and m_2 is stationary, after head on collision m_2 moves with twice the velocity of m_1.
(D) When the collision is oblique and $m_1 = m_2$ with m_2 stationary, after the collision the particle moves in opposite directions.

45. A metal ball hits a wall and does not rebound, whereas a rubber ball of the same mass on hitting the wall the same velocity rebounds back. It can be concluded that
(A) metal ball suffers greater change in momentum
(B) rubber ball suffers greater change in momentum
(C) the initial momentum of metal ball is greater than the initial momentum of rubber ball
(D) both suffer same change in momentum

46. Before a rubber ball bounces off from the floor, the ball is in contact with the floor for a fraction of second. Which of the following statements are correct?
(A) Conservation of energy is not valid during this period.
(B) Conservation of energy is valid during this period.

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III. COM of a solid hemisphere of radius $R = 4R/3\pi$ from the centre

IV. COM of a hemisphere shell of radius $R = R/2$ from the centre

Which statements are correct?

(A) I, II, IV (B) I, III, IV

(C) IV only (D) I, II only

5. If the external forces acting on a system have zero resultant, the centre of mass

(A) must not move

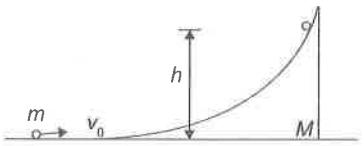
(B) must not accelerate

(C) may move

(D) may accelerate

Question No. 6 to 12

A particle of mass m moving horizontal with v_0 strikes a smooth wedge of mass M, as shown in the figure. After collision, the ball starts moving up the inclined face of the wedge and rises to a height h.



6. The final velocity of the wedge v_2 is

(A) $\frac{mv_0}{M}$ (B) $\frac{mv_0}{M+m}$

(C) v_0 (D) insufficient data

7. When the particle has risen to a height h on the wedge, then choose the correct alternative(s).

(A) The particle is stationary with respect to ground.

(B) Both are stationary with respect to the centre of mass.

(C) The kinetic energy of the centre of mass remains constant.

(D) The kinetic energy with respect to centre of mass is converted into potential energy.

8. The maximum height h attained by the particle is

(A) $\left(\frac{m}{m+M}\right)\frac{v_0^2}{2g}$ (B) $\left(\frac{m}{M}\right)\frac{v_0^2}{2g}$

(C) $\left(\frac{M}{m+M}\right)\frac{v_0^2}{2g}$ (D) none of these

9. Identify the correct statement(s) related to the situation when the particle starts moving downwards.

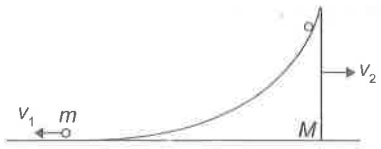
(A) The centre of mass of the system remains stationary.

(B) Both the particle and the wedge remain stationary with respect to centre of mass.

(C) When the particle reaches the horizontal surface its velocity relative to the wedge is v_0.

(D) None of these

10. Suppose the particle reaches the horizontal surfaces, its velocity with respect to ground is v_1 and that of wedge is v_2. Choose the correct statement(s)



(A) $mv_1 = Mv_2$ (B) $Mv_2 - mv_1 = mv_0$

(C) $v_1 + v_2 = v_0$ (D) $v_1 + v_2 < v_0$

11. Choose the correct statement(s) related to particle m.

(A) Its kinetic energy is $K_f = \left(\frac{mM}{m+M}\right)gh$.

(B) $v_1 = v_0\left(\frac{M-m}{M+m}\right)$

(C) The ratio of its final kinetic energy to its initial kinetic energy is $\frac{K_f}{K_i} = \left(\frac{M}{m+M}\right)^2$.

(D) It moves opposite to its initial direction of motion.

12. Choose the correct statement related to the wedge M

(A) Its kinetic energy is $K_f = \left(\frac{4m^2}{m+M}\right)gh$

(B) $v_2 = \left(\frac{2m}{m+M}\right)v_0$

(C) Its gain in kinetic energy is

$$\Delta K = \left(\frac{4mM}{(m+M)^2}\right)\left(\frac{1}{2}mv_0^2\right).$$

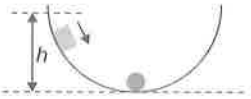
(D) Its velocity is more than that of the velocity of centre of mass.

13. Two blocks A (5 kg) and B (2 kg) attached to the ends of a spring constant 1120 N/m are placed on a

(C) As ball compressed kinetic energy is converted to compressed potential energy
(D) Both B and C

47. A ball of 0.1 kg strikes a wall at right angle with a speed of 6 m/s and rebounds along its original path at 4 m/s. The change in momentum in Newton-second is
(A) 10^3 (B) 10^2
(C) 10 (D) 1

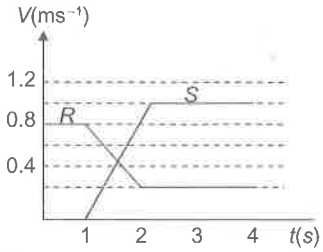
48. A block of mass m starts from rest and slides down a frictionless semi-circular track from a height h as shown in the figure. When it reaches the lowest point of the track, it collides with a stationary piece of putty also having mass m. If the block and the putty stick together and continue to slide, the maximum height that the block-putty system could reach is



(A) $h/4$ (B) $h/2$
(C) h (D) independent of h

49. Two billiard balls undergo a head-on collision. Ball 1 is twice as heavy as ball 2. Initially, ball 1 moves with a speed v towards ball 2 which is at rest. Immediately after the collision, ball 1 travels at a speed of $v/3$ in the same direction. What type of collision has occurred?
(A) Inelastic
(B) Elastic
(C) Completely inelastic
(D) Cannot be determined from the information given

50. The figure shows the velocity–time graph for two masses R and S that collided elastically. Which of the following statements is true?

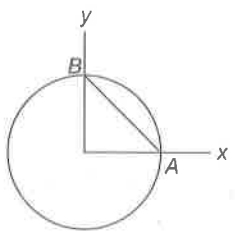


I. R and S moved in the same direction after the collision.
II. The velocities of R and S were equal at the mid time of the collision.
III. The mass of R was greater than mass of S.
(A) I only (B) II only
(C) I and II only (D) I, II and III

JEE Advanced

1. A body has its centre of mass at the origin. The x-coordinates of the particles
(A) may be all positive
(B) may be all negative
(C) may be all non-negative
(D) may be positive for some cases and negative in other cases

2. An object comprises a uniform ring of radius R and its uniform chord AB (not necessarily made of the same material) as shown in the figure. Which of the following cannot be the centre of mass of the object?

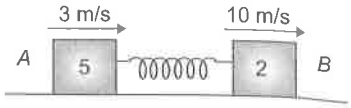


(A) $(R/3, R/3)$ (B) $(R/3, R/2)$
(C) $(R/4, R/4)$ (D) $\left(R/\sqrt{2}, R/\sqrt{2}\right)$

3. In which of the following cases, the centre of mass of a rod is certainly not at its centre?
(A) The density continuously increases from left to right
(B) The density continuously decreases from left to right
(C) The density decreases from left to right up to the centre and then increases
(D) The density increases from left to right up to the centre and then decreases

4. Consider following statements:
I. COM of a uniform semi-circular disc of radius $R = 2R/\pi$ from the centre
II. COM of a uniform semi-circular ring of radius $R = 4R/3\pi$ from the centre

smooth horizontal plane with the spring undeformed. Simultaneously velocities of 3 m/s and 10 m/s along the line of the spring in the same direction are imparted to A and B, then



(A) when the extension of the spring is maximum the velocities of A and B are zero.
(B) the maximum extension of the spring is 25 cm.
(C) maximum extension and maximum compression occur alternately.
(D) the maximum compression occur for the first time after $\frac{\pi}{56}$ s.

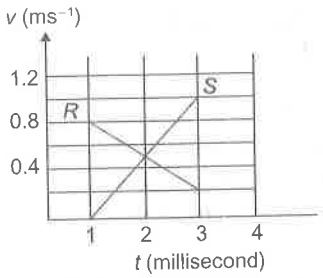
14. Two identical balls are interconnected with a massless and inextensible thread. The system is in gravity-free space with the thread just taut. Each ball is imparted a velocity v, one towards the other ball and the other perpendicular to the first, at $t = 0$. Then,
(A) the thread will become taut at $t = (L/v)$
(B) the thread will become taut at some time $t < (L/v)$
(C) the thread will always remain taut for $t > (L/v)$
(D) the kinetic energy of the system will always remain mv^2

15. A ball moving with a velocity v hits a massive wall moving towards the ball with a velocity u. An elastic impact lasts for a time Δt.
(A) The average elastic force acting on the ball is $\frac{m(u+v)}{\Delta t}$.
(B) The average elastic force acting on the ball is $\frac{2m(u+v)}{\Delta t}$.
(C) The kinetic energy of the ball increases by $2mu(u+v)$.
(D) The kinetic energy of the ball remains the same after the collision.

16. A particle moving with kinetic energy = 3 J makes an elastic head on collision with a stationary particle when has twice its mass during the impact.
(A) The minimum kinetic energy of the system is 1 J.
(B) The maximum elastic potential energy of the system is 2 J.
(C) Momentum and total kinetic energy of the system are conserved at every instant.
(D) The ratio of kinetic energy to potential energy of the system first decreases and then increases.

17. Two balls A and B having masses 1 kg and 2 kg, moving with speeds 21 m/s and 4 m/s, respectively, in opposite direction, collide head on. After collision, A moves with a speed of 1 m/s in the same direction, then correct statements is
(A) The velocity of B after collision is 6 m/s opposite to its direction of motion before collision.
(B) The coefficient of restitution is 0.2.
(C) The loss of kinetic energy due to collision is 200 J.
(D) The impulse of the force between the two balls is 40 Ns.

18. The figure shows the velocity–time graph for two masses R and S that collided elastically. Which of the following statements are true?



I. R and S moved in the same direction after the collision.
II. Kinetic energy of the system (R and S) is minimum at $t = 2$ millisecond.
III. The mass of R was greater than mass of S.
(A) I only (B) II only
(C) I and II only (D) I, II and III

19. In an inelastic collision,
(A) the velocity of both the particles may be same after collision.
(B) kinetic energy is not conserved.
(C) linear momentum of the system is conserved.
(D) velocity of separation will be less than velocity of approach.

20. In a one-dimensional collision between two particles, their relative velocity is $\vec{v}_1$ before the collision and $\vec{v}_2$ after the collision.

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(A) $\vec{v}_1 = \vec{v}_2$ if the collision is elastic

(B) $\vec{v}_1 = -\vec{v}_2$ if the collision is elastic

(C) $|\vec{v}_2| = |\vec{v}_1|$ in all cases

(D) $\vec{v}_1 = -k\vec{v}_2$ in all cases, where $k \geq 1$

21. A set of n-identical cubical blocks lie at rest parallel to each other along a line on a smooth horizontal surface. The separation between the near surfaces of any two adjacent blocks is L. The block at one end is given a speed V towards the next one at time $t = 0$. All collisions are completely inelastic, then

(A) the last block starts moving at $t = n(n-1)\dfrac{L}{2V}$

(B) the last block starts moving at $t = (n-1)\dfrac{L}{V}$

(C) the centre of mass of the system will have a final speed V/n

(D) the centre of mass of the system will have a final speed V

22. An isolated rail car original moving with speed v_0 on a straight, frictionless, level track contains a large amount of sand. A release value on the bottom of the car malfunctions, and sand begins to pour out straight down relative to the rail car.

(a) Is momentum conserved in this process?

(A) The momentum of the rail car along is conserved.

(B) The momentum of the rail car + sand remaining within the car is conserved.

(C) The momentum of the rail car + all of the sand, both inside and outside the rail car, is conserved.

(D) None of the three previous systems have momentum conservation.

(b) What happens to the speed of the rail car as the sand pours out?

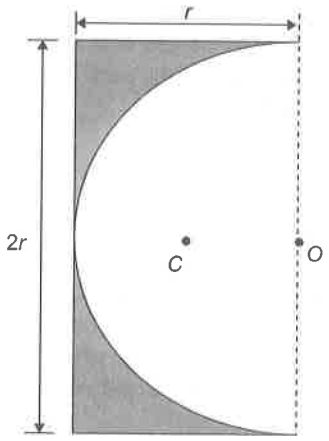
(A) The car begins to roll faster.

(B) The car maintains the same speed.

(C) The car begins to slow down.

(D) The problem cannot be solved since momentum is not conserved.

23. A semi-circular portion of radius 'r' is cut from a uniform rectangular plate as shown in the figure. The distance of centre of mass 'C' of remaining plate, from point 'O' is



(A) $\dfrac{2r}{(3-\pi)}$ (B) $\dfrac{3r}{2(4-\pi)}$

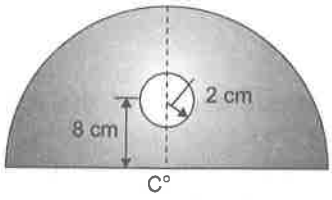
(C) $\dfrac{2r}{(4+\pi)}$ (D) $\dfrac{2r}{3(4-\pi)}$

24. From a circle of radius a, an isosceles right angled triangle with the hypotenuse as the diameter of the circle is removed. The distance of the centre of gravity of the remaining position from the centre of the circle is

(A) $3(\pi - 1)a$ (B) $\dfrac{(\pi-1)a}{6}$

(C) $\dfrac{a}{3(\pi-1)}$ (D) $\dfrac{a}{3(\pi+1)}$

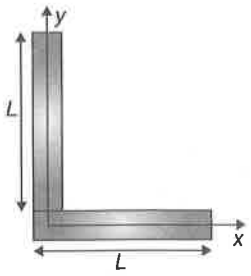
25. In the figure, a hole of radius 2 cm is made in semi-circular disc of radius 6π at a distance 8 cm from the centre C of the disc. The distance of the centre of mass of this system from point C is



(A) 4 cm (B) 8 cm

(C) 6 cm (D) 12 cm

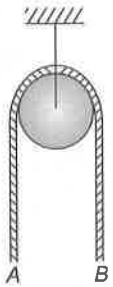
26. The centre of mass of two thin uniform rods of the same length but made up of different materials and kept as shown, can be, if the meeting point is the origin of coordinates



(A) $(L/2, L/2)$ (B) $(2L/3, L/2)$
(C) $(L/3, L/3)$ (D) $(L/3, L/6)$

Question No. 27 to 28

A uniform chain of length $2L$ is hanging in equilibrium position, if end B is given a slightly downward displacement the imbalance causes an acceleration. Here pulley is small and smooth and string is inextensible



27. The acceleration of end B when it has been displaced by distance x is

(A) $\frac{x}{L}g$ (B) $\frac{2x}{L}g$

(C) $\frac{x}{2}g$ (D) g

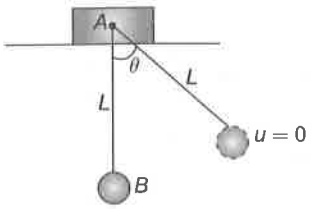
28. The velocity v of the string when it slips out of the pulley (height of pulley from floor $> 2L$) is

(A) $\sqrt{\frac{gL}{2}}$ (B) $\sqrt{2gL}$

(C) $\sqrt{gL}$ (D) none of these

Question No. 29 to 32

A small ball B of mass m is suspended with light inelastic string of length L from a block A of same mass m which can move on smooth horizontal surface as shown in the figure. The ball is displaced by angle θ from equilibrium position and then released.



29. The displacement of block when ball reaches the equilibrium position is

(A) $\frac{L\sin\theta}{2}$ (B) $L\sin\theta$

(C) L (D) none of these

30. Tension in string when it is vertical is

(A) mg (B) $mg(2 - \cos\theta)$
(C) $mg(3 - 2\cos\theta)$ (D) none of these

31. The maximum velocity of block during subsequent motion of the system after release of ball is

(A) $[gl(1 - \cos\theta)]^{1/2}$
(B) $[2gl(1 - \cos\theta)]^{1/2}$
(C) $[gl\cos q]^{1/2}$
(D) information are insufficient to decide

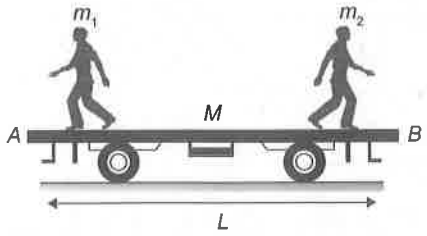
32. The displacement of centre of mass of $A + B$ system till the string becomes vertical is

(A) zero (B) $\frac{L}{2}(1-\cos\theta)$

(C) $\frac{L}{2}(1-\sin\theta)$ (D) none of these

Question No. 33 to 37

Two persons of masses m_1 and m_2 are standing at the two ends A and B, respectively, of a trolley of mass M as shown.



33. When the person standing at A jumps from the trolley towards left with u_{rel} with respect to the trolley, then

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(A) the trolley moves towards right

(B) the trolley rebounds with velocity $\frac{m_1 u_{rel}}{m_1 + m_2 + M}$

(C) the centre of mass of the system remains stationary

(D) all of these

34. When only the person standing at B jumps from the trolley towards right while the person at A keeps standing, then

(A) the trolley moves towards left

(B) the trolley moves with velocity $\frac{m_2 u_{rel}}{m_1 + m_2 + M}$

(C) the centre of mass of the system remains stationary

(D) all of these

35. When both the persons jump simultaneously with same speed, then

(A) the centre of mass of the system remains stationary

(B) the trolley remains stationary

(C) the trolley moves towards the end where the person with heavier mass is standing

(D) none of these

36. When both the persons jump simultaneously with u_{rel} with respect to the trolley, then the velocity of the trolley is

(A) $\frac{|m_1 - m_2| u_{rel}}{m_1 + m_2 + M}$ (B) $\frac{|m_1 - m_2| u_{rel}}{M}$

(C) $\left| \frac{m_1 u_{rel}}{m_2 + M} - \frac{m_2 u_{rel}}{m_1 + M} \right|$ (D) none of these

37. Choose the incorrect statement, if $m_1 = m_2 = m$ and both the persons jump one by one, then

(A) the centre of mass of the system remains stationary

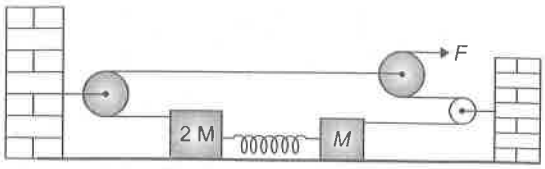
(B) the final velocity of the trolley is in the direction of the person who jumps first

(C) the final velocity of the trolley is

$$\left(\frac{mu_{rel}}{M + m} - \frac{mu_{rel}}{M + 2m} \right)$$

(D) none of these

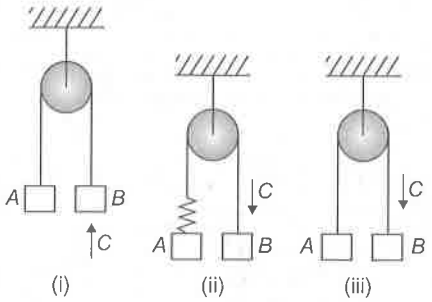
38. In the figure, there is no friction at any contact surface. Initially, the spring has no deformation. What will be the maximum deformation in the spring? Consider all the strings to be sufficiency large. Let the spring constant to be K.



(A) $4F/3K$ (B) $8F/3K$

(C) $F/3K$ (D) None of these

39. In the figures (i), (ii) amd (iii), the objects A, B and C are of same mass. String, spring and pulley are massless. C strikes B with velocity 'u' in each case and sticks to it. The ratio of velocity of B in case (i) to (ii) to (iii) is



(A) 1:1:1 (B) 3:3:2

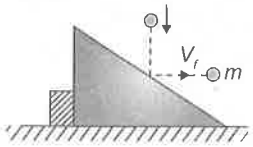
(C) 3:2:2 (D) none of these

40. A parallel beam of particles of mass m moving with velocity v impinges on a wall at an angle θ to its normal. The number of particles per unit volume in the beam is n. If the collision of particles with the wall is elastic, then the pressure exerted by this beam on the wall is

(A) $2\,mn\,v^2 \cos\theta$ (B) $2mn\,v^2 \cos^2\theta$

(C) $2\,mn\,v \cos\theta$ (D) $2mn\,v \cos^2\theta$

41. As shown in the figure, a body of mass m moving vertically with speed 3 m/s hits a smooth fixed inclined plane and rebounds with a velocity v_f in the horizontal direction. If the angle of inclined is 30°, the velocity v_f will be

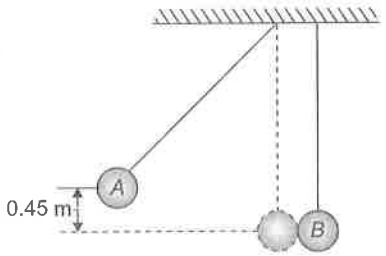


(A) 3 m/s (B) $\sqrt{3}$ m/s

(C) $1/\sqrt{3}$ m/s (D) this is not possible

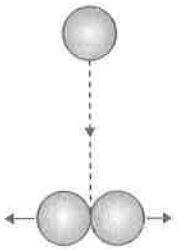
42. Two massless strings of length 5 m hang from the ceiling very near to each other as shown in the figure. Two balls A and B of masses 0.25 kg and 0.5 kg are

attached to the string. Ball A is released from rest at a height of 0.45 m as shown in the figure. The collision between two balls is completely elastic immediately after the collision. The kinetic energy of ball B is 1 J. The velocity of ball A just after the collision is



(A) 5 ms^{-1} to the right
(B) 5 ms^{-1} to the left
(C) 1 ms^{-1} to the right
(D) 1 ms^{-1} to the left

43. In the figure, the two identical balls of mass M and radius R each are placed in contact with each other on the frictionless horizontal surface. The third ball of mass M and radius $R/2$ is coming down vertically and has a velocity $= v_0$ when it simultaneously hits the two balls and itself comes to rest. Then each of the two bigger balls will move after collision with a speed equal to



(A) $4v_0/\sqrt{5}$ (B) $2v_0/\sqrt{5}$
(C) $v_0/\sqrt{5}$ (D) none of these

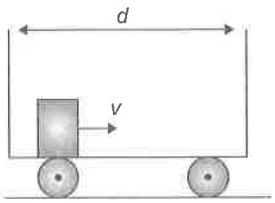
44. In the above, suppose that the smaller ball does not stop after collision, but continues to move downwards with a speed $= v_0/2$, after the collision. Then, the speed of each bigger ball after collision is

(A) $4v_0/\sqrt{5}$ (B) $2v_0/\sqrt{5}$
(C) $v_0/2\sqrt{5}$ (D) none of these

45. A body of mass 'm' is dropped from a height of 'h'. Simultaneously another body of mass $2m$ is thrown up vertically with a velocity v such that they collide at a height $h/2$. If the collision is perfectly inelastic, the velocity at the time of collision with the ground will be

(A) $\sqrt{\frac{5gh}{4}}$ (B) $\sqrt{gh}$
(C) $\sqrt{\frac{gh}{4}}$ (D) $\sqrt{\frac{10gh}{3}}$

46. In a smooth stationary cart of length d, a small block is projected along its length with velocity v towards front. Coefficient of restitution for each collision is e. The cart rests on a smooth ground and can move freely. The time taken by block to come to rest w.r.t. cart is



(A) $\frac{ed}{(1-e)v}$ (B) $\frac{ed}{(1+e)v}$
(C) $\frac{d}{e}$ (D) infinite

47. A ball is projected from ground with a velocity v at an angle θ to the vertical. On its path, it makes an elastic collision with a vertical wall and returns to ground. The total time of flight of the ball is

(A) $\frac{2v\sin\theta}{g}$ (B) $\frac{2v\cos\theta}{g}$
(C) $\frac{v\sin 2\theta}{g}$ (D) $\frac{v\cos\theta}{g}$

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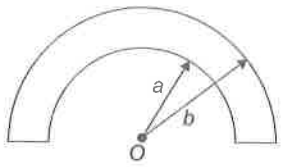
JEE Advanced

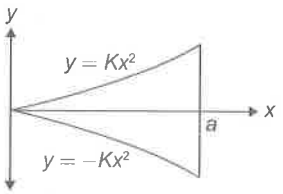
Level I

1. The mass of an uniform ladder of length l increases uniformly from one end A to the other end B.
   (A) Form an expression for linear mass density as a function of distance x from end A where linear mass density is λ_0. The density at one end being twice that of the other end.
   (B) Find the position of the centre of mass from end A.

2. Find the distance of centre of mass of a uniform plate having semi-circular inner and other boundaries of radii a and b from the centre O.

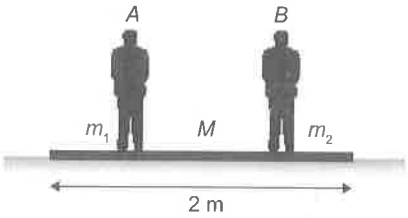


3. A thin sheet of metal of uniform thickness is cut into the shape bounded by the lines $x = a$ and $y = \pm Kx^2$, as shown. Find the coordinates of the centre of mass.



4. Two balls of equal masses are projected upwards simultaneously, one from the ground with speed 50 m/s and the other from a 40 m high tower with an initial speed 30 m/s. Find the maximum height attained by their centre of mass.

5. In the figure, when the persons A and B exchange their positions, then

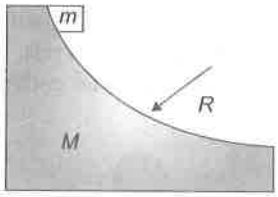


$m_1 = 50$ kg, $m_2 = 70$ kg, $M = 80$ kg

(A) the distance moved by the centre of mass of the system is _____.
(B) the plank moves towards _____.
(C) the distance moved by the plank is _____.
(D) the distance moved by A with respect to ground is _____.
(E) the distance moved by B with respect to ground is _____.

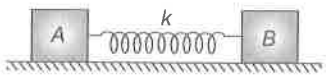
6. In the arrangement, $m_A = 2$ kg and $m_B = 1$ kg. String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.

7. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M. M rests on a table and both blocks move without friction. The blocks initially are at rest and m starts from the top of the path. Find the velocity v of the cube as it leaves the block.



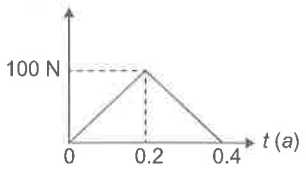
8. A (trolley + child) of total mass 200 kg is moving with a uniform speed of 36 km/h on a frictionless track. The child of mass 20 kg starts running on the trolley from one end to the other (10 m away) with a speed of 10 ms^{-1} relative to the trolley in the direction of the trolley's motion and jumps out of the trolley with the same relative velocity. What is the final speed of the trolley? How much has the trolley moved from the time the child begins to run?

9. In the figure, the spring is compressed by 'x_0' and released. Two blocks 'A' and 'B' of masses 'm' and '$2m$', respectively, are attached at the ends of the spring. Blocks are kept on a smooth horizontal surface and released.

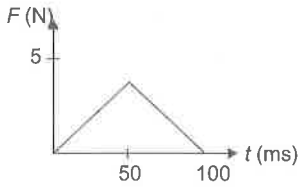


(A) Find the speed of block A by the time compression of the spring is reduced to $x_0/2$.
(B) Find the work done by the spring on 'A' by the time compression of the spring reduced to $x_0/2$.

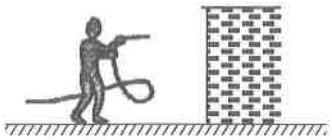
10. The figure shows the force versus time graph for a particle.
(A) Find the change in momentum Δp of the particle.
(B) Find the average force acting on the particle.



11. A force F acts on an object (mass = 1 kg) which is initially at rest as shown in the figure. Draw the graph showing the momentum of the object varying during the time for which the force acts.



12. A man hosing down his driveway hits the wall by mistake. Knowing that the velocity of the stream is 25 m/s and the cross-sectional area of the stream is 300 mm^2, determine the force exerted on the wall. Assume that stream strikes wall horizontally and after striking the wall, stream comes to rest. Also find the pressure exerted on the wall by stream.



13. A bullet of mass m strikes an obstruction and deviates off at 60° to its original direction. If its speed is also changed from u to v, find the magnitude of the impulse acting on the bullet.

14. A neutrino initially at rest, decays into a proton, an electron and an anti-neutrino. The ejected electron has a momentum of $p_1 = 1.4 \times 10^{-28}$ kg-m/s and the anti-neutrino has $p_2 = 6.5 \times 10^{-27}$ kg-m/s. Find the recoil speed of the proton
(A) if the electron and the anti-neutrino are ejected along the same direction and
(B) if they are ejected along perpendicular direction, the mass of the proton $m_p = 1.67 \times 10^{-27}$ kg.

15. A steel ball of mass 0.5 kg is dropped from a height of 4 m on to a horizontal heavy steel slab. The collision is elastic and the ball rebounds to its original height.
(A) Calculate the impulse delivered to the ball during impact.
(B) If the ball is in contact with the slab for 0.002 s, find the average reaction force on the ball during impact.

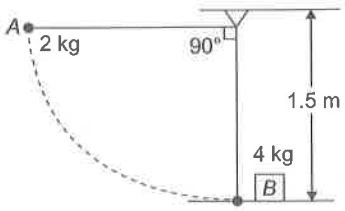
16. A particle A of mass 2 kg lies on the edge of a table of height 1 m. It is connected by a light inelastic string of length 0.7 m to a second particle B of mass 3 kg which is lying on the table 0.25 m from the edge (line joining A and B is perpendicular to the edge). If A is pushed gently so that it starts falling from the table, then find the speed of B when it starts to move. Also find the impulsive tension in the string at that moment.

17. Two particles, each of mass m, are connected by a light inextensible string of length $2l$. Initially, they lie on a smooth horizontal table at points A and B distant l apart. The particle at A is projected across the table with velocity u. Find the speed with which the second particle begins to move if the direction of u is (A) along BA, (B) at an angle of 120° with AB, (C) perpendicular to AB. In each case, calculate (in terms of m and u) the impulsive tension in the string.

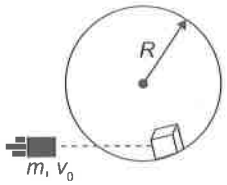
18. Two particles P of mass $2m$ and Q of mass m are subjected to mutual force of attraction and no other act on them. At time $t = 0$, P is at rest at a fixed O and Q is directly moving away from O with a speed $5u$. At a later instant when $t = T$ before any collision has taken place Q is moving towards O with speed u.
(A) Find in terms of m and u the total work done by the forces of attraction during the time interval $0 \le t \le T$.
(B) At the instant = T, impulses of magnitude J and K and applied to P and Q bringing them to rest. Find the values of J and K.

19. A block of mass m moving with a velocity v, enters a region where it starts colliding with the stationary dust

27. Sphere A is released from rest in the position shown in the figure and strikes block B which is at rest. If $e = 0.75$ between A and B and $\mu_k = 0.5$ between B and the support, determine
(A) the velocity of A just after the impact,
(B) the maximum displacement of B after the impact.

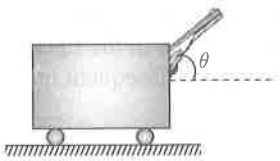


28. A small block of mass $2m$ initially rests at the bottom of a fixed circular, vertical track, which has a radius of R. The contact surface between the mass and the loop is frictionless. A bullet of mass m strikes the block horizontally with initial speed v_0 and remain embedded in the block as the block and the bullet circle the loop. Determine each of the following in terms of m, v_0, R and g.



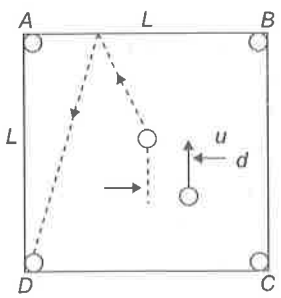
(A) The speed of the masses immediately after the impact.
(B) The minimum initial speed of the bullet if the block and the bullet are to successfully execute a complete ride on the loop.

29. A cart of total mass M_0 is at rest on a rough horizontal road. It ejects bullets at a rate of λ kg/s at an angle θ with the horizontal and at velocity 'u' (constant) relative to the cart. The coefficient of friction between the cart and the ground is μ. Find the velocity of the cart in terms of time 't'. The cart moves with sliding.



Level II

1. In a game of Carom Board, the Queen (a wooden disc of radius 2 cm and mass 50 g) is placed at the exact centre of the horizontal board. The striker is a smooth plastic disc of radius 3 cm and mass 100 g. The board is frictionless. The striker is given an initial velocity 'u' parallel to the sides BC or AD so that it hits the Queen inelastically with the same coefficient of restitution = 2/3. The impact parameter for the collision is 'd' (shown in the figure). The Queen rebounds from the edge AB of the board inelastically with the same coefficient of restitution = 2/3 and enters the hole D following the dotted path shown. The side of the board is L. Find the value of impact parameter 'd' and the time which the Queen takes to enter hole D after collision with the striker.



2. A flexible chain has a length l and mass m. It is lowered on the table top with constant velocity v. Find the force that the chain exerts on the table as a function of time.

3. A 24-kg projectile is fired at an angle of 53° above the horizontal with an initial speed of 50 m/s. At the highest point in its trajectory, the projectile explodes into two fragments of equal mass, the first of which falls vertically with zero initial speed.
(A) How far from the point of firing does the second fragment strike the ground? (Assume the ground is level.)
(B) How much energy was released during the explosion?

4. A particle is projected from point O on level ground towards a smooth vertical wall 50 m from O and hits the wall. The initial velocity of the particle is 30 m/s at 45° to the horizontal and the coefficient of restitution between the particle and the wall is e. Find the distance from O of the point at which the particle hits the ground again if (A) $e = 0$, (B) $e = 1$ and (C) $e = 1/2$.

5. A massive vertical wall is approaching a man at a speed u. When it is at a distance of 10 m, the man throws a ball with speed 10 m/s at an angle of 37° which after

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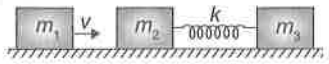
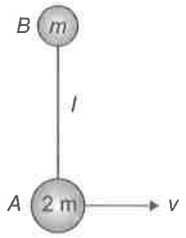
completely elastic rebound reaches back directly into his bands. Find the velocity u of the wall.

6. Mass m_1 hits and sticks with m_2 while sliding horizontally with velocity v along the common line of centres of the three equal masses ($m_1 = m_2 = m_3 = m$). Initially, masses m_2 and m_3 are stationary and the spring is unstretched. Find

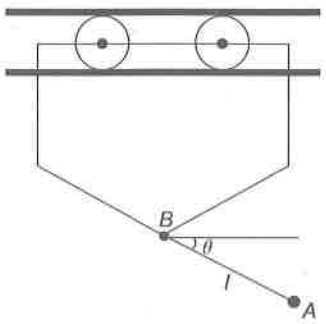


(A) the velocities of m_1, m_2 and m_3 immediately after impact,
(B) the maximum kinetic energy of m_3,
(C) the minimum kinetic energy of m_2,
(D) the maximum compression of the spring.

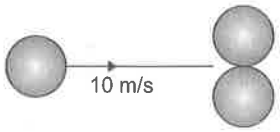
7. Two masses A and B connected with an inextensible string of length l lie on a smooth horizontal plane. A is given a velocity of v m/s along the ground perpendicular to the line AB as shown in the figure. Find the tension in string during their subsequent motion.



8. The simple pendulum A of mass m_A and length l is suspended from the trolley B of mass m_B. If the system is released from rest at $\theta = 0$, determine the velocity v_B of the trolley and tension in the string when $\theta = 90°$. Friction is negligible.

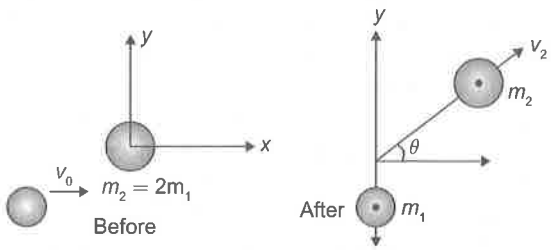


9. A ball with initial speed 10 m/s collides elastically with two other identical ball whose centres are on a line perpendicular to the initial velocity and which are initially in contact with each other. All the three balls are lying on a smooth horizontal table. The first ball is aimed directly at the contact point of the other two balls. All the balls are smooth. Find the velocities of the three balls after the collision.



10. A mass m_1 with initial speed v_0 in the positive x-direction collides with a mass $m_2 = 2m_1$ which is initially at rest at the origin, as shown in the figure. After the collision, m_1 moves off with speed $v_1 = v_0/2$ in the negative y-direction, and m_2 moves off with speed v_2 at angle θ.

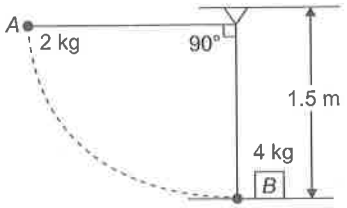
(A) Find the velocity (magnitude and direction) of the centre of mass before the collision, as well as its velocity after the collision.
(B) Write down the x- and y-components of the equation of conservation of momentum for the collision.
(C) Determine $\tan\theta$, and find v_2 in terms of v_0.
(D) Determine how much (if any) energy was gained or lost in the collision, and state whether the collision was elastic or inelastic.



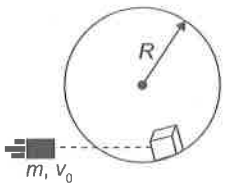
11. A cart is moving along $+x$ direction with a velocity of 4 m/s. A person in the cart throws a stone with a velocity of 6 m/s relative to himself. In the frame of reference of the cart, the stone is thrown in the yz plane making an angle of 30° with the vertical z-axis. At the highest point of its trajectory, the stone hits an object of equal mass hung vertically from branch of a tree by means of a string of length L. A completely inelastic collision occurs, in which the stone gets embedded in the object. Determine

(A) the speed of the combined mass immediately after collision with respect to an observer on the ground.

27. Sphere A is released from rest in the position shown in the figure and strikes block B which is at rest. If $e = 0.75$ between A and B and $\mu_k = 0.5$ between B and the support, determine
(A) the velocity of A just after the impact,
(B) the maximum displacement of B after the impact.

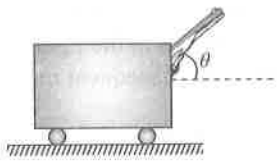


28. A small block of mass $2m$ initially rests at the bottom of a fixed circular, vertical track, which has a radius of R. The contact surface between the mass and the loop is frictionless. A bullet of mass m strikes the block horizontally with initial speed v_0 and remain embedded in the block as the block and the bullet circle the loop. Determine each of the following in terms of m, v_0, R and g.



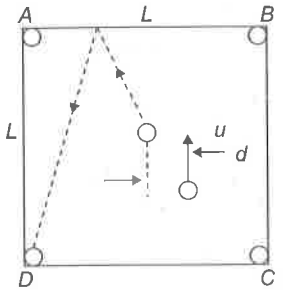
(A) The speed of the masses immediately after the impact.
(B) The minimum initial speed of the bullet if the block and the bullet are to successfully execute a complete ride on the loop.

29. A cart of total mass M_0 is at rest on a rough horizontal road. It ejects bullets at a rate of λ kg/s at an angle θ with the horizontal and at velocity 'u' (constant) relative to the cart. The coefficient of friction between the cart and the ground is μ. Find the velocity of the cart in terms of time 't'. The cart moves with sliding.



Level II

1. In a game of Carom Board, the Queen (a wooden disc of radius 2 cm and mass 50 g) is placed at the exact centre of the horizontal board. The striker is a smooth plastic disc of radius 3 cm and mass 100 g. The board is frictionless. The striker is given an initial velocity 'u' parallel to the sides BC or AD so that it hits the Queen inelastically with the same coefficient of restitution = 2/3. The impact parameter for the collision is 'd' (shown in the figure). The Queen rebounds from the edge AB of the board inelastically with the same coefficient of restitution = 2/3 and enters the hole D following the dotted path shown. The side of the board is L. Find the value of impact parameter 'd' and the time which the Queen takes to enter hole D after collision with the striker.



2. A flexible chain has a length l and mass m. It is lowered on the table top with constant velocity v. Find the force that the chain exerts on the table as a function of time.

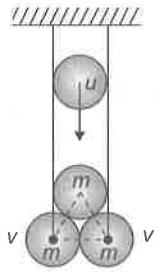
3. A 24-kg projectile is fired at an angle of 53° above the horizontal with an initial speed of 50 m/s. At the highest point in its trajectory, the projectile explodes into two fragments of equal mass, the first of which falls vertically with zero initial speed.
(A) How far from the point of firing does the second fragment strike the ground? (Assume the ground is level.)
(B) How much energy was released during the explosion?

4. A particle is projected from point O on level ground towards a smooth vertical wall 50 m from O and hits the wall. The initial velocity of the particle is 30 m/s at 45° to the horizontal and the coefficient of restitution between the particle and the wall is e. Find the distance from O of the point at which the particle hits the ground again if (A) $e = 0$, (B) $e = 1$ and (C) $e = 1/2$.

5. A massive vertical wall is approaching a man at a speed u. When it is at a distance of 10 m, the man throws a ball with speed 10 m/s at an angle of 37° which after

(B) the length L of the string such that tension in the string becomes zero when the string becomes horizontal during the subsequent motion of the combined mass.

12. Two equal spheres of mass 'm' are suspended by vertical strings so that they are in contact with their centres at same level. A third equal sphere of mass m falls vertically and strikes the other two simultaneously so that their centres at the instant of impact form an equilateral triangle in a vertical plane. If u is the velocity of m just before impact, find the velocities just after impact and the impulsive tension of the strings.



Previous Year Questions

JEE Main

1. Two identical particles move towards each other with velocity $2v$ and v, respectively. The velocity of centre of mass is **[AIEEE 2002]**

(A) v (B) $\frac{v}{3}$

(C) $\frac{v}{2}$ (D) zero

2. Consider the following two statements:
I. Linear momentum of a system of particles is zero.
II. Kinetic energy of a system of particles is zero.
Then **[AIEEE 2003]**
(A) A does not imply B and B does not imply A
(B) A implies B but B does not imply A
(C) A does not imply B but B implies A
(D) A implies B and B implies A

3. A body A of mass M while falling vertically downwards under gravity breaks into two parts; a body B of mass $\frac{1}{3}M$ and a body C of mass $\frac{2}{3}M$. The centre of mass of bodies B and C taken together shifts compared to that of body A towards **[AIEEE 2005]**
(A) depends on height of breaking
(B) does not shift
(C) body C
(D) body B

4. A mass m moves with a velocity v and collides inelastically with another identical mass. After collision, the first mass moves with velocity $\frac{v}{\sqrt{3}}$ in a direction perpendicular to the initial direction of motion. Find the speed of the second mass after collision. **[AIEEE 2005]**

(A) v (B) $\sqrt{3}v$

(C) $\frac{2}{\sqrt{3}}v$ (D) $\frac{v}{\sqrt{3}}$

5. A player caught a cricket ball of mass 150 g moving at a rate of 20 m/s. If the catching process is completed in 0.1 s, the force of the blow exerted by the ball on the hand of the player is equal to **[AIEEE 2006]**
(A) 150 N (B) 3 N
(C) 30 N (D) 300 N

6. Consider a two-particle system with particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved, so as to keep the centre of mass at the same position? **[AIEEE 2006]**

(A) $\frac{m_2}{m_1}d$ (B) $\frac{m_1}{m_1+m_2}d$

(C) $\frac{m_1}{m_2}d$ (D) d

7. A bomb of mass 16 kg at rest explodes into two pieces of masses 4 kg and 12 kg. The velocity of the 12 kg

mass is 4 ms^{-1}. The kinetic energy of the other mass is **[AIEEE 2006]**

(A) 144 J (B) 288 J
(C) 192 J (D) 96 J

8. A player caught a cricket ball of mass 150 g moving at a rate of 20 m/s. If the catching process is completed in 0.1 s, the force of the blow exerted by the ball on the hand of the player is equal to **[AIEEE 2006]**

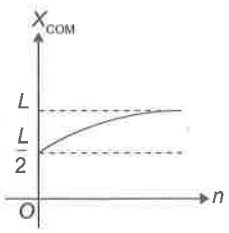
(A) 150 N (B) 3 N
(C) 30 N (D) 300 N

9. A circular disc of radius R is removed from a bigger circular disc of radius $2R$, such that the circumference of the discs coincide. The centre of mass of the new disc is αR from the centre of the bigger disc. The value of α is **[AIEEE 2007]**

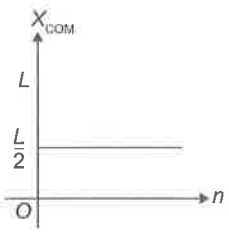
(A) $\frac{1}{3}$ (B) $\frac{1}{2}$
(C) $\frac{1}{6}$ (D) $\frac{1}{4}$

10. A thin rod of length L is lying along the x-axis with its ends at $x = 0$ and $x = L$. Its linear density (mass/length) varies with x as $k\left(\frac{x}{L}\right)$ where n can be zero or any positive number. If the position x_{COM} of the centre of mass of the rod is plotted against n, which of the following graphs best approximates the dependence of x_{COM} on n? **[AIEEE 2008]**

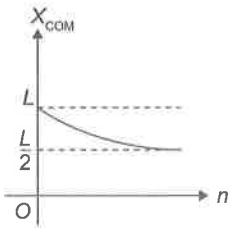
(A)



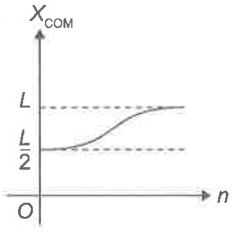
(B)



(C)



(D)



11. A block of mass 0.50 kg is moving with a speed of 2.00 ms^{-1} on a smooth surface. It strikes another mass of 1.00 kg and then they move together as a single body. The energy loss during the collision is **[AIEEE 2008]**

(A) 0.16 J (B) 1.00 J
(C) 0.67 J (D) 0.34 J

12. **Statement I:** Two particles moving in the same direction do not lose all their energy in a completely inelastic collision.
Statement II: Principle of conservation of momentum holds true for all kinds of collisions. **[AIEEE 2010]**

(A) Statement I is true, Statement II is true; Statement II is the correct explanation of Statement I.
(B) Statement I is true, Statement II is true; Statement II is not correct explanation of Statement I.
(C) Statement I is false, Statement II is true.
(D) Statement I is true, Statement II is false.

13. **Statement I:** A point particle of mass m moving with speed v collides with stationary point particle of mass M. If the maximum energy loss possible is given as $f\left(\frac{1}{2}mv^2\right)$, then $f = \left(\frac{m}{M+m}\right)$.

Statement II: Maximum energy loss occurs when the particles get stuck together as a result of the collision. **[JEE Main 2013]**

(A) Statement I is true, Statement II is false.
(B) Statement I is false, Statement II is true.
(C) Statement I is true, Statement II is a correct explanation of Statement I.
(D) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I.

14. Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h, then z_0 is equal to **[JEE Main 2015]**

(A) $\frac{5h}{8}$ (B) $\frac{3h^2}{8R}$
(C) $\frac{h^2}{4R}$ (D) $\frac{3h}{4}$

15. A particle of mass m moving in the x direction with speed $2v$ is hit by another particle of mass $2m$ moving in the y direction with speed v if the collision is perfectly inelastic, the percentage loss in the energy during the collision is close to **[JEE Main 2015]**

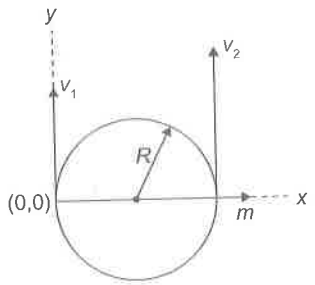
(A) 56% (B) 62%
(C) 44% (D) 50%

JEE Advanced

1. Two blocks of masses 10 kg and 4 kg are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of 14 m/s to the heavier block in the direction of the lighter block. The velocity of the centre of mass is **[IIT(Scr.) 2002]**

(A) 30 m/s (B) 20 m/s
(C) 10 m/s (D) 5 m/s

2. A particle of mass m, moving in a circular path of radius R with a constant speed is located at point $(2R, 0)$ at time $t = 0$ and a man starts moving with velocity v_1 along the +ve y-axis from origin at time $t = 0$. Calculate the linear momentum of the particle w.r.t. the man as a function of time. **[JEE 2003]**



3. **Statement 1**
In an elastic collision between two bodies, the relative speed of the bodies after collision is equal to the relative speed before the collision.
Because

Statement 2
In an elastic collision, the linear momentum of the system is conserved.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

4. The balls, having linear momenta $\vec{p}_1 = p\hat{i}$ and $\vec{p}_2 = -p\hat{i}$, undergo a collision in free space. There is no external force acting on the balls. Let $\vec{p}_1'$ and $\vec{p}_2'$ be their final momenta. The following option(s) is (are) NOT allowed for any non-zero value of p, a_1, a_2, b_1, b_2, c_1 and c_2. **[JEE 2008]**

(A) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$; $\vec{p}_2' = a_2\hat{i} + b_2\hat{j}$

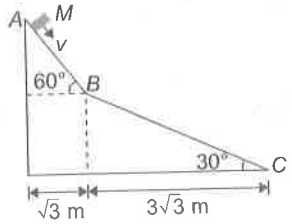
(B) $\vec{p}_1' = c_1\hat{k}$; $\vec{p}_2' = c_2\hat{k}$

(C) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$; $\vec{p}_2' = a_2\hat{i} + b_2\hat{j} - c_1\hat{k}$

(D) $\vec{p}_1' = a_1\hat{i} + b_1\hat{j}$; $\vec{p}_2' = a_2\hat{i} + b_1\hat{j}$

Question No. 5 to 7

A small block of mass M moves on a frictionless surface of an inclined plane, as shown in the figure. The angle of the incline suddenly changes from 60° to 30° at point B. The block is initially at rest at A. Assume that collisions between the block and the incline are totally inelastic ($g = 10$ m/s^2). **[JEE 2008]**



5. The speed of the block at point B immediately after it strikes the second incline is

(A) $\sqrt{60}$ m/s (B) $\sqrt{45}$ m/s
(C) $\sqrt{30}$ m/s (D) $\sqrt{15}$ m/s

6. The speed of the block at point C, immediately before it leaves the second incline is

(A) $\sqrt{120}$ m/s (B) $\sqrt{105}$ m/s
(C) $\sqrt{90}$ m/s (D) $\sqrt{75}$ m/s

7. If collision between the block and the incline is completely elastic, then the vertical (upward) component of the velocity of the block at point B, immediately after it strikes the second incline is

(A) $\sqrt{30}$ m/s (B) $\sqrt{15}$ m/s
(C) 0 (D) $-\sqrt{15}$ m/s

8. Look at the drawing given in the figure which has been drawn with ink of uniform line thickness. The mass of ink used to draw each of the two inner circles, and each of the two line segments is m. The mass of the ink used to draw the outer circle is 6 m. The coordinates

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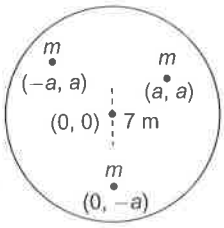
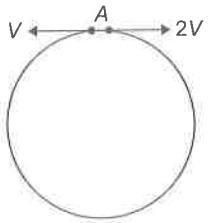
of the centres of the different parts are outer circle (0, 0), left inner circle $(-a, a)$, right inner circle (a, a), vertical line (0, 0) and horizontal line $(0, -a)$. The y-coordinate of the centre of mass of the ink in this drawing is **[JEE 2009]**

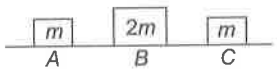


9. Two small particles of equal masses start moving in opposite directions from a point A in a horizontal circle orbit. Their tangential velocities are V and $2V$, respectively, as shown in the figure. Between collisions, the particles move with constant speeds. After making how many elastic collisions, other than that at A, these two particles will again reach point A? **[JEE 2009]**



(A) 4 (B) 3
(C) 2 (D) 1

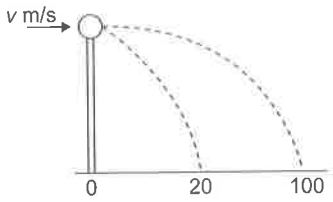
10. Three objects A, B and C are kept in a straight line on a frictionless horizontal surface. These have masses m, $2m$ and m, respectively. The object A moves towards B with the speed 9 ms^{-1} and makes an elastic collision with it. Thereafter, B makes completely inelastic collision with C. All motions occur on the same straight line. Find the final speed (in ms^{-1}) of the object C. **[JEE 2009]**



11. A point mass of 1 kg collides elastically with a stationary point mass of 5 kg. After their collision, the 1 kg mass reverse its direction and moves with the speed of 2 ms^{-1}. Which of the following statement(s) is (are) correct for the system of these two masses? **[JEE 2010]**

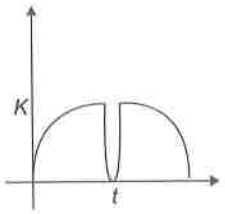
(A) Total momentum of the system is 3 kg ms^{-1}.
(B) Momentum of 5 kg mass after collision is 4 kg ms^{-1}.
(C) Kinetic energy of the centre of mass is 0.75 J.
(D) Total kinetic energy of the system is 4 J.

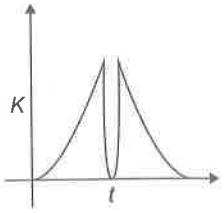
12. A ball of mass 0.2 kg rests on a vertical post of height 5 m. A bullet of mass 0.01 kg travelling with the velocity V m/s in a horizontal direction, hits the centre of the ball. After the collision, the ball and the bullet travel independently. The ball hits the ground at a distance of 20 m and the bullet at a distance of 100 m from the foot of the post. The initial velocity V of the bullet is **[JEE 2011]**

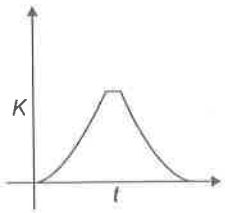


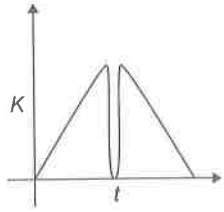
(A) 250 m/s (B) $250\sqrt{2}$ m/s
(C) 400 m/s (D) 500 m/s

13. A tennis ball is dropped on a horizontal smooth surface. It bounces back to its original position after hitting the surface. The force on the ball during the collision is proportional to the length of compression of the ball. Which one of the following sketches describes the variation of its kinetic energy K with time t most appropriately? The figures are only illustrative and not to the scale.

(A)


(B)


(C)


(D)


ANSWER KEYS

Exercises

JEE Main

1. B	**2.** C	**3.** A	**4.** C	**5.** C	**6.** C	**7.** C	**8.** B	**9.** D	**10.** D
11. D	**12.** B	**13.** A	**14.** A	**15.** C	**16.** A	**17.** A	**18.** C	**19.** (a) B, (b) C	
20. (a) C, (b) B		**21.** B	**22.** B	**23.** C	**24.** D	**25.** B	**26.** A	**27.** C	**28.** C
29. C	**30.** C	**31.** B	**32.** B	**33.** A	**34.** A	**35.** A	**36.** B	**37.** C	**38.** A
39. C	**40.** A	**41.** C	**42.** B	**43.** D	**44.** D	**45.** B	**46.** D	**47.** D	**48.** A
49. B	**50.** D								

JEE Advanced

1. C, D	**2.** B, D	**3.** A, B	**4.** C	**5.** B, C	**6.** B	**7.** B, D	**8.** C	**9.** C	**10.** B, C
11. B	**12.** A, B, C, D		**13.** B, C	**14.** A, C	**15.** B, C	**16.** A, B, D	**17.** A, B, C	**18.** D	
19. A, B, C, D		**20.** B, D	**21.** A, C	**22.** (a) A, C; (b) B		**23.** D	**24.** C	**25.** B	**26.** D
27. A	**28.** C	**29.** A	**30.** D	**31.** A	**32.** B	**33.** D	**34.** D	**35.** A	**36.** A
37. D	**38.** B	**39.** B	**40.** B	**41.** B	**42.** D	**43.** C	**44.** C	**45.** D	**46.** D
47. B									

JEE Advanced

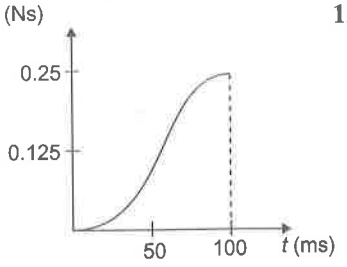
Level I

1. (A) $\lambda(x)=\lambda+\frac{\lambda x}{L}$, (B) $\frac{5}{9}L$ **2.** $y=\frac{4}{3\pi}\left[\frac{b^3-a^3}{b^2-a^2}\right]$ **3.** $\frac{3}{4}a$ **4.** 100 m

5. (A) zero; (B) right; (C) 20 cm; (D) 2.2 m; (E) 1.8 m **6.** $g/9$ downwards **7.** $\sqrt{\frac{2gR}{1+\frac{m}{M}}}$ **8.** 9 m/s, 9 m

9. (A) $\sqrt{\frac{kx_0^2}{2m}}$, (B) $\frac{Kx_0^2}{4}$ **10.** (A) 20 Ns, (B) 50 N

11.



12. 187.5 N, 625 kPa **13.** $m\times\sqrt{u^2-uv+v^2}$

14. (A) $\frac{p_1+p_2}{m_p}=12.3\,\text{m/s}$, (B) $\sqrt{\frac{p_1^2+p_2^2}{m_p}}=9.4\,\text{m/s}$ **5.** (A) $4\sqrt{5}$ N, (B) $2000\sqrt{5}$ N

16. 1.5 m/s, 3.6 Ns **17.** (A) $u/2, mu/2$; (B) $u\sqrt{13}/8$, $mu\sqrt{13}/8$, (C) $u\sqrt{3/4}$, $mu\sqrt{3/4}$

18. $W=-3\,mu^2, J=6\,mu, K=mu$ **19.** $\frac{mv}{(m+\rho Ax)}$ **20.** 6 m/s in the direction of football's velocity

21. (A) 3 J, (B) $\frac{12}{5}$ Ns **22.** 2:1 **23.** (A) $\frac{40}{3}$ m, (B) $t_e=3.25$ s

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24. (A) $v = 3\sqrt{gl}$, (B) $v' = \dfrac{\sqrt{gl}}{2}$ **25.** (A) $10\sqrt{3}$, (B) $\tan^{-1}\sqrt{2}$, (C) 45°, (D) 20 m

26. (A) $e = \cot^2\alpha + \dfrac{m}{M} = \dfrac{3}{4}$, $(g)t = 3$ s **27** $v_A = \sqrt{g/12}$m/s, $S_{max} = 49/48$ m **28.** (A) $v_0/3$, (B) $3\sqrt{5gR}$

29. $v = (u\cos\theta - \mu u\sin\theta)\ln\left(\dfrac{M_0}{M_0 - \lambda t}\right) - mgt$

Level II

1. $(5/\sqrt{17}$ cm, $153\ L/80\ u)$ **2.** $\dfrac{m}{\ell}v(v+gt)$ **3.** (A) 360 m, (B) 10800 J **4.** (A) 50 m, (B) 10 m, (C) 30 m

5. 13/3 m/s **6.** (A) $v/2$, $v/2$, 0; (B) $2mv^2/9$, (C) $mv^2/72$, (D) $x = \sqrt{m/6k}\,v$

7. $2mv^2/3l$ **8.** $v_B = \dfrac{m_A}{m_B}\sqrt{\dfrac{2gl}{1+m_A/m_B}}$; $T = 3m_Ag + \dfrac{2m_A^2 g}{m_B}$ **9.** −2 m/s, 6.93 m/s ∠30°

10. (A) $v_0/3$, (B) $mv_0 = 2mv_2\cos\theta$, $0 = 2mv_2\sin\theta - \dfrac{mv_0}{2}$, (C) $\dfrac{1}{2}$, $\dfrac{\sqrt{5}}{4}v_0$, (D) $\dfrac{mv_0^2}{16}$

11. 2.5 m/s, 0.312 m **12.** $v = \dfrac{2\sqrt{3u}}{7}$, $u' = \dfrac{5u}{7}$, $T = \dfrac{6}{7}mv$

Previous Year Questions

JEE Main

1. C **2.** C **3.** B **4.** C **5.** C **6.** C **7.** B **8.** C **9.** A **10.** A
11. C **12.** A **13.** B **14.** D **15.** A

JEE Advanced

1. C **2.** $\vec{P}_{PM} = m\vec{v}_{PM} = -mv_2\sin\omega t\,\hat{i} + m(v_2\cos\omega t - v_1)\hat{j}$ **3.** B **4.** A, D **5.** B **6.** B
7. C **8.** $a/10$ **9.** C **10.** 4 **11.** A, C **12.** D **13.** B

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CHAPTER 7

Rotational Dynamics

RIGID BODY

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time). This means that the shape and size do not change, during the motion, e.g., fan, pen, table, stone and so on.

Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body. For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles. In Fig. 7.1, velocities of A and B with respect to ground are $\vec{V}_A$ and $\vec{V}_B$, respectively.

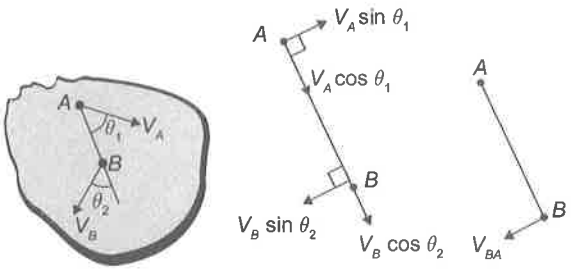


Figure 7.1

If the above body is rigid,

$$V_A \cos \theta_1 = V_B \cos \theta_2$$

Note

With respect to any particle of rigid body, the motion of any other particle of that rigid body is circular.

V_{BA} = relative velocity of B with respect to A.

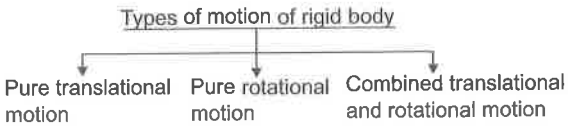


Figure 7.2

Pure Translational Motion

A body is said to be in pure translational motion if the displacement of each particle is the same during any time interval, however small or large. In this motion, all the particles have the same $\vec{s}$, $\vec{v}$ and $\vec{a}$ at an instant.

Example

A box is being pushed on a horizontal surface.

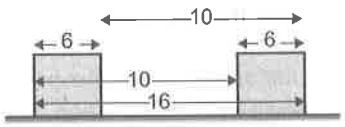


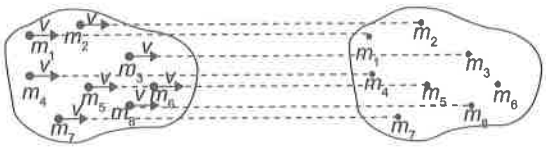
Figure 7.3

$$\vec{V}_{COM} = \vec{V} \text{ of any particle,}$$

$$\vec{a}_{COM} = \vec{a} \text{ of any particle}$$

$$\Delta\vec{S}_{COM} = \Delta\vec{S} \text{ of any particle.}$$

For pure translational motion,



$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots,$$

where m_1, m_2, m_3,... are the masses of different particles of the body having accelerations $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots$, respectively.

But acceleration of all the particles are the same.

So, $$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \cdots = \vec{a}$$

$$\vec{F}_{ext} = M\vec{a},$$

where M = total mass of the body

$\vec{a}$ = acceleration of any particle or of centre of mass of the body.

$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$, where m_1, m_2, m_3, ... are the masses of different particles of the body having velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$, respectively.

But velocities of all the particles are the same, so $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \cdots = \vec{v}$.

$\vec{P} = M\vec{v}$, where $\vec{v}$ = velocity of any particle or of centre of mass of the body.

Total kinetic energy of body

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots = \frac{1}{2}Mv^2$$

Pure Rotational Motion

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constant from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion, all the particles have the same $\vec{\theta}$, $\vec{\omega}$ and $\vec{\alpha}$ at an instant, e.g., a rotating ceiling fan, arms of a clock.

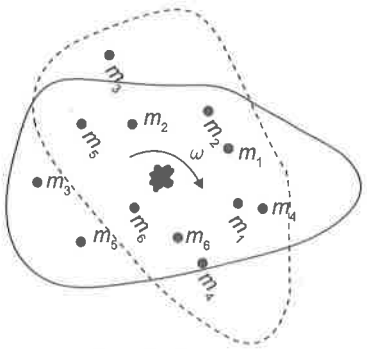


Figure 7.4

For pure rotational motion

$$\theta = \frac{s}{r},$$

where

θ = angle rotated by the particle

s = length of the arc traced by the particle

r = distance of the particle from the axis of rotation.

$$\omega = \frac{d\theta}{dt},$$

where ω = angular speed of the body.

$$\alpha = \frac{d\omega}{dt},$$

where α = angular acceleration of the body.

All the parameters θ, ω and α are the same for all the particles. Axis of rotation is perpendicular to the plane of rotation of particles.

Special case

If α = constant,

$$\omega = \omega_0 + \alpha t,$$

where ω_0 = initial angular speed

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2,$$

where t = time interval

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Total kinetic energy $= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots$

$$= \frac{1}{2}[m_1r_1^2 + m_2r_2^2 + \cdots]\omega^2$$

$$= \frac{1}{2}I\omega^2,$$

where I = moment of inertia = $m_1r_1^2 + m_2r_2^2 + \cdots$

ω = angular speed of the body.

Combined Translational and Rotational Motion

A body is said to be in translational and rotational motion if all the particles rotate about an axis of rotation and the axis of rotation moves with respect to the ground.

MOMENT OF INERTIA

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating. The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion. If a body is at rest, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis).

Moment of inertia depends on:

1. density of the material of body
2. shape and size of the body
3. axis of rotation

In totality, we can say that it depends upon the distribution of mass relative to the axis of rotation.

Notes

Moment of inertia does not change if the mass is
1. shifted parallel to the axis of the rotation
2. rotated with constant radius about axis of rotation.

Moment of Inertia of a Single Particle

For a very simple case, the moment of inertia of a single particle about an axis is given by

$$I = mr^2 \qquad (1)$$

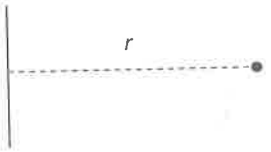


Figure 7.5

Here, m is the mass of the particle and r its distance from the axis under consideration.

Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by

$$I = \sum_i m_i r_i^2 \qquad (2)$$

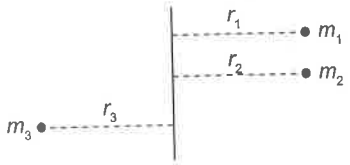


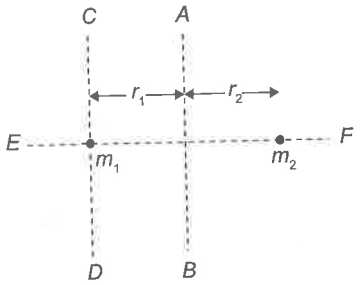
Figure 7.6

where r_i is the perpendicular distance from the axis to the i-th particle, which has a mass m_i.

SOLVED EXAMPLES

EXAMPLE 1

Two heavy particles having masses m_1 and m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2, respectively.



(a) What is the moment of inertia of the system about the axis AB?

(b) What is the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2?

(c) What is the moment of inertia of the system about an axis passing through m_1 and m_2?

SOLUTION

(a) Moment of inertia of the particle on the left is $I_1 = m_1 r_1^2$.

Moment of inertia of the particle on the right is $I_2 = m_2 r_2^2$.

Moment of inertia of the system about AB is

$$I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2$$

(b) Moment of inertia of the particle on the left is $I_1 = 0$

Moment of inertia of the system about CD is

$$I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$$

(c) Moment of inertia of the particle on the left is $I_1 = 0$

Moment of inertia of the particle on the right is $I_2 = 0$

Moment of inertia of the system about EF is

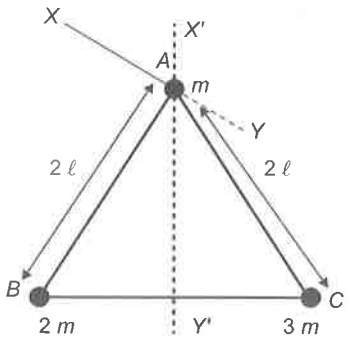
$$I = I_1 + I_2 = 0 + 0$$ ■

EXAMPLE 2

Three light rods, each of length 2ℓ, are joined together to form a triangle. Three particles A, B and C *of* masses m, $2m$ and $3m$ are fixed to the vertices of the triangle. Find the moment of inertia of the resulting body about

(a) an axis through A perpendicular to the plane ABC,

(b) an axis passing through A and the midpoint of BC.



SOLUTION

(a) B is at a distant 2ℓ from the axis XY, so the moment of inertia of B (I_B) about XY is $2m(2\ell)^2$.

Similarly, I_c about XY is $3m(2\ell)^2$ and I_A about XY is $m(0)^2$.

Therefore, the moment of inertia of the body about XY is

$$2m(2\ell)^2 + 3m(2\ell)^2 + m(0)^2 = 20m\ell^2.$$

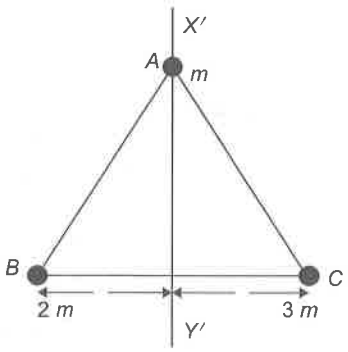
(b) I_A about $X'Y' = m(0)^2$

I_B about $X'Y' = 2m(\ell)^2$

I_C about $X'Y' = 3m(\ell)^2$.

Therefore, the moment of inertia of the body about $X'Y'$ is

$$m(0)^2 + 2m(\ell)^2 + 3m(\ell)^2 = 5m\ell^2.$$

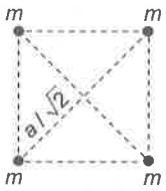


■

EXAMPLE 3

Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.

SOLUTION



The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$. ■

Moment of Inertia of Rigid Bodies

For a continuous mass distribution such as that found in a rigid body, we replace the summation of $I = \sum_i m_i r_i^2$ by an integral. If the system is divided into infinitesimal elements of mass dm and if r is the distance from a mass element to the axis of rotation, the moment of inertia is,

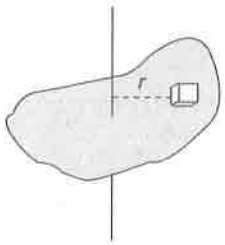


Figure 7.7

$$I = \int r^2 dm,$$

where the integral is taken over the system.

(A) *Uniform rod about a perpendicular bisector*

Consider a uniform rod of mass M and length l as shown in Fig. 7.8 and suppose the moment of inertia is to be calculated about the bisector AB. Take the origin at the middle point O of the rod. Consider the element of the rod between a distance x and $x + dx$ from the origin. As the rod is uniform,

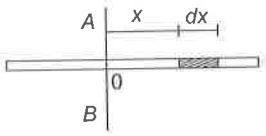


Figure 7.8

Mass per unit length of the rod $= M/l$
so that the mass of the element $= (M/l)dx$.

The perpendicular distance of the element from the line AB is x. The moment of inertia of this element about AB is

$$dI = \frac{M}{l} dx \cdot x^2$$

when $x = -l/2$, the element is at the left end of the rod. As x is changed from $-l/2$ to $l/2$, the elements cover the whole rod.

Thus, the moment of inertia of the entire rod about AB is

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx$$

$$= \left[\frac{Mx^3}{l3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

(B) *Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre*

The situation is shown in Fig. 7.9. Draw a line parallel to AB at a distance x from it and another at a distance $x + dx$. We can take the strip enclosed between the two lines as the small element.

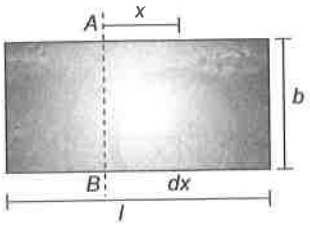


Figure 7.9

It is 'small' because the perpendiculars from the different points of the strip to AB differ by not more than dx. As the plate is uniform, its mass per unit area $= \frac{M}{bl}$.

Mass of the strip $= \frac{M}{bl} b\,dx = \frac{M}{l} dx$.

The perpendicular distance of the strip from $AB = x$.

The moment of inertia of the strip about $AB = dI = \frac{M}{l} dx \cdot x^2$. The moment of inertia of the given plate is, therefore, $I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx = \frac{Ml^2}{12}$.

The moment of inertia of the plate about the line parallel to the other edge and passing through the centre may be obtained from the above formula by replacing l by b, and thus,

$$I = \frac{Mb^2}{12}$$

(C) *Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre)*

Suppose the radius of the ring is R and its mass is M. As all the elements of the ring are at the same perpendicular distance R from the axis, the moment of inertia of the ring is

$$I = \int r^2 dm = \int R^2 dm$$

$$= R^2 \int dm = MR^2$$

(D) *Moment of inertia of a uniform circular plate about its axis*

Let the mass of the plate be M and its radius R. The centre is at O and the axis OX is perpendicular to the plane of the plate.

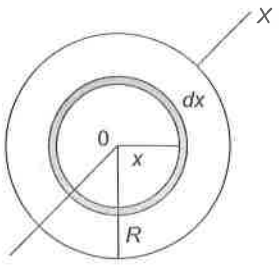


Figure 7.10

Draw two concentric circles of radii x and $x + dx$, both centred at O and consider the area of the plate between the two circles.

This part of the plate may be considered to be a circular ring of radius x. As the periphery of the ring is $2\pi x$ and its width is dx, the area of this elementary ring is $2\pi x dx$. The area of the plate is πR^2. As the plate is uniform, its mass per unit area $= \dfrac{m}{\pi R^2}$.

$$\text{Mass of the ring} = \frac{M}{\pi R^2} 2\pi x dx = \frac{2Mxdx}{R^2}$$

Using the result obtained above for a circular ring, the moment of inertia of the elementary ring about OX is

$$dI = \left[\frac{2Mxdx}{R^2}\right] x^2$$

The moment of inertia of the plate about OX is

$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{MR^2}{2}$$

(E) *Moment of inertia of a hollow cylinder about its axis*

Suppose the radius of the cylinder is R and its mass is M. As every element of this cylinder is at the same perpendicular distance R from the axis, the moment of inertia of the hollow cylinder about its axis is

$$I = \int r^2 dm$$

$$= R^2 \int dm = MR^2$$

(F) *Moment of inertia of a uniform solid cylinder about its axis*

Let the mass of the cylinder be M and its radius R. Draw two cylindrical surfaces of radii x and $x + dx$ coaxial with the given cylinder. Consider the part of the cylinder between the two surfaces. This part of the cylinder may be considered to be a hollow cylinder of radius x. The area of cross-section of the wall of this hollow cylinder is $2\pi x dx$. If the length of the cylinder is l, the volume of the material of this elementary hollow cylinder is $2\pi x dx l$.

The volume of the solid cylinder is $\pi R^2 l$ and it is uniform; hence, its mass per unit volume is

$$\rho = \frac{M}{\pi R^2 l}.$$

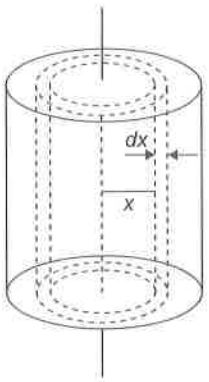


Figure 7.11

The mass of the hollow cylinder considered is

$$\frac{M}{\pi R^2 l} 2\pi x dx l = \frac{2M}{R^2} x dx$$

As its radius is x, its moment of inertia about the given axis is $dI = \left[\dfrac{2M}{R^2} x dx\right] x^2$.

The moment of inertia of the solid cylinder is, therefore,

$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{MR^2}{2}$$

Note that the formula does not depend on the length of the cylinder.

(G) *Moment of inertia of a uniform hollow sphere about a diameter*

Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis (Figure 7.12). The mass is spread over the surface of the sphere and the inside is hollow.

Let us consider a radius OA of the sphere at an angle θ with the axis OX and rotate this radius about OX. The point A traces a circle on the sphere. Now change θ to $\theta + d\theta$ and

get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in Fig. 7.12, forms a ring of radius $R \sin\theta$. The width of this ring is $Rd\theta$ and its periphery is $2\pi R \sin\theta$. Hence, the area of the ring $= (2\pi R \sin\theta)(Rd\theta)$.

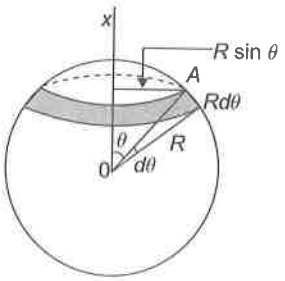


Figure 7.12

Mass per unit area of the sphere $= \dfrac{M}{4\pi R^2}$.

The mass of the ring $= \dfrac{M}{4\pi R^2}(2\pi R\sin\theta)(R\,d\theta)$

$$= \frac{M}{2}\sin\theta\, d\theta .$$

The moment of inertia of this elemental ring about OX is

$$dI = \left(\frac{M}{2}\sin\theta\, d\theta\right)(R\sin\theta)^2$$

$$= \frac{M}{2}R^2\sin^3\theta\, d\theta.$$

As θ increases from 0 to π, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$I = \int_0^{\pi} \frac{M}{2}R^2\sin^3\theta\, d\theta$$

$$= \frac{MR^2}{2}\left[\int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta\right]$$

$$= \frac{MR^2}{2}\left[\int_{\theta=0}^{\pi} -(1-\cos^2\theta)\, d(\cos\theta)\right]$$

$$= \frac{-MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{2}{3}MR^2.$$

(H) *Moment of inertia of a uniform solid sphere about a diameter*

Let M and R be the mass and radius of the given solid sphere. Let O be centre and OX the given axis. Draw two spheres of radii x and $x + dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius x.

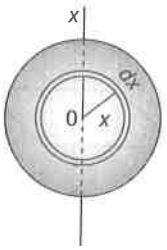


Figure 7.13

The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The thin hollow sphere considered above has a surface area $4\pi x^2$ and thickness dx. Its volume is $4\pi x^2 dx$, and hence, its mass is $= \left(\dfrac{3M}{4\pi R^3}\right)(4\pi x^2 dx) = \dfrac{3M}{R^3}x^2 dx$.

Its moment of inertia about the diameter OX is, therefore,

$$dI = \frac{2}{3}\left[\frac{3M}{R^3}x^2 dx\right]x^2$$

$$= \frac{2M}{R^3}x^4 dx$$

If $x = 0$, the shell is formed at the centre of the solid sphere. As x increases from 0 to R, the shells cover the whole solid sphere.

The moment of inertia of the solid sphere about OX is, therefore,

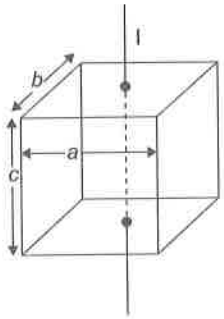
$$I = \int_0^R \frac{2M}{R^3}x^4 dx$$

$$= \frac{2}{5}MR^2$$

SOLVED EXAMPLE

EXAMPLE 4

Find the moment of inertia of a cuboid along the axis as shown in the figure.



SOLUTION

After compressing the cuboid parallel to the axis, $I = \frac{M(a^2+b^2)}{12}$.

THEOREMS OF MOMENT OF INERTIA

There are two important theorems on moment of inertia, which, in some cases enable the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

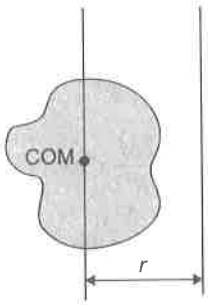


Figure 7.14

Theorem of Parallel Axes

A very useful theorem, called the parallel axes theorem, relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.

Two such axes are shown in Fig. 7.14 for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them, these moments are related by,

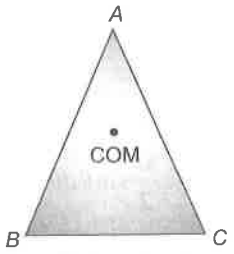
$$I = I_{COM} + Mr^2.$$

- Theorem of parallel axis is applicable for any type of rigid body be it a two-dimensional or three-dimensional body.

SOLVED EXAMPLES

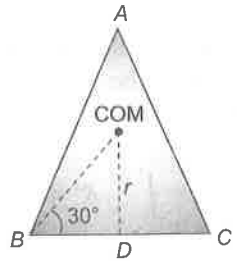
EXAMPLE 5

Three rods each of mass m and length l are joined together to form an equilateral triangle as shown in the figure. Find the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of triangle.



SOLUTION

Moment of inertia of rod BC about an axis perpendicular to the plane of the triangle ABC and passing through the mid-point of rod BC (i.e., D) is



$$I_1 = \frac{ml^2}{12}$$

From the theorem of parallel axes, moment of inertia of this rod about the asked axis is

$$I_2 = I_1 + mr^2$$

$$= \frac{ml^2}{12} + m\left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}$$

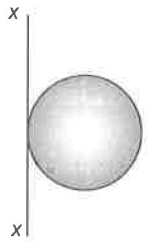
∴ Moment of inertia of all the three rods is

$$I = 3I_2$$

$$= 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}$$ ■

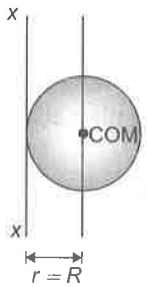
EXAMPLE 6

Find the moment of inertia of a solid sphere of mass M and radius R about an axis XX shown in the figure.



SOLUTION

From the theorem of parallel axis,



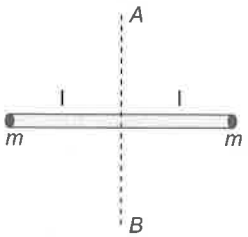
$$I_{XX} = I_{COM} + Mr^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2$$ ■

EXAMPLE 7

Consider a uniform rod of mass m and length $2l$ with two particles of mass m each at its ends. Let AB be a line perpendicular to the length of the rod passing through its centre. Find the moment of inertia of the system about AB.



SOLUTION

$$I_{AB} = I_{rod} + I_{both\ particles}$$

$$= \frac{m(2l)^2}{12} + 2(ml^2)$$

$$= \frac{7}{3}ml^2$$ ■

Theorem of Perpendicular Axes

The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it.

Let x and y axes be chosen in the plane of the body and z-axis perpendicular to this plane (Fig. 7.15), with the three axes being mutually perpendicular, then the theorem states that

$$I_z = I_x + I_y$$

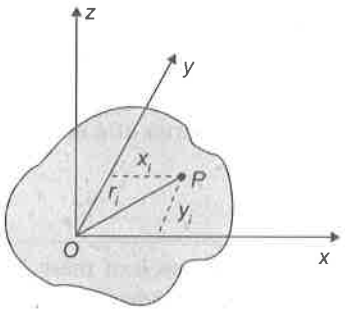


Figure 7.15

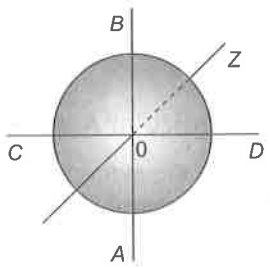
Important Points in Perpendicular Axis Theorem

1. This theorem is applicable only for the plane bodies (two dimensional).
2. In theorem of perpendicular axes, all the three axes (x, y and z) intersect each other and this point may be any point on the plane of the body (it may even lie outside the body).
3. Intersection point may or may not be the centre of mass of the body.

SOLVED EXAMPLES

EXAMPLE 8

Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.



SOLUTION

Let AB and CD be two mutually perpendicular diameters of the ring. Consider these as the x- and y-axes and the line perpendicular to the plane of the ring through the centre as the z-axis. The moment of inertia of the ring about the z-axis is $I = MR^2$. As the ring is uniform, all the diameters are equivalent and so $I_x = I_y$. From the perpendicular axes theorem,

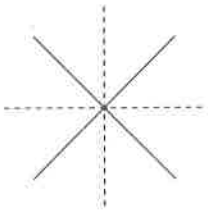
$$I_z = I_x + I_y.$$

Hence,
$$I_x = \frac{I_z}{2} = \frac{MR^2}{2}.$$

Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$. ■

EXAMPLE 9

Two uniform identical rods each of mass M and length ℓ are joined to form a cross as shown in the figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.

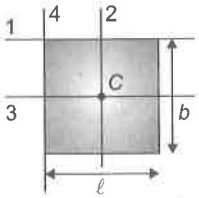


SOLUTION

Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\frac{M\ell^2}{12}$, and hence, the moment of inertia of the cross is $\frac{M\ell^2}{6}$. The moment of inertia of the cross about the two bisectors is equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{M\ell^2}{12}$. ■

EXAMPLE 10

In the figure shown find the moment of inertia of a plate having mass M, length ℓ and width b about axis 1, 2, 3 and 4. Assume that C is the centre and mass is uniformly distributed.



SOLUTION

Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$$I_1 = Mb^2/3.$$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$$I_2 = M\ell^2/12.$$

Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

$$I_3 = \frac{Mb^2}{12}.$$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$$I_4 = M\ell^2/3.$$ ■

Moment of Inertia of Compound Bodies

Consider two bodies A and B rigidly joined together. The moment of inertia of this compound body, about an axis XY, is required. Let I_A is the moment of inertia of body A about XY and I_B is the moment of inertia of body B about XY. Then, the moment of inertia of the compound body $I = I_A + I_B$.

Extending this argument to cover any number of bodies rigidly joined together, we see that the moment of inertia of the compound body, about a specified axis, is the

sum of the moments of inertia of the separate parts of the body about the same axis.

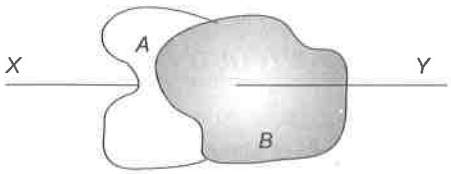
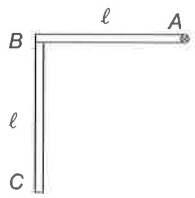


Figure 7.16

SOLVED EXAMPLES

EXAMPLE 11

Two rods each having length ℓ and mass m are joined together at point B as shown in the figure. Then find out the moment of inertia about the axis passing through A and perpendicular to the plane of the page as shown in the figure.



SOLUTION

We find the resultant moment of inertia I by dividing in two parts such as

I = MI of rod AB about A + MI of rod BC about A

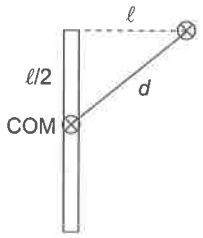
$$I = I_1 + I_2 \tag{1}$$

We first calculate I_1:

$$I_1 = \frac{m\ell^2}{3} \tag{2}$$

Calculation of I_2:

Using the parallel axis theorem,



$$I_2 = I_{COM} + md^2$$

$$= \frac{m\ell^2}{12} + m\left(\frac{\ell^2}{4} + \ell^2\right)$$

$$= \frac{m\ell^2}{12} + \frac{5\ell^2}{4}m \tag{3}$$

Inserting the values from Eqs. (2) and (3) into (1),

$$I = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + \frac{5\ell^2 m}{4}$$

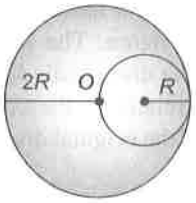
$$I = \frac{m\ell^2}{12}(4+1+15)$$

$$\Rightarrow \quad I = \frac{5m\ell^2}{3}$$ ■

CAVITY PROBLEMS

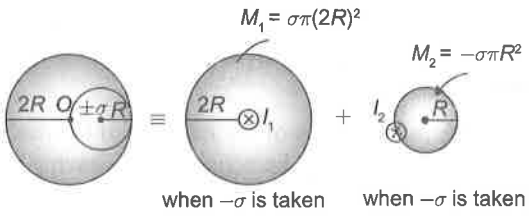
EXAMPLE 12

A uniform disc having radius $2R$ and mass density σ is shown in the figure. If a small disc of radius R is cut from the disc as shown, then find out the moment of inertia of the remaining disc around the axis that passes through O and is perpendicular to the plane of the page.



SOLUTION

We assume that in the remaining part a disc of radius R and mass density $\pm\sigma$ is placed. Then,



Total moment of inertia

$$I = I_1 + I_2$$

$$I_1 = \frac{M_1(2R)^2}{2}$$

$$I_1 = \frac{\sigma\pi 4R^2 \cdot 4R^2}{2} = 8\pi\sigma R^4$$

To calculate I_2, we use the parallel axis theorem,

$$I_2 = I_{COM} + M_2R^2$$

$$I_2 = \frac{M_2R^2}{2} + M_2R^2$$

$$I_2 = \frac{3}{2}M_2R^2 = \frac{3}{2}(-\sigma\pi R^2)R^2$$

$$I_2 = -\frac{3}{2}\sigma\pi R^4$$

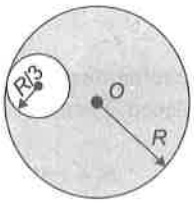
Now $I = I_1 + I_2$

$$I = 8\pi\sigma R^4 - \frac{3}{2}\sigma\pi R^4$$

$$I = \frac{13}{2}\sigma\pi R^4$$ ■

EXAMPLE 13

A uniform disc of radius R has a round disc of radius $R/3$ cut as shown in the figure. The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through the geometrical centre of the original disc and perpendicular to the plane of the disc.



SOLUTION

Let the mass per unit area of the material of the disc be σ. Now the empty space can be considered as having density $-\sigma$ and σ.

Now, $I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma\pi R^2)R^2/2$

= MI of σ about O

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2](2R/3)^2]$$

= MI of $-\sigma$ about 0

$\therefore$ $I_0 = \frac{4}{9}\sigma\pi R^4$ ■.

TORQUE

Torque represents the capability of a force to produce change in the rotational motion of the body

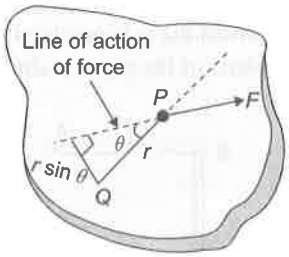


Figure 7.17

Torque about a Point

Torque of force $\vec{F}$ about a point is

$$\vec{\tau} = \vec{r} \times \vec{F},$$

where

$\vec{F}$ = force applied
P = point of application of force
Q = point about which we want to calculate the torque
$\vec{r}$ = position vector of the point of application of force from the point about which we want to determine the torque.

$$|\vec{\tau}| = rF\sin\theta$$

$$= r_\perp F = rF_\perp,$$

where θ = angle between the direction of force and the position vector of P with respect to Q. $r_\perp$ = perpendicular distance of line of action of force from point Q.

$F_\perp$ = force arm

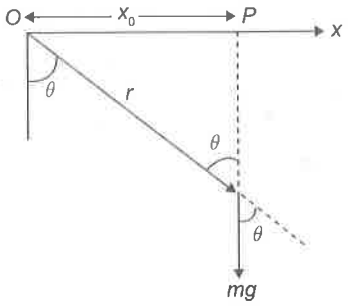
SI unit of torque is Nm.

Torque is a vector quantity and its direction is determined using the right-hand thumb rule.

SOLVED EXAMPLES

EXAMPLE 14

A particle of mass M is released in the vertical plane from a point P at $x = x_0$ on the x-axis and it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about the origin.



SOLUTION

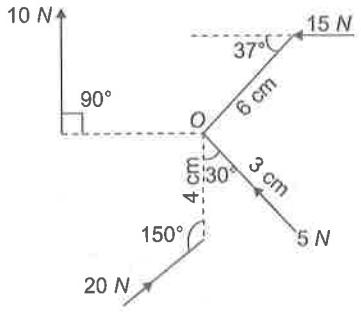
Torque is produced by the force of gravity

$$\vec{\tau} = rF \sin\theta \hat{k}$$

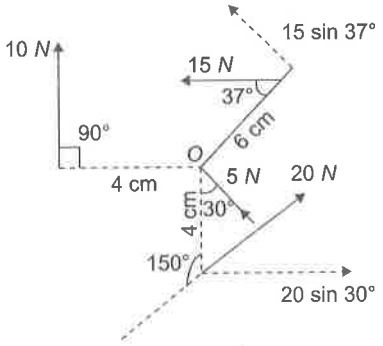
or $$\tau = r_\perp F = x_0 mg$$ ■

EXAMPLE 15

Calculate the total torque acting on the body shown in the figure about the point O.



SOLUTION



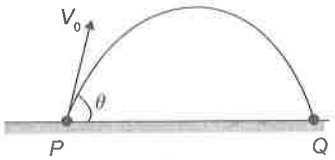
$$\tau_0 = 15 \sin 37 \times 6\odot + 20 \sin 30° \times 4\odot - 10 \times 4\odot$$

$$= 54 + 40 - 40 = 54 \text{ Ncm}$$

$$\tau_0 = 0.54 \text{ Nm}$$ ■

EXAMPLE 16

A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with the horizontal. Find out the torque about the point of projection acting on the particle when

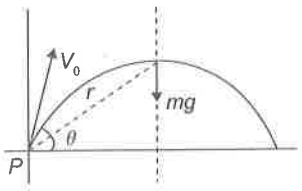


(a) it is at its maximum height.

(b) it is just about to hit the ground back.

SOLUTION

(a) Particle is at maximum height, then τ about point P is $\tau_P = r_\perp F$.



$$F = mg;$$

$$r_\perp = \frac{R}{2}$$

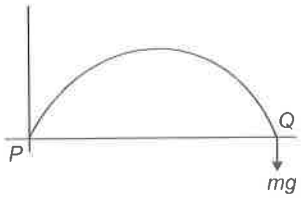
$$\Rightarrow \quad \tau_P = \frac{R}{2}mg$$

$$= mg \times \frac{v_0^2 \sin 2\theta}{2g}$$

$$\tau_P = \frac{mv_0^2 \sin 2\theta}{2}$$

(b) When the particle is at point Q, then τ about point P is

$$\tau_P' = r_\perp F$$



$$r_\perp = R; F = mg$$

$$\tau_P' = mgR$$

$$= mg\frac{v_0^2 \sin 2\theta}{g}$$ ■

EXAMPLE 17

In the previous question, during the motion of a particle from P to Q. torque of gravitational force about P is

(A) increasing **(B)** decreasing

(C) remains constant **(D)** first increasing then decreasing

SOLUTION

Torque of gravitational force about P is increasing because $r_\perp$ is increasing from O to R (range). ■

Torque about an Axis

It is given as

$$\vec{\tau} = \vec{r} \times \vec{F},$$

where

$\vec{\tau}$ = torque acting on the body about the axis of rotation

$\vec{r}$ = position vector of the point of application of force about the axis of rotation

$\vec{F}$ = force applied on the body

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots$$

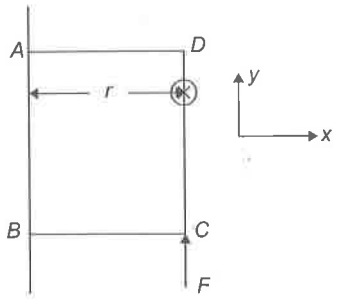


Figure 7.18

To understand the concept of torque about an axis, we take a general example which we come across in our daily life. Fig. 7.18 shows a door $ABCD$, which can rotate about the axis AB. Now if we apply force F at the point along the inward direction, then $\tau_{AB} = rF$ and direction of this τ_{AB} is along the y-axis from the right-hand thumb rule, which is parallel to AB and so gives the resultant torque.

Now we apply force at point C in the direction as shown the figure. At this time, $\vec{r}$ and $\vec{F}$ are perpendicular to each other, which gives $\tau_{AB} = rF$.

But the door cannot move when the force is applied in this direction because the direction of τ_{AB} is perpendicular to AB according to the right-hand thumb rule.

So there is no component of τ along AB which gives $\tau_{res} = 0$.

Now, we conclude that the torque about an axis is the component of $\vec{r} \times \vec{F}$ parallel to the axis of rotation.

Note

The direction of the torque is calculated using the right-hand thumb rule and it is always perpendicular to the plane of rotation of the body.

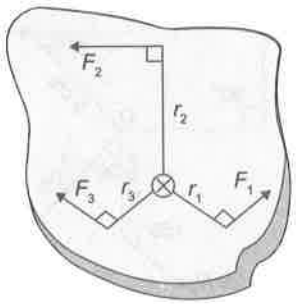


Figure 7.19

Note (contd)

If F_1 or F_2 is applied to the body, the body revolves in the anti-clockwise direction and F_3 makes body revolve in the clockwise direction. If all the three are applied,

$$\vec{\tau}_{\text{resultant}} = F_1 r_1 + F_2 r_2 - F_3 r_3$$

(in the anti-clockwise direction).

BODY IS IN EQUILIBRIUM

We can say a rigid body is in equilibrium when it is in

1. *Translational equilibrium*

i.e., $\vec{F}_{\text{net}} = 0$

$F_{\text{net}\,x} = 0$

and $F_{\text{net}\,y} = 0$ and

2. *Rotational equilibrium*

$\vec{\tau}_{\text{net}} = 0$,

i.e., torque about any point is zero.

Note

1. If the net force on the body is zero, then the net torque of the forces may or may not be zero.

   Example

   A pair of forces each of the same magnitude are acting in the opposite direction on the rod (Fig. 7.20).

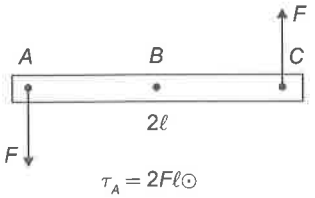


Figure 7.20

2. If the net force on the body is zero, then the torque of the forces about each and every point is the same

Note (contd)

τ about B $\quad \tau_B = F\ell + F\ell\odot$

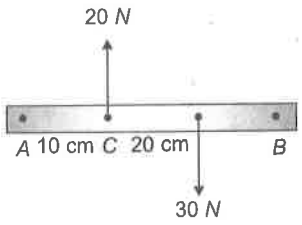
$\tau_B = 2F\ell\odot$

τ about C $\quad \tau_C = 2F\ell\odot$

SOLVED EXAMPLES

EXAMPLE 18

Determine the point of application of the third force for which the body is in equilibrium when forces of 20 N and 30 N are acting on the rod as shown in the figure.



SOLUTION

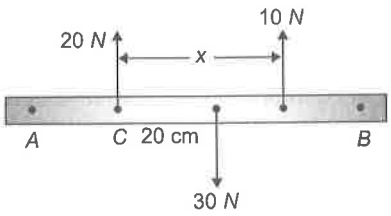
Let the magnitude of third force F is applied in the upward direction, then the body is in equilibrium when

(a) $\vec{F}_{\text{net}} = 0$ (translational equilibrium)

$\Rightarrow \quad 20 + F = 30$

$\Rightarrow \quad F = 10\ N.$

So the body is in translational equilibrium when 10 N force acts on it in the upward direction.



(b) Let us assume that this 10 N force acts on the body. Then keep the body in rotational equilibrium.

So torque about $C = 0$

i.e., $\tau_C = 0$

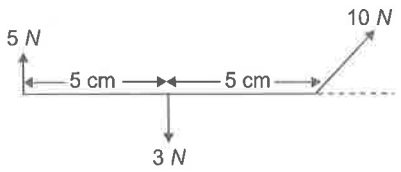
$\Rightarrow \quad 30 \times 20 = 10x$

$$x = 60 \text{ cm}$$

So the 10 N force is applied at 70 cm from point A to keep the body in equilibrium. ■

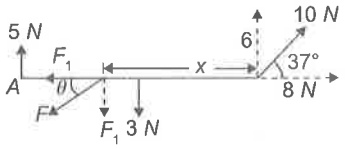
EXAMPLE 19

Determine the point of application of force, when the forces are acting on the rod as shown in the figure.



SOLUTION

Since the body is in equilibrium, we conclude $\vec{F}_{net} = 0$ and torque about any point is zero, i.e., $\vec{\tau}_{net} = 0$.



Let us assume that we apply F force downwards at A at angle θ from the horizontal, at x distance from B

From $\vec{F}_{net} = 0$

$\Rightarrow$ $F_{net}x = 0,$

which gives $F_2 = 8\ N$

From $F_{net\,y} = 0$

$\Rightarrow$ $5 + 6 = F_1 + 3$

$\Rightarrow$ $F_1 = 8\ N$

If the body is in equilibrium, then torque about point B is zero.

$\Rightarrow$ $3 \times 5 + F_1 x - 5 \times 10 = 0$

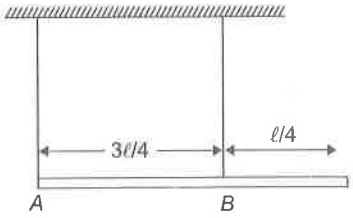
$\Rightarrow$ $15 + 8x - 50 = 0$

$$x = \frac{35}{9}$$

$\Rightarrow$ $x = 4.375$ cm. ■

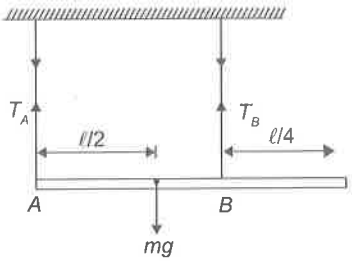
EXAMPLE 20

A uniform rod of length ℓ and mass m is hung from two strings of equal length from a ceiling as shown in the figure. Determine the tensions in the strings.



SOLUTION

Let us assume that the tension in the left and right strings is T_A and T_B, respectively. Then the rod is in equilibrium, then $\vec{F}_{net} = 0$ and $\vec{\tau}_{net} = 0$.



From $\vec{F}_{net} = 0$,

$$mg = T_A + T_B. \quad (1)$$

From $T_{net} = 0$ about A,

$$mg\frac{\ell}{2} - \frac{3\ell}{4}T_B = 0$$

$\Rightarrow$ $$T_B = \frac{2mg}{3}$$

From Eq. (1), $T_A + \frac{2mg}{3} = mg$

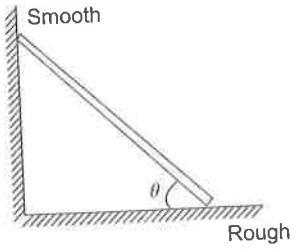
$\Rightarrow$ $$T_A = \frac{mg}{3}.$$ ■

LADDER PROBLEMS

EXAMPLE 21

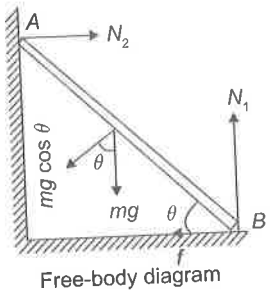
A stationary uniform rod of mass m and length ℓ leans against a smooth vertical wall making an angle θ with the

rough horizontal floor. Find the normal force and frictional force that are exerted by the floor on the rod.



SOLUTION

As the rod is stationary, the linear acceleration and angular acceleration of the rod are zero.



i.e., $$a_{COM} = 0; \alpha = 0.$$

$$\left.\begin{aligned} N_2 &= f \\ N_1 &= mg \end{aligned}\right\} \because a_{COM} = 0.$$

Torque about any point of the rod should also be zero

$$\therefore \quad \alpha = 0$$

$$\tau_A = 0$$

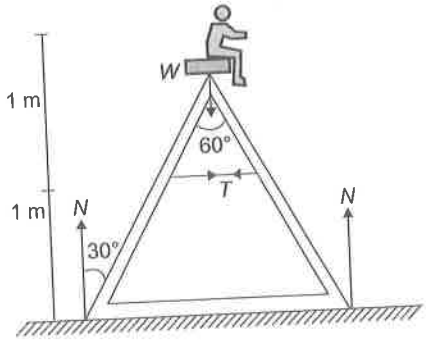
$$\Rightarrow \quad mg\cos\theta\,\frac{\ell}{2} + f\ell\sin\theta = N_1\cos\theta\ell$$

$$N_1\cos\theta = \sin\theta f + \frac{mg\cos\theta}{2}$$

$$f = \frac{mg\cos\theta}{2\sin\theta} = \frac{mg\cot\theta}{2}$$ ■

EXAMPLE 22

The ladder shown in the figure has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the contact force exerted by the floor on each leg and the tension in the crossbar.



SOLUTION

The forces acting on the different parts are shown in the figure. Consider the vertical equilibrium of 'the ladder plus the person' system. The forces acting on this system are its weight (80 kg) g and the contact force $N + N = 2N$ due to the floor. Thus, $2N = (80\text{ kg})g$

or $$N = (40\text{ kg})(9.8\text{ m/s}^2) = 392\text{ N}.$$

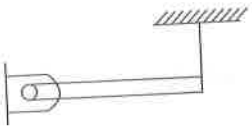
Next, the consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end, $N(2\text{ m})\tan 30° = T(1\text{ m})$

or $$T = N\frac{2}{\sqrt{3}}$$

$$= (392\ N) \times \frac{2}{\sqrt{3}} = 450\ N$$ ■

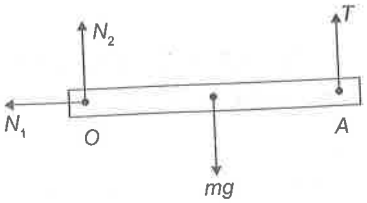
EXAMPLE 23

A thin plank of mass m and length ℓ is pivoted at one end and is held stationary in the horizontal position by means of a light thread as shown in the figure. Then find out the force on the pivot.



SOLUTION

The free-body diagram of the plank is shown in the figure.



$\therefore$ Plank is in equilibrium condition.

So, F_{net} and t_{net} on the plank are zero.

From $F_{net} = 0$

$\Rightarrow$ $F_{net\,x} = 0$

$N_1 = 0$

Now $F_{net\,y} = 0$

$\Rightarrow$ $N_2 + T = mg$

From $\tau_{net} = 0$

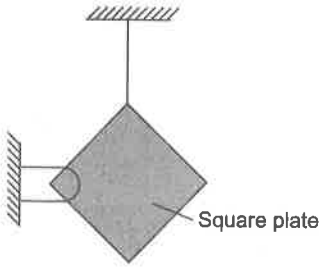
$\Rightarrow$ τ_{net} about point A is zero.

so $N_2 \cdot \ell = mg \cdot \ell/2$

$$N_2 = \frac{mg}{2}.$$ ■

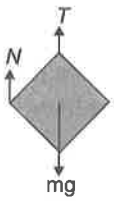
EXAMPLE 24

A square plate is hinged as shown in the figure and is held stationary by means of a light thread as shown in the figure. Then find out the force exerted by the hinge.



SOLUTION

Free-body diagram is shown.



$\because$ Body is in equilibrium and T and mg forces pass through one line. Thus, from $\tau_{net} = 0$,

$N = 0.$ ■

RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}.$$

Thus, for a rigid body, we have the rotational analogy of Newton's second law:

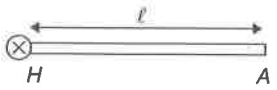
$$\Sigma\tau = I\alpha$$

Following two points are important regarding the above equation:

1. The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration α is different for different particles.
2. The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

EXAMPLE 25

A uniform rod of mass m and length ℓ can rotate in the vertical plane about a smooth horizontal axis hinged at point H.



(a) Find the angular acceleration α of the rod just after it is released from the initial horizontal position from rest.

(b) Calculate the acceleration (tangential and radial) of point A at this moment.

SOLUTION

(a) $\tau_H = I_H\alpha$

$$mg \cdot \frac{\ell}{2} = \frac{m\ell^2}{3}\alpha$$

$\Rightarrow$ $$\alpha = \frac{3g}{2\ell}$$

(b) $a_{\tau A} = a\ell$

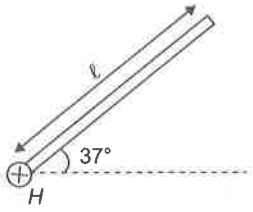
$$= \frac{3g}{2\ell} \cdot \ell = \frac{3g}{2}$$

$$a_{CA} = \omega^2 r = 0 \cdot \ell = 0$$

($\because$ $\omega = 0$ just after release). ■

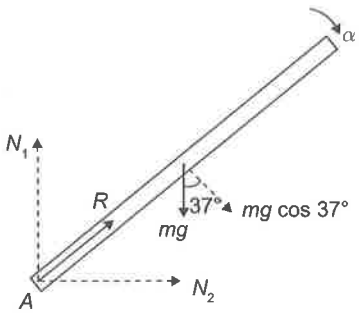
EXAMPLE 26

A uniform rod of mass m and length ℓ hinged at point H can rotate in the vertical plane about a smooth horizontal axis. Find the force exerted by the hinge just after the rod is released from rest, from an initial position making an angle of 37° with horizontal.



SOLUTION

Just after releasing at 37° from the horizontal, the free-body diagram of the plank is as shown below.



From $\tau_{net} = I\alpha$,

τ about point $A = \tau_A = mg \cos 37° \frac{\ell}{2} = \frac{m\ell^2}{3} \cdot \alpha$

$\Rightarrow \quad \alpha = \frac{6g}{5\ell}$ rad/s².

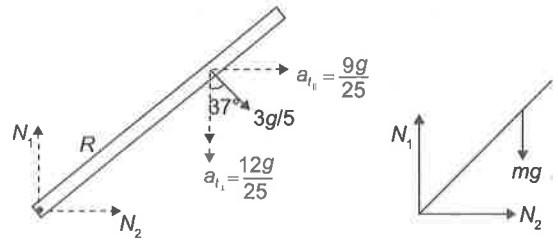
Now tangential acceleration of the centre of mass

$$a_t = \alpha \cdot \frac{\ell}{2} = \frac{3g}{5} \text{m/s}^2$$

just after release, $v_{COM} = 0$

$\Rightarrow \quad a_r = 0.$

Now resolving a_t in the horizontal and vertical directions as shown in the figure.



From $F_{net} = ma$ in both the horizontal and vertical directions

$$N_2 = m\left(\frac{9g}{25}\right)$$

$\Rightarrow \quad N_1 = \frac{13mg}{25}$

Now $\quad R = \sqrt{N_1^2 + N_2^2}$

$$R = \frac{mg\sqrt{10}}{5}$$

■

Pulley Block System

If there is friction between the pulley and string and if the pulley has some mass, then tension is different on two sides of the pulley.

Reason

To understand this concept, we take a pulley block system as shown in Fig. 7.21.

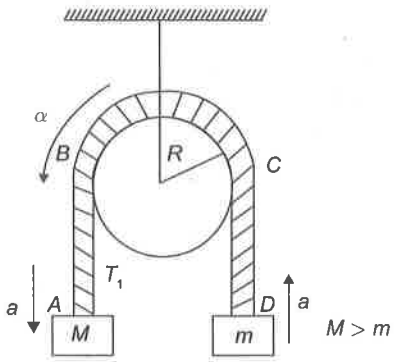


Figure 7.21

Let us assume that the tension induced in part AB of the string is T_1 and block M moves downward. If friction is present between the pulley and the string, then it opposes the relative slipping between the pulley and the string. Take two points e and f on the pulley and string, respectively. If friction is present, then due to this, both the points should move together. So frictional force acts on e and d in the direction as shown in Fig. 7.22.

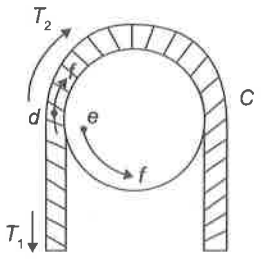


Figure 7.22

This frictional force f acting on point d increases the tension T_1 by a small amount dT.

Then $$T_1 = T_2 + dT$$

or we can say $T_2 = T_1 - f$.

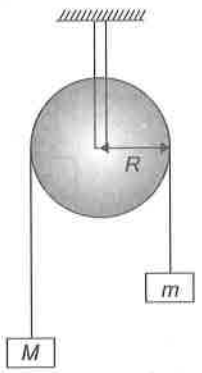
Thus, the tension on the two sides of the pulley is different. If there is no relative slipping between the pulley and the string, then $\alpha = \frac{a_t}{R} = \frac{a}{R}$.

EXAMPLE 27

The pulley shown in the figure has moment of inertia I about its axis and radius R. Find the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.

SOLUTION

Suppose the tension in the left string is T_1 and that in the right string is T_2. Suppose the block of mass M goes down with an acceleration a and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim.



The angular acceleration of the wheel $\alpha = \frac{a}{R}$.

The equations of motion for the mass M, the mass m and the pulley are as follows:

$$Mg - T_1 = Ma \quad (1)$$

$$T_2 - mg = ma \quad (2)$$

$$T_1R - T_2R = I\alpha = \frac{Ia}{R} \quad (3)$$

Substituting for T_1 and T_2 from Eqs. (1) and (2) in Eq. (3),

$$[M(g - a) - m(g + a)]R = \frac{Ia}{R}.$$

Solving, we get

$$a = \frac{(M - m)gR^2}{I + (M + m)R^2}$$

ANGULAR MOMENTUM

Angular Momentum of a Particle about a Point

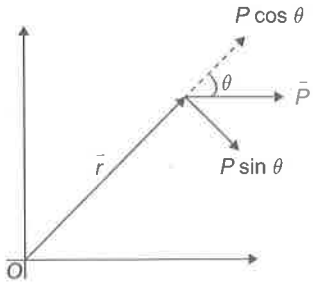


Figure 7.23

$$\vec{L} = \vec{r} \times \vec{P}$$

$$\Rightarrow \quad L = rp \sin\theta$$

$$|\vec{L}| = r_\perp \times P$$

$$\Rightarrow \quad |\vec{L}| = P_\perp \times r,$$

where

$\vec{P}$ = momentum of particle

$\vec{r}$ = position vector of particle with respect to a point about which the angular momentum is to be calculated

θ = angle between vectors $\vec{r}$ and $\vec{P}$

$r_\perp$ = perpendicular distance of the line of motion of the particle from point O.

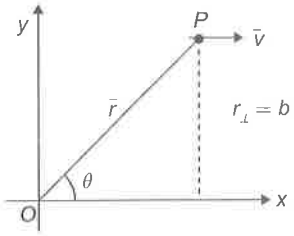
$P_\perp$ = perpendicular component of momentum.

SI unit of angular momentum is kgm²/s.

SOLVED EXAMPLES

EXAMPLE 28

A particle of mass m is moving along the line $y = b, z = 0$ with constant speed v. State whether the angular momentum of the particle about the origin is increasing, decreasing or constant.



SOLUTION

$$|\vec{L}| = mvr \sin\theta$$

$$= mvr_\perp$$

$$= mvb$$

$\therefore$ $|\vec{L}|$ = constant as m, v and b all are constants.

Direction of $\vec{r}\times\vec{v}$ also remains the same. Therefore, angular momentum of the particle about the origin remains constant with due course of time. ■

Note

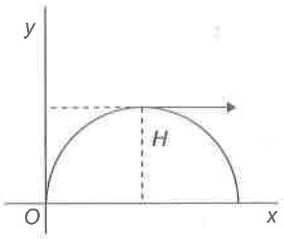
In this problem, $|\vec{r}|$ is increasing, θ is decreasing but $r \sin\theta$, i.e., b remains constant. Hence, the angular momentum remains constant.

EXAMPLE 29

A particle of mass m is projected with velocity v at an angle θ with the horizontal. Find its angular momentum about the point of projection when it is at the highest point of its trajectory.

SOLUTION

At the highest point, it has only horizontal velocity $v_x = v\cos\theta$. Length of the perpendicular to the horizontal velocity from O is the maximum height, where



$$H_{max} = \frac{v^2\sin^2\theta}{2g}$$

$$\Rightarrow \text{ Angular momentum } L = \frac{mv^3\sin^2\theta\cos\theta}{2g}.$$ ■

Angular Momentum of a Rigid Body Rotating about a Fixed Axis

Suppose a particle P of mass m is going in a circle of radius r and at some instant the speed of the particle is v. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case, $\vec{r}$ and $\vec{P}$ are perpendicular to each other and $\vec{r}\times\vec{P}$ is along the axis. Thus, component of $\vec{r}\times\vec{P}$ along the axis is mvr itself. The angular momentum of the whole rigid body about AB is the sum of components of all particles, i.e.,

$$L = \sum_i m_i r_i v_i$$

Here, $v_i = r_i\omega$

$\therefore$ $$L = \sum_i m_i r_i^2 \omega_i$$

or $$L = \omega\sum_i m_i r_i^2$$

or $$L = I\omega.$$

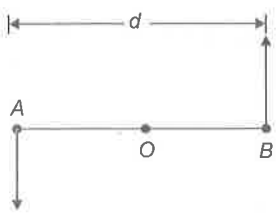
Here, I is the moment of inertia of the rigid body about AB.

Note

Angular momentum about the axis is the component of $I\vec{\omega}$ along the axis. In most of the cases, angular momentum about axis is $I\omega$.

EXAMPLE 30

Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.



SOLUTION

Consider the situation shown in the figure. The velocity of the ball A with respect to the centre O is $v = \frac{\omega d}{2}$.

The angular momentum of the ball with respect to the axis is

$$L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right)$$

$$= \frac{1}{4} m\omega d^2$$

The angular momentum L_2 of the second ball is the same. The angular momentum of the system is equal to the sum of these two angular momenta, i.e., $L = 1/2\ m\omega d^2$. ■

CONSERVATION OF ANGULAR MOMENTUM

The time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it.

or $$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \quad (1)$$

Now, suppose that $\vec{\tau}_{net} = 0$,

then $$\frac{d\vec{L}}{dt} = 0,$$

so that $$\vec{L} = \text{constant}.$$

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the z-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega.$$

It is possible for the moment of inertia I of a rotating body to change by rearrangement of its parts. If no net external torque acts, then L_z must remain constant and if I does change, there must be a compensating change in ω. The principle of conservation of angular momentum in this case is expressed as $I\omega$ = constant.

SOLVED EXAMPLES

EXAMPLE 31

A wheel of moment of inertia I and radius R is rotating about its axis at an angular speed ω_0. It picks up a stationary particle of mass m at its edge. Find the new angular speed of the wheel.

SOLUTION

Net external torque on the system is zero. Therefore, angular momentum will remain conserved. Thus,

$$I_1\omega_1 = I_2\omega_2$$

or $$\omega_2 = \frac{I_1\omega_1}{I_2}$$

Here, $$I_1 = I,\ \omega_1 = \omega_0,\ I_2 = I + mR^2$$

$$\therefore \quad \omega_2 = \frac{I\omega_0}{I + mR^2}$$ ■

Note

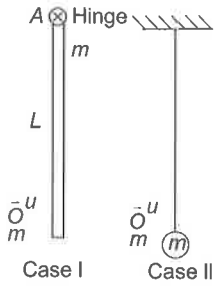


Figure 7.24

Comments on linear momentum

Case I: Linear momentum is not conserved just before and just after collision because during collision hinge force acts as an external force.

Case II: Linear momentum is conserved just before and just after collision because no external force acts on the string.

Comments on angular momentum

Case I: Hinge force acts as an external force during collision but except point A all the other reference points give $\tau_{net} \neq 0$. So angular momentum is conserved only for point A.

Case II: Angular momentum is conserved at all points in the world.

EXAMPLE 32

A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at

point H. A point mass having the same mass m coming with an initial speed u perpendicular to the rod strikes the rod inelastically at its free end. Find out the angular velocity of the rod just after collision.

SOLUTION

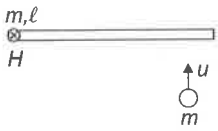
Angular momentum is conserved about H because no external force is present in the horizontal plane which is producing a torque about H.

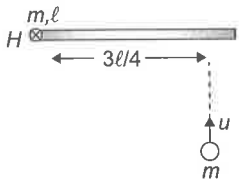
$$mu\ell = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega$$

$$\omega = \frac{3u}{4\ell}$$ ■

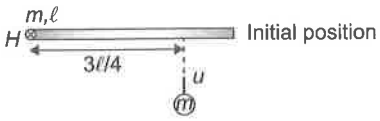
EXAMPLE 33

A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having the same mass m coming with an initial speed u perpendicular to the rod strikes the rod and sticks to it at a distance of $3\ell/4$ from the hinge point. Find out the angular velocity of the rod just after collision.



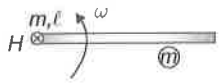


SOLUTION



From angular momentum conservation about H,

initial angular momentum = final angular momentum



$$mu\frac{3\ell}{4} = m\left(\frac{3\ell}{4}\right)^2\omega + \frac{m\ell^2}{3}\omega$$

$$\Rightarrow \quad \frac{3mu\ell}{4} = m\ell^2\left[\frac{1}{3} + \frac{9}{16}\right]\omega$$

$$\frac{3u}{4\ell} = \left[\frac{16+27}{48}\right]\omega$$

$$\Rightarrow \quad \omega = \frac{36u}{43\ell}$$ ■

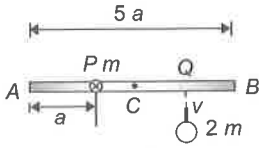
EXAMPLE 34

A uniform rod AB of mass m and length $5a$ is free to rotate on a smooth horizontal table about a pivot through P, a point on AB such that $AP = a$. A particle of mass $2m$ moving on the table strikes AB perpendicularly at the point $2a$ from P with speed v, the rod being at rest. If the coefficient of restitution between them is $\frac{1}{4}$, find their speeds immediately after impact.

SOLUTION

Let the point of impact be Q so that $PQ = 2a$.

Let P be the point of pivot such that $AP = a$.

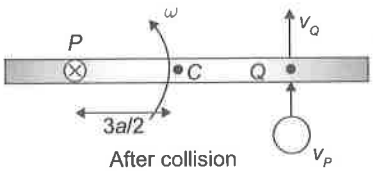


Before collision

Let the velocities of point Q and the particle after impact be v_Q and v_P, respectively then from momentum conservation about point P.

$$L_i = L_f$$

$$2a(2mv) = I_P\omega + (2a)(2mv_P) \qquad (1)$$



After collision

$$I_P = \frac{1}{3}m\left(\frac{5a}{2}\right)^2 + m\left(\frac{3a}{2}\right)^2 \left\{\begin{matrix}\text{use parallel}\\ \text{axis theorem}\end{matrix}\right\}$$

$$= \frac{13ma^2}{3} \qquad (2)$$

Using Eq. (2) in Eq. (1)

$$4ma(v - v_P) = \frac{13ma^2}{3}\omega$$

$$12(v - v_P) = 13a\omega \quad (3)$$

Coefficient of restitution,

$$e = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

$$\frac{1}{4} = \frac{v_Q - v_P}{v}$$

$$\Rightarrow \quad v_Q - v_P = \frac{v}{4} \quad (4)$$

$$v_Q = 2a\omega \quad (5)$$

Using the value of ω from Eq. (3) in Eq. (5),

$$v_Q = 2\left(\frac{12}{13}\right)(v - v_P)$$

So now from Eq. (4),

$$\frac{24}{13}(v - v_P) - v_P = \frac{v}{4}$$

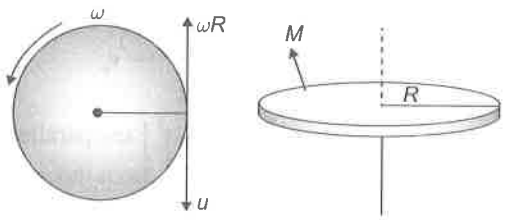
$$\Rightarrow \quad v_P = \frac{83v}{148}$$

Thus, we get $\omega = \frac{15v}{37a}$ ■

EXAMPLE 35

A person of mass m stands at the edge of a circular platform of radius R and moment of inertia. A platform is at rest initially. But the platform rotates when the person jumps off from the platform tangentially with velocity u with respect to the platform. Determine the angular velocity of the platform.

SOLUTION



Let the angular velocity of the platform is ω. Then the velocity of the person with respect to ground v,

$$v_{mD} = v_{mG} - V_{DG}$$

$$u = v_m + \omega R$$

$$v_m = u - \omega R$$

Now from angular momentum conservation,

$$L_i = L_f$$

$$0 = mv_m R - I\omega$$

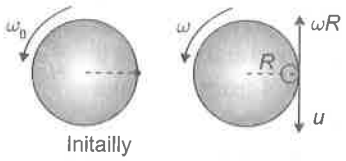
$$\Rightarrow \quad I\omega = m(u - \omega R)R$$

$$\Rightarrow \quad \omega = \frac{muR}{1 + mR^2}$$ ■

EXAMPLE 36

Consider the situation of previous example. If the platform is rotating initially with angular velocity ω_0 and the person jumps off tangentially, determine the new angular velocity of the platform.

SOLUTION



Let the angular velocity of the platform after the person jumps off the mass be ω. Then, velocity of the man,

$$v_m = v_{mp} + v_p$$

$$v_m = u - \omega R.$$

From angular momentum conservation,

$$(I + mR^2)\omega_0 = I\omega - m(u - \omega R)R$$

$$I\omega_0 + mR^2\omega_0 = I\omega - muR + mwR^2$$

$$\Rightarrow \quad \omega = \frac{(1 + mR^2)\omega_0 + muR}{(1 + mR^2)}.$$

Angular Impulse

The angular impulse of a torque in a given time interval is defined as $\int_{t_1}^{t_2} \vec{\tau} dt$.

Here, $\vec{\tau}$ is the resultant torque acting on the body. Further, since

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\therefore \quad \vec{\tau}dt = d\vec{L}$$

or $$\int_{t_1}^{t_2} \vec{\tau}dt = \text{angular impulse}$$

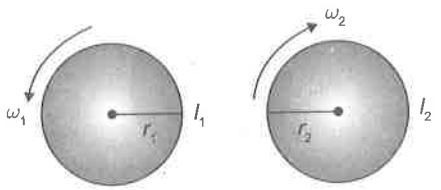
$$= \vec{L}_2 - \vec{L}_1 .$$

Thus, the angular impulse of the resultant torque is equal to the change in angular momentum. Let us take a few examples based on the angular impulse.

SOLVED EXAMPLES

EXAMPLE 37

The figure shows two cylinders of radii r_1 and r_2 having moments of inertia I_1 and I_2 about their respective axes. Initially, the cylinders rotate about their axes with angular speeds ω_1 and ω_2 as shown in the figure. The cylinders are moved closer to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact, but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.



SOLUTION

When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω_1' and ω_2' be the respective angular speeds, we have

$$\omega_1' r_1 = \omega_2' r_2 . \qquad (1)$$

The change in the angular speed is brought about by the frictional force, which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fr_1 and that on the second is fr_2. Assuming $\omega_1 r_1 > \omega_2 r_2$, the corresponding angular impulses are $-fr_1 t$ and $fr_2 t$. We, therefore, have

$$-fr_1 t = I_1(\omega_1' - \omega_1)$$

and $$fr_2 t = I_2(\omega_2' - \omega_2)$$

or, $$-\frac{I_1}{r_1}(\omega_1' - \omega_1)$$

$$= \frac{I_2}{r_2}(\omega_2' - \omega_2) \qquad (2)$$

Solving Eqs. (1) and (2),

$$\omega_1' = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2$$

and $$\omega_2' = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1$$

Kinetic Energy of a Rigid Body Rotating about a Fixed Axis

Suppose a rigid body is rotating about a fixed axis with angular speed ω.

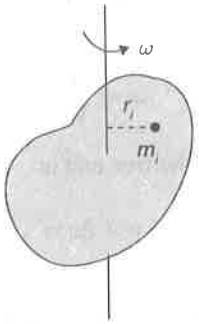


Figure 7.25

Then, kinetic energy of the rigid body will be

$$K = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \sum_i \frac{1}{2} m_i (\omega r_i)^2$$

$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2 = \frac{1}{2} I\omega^2$$

as $(\sum_i m_i r_i^2 = I)$.

Thus, $$KE = \frac{1}{2} I\omega^2$$

Sometimes, it is called the rotational kinetic energy.

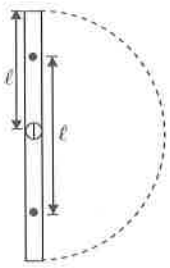
EXAMPLE 38

A uniform rod of mass m and length ℓ is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant?

375

SOLUTION

As the rod reaches its lowest position, the centre of mass is lowered by a distance ℓ. Its gravitational potential energy is decreased by $mg\ell$. As no energy is lost against friction, this should be equal to the increase in the kinetic energy. As the rotation occurs about the horizontal axis through the clamped end, the moment of inertia is $I = ml^2/3$. Thus,



$$\frac{1}{2}I\omega^2 = mg\ell$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2 = mg\ell$$

or $$\omega = \sqrt{\frac{6g}{\ell}}$$

The linear speed of the free end is

$$v = \ell\omega = \sqrt{6g\ell}\,.$$ ■

COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

We have already learnt about translational motion caused by a force and rotational motion about a fixed axis caused by a torque. Now we are going to discuss a motion in which a body undergoes translational as well as rotational motion. Rolling is an example of such motion. If the axis of rotation is moving, then the motion is combined translational and rotational motion.

To understand the concept of combined translational and rotational motion, we consider a uniform disc rolling on a horizontal surface. Velocity of its centre of mass is V_{COM} and its angular speed is ω as shown in the figure.

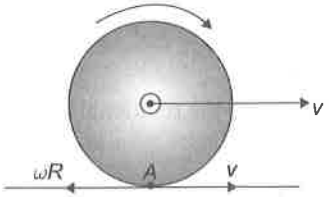


Figure 7.26

Let us take a point A on the disc and concentrate on its motion.

Path of point A with respect to the ground will be a cycloid as shown in Fig. 7.27.

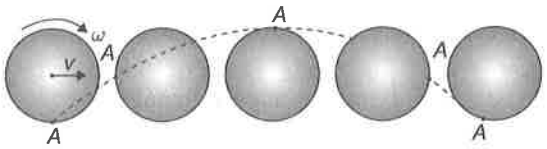


Figure 7.27

Motion of point A with respect to the centre of mass is pure rotational while the centre of mass itself is moving in a straight line. So for the analysis of rolling motion, we deal with translational motion separately and rotational motion separately and then we combine the result to analyze the overall motion.

The velocity of any point A on the rigid body can be obtained as

$$\vec{V}_A = \vec{V}_{COM} + \vec{V}_{A\text{-}COM}$$

$$\left|\vec{V}_{COM}\right| = V$$

$\left|\vec{V}_{A\text{-}COM}\right| = r\omega$ in the direction $\perp$ to line OA.

Thus, the velocity of point A is the vector sum of $\vec{V}_{COM}$ and $\vec{V}_{P\text{-}COM}$ as shown in the Fig. 7.28.

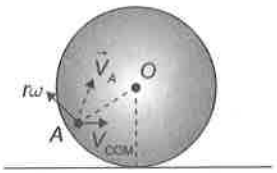


Figure 7.28

Important Points in Combined Rotational + Translational Motion

1. **Velocity of any point of the rigid body in combined $R + T$ motion is the vector sum of v (velocity of centre of mass) and $r\omega$**

   For example

   A disc of radius r has linear velocity v and angular velocity ω as shown in Fig. 7.29, then find the velocity of points A, B, C and D on the disc.

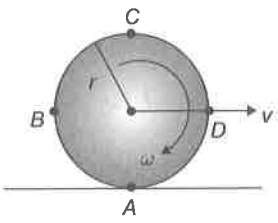


Figure 7.29

We divide our problem in two parts:

1. Pure rotational
2. Pure translational about the centre of mass.

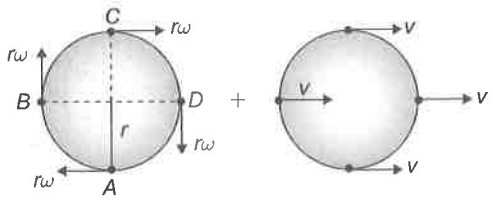


Figure 7.30

Combining the above results,

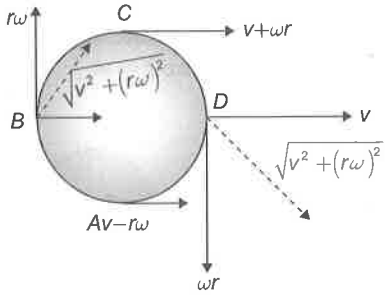


Figure 7.31

2. **In combined rotational and translational motion, angular velocity of any point of a rigid body with respect to another point in the rigid body is always the same.**
   For example

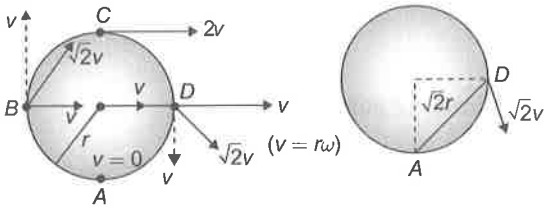


Figure 7.32

Now for ω_{DA} $\quad \omega_{DA} = \dfrac{\sqrt{2}v}{\sqrt{2}r} = \dfrac{v}{r}$.

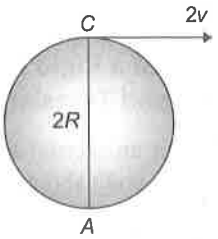


Figure 7.33

For ω_{CA},

$$\omega_{CA} = \frac{2v}{2r} = \frac{v}{r}.$$

For ω_{DB},

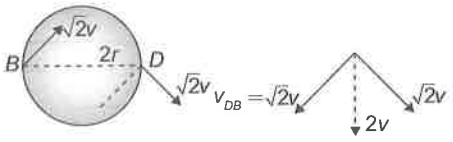


Figure 7.34

$$\Rightarrow \quad \omega_{DB} = \frac{2v}{2r} = \frac{v}{r}.$$

3. **Distance moved by the centre of mass of the rigid body in one full rotation is $2\pi R$.**

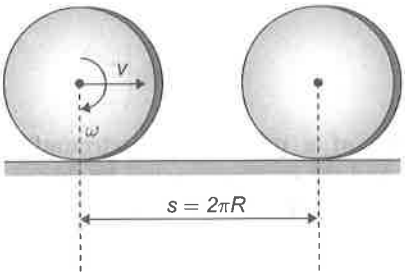


Figure 7.35

This can be shown as follows:
In one rotation angular displacement $\theta = 2\pi = \omega t$,

$$s = vT$$

$$= (\omega R)\left(\frac{2\pi}{\omega}\right) = 2\pi R.$$

In forward slipping

$$s > 2\pi R \text{ (as } v > \omega R)$$

and in backward slipping,

$$s < 2\pi R \text{ (as } v < \omega R).$$

4. **The speed of a point on the circumference of the body at any instant t is $2R\omega \sin \dfrac{\omega t}{2}$.**

Proof

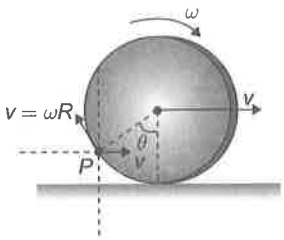


Figure 7.36

$$v_{xP} = v - v\cos\theta$$

$$= v[1 - \cos\theta]$$

$$v_{yP} = v\sin\theta$$

$$|\vec{v}_P| = \sqrt{v^2\sin^2\theta + v^2(1-\cos\theta)^2}$$

$$v = \sqrt{2v^2 - 2v^2\cos\theta}$$

$$= \sqrt{2v}(1-\cos\theta)^{1/2}$$

$$= 2v\sin\left(\frac{\theta}{2}\right)$$

$$= 2v\sin\left(\frac{\omega t}{2}\right)$$

$$= 2R\omega\sin\left(\frac{\omega t}{2}\right).$$

5. The path of a point on the circumference is a cycloid and the distance moved by this point in one full rotation is $8R$.

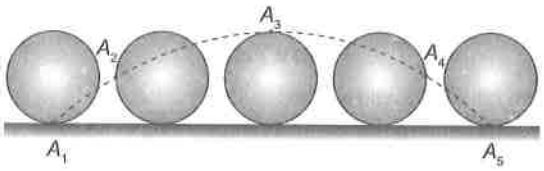


Figure 7.37

In Fig. 7.37, the dotted line is a cycloid and the distance $A_1A_2\ldots A_5$ is $8R$. This can be proved as follows.

According to point (3), speed of point A at any moment is

$$v_A = 2R\omega\sin\left(\frac{\omega t}{2}\right)$$

Distance moved by A in time dt is,

$$ds = v_A\,dt$$

$$= 2R\omega\sin\left(\frac{\omega t}{2}\right)dt$$

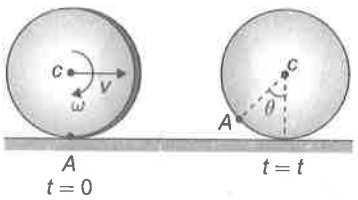


Figure 7.38

Therefore, the total distance moved in one full rotation is

$$S = \int_0^{T=2\pi/\omega} ds$$

or

$$S = \int_0^{T=2\pi/\omega} 2R\omega\sin\left(\frac{\omega t}{2}\right)dt$$

On integration, we get

$$s = 8R.$$

6. x- and y-coordinates of the bottommost point at any time t

At time t, the bottommost point will rotate at an angle $\theta = \omega t$ with respect to the centre of the disc C. The centre C will travel a distance $s = vt$.

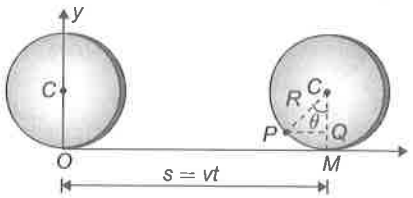


Figure 7.39

In Fig. 7.39,

$$PQ = R\sin\theta$$

$$= R\sin\omega t$$

$$CQ = R\cos\theta$$

$$= R\cos\omega t.$$

Coordinates of point P at time t are

$$x = OM - PQ$$

$$= vt - R\sin\omega t$$

and

$$y = CM - CQ$$

$$= R - R\cos\omega t$$

$$\therefore \quad (x, y) \equiv (vt - R\sin\omega t,\ R - R\cos\omega t).$$

Angular Momentum of a Rigid Body in Combined Rotation and Translation

Let O be a fixed point in an inertial frame of reference. Angular momentum of the body about O is

$$\vec{L} = \vec{L}_{COM} + M(\vec{r}_0 \times \vec{v}_0)$$

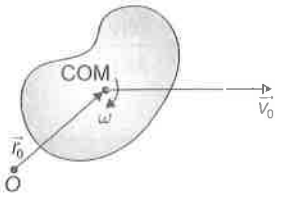


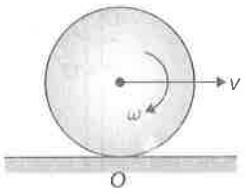
Figure 7.40

The first term $\vec{L}_{COM}$ represents the angular momentum of the body as seen from the centre of mass frame. The second term $M(\vec{r}_0 \times \vec{v}_0)$ equals the angular momentum of centre of mass about point O.

SOLVED EXAMPLES

EXAMPLE 39

A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = \frac{v}{R}$ in the clockwise direction as shown in the figure. Find the magnitude of the total angular momentum of the disc about the bottommost point O of the disc.



SOLUTION

$$\vec{L} = \vec{L}_{COM} + m(\vec{r}_0 \times \vec{v}_0) \qquad (1)$$

Here,

$$\vec{L}_{COM} = I\omega$$

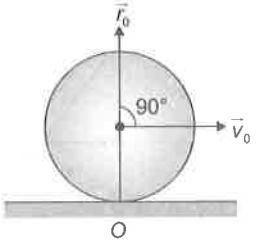
(perpendicular to paper inwards)

$$= \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)$$

$$= \frac{1}{2}mvR$$

and $m(\vec{r}_0 \times \vec{v}_0) = mRv$

(perpendicular to paper inwards).



Since, both the terms of the right-hand side of Eq. (1) are in the same direction,

$$|\vec{L}| = \frac{1}{2}mvR + mvR$$

or $$|\vec{L}| = \frac{3}{2}mvR$$ ■

Kinetic Energy of a Rolling Body

If a body of mass M is rolling on a plane such that velocity of its centre of mass is V and its angular speed is ω, its kinetic energy is given by

$$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2,$$

I is the moment of inertia of the body about the axis passing through the centre of mass.

In case of rolling without slipping,

$$KE = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}I\omega^2$$

$$\left[\because \quad v = \omega R \frac{F(x-RC)}{R(C+1)}\right.$$

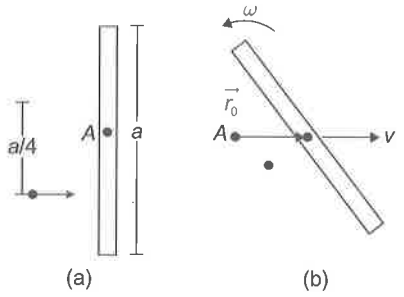
$$= \frac{1}{2}[MR^2 + I]\omega^2$$

$$= \frac{1}{2}I_c\omega^2.$$

I_c is the moment of inertia of the body about the axis passing through the point of contact.

EXAMPLE 40

A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance $a/4$ from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.



SOLUTION

The situation is shown in the figure. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the angular momentum of the system about any line will remain constant.

Suppose the velocity of the centre of the rod is V and the angular velocity about the centre is ω.

(a) The linear momentum before the collision is mv and that after the collision is MV. Thus,

$$mv = MV$$

or $$V = \frac{m}{M}v.$$

(b) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of 'the rod plus the particle' system about AB. Initially, the rod is at rest. The angular momentum of the particle about AB is

$$L = mv(a/4).$$

After the collision, the particle mass to rest. The angular momentum of the rod about A is

$$\vec{L} = \vec{L}_{COM} + M\left(\vec{r}_0 \times \vec{V}\right).$$

As $$\vec{r}_0 \parallel \vec{V},$$

$$\vec{r}_0 \times \vec{V} = 0$$

Thus, $$\vec{L} = \vec{L}_{COM}.$$

Hence, the angular momentum of the rod about AB is

$$L = I\omega = \frac{Ma^2}{12}\omega.$$

Thus, $$\frac{mva}{4} = \frac{Ma^2}{12}\omega$$

or $$\omega = \frac{3mv}{Ma}.$$

EXAMPLE 41

A uniform rod of length λ lies on a smooth horizontal table. A particle moving on the table has a mass m and a speed v before the collision and it sticks to the rod after the collision. The rod has a mass M, then find out

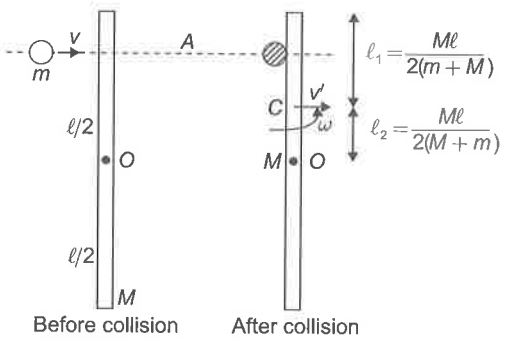
(a) the moment of inertia of the system about the vertical axis passing through the centre of mass C after the collision.

(b) the velocity of the centre of mass C and the angular velocity of the system about the centre of mass after the collision.

SOLUTION.

The figure shows the situation of the system just before and just after collision.

Initially, the centre of mass of the rod is at point O. After collision when the particle sticks to the rod, centre of mass is shifted from point O to C as shown in the figure. Now the system is rotated about the axis passing through C.



Now from linear momentum conservation,

$$mv = (M+m)v'$$

$$\Rightarrow \quad v' = \frac{mv}{M+m}.$$

(a) Let us assume that the moment of inertia of the system about C is 1. Then,

$$I = I_{(rod)C} + I_{(part)C},$$

$$I = I_0 + M\ell_2^2 + m\ell_1^2$$

$$I = \frac{M\ell^2}{12} + \frac{Mm^2\ell^2}{4(m+M)^2} + \frac{mM^2\ell^2}{4(m+M)^2}$$

$$\Rightarrow \qquad I = \frac{M(M+4m)}{12(m+M)}\ell^2$$

(b) From angular momentum conservation about A,

$$L_i = L_f$$

$$0 + 0 = I\omega - (m+M)v'\ell_1$$

$$\Rightarrow \qquad I\omega = (m+M)\,v'\ell_1$$

Subsituting the value of I, v', and ℓ_1, we get

$$\omega = \frac{6mv}{(M+4m)\ell}.$$ ■

Acceleration of a Point on the Circumference of the Body in $R + T$ Motion

(A) Both ω and v are constant

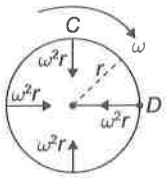


Figure 7.41

(B) When ω is constant and v is variable

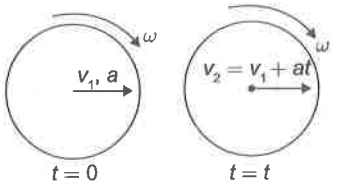


Figure 7.42

So acceleration of different points on the body is given by following Fig. 7.43.

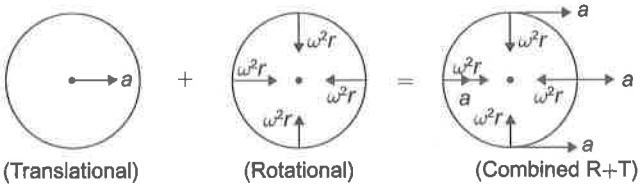


Figure 7.43

(C) When ω is variable and v is constant

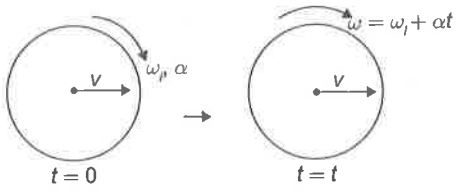


Figure 7.44

So acceleration of different points on the body is given by the following way:

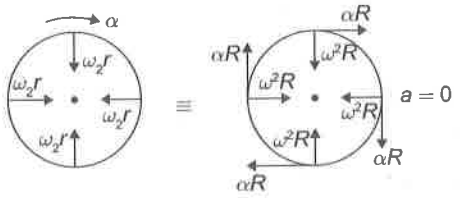


Figure 7.45

(D) When both ω & v are variables

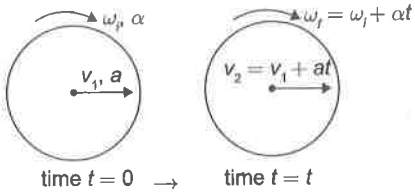


Figure 7.46

Now the net acceleration of different points on the rigid body is given by the following way.

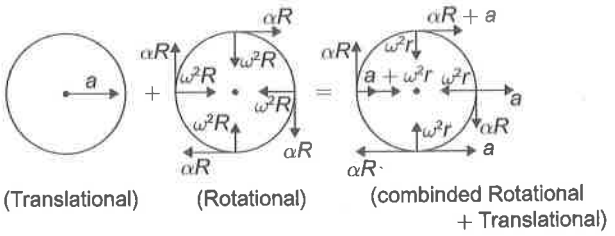


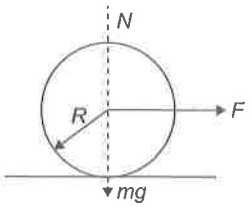
Figure 7.47

SOLVED EXAMPLES

EXAMPLE 42

A force F acts at the centre of a thin spherical shell of mass m and radius R.

Find the acceleration of the shell if the surface is smooth.



SOLUTION

$\because$ Force F, mg and N pass through the centre, $\tau_{net} = 0$, i.e., body is in rotational equilibrium.

But $\vec{F}_{net} = F$, so the body moves with a constant acceleration $a = \dfrac{F}{m}$. ■

EXAMPLE 43

In a previous problem, if force F is applied at a distance x above the centre, then find out the linear and angular accelerations.

SOLUTION

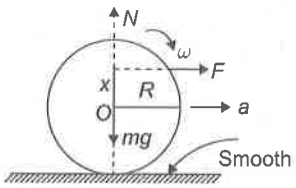
This force F translates the body linearly as well as rotates it. So,

Net torque about O is $\tau_0 = Fx$.

From rotational motion, $\tau_0 = I\alpha$.

$$\alpha = \frac{\tau}{I} = \frac{Fx}{\frac{2MR^2}{3}}$$

$$\Rightarrow \quad \alpha = \frac{3Fx}{2MR^2}.$$



From linear motion of a sphere,

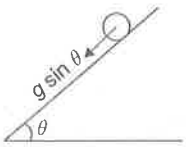
$$F = ma$$

$$\Rightarrow \quad a = \frac{F}{m}.$$ ■

EXAMPLE 44

A rigid body of mass m and radius r starts coming down an inclined plane of inclination θ. Then find out the acceleration of centre of mass if friction is absent.

SOLUTION



Friction is absent, so the body is moving down the incline without rolling, so acceleration of the centre of mass is $g \sin \theta$. ■

UNIFORM PURE ROLLING

Pure rolling means no relative motion (or no slipping at point of contact between two bodies).

For example, consider a disc of radius R moving with a linear velocity v and an angular velocity ω on a horizontal ground. The disc is said to be moving without slipping if velocities of points P and Q (shown in Fig. 7.48b) are equal, i.e.,

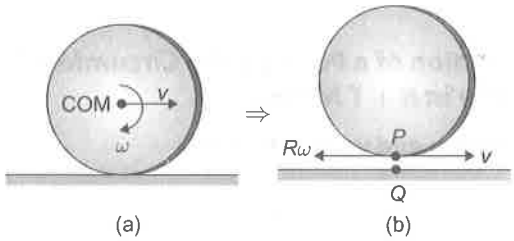


Figure 7.48

$$v_P = v_Q$$

or $$v - R\omega = 0$$

or $$v = R\omega.$$

If $v_P > v_Q$ or $v > R\omega$, the motion is said to be forward slipping and if $v_P < v_Q < R\omega$, the motion is said to be backward slipping.

Now, suppose an external force is applied to the rigid body, the motion will no longer remain uniform. The condition of pure rolling on a stationary ground is

$$a = R\alpha.$$

Thus, $v = R\omega$, $a = R\alpha$ is the condition of pure rolling on a stationary ground. Sometimes it is said to be simply rolling.

Note

We can represent the moment of inertia of a different rigid body in the following way.

$$I = CMR^2,$$

value of $C = 1$ for circular ring (R),

$C = \dfrac{1}{2}$ for a circular disc (D) and solid cylinder (SC).

Note (contd)

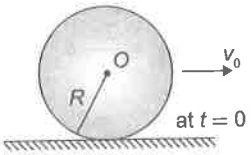
$C = \frac{2}{3}$ for a hollow sphere (HS),

$C = \frac{2}{5}$ for a solid sphere (SS).

SOLVED EXAMPLE

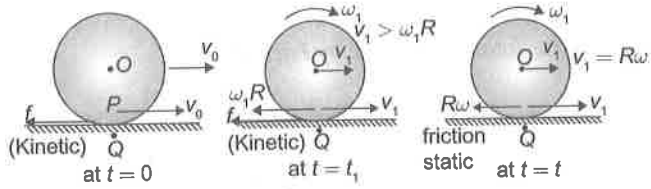
EXAMPLE 45

A rigid body $I = CMR^2$ is set into motion on a rough horizontal surface with a linear speed v_0 in the forward direction at time $t = 0$ as shown in the figure. After what time slipping finally stops and pure rolling starts. Find the linear speed of the body after it starts pure rolling on the surface.



SOLUTION

According to the given condition in the problem, the point P in the body moves with speed v_0 while the point Q on the ground is at rest. So the friction that acts on the body is in the backward direction, which produces the resultant torque on the body and increases the angular speed ω as shown in the figure.



As shown in the above figure, initially $v > R\omega$ so forward slipping takes place. After introducing the friction, speed decreases and ω increases and at time $t = t$, the relation $v = r\omega$ is satisfied. Therefore, pure rolling starts. Initially, the friction is kinetic until the motion is in slipping condition. Afterwards, at $v = r\omega$, friction is static. We divide the above problem into two parts.

1. *Translational Motion*

Linear acceleration

$$a = -mg.$$

So after time t,

$$v = v_0 - mgt \tag{1}$$

2. *Rotational Motion*

From $\tau_{net} = I\alpha$,

only frictional force is responsible for providing torque. So torque about O is

$$f \cdot R = I\alpha$$

$$\mu mgR = CmR^2\alpha \tag{2}$$

α is angular acceleration of the body.

From Eq. (2),

$$\alpha = \frac{\mu g}{CR}.$$

From $\omega_f = \omega_i + at$,

$$\omega = at$$

$$\Rightarrow \quad \omega = \frac{\mu g}{CR} \cdot t$$

$\therefore \quad \omega = \frac{v}{R}$ is at pure rolling condition.

So, $$v = \frac{\mu g t}{C} \tag{3}$$

From Eqs. (1) and (3),

$$v_0 - \mu gt = \frac{\mu gt}{C}$$

$$t = \frac{v_0 C}{\mu g(1+C)} \tag{4}$$

Equation (4) gives the time after the pure rolling starts. Substituting the values from Eq. (4) to Eq. (1),

$$v = v_0 - \frac{v_0 C}{(1+C)}$$

$$\Rightarrow \quad v = \frac{v_0}{1+C} \tag{5}$$

Equation (5) gives the linear speed at pure rolling situation.

Alternate Solution

Net torque on the body about the bottommost point A is zero. Therefore, angular momentum of the body will remain conserved about the bottommost point.

Net torque about A, $\tau_A = 0$,

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⇒ From angular momentum conservation,

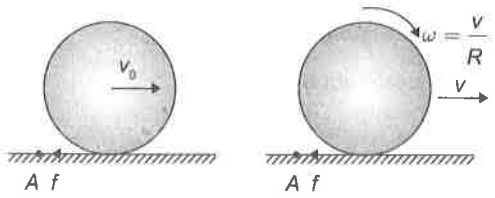
$$L_i = L_f$$

$$mv_0R = I\omega + mvR$$

$$mv_0R = CmR^2\frac{v}{R} + mvR$$

$$v_0 = Cv + v$$

⇒
$$v = \frac{v_0}{1+C}$$



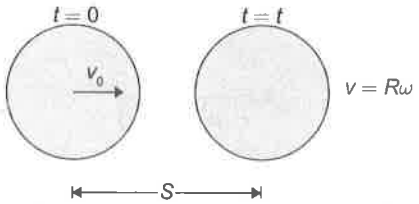
SOLVED EXAMPLES

EXAMPLE 46

In the previous problem consider a rigid body such as a solid cylinder, then find out the work done by friction from time $t = 0$ to $t = t$ (at $v = r\omega$).

SOLUTION.

Let us suppose that in between time $t = 0$ to $t = t$, the cylinder is displaced by a distance s.



Translational work done by friction + rotational work done by friction

Now calculate each type of work done one by one.

1. *Translational Work done by Friction*

For solid cylinder,

$$c = \frac{1}{2}.$$

From Eq. (5), $$v = \frac{v_0}{1+\frac{1}{2}} = \frac{2}{3}v_0.$$

From the equation

$$v_f^2 = u_i^2 + 2as$$

⇒
$$\left(\frac{2}{3}v_0\right)^2 = (v_0)^2 - 2\mu gs$$

$$s = \frac{5v_0^2}{18\mu g}.$$

Translational work done by friction $= -fs$

$$(WD)_{fT} = -\mu mg \cdot \frac{5v_0^2}{18\mu g} = -\frac{5mv_0^2}{18}.$$

2. *Rotational Work done by Friction*

We know that

$$\tau = I\alpha$$

$$\alpha = \frac{\tau}{I}$$

From $$\omega_f^2 - \omega_i^2 = 2\alpha\theta,$$

$$\left(\frac{2v_0}{3R}\right)^2 = \frac{2\tau}{I}\cdot\theta.$$

Put $$I = \frac{mR^2}{2}$$

$$\theta = \frac{v_0^2 m}{9\tau}.$$

Rotational work done by friction

$$W = \tau\theta$$

$$W_{fR} = \frac{v_0^2 m}{9}.$$

3. *So the Total Work done by Friction*

$$W = W_{fT} + W_{fR}$$

$$= -\frac{5}{18}mv_0^2 + \frac{v_0^2 m}{9}$$

$$W = -\frac{mv_0^2}{6}.$$

Alternative method

From work–energy theorem,

work done by friction = change in kinetic energy.

$$(WD)_f = \Delta K = k_f - k_i.$$

Now, $$k_f = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega^2$$

$$k_f = \frac{1}{2}m\left(\frac{2v_0}{3}\right)^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{2v_0}{3R}\right)^2$$

$$\left\{\because\ v_f = \frac{2v_0}{3}\right\}$$

$$k_f = \frac{mv_0^2}{3}$$

$$k_i = \frac{1}{2}mv_0^2$$

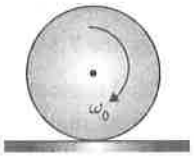
So, $$(\text{WD})_f = \frac{mv_0^2}{3} - \frac{1}{2}mv_0^2$$

$\Rightarrow$ $$(\text{WD})_f = -\frac{mv_0^2}{6}.$$

To calculate work done, we mostly prefer the alternative method. ■

EXAMPLE 47

A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed w_0 and no linear velocity. If the coefficient of friction is μ, find the linear velocity v and angular velocity ω at the end of slipping.

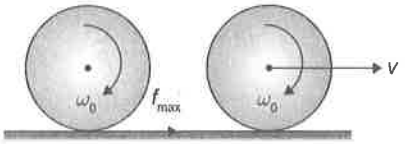


SOLUTION

Let m be the mass of the sphere.

Since it is a case of backward slipping, force of friction is in the forward direction. Limiting friction will act in this case.

Net torque on the sphere about the bottommost point is zero. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.



$$L_i = L_f$$

$\therefore$ $$I\omega_0 = I\omega + mrv$$

or $$\frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

$\therefore$ $$\omega = \frac{2}{7}\omega_0$$

and $$v = r\omega = \frac{2}{7}r\omega_0.$$ ■

Pure Rolling When Force *F* acts on a Body

Suppose a force F is applied at a distance x above the centre of a rigid body of radius R, mass M and moment of inertia CMR^2 about an axis passing through the centre of mass. Now, the applied force F can produce by itself

1. a linear acceleration a and
2. an angular acceleration α

If $a = R\alpha$, then there is no need of friction and force of friction $f = 0$.

If $a < R\alpha$, then to support the linear momentum, the force of friction f will act in the forward direction.

Similarly, if $a > R\alpha$, then due to no support, the angular motion the force of friction will act in the backward direction.

So, in this case force of friction will be either backward, forward or even zero also. It all depends on M, I and R. For calculation, you choose any direction of friction. Let we assume it to be in the forward direction.

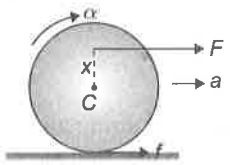


Figure 7.49

Let

a = linear acceleration,

α = angular acceleration

From linear motion,

$$F + f = Ma \qquad (1)$$

From rotational motion,

$$Fx - fR = I\alpha$$

$$Fx - fR = CMR^2\frac{a}{R}$$

$$Fx - fR = CMaR \qquad (2)$$

From Eqs. (1) and (2),

$$F(x + r) = MaR(C + 1)$$

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$$a = \frac{F(R+x)}{MR(C+1)} \quad (3)$$

Substituting the value from Eq. (3) to Eq. (1),

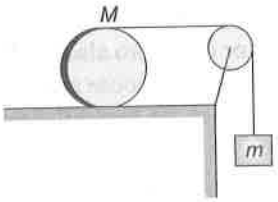
$$f = \frac{F(x-RC)}{R(C+1)}.$$

Thus, f should be $\leq \mu_s mg$ for pure rolling.

SOLVED EXAMPLE

EXAMPLE 48

Consider the arrangement shown in the figure. The string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a massless, frictionless pulley to a falling weight. Determine the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.



SOLUTION

Let T be the tension in the string and f the force of (static) friction, between the cylinder and the surface

a_1 = acceleration of centre of mass of the cylinder towards right

a_2 = downward acceleration of block m

α = angular acceleration of the cylinder (clockwise).

Equations of motion are given as

for block $\quad mg - T = ma_2 \quad (1)$

for cylinder, $\quad T + f = Ma_1 \quad (2)$

$$\alpha = \frac{(T-f)R}{\frac{1}{2}MR^2} \quad (3)$$

The string attaches the mass m to the highest point of the cylinder, hence

$$v_m = v_{COM} + R\omega.$$

Differentiating, we get

$$a_2 = a_1 + R\alpha \quad (4)$$

We also have (for rolling without slipping)

$$a_1 = R\alpha \quad (5)$$

Solving these equations, we get

$$a_2 = \frac{8mg}{3M+8m}$$

■

Note

Work done by friction in pure rolling on a stationary ground is zero as the point of application of the force is at rest. Therefore, mechanical energy can be conserved if all other dissipative forces are ignored.

Pure Rolling on an Inclined Plane

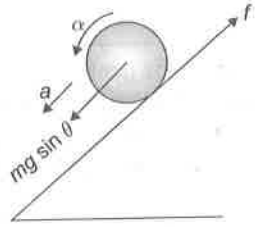


Figure 7.50

A rigid body of radius R and mass m is released at rest from height h on the incline whose inclination with the horizontal is θ and assume that friction is sufficient for pure rolling, then

$$a = \alpha R$$

and $\quad v = R\omega.$

From Fig. 7.50,

$$mg\sin\theta - f = ma \quad (1)$$

$$\{F_{net} = ma\}$$

$$fR = CmR^2 \cdot \frac{a}{R} \quad (2)$$

$$\{F_{net} = Ia\}.$$

From Eqs. (1) and (2),

$$a = \frac{g\sin\theta}{1+C}$$

So the body which has a low value of C has a greater acceleration.

Value of $C = 1$ for circular ring (R).

$C = \frac{1}{2}$ for circular disc (D) and solid cylinder (SC).

$C = \frac{2}{3}$ for hollow sphere (HS).

$C = \frac{2}{5}$ for solid sphere (SS).

So, descending order of a is given as

$$a_{SS} > a_D = a_{SC} > a_{HS} > a_R$$

and order of time of descent is

$$t_{SS} < t_D = t_{SC} < t_{HS} < t_R.$$

Kinetic Energy

Work done by friction in pure rolling is zero. Therefore,
Increase in kinetic energy = change in potential energy

$$\Rightarrow \qquad KE = mgh,$$

i.e., kinetic energy is constant for all rigid bodies rolling down the incline.

Requirement of Friction

From Eq. (2),

$$f = Cma$$

$$f = \frac{mg\sin\theta}{\left(1+\frac{1}{C}\right)} \qquad (3)$$

From Eq. (3), as the value of C increases, requirement of friction increases.

SOLVED EXAMPLES

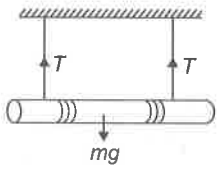
EXAMPLE 49

A cylinder of mass M is suspended through two strings wrapped around it as shown in the figure. Find the tension in the string and the speed of the cylinder as it falls through a distance h.

SOLUTION

The portion of the strings between the ceiling and cylinder are at rest. Hence, the points of the cylinder where the strings leave it are also at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is given by

$$\alpha = \frac{a}{R} \qquad (1)$$



as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$Mg - 2T = Ma$$

and for the motion about the centre of mass, it is

$$2TR = \left(\frac{MR^2}{2}\right)\alpha,$$

where $$I = \frac{MR^2}{2}$$

$$2TR = \frac{MR^2}{2}\,\frac{a}{R}$$

$$\Rightarrow \qquad 2T = \frac{Ma}{2} \qquad (2)$$

On adding Eqs.(1) and (2),

$$Mg = \frac{Ma}{2} + Ma; \quad \frac{3a}{2} = g$$

$$a = \frac{2g}{3}$$

$$\therefore \qquad 2T = \frac{M}{2}\cdot\frac{2g}{3}$$

$$\Rightarrow \qquad T = \frac{Mg}{6}.$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen a height h is given by

$$v^2 = 2\left[\frac{2g}{3}\right]h$$

or $$v = \sqrt{\frac{4gh}{3}}.$$ ■

EXAMPLE 50

A thin massless thread is wound on a reel of mass 3 kg and moment of inertia 0.6 kgm^2. The hub radius is $R = 10$ cm and peripheral radius is $2R = 20$ cm. The reel is placed on

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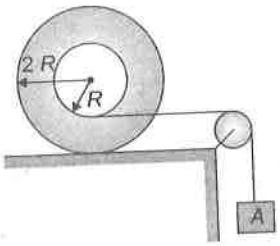
a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of 1 kg.



SOLUTION.

Let

a_1 = acceleration of centre of mass of reel

a_2 = acceleration of 1 kg block

α = angular acceleration of reel (clockwise)

T = tension in the string

and f = force of friction.

Free-body diagram of the reel is as shown below (only horizontal forces are shown).

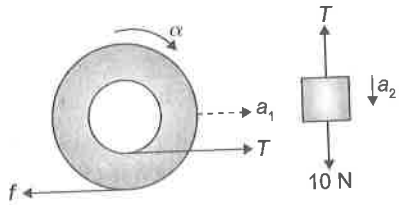
Equations of motion are given as

$$T - f = 3a_1 \quad (1)$$

$$\sigma = \frac{\tau}{I} = \frac{f(2R) - T.R}{I}$$

$$= \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \quad (2)$$

Free-body diagram of the mass



Equation of motion is

$$10 - T = a_2 \quad (3)$$

For no slipping condition,

$$a_1 = 2R\alpha$$

or $$= a_1 = 0.2\alpha \quad (4)$$

and $$a_2 = a_1 - R\alpha$$

or $$a_2 = a_1 - 01\alpha \quad (5)$$

Solving the above five equations, we get

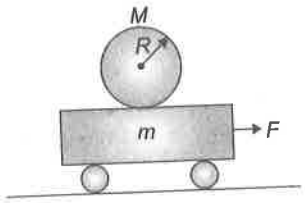
$$a_1 = 0.27 \text{ m/s}^2$$

and $$a_2 = 0.135 \text{ m/s}^2.$$

■

EXAMPLE 51

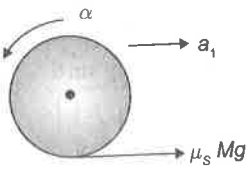
Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are m_s and m_k, respectively.

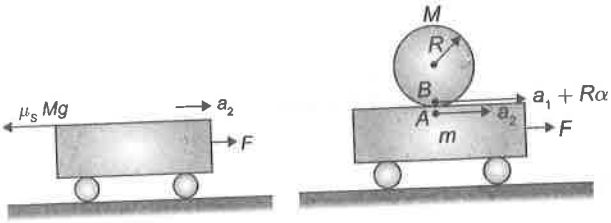


SOLUTION

The freebody diagrams of the sphere and the plank are as shown below.

Writing the equations of motion:





For sphere: Linear acceleration

$$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \quad (1)$$

Angular acceleration

$$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2} = \frac{5}{2}\frac{\mu_s g}{R} \quad (2)$$

For plank: Linear acceleration

$$a_2 = \frac{F - \mu_s Mg}{m} \quad (3)$$

For no slipping, acceleration of point B and A is the same.

So, $$a_2 = a_1 + R\alpha \quad (4)$$

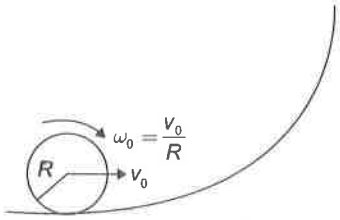
Solving the above four equations, we get

$$F = \mu_s g\left(M + \frac{7}{2}m\right).$$

Thus, maximum value of F can be $\mu_s g\left(M + \frac{7}{2}m\right)$. ■

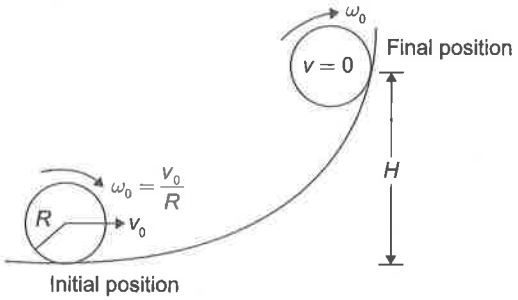
EXAMPLE 52

Find out the maximum height attained by the solid sphere on a frictionless track as shown in the figure.



SOLUTION.

Let us assume that the sphere attains a maximum height H on the track.



As the sphere moves upwards, the speed is decreased due to gravity but there is no force to change ω_0 (frictionless track). So from energy conservation,

$$\frac{1}{2}mv_0^2 + \frac{1}{2}I\mu v_0^2 = mgH_{max} + \frac{1}{2}I\omega_0^2,$$

$$H_{max} = \frac{v_0^2}{2g}.$$ ■

TOPPLING

You might have seen in your practical life that if a force F is applied to a block A of smaller width, it is more likely to topple down before sliding, while if the same force F is applied to an another block B of broader base, chances of its sliding are more compared to its toppling. Have you ever thought why it happens so. To understand it better, let us take an example.

Figure 7.51

Suppose a force F is applied at a height b above the base AE of the block. Further, suppose the friction f is sufficient to prevent sliding. In this case, if the normal reaction N also passes through C, then despite the fact that the block is in translational equilibrium ($F = f$ and $N = mg$), an unbalanced torque (due to the couple of forces F and f) is there. This torque has a tendency to topple the block about point E. To cancel the effect of this unbalanced torque, the normal reaction N is shifted towards right by a distance a such that the net anticlockwise torque is equal to the net clockwise torque or

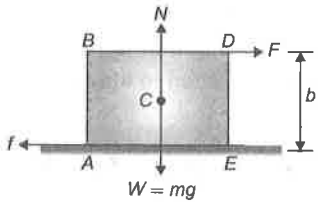


Figure 7.52

$$Fb = (mg)a$$

or $$a = \frac{Fb}{mg}.$$

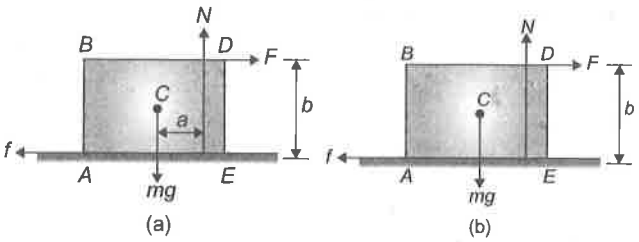


Figure 7.53

Now, as F or b (or both) are increased, distance a also increases. But it cannot go beyond the right edge of the block. So, in extreme cases (beyond which the block will topple down), the normal reaction passes through E as shown in Fig. 7.53.

Now, if F or b are further increased, the block will topple down. This is why the block having the broader base has less chances of toppling in comparison to a block of smaller base because the block of larger base has more margin for the normal reaction to shift.

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Why Rolling is so Easy on the Ground

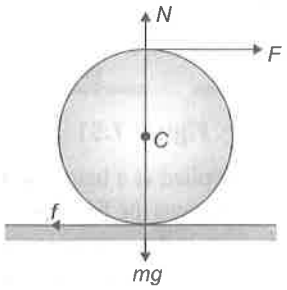


Figure 7.54

Because in this case the normal reaction has zero margin to shift, so even if the body is in translational equilibrium ($F = f$, $N = mg$), an unbalanced torque is left behind and the body starts rolling clockwise. As soon as the body starts rolling, the force of friction is so adjusted (both in magnitude and direction) that either the pure rolling starts (if friction is sufficient enough) or the body starts sliding. Let us take few examples related to toppling.

SOLVED EXAMPLE

EXAMPLE 53

A uniform block of height h and width a is placed on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under these conditions the block will

(a) Slide before toppling:
The block will slide when

$$mg \sin\theta > f$$

$$mg \sin\theta > mmg \cos\theta$$

$$\tan\theta > \mu,$$

i.e., block is at rest when

$$\tan\theta \leq \mu \qquad (1)$$

(b) Now suppose the friction f is sufficient to prevent sliding. Then we assume that N is shifted downward to a distance x to prevent toppling. Therefore, torque about O is zero.

$$f\cdot\frac{h}{2} = Nx$$

$$mg \sin\theta \frac{h}{2} = mg \cos\theta\, x$$

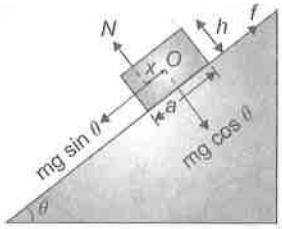
$$x = \frac{\tan\theta \cdot h}{2}.$$

Maximum value of x is $a/2$.
So to prevent toppling,

$$x \leq \frac{a}{2}$$

$$\Rightarrow \quad \frac{\tan\theta \cdot h}{2} \leq \frac{a}{2}$$

$$\Rightarrow \quad \tan\theta \leq \frac{a}{h} \qquad (2)$$



So, the block topples before sliding.
From Eqs. (1) and (2),

$$m_s > \frac{a}{h}$$

INSTANTANEOUS AXIS OF ROTATION

The combined effects of translation of the centre of mass and rotation about an axis through the centre of mass are equivalent to a pure rotation with the same angular speed about an axis passing through a point of zero velocity. Such an axis is called the instantaneous axis of rotation (IAOR). This axis is always perpendicular to the plane used to represent the motion and the intersection of the axis with this plane defines the location of instantaneous centre of zero velocity (IC).

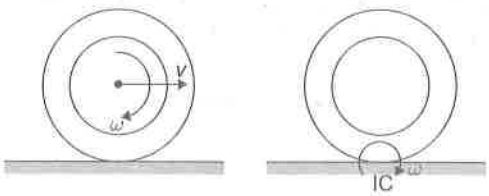


Figure 7.55

For example, consider a wheel which rolls without slipping. In this case, the point of contact with the ground has zero velocity. Hence, this point represents the IC for the wheel. If it is imagined that the wheel is momentarily pinned at this point, the velocity of any point on the wheel can be found using $v = r\omega$. Here, r is the distance of the point from IC.

Similarly, the kinetic energy of the body can be assumed to be pure rotational about IAOR or

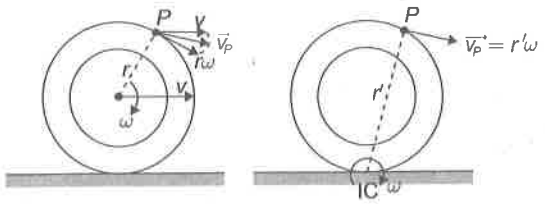


Figure 7.56

$$KE = \frac{1}{2} I_{IAOR} \omega^2$$

Rotation + Translation ⇒ Pure rotation about IAOR passing through *IC*

$$KE = \frac{1}{2} m v_{COM}^2 + \frac{1}{2} I_{COM} \omega^2$$

$$\Rightarrow \qquad KE = \frac{1}{2} I_{IAOR} \omega^2 .$$

Location of the *IC*

If the location of the *IC* is unknown, it may be determined by using the fact that the relative position vector extending from the *IC* to a point is always perpendicular to the velocity of the point. Following three possibilities exist.

SOLVED EXAMPLES

1. Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body

If v and ω are known, the IC is located along the line drawn perpendicular to $\vec{v}$ at P, such that the distance from P to IC is $r = \frac{v}{\omega}$. Note that IC lies on that side of P which causes rotation about the IC, which is consistent with the direction of motion caused by $\vec{\omega}$ and $\vec{v}$.

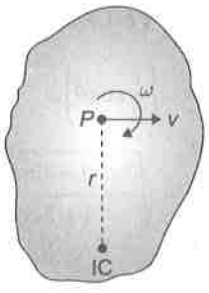
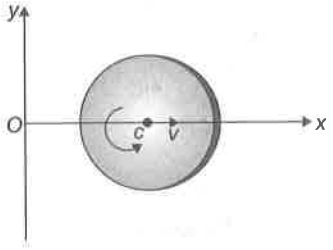


Figure 7.57

EXAMPLE 54

A rotating disc moves in the positive direction of the x-axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation if at the initial moment the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counterclockwise with a constant angular acceleration α. The initial angular velocity is equal to zero.



SOLUTION

$$t = \frac{x}{v}$$

and $$\omega = \alpha t = \frac{\alpha x}{v} .$$

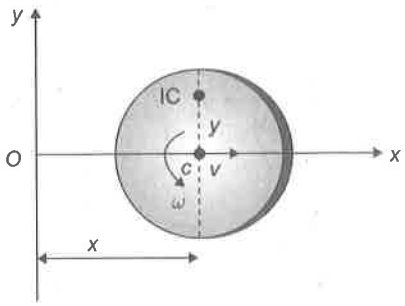
The position of IAOR will be at a distance

$$y = \frac{v}{\omega}$$

or $$y = \frac{v}{\frac{\alpha x}{v}}$$

or $$y = \frac{v^2}{\alpha x}$$

or $$xy = \frac{v^2}{\alpha} = \text{constant.}$$



This is the desired x–y equation. This equation represents a rectangular hyperbola. ■

2. Given the lines of action of two non-parallel velocities

Consider the body shown in Figure 7.58 where the line of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known. Draw perpendiculars at A and B to these lines of action. The point of intersection of these perpendiculars as shown locates the IC at the instant considered.

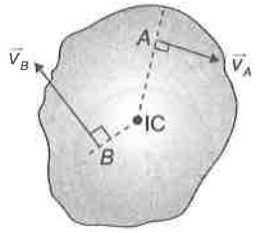


Figure 7.58

3. Given the magnitude and direction of two parallel velocities

When the velocities of points A and B are parallel and have known magnitudes v_A and v_B, then the location of the IC is determined by proportional triangles as shown in Fig. 7.59.

In both the cases,

$$r_{A,IC} = \frac{v_A}{\omega}$$

and $$r_{B,IC} = \frac{v_B}{\omega}.$$

In Fig. 7.59(a),

$$r_{A,IC} + r_{B,IC} = d$$

and in Fig. 7.59(b),

$$r_{B,IC} - r_{A,IC} = d.$$

As a special case, if the body is translating, $v_A = v_B$ and the IC would be located at infinity, in which case $\omega = 0$.

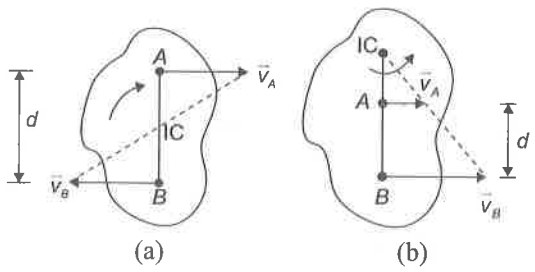


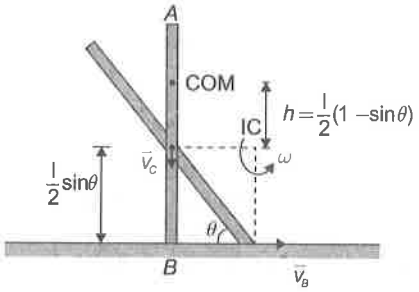
Figure 7.59

EXAMPLE 55

A uniform thin rod of mass m and length ℓ is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with the horizontal.

SOLUTION

As the floor is smooth, mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod; hence, the centre of mass moves vertically downwards in a straight line. Thus, velocities of COM and the lower end of B are in the direction shown in the figure. The location of IC at this instant can be found by drawing perpendiculars to $\vec{v}_C$ and $\vec{v}_B$ at the respective points. Now, the rod may be assumed to be in pure rotational motion about IAOR passing through IC with angular speed ω.



Applying conservation of mechanical energy,
decrease in gravitational potential energy of the rod = increase in rotational kinetic energy about IAOR.

$$\therefore \quad mgh = \frac{1}{2} I_{\text{IAOR}} \omega^2$$

or $$mg\frac{l}{2}(1-\sin\theta) = \frac{1}{2}\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega = \sqrt{\frac{12g(1-\sin\theta)}{l(1+3\cos^2\theta)}}$$

Now, $$|\vec{v}_C| = \left(\frac{l}{2}\cos\theta\right)\omega$$

$$= \sqrt{\frac{3gl(1-\sin\theta)\cos^2\theta}{1+3\cos^2\theta}}.$$

EXERCISES

JEE Main

(A) Moment of Inertia

1. The moment of inertia of a body depends on
   (A) mass only
   (B) angular velocity only
   (C) distribution of particles only
   (D) mass and distribution of mass about the axis

2. Two spheres of the same mass and radius are in contact with each other. If the moment of inertia of a sphere about its diameter is I, then the moment of inertia of both the spheres about the tangent at their common point would be
   (A) 3 (B) $7I$
   (C) $4I$ (D) $5I$

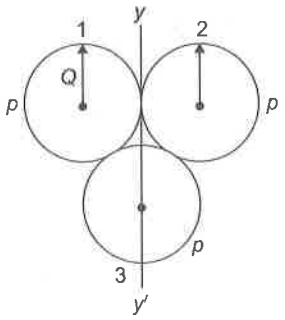
3. A disc of metal is melted to recast in the form of a solid sphere. The moment of inertias about a vertical axis passing through the centre would
   (A) decrease
   (B) increase
   (C) remains same
   (D) nothing can be said

4. The M.I. of a disc about its diameter is 2 units. Its M.I. about axis through a point on its rim and in the plane of the disc is
   (A) 4 units (B) 6 units
   (C) 8 units (D) 10 units

5. A solid sphere and a hollow sphere of the same mass have the same moments of inertia about their respective diameters. The ratio of their radii is
   (A) $(5)^{1/2}:(3)^{1/2}$ (B) $(3)^{1/2}:(5)^{1/2}$
   (C) 3:2 (D) 2:3

6. A stone of mass 4 kg is whirled in a horizontal circle of radius 1 m and makes 2 rev/s The moment of inertia of the stone about the axis of rotation is
   (A) 64 kg × m^2 (B) 4 kg × m^2
   (C) 16 kg × m^2 (D) 1 kg × m^2

7. Three rings, each of mass P and radius Q, are arranged as shown in the figure. The moment of inertia of the arrangement about the yy' axis will be



   (A) $\frac{7}{2}PQ^2$ (B) $\frac{2}{7}PQ^2$
   (C) $\frac{2}{5}PQ^2$ (D) $\frac{5}{2}PQ^2$

8. A circular disc A of radius r is made from an iron plate of thickness t and another circular disc B of radius $4r$ is made from an iron plate of thickness $t/4$. The relation between the moments of inertia I_A and I_B is
   (A) $I_A > I_B$
   (B) $I_A = I_B$
   (C) $I_A < I_B$
   (D) depends on the actual values of t and r

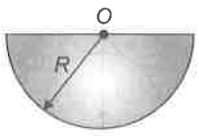
9. The moment of inertia of a uniform semi-circular wire of mass M and radius r about a line perpendicular to the plane of the wire through the centre is
   (A) Mr^2 (B) $\frac{1}{2}Mr^2$
   (C) $\frac{1}{4}Mr^2$ (D) $\frac{2}{5}Mr^2$

10. Let I_A and I_B be moments of inertia of a body about two axes A and B, respectively. The axis A passes through the centre of mass of the body but B does not.
    (A) $I_A < I_B$
    (B) If $I_A < I_B$, the axes are parallel.
    (C) If the axes are parallel, $I_A < I_B$
    (D) If the axes are not parallel, $I_A \geq I_B$

11. For the same total mass, which of the following will have the largest moment of inertia about an axis passing through its centre of mass and perpendicular to the plane of the body?

(A) A disc of radius a
(B) A ring of radius a
(C) A square lamina of side $2a$
(D) Four rods forming a square of side $2a$

12. Moment of inertia of a thin semi-circular disc (mass = M and radius = R) about an axis through point O and perpendicular to plane of disc, is given by



(A) $\frac{1}{4}MR^2$ (B) $\frac{1}{2}MR^2$
(C) $\frac{1}{8}MR^2$ (D) MR^2

13. A rigid body can be hinged about any point on the x-axis. When it is hinged such that the hinge is at x, the moment of inertia is given by $I = 2x^2 - 12x + 27$. The x-coordinate of centre of mass is
(A) $x = 2$ (B) $x = 0$
(C) $x = 1$ (D) $x = 3$

14. Consider the following statements.
Assertion (A): The moment of inertia of a rigid body reduces to its minimum value as compared to any other parallel axis when the axis of rotation passes through its centre of mass.
Reason (R): The weight of a rigid body always acts through its centre of mass in uniform gravitational field. Of these statements:
(A) both A and R are true and R is the correct explanation of A
(B) both A and R are true but R is not a correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

15. A body is rotating uniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(A) vertical
(B) horizontal and skew with the axis
(C) horizontal and intersecting the axis
(D) none of these

16. One end of a uniform rod of mass m and length l is clamped. The rod lies on a smooth horizontal surface and rotates on it about the clamped end at a uniform angular velocity ω. The force exerted by the clamp on the rod has a horizontal component
(A) mw^2l (B) zero
(C) mg (D) $\frac{1}{2}m\omega^2 l$

17. A rod of length 'L' is hinged from one end. It is brought to a horizontal position and released. The angular velocity of the rod when it is in vertical position is
(A) $\sqrt{\frac{2g}{L}}$ (B) $\sqrt{\frac{3g}{L}}$
(C) $\sqrt{\frac{g}{2L}}$ (D) $\sqrt{\frac{g}{L}}$

(B) Torque and Pure Rotational Motion

18. A disc of radius 2 m and mass 200 kg is acted upon by a torque 100 N-m. Its angular acceleration would be
(A) 1 rad/s^2 (B) 0.25 rad/s^2
(C) 0.5 rad/s^2 (D) 2 rad/s^2

19. On applying a constant torque on a body,
(A) linear velocity may be increased
(B) angular velocity may be increased
(C) it will rotate with constant angular velocity
(D) it will move with constant velocity

20. A wheel starting with angular velocity of 10 radian/s acquires angular velocity of 100 radian/sec in 15 s. If moment of inertia is 10 kg-m^2, then applied torque (in newton-metre) is
(A) 900 (B) 100
(C) 90 (D) 60

21. An automobile engine develops 100 H.P. when rotating at a speed of 1800 rad/min. The torque it delivers is
(A) 3.33 W-s (B) 200 W-s
(C) 248.7 W-s (D) 2487 W-s

22. The moment of inertia and rotational kinetic energy of a fly wheel are 20 kg-m^2 and 1000 joule, respectively. Its angular frequency per minute would be
(A) $\frac{600}{\pi}$ (B) $\frac{25}{\pi^2}$
(C) $\frac{5}{\pi}$ (D) $\frac{300}{\pi}$

23. The angular velocity of a body is $\vec{\omega} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and a torque $\vec{\tau} = \hat{i} + 2\hat{j} + 3\hat{k}$ acts on it. The rotational power will be

(A) 20 watt (B) 15 watt

(C) $\sqrt{17}$ watt (D) $\sqrt{14}$ watt

24. A torque of 2 N-m produces an angular acceleration of 2 rad/s^2 a body. If its radius of gyration is 2 m, its mass will be:

(A) 2 kg (B) 4 kg

(C) 1/2 kg (D) 1/4 kg

25. A particle is at a distance r from the axis of rotation. A given torque τ produces some angular acceleration in it. If the mass of the particle is doubled and its distance from the axis is halved, the value of torque to produce the same angular acceleration is

(A) $\tau/2$ (B) τ

(C) 2τ (D) 4τ

26. A weightless rod is acted on by upward parallel forces of 2 N and 4 N ends A and B, respectively. The total length of the rod $AB = 3$ m. To keep the rod in equilibrium, a force of 6 N should act in the following manner:

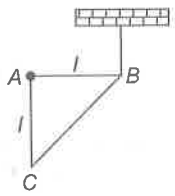
(A) Downwards at any point between A and B.

(B) Downwards at mid-point of AB.

(C) Downwards at a point C such that $AC = 1$ m.

(D) Downwards at a point D such that $BD = 1$ m.

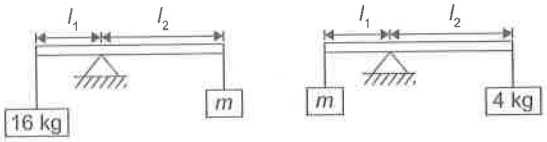
27. A right triangular plate ABC of mass m is free to rotate in the vertical plane about a fixed horizontal axis through A. It is supported by a string such that the side AB is horizontal. The reaction at the support A is



(A) $\frac{mg}{3}$ (B) $\frac{2mg}{3}$

(C) $\frac{mg}{2}$ (D) mg

28. In an experiment with a beam balance on unknown mass m is balanced by two known mass m is balanced by two known masses of 16 kg and 4 kg as shown in the figure.

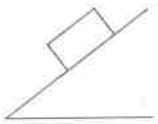


The value of the unknown mass m is

(A) 10 kg (B) 6 kg

(C) 8 kg (D) 12 kg

29. A homogeneous cubical brick lies motionless on a rough inclined surface. The half of the brick which applies greater pressure on the plane is



(A) left half

(B) right half

(C) both applies equal pressure

(D) the answer depends on coefficient of friction

30. Consider the following statements.

Assertion (*A*): A cyclist always bends inwards while negotiating a curve.

Reason (*R*): By bending he lowers his centre of gravity. Of these statements,

(A) both A and R are true and R is the correct explanation of A

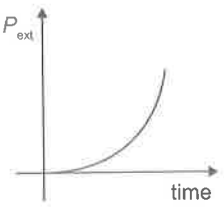
(B) both A and R are true but R is not the correct explanation of A

(C) A is true but R is false

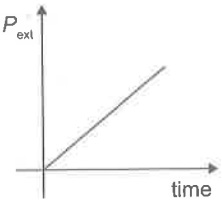
(D) A is false but R is true

31. A rod is hinged at its centre and rotated by applying a constant torque starting from rest. The power developed by the external torque as a function of time is

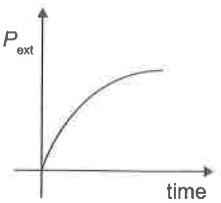
(A)



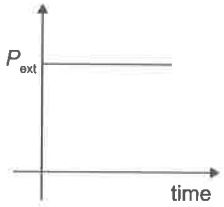
(B)



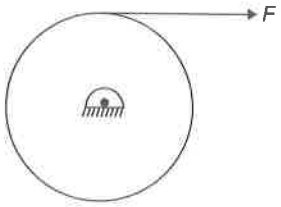
(C)



(D)



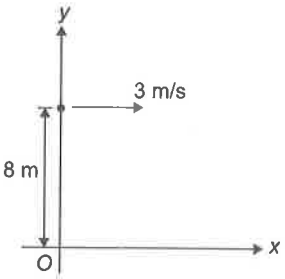
32. A pulley is hinged at the centre and a massless thread is wrapped around it. The thread is pulled with a constant force F starting from rest. As the time increases,



(A) its angular velocity increases, but force on hinge remains constant
(B) its angular velocity remains same, but force on hinge increases
(C) its angular velocity increases and force on hinge increases
(D) its angular velocity remains same and force on hinge is constant

33. The angular momentum of a flywheel having a moment of inertia of 0.4 kg m^2 decreases from 30 to 20 kg m^2/s in a period of 2 s. The average torque acting on the flywheel during this period is
(A) 10 Nm (B) 2.5 Nm
(C) 5 Nm (D) 1.5 Nm

34. A particle starts from the point (0 m, 8 m) and moves with uniform velocity of $3\hat{i}$ m/s. After 5 s, the angular velocity of the particle about the origin will be



(A) $\frac{8}{289}$ rad/s (B) $\frac{3}{8}$ rad/s
(C) $\frac{24}{289}$ rad/s (D) $\frac{8}{17}$ rad/s

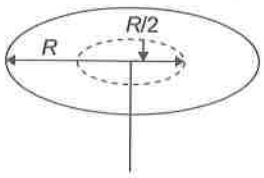
(C) Angular Momentum

35. The rate of change of angular momentum is called
(A) angular velocity
(B) angular acceleration
(C) force
(D) torque

36. The rotational kinetic energy of a rigid body of moment of inertia 5 kg-m^2 is 10 J. The angular momentum about the axis of rotation would be
(A) 100 J-s (B) 50 J-s
(C) 10 J-s (D) 2 J-s

37. The angular velocity of a body changes from one revolution per 9 second to 1 revolution per second without applying any torque. The ratio of its radius of gyration in the two cases is
(A) 1:9 (B) 3:1
(C) 9:1 (D) 1:3

38. A dog of mass m is walking on a pivoted disc of radius R and mass M in a circle of radius $R/2$ with an angular frequency n: the disc will revolve in opposite direction with frequency



(A) $\frac{mn}{M}$ (B) $\frac{mn}{2M}$
(C) $\frac{2mn}{M}$ (D) $\frac{2Mn}{M}$

39. A particle moves with a constant velocity parallel to the x-axis. Its angular momentum with respect to the origin
(A) is zero
(B) remains constant
(C) goes on increasing
(D) goes on decreasing

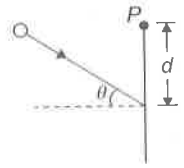
40. A person sitting firmly over a rotating stool has his arms stretched. If he folds his arms, his angular momentum about the axis of rotation

(A) increases (B) decreases
(C) remains unchanged (D) doubles

41. A man, sitting firmly over a rotating stool has his arms stretched. If he folds his arms, the work done by the man is
(A) zero
(B) positive
(C) negative
(D) may be positive or negative

42. A particle of mass 2 kg located at the position $(\hat{i}+\hat{j})$m has a velocity $2(+\hat{i}-\hat{j}+\hat{k})$m/s. Its angular momentum about the z-axis in kg-m^2/s is
(A) zero (B) +8
(C) 12 (D) −8

43. A ball of mass m moving with velocity v, collide with the wall elastically as shown in the figure. After impact the change in angular momentum about P is



(A) $2mvd$ (B) $2mvd\cos\theta$
(C) $2mvd\sin\theta$ (D) zero

44. A uniform rod of mass M has an impulse applied at right angles to one end. If the other end begins to move with speed V, the magnitude of the impulse is
(A) MV (B) $\frac{MV}{2}$
(C) $2MV$ (D) $\frac{2MV}{3}$

(D) Combined Translational + Rotational Motion

45. A circular ring of wire of mass M and radius R is making n revolutions/s about an axis passing through a point on its rim and perpendicular to its plane. The kinetic energy of rotation of the ring is given by
(A) $4p^2MR^2n^2$ (B) $2p^2MR^2n^2$
(C) $\frac{1}{2}p^2MR^2n^2$ (D) $8p^2MR^2n^2$

46. Rotational kinetic energy of a disc of constant moment of inertia is
(A) directly proportional to angular velocity
(B) inversely proportional to angular velocity
(C) inversely proportional to square of angular velocity
(D) directly proportional to square of angular velocity

47. A circular disc has a mass of 1 kg and radius 40 cm. It is rotating about an axis passing through its centre and perpendicular to its plane with a speed of 10 rev/s. The work done in joules in stopping it would be
(A) 4 (B) 47.5
(C) 79 (D) 158

48. A disc rolls on a table. The ratio of its K.E. of rotation to the total K.E. is
(A) 2/5 (B) 1/3
(C) 5/6 (D) 2/3

49. A disc and a ring of the same mass are rolling to have the same kinetic energy. What is ratio of their velocities of centre of mass?
(A) $(4:3)^{1/2}$ (B) $(3:4)^{1/2}$
(C) $(2)^{1/2}:(3)^{1/2}$ (D) $(3)^{1/2}:(2)^{1/2}$

50. If the applied torque is directly proportional to the angular displacement θ, then the work done in rotating the body through an angle θ would be (C is constant of proportionality)
(A) $C\theta$ (B) $\frac{1}{2}C\theta$
(C) $\frac{1}{2}Cq^2$ (D) Cq^2

51. The centre of a wheel rolling without slipping in a plane surface moves with speed v_0. A particle on the rim of the wheel at the same level as the centre will be moving at a speed
(A) zero (B) v_0
(C) $\sqrt{2}v_0$ (D) $2v_0$

52. A solid sphere, a hollow sphere and a disc, all having smooth incline and released. Least time will be taken in reaching the bottom by
(A) the solid sphere
(B) the hollow sphere
(C) the disc
(D) all will take same time

53. A wheel of radius r rolling on a straight line, the velocity of its centre being v. At a certain instant, the

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point of contact of the wheel with the grounds is M and N is the highest point on the wheel (diametrically opposite to M). The incorrect statement is

(A) The velocity of any point P of the wheel is proportional to MP.

(B) The points of the wheel moving with velocity greater than v form a larger area of the wheel than points moving with velocity less than v.

(C) The point of contact M is instantaneously at rest.

(D) The velocities of any two parts of the wheel which are equidistant from centre are equal.

54. There is a rod of length l. The velocities of its two ends are v_1 and v_2 in opposite directions normal to the rod. The distance of the instantaneous axis of rotation from v_1 is

(A) zero

(B) $\frac{v_2}{v_1 + v_2} l$

(C) $\frac{v_1 l}{v_1 + v_2}$

(D) $l/2$

55. A ring of radius R rolls without sliding with a constant velocity. The radius of curvature of the path followed by any particle of the ring at the highest point of its path will be

(A) R

(B) $2R$

(C) $4R$

(D) none of these

56. The linear speed of a uniform spherical shell after rolling down an inclined plane of vertical height h from rest is

(A) $\sqrt{\frac{10gh}{7}}$

(B) $\sqrt{\frac{4gh}{5}}$

(C) $\sqrt{\frac{6gh}{5}}$

(D) $\sqrt{2gh}$

57. A body kept on a smooth horizontal surface is pulled by a constant horizontal force applied at the top point of the body. If the body rolls purely on the surface, its shape can be

(A) thin pipe

(B) uniform cylinder

(C) uniform sphere

(D) thin spherical shell

58. A solid sphere with a velocity (of centre of mass) v and angular velocity ω is gently placed on a rough horizontal surface. The frictional force on the sphere

(A) must be forward (in direction of v)

(B) must be backward (opposite to v)

(C) cannot be zero

(D) none of these

59. A cylinder is pure rolling up an inclined plane. It stops momentarily and then rolls back. The force of friction

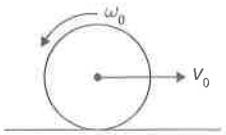
(A) on the cylinder is zero throughout the journey

(B) is directed opposite to the velocity of the centre of mass throughout the journey

(C) is directed up the plane throughout the journey

(D) is directed down the plane throughout the journey

60. A uniform circular disc placed on a rough horizontal surface has initially a velocity v_0 and an angular velocity w_0 as shown in the figure. The disc comes to rest after moving some distance in the direction of motion. Then $\frac{v_0}{r\omega_0}$ is



(A) $\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 2

61. A cubical block of mass M and edge a slides down a rough inclined plane of inclination θ with a uniform velocity. The torque of the normal force on the block about its centre has a magnitude

(A) zero

(B) Mga

(C) $Mga \sin\theta$

(D) $\frac{1}{2} Mga \sin\theta$

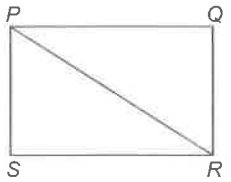
JEE Advanced

(A) Moment of Inertia

1. Three bodies have equal masses m. Body A is solid cylinder of radius R, body B is a square lamina of side R and body C is a solid sphere of radius R. Which body has the smallest moment of inertia about an axis passing through their centre of mass and perpendicular to the plane (in case of lamina)?
(A) A (B) B
(C) C (D) Both A and C

2. Two rods of equal mass m and length l lie along the x- and y-axis with their centres at the origin. What is the moment of inertia of both about the line $x = y$?
(A) $\frac{ml^2}{3}$ (B) $\frac{ml^2}{4}$
(C) $\frac{ml^2}{12}$ (D) $\frac{ml^2}{6}$

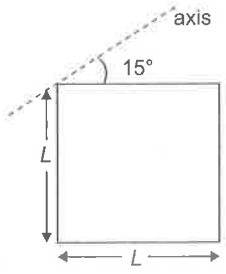
3. Moment of inertia of a rectangular plate about an axis passing through P and perpendicular to the plate is I. Then moment of PQR about an axis perpendicular to the plane of the plate



(A) about $P = I/2$ (B) about $R = I/2$
(C) about $P > I/2$ (D) about $R > I/2$

4. A thin uniform rod of mass M and length L has its moment of inertia I_1 about its perpendicular bisector. The rod is bend in the form of a semi-circular arc. Now its moment of inertia through the centre of the semi-circular arc and perpendicular to its plane is I_2. The ratio of $I_1 : I_2$ will be _____.
(A) < 1 (B) > 1
(C) $= 1$ (D) cannot be said

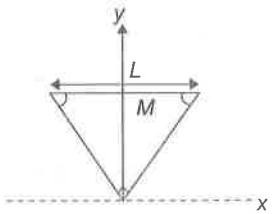
5. A square plate of mass M and edge L is shown in the figure. The moment of inertia of the plate about the axis in the plane of plate passing through one of its vertex making an angle 15° from horizontal is



(A) $\frac{ML^2}{12}$ (B) $\frac{11ML^2}{24}$
(C) $\frac{7ML^2}{12}$ (D) none of these

Question No. 6 to 9

The figure shows an isosceles triangular plate of mass M and base L. The angle at the apex is 90°. The apex lies at the origin and the base is parallel to x-axis.



6. The moment of inertia of the plate about the z-axis is
(A) $\frac{ML^2}{12}$ (B) $\frac{ML^2}{24}$
(C) $\frac{ML^2}{6}$ (D) none of these

7. The moment of inertia of the plate about the x-axis is
(A) $\frac{ML^2}{8}$ (B) $\frac{ML^2}{32}$
(C) $\frac{ML^2}{24}$ (D) $\frac{ML^2}{6}$

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8. The moment of inertia of the plate about its base parallel to the x-axis is

(A) $\frac{ML^2}{18}$ (B) $\frac{ML^2}{36}$

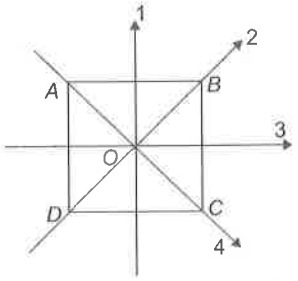
(C) $\frac{ML^2}{24}$ (D) none of these

9. The moment of inertia of the plate about the y-axis is

(A) $\frac{ML^2}{6}$ (B) $\frac{ML^2}{8}$

(C) $\frac{ML^2}{24}$ (D) none of these

10. $ABCD$ is a square plate with centre O. The moment of inertia of the plate about the perpendicular axis through O is I and about the axes 1, 2, 3 and 4 are I_1, I_2, I_3 and I_4, respectively. It follows that



(A) $I_2 = I_3$ (B) $I = I_1 + I_4$

(C) $I = I_2 + I_4$ (D) $I_1 = I_3$

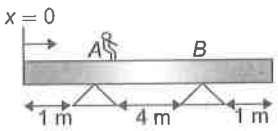
(B) Torque and Pure Rotational Motion

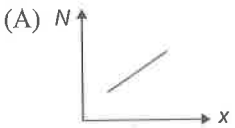
11. A horizontal force $F = mg/3$ is applied on the upper surface of a uniform cube of mass 'm' and side 'a' which is resting on a rough horizontal surface having $m_s = 1/2$. The distance between lines of action of 'mg' and normal reaction 'N' is

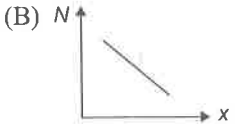
(A) $a/2$ (B) $a/3$

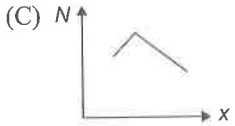
(C) $a/4$ (D) none of these

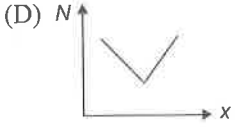
12. A man can move on a horizontal plank supported symmetrically as shown in the figure. The variation of normal reaction on support A with distance x of the man from the end of the plank is best represented by



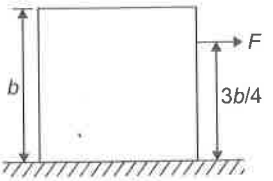
(A) 

(B) 

(C) 

(D) 

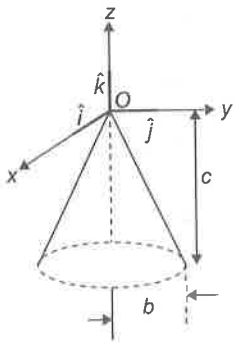
13. A uniform cube of side 'b' and mass M rest on a rough horizontal table. A horizontal force F is applied normal to one of the face at a point, at a height $3b/4$ above the base. What should be the coefficient of friction (μ) between cube and table so that it will tip about an edge before it starts slipping?



(A) $\mu > \frac{2}{3}$ (B) $\mu > \frac{1}{3}$

(C) $\mu > \frac{3}{2}$ (D) None of these

14. A solid cone hangs from a frictionless pivot at the origin O, as shown in the figure. If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors, and a, b and c are positive constants, which of the following forces F applied to the rim of the cone at a point P result in a torque τ on the cone with a negative component t_z?

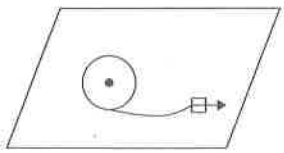


(A) $F = a\hat{k}$, P is $(0, b, -c)$

(B) $F = -a\hat{k}$, P is $(0, -b, -c)$

(C) $F = a\hat{j}$, P is $(-b, 0, -c)$
(D) None of these

15. A block of mass m is attached to a pulley disc of equal mass m, radius r by means of a slack string as shown in the figure. The pulley is hinged about its centre on a horizontal table and the block is projected with an initial velocity of 5 m/s. Its velocity when the string becomes taut will be



(A) 3 m/s (B) 2.5 m/s
(C) 5/3 m/s (D) 10/3 m/s

16. A rod of weight w is supported by two parallel knife edges A and B and is in equilibrium in a horizontal position. The knives are at a distance d from each other. The centre of mass of the rod is at a distance x from A.

(A) The normal reaction at A is $\frac{wx}{d}$

(B) The normal reaction at A is $\frac{w(d-x)}{d}$

(C) The normal reaction at B is $\frac{wx}{d}$

(D) The normal reaction at B is $\frac{w(d-x)}{d}$

17. A block with a square base measuring $a \times a$ and height h is placed on an inclined plane. The coefficient of friction is μ. The angle of inclination (θ) of the plane is gradually increased. The block will

(A) topple before sliding if $\mu > \frac{a}{h}$

(B) topple before sliding if $\mu < \frac{a}{h}$

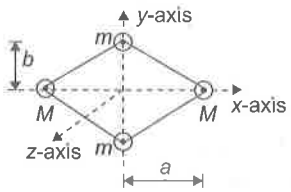
(C) slide before toppling if $\mu > \frac{a}{h}$

(D) slide before toppling if $\mu < \frac{a}{h}$

18. A body is in equilibrium under the influence of a number of forces. Each force has a different line of action. The minimum number of forces required is
(A) 2, if their lines of action pass through the centre of mass of the body
(B) 3, if their lines of action are not parallel
(C) 3, if their lines of action are parallel
(D) 4, if their lines of action are parallel and all the forces have the same magnitude

19. A block of mass m moves on a horizontal rough surface with initial velocity v. The height of the centre of mass of the block is h from the surface. Consider a point A on the surface.
(A) Angular momentum about A is mvh initially.
(B) The velocity of the block decreases at time passes.
(C) Torque of the forces acting on block is zero about A.
(D) Angular momentum is not conserved about A.

20. Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane. Let w be the angular speed of rotation. Then



(A) rotational kinetic energy associated with a given angular speed depends on the axis of rotation.
(B) rotational kinetic energy about the y-axis is independent of m and its value is Ma^2w^2.
(C) rotational kinetic energy about the z-axis depends on m and its value is $(Ma^2 + mb^2)w^2$.
(D) rotational kinetic energy about z-axis is independent of m and its value is Mb^2w^2.

21. A particle falls freely near the surface of the earth. Consider a fixed point O (not vertically below the particle) on the ground.
(A) Angular momentum of the particle about O is increasing.
(B) Torque of the gravitational force on the particle about O is decreasing.
(C) The moment of inertia of the particle about O is decreasing.
(D) The angular velocity of the particle about O is increasing.

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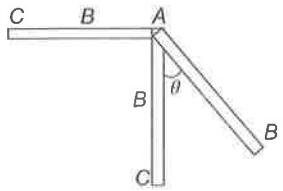
22. A rod hinged at one end is released from the horizontal position as shown in the figure. When it becomes vertical, its lower half separates without exerting any reaction at the breaking point. Then the maximum angle 'θ' made by the hinged upper half with the vertical is



(A) 30° (B) 45°
(C) 60° (D) 90°

(C) Angular Momentum

23. If a person sitting on a rotating stool with his hands outstretched, suddenly lowers his hands, then his
(A) kinetic energy will decrease
(B) moment of inertia will decrease
(C) angular momentum will increase
(D) angular velocity will remain constant

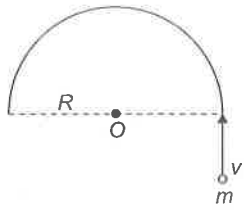
24. A man spinning in free space changes the shape of his body, e.g., by spreading his arms or curling up. By doing this, he can change his
(A) moment of inertia
(B) angular momentum
(C) angular velocity
(D) rotational kinetic energy

25. A thin circular ring of mass 'M" and radius 'R' is rotating about its axis with a constant angular velocity ω. Two objects each of mass m are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity

(A) $\dfrac{\omega M}{(M+m)}$ (B) $\dfrac{\omega M}{(M+2m)}$

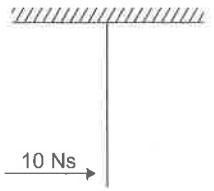
(C) $\dfrac{\omega M}{(M-2m)}$ (D) $\dfrac{\omega(M+3m)}{M}$

26. A small bead of mass m moving with velocity v gets threaded on a stationary semi-circular ring of mass m and radius R kept on a horizontal table. The ring can freely rotate about its centre. The bead comes to rest relative to the ring. What will be the final angular velocity of the system?



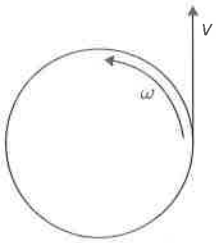
(A) v/R (B) $2v/R$
(C) $v/2R$ (D) $3v/R$

27. A thin uniform straight rod of mass 2 kg and length 1 m is free to rotate about its upper end when at rest. It receives an impulsive blow of 10 Ns at its lowest point, normal to its length as shown in the figure. The kinetic energy of rod just after impact is



(A) 75 J (B) 100 J
(C) 200 J (D) none of these

28. A child with mass m is standing at the edge of a disc with moment of inertia I, radius R and initial angular velocity ω. See the figure given below. The child jumps off the edge of the disc with tangential velocity v with respect to the ground. The new angular velocity of the disc is



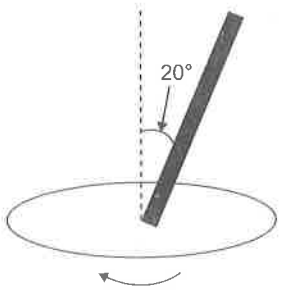
(A) $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$ (B) $\sqrt{\dfrac{(I+mR^2)\omega^2 - mv^2}{I}}$

(C) $\dfrac{I\omega - mvR}{I}$ (D) $\dfrac{(I+mR^2)\omega - mvR}{I}$

Question No. 29 and 30

A uniform rod is fixed to a rotating turntable so that its lower end is on the axis of the turntable and it makes an angle of 20° to the vertical. (The rod is thus rotating with

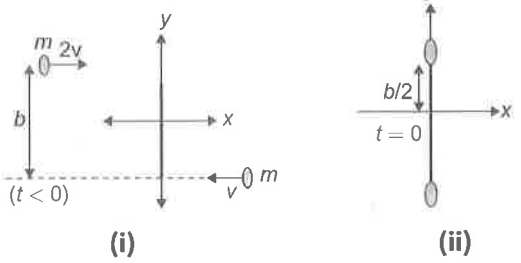
uniform angular velocity about a vertical axis passing through one end.) If the turntable is rotating clockwise as seen from the above:



29. What is the direction of the rod's angular momentum vector (calculated about its lower end)?
(A) Vertically downwards
(B) Down at 20° to the horizontal
(C) Up at 20° to the horizontal
(D) Vertically upwards

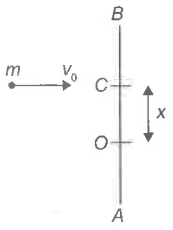
30. Is there a torque acting on it, and if so in what direction?
(A) Yes, vertically
(B) Yes, horizontally
(C) Yes at 20° to the horizontal
(D) No

31. One ice skater of mass m moves with speed $2v$ to the right, while another of the same mass m moves with speed v towards the left, as shown in Fig. (i). Their paths are separated by distance b. At $t = 0$, when they are both at $x = 0$, they grasp a pole of length b and negligible mass. For $t > 0$, consider the system as a rigid body of two masses m separated by distance b, as shown in Fig. (ii). Which of the following is the correct formula for the motion after $t = 0$ of the skater initially at $y = b/2$?



(A) $x = 2vt, y = b/2$
(B) $x = vt + 0.5b \sin(3vt/b), y = 0.5b \cos(3vt/b)$
(C) $x = 0.5vt + 0.5b \sin(3vt/b), y = 0.5b \cos(3vt/b)$
(D) $x = 0.5vt + 0.5b \sin(6vt/b), y = 0.5b \cos(6vt/b)$

32. A uniform rod AB of length L and mass M is lying on a smooth table. A small particle of mass m strike the rod with a velocity v_0 at point C at a distance x from the centre O. The particle comes to rest after collision. The value of x, so that point A of the rod remains stationary just after collision is

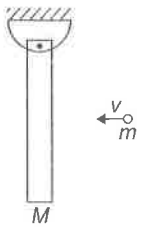


(A) $L/3$ (B) $L/6$
(C) $L/4$ (D) $L/12$

33. A uniform rod AB of mass m and length l is at rest on a smooth horizontal surface. An impulse J is applied to the end B, perpendicular to the rod in the horizontal direction. Speed of particle P at a distance $\frac{l}{6}$ from the centre towards A of the rod after time $t = \frac{\pi ml}{12J}$ is

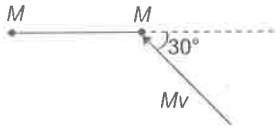
(A) $2\frac{J}{m}$ (B) $\frac{J}{\sqrt{2m}}$
(C) $\frac{J}{m}$ (D) $\sqrt{2}\frac{J}{m}$

34. A uniform rod of mass M is hinged at its upper end. A particle of mass m moving horizontally strikes the rod at its mid-point elastically. If the particle comes to rest after collision, find the value of M/m.



(A) 3/4 (B) 4/3
(C) 2/3 (D) None of these

35. Two equal masses each of mass M are joined by a massless rod of length L. Now an impulse Mv is given to the mass M making an angle of 30° with the length of the rod. The angular velocity of the rod just after imparting the impulse is



(A) $\frac{v}{L}$ (B) $\frac{2v}{L}$

(C) $\frac{v}{2L}$ (D) none of these

36. Two particles of equal mass m at A and B are connected by a rigid light rod AB lying on a smooth horizontal table. An impulse J is applied at A in the plane of the table and perpendicular at AB. Then the velocity of the particle at A is

(A) $\frac{J}{2m}$ (B) $\frac{J}{m}$

(C) $\frac{2J}{m}$ (D) zero

(D) Combined Translational + Rotational Motion

37. A ring rolls without slipping on the ground. Its centre C moves with a constant speed u. P is any point on the ring. The speed of P with respect to the ground is v.

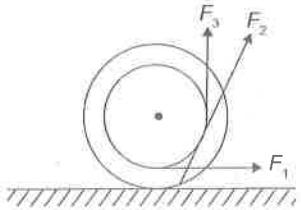
(A) $0 \leq v \leq 2u$

(B) $v = u$, if CP is horizontal

(C) $v = u$, if CP makes an angle of 30° with the horizontal and P is below the horizontal level of C

(D) $v = \sqrt{2}\,u$, if CP is horizontal

38. A yo-yo is resting on a perfectly rough horizontal table. Forces F_1, F_2 and F_3 are applied separately as shown in the figure. The correct statement is



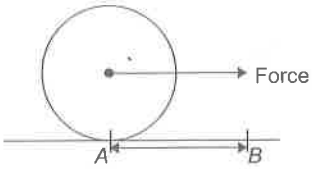
(A) when F_3 is applied the centre of mass will move to the right

(B) when F_2 is applied the centre of mass will move to the left

(C) when F_1 is applied the centre of mass will move to the right

(D) when F_2 is applied the centre of mass will move to the right

39. A disc of circumference s is at rest at a point A on a horizontal surface when a constant horizontal force begins to act on its centre. Between A and B there is sufficient friction to prevent slipping, and the surface is smooth to the right of B. $AB = s$. The disc moves from A to B in time T. To the right of B,



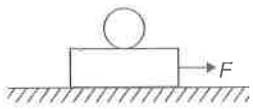
(A) the angular acceleration of the disc will disappear, linear acceleration will remain unchanged

(B) linear acceleration of the disc will increase

(C) the disc will make one rotation in time $T/2$

(D) the disc will cover a distance greater than s in further time T

40. A plank with a uniform sphere placed on it, rests on a smooth horizontal plane. Plank is pulled to right by a constant force F. If the sphere does not slip over the plank,



(A) acceleration of the centre of sphere is less than that of the plank

(B) acceleration of the centre of sphere is greater than the plank because friction acts rightwards on the sphere

(C) acceleration of the centre of sphere may be towards left

(D) acceleration of the centre of sphere relative to plank may be greater than that of the plank relative to floor

41. A hollow sphere of radius R and mass m is fully filled with water of mass m. It is rolled down a horizontal plane such that its centre of mass moves with a velocity v. If it purely rolls

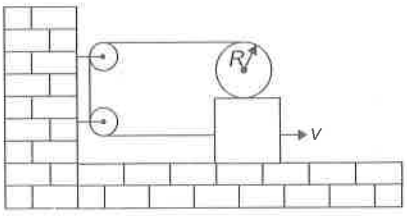
(A) kinetic energy of the sphere is $\frac{5}{6}mv^2$

(B) kinetic energy of the sphere is $\frac{4}{5}mv^2$

(C) angular momentum of the sphere about a fixed point on ground is $\frac{8}{3}mvR$

(D) angular momentum of the sphere about a fixed point on ground is $\frac{14}{5}mvR$

42. In the figure shown, the plank is being pulled to the right with a constant speed v. If the cylinder does not slip, then



(A) the speed of the centre of mass of the cylinder is $2v$
(B) the speed of the centre of mass of the cylinder is zero
(C) the angular velocity of the cylinder is v/R
(D) the angular velocity of the cylinder is zero

43. If a cylinder is rolling down the incline with sliding
(A) after some time it may start pure rolling
(B) after sometime it will start pure rolling
(C) it may be possible that it will never start pure rolling
(D) none of these

44. Which of the following statements are correct?
(A) Friction acting on a cylinder without sliding on an inclined surface is always upwards along the incline irrespective of any external force acting on it.
(B) Friction acting on a cylinder without sliding on an inclined surface is, may be upwards or may be downwards, depending on the external force acting on it.
(C) Friction acting on a cylinder rolling without sliding may be zero depending on the external force acting on it.
(D) Nothing can be said exactly about it as it depends on the friction coefficient on inclined plane.

Question No. 45 to 47

A cylinder and a ring of same mass M and radius R are placed on the top of a rough inclined plane of inclination θ. Both are released simultaneously from the same height h.

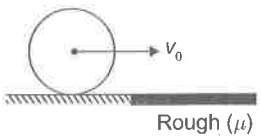
45. Choose the correct statement(s) related to the motion of each body.
(A) The frictional force acting on each body opposes the motion of its centre of mass.
(B) The frictional force provides the necessary torque to rotate the body about its centre of mass.
(C) Without friction none of the two bodies can roll.
(D) The friction force ensures that the point of contact must remain stationary.

46. Identify the correct statement(s).
(A) The frictional force acting on the cylinder may be more than that acting on the ring.
(B) The frictional force acting on the ring may be more than that acting on the cylinder.
(C) If the friction is sufficient to roll the cylinder, then the ring will also roll.
(D) If the friction is sufficient to roll the ring, then the cylinder will also roll.

47. When these bodies roll down to the foot of the inclined plane, then
(A) the mechanical energy of each body is conserved
(B) the velocity of the centre of mass of the cylinder is $2\sqrt{\frac{gh}{3}}$
(C) the velocity of centre of mass of the ring is $\sqrt{gh}$
(D) the velocity of centre of mass of each body is $\sqrt{2gh}$

Question No. 48 to 51

A ring of mass M and radius R sliding with a velocity v_0 suddenly enters into rough surface where the coefficient of friction is μ, as shown in the figure.



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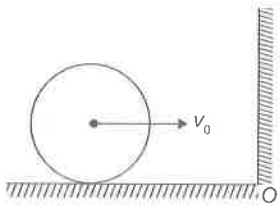
48. Choose the correct statement(s).
(A) As the ring enters on the rough surface, the kinetic friction force acts on it.
(B) The direction of friction is opposite to the direction of motion.
(C) The frictional force accelerates the ring in the clockwise sense about its centre of mass.
(D) As the ring enters on the rough surface, it starts rolling.

49. Choose the correct statement(s).
(A) The momentum of the ring is conserved.
(B) The angular momentum of the ring is conserved about its centre of mass.
(C) The angular momentum of the ring conserved about any point on the horizontal surface.
(D) The mechanical energy of the ring is conserved.

50. Choose the correct statement(s).
(A) The ring starts its rolling motion when the centre of mass stationary.
(B) The ring starts rolling motion when the point of contact becomes stationary.
(C) The time after which the ring starts rolling is $\frac{v_0}{2\mu g}$.
(D) The rolling velocity is $\frac{v_0}{2}$.

51. Choose the correct alternative(s).
(A) The linear distance moved by the centre of mass before the ring starts rolling is $\frac{3v_0^2}{8\mu g}$.
(B) The net work done by frictional force is $-\frac{3}{8}mv_0^2$.
(C) The loss in kinetic energy of the ring is $\frac{mv_0^2}{4}$.
(D) The gain in rotational kinetic energy is $+\frac{mv_0^2}{8}$.

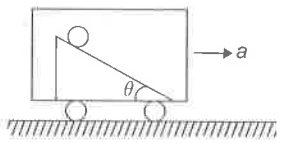
52. Consider a sphere of mass 'm' and radius 'R' doing pure rolling motion on a rough surface having velocity $\vec{v}_0$ as shown in the figure. It makes an elastic impact with the smooth wall and moves back and starts pure rolling after sometime again.



(A) Change in angular momentum about O in the entire motion equals $2mv_0R$ in magnitude.
(B) Moment of impulse provided by wall during impact about O equals $2mv_0R$ in magnitude.
(C) Final velocity of the ball will be $\frac{3}{7}\vec{v}_0$.
(D) Final velocity of the ball will be $-\frac{3}{7}\vec{v}_0$.

53. A solid sphere, a hollow sphere and a disc, all having the same mass and radius, are placed at the top of an incline and released. The friction coefficients between the objects and the incline are same and not sufficient to allow pure rolling. The smallest kinetic energy at the bottom of the incline will be achieved by
(A) the solid sphere
(B) the hollow sphere
(C) the disc
(D) all will achieve same kinetic energy

54. The following figure shows a smooth inclined plane fixed in a car accelerating on a horizontal road. The angle of incline θ is related to the acceleration a of the car as $a = g \tan \theta$. If the sphere is set in pure rotation on the incline,



(A) it will continue pure rolling
(B) it will slip down the plane
(C) its linear velocity will increase
(D) its linear velocity will decrease

55. A straight rod of length L is released on a frictionless horizontal floor in a vertical position. As it falls and slips, the distance of a point on the rod from the lower end, which follows a quarter circular locus is
(A) $L/2$ (B) $L/4$
(C) $L/8$ (D) None of these

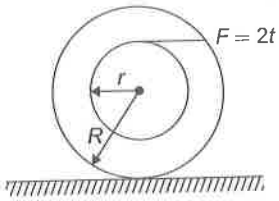
56. A ladder of length L slips with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\alpha = 30°$ with the horizontal. Then the speed of the ladder's centre must be

(A) $2v/\sqrt{3}$ (B) $v/2$
(C) v (D) none of these

57. In the previous question, if $dv/dt = 0$, then the angular acceleration of the ladder when $\alpha = 45°$ is

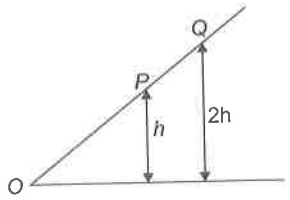
(A) $2v^2/L^2$ (B) $v^2/2L^2$
(C) $\sqrt{2}[v^2/L^2]$ (D) none of these

58. A time varying force $F = 2t$ is applied on a spool rolling as shown in the figure. The angular momentum of the spool at time t about bottom mostpoint is



(A) $\frac{r^2t^2}{R}$ (B) $\frac{(R+r)^2}{r}t^2$
(C) $(R+r)t^2$ (D) data is insufficient

59. A ball rolls down an inclined plane as shown in the figure. The ball is first released from rest from P and then later from Q. Which of the following statement is/ are correct?



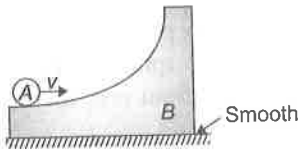
(i) The ball takes twice as much time to roll from Q to O as it does to roll from P to O.
(ii) The acceleration of the ball at Q is twice as large as the acceleration at P.
(iii) The ball has twice as much K.E. at O when rolling from Q as it does when rolling from P.

(A) i, ii only (B) ii, iii only
(C) i only (D) iii only

60. Starting from the rest, at the same time, a ring, a coin and a solid ball of same mass roll down an incline without slipping. The ratio of their translational kinetic energies at the bottom will be

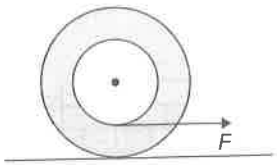
(A) 1:1:1 (B) 10:5:4
(C) 21:28:30 (D) none of these

61. In the figure, a ring A is initially rolling without sliding with a velocity v on the horizontal surface of the body B (of same mass as A). All surfaces are smooth. B has no initial velocity. What will be the maximum height reached by A on B?



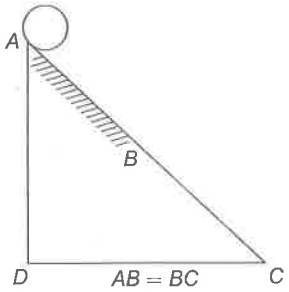
(A) $\frac{3v^2}{4g}$ (B) $\frac{v^2}{4g}$
(C) $\frac{v^2}{2g}$ (D) $\frac{v^2}{3g}$

62. Inner and outer radii of a spool are r and R, respectively. A thread is wound over its inner surface and placed over a rough horizontal surface. Thread is pulled by a force F as shown in the figure. Then in case of pure rolling



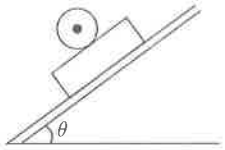
(A) thread unwinds, spool rotates anti-clockwise and friction acts leftwards
(B) thread winds, spool rotates clockwise and friction acts leftwards
(C) thread winds, spool moves to the right and friction acts rightwards
(D) thread winds, spool moves to the right and friction does not come into existence

63. Portion AB of the wedge shown in the figure is rough and BC is smooth. A solid cylinder rolls without slipping from A to B. The ratio of translational kinetic energy to rotational kinetic energy, when the cylinder reaches point C is



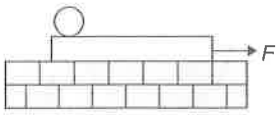
(A) 3/4 (B) 5
(C) 7/5 (D) 8/3

64. A plank of mass M is placed over smooth inclined plane and a sphere is also placed over the plank. Friction is sufficient between sphere and plank. If plank and sphere are released from rest, the frictional force on sphere is



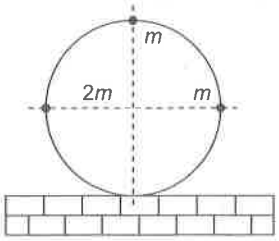
(A) up the plane (B) down the plane
(C) horizontal (D) zero

65. A plank with a uniform sphere placed on it rests on a smooth horizontal plane. Plank is pulled to right by a constant force F. If sphere does not slip over the plank. Which of the following is incorrect?



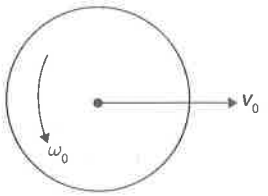
(A) Acceleration of the centre of sphere is less than that of the plank.
(B) Work done by friction acting on the sphere is equal to its total kinetic energy.
(C) Total kinetic energy of the system is equal to work done by the force F.
(D) None of these

66. A ring of mass m and radius R has three particles attached to the ring as shown in the figure. The centre of the ring has speed v_0. The kinetic energy of the system is (slipping is absent)



(A) $6\,mv_0^2$ (B) $12\,mv_0^2$
(C) $4\,mv_0^2$ (D) $8\,mv_0^2$

67. A uniform sphere of radius R is placed on a rough horizontal surface and given a linear velocity v_0 and angular velocity w_0 as shown. The sphere comes to rest after moving some distance to the right. It follows that



(A) $v_0 = w_0R$ (B) $2\,v_0 = 5\,w_0R$
(C) $5\,v_0 = 2\,w_0R$ (D) $2\,v_0 = w_0R$

JEE Advanced

Level I

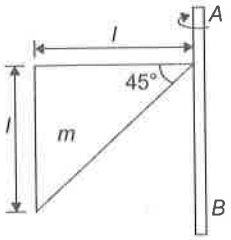
(A) *Moment of Inertia*

1. Find the moment of inertia of a uniform half-disc about an axis perpendicular to the plane and passing through its centre of mass. Mass of this disc is M and radius is R.

2. Find the moment of inertia of a pair of solid spheres, each having a mass m and radius r, kept in contact about the tangent passing through the point of contact.

3. Find the radius of gyration of a circular ring of radius r about a line perpendicular to the plane of this ring and tangent to the ring.

4. Find the moment of inertia of a triangular plane of mass M shown in the figure about vertical axis AB.



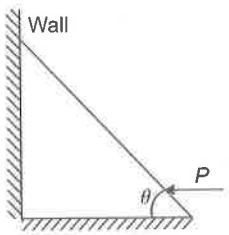
5. A uniform rod of mass m is bent into the form of a semi-circle of radius R. Find the moment of inertia of the rod about an axis passing through A and perpendicular to the plane of the paper.

(B) *Torque and Pure Rotational Motion*

6. A simple pendulum of length ℓ is pulled aside to made an angle θ with the vertical. Find the magnitude of the torque of the weight w of the bob about the point of suspension. When is the torque zero?

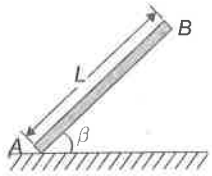
7. Two forces $\vec{F}_1 = 2\hat{i} - 5\hat{j} - 6\hat{k}$ and $\vec{F}_2 = -\hat{i} + 2\hat{j} - \hat{k}$ are acting on a body at the points (1, 1, 0) and (0, 1, 2). Find the torque acting on the body about the point $(-1, 0, 1)$.

8. Assuming frictionless contacts, determine the magnitude of external horizontal force P applied at the lower end for equilibrium of the rod. The rod is uniform and its mass is 'm'.



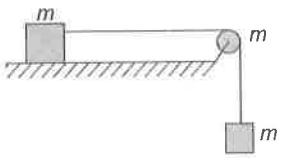
9. A rod of mass m and length L, lying horizontally, is free to rotate about a vertical axis through its centre. A horizontal force of constant magnitude F acts on the rod at a distance of $L/4$ from the centre. The force is always perpendicular to the rod. Find the angle rotated by the rod during the time t after the motion starts.

10. The uniform rod AB of mass m is released from rest when $\beta = 60°$. Assuming that the frictional force between end A and the surface is large enough to prevent sliding, determine (for the instant just after release)

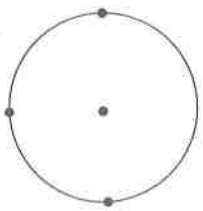


(A) the angular acceleration of the rod,
(B) the normal reaction and the friction force at A,
(C) the minimum value of μ, compatible with the described motion.

11. Figure shows two blocks of mass m and m connected by a string passing over a pulley. The horizontal table over which the mass m slides is smooth. The pulley (uniform disc) has mass m and it can freely rotate about this axis. Find the acceleration of the mass m assuming that the string does not slip on the pulley.



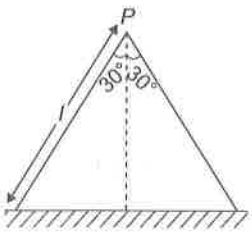
12. A solid cylinder of mass $M = 1$ kg and radius $R = 0.5$ m is pivoted at its centre and has three particles of mass $m = 0.1$ kg mounted at its perimeter as shown in the figure. The system is originally at rest. Find the angular speed of the cylinder, when it has swung through 90° in the anti-clockwise direction.



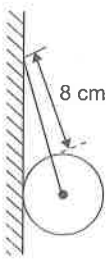
13. A cube is in limiting equilibrium on an inclined plane forming an angle of 30° with the horizontal. Find the line of action of the normal reaction of the plane on the cube.

14. A body weighs 6 g when placed in one pan and 24 g when placed on the other pan of a false balance. If the beam is horizontal when both the pans are empty, find the true weight of the body.

15. An inverted '*V*' is made up of two uniform boards each weighing 200 N. Each side has the same length and makes an angle 30° with the vertical as shown in the figure. Find the magnitude of the static frictional force that acts on each of the lower end of *V*.

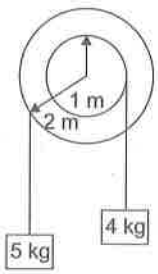


16. A uniform sphere of weight *W* and radius 5 cm is being held by a string as shown in the figure. The wall is smooth. Find the tension in the string.

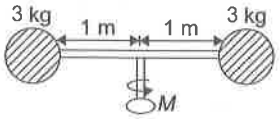


17. A light string is wrapped around a cylinder of mass '*m*' and radius '*R*'. The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder unwinds the string. Then find the length of the string unwound when the cylinder has reached a speed ω.

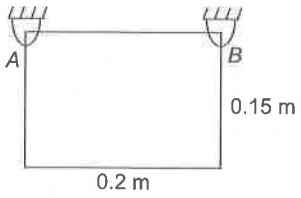
18. The moment of inertia of the pulley system as shown in the figure is 4 kgm^2. The radii of bigger and smaller pulleys are 2 m and 1 m, respectively. Find the angular acceleration of the pulley system.



19. The two small spheres each have a mass of 3 kg and are attached to the rod of negligible mass. A torque $M = 8t$ Nm, where t is in seconds is applied to the rod. Find the value of time when each sphere attains a speed of 3 m/s starting from rest.

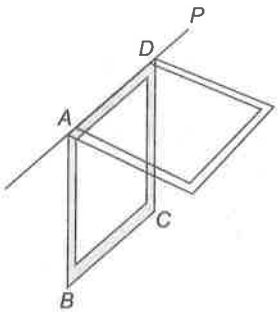


20. A rectangular plate of mass 20 kg is suspended from points *A* and *B* as shown. If pin *B* is removed, determine the initial angular acceleration (in rad/s^2) of plate. (g = 10 m/s^2)



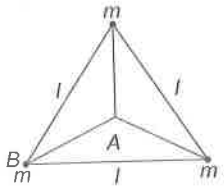
21. A solid homogeneous cylinder of height h and base radius r is kept vertically on a conveyer belt moving horizontally with an increasing velocity $v = a + bt^2$. If the cylinder is not allowed to slip, find the time when the cylinder is about to topple.

22. A square frame made up of a wire of mass m and length l is held in the horizontal plane. It is free to rotate about *AD*. If the frame is released, the work done by gravity during the time frame rotates through an angle of 90° is equal to

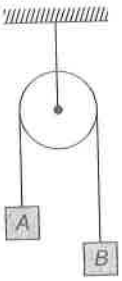


23. Three equal masses m are rigidly connected to each other by massless rods of length l forming an equilateral triangle, as shown in the figure. The assembly is to be given an angular velocity ω about an axis perpendicular to the triangle. For fixed ω, the the

ratio of the kinetic energy of the assembly for an axis through B compared with that for an axis through A is equal to

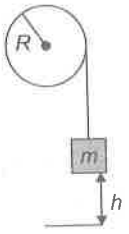


24. In the figure, A and B are two blocks of masses 4 kg and 2 kg, respectively, attached to the two ends of a light string passing over a disc C of mass 40 kg and radius 0.1 m. The disc is free to rotate about a fixed horizontal axes, coinciding with its own axis. The system is released from rest and the string does not slip over the disc. Find:



(A) the linear acceleration of mass B,
(B) the number of revolutions made by the disc at the end of 10 s from the start,
(C) the tension in the string segment supporting the block A.

25. A mass m is attached to a pulley through a cord as shown in the figure. The pulley is a solid disc with radius R. The cord does not slip on the disc. The mass is released from rest at a height h from the ground and at the instant the mass reaches the ground, the disc is rotating with angular velocity ω. Find the mass of the disc.



(C) *Angular Momentum*

26. A particle having mass 2 kg is moving along the straight line $3x + 4y = 5$ with speed 8 m/s. Find the angular momentum of the particle about the origin, x and y are in metres.

27. A particle having mass 2 kg is moving with velocity $\left(2\hat{i} + 3\hat{j}\right)$ m/s. Find the angular momentum of the particle about the origin when it is at (1, 1, 0).

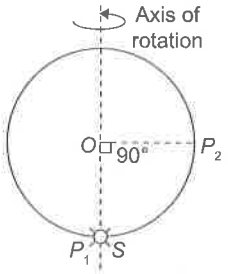
28. A uniform square plate of mass 2.0 kg and edge 10 cm rotates about one of its diagonals under the action of a constant torque of 0.10 Nm. Calculate the angular momentum and the kinetic energy of the plate at the end of the fifth second after the start.

29. A wheel of moment of inertia 0.500 kg-m^2 and radius 20.0 cm is rotating about its axis at an angular speed of 20.0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find new angular speed of the wheel.

30. A uniform circular disc can rotate freely about a rigid vertical axis through its centre O. A man stands at rest at A on the edge due east of O. The mass of the disc is 22 times the mass of the man. The man starts walking anti-clockwise. What will be his position, when he reaches the point A after completing one rotation relative to the disc?

31. Two identical discs are positioned on a vertical axis. The bottom disc is rotating at angular velocity ω_0 and has rotational kinetic energy KE_0. The top disc is initially at rest. It is allowed to fall and sticks to the bottom disc. What is the rotational kinetic energy of the system after the collision?

32. A uniform ring is rotating about vertical axis with angular velocity ω initially. A point insect (S) having the same mass as that of the ring starts walking from the lowest point P_1 and finally reaches the point P_2 (as shown in the figure). The final angular velocity of the ring will be equal to



33. A particle of mass 10 kg is moving with a uniform speed of 6 m/s in the x–y plane along the line $3y = 4x + 10$. Find the magnitude of its angular momentum about the origin in kg-m^2/s.

(D) *Combined Translational + Rotational Motion*

34. A sphere of mass m rolls on a plane surface. Find its kinetic energy at an instant when its centre moves with speed v.

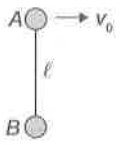
35. A cylinder rolls on a horizontal plane surface. If the speed of the centre is 25 m/s, what is the speed of the highest point?

36. A small spherical ball is released from a point at a height h on a rough track shown in the figure. Assuming that it does not slip anywhere, find its linear speed when it rolls on the horizontal part of the track.

37. A sphere starts rolling down an incline plane of inclination θ. Find the speed of its centre when it has covered a distance ℓ.

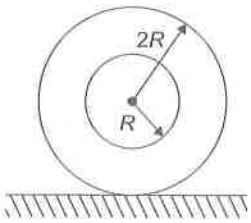
38. A solid uniform sphere of mass m is released from rest from the rim of a hemispherical cup so that it rolls without sliding along the surface. If the rim of the hemisphere is kept horizontal, find the normal force exerted by the cup on the ball when the ball reaches the bottom of the cup.

39. Two small spheres A and B, respectively, of masses m and 2 m are connected by a rigid rod of length ℓ and negligible mass. The two spheres are resting on a horizontal, frictionless surface. When A is suddenly given the velocity v_0 as shown in the figure. Find the velocities of A and B after the rod has rotated through 180°.

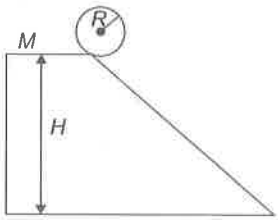


40. A uniform rod of mass m and length ℓ is struck at an end by a force F perpendicular to the rod for a short time interval t. Calculate
    (A) the speed of the centre of mass,
    (B) the angular speed of the rod about the centre of mass,
    (C) the kinetic energy of the rod and
    (D) the angular momentum of the rod about the centre of mass after the force has stopped to act. Assume that t is so small that the rod does not appreciably change its direction while the force acts.

41. A hollow cylinder with inner radius R, outer radius $2R$ and mass M is rolling with speed of its axis v. Find its kinetic energy.



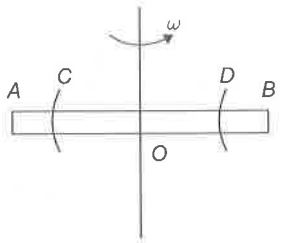
42. The cylinder shown, with mass M and radius R, has a radially dependent density. The cylinder starts from rest and rolls without slipping down an inclined plane of height H. At the bottom of the plane, its translational speed is $(8gH/7)^{1/2}$. Which of the following is the rotational inertia of the cylinder?



Level II

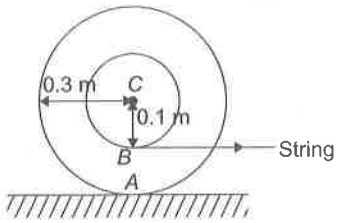
1. A thin uniform rod of mass M and length L is hinged at its upper end and released from rest in a horizontal position. The tension at a point located at a distance $L/3$ from the hinge point, when the rod becomes vertical, will be

2. A rigid horizontal smooth rod AB of mass 0.75 kg and length 40 cm can rotate freely about a fixed vertical axis through its mid-point O. Two rings each of mass 1 kg are initially at rest at a distance of 10 cm from O on either side of the rod. The rod is set in rotation with an angular velocity of 30 rad/s. The velocity of each ring along the length of the rod in m/s when they reach the ends of the rod is



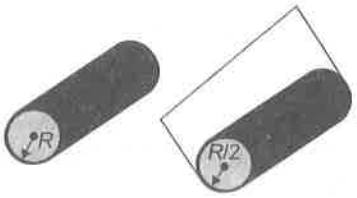
3. A straight rod AB of mass M and length L is placed on a frictionless horizontal surface. A horizontal force having constant magnitude F and a fixed direction starts acting at the end A. The rod is initially perpendicular to the force. The initial acceleration of end B is

4. A wheel is made to roll without slipping, towards right, by pulling a string wrapped around a coaxial spool as shown in the figure. With what velocity the string should be pulled so that the centre of the wheel moves with a velocity of 3 m/s?

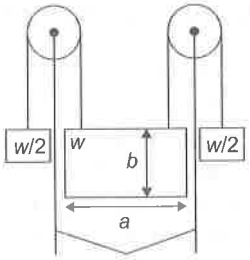


5. A solid uniform disc of mass m rolls without slipping down a fixed inclined plane with an acceleration a. The frictional force on the disc due to surface of the plane is

6. A carpet of mass 'M' made of inextensible material is rolled along its length in the form of a cylinder of radius 'R' and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. The horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $R/2$ will be

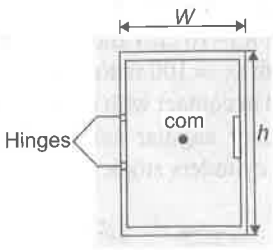


7. A slightly loosely fit window is balanced by two strings which are connected to weights $w/2$ each. The strings pass over the frictionless pulleys as shown in the figure. The strings are tied almost at the corner of the window. The string on the right is cut and then the window accelerates downwards. If the coefficient of friction between the window and the side supports is μ, then calculate the acceleration of the window in terms of μ, a, b and g, where a is the width and b is the length of the window.



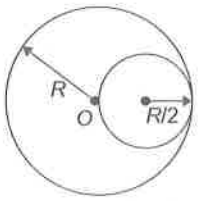
Fixed window support

8. A uniform wood door has mass m, height h and width w. It is hanging from two hinges attached to one side; the hinges are located at $h/3$ and $2h/3$ from the bottom of the door. Suppose that $m = 20.0$ kg, $h = 2.20$ m and $w = 1.00$ m and the bottom smooth hinge is not screwed into the door frame. Find the forces acting on the door.

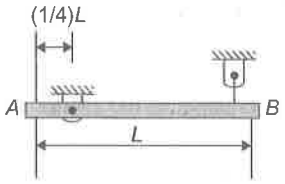


9. A hole of radius $R/2$ is cut from a solid sphere of radius R. If the mass of the remaining plate is M, then moment of inertia of the body about an axis through O perpendicular to plane is _____.

413



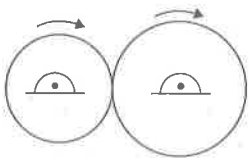
10. A uniform beam of length L and mass m is supported as shown in the figure. If the cable suddenly breaks, determine:



(A) the acceleration of end B,
(B) the reaction at the pin support.

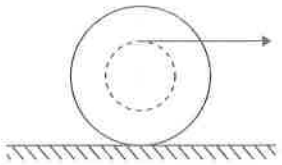
11. A thin rod AB of length a has variable mass per unit length $\rho_0\left(1+\frac{x}{a}\right)$, where x is the distance measured from A and ρ_0 is a constant.
(A) Find the mass M of the rod.
(B) Find the position of centre of mass of the rod.
(C) Find the moment of inertia of the rod about an axis passing through A and perpendicular to AB. Rod is freely pivoted at A and is hanging in equilibrium when it is struck by a horizontal impulse of magnitude P at the point B.
(D) Find the angular velocity with which the rod begins to rotate.
(E) Find the minimum value of impulse P if B passes through a point vertically above A.

12. Two separate cylinders of masses m (= 1 kg) and $4m$ and radii R (=10 cm) and $2R$ rotating in clockwise direction with $\omega_1 = 100$ rad/s and $\omega_2 = 200$ rad/s. Now they are held in contact with each other as in the figure. Determine their angular velocities after the slipping between the cylinders stops.

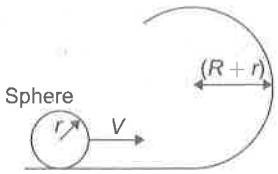


13. A circular disc of mass 300 g and radius 20 cm can rotate freely about a vertical axis passing through its centre of O. A small insect of mass 100 g is initially at a point A on the disc (which is initially stationary) the insect starts walking from rest along the rim of the disc with such a time varying relative velocity that the disc rotates in the opposite direction with a constant angular acceleration = 2π rad/s^2. After some time T, the insect is back at the point A. By what angle has the disc rotated till now, as seen by a stationary earth observer? Also find the time T.

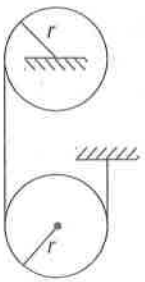
14. A spool of inner radius R and outer radius $3R$ has a moment of inertia = MR^2 about an axis passing through its geometric centre, where M is the mass of the spool. A thread wound on the inner surface of the spool is pulled horizontally with a constant force = Mg. Find the acceleration of the point on the thread which is being pulled assuming that the spool rolls purely on the floor.



15. A sphere of mass m and radius r is pushed onto the fixed horizontal surface such that it rolls without slipping from the beginning. Determine the minimum speed v of its mass centre at the bottom so that it rolls completely around the loop of radius $(R + r)$ without leaving the track in between.

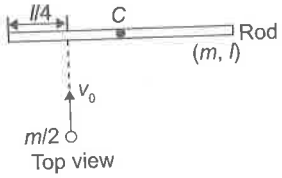


16. Two uniform cylinders, each of mass m = 10 kg and radius r = 150 mm, are connected by a rough belt as shown in the figure. If the system is released from rest, determine:

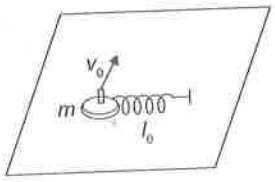


(A) the velocity of the centre of cylinder A after it has moved through 1.2 m,
(B) the tension in the portion of the belt connecting the two cylinders.

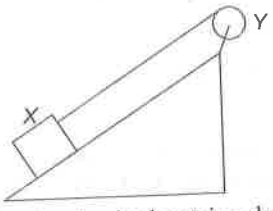
17. A uniform rod of mass m and length l is resting on a smooth horizontal surface. A particle of mass $m/2$ travelling with a speed v_0 hits the rod normally and elastically. Find final velocity of the particle and the angular velocity of the rod.



18. One side of a spring of initial, upstretched length $l_0 =$ 1 m, lying on a frictionless table, is fixed; the other one is fastened to a small puck of mass $m = 0.1$ kg. The puck is given velocity in a direction perpendicular to the spring, at an initial speed $v_0 = 11$ m/s. In the course of the motion, the maximum elongation of the spring is $l = l_0/10$. What is the force constant of the spring (in SI units)?



19. A block X of mass 0.5 kg is held by a long massless string on a frictionless inclined plane of inclination 30° to the horizontal. The string is wound on a uniform solid cylindrical drum Y of mass 2 kg and of radius 0.2 m as shown in the figure. The drum is given an initial angular velocity such that the block X starts moving up the plane.



(A) Find the tension in the string during the motion.
(B) At a certain instant of time, the magnitude of the angular velocity of Y is 10 rad/s. Calculate the distance travelled by X from that instant of time until it comes to rest.

Previous Year Questions

JEE Main

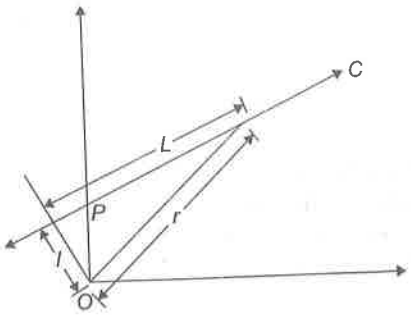
1. A solid sphere, a hollow sphere and a ring are released from top of an inclined plane (frictionless) so that they slide down the plane. Then the maximum acceleration down the plane is for (no rolling) **[AIEEE 2002]**

(A) solid sphere (B) hollow sphere
(C) ring (D) all same

2. Moment of inertia of a circular wire of mass M and radius R about its diameter is **[AIEEE 2002]**

(A) $\frac{MR^2}{2}$ (B) MR^2
(C) $2MR^2$ (D) $\frac{MR^2}{4}$

3. A particle of mass m moves along the line PC with velocity v as shown in the figure. What is the angular momentum of the particle about O? **[AIEEE 2002]**



(A) mvL (B) mvl
(C) mvr (D) zero

4. Initial angular velocity of a circular disc of mass M is ω_1. Then two small spheres of mass m are attached gently to two diametrically opposite points on the edge of the disc. What is the final angular velocity of the disc? **[AIEEE 2002]**

(A) $\left(\frac{M+m}{M}\right)\omega_1$ (B) $\left(\frac{M+m}{m}\right)\omega_1$

(C) $\left(\frac{M}{M+4m}\right)\omega_1$ (D) $\left(\frac{M}{M+2m}\right)\omega_1$

5. Let $\vec{F}$ be the force acting on a particle having position vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force about the origin. Then **[AIEEE 2003]**

(A) $\vec{r}\cdot\vec{\tau}=0$ and $\vec{F}\cdot\vec{\tau}\neq 0$

(B) $\vec{r}\cdot\vec{\tau}\neq 0$ and $\vec{F}\cdot\vec{\tau}=0$

(C) $\vec{r}\cdot\vec{\tau}\neq 0$ and $\vec{F}\cdot\vec{\tau}\neq 0$

(D) $\vec{r}\cdot\vec{\tau}=0$ and $\vec{F}\cdot\vec{\tau}=0$

6. A particle performing uniform circular motion has angular momentum L. If its angular frequency is doubled and its kinetic energy halved, then the new angular momentum is **[AIEEE 2003]**

(A) $\frac{L}{4}$ (B) $2L$

(C) $4L$ (D) $L/2$

7. A circular disc X of radius R is made from an iron plate of thickness t, and another disc y of radius $4R$ is made from an iron plate of thickness $t/4$. Then the relation between the moment of inertia I_X and I_Y is **[AIEEE 2003]**

(A) $I_Y=32I_x$ (B) $I_Y=16I_X$

(C) $I_Y=I_X$ (D) $I_Y=64I_X$

8. A solid sphere is rotating in free space. If the radius of the sphere is increased keeping mass same, which one of the following will not be affected? **[AIEEE 2004]**

(A) Moment of inertia

(B) Angular momentum

(C) Angular velocity

(D) Rotational kinetic energy

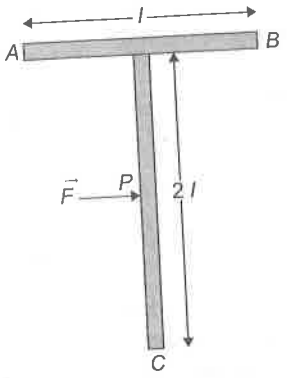
9. One solid sphere A and another hollow sphere B are of same mass and same outer radii. Their moment of inertia about their diameters are respectively I_A and I_B such that **[AIEEE 2004]**

(A) $I_A=I_B$ (B) $I_A>I_B$

(C) $I_A<I_B$ (D) $\frac{I_A}{I_B}=\frac{d_A}{d_B}$

Where d_A and d_B are their densities.

10. A T-shaped object with dimensions shown in the figure is lying on a smooth floor. A force $\vec{F}$ is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C. **[AIEEE 2005]**



(A) $\frac{2}{3}l$ (B) $\frac{3}{2}l$

(C) $\frac{4}{3}l$ (D) l

11. The moment of inertia of uniform semi-circular disc of mass M and radius r about a line perpendicular to the plane of the disc through the centre is **[AIEEE 2005]**

(A) $\frac{1}{4}Mr^2$ (B) $\frac{2}{5}Mr^2$

(C) Mr^2 (D) $\frac{1}{2}Mr^2$

12. A thin circular ring of mass m and radius R is rotating about its axis with a constant angular velocity ω. Two objects each of mass M are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity $\omega'=$ **[AIEEE 2006]**

(A) $\frac{\omega(m+2M)}{m}$ (B) $\frac{\omega(m-2M)}{(m+2M)}$

(C) $\frac{\omega m}{(m+M)}$ (D) $\frac{\omega m}{(m+2M)}$

13. Four point masses, each of value m, are placed at the corners of a square $ABCD$ of side l. The moment of inertia of this system about an axis passing through A and parallel to BD is **[AIEEE 2006]**

(A) $2ml^2$ (B) $\sqrt{3}ml^2$
(C) $3ml^2$ (D) ml^2

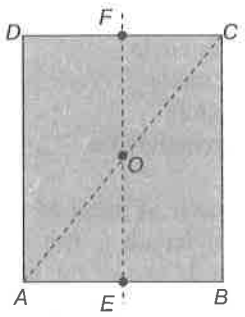
14. Angular momentum of the particle rotating with a central force is constant due to **[AIEEE 2007]**
(A) constant force
(B) constant linear momentum
(C) zero torque
(D) constant torque

15. A round uniform body of radius R, mass M and moment of inertia I, rolls down (without slipping) an inclined plane making an angle θ with the horizontal. Then its acceleration is **[AIEEE 2007]**

(A) $\frac{g\sin\theta}{1+I/MR^2}$ (B) $\frac{g\sin\theta}{1+MR^2/I}$

(C) $\frac{g\sin\theta}{1-I/MR^2}$ (D) $\frac{g\sin\theta}{1-MR^2/I}$

16. For the given uniform square lamina $ABCD$, whose centre is O **[AIEEE 2007]**



(A) $\sqrt{2}I_{AC} = I_{EF}$ (B) $I_{AD} = 3I_{EF}$
(C) $I_{AD} = 4I_{EF}$ (D) $I_{AD} = \sqrt{2}I_{EF}$

17. Consider a uniform square plate of side a and mass m. The moment of inertia of this plate about an axis perpendicular to its plane and passing through one of its corners is **[AIEEE 2008]**

(A) $\frac{5}{6}ma^2$ (B) $\frac{1}{12}ma^2$
(C) $\frac{7}{12}ma^2$ (D) $\frac{2}{3}ma^2$

18. A thin uniform rod of length l and mass m is swinging freely about a horizontal axis passing through its end. Its maximum angular speed is ω. Its centre of mass rises to the maximum height of **[AIEEE 2009]**

(A) $\frac{1}{3}\frac{l^2\omega^2}{g}$ (B) $\frac{1}{6}\frac{l\omega}{g}$
(C) $\frac{1}{2}\frac{l^2\omega^2}{g}$ (D) $\frac{1}{6}\frac{l^2\omega^2}{g}$

19. A pulley of radius 2 m is rotated about its axis by a force $F = (20t - 5t^2)$ N (where t is measured in seconds) applied tangentially. If the moment of inertia of the pulley about its axis of rotation is 10 kg-m^2 the number of rotations made by the pulley before its direction of motion, if reserved, is **[AIEEE 2011]**
(A) more than 3 but less than 6
(B) more than 6 but less than 9
(C) more than 9
(D) less than 3

20. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect is at rest at a point near the rim of the disc. The insect now moves along a diameter of the disc to reach its other end. During the journey of the insect, the angular speed of the disc **[AIEEE 2011]**
(A) continuously decreases
(B) continuously increases
(C) first increases and then decreases
(D) remains unchanged

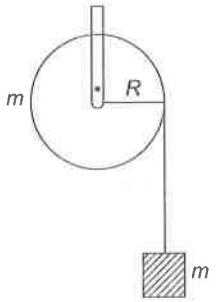
21. A hoop of radius r and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of the hoop when it cases to slip? **[JEE Mains 2013]**

(A) $\frac{r\omega_0}{2}$ (B) $r\omega_0$
(C) $\frac{r\omega_0}{4}$ (D) $\frac{r\omega_0}{3}$

22. A bob of mass m attached to an inextensible string of length ℓ is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical. About the point of suspension:
(A) Angular momentum changes in direction but not in magnitude.
(B) Angular momentum changes both in direction and in magnitude.
(C) Angular momentum is conserved.
(D) Angular momentum changes in magnitude but not in direction. **[JEE Main 2014]**

23. A mass 'm' is supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release?

[JEE Main 2014]



(A) $\frac{5g}{6}$ (B) g

(C) $\frac{2g}{3}$ (D) $\frac{g}{2}$

24. A block of mass m is placed on a surface with a vertical cross-section given by $y = \frac{x^3}{6}$. If the coefficient of friction is 0.5, the maximum height above the ground at which the block can be placed without slipping is

[JEE Main 2014]

(A) $\frac{1}{3}$ m (B) $\frac{1}{2}$ m

(C) $\frac{1}{6}$ m (D) $\frac{2}{3}$ m

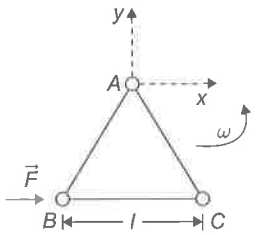
25. From a solid sphere of mass M and radius R, a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its centre and perpendicular to one of its faces is **[JEE Main 2014]**

(A) $\frac{4MR^2}{9\sqrt{3}\pi}$ (B) $\frac{4MR^2}{3\sqrt{3}\pi}$

(C) $\frac{MR^2}{32\sqrt{2}\pi}$ (D) $\frac{MR^2}{16\sqrt{2}\pi}$

JEE Advanced

1. Three particles A, B and C, each of mass m, are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side l. This body is placed on a horizontal frictionless table (x–y plane) and is hinged to it at the point A so that it can move without friction about the vertical axis through A (see the figure). The body is set into rotational motion on the table about A with a constant angular velocity ω. **[JEE'(Scr) 2002]**



(A) Find the magnitude of the horizontal force exerted by the hinge on the body.

(B) At time T, when the side BC is parallel to the x-axis, a force F is applied on B along BC (as shown). Obtain the x-component and the y-component of the force exerted by the hinge on the body, immediately after time T.

2. A particle is moving in a horizontal uniform circular motion. The angular momentum of the particle is conserved about the point: **[JEE'(Scr) 2003]**

(A) Centre of the circle
(B) Outside the circle
(C) Inside the circle
(D) Point on circumference

3. Two particles each of mass M are connected by a massless rod of length l. The rod is lying on the smooth surface. If one of the particles is given an impulse MV as shown in the figure, then the angular velocity of the rod would be

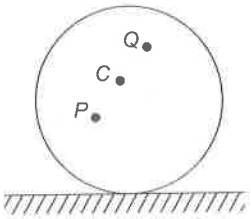
[JEE'(Scr) 2003]

(A) v/l (B) $2v/l$

(C) $v/2l$ (D) none of these

4. A disc is rolling (without slipping) on a horizontal surface. C is its centre, and Q and P are two points equidistant from C. Let V_P, V_Q and V_C be the magnitude of velocities of points P, Q and C, respectively, then

[JEE'(Scr) 2004]



(A) $V_Q > V_C > V_D$ (B) $V_Q < V_C < V_P$

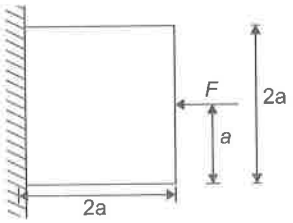
(C) $V_Q = V_p$, $V_C = \frac{1}{2}V_p$ (D) $V_Q < V_C > V_P$

5. A child is standing with folded hands at the centre of a platform rotating about its central axis. The kinetic energy of the system is K. The child now stretches his arms so that the moment of inertia of the system doubles. The kinetic energy of the system now is **[JEE'(Scr) 2004]**

(A) $2K$ (B) $K/2$

(C) $K/4$ (D) $4K$

6. A block of mass m is held fixed against a wall by a applying a horizontal force F. Which of the following option is incorrect? **[JEE'(Scr) 2005]**



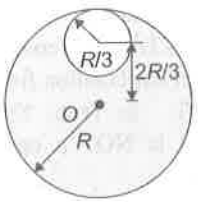
(A) Friction force $= mg$

(B) F will not produce torque

(C) Normal will not produce torque

(D) Normal reaction $= F$

7. A disc has mass $9m$. A hole of radius $R/3$ is cut from it as shown in the figure. The moment of inertia of the remaining part about an axis passing through the centre 'O' of the disc and perpendicular to the plane of the disc is **[JEE'(Scr) 2005]**



(A) $8mR^2$ (B) $4mR^2$

(C) $\frac{40}{9}mR^2$ (D) $\frac{37}{9}mR^2$

8. A particle moves in circular path with decreasing speed. Which of the following is correct? **[JEE'(Scr) 2005]**

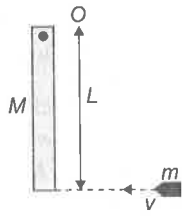
(A) $\vec{L}$ is constant

(B) Only direction of $\vec{L}$ is constant

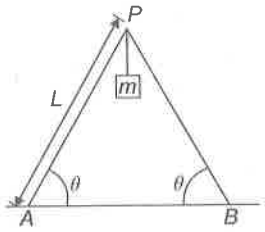
(C) Acceleration $\vec{a}$ is towards the centre

(D) It will move in a spiral and finally reach the centre

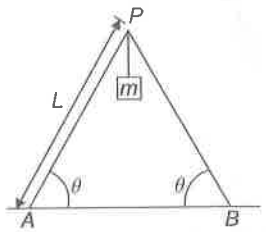
9. A wooden log of mass M and length L is hinged by a frictionless nail at O. A bullet of mass m strikes with velocity v and sticks to it. Find angular velocity of the system immediately after the collision about O. **[JEE' 2005]**



10. A cylinder of mass m and radius R rolls down an inclined plane of inclination θ. Calculate the linear acceleration of the axis of cylinder. **[JEE' 2005]**



11. Two identical ladders, each of mass M and length L, are resting on the rough horizontal surface as shown in the figure. A block of mass m hangs from P. If the system is in equilibrium, find the magnitude and the direction of frictional force at A and B. **[JEE' 2005]**



12. A solid sphere of mass M, radius R and having moment of inertia about an axis passing through the centre of

mass as I, is recast into a disc of thickness t, whose moment of inertia about an axis passing through its edge and perpendicular to its plane remains I. Then, the radius of the disc will be **[JEE' 2006]**

(A) $2R/\sqrt{15}$ (B) $R\sqrt{2/15}$

(C) $4R/\sqrt{15}$ (D) $R/4$

13. A solid cylinder of mass m and radius r is rolling on a rough inclined plane of inclination θ. The coefficient of friction between the cylinder and incline is μ. Then **[JEE' 2006]**

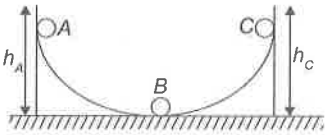
(A) frictional force is always $\mu mg \cos\theta$

(B) friction is a dissipative force

(C) by decreasing θ, frictional force decreases

(D) friction opposes translation and supports rotation

14. A ball moves over a fixed track as shown in the figure. From A to B, the ball rolls without slipping. Surface BC is frictionless. K_A, K_B and K_C are kinetic energies of the ball at A, B and C, respectively. Then **[JEE' 2006]**



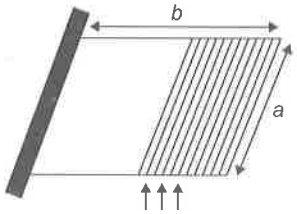
(A) $h_A > h_C$; $K_B > K_C$

(B) $h_A > h_C$; $K_C > K_A$

(C) $h_A = h_C$; $K_B = K_C$

(D) $h_A < h_C$; $K_B > K_C$

15. There is a rectangular plate of mass M kg of dimensions $(a \times b)$. The plate is held in horizontal position by striking n small balls each of mass m per unit area per unit time. These are striking in the shaded half region of the plate. The balls are colliding elastically with velocity v. What is v? **[JEE' 2006]**



It is given $n = 100$, $M = 3$ kg, $m = 0.01$ kg; $b = 2$ m, $a = 1$ m; $g = 10$ m/s^2.

Question No. 16 to 18

Two discs A and B are mounted coaxially on a vertical axle. The discs have moments of inertia I and $2I$, respectively, about the common axis. Disc A is imparted an initial angular velocity 2ω using the entire potential energy of a spring compressed by a distance x_1. Disc B is imparted an angular velocity ω by a spring having the same spring constant and compressed by a distance x_2. Both the discs rotate in the clockwise direction.

16. The ratio x_1/x_2 is **[JEE' 2007]**

(A) 2 (B) 1/2

(C) $\sqrt{2}$ (D) $1\sqrt{2}$

17. When disc B is brought in contact with disc A, they acquire a common angular velocity in time t. The average frictional torque on one disc by the other during this period is **[JEE' 2007]**

(A) $2I\omega/(3t)$ (B) $9I\omega/(2t)$

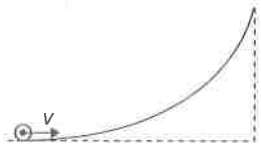
(C) $9I\omega/(4t)$ (D) $3I\omega/(2t)$

18. The loss of kinetic energy during the above process is **[JEE' 2007]**

(A) $Iw^2/2$ (B) $Iw^2/3$

(C) $Iw^2/4$ (D) $Iw^2/6$

19. A small object of uniform density rolls up a curved surface with an initial velocity v. It reaches up to a maximum height of $3v^2/(4\ g)$ with respect to the initial position. The object is **[JEE' 2007]**



(A) ring (B) solid sphere

(C) hollow sphere (D) disc

20. **Statement 1:** If there is no external torque on a body about its centre of mass, then the velocity of the centre of mass remains constant. **[JEE 2007]**

because

Statement 2: The linear momentum of an isolated system remains constant.

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.

(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.

(C) Statement 1 is True, Statement 2 is False.

(D) Statement 1 is False, Statement 2 is True.

21. **Statement 1:** Two cylinders, one hollow (metal) and the other solid (wood) with the same mass and identical dimensions are simultaneously allowed to roll without slipping down an inclined plane from the same height. The hollow cylinder will reach the bottom of the inclined plane first.

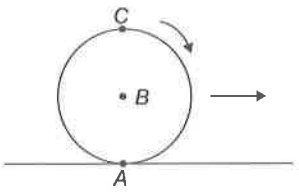
Statement 2: By the principle of conservation of energy, the total kinetic energies of both the cylinders are identical when they reach the bottom of the incline. **[JEE 2008]**

(A) Statement 1 is True, Statement 2 is True; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is True, Statement 2 is True; Statement 2 is NOT a correct explanation for Statement 1.
(C) Statement 1 is True, Statement 2 is False.
(D) Statement 1 is False, Statement 2 is True.

22. If the resultant of all the external forces acting on a system of particles is zero, then from an inertial frame, one can surely say that **[JEE 2009]**

(A) linear momentum of the system does not change in time
(B) kinetic energy of the system does not change in time
(C) angular momentum of the system does not change in time
(D) potential energy of the system does not change in time

23. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, A is the point of contact, B is the centre of the sphere and C is its topmost point. Then, **[JEE 2009]**



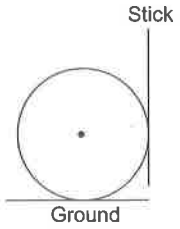
(A) $\vec{V}_C - \vec{V}_A = 2(\vec{V}_B - \vec{V}_C)$

(B) $\vec{V}_C - \vec{V}_B = \vec{V}_B - \vec{V}_A$

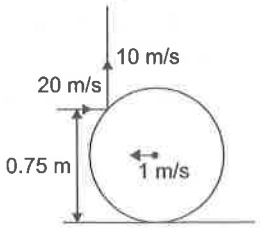
(C) $|\vec{V}_C - \vec{V}_A| = 2|\vec{V}_B - \vec{V}_C|$

(D) $|\vec{V}_C - \vec{V}_A| = 4|\vec{V}_B|$

24. A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shown in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an acceleration of 0.3 m/s^2. The coefficient of friction between the ground and the ring is large enough that rolling always occurs and the coefficient of friction between the stick and the ring is (P/10). The value of P is **[JEE 2011]**

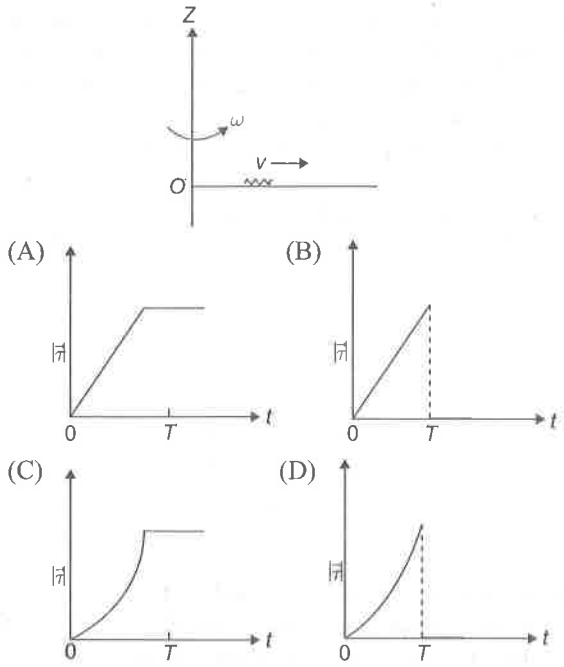


25. A thin ring of mass 2 kg and radius 0.5 m is rolling without slipping on a horizontal plane with velocity 1 m/s. A small ball of mass 0.1 kg, moving with velocity 20 m/s in the opposite direction, hits the ring at a height of 0.75 m and goes vertically up with velocity 10 m/s. Immediately after the collision **[JEE 2011]**

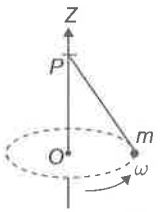


(A) the ring has pure rotation about its stationary CM
(B) the ring comes to a complete stop
(C) friction between the ring and the ground is to the left
(D) there is no friction between the ring and the ground

26. A thin uniform rod, pivoted at O is rotating in the horizontal plane with constant angular speed ω, as shown in the figure. At time $t = 0$, small insect starts from O and moves with constant speed v with respect to the rod towards the other end. It reaches the end of the rod at $t = T$ and stops. The angular speed of the system remains ω throughout. The magnitude of the torque $(|\vec{\tau}|)$ on the system about O, as a function of time is best represented by which plot? **[JEE 2012]**



27. A small mass m is attached to a massless string whose other end is fixed at P as shown in the figure. The mass is undergoing circular motion in the x–y plane with centre at O and constant angular speed ω. If the angular momentum of the system, calculated about O and P are denoted by $\vec{L}_O$ and $\vec{L}_P$, respectively, then **[JEE 2012]**



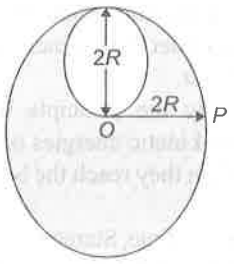
(A) $\vec{L}_O$ and $\vec{L}_P$ do not vary with time

(B) $\vec{L}_O$ varies with time while $\vec{L}_P$ remains constant

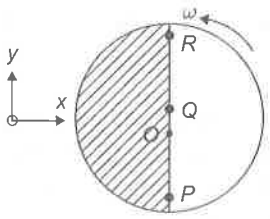
(C) $\vec{L}_O$ remains constant while $\vec{L}_P$ varies with time

(D) $\vec{L}_O$ and $\vec{L}_P$ both vary with time

28. A lamina is made by removing a small disc of diameter $2R$ from a bigger disc of uniform mass density and radius $2R$, as shown in the figure. The moment of inertia of this lamina about axes passing through O and P is I_O and I_P, respectively. Both these axes are perpendicular to the plane of the lamina. The ratio $\frac{I_P}{I_O}$ to the nearest integer is **[JEE 2012]**



29. Consider a disc rotating in the horizontal plane with a constant angular speed ω about its centre O. The disc has a shaded region on one side of the diameter and an unshaded region on the other side as shown in the figure. When the disc is in the orientation as shown, two pebbles P and Q are simultaneously projected at an angle towards R. The velocity of projection is in the y–z plane and is same for both pebbles with respect to the disc. Assume that (i) they land back on the disc before the disc has completed 1/8 rotation, (ii) their range is less than half the disc radius and (iii) ω remains constant throughout. Then **[JEE 2012]**



(A) P lands in the shaded region and Q in the unshaded region

(B) P lands in the unshaded region and Q in the shaded region

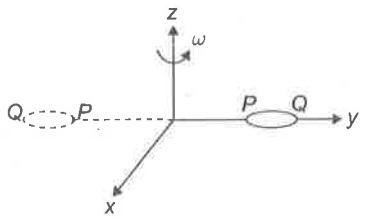
(C) Both P and Q land in the unshaded region

(D) Both P and Q land in the shaded region

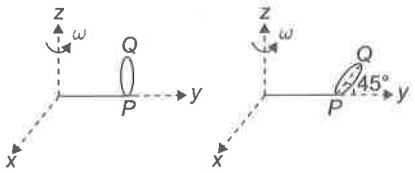
Question No. 30 to 31

The general motion of a rigid body can be considered to be a combination of (i) a motion of its centre of mass about an axis, and (ii) its motion about an instantaneous axis passing through the centre of mass. These axes need not be stationary. Consider, for example, a thin uniform disc welded (rigidly fixed) horizontally at its rim to a massless stick, as shown in the figure. When the disc-stick system is rotated about the origin on a horizontal frictionless plane with angular speed ω, the motion at any instant can be taken as a combination

of (i) a rotation of the centre of mass of the disc about the z-axis, and (ii) a rotation of the disc through an instantaneous vertical axis passing through its centre of mass (as is seen from the changed orientation of points P and Q). Both these motions have the same angular speed ω in this case.



Now consider two similar systems as shown in the figure: Case (A) The disc with its face vertical and parallel to the x–z plane. Case (B) The disc with its face making an angle of 45° with the x–y plane and its horizontal diameter parallel to the x-axis. In both the cases, the disc is welded at point P, and the systems are rotated with constant angular speed ω about the z-axis.



Case (A) **Case (B)**

30. Which of the following statements about the instantaneous axis (passing through the centre of mass) is correct? **[JEE 2012]**

(A) It is vertical for both the cases (A) and (B).

(B) It is vertical for case (A), and is at 45° to the x–z plane and lies in the plane of the disc for case (B).

(C) It is horizontal for case (A); and is at 45° to the x–z plane and is normal to the plane of the disc for case (B).

(D) It is vertical for case (A); and is at 45° to the x–z plane and is normal to the plane of the disc for case (B).

31. Which of the following statements regarding the angular speed about the instantaneous axis (passing through the centre of mass) is correct? **[JEE 2012]**

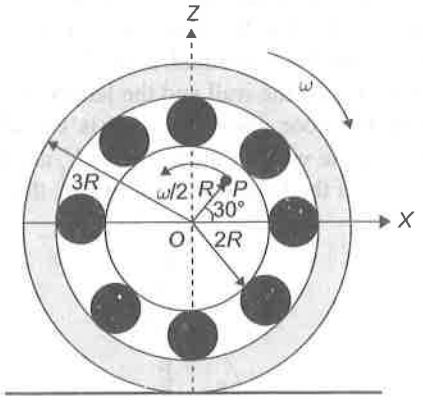
(A) It is $\sqrt{2}\omega$ for both the cases.

(B) It is ω for case (A), and $\dfrac{\omega}{\sqrt{2}}$ for case (B).

(C) It is ω for case (A), and $\sqrt{2}\omega$ for case (B).

(D) It is ω for both the cases.

32. The figure shows a system consisting of (i) a ring of outer radius $3R$ rolling clockwise without slipping on a horizontal surface with angular speed ω and (ii) an inner disc of radius $2R$ rotating anti-clockwise with angular speed $\omega/2$. The ring and disc are separated by frictionless ball bearings. The system is in the x–z plane. The point P on the inner disc is at a distance R from the origin, where OP makes an angle of 30° with the horizontal. Then with respect to the horizontal surface, **[JEE 2012]**



(A) the point O has a linear velocity $3R\omega\hat{i}$

(B) the point P has a linear velocity $\dfrac{11}{4}R\omega\hat{i}+\dfrac{\sqrt{3}}{4}R\omega\hat{k}$

(C) the point P has a linear velocity $\dfrac{13}{4}R\omega\hat{i}-\dfrac{\sqrt{3}}{4}R\omega\hat{k}$

(D) the point P has a linear velocity $\left(3-\dfrac{\sqrt{3}}{4}\right)R\omega\hat{i}+\dfrac{1}{4}R\omega\hat{k}$

33. Two solid cylinders P and Q of same mass and same radius start rolling down a fixed inclined plane from the same height at the same time. Cylinder P has most of its mass concentrated near its surface, while Q has most of its mass concentrated near the axis. Which statement(s) is (are) correct? **[JEE 2012]**

(A) Both cylinders P and Q reach the ground at the same time.

(B) Cylinder P has larger linear acceleration than cylinder Q.

(C) Both cylinders reach the ground with same translational kinetic energy.

(D) Cylinder Q reaches the ground with larger angular speed.

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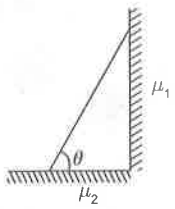
34. A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad s^{-1} about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is

[JEE 2013]

35. In the figure, a ladder of mass m is shown leaning against a wall. It is in static equilibrium making an angle θ with the horizontal floor. The coefficient of friction between the wall and the ladder is m_1 and that between the floor and the ladder is m_2. The normal reaction of the wall on the ladder is N_1 and that of the floor is N_2. If the ladder is about to slip, then



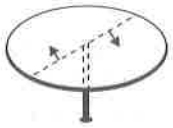
(A) $m_1 = 0, m_2 \neq 0$ and $N_2 \tan\theta = \frac{mg}{2}$

(B) $m_1 \neq 0, m_2 = 0$ and $N_1 \tan\theta = \frac{mg}{2}$

(C) $m_1 \neq 0, m_2 \neq 0$ and $N_2 = \frac{mg}{1+\mu_1\mu_2}$

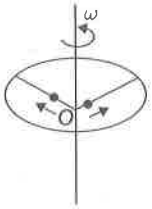
(D) $m_1 = 0, m_2 \neq 0$ and $N_1 \tan\theta = \frac{mg}{2}$

36. A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy-guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance $0.25b$ m from the centre on its either sides along its diameter (see the figure). Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform, the balls have horizontal speed of 9 ms^{-1} with respect to the ground. The rotational speed of the platform in rad s^{-1} after the balls leave the platform is



37. A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $\frac{M}{8}$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant, the angular speed of the system is $\frac{8}{9}\omega$ and one of the masses is at a distance of $\frac{3}{5}R$ from O. At this instant, the distance of the other mass from O is

[JEE Advanced 2015]

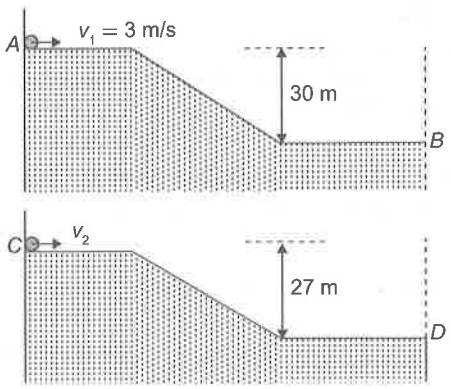


(A) $\frac{2}{3}R$ (B) $\frac{1}{3}R$

(C) $\frac{3}{5}R$ (D) $\frac{4}{5}R$

38. Two identical uniform discs roll without slipping on two different surfaces AB and CD (see the figure) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces. If they reach B and D with the same linear speed and $v_1 = 3$ m/s, then v_2 in m/s is ($g = 10$ m/s^2)

[JEE Advanced 2015]



ANSWER KEYS

Exercises

JEE Main

1. D	2. B	3. A	4. D	5. A	6. B	7. A	8. C	9. A	10. C
11. D	**12.** B	**13.** **D**	**14.** B	**15.** C	**16.** D	**17.** B	**18.** B	**19.** A	**20.** D
21. D	**22.** D	**23.** A	**24.** D	**25.** A	**26.** D	**27.** B	**28.** C	**29.** A	**30.** B
31. B	**32.** A	**33.** C	**34.** C	**35.** D	**36.** C	**37.** B	**38.** B	**39.** B	**40.** C
41. B	42. D	43. B	44. B	45. A	46. D	47. D	48. B	49. A	50. C
51. C	**52.** D	**53.** D	**54.** C	**55.** C	**56.** C	**57.** A	**58.** D	**59.** C	**60.** A
61. D									

JEE Advanced

1. B	**2.** C	**3.** C	**4.** A	**5.** B	**6.** C	**7.** A	**8.** C	**9.** C	
10. ABCD	**11.** B	**12.** B	**13.** A	**14.** C	**15.** D	**16.** BC	**17.** AD	**18.** BCD	**19.** ABD
20. ABC	**21.** ACD	**22.** C	**23.** B	**24.** ACD	**25.** B	**26.** C	**27.** A	**28.** D	**29.** B
30. B	**31.** C	**32.** B	**33.** D	**34.** A	**35.** C	**36.** B	**37.** ACD	**38.** C	**39.** BCD
40. A	**41.** C	**42.** BC	**43.** AC	**44.** BC	**45.** ABCD	**46.** BD	**47.** ABC	**48.** ABC	**49.** C
50. BCD	**51.** ACD	**52.** ABD	**53.** B	**54.** A	**55.** B	**56.** C	**57.** A	**58.** C	**59.** D
60. C	**61.** B	**62.** B	**63.** B	**64.** D	**65.** D	**66.** A	**67.** C		

JEE Advanced

Level I

1. $\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2$ **2.** $\frac{14mr^2}{5}$ **3.** $\sqrt{2}r$ **4.** $\frac{Ml^2}{2}$ **5.** $2mR^2$

6. $w\ell\sin\theta$, when the bob is at the lowest point **7.** $-14\hat{i}+10\hat{j}-9\hat{k}$ **8.** $P=\frac{mg}{2}\cot\theta$ **9.** $\frac{3Ft^2}{2m\ell}$

10. (A) $\frac{3g}{4L}(cw)$ (B) $N=\frac{13mg}{16}\uparrow,\ F=\left(\frac{3\sqrt{3}}{16}\right)mg\rightarrow$ (C) $\frac{3\sqrt{3}}{16}$

11. $\frac{2g}{5}\downarrow$ **12.** $w=\sqrt{5}$ rad/s **13.** At a distance $a/2\sqrt{3}$ from the centre down the plane **14.** 12 g

15. $\frac{100}{\sqrt{3}}$ N **16.** 13 $W/12$ **17.** $\frac{R^2\omega^2}{4g}$ **18.** 2.1 rad/s^2 **19.** $1.5\sqrt{2}$ s **20.** 48 **21.** gr/bh **22.** $\frac{mgl}{8}$ **23.** 2

24. (A) 10/13 m/s^2, (B) $5000/26\pi$, (C) 480/13 N **25.** $M=2m\left(\frac{2gh}{R^2\omega^2}-1\right)$ **26.** 16 kg-m^2/s **27.** $2\hat{k}$ kg-m^2/s

28. 0.5 kg-m^2/s, 75 J **29.** 19.7 rad/s **30.** 60° east of south, 30° south of east **31.** $(1/2)KE_0$ **32.** $\frac{\omega}{3}$

33. 120 **34.** $\frac{7}{10}mv^2$ **35.** 50 m/s **36.** $\sqrt{\frac{10gh}{7}}$ **37.** $\sqrt{\frac{10}{7}g\ell\sin\theta}$ **38.** $\frac{17}{7}mg$ **39.** $\frac{v_0}{3}(\leftarrow),\ \frac{2v_0}{3}(\rightarrow)$

40. (A) $\frac{Ft}{m}$, (B) $\frac{6Ft}{m\ell}$, (C) $\frac{2F^2t^2}{m}$, (D) $\frac{F\ell t}{2}$ **41.** $\frac{13}{16}Mv^2$ **42.** $\frac{3}{4}MR^2$

Level II

1. $2mg$ **2.** 3 **3.** 2 F/M **4.** 2 m/s **5.** $1/2\ ma$ **6.** $v=\sqrt{\frac{14gR}{3}}$ **7.** $a=\left[\frac{b-\mu a}{3b+\mu a}\right]g$

8. $F_{dlx}=\frac{3mgw}{2h}$ and $F_{duy}=mg$ **9.** $57/140\ MR^2$ **10.** (A) $\frac{9g}{7}\downarrow$, (B) $\frac{4mg}{7}\uparrow$

11. (A) $\frac{3\rho_0 a}{2}$, (B) $\frac{5a}{9}$, (C) $\frac{7a^3\rho_0}{12}$, (D) $\frac{18p}{7Ma}$, (E) $\frac{M}{9}\sqrt{70ag}$ **12.** 300 rad/s, 150 rad/s

13. $t=2/\sqrt{5}$ s, $q=4p/5$ rad **14.** 16 m/s² **15.** $v=\sqrt{\frac{27}{7}gR}$ **16.** (A) $4\sqrt{\frac{3}{7}}$m/s, (B) $\frac{200}{7}$N **17.** $-\frac{1}{15}v_0$

18. 210 **19.** 1.63 N, 1.224 m

Previous Year Questions

JEE Main

1. D	**2.** A	**3.** B	**4.** C	**5.** D	**6.** A	**7.** D	**8.** B	**9.** C	**10.** C
11. D	**12.** D	**13.** C	**14.** C	**15.** A	**16.** C	**17.** D	**18.** D	**19.** A	**20.** C
21. A	**22.** A	**23.** D	**24.** C	**25.** A					

JEE Advanced

1. (A) $\sqrt{3}$ m w^2 l, (B) $F_x=F/4, F_y=\sqrt{3}$ m w^2 l **2.** A **3.** A **4.** A **5.** B **6.** C

7. B **8.** B **9.** $\omega=\frac{3mv}{(3m+M)L}$ **10.** $a_{axis}=\frac{2g\sin\theta}{3}$ **11.** $f=(M+m)g\frac{\cot\theta}{2}$ **12.** A **13.** C, D

14. A, B	**15.** 10 m/s	**16.** C	**17.** A	**18.** B	**19.** D	**20.** D	**21.** D	**22.** A	**23.** B, C
24. 0004	**25.** A	**26.** B	**27.** C	**28.** 0003	**29.** C	**30.** A	**31.** D	**32.** A,B	**33.** D
34. 0008	**35.** C, D	**36.** 4	**37.** D	**38.** 7					

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