Day 1

Sets, Relations and Functions

Day 1 Outlines ...

- Sets
- Types of Sets
- Operations on Sets
- Laws of Algebra of Sets
- Relation
- Types of Relations
- Function or Mapping
- Different Types of Function

Sets

Sets are one of the most fundamental concepts in mathematics. A **set** is a well defined **class** or collection of the objects. Set is denoted by the symbol $A, B, C, \ldots$ and its elements are denoted by $a, b, c, \ldots$ etc.

e.g., The numbers 2, 4, and 6 are distinct objects when considered separately but when they are considered collectively they form a single set of size three, written as $\{2,4,6\}$.

Roster Method (Listing Method)

In this method, a set is described by listing elements, separated by commas, within braces.

e.g., A set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.

Set Builder Method (Rule Method)

In this method, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x) \text{ holds}\}$ or $\{x | P(x) \text{ holds}\}$ which is read as the set of all x such that $P(x)$ holds.

The symbol '|' or ':' is read as such that

e.g., The set $P = \{0, 1, 4, 9, 16, \ldots\}$ can be written as $P = \{x^2 \mid x \in Z\}$.

Types of Sets

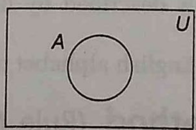
- The set which contains no element at all is called the **null set** (empty set or void set) and it is denoted by the symbol 'ϕ' or '{}' and if it contains single element in a set is called **singleton set**.
- A set in which the process of counting of elements is definitely comes to an end, is called a **finite set**, otherwise it is an **infinite set**.
- Two sets A and B are said to be **equal set** iff every element of A is an element of B and also every element of B is an element of A. *i.e.*, $A = B$, if $x \in A \Leftrightarrow x \in B$.
- But in **equivalent set**, they have the same number of elements, not exactly the same elements.
- Two sets A and B are **comparable**, if one of them is a subset of the other *i.e.*, either $A \subseteq B$ or $B \subseteq A$.
- A set that contains all sets in a given context is called **universal set** (U).
- Let A and B be two sets. If every element of A is an element of B, then A is called a **subset** of B, *i.e.*, $A \subseteq B$.
- If A is a subset of B and $A \neq B$, then A is a **proper subset** of B. *i.e.*, $A \subset B$.
- The null set ϕ is a subset of every set. Every set is a subset of itself *i.e.*, $\phi \subset A$ and $A \subseteq A$ for every set A. They are called **improper subsets** of A.
- If S is any set, then the family of all the subsets of S is called the **power set** of S and it is denoted by $P(S)$. Power set of a given set is always non-empty. If A has n elements, then $P(A)$ has 2^n elements.

> - *The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ.*
> - *Whenever, we have to show that two sets A and B are equal show that $A \subseteq B$ and $B \subseteq A$, then $A = B$.*
> - *Total number of subsets of a finite set containing n elements is 2^n.*

Venn Diagram

The combination of rectangles and circles is called **Venn Euler diagram** or Venn diagram.

In Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle on some closed geometrical figure.



where A is the set and U is the universal set.

Operations on Sets

1. Union

Let A and B are two sets. The union of A and B is the set of all elements which are in set A or in B or both A and B.

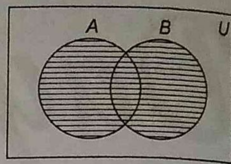
i.e., $A \cup B = \{x : x \in A \text{ or } x \in B\}$

If $A_1, A_2, \ldots, A_n$ is a finite family of sets.

Then, union of sets is denoted by

$$\bigcup_{i=1}^{n} A_i$$

or $A_1 \cup A_2 \cup \ldots \cup A_n$.

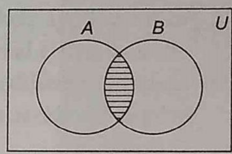


$A \cup B$

2. Intersection

The intersection of A and B is the set of all those elements that belongs to both A and B.

i.e., $A \cap B = \{x : x \in A \text{ and } x \in B\}$.



$A \cap B$

If $A_1, A_2, \ldots, A_n$ is a finite family of sets.

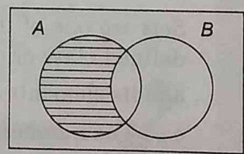
Then, their intersection of sets is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.

3. Difference

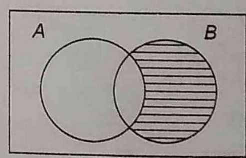
The difference of set A and set B *i.e.*, $A - B$, is the set of all those elements of A which do not belong to B.

i.e., $A - B = \{x : x \in A \text{ and } x \notin B\}$

and $B - A = \{x \in B : x \notin A\}$.



$A - B$

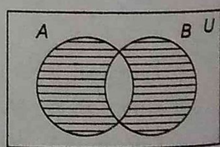


$B - A$

4. Symmetric Difference

The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.

$$A \Delta B = (A - B) \cup (B - A)$$

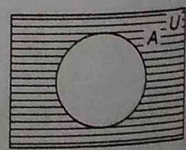


$A \Delta B$

5. Complement

Let U be the universal set and A be a set such that $A \subset U$.

Then, complement of A with respect to U is denoted by A' or A^c or $C(A)$ or $U - A$. It is defined as the set of all those elements of U which are not in A.



A'

Laws of Algebra of Sets

If A, B and C are any three sets, then

1. Idempotent Laws

(i) $A \cup A = A$
(ii) $A \cap A = A$

2. Identity Laws

(i) $A \cup \phi = A$
(ii) $A \cap U = A$

3. Distributive Laws

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. De-Morgan's Laws

(i) $(A \cup B)' = A' \cap B'$
(ii) $(A \cap B)' = A' \cup B'$
(iii) $A - (B \cap C) = (A - B) \cup (A - C)$
(iv) $A - (B \cup C) = (A - B) \cap (A - C)$

5. Associative Laws

(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Some of the laws are also given as

(i) $A - B = A \cap B'$
(ii) $B - A = B \cap A'$
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$
(iv) $(A - B) \cup B = A \cup B$
(v) $(A - B) \cap B = \phi$
(vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$
(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
(viii) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(ix) $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Cartesian Product of Sets

Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

Thus, $A \times B = [(a, b) : a \in A \text{ and } b \in B]$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

If A and B be two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

Number of Elements in Sets

(Important Results)

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A$ and B are disjoint non-void sets.
(iii) $n(A - B) = n(A) - n(A \cap B)$
(iv) $n(A \,\Delta\, B) = n(A) + n(B) - 2n(A \cap B)$
(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
(vi) n (number of elements in exactly two of the sets A, B and C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
(vii) n (number of elements in exactly one of the sets A, B and C)
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
(viii) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
(ix) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Relation

Let A and B be two non-empty sets, then relation R from A to B is a subset of $A \times B$. Let $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R *i.e.*, aRb. If $(a, b) \in R$, then we write it as aRb.

Domain and Range of a Relation

Let R be a relation from a set A to set B. Then, the set of all first components or coordinates of the ordered pairs belonging to R is called the **domain of R** while the set of all second components or coordinates of the ordered pairs in R is called the **range of R**. Thus,

Domain $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$

Types of Relations

The different types of relations are as given below

1. Reflexive Relation

A relation R on a set A is said to be reflexive, if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R, \forall\ a \in A$.

2. Symmetric Relation

A relation R on a set A is said to be symmetric relation iff

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall\ a, b \in A$$

or R on a set A is symmetric iff $R^{-1} = R$, where R^{-1} is a inverse relation of R.

3. Anti-symmetric Relation

Let A be any set. Then, a relation R on set A is said to be an anti-symmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall\ a, b \in A$.

4. Transitive Relation

Let A be any set. Then, a relation R on set A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall\ a, b, c \in A$

i.e., aRb and $bRc \Rightarrow aRc, \forall\ a, b, c \in A$.

5. Identity Relation

Let A be a non-empty set. Then, the relation $I_A = \{(a, a) : a \in A\}$ on A is called the **identity relation** on A.

6. Equivalence Relation

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e., $(a, a) \in R, \forall\ a \in A$.

(ii) it is symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R, \forall\ a, b \in A$

(iii) it is transitive i.e., $(a, b) \in R$ and $(b, c) \in R$ $\Rightarrow (a, c) \in R, \forall\ a, b, c \in A$

- *The intersection of two equivalence relations on a set is an equivalance relation on the set.*
- *The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.*
- *If R and S are two equivalence relations on set A, then $R \cap S$ is also an equivalence relation on A.*
- *If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation A.*

7. Inverse Relation

Let A and B be two sets and R be a relation from a set A to set B. Then, the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.

Composition of Relations

Let R and S be two relations from set A to B and B to C respectively, then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This relation is called the **composition of R and S.**

But $RoS \neq SoR$. Also, $(SoR)^{-1} = R^{-1}oS^{-1}$

Function or Mapping

If A and B are two non-empty sets, then a rule f which associated to each $x \in A$, a unique number $y \in B$, is called a function from A to B. i.e., $f : A \to B$. The set of A is called the **domain** of $f(D_f)$. The set of B is called the **codomain** of $f(C_f)$. The set consisting of all the images of the elements of the domain $A(R_f)$.

Different Types of Function

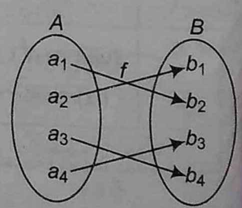
1. One-One Function (Injection)

A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus, $f : A \to B$ is one-one, if there exist $a, b \in A$ such that

$$a \neq b \Rightarrow f(a) \neq f(b), \forall\ a, b \in A$$

or $$f(a) = f(b) \Rightarrow a = b, \forall\ a, b \in A$$

Graphically Any line parallel to x-axis cuts the graph of the function atmost at one point.



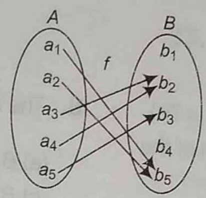
- *The number of functions from a finite set A into finite set $B = \{n(B)\}^{n(A)}$.*
- *The number of one-one function that can be defined from a finite set A into finite set B is* $\begin{cases} {}^{n(B)}P_{n(A)}, & \text{if } n(B) \geq n(A) \\ 0, & \text{otherwise} \end{cases}$
- *Any function which is entirely increasing or decreasing in the whole of a domain is one-one function.*

Methods to Check One-One Function

- ***Method 1*** If $f(x) = f(y) \Rightarrow x = y$, then f is one-one.
- ***Method 2*** A function is one-one iff no line parallel to x-axis meets the graph of function at more than one point.
- ***Method 3*** If $f'(x) \le 0$, or $f'(x) \ge 0, \forall x \in$ domain, then f is one-one.

2. Many-One Function

A function $f : A \to B$ is said to be a many-one function, if two or more elements of set A have the same image in B.



Thus, $f : A \to B$ is a many-one function, if there exist $a, b \in A$ such that $a \ne b$ but $f(a) = f(b)$.

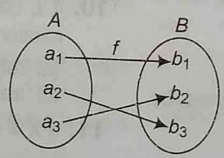
i.e., $f : A \to B$ is a many-one function, if it is not a one-one function.

Graphically Any line parallel to x-axis, cuts the graph of the function atleast two points.

In three consecutive quadrants, trigonometrical functions are always many-one function.

3. Onto Function (Surjection)

A function $f : A \to B$ is said to be an onto, if each element of B has its pre-image in A.



$\therefore$ If $f^{-1}(y) \in A, \forall y \in B$, then function is onto, where f^{-1} is an inverse of f Range of f = Codomain of f.

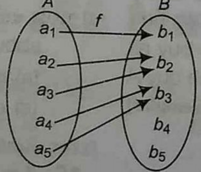
> *The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing 2 elements = $2^n - 2$.*
>
> *The number of onto functions that can be defined from a finite set A out a finite set B = Number of ways of dividing n(A) things into n(B) groups. So, that no group is empty, if n(A) ≥ n(B) and is 0, otherwise.*

Method to Check Onto Function

Find the range $y = f(x)$ and show that range $f(x) =$ codomain of $f(x)$.

4. Into Function

A function $f : A \to B$ is said to be an into function, if there exists atleast one element in B having no pre-image in A. i.e., $f : A \to B$ is an into function, if it is not an onto function.



5. One-One and Onto Function (Bijection)

A function $f : A \to B$ is said to be a bijection, if it is one-one as well as onto. Thus, $f : A \to B$ is a bijection, if

(i) it is one-one

i.e., $\quad f(x) = f(y)$

$\Rightarrow \quad x = y, \forall x, y \in A.$

(ii) it is onto

i.e., $\forall y \in B$, there exists $x \in A$ such that $f(x) = y$.

6. Inverse Function

Let f be defined a function from A to B such that for every element of B their exist a image. Let y be an arbitrary element of B.

Then, f being onto, there exists an element $x \in A$ such that $f(x) = y$. Also, f being one-one, this x must be unique.

Thus, for each $y \in B$, there exists a unique element $x \in A$ such that $f(x) = y$. So, we may define a function,

$$f^{-1} : B \to A$$

$$\therefore \quad f^{-1}(y) = x$$

$$\Leftrightarrow f(x) = y$$

The above function f^{-1} is called the **inverse** of f.

- Inverse of bijective function is also bijective function.
- If the inverse of f exist, then f is called an invertible function i.e., A function f is invertible if and only if f is one-one onto.

7. Composite Function

If $f : A \to B$ and $g : B \to C$ are two functions, then the composite function of f and g

i.e., $gof : A \to C$ will be defined as,

$$gof\ (x) = g[f(x)], \forall x \in A.$$

Generally, $\quad gof \ne fog$.

Practice Zone DAY 1

1. If $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{x \in N : 30 < x^2 < 70\}$, $B = \{x : x \text{ is a prime number less than } 10\}$, then which of the following is incorrect?
(a) $A \cup B = \{2, 3, 5, 6, 7, 8\}$ (b) $A \cap B = \{7, 8\}$
(c) $A - B = \{6, 8\}$ (d) $A \Delta B = \{2, 3, 5, 6, 8\}$

2. Let X be the universal set for sets A and B, if $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to 300 provided $n(X)$ is equal to
(a) 600 (b) 700 (c) 800 (d) 900

3. If two sets A and B are having 80 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are
(a) 2^{80} (b) 80^2 (c) 81 (d) 79

4. If R is relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is
(a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 18), (13, 10)\}$
(c) $\{(10, 13), (8, 11)\}$ (d) None of these

5. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 3 elements and $B_1, B_2, \ldots, B_n$ are n sets each having 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of A_i's and exactly 9 of B_j's. The value of n is equal to **[NCERT Exemplar]**
(a) 15 (b) 3
(c) 45 (d) None of these

6. In a town of 10000 families it was found that 40% family buy newspaper A, 20% family buy newspaper B and 10% family buy newspaper C. 5% families buy A and B, 3% families buy B and C and 4% families buy A and C. If 2% families buy all the three newspapers. Then, number of families which buy A only is **[NCERT Exemplar]**
(a) 3100 (b) 3300
(c) 2900 (d) 1400

7. If A and B be two universal sets and $A \cup B \cup C = U$. Then, $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to
(a) $A \cup B \cup C$ (b) $A \cup (B \cap C)$
(c) $A \cap B \cap C$ (d) $A \cap (B \cup C)$

8. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to **[NCERT Exemplar]**
(a) $B \cap C'$ (b) $A \cap C$
(c) $B' \cap C'$ (d) None of these

9. If there are three athelitic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is
(a) 42 (b) 43
(c) 45 (d) None of these

10. If L denotes the set of all straight lines in a plane and a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. Then, R is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

11. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$. Then,
(a) $X \subseteq Y$ (b) $Y \subseteq X$ **[NCERT Exemplar]**
(c) $X \neq Y$ (d) None of these

12. If $P(A)$ denotes the power set of A and A is the void set, then what is number of elements in $P\{P\{P\{P(A)\}\}\}$?
(a) 0 (b) 1
(c) 4 (d) 16

13. If $g(x) = 1 + \sqrt{x}$ and $f\{g(x)\} = 3 + 2\sqrt{x} + x$, then $f(x)$ is equal to
(a) $1 + 2x^2$ (b) $2 + x^2$
(c) $1 + x$ (d) $2 + x$

14. Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0, c \neq 0$. Assume $a = 1, b = 2$, if $(fog)(x) = (gof)(x)$ for all x. What can you say about c and d?
(a) c and d both arbitrary (b) $c = 1$ and d is arbitrary
(c) c is arbitrary and $d = 1$ (d) $c = 1, d = 1$

15. If $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, $\forall x$, then $f\{g(x)\}$ is equal to
(a) x (b) 1
(c) $f(x)$ (d) $g(x)$

16. If $f(x) = 2[x] + \cos x$, then $f : R \to R$ is
(a) one-one and onto
(b) one-one and into
(c) many-one and into
(d) many-one and onto

Directions (Q. Nos. 17 to 19) *Let* $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 9, 16\}$, $U = \{1, 2, 3, 4, 9, 16\}$ *and R be a relation defined on A such that* $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (3, 1), (1, 3)\}$. *Also, define a function* $f : A \to B$ *is* $f(x) = x^2$.

17. Find the set $(A \cap B)' \cap U$.
(a) $\{4\}$
(b) $\{2, 3, 9, 16\}$
(c) $\{1, 2, 3, 4, 9, 16\}$
(d) None of these

18. Function f is a
(a) one-one
(b) one-one-onto
(c) many-one
(d) None of these

19. Relation R is a
(a) reflexive
(b) symmetric
(c) transitive
(d) None of these

Directions (Q. Nos. 20 to 23) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

20. Let R be a relation from set $A = \{1, 2, 4\}$ to set $B = \{1, 2, 3, 4, 6, 8\}$ defined by $x R y$, if and only if x divides y, then

Statement I Domain and range of relation are respectively the sets.

Statement II All subsets of A and B are the domain and range of the relation.

21. If $f : R \to R$ and $g : R \to R$ be two mappings such that

Statement I $f(x) = \sin x$ and $g(x) = x^2$, then $fog \neq gof$.

Statement II $(fog)\,x = f(x)\,g(x) = (gof)x$.

22. Statement I A relation is defined by

$$f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 2x, & 3 \le x \le 9 \end{cases}$$ is a function.

Statement II In a function, every pre-image must have an unique image.

23. Statement I Let n be a fixed positive integer and a relation R be defined in I (the set of all integers) as follows : aRb iff $\frac{n}{(a-b)}$, then $(a - b)$ is divisible by n. Then, relation is an equivalence.

Statement II If R and R' are symmetric relation, then the relation $R \cap R'$ is not symmetric.

24. Two sets A and B are defined as follows
$A = \{(x, y) : y = e^{2x}, x \in R\}$ and
$B = \{(x, y) : y = x^2, x \in R\}$, then
(a) $A \subset B$
(b) $B \subset A$
(c) $A \cup B$
(d) $A \cap B = \phi$

25. Let $f : R \sim \{n\} \to R$ be a function defined by $f(x) = \frac{x - m}{x - n}$, where $m \neq n$. Then
(a) f is one-one-onto
(b) f is one-one-into
(c) f is many-one-onto
(d) f is many-one-into

26. If $f(x) = \left(\frac{1 + x}{1 - x}\right)$ and $g(x) = \frac{1 + x^3}{1 - x^3}$, then $fog\,(x)$ is equal to
(a) $\frac{1}{x^2}$
(b) $-\frac{1}{x^2}$
(c) $\frac{1}{x^3}$
(d) $-\frac{1}{x^3}$

27. If $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

and $g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$. Then, $f - g$ is
(a) one-one and into
(b) neither one-one nor onto
(c) many-one and onto
(d) one-one and onto

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28. Let $A = \{1, 2, 3, 4\}$ and $R : A \to A$ be the relation defined by $R = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$. The correct statement is **[JEE Main 2013]**
(a) R does not have an inverse
(b) R is not a one to one function
(c) R is an onto function
(d) R is not a function

29. Let $R = \{(3, 3), (5, 5), (9, 9), (12, 12), (5, 12), (3, 9), (3, 12)\ (3, 5)\}$ be a relation on the set $A = \{3, 5, 9, 12\}$. Then, R is **[JEE Main 2013]**
(a) reflexive, symmetric but not transitive
(b) symmetric, transitive but not reflexive
(c) an equivalence relation
(d) reflexive, transitive but not symmetric.

30. Let $R = \{(x, y) : x, y \in N \text{ and } x^2 - 4xy + 3y^2 = 0\}$, where N is the set of all natural numbers. Then, the relation R is **[JEE Main 2013]**
(a) reflextive but neither symmetric nor transitive
(b) symmetric and transitive
(c) reflexive and symmetric
(d) reflexive and transitive

31. Let $A = \{\theta : \sin(\theta) = \tan(\theta)\}$ and $B = \{\theta : \cos(\theta) = 1\}$ be two sets. Then, **[JEE Main 2013]**
(a) $A = B$
(b) $A \not\subset B$
(c) $B \not\subset A$
(d) $A \subset B$ and $B - A \neq \phi$

32. Let R be the set of real numbers.
Statement I $A = \{(x, y) \in R \times R : y - x \text{ is an integer}\}$ is an equivalence relation on R.
Statement II $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R. **[AIEEE 2011]**
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

33. Consider the following relations
$R = \{(x, y) \mid x$ and y are real numbers and $x = wy$ for some rational number $w\}$;
$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \text{ are integers such that } n, q \neq 0 \text{ and } qm = pn\right\}$. Then, **[AIEEE 2010]**
(a) R is an equivalence relation but S is not an equivalence relation.
(b) neither R nor S is an equivalence relation.
(c) S is an equivalence relation but R is not an equivalence relation.
(d) R and S both are equivalence relations.

34. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[AIEEE 2009]**
(a) $A = C$ (b) $B = C$ (c) $A \cap B = \phi$ (d) $A = B$

35. For real x, let $f(x) = x^3 + 5x + 1$, then **[AIEEE 2009]**
(a) f is one-one but not onto in R
(b) f is onto in R but not one-one
(c) f is one-one and onto in R
(d) f is neither one-one nor onto in R

36. Let $f(x) = (x + 1)^2 - 1, x \geq -1$
Statement I The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$
Statement II f is a bijection. **[AIEEE 2009]**
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

37. Let R be the real line. Consider the following subsets of the plane $R \times R$.
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
and $T = \{(x, y) : x - y \text{ is an integer}\}$
Which one of the following is true? **[AIEEE 2008]**
(a) T is an equivalence relation on R but S is not
(b) Neither S nor T is an equivalence relation on R
(c) Both S and T are equivalence relations on R
(d) S is an equivalence relation on R but T is not

38. Let W denotes the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W :$ the words x and y have at least one letter in common$\}$. Then, R is **[AIEEE 2006]**
(a) reflexive, symmetric and not transitive
(b) reflexive, symmetric and transitive
(c) reflexive, not symmetric and transitive
(d) not reflexive, symmetric and transitive

39. If $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is **[AIEEE 2005]**
(a) an equivalence relation (b) reflexive and symmetric
(c) reflexive and transitive (d) only reflexive

40. If $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. Then, the relation R is **[AIEEE 2004]**
(a) reflexive (b) transitive
(c) not symmetric (d) a function

41. A function f from the set of natural numbers to integers defined by $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$ is **[AIEEE 2003]**
(a) one-one but not onto (b) onto but not one-one
(c) one-one and onto both (d) neither one-one nor onto

Answers

1. (d)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (a)	**9.** (b)	**10.** (b)
11. (a)	**12.** (d)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (b)	**18.** (b)	**19.** (b)	**20.** (c)
21. (c)	**22.** (d)	**23.** (b)	**24.** (d)	**25.** (b)	**26.** (d)	**27.** (d)	**28.** (c)	**29.** (d)	**30.** (a)
31. (b)	**32.** (b)	**33.** (c)	**34.** (b)	**35.** (c)	**36.** (c)	**37.** (a)	**38.** (a)	**39.** (c)	**40.** (c)
41. (c)									

Hints & Solutions

1. Given, $A = \{x \in N : 30 < x^2 < 70\} = \{6, 7, 8\}$

$B = \{x : x \text{ is a prime number less than } 10\}$
$= \{2, 3, 5, 7\}$

$\therefore \quad A \cup B = \{2, 3, 5, 6, 7, 8\}$ and $A \cap B = \{7\}$

$\therefore \quad A - B = \{6, 8\}$ and $B - A = \{2, 3, 5\}$

$\therefore \quad A \Delta B = (A - B) \cup (B - A)$
$= \{6, 8\} \cup \{2, 3, 5\} = \{2, 3, 5, 6, 8\}$

2. $\because \; n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore \; n(A \cup B) = 200 + 300 - 100 = 400$

$\therefore n(A' \cap B') = n(A \cup B)' = n(X) - n(A \cup B)$

$\Rightarrow \quad 300 = n(X) - 400$

$\therefore \quad n(X) = 700$

3. $n\{(A \times B) \cap (B \times A)\}$
$= n\{(A \cap B) \times (B \cap A)\}$
$= n(A \cap B) \times n(B \cap A)$
$= n(A \cap B) \times n(A \cap B)$
$= 80 \times 80 = 80^2$

4. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by

$y = x - 3 \Rightarrow x - y = 3$

$\therefore \quad R = \{(11, 8), (13, 10)\}$

Hence, $\quad R^{-1} = \{(8, 11), (10, 13)\}$

5. If elements are not repeated, then number of elements in $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{30}$ is 30×5 but each element is used 10 times, so

$$S = \frac{30 \times 5}{10} = 15 \quad \ldots(\text{i})$$

Similarly, if elements in $B_1, B_2, \ldots, B_n$ are not repeated, then total number of elements is $3n$ but each element is repeated 9 times, so

$$S = \frac{3n}{9} \Rightarrow 15 = \frac{3n}{9} \quad [\text{from Eq. (i)}]$$

$\Rightarrow \quad n = 45$

6. Given, $\; n(A) = 40\%$ of $10000 = 4000$

$n(B) = 20\%$ of $10000 = 2000$

$n(C) = 10\%$ of $10000 = 1000$

$n(A \cap B) = 5\%$ of $10000 = 500$

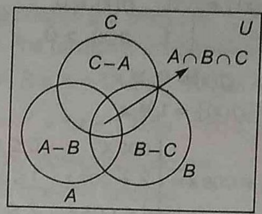
$n(B \cap C) = 3\%$ of $10000 = 300$

$n(C \cap A) = 4\%$ of $10000 = 400$

$n(A \cap B \cap C) = 2\%$ of $10000 = 200$

$\therefore n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$
$= 4000 - (500 + 400 - 200)$
$= 4000 - 700 = 3300$

7. From Venn Euler's diagram,



It is clear that, $\{(A - B) \cup (B - C) \cup (C - A)\}' = A \cap B \cap C$

8. $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$
$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C' = (\phi \cup B \cup C) \cap C'$
$= (B \cup C) \cap C' = (B \cap C') \cup \phi = B \cap C'$

9. Given, $n(B) = 21, n(H) = 26, n(F) = 29,$

$n(H \cap B) = 14, n(H \cap F) = 15, n(F \cap B) = 12$

and $n(B \cap H \cap F) = 8$

$\therefore \quad n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(B \cap H)$
$- n(H \cap F) - n(B \cap F) + n(B \cap H \cap F)$
$= 21 + 26 + 29 - 14 - 15 - 12 + 8 = 43$

10. Here, $\alpha R \beta \Leftrightarrow \alpha \perp \beta$

$\therefore \quad \alpha \perp \beta \Leftrightarrow \beta \perp \alpha$

Hence, R is symmetric.

11. Since, $8^n - 7n - 1 = (7 + 1)^n - 7n - 1$
$= 7^n + {}^nC_1 7^{n-1} + {}^nC_2 7^{n-2} + \ldots + {}^nC_{n-1} 7 + {}^nC_n - 7n - 1$
$= {}^nC_2 7^2 + {}^nC_3 7^3 + \ldots + {}^nC_n 7^n \quad (\because {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1})$
$= 49[{}^nC_2 + {}^nC_3(7) + \ldots + {}^nC_n 7^{n-2}]$

Thus, $8^n - 7n - 1$ is a multiple of 49 for all $n \in N$.

$\therefore X$ contains elements which are multiples of 49 and clearly Y contains all multiples of 49.

$\therefore \quad X \subseteq Y$

12. The number of elements in power set of A is 1.

$\therefore \quad P\{P(A)\} = 2^1 = 2$

$\Rightarrow \quad P\{P\{P(A)\}\} = 2^2 = 4$

$\Rightarrow \quad P\{P\{P\{P(A)\}\}\} = 2^4 = 16$

13. Given, $g(x) = 1 + \sqrt{x}$ and $f\{g(x)\} = 3 + 2\sqrt{x} + x \quad \ldots(\text{i})$

$\Rightarrow \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$

$\therefore \quad f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$

$\therefore \quad f(x) = 2 + x^2$

14. Now, $(fog)(x) = f\{g(x)\} = a(cx + d) + b$

and $\quad (gof)(x) = g\{f(x)\} = c(ax + b) + d$

Since, $\quad cx + d + 2 = cx + 2c + d \quad (\because a = 1, b = 2)$

Hence, $c = 1$ and d is arbitrary.

15. $\because g(x) = 1 + x - [x]$ (put $x = n \in Z$)

and $g(x) = 1 + n + k - n = 1 + k$ (put $x = n + k$)

(where, $n \in Z$, $0 < k < 1$)

Now, $f\{g(x)\} = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

Clearly, $g(x) > 0, \forall x$

So, $f\{g(x)\} = 1, \forall x$

16. Since, $f(x) = 2[x] + \cos x = \begin{cases} \cos x, & 0 \le x < 1 \\ 2 + \cos x, & 1 \le x < 2 \\ 4 + \cos x, & 2 \le x < 3 \end{cases}$

Since, $\cos x < 1$ and $2 + \cos x > 1$.

So, $f(x)$ never gives the value one. Hence, $f(x)$ is into.

If $0 < \alpha < \pi - 3$, then $f(\pi - \alpha) = f(\pi + \alpha)$

So, $f(x)$ is not one-one.

17. Now, $A \cap B = \{1, 4\}$

$\Rightarrow$ $(A \cap B)' = \{2, 3, 9, 16\}$

$\therefore$ $(A \cap B)' \cap U = \{2, 3, 9, 16\}$

18. $f : A \to B$, $f(x) = x^2$

Here, we see that for every element x of A, there exist a image in B. So, it is one-one mapping. Also, for every element of B there exist a pre-image, so it is onto. Hence, f is one-one-onto.

19. $A = \{1, 2, 3, 4\}$. Here, we see that $(4, 4) \notin R$, so it is not reflexive.

Also, $(1, 2) \in R \Rightarrow (2, 1) \in R$ and $(3, 1) \in R \Rightarrow (1, 3) \in R$

Hence, R is symmetric.

20. Given, $R = \{(x, y) : x \text{ divides } y, x \in A, y \in B\}$

$\therefore R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$ ...(i)

Domain of $R = \{2, 4\}$

Now, there are 8 subsets of A but every subset of A is not the domain of relation defined in Eq. (i).

Again, range of relation $= \{2, 4, 6, 8\}$

But there are $2^6 = 64$ subsets of B but every subset of B is not the range of the relation defined in Eq. (i).

21. Since, $(fog)x = f\{g(x)\} = f(x^2) = \sin x^2$

and $(gof)x = g\{f(x)\} = g(\sin x) = \sin^2 x$

$\therefore$ $fog \ne gof$

22. **Statement I** $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 2x, & 3 \le x \le 9 \end{cases}$

Now, $f(3) = 9$

Also, $f(3) = 2 \times 3 = 6$

Here, we see that for one value of x, we get two different values of $f(x)$. Hence, it is not a function but Statement II is true.

23. I. (a) ***R* is reflexive**

Let $a \in I$, then $a - a = 0$ which is divisible by n.

$\therefore aRa, \forall a \in I$

(b) ***R* is symmetric**

Let $a, b \in I$

$\therefore$ $a - b = nk, k \in I$

$\Rightarrow$ $(b - a) = -kn$, where $-k \in I$

i.e., $aRb = bRa$

(c) ***R* is transitive**

Now, $aRb \Rightarrow a - b = k_1 n$ and

$bRc \Rightarrow b - c = k_2 n, k_1, k_2 \in I$

$\therefore$ $a - c = (a - b) + (b - c) = k_1 n + k_2 n$

$= (k_1 + k_2)n$, where $k_1 + k_2 \in I$

II. As $R \cap R'$ are not disjoint, then there is atleast one ordered pair, say (a, b) in $R \cup R'$. But $(a, b) \in R \cap R'$ $\Rightarrow$ $(a, b) \in R$ and $(a, b) \in R'$.

Since, R and R' are symmetric. Hence, $R \cap R'$ is symmetric.

Hence, both statements are true but Statement II is not a correct explanation of Statement I.

24. Set A represents the set of points lying on the graph of an exponential function and set B represent the set of points lying on the graph of the polynomial.

Take $e^{2x} = x^2$, then the two curves does not intersect. Hence, there is no point common between them.

25. For any $x, y \in R$, we have

$$f(x) = f(y)$$

$$\Rightarrow \frac{x - m}{x - n} = \frac{y - m}{y - n} \Rightarrow x = y$$

So, f is one-one.

Let $\alpha \in R$ such that $f(x) = \alpha$

$$\therefore \frac{x - m}{x - n} = \alpha \Rightarrow x = \frac{m - n\alpha}{1 - \alpha}$$

Clearly, $x \in R$ for $\alpha = 1$

So, f is not onto.

26. We have, $f(x) = \dfrac{1 + x}{1 - x}$ and $g(x) = \dfrac{1 + x^3}{1 - x^3}$,

$$\therefore fog(x) = f\{g(x)\} = \frac{1 + \dfrac{1 + x^3}{1 - x^3}}{1 - \dfrac{1 + x^3}{1 - x^3}} = \frac{2}{-2x^3} = -\frac{1}{x^3}$$

27. Let $\phi(x) = f(x) - g(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$

Now, to check one-one

Take any straight line parallel to x-axis which will intersect $\phi(x)$ only at one point.

Now, to check onto Since, $f(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases} \Rightarrow y \in R$

28. Here, we see that every element of codomain there exist a pre-image, hence it is onto.

29. $\forall a \in A$, $(a, a) \in R$, so it is reflexive $(5, 12) \in R$ but $(12, 5) \notin R$, so it is not symmetric.

$(3, 5), (5, 12) \in R \Rightarrow (3, 12) \in R$, so it is transitive.

30. $\because a^2 - 4a\cdot a + 3a^2 = 4a^2 - 4a^2 = 0$

$\therefore$ $(a, a) \in N$ is reflexive.

Also, $(a, b) \in N$

$\Rightarrow$ $a^2 - 4ab + 3b^2 = 0 \Rightarrow b^2 - 4ba + 3a^2 \neq 0$

$\therefore$ R is not symmetric.

Now, $(a, b) \in N$ and $(b, c) \in N \not\Rightarrow (a, c) \in N$

$\therefore R$ is not transitive.

31. $A\{\ldots,-2\pi,-\pi,0,\pi,2\pi,\ldots\}$, $B=\{\ldots,-2\pi,0,2\pi,\ldots\}$

$\therefore$ $A \not\subset B$

32. I. $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$

(a) **Reflexive** $xRx : (x - x)$ is an integer

i.e., True

$\therefore$ Reflexive

(b) **Symmetric** $xRy : (x - y)$ is an integer

$\Rightarrow -(y - x)$ is an integer $\Rightarrow (y - x)$ is an integer $\Rightarrow yRx$

$\therefore$ Symmetric

(c) **Transitive**

xRy and $yRz \Rightarrow (x - y)$ is an integer and $(y - z)$ is an integer.

Now, $(x - y) + (y - z)$ is an integer

$\Rightarrow (x - z)$ is an integer $\Rightarrow xRz$

$\therefore$ Transitive

Hence, R is an equivalence relation.

II. $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$

If $\alpha = 1$, then $xRy : x = y$ (to check equivalence)

(a) **Reflexive** $xRx : x = x$ (true)

$\therefore$ Reflexive

(b) **Symmetric** $xRy : x = y \Rightarrow y = x \Rightarrow yRx$

$\therefore$ Symmetric

(c) **Transitive** xRy and $yRz \Rightarrow x = y$ and $y = z \Rightarrow x = z \Rightarrow xRz$

Hence R is an Equivalence relation.

Hence, both statements are true but Statement II is not a correct explanation of Statement I.

33. Since, the relation R is defined as $R = \{(x, y) | \; x, y$ are real numbers and $x = wy$ for some rational number $w\}$.

(a) **Reflexive** $xRx \Rightarrow x = wx$

Let $w = 1 \in$ Rational number

So, the relation R is reflexive.

(b) **Symmetric** $xRy \not\Rightarrow yRx$ as $0R1 \Rightarrow 0 = w$

but $1R0 \Rightarrow 1 = w\cdot(0)$

which is not true for any rational number.

So, the relation R is not symmetric.

Thus, R is not equivalence relation.

Now, for the relation S is defined as

$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right) \middle| m, n, p \text{ and } q \in \text{integers such that } n, q \neq 0\right.$

and $qm = pn\}$

(a) **Reflexive** $\frac{m}{n} R \frac{m}{n} \Rightarrow mn = mn$ (true)

Hence, the relation S is reflexive.

(b) **Symmetric** $\frac{m}{n} R \frac{p}{q} \Rightarrow mq = np \Rightarrow np = mq \Rightarrow \frac{p}{q} R \frac{m}{n}$

Hence, the relation S is symmetric.

(c) **Transitive** $\frac{m}{n} R \frac{p}{q}$ and $\frac{p}{q} R \frac{r}{s} \Rightarrow mq = np$ and $ps = rq$

$\Rightarrow mq\cdot ps = np\cdot rq \Rightarrow ms = nr \Rightarrow \frac{m}{n} = \frac{r}{s} \Rightarrow \frac{m}{n} R \frac{r}{s}$

So, the relation S is transitive.

Hence, the relation S is equivalence relation.

34. Since, $A \cap B = A \cap C$ and $A \cup B = A \cup C \Rightarrow B = C$

35. Given, $f(x) = x^3 + 5x + 1$. Now, $f'(x) = 3x^2 + 5 > 0, \forall x \in R$

$\therefore$ $f(x)$ is strictly increasing function.

Hence, $f(x)$ is one-one function.

Clearly, $f(x)$ is a continuous function and also increasing on R.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} = \infty$

$\therefore f(x)$ takes every value between $-\infty$ and ∞.

Thus, $f(x)$ is onto function.

36. Given, $f(x) = (x + 1)^2 - 1, x \geq -1$

$\Rightarrow$ $f'(x) = 2(x + 1) \geq 0$ for $x \geq -1$

$f(x)$ is one-one.

Since, codomian of the given function is not given, hence it can be considered as R, the set of reals and consequently R is not onto.

Hence, f is not bijective. Statement II is false.

Also, $f(x) = (x + 1)^2 - 1 \geq -1$ for $x \geq -1 \Rightarrow R_f = [-1, \infty)$

Clearly, $f(x) = f^{-1}(x)$ at $x = 0$ and $x = -1$

Hence, Statement I is true.

37. Since, $(1, 2) \in S$ but $(2, 1) \notin S$

$\therefore$ S is not symmetric.

Hence, S is not an equivalence relation.

Given, $T = \{(x, y) : (x - y) \in I\}$

Now, $x - x = 0 \in I$, it is reflexive relation.

Again now, $(x - y) \in I$

$\Rightarrow y - x \in I$, it is symmetric relation.

Let $x - y = I_1$ and $y - z = I_2$.

Now, $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$

$\therefore$ T is also transitive. Hence, T is an equivalence relation.

38. Let $W = \{$CAT, TOY, YOU, ...$\}$

Clearly, R is reflexive and symmetric but not transitive.

(since, ${}_{CAT}R_{TOY}$, ${}_{TOY}R_{YOU} \not\Rightarrow {}_{CAT}R_{YOU}$)

39. Since, for every element of A, there exist an element $(3, 3), (6, 6), (9, 9), (12, 12) \in R \Rightarrow R$ is reflexive relation.

Now, $(6, 12) \in R$ but $(12, 6) \notin R$. So, it is not a symmetric relation.

Also, $(3, 6), (6, 12) \in R \Rightarrow (3, 12) \in R \Rightarrow R$ is transitive.

40. Since, $(2, 3) \in R$ but $(3, 2) \notin R$. So, R is not symmetric.

41. Let $x, y \in N$ and both are even.

Then, $f(x) = f(y) \Rightarrow -\frac{x}{2} = -\frac{y}{2} \Rightarrow x = y$

Again, $x, y \in N$ and both are odd.

Then, $f(x) = f(y) \Rightarrow x = y$

Since, each negative integer is an image of even natural number and positive integer is an image of odd natural number.

Day 2

Complex Numbers

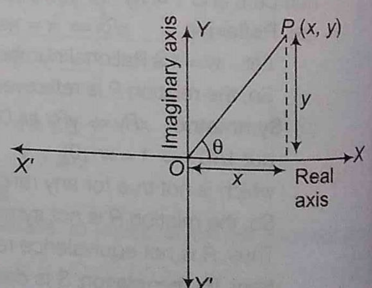
Day 2 Outlines ...

- Complex Number and Its Representation
- Algebra of Complex Numbers
- Conjugate and Modulus of a Complex Number
- Argument or Amplitude of a Complex Number
- De-Moivre's Theorem

Complex Number and Its Representation

A number in the form of $z = x + iy$, where $x, y \in R$, $i = \sqrt{-1}$ is called a **complex number**. The real numbers x and y are respectively called **real** and **imaginary** parts of complex number z.

i.e., $x = \text{Re}(z)$ and $y = \text{Im}(z)$



The complex number $z = x + iy$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called **real** and **imaginary axes**, respectively.

This plane is called **argand plane** or **Gaussian plane**.

The magnitude of the complex number z is $|z| = \sqrt{x^2 + y^2}$

and
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

A complex number z is said to be purely real or imaginary, if $y = 0$ or $x = 0$, respectively.

A complex number can be represented in the following forms

(i) Geometrical form $z = x + iy$

(ii) Vectorial form $\mathbf{OP} = z = x + iy$

(iii) Trigonometrical form $z = r(\cos\theta + i\sin\theta)$

(iv) Eulerian form $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$

- $\sqrt{-3}\,\sqrt{-5} \neq \sqrt{15}$ but $\sqrt{-3}\,\sqrt{-5} = \sqrt{3}i\,\sqrt{5}i = -\sqrt{15}$
- $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$
- If n is an integer, then $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$
- Iota (i) is neither 0 nor greater than 0 nor less than 0.

Algebra of Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then

(i) Addition of complex numbers is

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Its additive identity is $0 + 0i$.

(ii) Subtraction of complex numbers is

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

(iii) Multiplication of complex numbers is

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Its multiplicative identity is $1 + 0i$.

(iv) Division of complex numbers is

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Conjugate and Modulus of a Complex Number

If $z = x + iy$ is a complex number, then **conjugate** of z is denoted by $\bar{z}$ and is obtained by replacing y by $-y$.

i.e., $\bar{z} = x - iy$

Further, if $z = x + iy$, then **modulus** or **magnitude** of z is denoted by $|z|$ and is given by $|z| = |\bar{z}| = \sqrt{x^2 + y^2}$

Results on Conjugate and Modulus

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2\operatorname{Re}(z)$, $z - \bar{z} = 2i\operatorname{Im}(z)$
3. $z = \bar{z} \Leftrightarrow z$ is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.
5. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
6. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, if $|z_2| \neq 0$
8. If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$ where $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.
9. $|z| = 0 \Leftrightarrow z = 0$
10. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
11. $-|z| \leq \operatorname{Re}(z), \operatorname{Im}(z) \leq |z|$
12. $|z_1 z_2| = |z_1||z_2|$
13. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $|z_2| \neq 0$
14. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$
15. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \bar{z}_2)$
16. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
17. If a and b are real numbers and z_1, z_2 are complex numbers, then $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$
18. If $z_1, z_2 \neq 0$ and $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
19. $|z^n| = |z|^n$, $n \in N$
20. **Reciprocal of a Complex Number** For an existing non-zero complex number $z = x + iy$, the reciprocal is given by $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

- The sum and product of two complex numbers are real simultaneously, if and only if they are conjugate to each other.
- If $|z| = 1$, then $\frac{z+1}{z-1}$ is purely imaginary.

Triangle Inequality

In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side.

1. $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $|z_1 + z_2| \geq ||z_1| - |z_2||$
3. $|z_1 - z_2| \leq |z_1| + |z_2|$
4. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Complex numbers do not possess the inequality. i.e., $3 + 2i > 1 + 2i$ does not make any sense.

Square Roots of a Complex Number

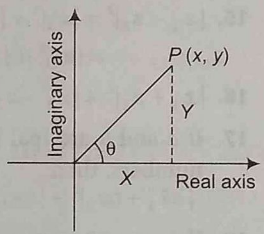
If $z = a + ib$ be a complex number, such that $\sqrt{a+ib} = x + iy$, where x, y are real numbers, then $a + ib = (x+iy)^2 \Rightarrow a + ib = (x^2 - y^2) + i2xy$ then square root of z by equating real and imaginary part can be given

$$\sqrt{z} = \pm\left[\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} + i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}\right], b > 0$$

$$= \pm\left[\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} - i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}\right], b < 0$$

Argument or Amplitude of a Complex Number

A complex number $z = x + iy$ can be represented geometrically in a plane, which is called an **argand plane**. Let θ be the argument of z, then



$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Argument of z is not unique.
General value of argument of z is $2n\pi + \theta$.

The principal value of argument θ satisfies the inequality $-\pi < \theta < \pi$.

(i) If $x > 0$ and $y > 0$, then argument of z is θ.
(ii) If $x < 0$ and $y > 0$, then argument of z is $\pi - \theta$.
(iii) If $x < 0$ and $y < 0$, then argument of z is $-(\pi - \theta)$.
(iv) If $x > 0$ and $y < 0$, then argument of z is $-\theta$.

Argument of 0 is not defined.

- *If* $amp(z) - amp(-z) = \begin{cases} \pi, & \text{if } amp(z) > 0 \\ -\pi, & \text{if } amp(z) < 0 \end{cases}$
- *The argument of a product is the sum of the arguments of the factors.*

Results on Argument

If z, z_1 and z_2 are complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$
(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
(iii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$
(iv) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
(v) the general value of $\arg(\bar{z})$ is $2n\pi - \arg(z)$.
(vi) If $|z_1 + z_2| = |z_1 - z_2|$, then $\arg\left(\dfrac{z_1}{z_2}\right)$ or $\arg(z_1) - \arg(z_2) = \dfrac{\pi}{2}$
(vii) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$
(viii) If $|z - 1| = |z + 1|$, then $\arg(z)$ is $\pm\dfrac{\pi}{2}$.
(ix) If $\arg\left(\dfrac{z-1}{z+1}\right)$ is $\dfrac{\pi}{2}$, then $|z| = 1$.
(x) If $\arg(z+1) - \arg(z-1)$ is $\dfrac{\pi}{2}$, then z lies on circle of radius one and centre origin.

- *If* $z = 1 + \cos\theta + i\sin\theta$; $\arg(z) = \dfrac{\theta}{2}$, $|z| = 2\cos\left(\dfrac{\theta}{2}\right)$
- *If* $z = 1 + \cos\theta - i\sin\theta$, $\arg(z) = -\dfrac{\theta}{2}$, $|z| = 2\cos\dfrac{\theta}{2}$
- *If* $z = 1 - \cos\theta + i\sin\theta$; $\arg(z) = \left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$, $|z| = 2\sin\dfrac{\theta}{2}$
- *If* $z = 1 - \cos\theta - i\sin\theta$; $\arg(z) = \left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$, $|z| = \sqrt{2}\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right)$

De-Moivre's Theorem

A simple formula for calculating integer powers of complex numbers in terms of $\cos\theta$ and $\sin\theta$ is known as **De-Moivre's theorem**. If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

If n is a rational number *i.e.*, $n = \dfrac{p}{q}, q \neq 0$

Then, $(\cos\theta + i\sin\theta)^{p/q}$

$$= \cos\left[\frac{p}{q}(2k\pi + \theta)\right] + i\sin\left[\frac{p}{q}(2k\pi + \theta)\right]$$

where, $k = 0, 1, 2, \ldots, q - 1$.

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$ or $\cos\theta$ *i.e.*,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$
- $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^n = \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$
- $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

nth Roots of Unity

By nth root of unity we mean any complex number z which satisfies the equation $z^n = 1$.

(i) The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

(ii) The sum of the nth roots of unity is always zero and product of nth roots of unity is $(-1)^{n-1}$.

(iii) The nth roots of unity form a GP with common ratio, $e^{\frac{i2\pi}{n}}$.

Cube Roots of Unity

The third or cube roots of unity satisfy the equation given as

$$x = \sqrt[3]{1} \quad \Rightarrow \quad x^3 - 1 = 0$$

$$\Rightarrow \quad (x-1)(x^2+x+1) = 0$$

$$\Rightarrow \quad x = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

If second root be represented by ω, then third root will be ω^2.

$$\therefore \quad \omega = \frac{-1+\sqrt{3}\,i}{2} \text{ and } \omega^2 = \frac{-1-\sqrt{3}\,i}{2}$$

ω and ω^2 are conjugate to each other.

Results on Cube Roots of Unity (ω)

1. $\omega^3 = 1,\ 1 + \omega + \omega^2 = 0$
2. $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$
3. ω and ω^2 are the roots of $x^2 + x + 1 = 0$
4. $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$
5. $a^2 + b^2 + c^2 - bc - ca - ab$
   $= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$
6. $a^3 + b^3 + c^3 - 3abc = (a+b+c)$
   $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$
7. $a^3 + 1 = (a+1)(a+\omega)(a+\omega^2)$
8. $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

Applications of Complex Numbers in Coordinate Geometry

(i) Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_2 - z_1|$.

(ii) The point $P(z)$ which divides the line segment joining $A(z_1)$ and $B(z_2)$ in the ratio $m:n$.

(a) For **internally** is given by $z = \dfrac{mz_2 + nz_1}{m+n}$ (b) For **externally** is given by $z = \dfrac{mz_2 - nz_1}{m-n}$

(iii) Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$, then (a) centroid $G(z)$ of the ΔABC is given by $z = \dfrac{1}{3}(z_1 + z_2 + z_3)$

(b) incentre $I(z)$ of the ΔABC is given by $z = \dfrac{az_1 + bz_2 + cz_3}{a+b+c}$ where a, b and c are the sides of a triangle.

(iv) Area of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by $\Delta = \dfrac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

For an equilateral triangle, $z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

(v) The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

(vi) If z_1 and z_2 are two fixed points, then $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the line segment joining $A(z_1)$ and $B(z_2)$.

(vii) (a) An equation of the circle with centre at z_0 and radius r is $|z - z_0| = r$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

(b) General equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$

where, a is a complex number and b is real number and whose centre is $-a$ and radius is $\sqrt{a\bar{a} - b}$.

(viii) If z_1 and z_2 are two fixed points and $k > 0$, $k \neq 1$ is a real number, then $\dfrac{|z - z_1|}{|z - z_2|} = k$ represents a circle. For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

(ix) If end points of diameter are $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle, then equation of circle in diameter form is $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

If three complex numbers are in AP, then they lie on a straight line in the complex plane.

Practice Zone

DAY 2

1. $\sum_{n=1}^{13}(i^n + i^{n+1})$ is equal to

(a) i (b) $i-1$ (c) $-i$ (d) 0

2. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1+2i$, then $|f(z)|$ is equal to **[NCERT Exemplar]**

(a) $\frac{|z|}{2}$ (b) $|z|$

(c) $2|z|$ (d) None of these

3. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then the value of $|z|$ is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{3}{4}$

4. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then z_1 is equal to **[NCERT Exemplar]**

(a) $2\bar{z}_2$ (b) $\bar{z}_2$

(c) $-\bar{z}_2$ (d) None of these

5. If $\frac{z-1}{z+1}$ is a purely imaginary number (where, $z \neq -1$), then the value of $|z|$ is **[NCERT Exemplar]**

(a) -1 (b) 1 (c) 2 (d) -2

6. The value of $\sqrt{-8-6i}$ is

(a) $1 \pm 3i$ (b) $\pm(1-3i)$

(c) $\pm(1+3i)$ (d) $\pm(3-i)$

7. If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then the value of $|z|$ is

(a) 1 (b) 2

(c) 1/2 (d) None of these

8. If $|z-1| = 1$, then $\arg(z)$ is equal to

(a) $\frac{1}{2}\arg(z)$ (b) $\frac{1}{3}\arg(z+1)$

(c) $\frac{1}{2}\arg(z-1)$ (d) None of these

9. The multiplicative inverse of $(6+5i)^2$ is

(a) $\frac{11}{61} - \frac{60}{61}i$ (b) $\frac{11}{61} + \frac{60}{61}i$

(c) $\frac{9}{61} - \frac{60}{61}i$ (d) None of these

10. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(a) a line not passing through the origin

(b) $|z| = \sqrt{2}$

(c) the X-axis

(d) the Y-axis

11. If a, b and c are integers not all equal and ω is a cube root of unity (where, $\omega \neq 1$), then minimum value of $|a + b\omega + c\omega^2|$ is equal to

(a) 0 (b) 1

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

12. If $z = 1 + \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, then $\{\sin(\arg(z))\}$ is equal to

(a) $\frac{\sqrt{10-2\sqrt{5}}}{4}$ (b) $\frac{\sqrt{5}-1}{4}$

(c) $\frac{\sqrt{5}+1}{4}$ (d) None of these

13. If $\text{Re}\left(\frac{1}{z}\right) = 3$, then z lies on

(a) circle with centre on Y-axis

(b) circle with centre on X-axis not passing through origin

(c) circle with centre on X-axis passing through origin

(d) None of the above

14. If a complex number z lies in the interior or on the boundary of a circle of radius 3 and centre at $(-4, 0)$, then the greatest and least value of $|z+1|$ are

(a) 5, 0 (b) 6, 1

(c) 6, 0 (d) None of these

15. If $\arg(\bar{z}_1) = \arg(z_2)$, then

(a) $z_2 = kz_1^{-1}$ $(k > 0)$ (b) $z_2 = kz_1$ $(k > 0)$

(c) $|z_2| = |\bar{z}_1|$ (d) None of these

16. If 1, ω and ω^2 are the three cube roots of unity α, β, γ are the cube roots of $p, q < 0$, then for any x, y, z the expression $\left(\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}\right)$ is equal to

(a) 1 (b) ω

(c) ω^2 (d) None of these

17. If z_1 and z_2 be two complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$, then
(a) z_1, z_2 are collinear
(b) z_1, z_2 and the origin form a right angled triangle
(c) z_1, z_2 and the origin form an equilateral triangle
(d) None of the above

18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ is equal to
(a) 0 (b) $\pi/2$ (c) $3\pi/2$ (d) π

19. The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=\frac{2}{\pi}(\sec^{-1}\frac{1}{x}+\sin^{-1}x)$, where $x\neq 0; -1\leq x\leq 1$ is
(a) 2 (b) 4 (c) 6 (d) 8

20. If $z=\sqrt{(5+12i)}+\sqrt{(12i-5)}$, then the principal value of $\arg(z)$ can be
(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{3\pi}{4}$ (d) All of these

21. If α and β are two different complex numbers such that $|\alpha|=1, |\beta|=1$, then the expression $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is equal to
(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these

22. If $z_r=\text{cis}\,\frac{\pi}{2^r}$, then $z_1 z_2 \ldots \infty$ is equal to
(a) 1 (b) -1 (c) -2 (d) $-\frac{1}{2}$

23. If $x^2+x+1=0$, then $\sum_{r=1}^{25}\left(x^r+\frac{1}{x^r}\right)^2$ is equal to
(a) 25 (b) 25ω
(c) $25\omega^2$ (d) None of these

24. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the nth roots of unity, then $(2-\alpha_1)(2-\alpha_2)\ldots(2-\alpha_{n-1})$ is equal to
(a) n (b) 2^n (c) 2^n+1 (d) 2^n-1

Directions (Q. Nos. 25 and 26) *Let $z=a+ib=(a,b)$ be any complex number, $\forall\, a,b\in R$ and $i=\sqrt{-1}$. If $(a,b)\neq(0,0)$, then arg $(z)=\tan^{-1}\left(\frac{b}{a}\right)$, where arg $(z)\leq\pi$ and arg $(\bar{z})$ + arg $(-z)$ $=\begin{cases}\pi, & \text{if arg } (z)<0\\ -\pi, & \text{if arg } (z)>0\end{cases}$.*

25. If $\arg(z)>0$, then $\arg(-z)-\arg(z)=\lambda_1$ and if $\arg(z)<0$, $\arg(z)-\arg(-z)=\lambda_2$, then
(a) $\lambda_1+\lambda_2=0$ (b) $\lambda_1-\lambda_2=0$
(c) $3\lambda_1-2\lambda_2=0$ (d) $2\lambda_1-3\lambda_2=0$

26. The value of $\sqrt{\{\arg(z)+\arg(-\bar{z})-2\pi\}\,\{\arg(-z)+\arg(\bar{z})\}}$, $\forall\, z=x+iy$, (where, $i=\sqrt{-1}$) $x,y>0$ is
(a) π (b) $-\pi$ (c) 0 (d) Not defined

Directions (Q. Nos. 27 to 29) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

27. **Statement I** If $|z_1+z_2|^2=|z_1|^2+|z_2|^2$, then $\frac{z_1}{z_2}$ is purely imaginary.

Statement II If z is purely imaginary, then $z+\bar{z}=0$.

28. **Statement I** If $|z|<\sqrt{2}-1$, then $|z^2+2z\sin\alpha|<1$.

Statement II $|z_1+z_2|<|z_1|+|z_2|$ and $|\sin\alpha|\leq 1$.

29. **Statement I** If $z=\sqrt{(5+12i)}+\sqrt{(12i-5)}$, then the principal values of arg (z) are $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, where $i=\sqrt{-1}$.

Statement II If $z=a+ib$, then

$$\sqrt{z}=\pm\left\{\sqrt{\left(\frac{|z|+a}{2}\right)}+i\sqrt{\left(\frac{|z|-a}{2}\right)}\right\}\text{ for } b>0$$

and $$\sqrt{z}=\pm\left\{\sqrt{\left(\frac{|z|+a}{2}\right)}-i\sqrt{\left(\frac{|z|-a}{2}\right)}\right\}\text{ for } b<0$$

30. The maximum value of $|z|$, when z satisfy the condition $|z+3/z|=3$ is
(a) $\frac{2-\sqrt{21}}{2}$ (b) $\frac{3-\sqrt{20}}{2}$ (c) $\frac{3+\sqrt{21}}{2}$ (d) $\frac{3+\sqrt{20}}{2}$

31. If $\alpha+i\beta=\cot^{-1}(z)$, where $z=x+iy$ and α is a constant, then the locus of z is
(a) $x^2+y^2-x\cot 2\alpha-1=0$
(b) $x^2+y^2-2x\cot\alpha-1=0$
(c) $x^2+y^2-2x\cot 2\alpha+1=0$
(d) $x^2+y^2-2x\cot 2\alpha-1=0$

32. The value $\begin{vmatrix}1+\omega & \omega^2 & 1+\omega^2\\ -\omega & -(1+\omega^2) & (1+\omega)\\ -1 & -(1+\omega^2) & 1+\omega\end{vmatrix}$, where ω is cube root of unity, is equal to
(a) 2ω (b) $3\omega^2$ (c) $-3\omega^2$ (d) 3ω

33. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is equal to
(a) 0 (b) 2 (c) 7 (d) 17

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34. If $z_1 \neq 0$ and z_2 be two complex numbers such that $\frac{z_2}{z_1}$ is a purely imaginary number, then $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right|$ is equal to **[JEE Main 2013]**

(a) 2 (b) 5 (c) 3 (d) 1

35. If a complex number z satisfies the equation $z + \sqrt{2}\,|z+1| + i = 0$, then $|z|$ is equal to **[JEE Main 2013]**

(a) 2 (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) 1

36. If $a = \text{Im}\left(\frac{1+z^2}{2iz}\right)$, where z is any non-zero complex number. Then, the set $A = \{a : |z| = 1 \text{ and } z \neq -1\}$ is equal to **[JEE Main 2013]**

(a) $(-1, 1)$ (b) $[-1, 1]$ (c) $[0, 1)$ (d) $(-1, 0]$

37. Let z satisfy $|z| = 1$ and $z = 1 - \bar{z}$.

Statement I z is a real number.

Statement II Principal argument of z is $\frac{\pi}{3}$. **[JEE Main 2013]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

38. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals to **[JEE Main 2013]**

(a) $-\theta$ (b) $\frac{\pi}{2} - \theta$ (c) θ (d) $\pi - \theta$

39. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies **[AIEEE 2012]**

(a) either on the real axis or on a circle passing through the origin.
(b) on a circle with centre at the origin.
(c) either on the real axis or on a circle not passing through the origin.
(d) on the imaginary axis.

40. If $\omega\ (\neq 1)$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$. Then, (A, B) is equal to **[AIEEE 2011]**

(a) $(1, 1)$ (b) $(1, 0)$
(c) $(-1, 1)$ (d) $(0, 1)$

41. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$, is **[AIEEE 2010]**

(a) 0 (b) 1
(c) 2 (d) ∞

42. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is **[AIEEE 2009]**

(a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$
(c) 2 (d) $2 + \sqrt{2}$

43. The conjugate of a complex number is $\frac{1}{i-1}$. Then, that complex number is **[AIEEE 2008]**

(a) $\frac{1}{i-1}$ (b) $-\frac{1}{i-1}$
(c) $\frac{1}{i+1}$ (d) $-\frac{1}{i+1}$

44. If $|z+4| \leq 3$, then the maximum value of $|z+1|$ is **[AIEEE 2007]**

(a) 4 (b) 10
(c) 6 (d) 0

45. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is **[AIEEE 2006]**

(a) 1 (b) -1
(c) $-i$ (d) i

46. If $\omega = \frac{z}{z - \frac{i}{3}}$ and $|\omega| = 1$, then z lies on **[AIEEE 2005]**

(a) a circle (b) an ellipse
(c) a parabola (d) a straight line

47. If $|z^2 - 1| = |z|^2 + 1$, then z lies on **[AIEEE 2004]**

(a) the real axis (b) an ellipse
(c) a circle (d) imaginary axis

Answers

1. (b)	2. (a)	3. (a)	4. (c)	5. (b)	6. (b)	7. (a)	8. (c)	9. (d)	10. (d)
11. (b)	12. (b)	13. (b)	14. (c)	15. (a)	16. (c)	17. (c)	18. (a)	19. (b)	20. (d)
21. (b)	22. (b)	23. (d)	24. (d)	25. (b)	26. (a)	27. (a)	28. (a)	29. (b)	30. (c)
31. (d)	32. (c)	33. (b)	34. (d)	35. (c)	36. (a)	37. (d)	38. (c)	39. (a)	40. (a)
41. (b)	42. (b)	43. (d)	44. (c)	45. (c)	46. (d)	47. (d)			

Hints & Solutions

1. $\sum_{n=1}^{13}(i^n + i^{n+1}) = (1+i)\sum_{n=1}^{13} i^n$

$$= (1+i)\frac{i(1-i^{13})}{1-i}$$

$$= i - 1 \qquad (\because i^{13} = i)$$

2. Given, $f(z) = \frac{7-z}{1-z^2}$ and $z = 1+2i$

$$\therefore \quad f(z) = \frac{7-(1+2i)}{1-(1+2i)^2} = \frac{6-2i}{1-(1-4+4i)} = \frac{6-2i}{4-4i}$$

$$= \frac{6-2i}{4(1-i)} \times \frac{1+i}{(1+i)} = \frac{6+4i+2}{4(1^2-i^2)}$$

$$= \frac{8+4i}{4(2)} = \frac{1}{2}(2+i)$$

Now, $|f(z)| = \frac{\sqrt{4+1}}{2} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$

$$(\because z = 1+2i, \text{ given} \Rightarrow |z| = \sqrt{5})$$

3. Given, $8iz^3 + 12z^2 - 18z + 27i = 0$

$$\Rightarrow \quad 4z^2(2iz+3) + 9i(2iz+3) = 0$$

$$\Rightarrow \quad (2iz+3)(4z^2+9i) = 0$$

$$\Rightarrow \quad 2iz + 3 = 0 \text{ or } 4z^2 + 9i = 0$$

$$\therefore \quad |z| = \frac{3}{2}$$

4. Let $z_1 = r(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r(\cos\theta_2 + i\sin\theta_2)$

Since, $|z_2| = |z_1|$

Also, $\arg(z_1) + \arg(z_2) = \pi$

$$\therefore \quad \arg(z_2) = \pi - \arg(z_1)$$

$$\Rightarrow \quad \arg(z_2) = \pi - \theta_1$$

$$\therefore \quad z_2 = r_1\{\cos(\pi-\theta_1) + i\sin(\pi-\theta_1)\}$$

$$= r_1(-\cos\theta_1 + i\sin\theta_1)$$

$$= -r_1(\cos\theta_1 - i\sin\theta_1) = -\bar{z}_1$$

$$\Rightarrow \quad z_1 = -\bar{z}_2$$

5. Let $z = x + iy$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 - i^2y^2}$$

$$= \frac{x^2 - 1 + iy(x+1-x+1) + y^2}{(x+1)^2 + y^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2+y^2-1)}{(x+1)^2+y^2} + \frac{i(2y)}{(x+1)^2+y^2}$$

Since, $\frac{z-1}{z+1}$ is purely imaginary.

Then, $\quad \text{Re}\left(\frac{z-1}{z+1}\right) = 0$

$$\Rightarrow \quad \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0$$

$$\Rightarrow \quad x^2 + y^2 - 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 = 1$$

$$\Rightarrow \quad |z|^2 = 1$$

$$\therefore \quad |z| = 1$$

6. Let $z = \sqrt{-8-6i}$. Here, $a = -8, b = -6$

Since, $b < 0$

$$\therefore \quad \sqrt{z} = \pm\left[\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)} - i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2})-a}\right]$$

$$= \pm\left[\sqrt{\frac{1}{2}(\sqrt{64+36}-8)} - i\sqrt{\frac{1}{2}(\sqrt{64+36}+8)}\right]$$

$$= \pm\left[\sqrt{\frac{1}{2}(10-8)} - i\sqrt{\frac{1}{2}(10+8)}\right]$$

$$= \pm(1-3i)$$

7. Since, $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then $\frac{z-1}{z+1}$ is purely imaginary.

Let $\quad w = \frac{z-1}{z+1}$

$$\therefore \quad w = -\bar{w} \quad (\text{if } z \text{ is imaginary, then } z = -\bar{z})$$

$$\Rightarrow \quad \frac{z-1}{z+1} = -\overline{\left(\frac{z-1}{z+1}\right)}$$

$$\Rightarrow \quad (z-1)(\bar{z}+1) = -(z+1)(\bar{z}-1)$$

$$\Rightarrow \quad z\bar{z} + z - \bar{z} - 1 = -(z\bar{z} - z + \bar{z} - 1)$$

$$\Rightarrow \quad |z|^2 = 1 \qquad (\because z\bar{z} = |z|^2)$$

$$\therefore \quad |z| = \pm 1$$

8. Given, $|z-1| = 1$

$$\Rightarrow \quad z - 1 = e^{i\theta}, \qquad \text{where } \arg(z-1) = \theta \quad \ldots(\text{i})$$

$$\Rightarrow \quad z = e^{i\theta} + 1$$

$$\Rightarrow \quad z = 1 + \cos\theta + i\sin\theta$$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2} \qquad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$\Rightarrow \quad \arg(z) = \frac{\theta}{2} = \frac{1}{2}\arg(z-1) \qquad [\text{from Eq. (i)}]$$

9. Let $z = (6+5i)^2 = 36 + 2\times 6\times 5i + 25i^2 = 11 + 60i$

Then, $\quad \bar{z} = 11 - 60i$

and $\quad |z| = \sqrt{(11)^2 + (60)^2} = \sqrt{121 + 3600}$

$$= \sqrt{3721} = 61$$

$\therefore$ Multiplicative inverse of $z = \frac{\bar{z}}{|z|^2}$

$$= \frac{11-60i}{(61)^2} = \frac{11}{(61)^2} - \frac{60}{(61)^2}i$$

10. Now, $\frac{z}{1-z^2}=\frac{z}{z\bar{z}-z^2}=\frac{1}{\bar{z}-z}$, which is an imaginary.

11. $|a+b\omega+c\omega^2|^2=(a+b\omega+c\omega^2)(a+b\bar{\omega}+c\bar{\omega}^2)$

$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$(\because \bar{\omega}=\omega^2 \text{ and } \bar{\omega}^2=\omega)$

$=a^2+b^2+c^2-ab-bc-ca$

$=\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$

So, it has minimum value 1 for $a=b=1$ and $c=2$.

12. If $z=1+\cos\theta+i\sin\theta$, then $\arg(z)=\frac{\theta}{2}$

$\therefore \quad \arg(z)=\frac{\pi/5}{2}=\frac{\pi}{10}$

$\Rightarrow \quad \sin(\arg z)=\sin\left(\frac{\pi}{10}\right)=\sin 18^\circ=\frac{\sqrt{5}-1}{4}$

13. Given, $\text{Re}\left(\frac{1}{z}\right)=3 \Rightarrow \text{Re}\left(\frac{\bar{z}}{|z|^2}\right)=3 \quad \left(\because \frac{1}{z}=\frac{\bar{z}}{|z|^2}\right)$

$\Rightarrow \quad \frac{x}{x^2+y^2}=3$

$\Rightarrow \quad 3x^2+3y^2-x=0$

So, it is a circle whose centre is on X-axis and passes through the origin.

14. Given, $|z+4|\le 3$

Now, $|z+1|=|z+4-3|\le|z+4|+|3|$

$=|z+4|+3\le 3+3=6$

Hence, greatest value of $|z+1|=6$

Since, least value of the modulus of a complex number is 0.

$\therefore \quad |z+1|=0 \Rightarrow z=-1$

Now, $\quad |z+4|=|-1+4|=3$

$|z+4|\le 3$ is satisfied by $z=-1$.

$\therefore$ Least value of $|z+1|=0$

15. Now, $\bar{z}_1=\frac{z_1\bar{z}_1}{z_1}=|z_1|^2 z_1^{-1}$

$\Rightarrow \quad \arg(z_1^{-1})=\arg(\bar{z}_1)=\arg(z_2)$

$\Rightarrow \quad z_2=kz_1^{-1}\ (k>0)$

16. $\because p<0$, take $\quad p=-q^3 (q>0)$

$\therefore \quad p^{1/3}=q(-1)^{1/3}=-q,-q\omega,-q\omega^2$

Thus, $\quad \alpha=-q, \beta=-q\omega, \gamma=-q\omega^2$

$\therefore$ Given expression $=\frac{x+y\omega+z\omega^2}{x\omega+y\omega^2+z}=\omega^2$

17. Given, $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1 \Rightarrow z_1^2+z_2^2=z_1z_2$

$\Rightarrow z_1^2+z_2^2+z_3^2=z_1z_2+z_1z_3+z_2z_3$, where $z_3=0$

So, z_1, z_2 and the origin form an equilateral triangle.

18. Since, $z_2=\bar{z}_1$ and $z_4=\bar{z}_3$

$\therefore \quad z_1z_2=z_1\bar{z}_1=|z_1|^2$ and $z_3z_4=z_3\bar{z}_3=|z_3|^2$

Now, $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)=\arg\left(\frac{z_1z_2}{z_4z_3}\right)=\arg\left(\frac{|z_1|^2}{|z_3|^2}\right)$

$=\arg\left(\left|\frac{z_1}{z_3}\right|^2\right)=0$

$\left\{\text{since, } \left|\frac{z_1}{z_3}\right| \text{ is a real number}\right\}$

19. For $-1\le x\le 1$ and $x\ne 0$, $\sec^{-1}\left(\frac{1}{x}\right)=\cos^{-1}x$

$\therefore \quad \sec^{-1}\left(\frac{1}{x}\right)+\sin^{-1}x=\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$

$\therefore \quad \left(\frac{1+i}{1-i}\right)^n=\frac{2}{\pi}\times\frac{\pi}{2}=1$

$\Rightarrow \quad \left[\frac{(1+i)^2}{2}\right]^n=1 \Rightarrow i^n=1$

So, the least positive integral value of n is 4.

20. Now, $\sqrt{(5+12i)}=\sqrt{5+2(3)(2i)}$

$=\sqrt{(3+2i)^2}$

$=\pm(3+2i)$

and $\quad \sqrt{(12i-5)}=\sqrt{-5+2(3)(2i)}$

$=\sqrt{(2+3i)^2}=\pm(2+3i)$

$\therefore \quad z=\sqrt{(5+12i)}+\sqrt{(12i-5)}$

$=\pm(3+2i)\pm(2+3i)$

$\Rightarrow \quad z=5+5i, -1+i, -5-5i, 1-i$

Hence, the principal values of arg (z) are $\frac{\pi}{4},\frac{3\pi}{4},\frac{-3\pi}{4},\frac{-\pi}{4}$.

21. $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\left|\frac{\beta-\alpha}{\beta\cdot\bar{\beta}-\bar{\alpha}\beta}\right| \quad \left(\because |\beta|=1 \text{ and } |\beta|^2=\beta\bar{\beta}=1\right)$

$=\left|\frac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right|=\frac{1}{|\beta|}\frac{|\beta-\alpha|}{|\bar{\beta}-\bar{\alpha}|}=\frac{|\beta-\alpha|}{|\beta-\alpha|}=1$

22. The argument of a product is the sum of the arguments of the factors.

$\therefore \quad \arg(z_1\cdot z_2 \ldots)=\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}\ldots$

$=\frac{\frac{\pi}{2}}{1-\frac{1}{2}}=\pi \quad \left(\because S_\infty=\frac{a}{1-r}\right)$

$\therefore \quad z_1z_2z_3\ldots=\text{cis}\,\pi=-1$

23. $x^2+x+1=0 \Rightarrow \quad x=\omega,\omega^2$

So, $x^r+\frac{1}{x^r}=\omega^r+\frac{1}{\omega^r}=-1$ or 2 according as r is not divisible by 3 or divisible by 3.

$\therefore$ Required sum $=17(-1)^2+8\cdot 2^2=49$

24. Since, $(x-\alpha)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})=x^n-1$

Put $x=2$, we get

$(2-\alpha_1)(2-\alpha_2)\ldots(2-\alpha_{n-1})=2^n-1$

25. Given, $\arg(z) > 0$ and $\arg(-z) - \arg(z) = \lambda_1$, then

$$(\arg \bar{z}) + \arg(-z) = \lambda_1$$

$$\Rightarrow \quad -\pi = \lambda_1 \Rightarrow \lambda_1 = -\pi$$

Again, given $\arg(z) < 0$, $\arg(z) - \arg(-z) = \lambda_2$

$$\Rightarrow \quad -[-\arg(z) + \arg(-z)] = \lambda_2$$

$$\Rightarrow \quad \lambda_2 = -\pi$$

$$\therefore \quad \lambda_1 = \lambda_2$$

26. $\sqrt{\{\arg(z) + \arg(-\bar{z}) - 2\pi\}\{\arg(-z) + \arg(\bar{z})\}}$

$$= \sqrt{\{\arg(z) + \pi - \arg(z) - 2\pi\}\{\arg(-z) - \arg(z)\}}$$

$$= \sqrt{(-\pi) - \{\arg(z) - \arg(-z)\}} = \sqrt{(-\pi)(-\pi)}$$

$$= \pi$$

27. Given, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$$\Rightarrow |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2$$

$$\Rightarrow \quad z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$$

$$\Rightarrow \quad \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0$$

$$\Rightarrow \quad \frac{z_1}{z_2} + \left(\overline{\frac{z_1}{z_2}}\right) = 0$$

Hence, $\frac{z_1}{z_2}$ is purely imaginary.

(if $z + \bar{z} = 0$, then z is purely imaginary)

28. Now, $|z^2 + 2z\sin\alpha| \le |z^2| + |2z\sin\alpha| = |z|^2 + 2|z||\sin\alpha|$

$$< (\sqrt{2} - 1)^2 + 2(\sqrt{2} - 1)\cdot 1 = 1$$

$$\therefore \quad |z^2 + 2z\sin\alpha| < 1$$

29. Now, $\sqrt{(5 + 12i)} = \pm\left\{\sqrt{\frac{13+5}{2}} + i\sqrt{\frac{13-5}{2}}\right\}$ $(\because |5 + 12i| = 13)$

$$= \pm(3 + 2i)$$

and $\sqrt{(-5 + 12i)} = \pm\left\{\sqrt{\left(\frac{13-5}{2}\right)} + i\sqrt{\left(\frac{13+5}{2}\right)}\right\}$

$$= \pm(2 + 3i)$$

$$\therefore \quad z = \pm(3 + 2i) \pm (2 + 3i)$$

$$\Rightarrow \quad z = (5 + 5i, 1 - i, -1 + i, -5 - 5i$$

Hence, principal values of z are $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$.

Also, Statement II is true, but it is not a correct explanation of Statement I.

30. $|z| = |z + 3/z - 3/z| \le |z + 3/z| + |-3/z| = 3 + \frac{3}{|z|}$

Let $|z| = t$, $\quad t \le 3 + \frac{3}{t} \Rightarrow \frac{t^2 - 3t - 3}{t} \le 0$

$$\Rightarrow \quad \left(t - \frac{1}{2}\right)^2 \le \frac{21}{4}$$

Hence, $\frac{3 - \sqrt{21}}{2} < t \le \frac{3 + \sqrt{21}}{2}$

Hence, the maximum value of $|z|$ is $\frac{3 + \sqrt{21}}{2}$.

31. $\alpha + i\beta = \cot^{-1}(z)$ or $\cot(\alpha + i\beta) = x + i\gamma$

and $\quad \cot(\alpha - i\beta) = x - i\gamma$

$$\therefore \quad \cot 2\alpha = \cot[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$= \frac{\cot(\alpha + i\beta)\cdot\cot(\alpha - i\beta) - 1}{\cot(\alpha + i\beta) + \cot(\alpha - i\beta)} = \frac{(x^2 + y^2 - 1)}{2x}$$

$$\therefore \quad x^2 + y^2 - 2x\cot 2\alpha - 1 = 0$$

32. Use $1 + \omega + \omega^2 = 0$ and $\Delta = \begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2$,

$$\Delta = \begin{vmatrix} 0 & \omega^2 & -\omega \\ 0 & \omega & -\omega^2 \\ \omega^2 + 2\omega & \omega & -\omega^2 \end{vmatrix} = (\omega^2 + 2\omega)(-\omega + \omega^2) = -3\omega^2$$

33. Two complex numbers z_1 and z_2 describe radii of 12 and 5 with centres O and (3, 4). The nearest distance between the circles is $12 - 10 = 2$.

34. Given, $\frac{z_2}{z_1}$ is a purely imaginary *i.e.*, ni.

Now, $\left|\frac{2 + 3\cdot\frac{z_2}{z_1}}{2 - 3\cdot\frac{z_2}{z_1}}\right| = \frac{|2 + 3ni|}{|2 - 3ni|} = \frac{\sqrt{4 + 9n^2}}{\sqrt{4 + 9n^2}} = 1$

35. We have, $\quad (x + iy) + \sqrt{2}|x + iy + 1| + i = 0$ $\quad$ (put $z = x + iy$)

$$\Rightarrow \quad (x + iy) + \sqrt{2}\sqrt{(x+1)^2 + y^2} + i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2 + y^2} = 0 \text{ and } y + 1 = 0$$

$$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2 + (-1)^2} = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x^2 = 2[(x+1)^2 + 1] \Rightarrow x^2$$

$$\Rightarrow \quad x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0 \Rightarrow x = -2$$

$$\therefore \quad z = -2 - i \Rightarrow |z| = \sqrt{4+1} = \sqrt{5}$$

36. On simplifying, we get the set (−1, 1)

37. Let $\quad z = x + iy$

$$\therefore \quad x^2 + y^2 = 1 \text{ and } x + iy = 1 - (x - iy)$$

$$\Rightarrow \quad x^2 + y^2 = 1 \text{ and } 2x = 1$$

$$\Rightarrow \quad x = \frac{1}{2} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \quad z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Now, $\quad z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$$

38. Given, $|z| = 1$, $\arg z = \theta$

$\therefore \ z = e^{i\theta}$ but $\bar{z} = \frac{1}{z}$

$$\therefore \quad \arg\left(\frac{1 + z}{1 + \frac{1}{z}}\right) = \arg(z) = \theta$$

39. Since, $\frac{z^2}{z-1}$ $(z \neq 1)$ is purely real.

Hence, $$\frac{z^2}{z-1} = \frac{\bar{z}^2}{\bar{z}-1}$$

$$\Rightarrow z^2(\bar{z}-1) = \bar{z}^2(z-1)$$
$$\Rightarrow z^2\bar{z} - z^2 = \bar{z}^2 z - \bar{z}^2$$
$$\Rightarrow zz\bar{z} - z^2 = \bar{z}\bar{z}z - \bar{z}^2$$
$$\Rightarrow z|z|^2 - z^2 = \bar{z}|z|^2 - \bar{z}^2$$
$$\Rightarrow z|z|^2 - \bar{z}|z|^2 = z^2 - \bar{z}^2$$
$$\Rightarrow |z|^2(z-\bar{z}) = (z-\bar{z})(\bar{z}+z)$$
$$\Rightarrow |z|^2(z-\bar{z}) - (z-\bar{z})(\bar{z}+z) = 0$$
$$\Rightarrow (z-\bar{z})(|z|^2 - (\bar{z}+z)) = 0$$

Either $(z-\bar{z}) = 0$ or $(|z|^2 - (\bar{z}+z)) = 0$

Now, $z = \bar{z} \Rightarrow$ Locus of z is real axis.

and $$(|z|^2 - (\bar{z}+z)) = 0$$
$$\Rightarrow z\bar{z} - (\bar{z}+z) = 0$$

So, the locus of z is a circle passing through origin.

Alternate Method

Put $z = x + iy$, then

$$\frac{z^2}{z-1} = \frac{(x+iy)^2}{(x+iy)-1} = \frac{(x^2-y^2)+i(2xy)}{(x-1)+iy}$$
$$= \frac{(x^2-y^2)+i(2xy)}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

Since, $\frac{z^2}{z-1}$ $(z \neq 1)$ is purely real, hence its imaginary part should be equal to zero.

$$\therefore \quad (x^2-y^2)(-y) + (2xy)(x-1) = 0$$
$$\Rightarrow y(x^2 - y^2 + 2x - 2x^2) = 0$$
$$\Rightarrow y(x^2 + y^2 - 2x) = 0$$

$\Rightarrow$ Either $y = 0$ or $x^2 + y^2 - 2x = 0$

Now, $y = 0 \Rightarrow$ Locus of z is real axis

and $x^2 + y^2 - 2x = 0$. Locus of z is a circle passing through origin.

Hence, the locus of z is either real axis or a circle passing through origin.

40. $(1+\omega)^7 = A + B\omega$, we know that, $1 + \omega + \omega^2 = 0$

$$\therefore \quad 1 + \omega = -\omega^2$$
$$\Rightarrow (-\omega^2)^7 = A + B\omega$$
$$\Rightarrow -\omega^{14} = A + B\omega \quad (\because \omega^{14} = \omega^{12} \cdot \omega^2 = \omega^2)$$
$$\Rightarrow -\omega^2 = A + B\omega$$
$$\Rightarrow 1 + \omega = A + B\omega$$

On comparing both sides, we get $A = 1, B = 1$

41. Let $z = x + iy$

$|z-1| = |z+1| \Rightarrow \text{Re } z = 0 \Rightarrow x = 0$

$|z-1| = |z-i| \Rightarrow x = y$

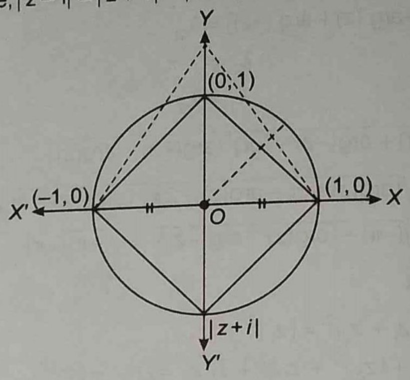
$|z+1| = |z-1| \Rightarrow y = -x$

Since, only (0, 0) will satisfy all conditions.

$\therefore$ Number of complex number $z = 1$

Alternate Method

We have, $|z-1| = |z+1| = |z-i|$



Clearly, z is the circumcentre of the triangle formed by the vertices (1, 0) and (0, 1) and (– 1, 0), which is unique.

Hence, the number of complex number z is one.

42. $|z| = \left|\left(z - \frac{4}{z}\right) + \frac{4}{z}\right| \Rightarrow |z| \leq \left|z - \frac{4}{z}\right| + \frac{4}{|z|} \Rightarrow |z| \leq 2 + \frac{4}{|z|}$

$$\Rightarrow \frac{|z|^2 - 2|z| - 4}{|z|} \leq 0$$

Since, $$|z| > 0$$
$$|z|^2 - 2|z| - 4 \leq 0$$
$$\Rightarrow [|z| - (\sqrt{5}+1)][|z| - (1-\sqrt{5})] \leq 0$$
$$\Rightarrow 1 - \sqrt{5} \leq |z| \leq \sqrt{5} + 1$$

43. Let $z = \frac{1}{i-1}$. Then, $\bar{z} = \overline{\left(\frac{1}{i-1}\right)} = \frac{1}{-i-1} = -\frac{1}{i+1}$

44. Now, $|z+1| = |z+4-3| \leq |z+4| + |-3| \leq 6$

45. $i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right) = i\sum_{k=1}^{10}\left(e^{-\frac{2k\pi}{11}i}\right)$

$$= i\left\{\sum_{k=0}^{10}\left(e^{-\frac{2k\pi}{11}i}\right) - 1\right\} = -i$$

46. $|\omega| = 1 \Rightarrow |z| = \left|z - \frac{i}{3}\right|$

It is the perpendicular bisector of the line segment joining $(0, 0)$ to $\left(0, \frac{-1}{3}\right)$ i.e., the line $y = -\frac{1}{6}$.

47. $\because z = re^{i\theta} \Rightarrow |r^2e^{2i\theta} - 1| = r^2 + 1$

$$\Rightarrow (r^2\cos 2\theta - 1)^2 + (r^2 \sin 2\theta)^2 = (r^2+1)^2$$
$$\Rightarrow r^4 - 2r^2\cos 2\theta + 1 = r^4 + 2r^2 + 1$$
$$\therefore \quad \cos 2\theta = -1 \Rightarrow \theta = \frac{\pi}{2}$$
$$\therefore \quad z = r\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = ir\right)$$

Day 3

Sequence and Series

Day 3 Outlines ...

- Sequence and Series
- Arithmetic Progression (AP)
- Geometric Progression (GP)
- Arithmetico-Geometric Progression (AGP)
- Harmonic Progression (HP)

Sequence and Series

Sequence is a function whose domain is a subset of N *i.e.*, a set of natural numbers. It displays the images of $1, 2, 3, \ldots, n, \ldots$ as $f_1, f_2, f_3, \ldots, f_n, \ldots$ where $f_n = f(n)$. If the terms of a sequence follows certain pattern, then it is called a **progression**. If $a_1, a_2, a_3, \ldots, a_n$ is a sequence, then the expression gives the **series** $a_1 + a_2 + a_3 + \ldots + a_n$.

Arithmetic Progression (AP)

It is a sequence in which the difference between two consecutive terms is same. *i.e.*, $a, a+d, a+2d, a+3d, \ldots$

where, a is the first term, d is the common difference.

(i) The nth term, $t_n = l = a + (n-1)d$

(ii) Sum of n terms, $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$, where l is the last term.

- $t_n = S_n - S_{n-1}$
- $d = S_n - 2S_{n-1} + S_{n-2}$
- $t_n = \frac{1}{2}[t_{n-k} + t_{n+k}]$, where $k < n$

Insertion of Arithmetic Mean (AM)

If a, A and b are in AP, then $A = \frac{a+b}{2}$ is the arithmetic of a and b.

If $a, A_1, A_2, \ldots, A_n, b$ are in AP, then $A_1, A_2, \ldots, A_n$ are the n arithmetic means between a and b.

Hence,
$$d = \frac{b-a}{n+1}$$
$$A_1 = a + d = \frac{na+b}{n+1}$$
$$A_2 = a + 2d = \frac{(n-1)a+2b}{n+1}$$
....

....
$$A_n = a + nd = \frac{a+nb}{n+1}$$

(i) Sum of n AM's between a and b is nA *i.e.*, $A_1 + A_2 + A_3 + \ldots + A_n = nA$

(ii) Any three numbers in AP can be taken as $a-d, a, a+d$.

(iii) Any four numbers in AP can be taken as $a-3d, a-d, a+d, a+3d$.

(iv) In arithmetic series containing even numbers, we assume, number of terms in such series as $2n$, nth and $(n+1)$th terms will be two middle terms.

(v) In arithmetic series containing odd numbers, we assume, number of terms in such series as $(2n+1)$, then $(n+1)$th term will be middle term.

(vi) Sum of terms from beginning and from end of arithmetic series is constant.

» *If pth term of an AP is q and qth term is p, then $T_{p+q} = 0$*

» *If $S_p = q$ and $S_q = p$ for an AP, then $S_{p+q} = -(p+q)$*

(i) *If $S_p = S_q$ for an AP, then $S_{p+q} = 0$*

(ii) *If pth terms is $\frac{1}{q}$, qth term is $\frac{1}{p}$, then $T_{pq} = 1$*

Important Results on AP

1. If the same number is added or subtracted to each term of an AP, then the resultant sequence is also an AP having same common difference.
2. If each term of an AP is multiplied or divided by the same number k, then the resulting sequence is also an AP.
3. In a finite AP, the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last term *i.e.*, $a_1 + a_n = a_k + a_{n-(k-1)}, \forall\, k = 1, 2, 3, \ldots, n-1$
4. A sequence is an AP, iff its nth term is a linear expression in n, *i.e.*, $a_n = An + B$, where A and B are constant.
5. A sequence is an AP, iff the sum of its first n terms is of the form $An^2 + Bn$. In such case, common difference is $2A$.
6. If the terms of an AP are choosen at regular intervals, then they form an AP.

Geometric Progression (GP)

It is a sequence in which the ratio of any two consecutive terms is same *i.e.*, $a, ar, ar^2, \ldots$

where, a is the first term and r is common ratio.

(i) The nth term, $t_n = ar^{n-1}$

(ii) Sum of n terms, $S_n = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & |r| < 1 \\ a\left(\frac{r^n-1}{r-1}\right), & |r| > 1 \end{cases}$

(iii) If $|r| < 1$, then sum of infinite GP is $S_\infty = \frac{a}{1-r}$

For a GP, $T_p = P, T_q = Q$, then $T_n = \left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$

Insertion of Geometric Mean (GM)

If a, x and b are in GP, then $x = \sqrt{ab}$ is the geometric mean of a and b.

If $a, x_1, x_2, \ldots, x_n, b$ are in GP, then $x_1, x_2, \ldots, x_n$ are the n geometric means between a and b.

where,
$$r = \left(\frac{b}{a}\right)^{1/(n+1)}$$
and
$$x_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$$

(i) Three numbers in GP can be taken as $\frac{a}{r}, a, ar$.

(ii) Four numbers in GP can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

(iii) Five numbers in GP can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.

» *If A and G be the AM and GM between two numbers a and b, then a, b are given by $[A \pm \sqrt{(A+G)(A-G)}]$.*

» *The product of first n terms of GP is*
$$P = aar \cdot ar^2 \ldots ar^{n-1} = a^n r^{\frac{n(n-1)}{2}}$$

Important Results on GP

1. If all terms of a GP be multiplied or divided by the same non-zero constant, then it remains a GP with the same common ratio.
2. The reciprocal of the terms of a given GP form a GP.
3. If each term of a GP be raised to same power, then resulting sequence also forms GP.
4. If $a_1, a_2, a_3, \ldots, a_n$ is a GP of non-zero negative terms, then $\log a_1, \log a_2, \ldots, \log a_n$ be an AP.
5. If $a_1, a_2, a_3, \ldots, a_n$ be non-zero, non-negative numbers, then their GM $= (a_1 a_2 a_3 \ldots a_n)^{1/n}$.
6. In a finite GP, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

Summation of Series by the Difference Method

Let $T_1 + T_2 + T_3 + \ldots$ be a given infinite series.

If $T_2 - T_1, T_3 - T_2, \ldots$ are in AP or GP, then T_n and S_n of series may be found by the method of differences

Let $\quad S_n = T_1 + T_2 + T_3 + \ldots + T_n \quad \ldots$(i)

Again, $\quad S_n = T_1 + T_2 + T_3 + \ldots T_{n-1} + T_n \quad \ldots$(ii)

$\therefore \quad S_n - S_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$

$\Rightarrow \quad T_n = T_1 + t_1 + t_2 + t_3 + t_{n-1}$

where $t_1, t_2, t_3, \ldots$ are terms of the new series.

Arithmetico-Geometric Progression (AGP)

A series in which every term is a product of a term of AP and GP is known as **arithmetico-geometric progression.**

Let the series of AGP be

$$a + (a+d)r + (a+2d)r^2 + \ldots + \{a + (n-1)d\}r^n.$$

Then,

(i) $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{\{a+(n-1)d\}r^n}{1-r}, r \neq 1$

(ii) $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}, |r| < 1$

where, a and d are the first term and common difference of an AP and r be the common ratio of a GP.

Harmonic Progression (HP)

A sequence is in HP, if the reciprocals of its terms form an AP *i.e.*, $3, 1, \frac{3}{5}$ is in HP, since $\frac{1}{3}, 1$ and $\frac{5}{3}$ is an AP.

General sequence of HP is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}$$

- *Sum of HP series can not be determined by any formula*
- *If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then*

$$\frac{a_1}{a_2 + a_3 + \ldots + a_n}, \frac{a_2}{a_1 + a_3 + a_4 + \ldots + a_n}, \frac{a_3}{a_1 + a_2 + a_4 + \ldots + a_n}, \ldots, \frac{a_n}{a_1 + a_2 + \ldots + a_{n-1}}$$ *are in HP.*

- *If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then*
  $a_1 a_2 + a_2 a_3 + \ldots + a_{n-1} a_n = (n-1) a_1 a_n$
- *If mth term of HP $= n$ and nth term of HP $= m$, then*

$$T_r = \frac{mn}{r},$$

$$T_{m+n} = \frac{mn}{m+n},$$

$$T_{mn} = 1$$

Insertion of Harmonic Mean (HM)

If a, x, b are in HP, then $x = \dfrac{2ab}{a+b}$ is the harmonic mean of a and b.

If $a, x_1, x_2, \ldots, x_n, b$ are in HP, then $x_1, x_2, \ldots, x_n$ are the n harmonic means between a and b.

These are the reciprocal of the n arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$.

Hence, $\quad x_1 = \dfrac{(n+1)ab}{a+nb}$

$\quad x_2 = \dfrac{(n+1)ab}{2a+(n-1)b}$

....

....

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Sum of Special Series

1. Sum of first n natural numbers,

$$1+2+\ldots+n=\Sigma n=\frac{n(n+1)}{2}$$

2. Sum of square of first n natural numbers,

$$1^2+2^2+\ldots+n^2=\Sigma n^2=\frac{n(n+1)(2n+1)}{6}$$

3. Sum of cube of first n natural numbers,

$$1^3+2^3+3^3+\ldots+n^3=\Sigma n^3=\left[\frac{n(n+1)}{2}\right]^2$$

4. Sum of product of first n natural numbers taken two at a time is

$$\frac{1}{2}[\Sigma n^3-\Sigma n^2]=\frac{n(n+1)(n-1)(3n+2)}{24}$$

5. (i) Sum of first even natural numbers

$$2+4+6+\ldots+2n=n(n+1)$$

(ii) Sum of first odd natural numbers

$$1+3+5+\ldots+(2n-1)=n^2$$

6. Sum of n terms of series

$$1^2-2^2+3^2-4^2+5^2-6^2+\ldots$$

***Case* I** When n is odd $\frac{n(n+1)}{2}$

***Case* II** When n is even $\frac{-n(n+1)}{2}$

Relation between Arithmetic, Geometric *and* Harmonic Means

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b.

Then, $$A=\frac{a+b}{2},\ G=\sqrt{ab}$$

and $$H=\frac{2ab}{a+b}$$

(i) $A\geq G\geq H$

(ii) A, G and H form a GP *i.e.*, $G^2=AH$.

(iii) The equation having a and b as its roots is $x^2-2Ax+G^2=0$

(iv) The equation having a, b and c as roots is $x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$

where, $A=\frac{a+b+c}{3}$ and $\frac{1}{H}=\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right)$

(v) If A, G, H be AM, GM, HM between a and b, then

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\begin{cases}A, & \text{when } n=0\\ G, & \text{when } n=-\frac{1}{2}\\ H, & \text{when } n=-1\end{cases}$$

(vi) If A_1, A_2 be two AM's; G_1 and G_2 be two GM's and H_1 and H_2 be two HM's between two numbers a and b, then $\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$

Practice Zone

DAY 3

1. The sum of n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ is
(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$
(c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

2. If $\log_3 2, \log_3 (2^x-5)$ and $\log_3\left(2^x-\frac{7}{2}\right)$ are in AP, then x is equal to
(a) 2 (b) 3
(c) 4 (d) 2, 3

3. The number of numbers lying between 100 and 500 that are divisible by 7 but not by 21 is
(a) 57 (b) 19
(c) 38 (d) None of these

4. If the sum of an infinite GP is $\frac{7}{2}$. Sum of the squares of its terms is $\frac{147}{16}$, then the sum of the cubes of its terms is
(a) $\frac{315}{19}$ (b) $\frac{700}{39}$
(c) $\frac{985}{13}$ (d) $\frac{1029}{38}$

5. If the pth and qth terms of a GP are q and p respectively, then $(p+q)$th term is **[NCERT Exemplar]**
(a) $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$ (b) $\left(\frac{q^q}{p^p}\right)^{\frac{1}{p-q}}$
(c) $\left(\frac{p^q}{q^p}\right)^{\frac{1}{p-q}}$ (d) None of these

6. If $x=111\ldots1$ (20 digits), $y=333\ldots3$ (10 digits) and $z=222\ldots2$ (10 digits), then $\frac{x-y^2}{z}$ is equal to
(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 3

7. If one GM, g and two AM's, p and q are inserted between two numbers a and b, then $(2p-q)(p-2q)$ is equal to
(a) g^2 (b) $-g^2$
(c) $2g$ (d) $3g^2$

8. If $5x-y, 2x+y$ and $x+2y$ are in AP and $(x-1)^2$, $(xy+1)$ and $(y+1)^2$ are in GP, $x\neq 0$, then $x+y$ is equal to
(a) 3/4 (b) 3
(c) −5 (d) 6

9. The sum of the first 10 terms of $\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\ldots$ is
(a) $10-2^{-10}$ (b) $9-2^{-10}$
(c) $11-2^{-10}$ (d) None of these

10. The sum of the series $1^3+3^3+5^3+\ldots$ to 20 terms is
(a) 319600 (b) 321760
(c) 306000 (d) 347500

11. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which are in AP. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. The value of the 8th instalment is
(a) ₹ 35 (b) ₹ 50
(c) ₹ 65 (d) None of these

12. The minimum value of 4^x+4^{1-x}, $x\in R$ is **[NCERT Exemplar]**
(a) 2 (b) 4
(c) 1 (d) 0

13. The sum to 50 terms of $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots$ is
(a) $\frac{50}{17}$ (b) $\frac{100}{17}$
(c) $\frac{150}{17}$ (d) $\frac{200}{17}$

14. The coefficient of x^8 in the polynomial $(x-1)(x-2)(x-3)\ldots(x-10)$ is
(a) 1025 (b) 1240
(c) 1320 (d) 1440

15. The sum of the infinite series $\frac{5}{13}+\frac{55}{13^2}+\frac{555}{13^3}+\ldots$ is
(a) $\frac{31}{18}$ (b) $\frac{65}{32}$
(c) $\frac{65}{36}$ (d) $\frac{75}{36}$

16. The successive terms of an AP are $a_1, a_2, a_3, \ldots$. If $a_6 + a_9 + a_{12} + a_{15} = 20$, then $\sum_{r=1}^{20} a_r$ is equal to

(a) 75 (b) 100 (c) 120 (d) 150

17. If 2, 7, 9 and 5 are subtracted respectively from 4 numbers forming a GP, the resulting numbers are in AP, then the smallest of the four numbers is

(a) −24 (b) −12 (c) 6 (d) 3

18. If x, $2y$ and $3z$ are in AP, where distinct numbers x, y and z are in GP. The common ratio of GP is

(a) 1 (b) 1/2 (c) 1/3 (d) 2/5

19. In an AP of which 1 is the first term, if the second, tenth and thirty fourth terms forms a GP, then fourth terms of AP is

(a) 1/2 (b) 1 (c) 3/2 (d) 2

20. The values of $x + y + z$ is 15. If a, x, y and z and b are in AP, while the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is $\frac{5}{8}$. If a, x, y and z and b are in HP, then $a^2 + b^2$ is equal to

(a) 48 (b) 50 (c) 52 (d) 60

21. If the function f satisfies the relation $f(x + y) = f(x) \cdot f(y)$ for all natural numbers $x, y, f(1) = 2$ and $\sum_{r=1}^{n} f(a + r) = 16(2^n - 1)$, then the natural number a is

(a) 2 (b) 3 (c) 4 (d) 5

22. If $S_1, S_2, S_3, \ldots$ are the sum of infinite geometric series whose first terms are $1, 2, 3, \ldots$ and whose common ratios $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ respectively, then $S_1^2 + S_2^2 + S_3^2 + \ldots + S_{10}^2$ is equal to

(a) 485 (b) 495 (c) 500 (d) 505

23. For $0 < \theta < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta$, then xyz is equal to

(a) $xz + y$ (b) $x + y + z$ (c) $yz + x$ (d) $x + y - z$

24. If x and y are positive real numbers and m, n are positive integers, then the maximum value of $\frac{x^m y^n}{(1 + x^{2m})(1 + y^{2n})}$ is

(a) 2 (b) 1/4 (c) 1/2 (d) 1

Directions (Q. Nos. 25 and 26) *If A, G and H are respectively arithmetic, geometric and harmonic means between a and b both being unequal and positive, then* $A = \frac{a+b}{2} \Rightarrow a + b = 2A$,

$$G = \sqrt{ab} \Rightarrow G^2 = ab,\ H = \frac{2ab}{a+b} \Rightarrow G^2 = AH.$$

25. If the geometric and harmonic means of two numbers are 16 and $12\frac{4}{5}$, then the ratio of one number to the other is

(a) 1 : 4 (b) 2 : 3 (c) 1 : 2 (d) 2 : 1

26. The sum of the AM and GM of two positive numbers equal to the difference between the numbers. The numbers are in the ratio

(a) 1 : 3 (b) 1 : 6 (c) 9 : 1 (d) 1 : 12

Directions (Q. Nos. 27 and 28) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

27. Let $a, r \in R - \{0, 1, -1\}$ and n be an even number.

Statement I $a, ar, ar^2, \ldots, ar^{n-1} = (a^2 r^{n-1})^{n/2}$

Statement II Product of ith term from the beginning and from the end in a GP is independent of i.

28. Statement I $0.3 + 0.03 + 0.003 + \ldots = 1/3$

Statement II For each positive integer n, let $a_n = a + nd$, a and d are real numbers. Then,

$$a_1 + a_2 + \ldots + a_n = \frac{n}{2}[2a + (n+1)d]$$

29. The length of a side of a square is a m. A second square is formed by joining the middle points of these squares. Then, a third square is formed by joining the middle points of the second square and so on. Then, sum of the area of the squares which carried upto infinity is

(a) a^2 (b) $2a^2$ (c) $3a^2$ (d) $4a^2$

30. If $\log_{\sqrt{3}} a^2 + \log_{(3)^{1/3}} a^2 + \log_{3^{1/4}} a^2 + \ldots$ upto 8th term = 44, then the value of a is

(a) $\pm\sqrt{3}$ (b) $2\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) None of these

31. In an equilateral triangle another equilateral triangle is drawn inside joining the mid-points of the sides of given equilateral triangle and the process is continued upto 7 times. If the side of a given equilateral triangle is a unit, then the ratio of first and fourth triangles is

(a) 254 : 15 (b) 256 : 1 (c) 256 : 7 (d) None of these

32. An infinite GP has first term x and sum 5, then x belongs to

(a) $x < -10$ (b) $-10 < x < 0$ (c) $0 < x < 10$ (d) $x > 10$

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33. Let $a_1, a_2, a_3, \ldots$ be an AP such that $\dfrac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + a_3 + \ldots + a_q} = \dfrac{p^3}{q^3}$; $p \neq q$, then $\dfrac{a_6}{a_{21}}$ is equal to **[JEE Main 2013]**

(a) $\dfrac{41}{11}$ (b) $\dfrac{121}{1681}$
(c) $\dfrac{11}{41}$ (d) $\dfrac{121}{1861}$

34. The sum of the series
$1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots$ upto 10 terms is **[JEE Main 2013]**

(a) $\dfrac{18}{11}$ (b) $\dfrac{22}{13}$
(c) $\dfrac{20}{11}$ (d) $\dfrac{16}{9}$

35. Given sum of the first n terms of an AP is $2n + 3n^2$. Another AP is formed with the same first term and double of the common difference, the sum of n terms of the new AP is

(a) $n + 4n^2$ (b) $6n^2 - n$
(c) $n^2 + 4n$ (d) $3n + 2n^2$

36. The sum $\dfrac{3}{1^2} + \dfrac{5}{1^2 + 2^2} + \dfrac{7}{1^2 + 2^2 + 3^2} + \ldots$ upto 11 terms is **[JEE Main 2013]**

(a) $\dfrac{7}{2}$ (b) $\dfrac{11}{4}$
(c) $\dfrac{11}{2}$ (d) $\dfrac{60}{11}$

37. The sum of the series $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$ upto 10 terms is **[JEE Main 2013]**

(a) 11300 (b) 11200 (c) 12100 (d) 12300

38. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in AP such that $a_4 - a_7 + a_{10} = m$, then the sum of first 13 terms of this AP is **[JEE Main 2013]**

(a) 10 m (b) 12 m (c) 13 m (d) 15 m

39. Given a sequence of 4 numbers, first three of which are in GP and the last three are in AP with common difference six. If first and last terms of this sequence are equal, then the last term is

(a) 16 (b) 8 (c) 4 (d) 2

40. The value of $1^2 + 3^2 + 5^2 + \ldots + 25^2$ is **[JEE Main 2013]**

(a) 2925 (b) 1469
(c) 1728 (d) 1456

41. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ..., is **[JEE Main 2013]**

(a) $\dfrac{7}{81}[179 - 10^{20}]$ (b) $\dfrac{7}{9}[99 - 10^{-20}]$

(c) $\dfrac{7}{81}[179 + 10^{-20}]$ (d) $\dfrac{7}{9}[99 + 10^{-20}]$

42. **Statement I** The sum of the series
$1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$ is 8000.

Statement II $\sum_{k=1}^{n} [k^3 - (k-1)^3] = n^3$, for any natural number n. **[AIEEE 2012]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

43. If 100 times the 100th term of an AP with non-zero common difference equals the 50 times its 50th term, then the 150th term of this AP is **[AIEEE 2012]**

(a) – 150 (b) 150 times its 50th term
(c) 150 (d) zero

44. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹ 40 more than the saving of immediately previous month. His total saving from the start of service will be ₹ 11040 after **[AIEEE 2011]**

(a) 19 months (b) 20 months
(c) 21 months (d) 18 months

45. A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the nth minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in AP with common difference –2, then the time taken by him to count all notes, is **[AIEEE 2010]**

(a) 24 min (b) 34 min
(c) 125 min (d) 135 min

46. The sum of the infinity of the series
$$1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots \text{ is}$$ **[AIEEE 2009]**

(a) 3 (b) 4 (c) 6 (d) 2

47. The first two terms of a geometric progression add upto 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then first term is **[AIEEE 2008]**

(a) 4 (b) – 4
(c) – 12 (d) 12

48. In a geometric progression consisting of positive term, each term equals to the next two terms. Then, the common ratio of this progression is equal to **[AIEEE 2007]**

(a) $\dfrac{1}{2}(1 - \sqrt{5})$ (b) $\dfrac{1}{2}\sqrt{5}$

(c) $\sqrt{5}$ (d) $\dfrac{1}{2}(\sqrt{5} - 1)$

49. Let $a_1, a_2, \ldots,$ be terms of an AP. If $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^2}{q^2}$, where $p \neq q$, then $\frac{a_6}{a_{21}}$ is equal to **[AIEEE 2007]**

(a) $\frac{7}{2}$ (b) $\frac{2}{7}$ (c) $\frac{11}{41}$ (d) $\frac{41}{11}$

50. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$, where a, b and c are in AP and $|a| < 1, |b| < 1, |c| < 1$, then x, y and z are in **[AIEEE 2005]**

(a) AP (b) GP (c) HP (d) AGP

51. The sum of the first n terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$ is $\frac{n}{2}(n+1)^2$, when n is even. When n is odd, then the sum is **[AIEEE 2004]**

(a) $\frac{n^2}{2}(n+1)$ (b) $\frac{n}{2}(n-1)^2$

(c) $\frac{n^2}{2}(n-1)$ (d) $\frac{n(n+1)}{2}$

Answers

1. (c)	**2.** (b)	**3.** (c)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (d)	**20.** (c)
21. (b)	**22.** (d)	**23.** (b)	**24.** (b)	**25.** (a)	**26.** (c)	**27.** (b)	**28.** (d)	**29.** (b)	**30.** (a)
31. (b)	**32.** (c)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (c)	**39.** (b)	**40.** (a)
41. (c)	**42.** (a)	**43.** (d)	**44.** (c)	**45.** (b)	**46.** (a)	**47.** (c)	**48.** (d)	**49.** (c)	**50.** (c)
51. (a)									

Hints & Solutions

1. $\therefore \sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \ldots$

$= \sqrt{2}\,(1 + 2 + 3 + 4 + \ldots n \text{ terms})$

$= \sqrt{2}\,\Sigma n = \sqrt{2}\,\frac{n(n+1)}{2} = \frac{n(n+1)}{\sqrt{2}}$

2. $\because \; 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \quad (2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \quad t^2 + 25 - 10t = 2t - 7 \qquad (\text{put } 2^x = t)$

$\Rightarrow \quad t^2 - 12t + 32 = 0$

$\Rightarrow \quad (t-8)(t-4) = 0$

$\Rightarrow \quad 2^x = 8 \text{ or } 2^x = 4$

$\therefore \quad x = 3 \text{ or } x = 2$

At, $x = 2$, $\log_3(2^x - 5)$ is not defined.

Hence, $x = 3$ is the only solution.

3. The numbers between 100 and 500 that are divisible by 7 are 105, 112, 119, 126, …, 490, 497.

Let such numbers be n.

$\therefore \quad t_n = a_n + (n-1)d$

$\Rightarrow \quad 497 = 105 + (n-1) \times 7$

$\therefore \quad n - 1 = 56$

$\therefore \quad n = 57$

The numbers between 100 and 500 that are divisible by 21 are 105, 126, 147, …, 483.

Let such numbers be m.

$\therefore \quad 483 = 105 + (m-1) \times 21$

$\Rightarrow \quad 18 = m - 1$

$\therefore \quad m = 19$

$\therefore$ Required number $= n - m = 57 - 19 = 38$

4. Let GP be $a, ar, ar^2, \ldots, |r| < 1$. Sequence

According to the question, $\frac{a}{1-r} = \frac{7}{2}, \frac{a^2}{1-r^2} = \frac{147}{16}$

Eliminating a, we get

$\frac{147}{16}(1 - r^2) = \left(\frac{7}{2}\right)^2 (1-r)^2$

$\Rightarrow \quad 3(1+r) = 4(1-r) \Rightarrow r = \frac{1}{7}, a = 3$

$\therefore$ Sum of cubes $= \frac{a^3}{1-r^3} = \frac{(3)^3}{1-\left(\frac{1}{7}\right)^3} = \frac{1029}{38}$

5. Let first term be A and common ratio be R.

Given, pth term, $T_p = q$ and qth term, $T_q = p$

$AR^{p-1} = q$ and $AR^{q-1} = p \qquad \ldots(\text{i})$

$\therefore \quad \frac{AR^{p-1}}{AR^{q-1}} = \frac{q}{p} \Rightarrow R^{p-q} = \frac{q}{p}$

$\Rightarrow \quad R = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$

On putting the value of R in Eq. (i), we get

$A \cdot \left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \Rightarrow A = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$

Now, $(p+q)$th term, $T_{p+q} = AR^{p+q-1} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \times \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$

$$= \frac{q^{1 - \frac{p-1}{p-q} + \frac{p+q-1}{p-q}}}{p^{\frac{p+q-1}{p-q} - \frac{p-1}{p-q}}} = \frac{q^{\frac{p-q-p+1+p+q-1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}} = \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}} = \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$$

6. Given, $x = \frac{1}{9}(999\ldots9) = \frac{1}{9}(10^{20} - 1)$

$y = \frac{1}{3}(999\ldots9) = \frac{1}{3}(10^{10} - 1)$

and $z = \frac{2}{9}(999\ldots9) = \frac{2}{9}(10^{10} - 1)$

$\therefore \frac{x - y^2}{z} = \frac{10^{20} - 1 - (10^{10} - 1)^2}{2(10^{10} - 1)} = \frac{10^{10} + 1 - (10^{10} - 1)}{2} = 1$

7. Since, $g = \sqrt{ab}$. Also, a, p, q and b are in AP.

So, common difference d is $\frac{b-a}{3}$.

$\therefore \quad p = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$

$q = b - d = b - \frac{b-a}{3} = \frac{a+2b}{3}$

Now, $(2p - q)(p - 2q) = \frac{(4a + 2b - a - 2b)}{3}\frac{(2a + b - 2a - 4b)}{3}$

$= -ab = -g^2$

8. Since, $5x - y + x + 2y = 2(2x + y) \Rightarrow 2x = y$

and $(x-1)^2(y+1)^2 = (xy+1)^2$

$\Rightarrow (x-1)(y+1) = \pm(xy+1)$

$\Rightarrow (x-1)(2x+1) = \pm(2x^2+1)$

$\Rightarrow 2x^2 - x - 1 = +(2x^2+1)$ (taking +ve sign)

$\therefore x = -2, y = -4$

Also, $2x^2 - x - 1 = -2x^2 - 1$ (taking –ve sign)

$\Rightarrow x = \frac{1}{4}, y = \frac{1}{2}$ $(\because x \neq 0, \text{ given})$

$\therefore x + y = -6$ or $\frac{3}{4}$

9. $\frac{3}{2} + \frac{5}{4} + \frac{9}{8} + \ldots$ upto 10 terms

$= \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{4}\right) + \left(1 + \frac{1}{8}\right) + \ldots 10$ terms

$= 10 + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots 10 \text{ terms}\right)$

$= 10 + \frac{1}{2}\left(\frac{1 - \frac{1}{2^{10}}}{1 - \frac{1}{2}}\right) = 10 + \frac{1}{2}\left(1 - \frac{1}{2^{10}}\right)\Big/\frac{1}{2} = 11 - 2^{-10}$

10. $1^3 + 3^3 + \ldots + 39^3 = 1^3 + 2^3 + 3^3 + \ldots + 40^3$

$- (2^3 + 4^3 + 6^3 + \ldots + 40^3)$

$= \left(\frac{40 \times 41}{2}\right)^2 - 8(1^3 + 2^3 + 3^3 + \ldots + 20^3)$

$= (20 \times 41)^2 - 8\left(\frac{20 \times 21}{2}\right)^2 = 20^2[41^2 - 2(21)^2] = 319600$

11. Given, $3600 = \frac{40}{2}[2a + (40 - 1)d]$

$\Rightarrow 3600 = 20(2a + 39d)$

$\Rightarrow 180 = 2a + 39d$ ...(i)

After 30 instalments one-third of the debt is unpaid

Hence, $\frac{3600}{3} = 1200$ is unpaid and 2400 is paid.

Now, $2400 = \frac{30}{2}\{2a + (30 - 1)d\}$

$\therefore 160 = 2a + 29d$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a = 51, d = 2$

Now, the value of 8th instalment $= a + (8 - 1)d$

$= 51 + 7 \cdot 2 = ₹\, 65$

12. We know that, AM $\geq$ GM

$\therefore \frac{4^x + \frac{4}{4^x}}{2} \geq \sqrt{4^x \times \frac{4}{4^x}}$

$\Rightarrow 4^x + \frac{4}{4^x} \geq 2\sqrt{4} \Rightarrow 4^x + \frac{4}{4^x} \geq 4$

13. Here, $t_r = \frac{2r+1}{1^2 + 2^2 + \ldots + r^2} = \frac{(2r+1)}{\frac{r(r+1)(2r+1)}{6}} = \frac{6}{r(r+1)}$

$\therefore \sum_{r=1}^{50} t_r = 6\sum_{r=1}^{50} \frac{1}{r(r+1)} = 6\sum_{r=1}^{50}\left(\frac{1}{r} - \frac{1}{r+1}\right) = 6\left(1 - \frac{1}{51}\right) = \frac{100}{17}$

14. Coefficient of x^8 is

$1 \cdot 2 + 1 \cdot 3 + \ldots + 2 \cdot 3 + 2 \cdot 4 + \ldots$

$= \frac{1}{2}[(1 + 2 + \ldots + 10)^2 - (1^2 + 2^2 + \ldots + 10^2)]$

$= \frac{1}{2}\left[\left(\frac{10 \times 11}{2}\right)^2 - \frac{10 \times 11 \times 21}{6}\right]$

$= \frac{1}{2}[55^2 - 55 \times 7] = 55 \times 24 = 1320$

15. Let $S = \frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \ldots$ ...(i)

and $\frac{S}{13} = \frac{5}{13^2} + \frac{55}{13^3} + \ldots$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$\frac{12}{13}S = \frac{5}{13} + \frac{50}{13^2} + \frac{500}{13^3} + \ldots$

which is a GP with common ratio $\frac{10}{13}$.

$\therefore S = \frac{13}{12} \times \left[\frac{5}{13} \div \left(1 - \frac{10}{13}\right)\right]$ $\left(\because S_\infty = \frac{a}{1-r}\right)$

$= \frac{65}{36}$

16. Let d be the common difference.

Given, $a_6 + a_9 + a_{12} + a_{15} = 20$

$\Rightarrow a_1 + 5d + a_1 + 8d + a_1 + 11d + a_1 + 14d = 20$

$\Rightarrow 4a_1 + 38d = 20$

$\Rightarrow 2a_1 + 19d = 10$

$\therefore S_{20} = \frac{20}{2}[2a + 19d] = 10 \times 10 = 100$

17. Let the four numbers in GP are a, ar, ar^2 and ar^3.

So, $a - 2, ar - 7, ar^2 - 9$ and $ar^3 - 5$ are in AP.

$\Rightarrow a - 2 + ar^2 - 9 = 2(ar - 7)$ $(\because 2A = B + C)$

$\Rightarrow a(r^2 + 1) - 11 = 2ar - 14$

$\therefore r^2 + 1 = 2r - \frac{3}{a}$ ...(i)

Further, $ar - 7 + ar^3 - 5 = 2(ar^2 - 9)$

$\Rightarrow \quad ar(r^2+1) - 12 = 2ar^2 - 18$

$\therefore \quad r^2 + 1 = 2r - \frac{6}{ar}$ ...(ii)

From Eqs. (i) and (ii), $r = 2, a = -3$

Hence, numbers are $-3, -6, -12, -24$.

Hence, the smallest number is -24.

18. Since, x, y and z are in GP.

$\therefore \quad x = \frac{y}{r}, z = yr$

Also, $x, 2y$ and $3z$ are in AP. So, $\frac{y}{r}, 2y$ and $3yr$ are in AP.

$\Rightarrow \quad \frac{y}{r} + 3yr = 4y$ $(\because 2B = A + C)$

$\Rightarrow \quad 3r^2 - 4r + 1 = 0$

$\therefore \quad r = 1, \frac{1}{3}$

Since, terms of a GP are distinct, so we do not consider $r = 1$.

Therefore, we consider only $r = \frac{1}{3}$

19. Since, T_2, T_{10} and T_{34} are in GP. Also, $1 + d, 1 + 9d, 1 + 33d$ are in GP.

$\Rightarrow \quad (1+9d)^2 = (1+d)(1+33d)$

$\Rightarrow \quad 16(3d-1)d = 0$

$\therefore \quad d = \frac{1}{3}$

Then, $T_4 = 1 + 3d = 1 + 1 = 2$

$\therefore \quad T_4 = 2$

20. Since, a, x, y, z and b are in AP.

Then, $y = \frac{a+b}{2}$

Now, $x + z = a + d + b - d = a + b$

where, d is common difference.

$\because \quad x + y + z = 15$

$\Rightarrow \quad \frac{a+b}{2} + (a+b) = 15$

$\therefore \quad a + b = 10$ ...(i)

Also, a, x, y, z, b are in HP and $\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in AP.

Now, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + \left(\frac{1}{a} + \frac{1}{b}\right)$

$= \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

$= \frac{3}{2}\frac{(a+b)}{ab}$

$= \frac{5}{8}$ (given)

$\therefore \quad ab = \frac{8}{5} \times \frac{3}{2} \times 10 = 24$ [from Eq. (i)] ...(ii)

From Eqs. (i) and (ii),

$(a, b) = (4, 6)$

$\therefore \quad a^2 + b^2 = 52$

21. Now, $f(2) = f(1+1) = f(1) \cdot f(1) = 2^2$ and $f(3) = 2^3$

Similarly, $f(n) = 2^n$

$\therefore \quad 16(2^n - 1) = \sum_{r=1}^{n} f(a+r) = \sum_{r=1}^{n} 2^{a+r} = 2^a(2 + 2^2 + \dots + 2^n)$

$= 2^a \cdot 2\left(\frac{2^n - 1}{2 - 1}\right)$ (GP series)

$= 2^{a+1}(2^n - 1)$

$\therefore \quad 2^{a+1} = 16 = 2^4$

$\therefore \quad a = 3$

22. Here, S_r is sum of an infinite GP, r is first term and $\frac{1}{r+1}$ is common ratio.

$\therefore \quad S_r = \frac{r}{1 - \frac{1}{r+1}} = r + 1$

$\therefore \quad \sum_{r=1}^{10} S_r^2 = 2^2 + 3^2 + \dots + 11^2$

$= 1^2 + 2^2 + 3^2 + \dots + 11^2 - 1$

$= \frac{11 \times 12 \times 23}{6} - 1 = 505$

23. Sum of three infinite GP's are $x = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}$

$y = \frac{1}{\cos^2\theta}$ and $z = \frac{1}{1 - \cos^2\theta \sin^2\theta}$

Now, $\frac{1}{x} + \frac{1}{y} = 1$ $(\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow \quad x + y = xy$

$\frac{1}{z} = 1 - \cos^2\theta \sin^2\theta = 1 - \frac{1}{xy} = \frac{xy - 1}{xy} \Rightarrow xy = xyz - z$

$\therefore \quad xyz = xy + z = x + y + z$

24. Using AM $\geq$ GM, $\frac{1 + x^{2m}}{2} \geq x^m$ and $\frac{1 + y^{2m}}{2} \geq y^n$

$\therefore \quad \frac{x^m}{1 + x^{2m}} \cdot \frac{y^n}{1 + y^{2n}} \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

25. Since, $\sqrt{ab} = 16$ and $\frac{2ab}{a+b} = 12\frac{4}{5}$

$\therefore \quad 2 \times \frac{256}{a+b} = \frac{64}{5} \Rightarrow a + b = 40 = 8 + 32$

$\therefore \quad \frac{a}{b} = \frac{1}{4}$

26. Since, $\frac{a+b}{2} + \sqrt{ab} = a - b \Rightarrow 2\sqrt{ab} = a - 3b$

$\Rightarrow \quad 4ab = a^2 - 6ab + 9b^2$

$\Rightarrow \quad a^2 - 10ab + 9b^2 = 0$

$\Rightarrow \quad (a - 9b)(a - b) = 0$

$\therefore \quad a = 9b \Rightarrow a : b = 9 : 1$ $(\because a \neq b)$

27. **Statement I** We have, $a \cdot ar \dots ar^{n-1} = a^n r^{1+2+\dots+(n-1)}$

$= a^n r^{\frac{n(n-1)}{2}} = (a^2 r^{n-1})^{n/2}$

Statement II Also, $(a \cdot r^{ik-1})(a \cdot r^{n-ik}) = a^2 r^{n-1}$

which is independent of k.

28. **Statement I** $0.3 + 0.03 + 0.003 + \ldots = 3\,[0.1 + 0.01 + 0.001 + \ldots]$

$$= \frac{3}{9}[0.9 + 0.09 + 0.009 + \ldots]$$

$$= \frac{1}{3}[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \ldots]$$

$$= \frac{1}{3}[n - \{0.1 + 0.01 + 0.001 + \ldots\}]$$

$$= \frac{1}{3}\left[n - \frac{0.1\{1 - (0.1)^n\}}{0.9}\right]$$

$$= \frac{n}{3} - \frac{1}{27}\{1 - (0.1)^n\}$$

Statement II Also, $a_n = a + nd$

$$\therefore a_1 + a_2 + \ldots + a_n = a + d + a + 2d + \ldots + a + nd$$

$$= na + \frac{n(n+1)}{2}d$$

$$= \frac{n}{2}[2a + (n+1)d]$$

29. $\therefore$ The sum of all of the squares

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \ldots = \frac{a^2}{1 - \frac{1}{2}} = 2a^2$$

30. $S_n = \log a^2\left[\frac{1}{\frac{1}{2}\log 3} + \frac{1}{\frac{1}{3}\log 3} + \frac{1}{\frac{1}{4}\log 3} + \ldots \text{upto 8th term}\right]$

$$\Rightarrow \quad \frac{\log a^2}{\log 3}[2 + 3 + 4 + \ldots 9] = 44 \qquad \text{(given)}$$

$$\Rightarrow \quad 44\log a^2 = 44\log 3$$

$$\therefore \quad a = \pm\sqrt{3}$$

31. The area of an equilateral triangle is $\frac{a^2\sqrt{3}}{4}$, where a is side.

The side of the subsequent triangle will be $\frac{a}{2}, \frac{a}{4}, \ldots, \frac{a}{8}, \frac{a}{16}$.

So, area of fourth triangle is $\frac{a^2}{256}\frac{\sqrt{3}}{4}$.

$\therefore$ Required ratio $= \frac{S_1}{S_4} = \frac{a^2\sqrt{3}}{4} \times \frac{256 \times 4}{a^2\sqrt{3}} = 256 : 1$

32. Since, $S_\infty = \frac{x}{1-r} = 5 \Rightarrow r = \frac{5-x}{5}$

For infinite GP, $|r| < 1$

$$\Rightarrow \quad -1 < \frac{5-x}{5} < 1 \Rightarrow -10 < -x < 0$$

$$\therefore \quad 0 < x < 10$$

33. Given that, $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + \ldots + a_q} = \frac{p^3}{q^3}$

$$\Rightarrow \quad \frac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_2 + (q-1)d]} = \frac{p^3}{q^3}$$

where, d be a common difference of an AP.

$$\Rightarrow \quad \frac{(2a_1 - d) + pd}{(2a_1 - d) + qd} = \frac{p^2}{q^2}$$

34. nth term of the series is, $T_n = \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)}$

$$\Rightarrow \quad T_n = 2\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$$\Rightarrow \quad T_1 = 2\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right), T_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$\ldots T_{10} = 2\left(\frac{1}{10} - \frac{1}{11}\right)$$

$$\therefore \quad S_{10} = T_1 + T_2 + \ldots + T_{20}$$

$$S_{10} = 2\left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{10} - \frac{1}{11}\right]$$

$$= 2\left(1 - \frac{1}{11}\right) = 2 \cdot \frac{10}{11} = \frac{20}{11}.$$

35. Here, $T_1 = S_1 = 2(1) + 3(1)^2 = 5$

$$T_2 = S_2 - S_1 = 16 - 5 = 11$$

$$T_3 = S_3 - S_2 = 33 - 16 = 17$$

$\therefore$ Sequence 5, 11, 17

$$a = 5, d = 6$$

For new AP

$$A = 5, D = 2 \times 6 = 12$$

$$S'_n = \frac{n}{2}[2 \times 5 + (n-1)12] = 6n^2 - n$$

36. $T_n = \frac{2n+1}{(1^2 + 2^2 + \ldots + n^2)} = \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$

$$= 6\left(\frac{1}{n} - \frac{1}{(n+1)}\right)$$

$$T_1 = 6\left(\frac{1}{1} - \frac{1}{2}\right), T_2 = 6\left[\frac{1}{2} - \frac{1}{3}\right] \ldots T_{11} = 6\left[\frac{1}{11} - \frac{1}{12}\right]$$

$$\therefore \quad S = 6\left[\frac{1}{1} - \frac{1}{12}\right] = \frac{6 \times 11}{12} = \frac{11}{2}$$

37. Series $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$

$$= 4\{1 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + \ldots\}$$

$$T_n = 4n \cdot n^2$$

$$S_n = \Sigma T_n = 4\Sigma n^2 = 4\left[\frac{n(n+1)}{2}\right]^2$$

$$\Rightarrow \quad S_{10} = [10 \cdot (10+1)]^2 = (110)^2 = 12100$$

38. Let 'a' and 'd' be the first term and common difference of on AP, respectively.

$$\because \quad a_4 - a_7 + a_{10} = m$$

$$\Rightarrow \quad (a + 3d) - (a + 6d) + (a + 9d) = m$$

$$\Rightarrow \quad a + 6d = m \qquad \ldots(i)$$

$$\because \quad S_{13} = \frac{13}{2}[2a + 12d] = 13(a + 6d) = 13m$$

39. Let the numbers $\frac{a}{r}, a, ar, 2ar-a$ ...(i)

Given, common difference of an AP is 6.

$\therefore \quad ar - a = 6$ ...(ii)

Also, $\frac{a}{r} = 2ar - a \Rightarrow \frac{a}{r} = 2(ar-a) + a$

$\Rightarrow \frac{a}{r} - a = 12 \Rightarrow a(1-r) = 12r$

$\therefore \quad r = -\frac{1}{2}$

From Eq. (ii), we get $a\left[\left(-\frac{1}{2}\right)-1\right] = 6 \Rightarrow a = -4$

$\therefore$ Required series are 8, – 4, 2, 8

40. $T_n = (2n-1)^2 = 4n^2 + 1 - 4n$

$$S_n = \sum_{n=1}^{13} (4n^2 + 1 - 4n)$$

$$= 4\left[\frac{13(13+1)(26+1)}{6}\right] + 13 - \frac{4\times 13\times 14}{2}$$

$$= 4(819) + 13 - 364 = 2925$$

41. Let $S = 0.7 + 0.77 + 0.777 + \ldots = \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \ldots +$ upto 20 term

$$= 7\left[\frac{1}{10} + \frac{11}{10^2} + \frac{111}{10^3} + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{10^2}\right) + \left(1-\frac{1}{10^3}\right) + \ldots + \text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[(1+1+\ldots+\text{upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots + \text{upto 10 terms}\right)\right]$$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^{20}\right\}}{1-\frac{1}{10}}\right]$$

$$\left[\because \sum_{i=1}^{\infty} = 20 \text{ and sum of } n \text{ terms of GP, } S_n = \frac{a(1-r^n)}{1-r}, \text{ where } r < 1\right]$$

$$= \frac{7}{9}\left[20 - \frac{1}{9}\left\{1-\left(\frac{1}{10}\right)^{20}\right\}\right]$$

$$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + 10)^{-20}]$$

42. Statement I

$S = (1) + (1+2+4) + (4+6+9) + (9+12+16) + \ldots + (361+380+400)$

$S = (0+0+1) + (1+2+4) + (4+6+9) + (9+12+16) + \ldots + (361+380+400)$

Now, we can clearly observe the first elements in each bracket.

In second bracket, the first element is $1 = 1^2$

In third bracket, the first element is $4 = 2^2$

In fourth bracket, the first element is $9 = 3^2$

... ...

In last bracket, the first element is $361 = 19^2$

Hence, we can conclude that there are 20 brackets in all.

Also, in each of the brackets there are 3 terms out of which the first and last terms are perfect squares of consecutive integers and the middle term is their product.

$\therefore$ The general term of the series is $T_r = (r-1)^2 + (r-1)r + (r^2)$

The sum of the n terms of the series is

$$S_n = \sum_{r=1}^{n} [(r-1)^2 + (r-1)r + (r)^2]$$

$$S_n = \sum_{r=1}^{n} \left(\frac{r^3 - (r-1)^3}{r-(r-1)}\right)$$

$$[\because (a^3 - b)^3 = (a-b)(a^2 + ab + b^2)]$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} (r^3 - (r-1)^3)$$

$$\therefore \quad S_n = (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \ldots + (n^3 - (n-1)^3)$$

Rearranging the terms, we get

$$S_n = -0^3 + (1^3 - 1^3) + (2^3 - 2^3) + (3^3 - 3^3) + \ldots + [(n-1)^3 - (n-1)^3] + n^3$$

$$S_n = n^3$$

Since, the number of terms is 20, hence substituting $n = 20$, we get $S_{20} = 8000$

Hence, Statement I is correct.

Statement II We have, already proved in the Statement I that

$$S_n = \sum_{r=1}^{n} (r^3 - (r-1)^3) = n^3$$

Hence, Statement II is also correct and it is a correct explanation of Statement I.

43. Let a be the first term and d ($d \neq 0$) be the common difference of a given AP, then

$$T_{100} = a + (100-1)d = a + 99d$$
$$T_{50} = a + (50-1)d = a + 49d$$
$$T_{150} = a + (150-1)d = a + 149d$$

Now, according to the given condition

$$100 \times T_{100} = 50 \times T_{50}$$
$$\Rightarrow \quad 100(a + 99d) = 50(a + 49d)$$
$$\Rightarrow \quad 2(a + 99d) = (a + 49d)$$
$$\Rightarrow \quad 2a + 198d = a + 49d$$
$$\Rightarrow \quad a + 149d = 0$$
$$\Rightarrow \quad T_{150} = 0$$

44. Let the time taken to save ₹ 11040 be $(n+3)$ months.
For first 3 month he saves ₹ 200 each month.
In $(n+3)$ months,

$$3\times200+\frac{n}{2}\{2(240)+(n-1)\times40\}=11040$$

$$\Rightarrow\quad 600+\frac{n}{2}\{40(12+n-1)\}=11040$$

$$\Rightarrow\quad 600+20n(n+11)=11040$$

$$\Rightarrow\quad 30+n^2+11n=552$$

$$\Rightarrow\quad n^2+11n-522=0$$

$$\Rightarrow\quad n^2+29n-18n-522=0$$

$$\Rightarrow\quad n(n+29)-(n+29)=0$$

$$\Rightarrow\quad (n-18)(n+29)=0$$

$$\therefore\quad n=18,\text{ neglecting } n=-29$$

$\therefore$ Total time $=(n+3)=21$ months

45. Number of notes that the person counts in 10 min

$$=10\times150=1500$$

Since, $a_{10}, a_{11}, a_{12},\ldots$ are in AP with common difference -2.
Let n be the time taken to count remaining 3000 notes.

Then, $\frac{n}{2}[2\times148+(n-1)\times-2]=3000$

$$\Rightarrow\quad n^2-149n+3000=0$$

$$\Rightarrow\quad (n-24)(n-125)=0$$

$$\therefore\quad n=24\text{ and }125$$

Then, the total time taken by the person to count all notes

$$=10+24=34\text{ min}$$

46. Let $S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$

$$\Rightarrow\quad S-1=\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots\quad\ldots\text{(i)}$$

$$\Rightarrow\quad \frac{S-1}{3}=\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\frac{14}{3^5}+\ldots\quad\ldots\text{(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}(S-1)=\frac{2}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots$$

$$\Rightarrow\quad S-1=1+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\ldots$$

$$\Rightarrow\quad S=2+\frac{\frac{2}{3}}{1-\frac{1}{3}}=2+1=3$$

47. Since, $a+ar=a(1+r)=12\quad\ldots\text{(i)}$
and $ar^2+ar^3=ar^2(1+r)=48\quad\ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$$r^2=4$$

$$\therefore\quad r=-2$$

(since, the series is alternately sign, so we take negative values)
On putting the value of r in Eq. (i), we get

$$\therefore\quad a=-12$$

48. Since, $ar^{n-1}=ar^n+ar^{n+1}$

$$\Rightarrow\quad \frac{1}{r}=1+r$$

$$\Rightarrow\quad r^2+r-1=0$$

$$\therefore\quad r=\frac{-1\pm\sqrt{1+4}}{2}$$

$$\Rightarrow\quad r=\frac{\sqrt5-1}{2}\quad\text{(since, } r \text{ cannot be negative)}$$

49. Since, $\frac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\frac{p^2}{q^2}$

$$\Rightarrow\quad \frac{(2a_1-d)+pd}{(2a_1-d)+qd}=\frac{p}{q}$$

$$\Rightarrow\quad (2a_1-d)(p-q)=0\quad(\because p=q)$$

$$\therefore\quad a_1=\frac{d}{2}$$

Now, $\frac{a_6}{a_{21}}=\frac{a_1+5d}{a_1+20d}=\frac{\frac{d}{2}+5d}{\frac{d}{2}+20d}=\frac{11}{41}$

50. Here, $x=\frac{1}{1-a}, y=\frac{1}{1-b}$ and $z=\frac{1}{1-c}$

$$\Rightarrow\quad 1-a=\frac{1}{x}, 1-b=\frac{1}{y}, 1-c=\frac{1}{z}$$

$$\Rightarrow\quad a=1+\frac{1}{x}, b=1+\frac{1}{y}, c=1+\frac{1}{z}$$

Since, a, b and c are in AP.

$\therefore$ $1+\frac{1}{x}, 1+\frac{1}{y}, 1+\frac{1}{z}$ are in AP.

$\Rightarrow$ $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ are in AP.

Hence, x, y and z are in HP.

51. Let $S=1^2+2\cdot2^2+3^2+2\cdot4^2+\ldots+2(n-1)^2+n^2$,
when n is odd, the last term *i.e.*, the nth term will be n^2 in this case $n-1$ is even and so the sum of the first $n-1$ terms of the series is obtained by replacing n by $n-1$ in the given formula

$$\therefore\quad \frac{n}{2}(n+1)^2=\frac{(n-1)}{2}(n-1+1)^2=\frac{(n-1)n^2}{2}$$

$\therefore$ Required sum = [Sum of $(n-1)$ terms] + nth term

$$=\frac{1}{2}(n-1)n^2+n^2=\frac{1}{2}(n+1)n^2$$

Day 4

Quadratic Equation and Inequalities

Day 4 Outlines ...

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, a, b and c, $x \in R$, is called a **real quadratic equation**. The numbers a, b and c are called the **coefficients** of the equation. The quantity $D = b^2 - 4ac$ is known as the **discriminant** of the equation $ax^2 + bx + c = 0$ and its roots are given by $x = \frac{-b \pm \sqrt{D}}{2a}$

An equation of the form $az^2 + bz + c = 0$, where $a \neq 0$, a, b, c, and z, $\notin C$ (complex) is called a **complex quadratic equation** and its roots are given by $z = \frac{-b \pm \sqrt{D}}{2a}$.

Nature of Roots of Quadratic Equation

Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ implies whether the roots are real or imaginary by analysing the quantity D.

1. Let $a, b, c \in R$ and $a \neq 0$, then the equation $ax^2 + bx + c = 0$
   (i) has real and distinct roots if and only if $D > 0$
   (ii) has real and equal roots if and only if $D = 0$
   (iii) has complex roots with non-zero imaginary parts if and only if $D < 0$. Since, if $p + iq$ (where, $p, q \in R$, $q \neq 0$) is one root of $ax^2 + bx + c = 0$, then second root will be $p - iq$
2. If $a, b, c \in Q$ and D is a perfect square, then $ax^2 + bx + c = 0$ has **rational roots.**

3. If $a, b, c \in Q$ and $p+\sqrt{q}\,(p, q \in Q)$ is an **irrational root** of $ax^2+bx+c=0$, then other root will be $p-\sqrt{q}$.
4. If $a=1$, b, $c \in I$ and roots of $ax^2+bx+c=0$ are rational numbers, then these roots must be **integers**.
5. If $ax^2+bx+c=0$ is satisfied by more than two distinct complex numbers, then it becomes an identity *i.e.*, $a=b=c=0$.
6. If the roots of $ax^2+bx+c=0$ are both positive, then the signs of a and c should be like and opposite to the sign of b.
7. If the roots of $ax^2+bx+c=0$ are both negative, then signs of a,b and c should be like.
8. If the roots of $ax^2+bx+c=0$ are equal in magnitude but opposite in sign, then $b=0$ and $c<0$.
9. If the roots of $ax^2+bx+c=0$ are reciprocal to each other, then $c=a$.
10. In the equation $ax^2+bx+c=0$ $(a,b,c \in R)$. If $a+b+c=0$, then the roots are $1, \frac{c}{a}$ and if $a-b+c=0$, then the roots are -1 and $-\frac{c}{a}$.

- *If a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.*
- *A quadratic equation cannot have more than two roots.*

Relation between Roots and Coefficients

1. Quadratic Roots

If α and β are the roots of quadratic equation $ax^2+bx+c=0$, then Sum of roots $=\alpha+\beta=-\frac{b}{a}$ and product of roots $=\alpha\beta=\frac{c}{a}$

2. Cubic Roots

If α, β and γ are the roots of cubic equation $ax^3+bx^2+cx+d=0; a \neq 0$, then

$$\alpha+\beta+\gamma=-\frac{b}{a}$$

$$\beta\gamma+\gamma\alpha+\alpha\beta=\frac{c}{a} \quad \text{and} \quad \alpha\beta\gamma=-\frac{d}{a}$$

3. Biquadratic Roots

Also, if α, β, γ and δ are the roots of biquadratic equation $ax^4+bx^3+cx^2+dx+e=0; a \neq 0$, then

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$$

$$(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{c}{a}$$

$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-\frac{d}{a}$$

and $$\alpha\beta\gamma\delta=\frac{e}{a}$$

4. Symmetric Roots

If α, β are roots of quadratic equation $ax^2+bx+c=0$, then to find the symmetric function of α and β use the following results.

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

(ii) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

(iii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

(iv) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

5. Common Roots (Conditions)

Suppose that the quadratic equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$.

(i) When **one root** is common, then the condition is $(a'c-ac')^2=(bc'-b'c)(ab'-a'b)$.

(ii) When **both roots** are common, then the condition is $\frac{a}{a'}=\frac{b}{b'}$.

Position of Roots

Let $ax^2+bx+c=0$ has roots α and β

(i) with respect to one real number (k).

S.No.	Situation	Required Conditions
(a)	$\alpha<\beta<k$	$D \geq 0$, $af(k)>0$, $k>-\frac{b}{2a}$
(b)	$k<\alpha<\beta$	$D \geq 0$, $af(k)>0$, $k<-\frac{b}{2a}$
(c)	$\alpha<k<\beta$	$D \geq 0$, $af(k)<0$

(ii) with respect to two real numbers k_1 and k_2.

S.No.	Situation	Required Conditions
(a)	$k_1<\alpha<\beta<k_2$	$D \geq 0$, $af(k_1)>0$, $af(k_2)>0$, $k_1<-\frac{b}{2a}<k_2$
(b)	$\alpha<k_1<k_2<\beta$	$D \geq 0$, $af(k_1)<0$, $af(k_2)<0$
(c)	$k_1<\alpha<k_2<\beta$	$D \geq 0$, $f(k_1)f(k_2)<0$

Formation of an Equation

1. **Quadratic Equation**

If the roots of a quadratic equation are α and β, then the equation will be form of

$$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

2. **Cubic Equation**

If α, β and γ are the roots of the cubic equation, then the equation will be form of

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)\,x-\alpha\beta\gamma=0.$$

3. **Biquadratic Equation**

If α, β, γ and δ are the roots of the biquadratic equation, then the equation will be form of

$$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\delta+\beta\delta+\gamma\delta)x^2$$
$$-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\,x+\alpha\beta\gamma\delta=0.$$

Sign of Quadratic Equation

Let $f(x)=ax^2+bx+c$ or $y=ax^2+bx+c$, where a,b and $c\in R$ and $a\neq 0$.

(i) If $a>0$ and $D<0$, then $f(x)>0, \forall x\in R$.

(ii) If $a<0$ and $D<0$, then $f(x)<0, \forall x\in R$.

(iii) If $a>0$ and $D=0$, then $f(x)\geq 0, \forall x\in R$.

(iv) If $a<0$ and $D=0$, then $f(x)\leq 0, \forall x\in R$.

(v) If $a>0$, $D>0$ and $f(x)=0$ have two real roots α and β, where $(\alpha<\beta)$, then $f(x)>0$, $\forall\ x\in(-\infty,\alpha)\cup(\beta,\infty)$ and $f(x)<0$, $\forall\ x\in(\alpha,\beta)$.

(vi) If $a<0$, $D>0$ and $f(x)=0$ have two real roots α and β, where $(\alpha<\beta)$, then $f(x)<0$, $\forall\ x\in(-\infty,\alpha)\cup(\beta,\infty)$ and $f(x)>0$, $\forall\ x\in(\alpha,\beta)$.

Condition for Formation of an Equation

Let equation be

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0 \quad \ldots(A)$$

Then, to form an equation

- whose roots are $k\ (\neq 0)$ times roots of the eq. (A), replace x by $\frac{x}{k}$ in eq. (A).
- whose roots are the negatives of the roots of eq. (A), replace x by $-x$ in eq. (A). Alternatively, change the sign of the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}, \ldots$ etc., in eq. (A).
- whose roots are k more than the roots of eq. (A), replace x by $(x-k)$ in eq. (A).
- whose roots are reciprocals of the roots of eq. (A), replace x by $\frac{1}{x}$ in eq. (A) and then multiply both the sides by x^n.

Inequalities

Let a and b are two real numbers. If $a-b$ is negative, we say that a is less than b ($a<b$) and if $a-b$ is positive, then a is greater than b $(a>b)$. This shows the inequalities concept.

Important Results on Inequalities

1. If $a>b$ and $b>c$, then $a>c$. Generally, if $a_1>a_2$, $a_2>a_3$, …, $a_{n-1}>a_n$, then $a_1>a_n$.
2. If $a>b$, then $a\pm c>b\pm c, \forall c\in R$.
3. If $a>b$, then
   (i) for $m>0$, $am>bm, \frac{a}{m}>\frac{b}{m}$
   (ii) and for $m<0$, $am<bm, \frac{a}{m}<\frac{b}{m}$
4. (i) If $a>b>0$, then
   (a) $a^2>b^2$ (b) $|a|>|b|$ (c) $\frac{1}{a}<\frac{1}{b}$
   (ii) If $a<b<0$, then
   (a) $a^2>b^2$ (b) $|a|>|b|$ (c) $\frac{1}{a}>\frac{1}{b}$
5. If $a<0<b$, then
   (i) $a^2>b^2$, if $|a|>|b|$ (ii) $a^2<b^2$, if $|a|<|b|$
6. If $a<x<b$ and a,b are positive real numbers, then $a^2<x^2<b^2$.
7. If $a<x<b$ and a is negative number and b is positive number, then
   (i) $0\leq x^2<b^2$, if $|b|>|a|$ (ii) $0\leq x^2\leq a^2$, if $|a|>|b|$

7. If $a < x < b$ and a is negative number and b is positive number, then

(i) $0 \le x^2 < b^2$, if $|b| > |a|$ (ii) $0 \le x^2 \le a^2$, if $|a| > |b|$

8. If $\frac{a}{b} > 0$, then

(i) $a > 0$, if $b > 0$ (ii) $a < 0$, if $b < 0$

9. If $a_i > b_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 a_2 a_3 \ldots a_n > b_1 b_2 b_3 \ldots b_n.$$

10. If $|x| < a$, then

(i) if a is positive, then $-a < x < a$.

(ii) if a is negative, then $x \in \phi$.

11. If $a_i > b_i$, where $i = 1, 2, 3, \ldots, n$, then

$$a_1 + a_2 + a_3 + \ldots + a_n > b_1 + b_2 + \ldots + b_n.$$

12. If $0 < a < 1$ and n is a positive rational number, then

(i) $0 < a^n < 1$ (ii) $a^{-n} > 1$

Arithmetico-Geometric Mean Inequality

The Arithmetico-Geometric Mean (AGM) of two positive real numbers a and b is defined as follows

(i) If $a, b > 0$ and $a \ne b$, then $\frac{a+b}{2} \ge \sqrt{ab} \ge \frac{2}{\frac{1}{a} + \frac{1}{b}}$,

i.e., Arithmetic Mean ≥ Geometric Mean ≥ Harmonic Mean

(ii) If $a_i > 0$, where $i = 1, 2, 3, \ldots, n$, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge (a_1 \cdot a_2 \ldots a_n)^{1/n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

Logarithm Inequality

If a is a positive real number other than 1 and $a^x = m$, then x is called the logarithm of m to the base a, written as $\log_a m$.

In $\log_a m$, m should be always positive.

(i) If $m < 0$, then $\log_a m$ will be imaginary and if $m = 0$, then $\log_a m$ will be meaningless.

(ii) $\log_a m$ exists, if $m, a > 0$ and $a \ne 1$.

Important Results on Logarithm

1. $a^{\log_a x} = x; a \ne 0, \ne 1, x > 0$

2. $a^{\log_b x} = x^{\log_b a}; a, b > 0, \ne 1, x > 0$

3. $\log_a a = 1, a > 0, \ne 1$

4. $\log_a x = \frac{1}{\log_x a}; x, a > 0, \ne 1$

5. $\log_a x = \log_a b \log_b x = \frac{\log_b x}{\log_b a}; a, b > 0, \ne 1, x > 0$

6. For $m, n > 0$ and $a > 0, \ne 1$, then

(i) $\log_a (m \cdot n) = \log_a m + \log_a n$

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a (m^n) = n \log_a (m)$

7. For $x > 0, a > 0, \ne 1$

(i) $\log_{a^n} (x) = \frac{1}{n} \log_a x$

(ii) $\log_{a^n} x^m = \left(\frac{m}{n}\right) \log_a x$

8. For $x > y > 0$

(i) $\log_a x > \log_a y$, if $a > 1$

(ii) $\log_a x < \log_a y$, if $0 < a < 1$

9. If $a > 1$ and $x > 0$, then

(i) $\log_a x > p \Rightarrow x > a^p$

(ii) $0 < \log_a x < p \Rightarrow 0 < x < a^p$

10. If $0 < a < 1$, then

(i) $\log_a x > p \Rightarrow 0 < x < a^p$

(ii) $0 < \log_a x < p \Rightarrow a^p < x < 1$

Application of Inequalities to find the Greatest and Least Values

- If $x_1, x_2, \ldots, x_n$ are n positive variables such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then the product $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is greatest when $x_1 = x_2 = \ldots = x_n = \frac{c}{n}$ and the greatest value is $\left(\frac{c}{n}\right)^n$.
- If $x_1, x_2, \ldots, x_n$ are positive variables such that $x_1, x_2, \ldots, x_n = c$ (constant), then the sum $x_1 + x_2 + \ldots + x_n$ is least when $x_1 = x_2 = \ldots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.
- If $x_1, x_2, \ldots, x_n$ are variables and $m_1, m_2, \ldots, m_n$ are positive real number such that $x_1 + x_2 + \ldots + x_n = c$ (constant), then $x_1^{m_1} \cdot x_2^{m_2} \cdot \ldots \cdot x_n^{m_n}$ is greatest, when

$$\frac{x_1}{m_1} = \frac{x_2}{m_2} = \ldots = \frac{x_n}{m_n} = \frac{x_1 + x_2 + \ldots + x_n}{m_1 + m_2 + \ldots + m_n}$$

Practice Zone DAY 4

1. The equation $(\cos\beta - 1)x^2 + (\cos\beta)x + \sin\beta = 0$ in the variable x has real roots, then β is in the interval

(a) $(0, 2\pi)$ (b) $(-\pi, 0)$
(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $(0, \pi)$

2. If $a + b + c = 0$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$, where $a, b, c \in R$ are

(a) real and distinct (b) imaginary
(c) real and equal (d) infinite

3. For the complex number $z = x + iy$, the number of solutions of the equation $z^2 + |z|^2 = 0$ is **[NCERT Exemplar]**

(a) 1 (b) 2
(c) 3 (d) infinitely many

4. If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root, then the ordered pair (a, b) is

(a) $(-6, -7)$ (b) $(-7, -8)$
(c) $(-6, -8)$ (d) $(-8, -7)$

5. If the equation $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ have both roots common, then $2r - p$ is equal to

(a) 2 (b) 1
(c) 0 (d) k

6. If the ratio of the roots of $\lambda x^2 + \mu x + \nu = 0$ is equal to the ratio of the roots of $x^2 + x + 1 = 0$, then λ, μ and ν are in

(a) AP (b) GP
(c) HP (d) None of these

7. The solution set of $\frac{|x-2|-1}{|x-2|-2} \le 0$ is **[NCERT Exemplar]**

(a) $[0, 1] \cup (3, 4)$ (b) $[0, 1] \cup [3, 4]$
(c) $[-1, 1) \cup (3, 4]$ (d) None of these

8. If $ax^2 + 2bx - 3c = 0$ has no real root and $\frac{3c}{4} < a + b$, then the range of c is

(a) $(-1, 1)$ (b) $(0, 1)$
(c) $(0, \infty)$ (d) $(-\infty, 0)$

9. If a, b and c are real numbers in AP, then the roots of $ax^2 + bx + c = 0$ are real for

(a) all a and c (b) no a and c
(c) $\left|\frac{c}{a} - 7\right| \ge 4\sqrt{3}$ (d) $\left|\frac{a}{c} + 7\right| \ge 2\sqrt{3}$

10. If $x^2 - 5x + 1 = 0$, then $\frac{x^{10} + 1}{x^5}$ is equal to

(a) 2424 (b) 3232
(c) 2525 (d) None of these

11. If a, b and c are in AP and if the equations $(b - c)x^2 + (c - a)x + (a - b) = 0$ and $2(c + a)x^2 + (b + c)x = 0$ have a common root, then

(a) a^2, b^2 and c^2 are in AP (b) a^2, c^2 and b^2 are in AP
(c) c^2, a^2 and b^2 are in AP (d) None of these

12. If a, b and c are distinct real numbers, then the expression
$$f(x) = a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}$$
is identically equal to

(a) $x^2 - (a + b + c)x + abc$ (b) $x^2 + x - abc$
(c) x^2 (d) None of these

13. If $x \in R$ and $b < c$, then $\frac{x^2 - bc}{2x - b - c}$ has no value

(a) in $(-\infty, b)$ (b) in (c, ∞)
(c) between b and c (d) between $-c$ and $-b$

14. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots is

(a) $-2(p^2 + q^2)$ (b) $-(p^2 + q^2)$
(c) $\frac{-(p^2 + q^2)}{2}$ (d) $-pq$

15. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lie in the interval

(a) $[1, 2]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left[-\frac{1}{2}, 1\right]$ (d) $[0, 1]$

16. The solution set of the inequality $x+1>\sqrt{x+3}$ is
(a) $\{x:-1<x\le -3\}$ (b) $\{x:x>1\}$
(c) $\{x:-3\le x\le -2\}$ (d) None of these

17. If x, y and z are three positive real numbers, then the minimum value of $\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}$ is
(a) 6 (b) 3 (c) 2 (d) 1

18. If α and β be the roots of $x^2+x+2=0$ and γ,δ be the roots of $x^2+3x+4=0$, then $(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)$ is equal to
(a) −18 (b) 18
(c) 24 (d) 44

19. If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+dx+c=0$; $ac\ne 0$, then the equation $P(x)\cdot Q(x)=0$ has
(a) four real roots (b) exactly two real roots
(c) either two or four real roots (d) atmost two real roots

20. The product of all the values of x satisfying the equation $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$ is
(a) 4 (b) 6 (c) 8 (d) 19

21. In $\Delta PQR, R=\frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2+bx+c=0$, then
(a) $a=b+c$ (b) $b=c+a$ (c) $c=a+b$ (d) $b=c$

22. If the roots of the equation $(a+1)x^2-3ax+4a=0$ $(a\ne -1)$ be greater than unity, then the values of a are
(a) $\left[-\frac{16}{7},-1\right)$ (b) $[0,-1]$
(c) $\left[-\frac{16}{7},1\right]$ (d) None of these

23. The solution set of $1\le|x-2|\le 3$ is **[NCERT Exemplar]**
(a) $(-1,1)\cup(3,5]$ (b) $[-1,1]\cup[3,5]$
(c) $[-1,1]\cup[3,5)$ (d) None of these

24. If $y=3^{x-1}+3^{-x-1}$ (x is real), then the least value of y is
(a) 2 (b) 6
(c) 2/3 (d) None of these

25. If a, b and c be the lengths of the sides of a triangle, then $(b+c-a)(c+a-b)(a+b-c)$ is less than or equal to
(a) bc (b) $\frac{1}{abc}$
(c) ca (d) abc

26. If $(\log_5 x)^2+\log_5 x<2$, then x belongs to
(a) $\left(\frac{1}{25},5\right)$ (b) $\left(\frac{1}{5},\frac{1}{\sqrt{5}}\right)$
(c) $(1,\infty)$ (d) None of these

27. What is the solution set of the following inequality?
$$\log_x\left(\frac{x+5}{1-3x}\right)>0$$
(a) $0<x<\frac{1}{3}$ (b) $x\ge 3$
(c) $\frac{1}{3}<x<1$ (d) None of these

28. If $\log_5 2, \log_5(2^x-5)$ and $\log_5\left(2^x-\frac{7}{2}\right)$ are in AP, then x is equal to
(a) $\frac{1}{2},\frac{3}{2}$ (b) 3
(c) 4, 5 (d) 8

29. If $\log_{0.3}(x-1)>\log_{0.09}(x-1)$, then x lies in
(a) $(1,2)$ (b) $(-\infty,1)$
(c) $(2,\infty)$ (d) None of these

30. The set of all real number's x for which $x^2-|x+2|+x>0$ is
(a) $(-\infty,-2)\cup(2,\infty)$ (b) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$
(c) $(-\infty,-1)\cup(1,\infty)$ (d) $(\sqrt{2},\infty)$

Directions (Q. Nos. 31 and 32) *Let $f(x)=x^2+b_1x+c_1$, $g(x)=x^2+b_2x+c_2$, real roots of $f(x)=0$ be α,β and real roots of $g(x)=0$ be $\alpha+\delta,\beta+\delta$.*

Also, assume that the least value of $f(x)$ be $-\frac{1}{4}$ and the least value of $g(x)$ occurs at $x=\frac{7}{2}$.

31. The least value of $g(x)$ is
(a) −1 (b) $-\frac{1}{2}$
(c) $-\frac{1}{4}$ (d) $-\frac{1}{3}$

32. The value of b_2 is
(a) 0 (b) −7 (c) 6 (d) 8

Directions (Q. Nos. 33 to 36) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

33. **Statement I** Let $f(x)$ be quadratic expression such that $f(0)+f(1)=0$. If −2 is one of the root of $f(x)=0$, then other root is 3/5.
Statement II If α and β are the zero's of $f(x)=ax^2+bx+c$, then sum of zero's $=-b/a$, product of zero's $=c/a$.

34. Suppose both roots of $f(x) = ax^2 + bx + c$ are in (k_1, k_2), then $D \geq 0$, $f(k_1) > 0$ and $f(k_2) > 0$.

Statement I If both roots of the equation $2x^2 - x + a = 0\ (a \in R)$ lies in $(1, 2)$, then $-1 < a \leq 1/8$.

Statement II If $f(x) = 2x^2 - x + a$, then $D \geq 0, f(1) > 0, f(2) > 0$ yield $-1 < a \leq 1/8$.

35. Suppose if α is any roots of the equation, then it will satisfy the equation.

Statement I If $\cos^2 \frac{\pi}{8}$ is a root of the equation $x^2 + ax + b = 0$, where $a, b \in R$ rational number, then ordered pair (a, b) is $(1, 1/8)$.

Statement II If $a + mb = 0$ and m is a irrational, then $a, b = 0$.

36. Consider $\log_a x$ is defined as $a > 0, a \neq 1$ and $x > 0$ and also $\log_{a^{-2}} x = -\frac{1}{2}\log_a x$

Statement I The equation $\log_{\frac{1}{2+|x|}}(5 + x^2) = \log_{(3 + x^2)}(15 + \sqrt{x})$ has no solution.

Statement II $\log a^{2m} = 2m \log |a|, \forall\, a > 0, m \in N$

37. If one root of the equation $x^2 - \lambda x + 12 = 0$ is even prime while $x^2 + \lambda x + \mu = 0$ has equal roots, then μ is equal to

(a) 8 (b) 16
(c) 24 (d) 32

38. If λ be an integer and α, β be the roots of $4x^2 - 16x + \frac{\lambda}{4} = 0$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then the possible values of λ are

(a) {60, 64, 68} (b) {61, 62, 63}
(c) {49, 50,..., 62, 63} (d) {62, 65, 68, 71, 75}

39. If $S = \{a \in N, 1 \leq a \leq 100\}$ and $[\tan^2 x] - \tan x - a = 0$ has real roots, where [.] denotes the greatest integer function, then number of elements in set S equals to

(a) 2 (b) 5
(c) 6 (d) 9

40. The roots of the equation $2^{x+2} \cdot 3^{3x/(x-1)} = 9$ are given by

(a) $1 - \log_2 3, 2$ (b) $\log_2\left(\frac{2}{3}\right), 1$
(c) $2, -2$ (d) $-2, 1 - \frac{(\log 3)}{(\log 2)}$

41. If α and β are roots of $375x^2 - 25x - 2 = 0$ and $S_n = \alpha^n + \beta^n$, then $\lim_{n \to \infty} \sum_{r=1}^{n} S_r$ is equal to

(a) $\frac{7}{116}$ (b) $\frac{1}{12}$ (c) $\frac{29}{358}$ (d) None of these

42. Let a, b and c be the sides of a scalene triangle. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0, \lambda \in R$ are real, then

(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$ (c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

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43. The values of a for which one root of the equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is lesser than 2, are given by **[JEE Main 2013]**

(a) $3 < a < 10$ (b) $a \geq 10$
(c) $-2 < a < 3$ (d) $a \leq -2$

44. If α and β are roots of the equation $x^2 + px + 3 \cdot \frac{p}{4} = 0$, such that $|\alpha - \beta| = \sqrt{10}$, then p belongs to **[JEE Main 2013]**

(a) {2, – 5} (b) {– 3, 2}
(c) {– 2, 5} (d) {3, – 5}

45. The least integral value α of x such that $\frac{x - 5}{x^2 + 5x - 14} > 0$, satisfies **[JEE Main 2013]**

(a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$
(c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$

46. If p and q are non-zero real numbers and $\alpha^3 + \beta^3 = -p, \alpha\beta = q$, then a quadratic equation, whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, is **[JEE Main 2013]**

(a) $px^2 - qx + p^2 = 0$ (b) $qx^2 + px + q^2 = 0$
(c) $px^2 + qx + p^2 = 0$ (d) $qx^2 - px + q^2 = 0$

47. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is **[AIEEE 2013]**

(a) 1 : 2 : 3 (b) 3 : 2 : 1
(c) 1 : 3 : 2 (d) 3 : 1 : 2

48. Let α and β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $(z) = 1$, then it is necessary that **[AIEEE 2011]**

(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in [1, \infty)$ (d) $\beta \in (0, 1)$

49. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ is equal to **[AIEEE 2010]**

(a) -2 (b) -1 (c) 1 (d) 2

50. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ **[AIEEE 2009]**

(a) greater than $4ab$ (b) less than $4ab$
(c) greater than $-4ab$ (d) less than $-4ab$

51. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then, the common root is **[AIEEE 2008]**

(a) 2 (b) 1
(c) 4 (d) 3

52. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is **[AIEEE 2007]**

(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$

53. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4 lie in the interval **[AIEEE 2006]**

(a) $m > 3$ (b) $-1 < m < 3$ (c) $1 < m < 4$ (d) $-2 < m < 0$

54. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assumes the least value is **[AIEEE 2005]**

(a) 0 (b) 1 (c) 2 (d) 3

55. If both the roots of the equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k belongs to **[AIEEE 2005]**

(a) $(6, \infty)$ (b) $(5, 6]$ (c) $[4, 5]$ (d) $(-\infty, 4)$

56. If $1 - p$ is a root of $x^2 + px + 1 - p = 0$, then its roots are **[AIEEE 2004]**

(a) 0, 1 (b) $-1, 2$
(c) $0, -1$ (d) $-1, 1$

57. If one root of $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice the other, then a is equal to **[AIEEE 2003]**

(a) 2/3 (b) $-2/3$ (c) 1/3 (d) $-1/3$

Answers

1. (d)	2. (a)	3. (d)	4. (b)	5. (c)	6. (b)	7. (d)	8. (d)	9. (c)	10. (c)
11. (b)	12. (c)	13. (c)	14. (c)	15. (c)	16. (b)	17. (a)	18. (d)	19. (c)	20. (c)
21. (c)	22. (a)	23. (b)	24. (c)	25. (d)	26. (a)	27. (d)	28. (b)	29. (a)	30. (b)
31. (c)	32. (b)	33. (a)	34. (a)	35. (c)	36. (b)	37. (b)	38. (c)	39. (d)	40. (d)
41. (b)	42. (a)	43. (c)	44. (c)	45. (d)	46. (b)	47. (a)	48. (c)	49. (c)	50. (c)
51. (a)	52. (a)	53. (b)	54. (b)	55. (d)	56. (c)	57. (a)			

Hints & Solutions

1. Discriminant, $D = b^2 - 4ac = \cos^2\beta - 4(\cos\beta - 1)\sin\beta \ge 0$

$= \cos^2\beta + 4(1 - \cos\beta)\sin\beta \ge 0$

So, it should be $\sin\beta > 0$. $(\because \cos^2\beta \ge 0, 1 - \cos\beta \ge 0)$

$\Rightarrow \beta \in (0, \pi)$

2. Here, $D = (3b)^2 - 4(4a)(2c)$

$= 9b^2 - 32ac = 9(-a-c)^2 - 32ac$

$= 9a^2 - 14ac + 9c^2 = 9c^2\left[\left(\frac{a}{c}\right)^2 - \frac{14}{9}\cdot\frac{a}{c} + 1\right]$

$= 9c^2\left[\left(\frac{a}{c} - \frac{7}{9}\right)^2 - \frac{49}{81} + 1\right] > 0$

Hence, the roots are real and distinct.

3. $z^2 + |z|^2 = 0, z \ne 0$

$\Rightarrow x^2 - y^2 + i2xy + x^2 + y^2 = 0$ (put $z = x + iy$)

$\Rightarrow 2x^2 + i2xy = 0 \Rightarrow 2x(x + iy) = 0$

$\Rightarrow x = 0$ or $x + iy = 0$ (not possible)

Therefore, $x = 0$ and $z \ne 0$

So, y can have any real value. Hence, infinitely many solutions.

4. On adding given equations first and second, we get

$2x^2 + (a+b)x + 27 = 0$

Above equation is subtracting from given third equation, we get

$x^2 - 9 = 0 \Rightarrow x = 3, -3$

Thus, common positive root is 3.

$\therefore (3)^2 + 3a + 12 = 0 \Rightarrow a = -7$

and $9 + 3b + 15 = 0 \Rightarrow b = -8$

Hence, the order pair (a, b) is $(-7, -8)$.

5. Given, $(6k+2)x^2 + rx + 3k - 1 = 0$

and $(12k+4)x^2 + px + 6k - 2 = 0$

For both common roots, $\frac{6k+2}{12k+4} = \frac{r}{p} = \frac{3k-1}{6k-2}$

$\Rightarrow \frac{r}{p} = \frac{1}{2} \Rightarrow 2r - p = 0$

6. Let α, β and α', β' are the roots of the equation of the given equations.

$\therefore \alpha + \beta = -\frac{\mu}{\lambda}, \quad \alpha\beta = \frac{\nu}{\lambda}$ ...(i) and $\alpha' = \omega$ and $\beta' = \omega^2$

$\therefore \quad \frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$ (given)

$\Rightarrow \quad \frac{\alpha}{\beta} = \frac{\omega}{\omega^2} \Rightarrow \beta = \alpha\omega$

From Eq. (i), $\alpha + \alpha\omega = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda}$

$\Rightarrow \quad -\alpha\omega^2 = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda} \quad (\because -\omega^2 = 1 + \omega)$

$\Rightarrow \quad \frac{\mu^2}{\lambda^2} = \frac{\nu}{\lambda} \Rightarrow \mu^2 = \lambda\nu$

7. Given, $\frac{|x-2|-1}{|x-2|-2} \le 0$

Let $|x-2| = k$

$\therefore$ Given equation becomes, $\frac{k-1}{k-2} \le 0$

$\Rightarrow \quad \frac{(k-1)(k-2)}{(k-2)^2} \le 0$

$\Rightarrow \quad (k-1)(k-2) \le 0 \Rightarrow 1 \le k \le 2$

$\Rightarrow \quad 1 \le |x-2| \le 2$

Case I When $1 \le |x-2| \Rightarrow |x-2| \ge 1$

$\Rightarrow \quad x - 2 \ge 1$ or $x - 2 \le -1$

$\Rightarrow \quad x \ge 3$ and $x \le 1$ ...(i)

Case II When $|x-2| \le 2$

$\Rightarrow \quad -2 \le x - 2 \le 2$

$\Rightarrow \quad -2 + 2 \le x \le 2 + 2$

$\Rightarrow \quad 0 \le x \le 4$ ...(ii)

From Eqs. (i) and (ii),

$x \in [0, 1] \cup [3, 4]$

8. Here, $D = 4b^2 + 12ca < 0$

$\Rightarrow \quad b^2 + 3ca < 0$ ...(i)

$\Rightarrow \quad ca < 0$

If $c > 0$, then $a < 0$

Also, $\frac{3c}{4} < a + b \Rightarrow 3ca > 4a^2 + 4ab$

$\Rightarrow b^2 + 3ca > 4a^2 + 4ab + b^2 = (2a + b)^2 \ge 0$ ...(ii)

From Eqs. (i) and (ii), $c > 0$, which is not true.

$\therefore \quad c < 0$

9. Since, $D \ge 0$

$\therefore \quad b^2 - 4ac \ge 0$

$\Rightarrow \quad \left(\frac{c+a}{2}\right)^2 - 4ac \ge 0 \quad (\because 2b = a + c)$

$\Rightarrow \quad c^2 - 14ca + a^2 \ge 0$

$\Rightarrow \quad \left(\frac{c}{a}\right)^2 - 14\left(\frac{c}{a}\right) + 1 \ge 0$

$\Rightarrow \quad \left(\frac{c}{a} - 7\right)^2 \ge 48$

$\Rightarrow \quad \left|\frac{c}{a} - 7\right| \ge 4\sqrt{3}$

10. $x + \frac{1}{x} = 5 \Rightarrow x^2 + \frac{1}{x^2} = 5^2 - 2 = 23$

$x^3 + \frac{1}{x^3} = 5^3 - 3 \times 5 = 110$

Now, $\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = 23 \times 110 = 2530$

$\Rightarrow \quad x^5 + \frac{1}{x^5} = 2530 - \left(x + \frac{1}{x}\right) = 2525$

11. Since, $x = 1$ is a root of first equation. If α is a root of first equation, then $\alpha = 1, \alpha = \frac{a-b}{b-c}$ (product of roots)

$= \frac{2a - 2b}{2b - 2c} = \frac{2a - (a+c)}{a + c - 2c} = 1$

So, the roots of first equation are 1 and 1.

Both the equations will have a common root, if 1 is also a root of second.

$\Rightarrow \quad 2(c + a) + b + c = 0$

$\Rightarrow \quad 2(2b) + b + c = 0$ (since, a, b and c are in AP)

$\Rightarrow \quad c = -5b$

Also, $a + c = 2b$

$\Rightarrow \quad a = 2b - c = 2b + 5b = 7b$

$\therefore \quad a^2 = 49b^2, c^2 = 25b^2$

Hence, a^2, c^2 and b^2 are in AP.

12. Here, $f(a) = a^2, f(b) = b^2, f(c) = c^2$

If $g(x) = f(x) - x^2$; then $g(a) = g(b) = g(c) = 0$

i.e., $f(x) - x^2$ is a quadratic expression which vanishes at three distinct points a, b and c.

Thus, $f(x) - x^2 \equiv 0$ or $f(x) \equiv x^2$

13. Let $y = \frac{x^2 - bc}{2x - b - c} \Rightarrow x^2 - 2yx + (b + c)y - bc = 0$

Discriminant, $4y^2 - 4(b + c)y + 4bc \ge 0 \quad (\because \Delta \ge 0)$

$\Rightarrow \quad (y - b)(y - c) \ge 0 \Rightarrow y \in (-\infty, b] \cup [c, \infty)$

14. Simplified form of given equation $(2x + p + q)r = (x + p)(x + q)$

$\Rightarrow x^2 + (p + q - 2r)x - (p + q)r + pq = 0$

Since, sum of roots $= 0$

$\Rightarrow \quad -(p + q - 2r) = 0 \Rightarrow r = \frac{p+q}{2}$

and product of roots $= -(p + q)r + pq$

$= -\frac{(p+q)^2}{2} + pq = -\frac{1}{2}(p^2 + q^2)$

15. Since, $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\Sigma(a - b)^2 \ge 0$

$\therefore \quad 1 - \Sigma ab \ge 0 \Rightarrow \Sigma ab \le 1$ ...(i)

Now, $(a + b + c)^2 = \Sigma a^2 + 2\Sigma ab$

$\Rightarrow \quad \Sigma ab \ge -\frac{1}{2}$ ...(ii)

From Eqs. (i) and (ii), Σab lie in $\left[-\frac{1}{2}, 1\right]$.

16. Since, x cannot be less than -3.

Now, $x+1>\sqrt{x+3}$

$\Rightarrow x^2+x-2>0$

$\Rightarrow (x+2)(x-1)>0$

$\therefore x>1$

17. LHS $=\left(\frac{y}{x}+\frac{x}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)\ge 2+2+2$

$=6$ $(\because AM\ge GM)$

18. Since, $\alpha+\beta=-1, \alpha\beta=2, \gamma+\delta=-3, \gamma\delta=4$

$\therefore (\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta)$

$=(\alpha^2-3\alpha+4)(\beta^2-3\beta+4)$

$=4-3\alpha\beta^2+4\beta^2-3\alpha^2\beta+9\alpha\beta-12\beta^2+4\alpha^2-12\alpha+16$

$=4-3(2)\beta+4\beta^2+4\alpha^2-3(2)\alpha+9(2)-12(\beta+\alpha)+16$

$=4-6\beta+4(\alpha^2+\beta^2)-6\alpha+18+12+16$

$=50+6+4[(\alpha+\beta)^2-2\alpha\beta]$

$=56-12=44$

19. Let D_1 and D_2 be two discriminants, then

$D_1+D_2=b^2-4ac+d^2+4ac=b^2+d^2>0$

So, either D_1 and D_2 are positive or atleast one D's is positive.

Therefore, $P(x)Q(x)=0$ has either four or two real roots.

20. $\because 5-2\sqrt{6}=\frac{1}{5+2\sqrt{6}}$

$\therefore t+\frac{1}{t}=10$, where $t=(5+2\sqrt{6})^{x^2-3}$ ...(i)

$\Rightarrow t^2-10t+1=0$

$\Rightarrow t=5\pm 2\sqrt{6}$

or $t=(5+2\sqrt{6})^{\pm 1}$ ...(ii)

From Eqs. (i) and (ii),

$x^2-3=\pm 1 \Rightarrow x^2=2,4$

$\Rightarrow x=-\sqrt{2},\sqrt{2},-2,2$

$\therefore$ Required product $=8$

21. $R=\frac{\pi}{2}\Rightarrow P+Q=\frac{\pi}{2}\Rightarrow \frac{P}{2}+\frac{Q}{2}=\frac{\pi}{4}$

$\Rightarrow 1=\tan\frac{\pi}{4}=\frac{\tan\frac{P}{2}+\tan\frac{Q}{2}}{1-\tan\frac{P}{2}\tan\frac{Q}{2}}\Rightarrow 1=\frac{-\frac{b}{a}}{1-\frac{c}{a}}=\frac{b}{c-a}\Rightarrow a+b=c$

22. For roots greater than unity,

Discriminant ≥ 0,

sum of roots >2 and $(a+1)f(1)>0$

$\Rightarrow 9a^2-16a(a+1)\ge 0, \frac{3a}{a+1}>2$

and $(a+1)(a+1-3a+4a)>0$

$\Rightarrow a(7a+16)\le 0, \frac{a-2}{a+1}>0$

and $(a+1)(2a+1)>0$

$\Rightarrow \frac{-16}{7}\le a\le 0, a<-1$ or $a>2$

and $a<-1$ or $a>-\frac{1}{2}$

$\Rightarrow -\frac{16}{7}\le a<-1$

23. Given, $1\le|x-2|\le 3$

Case I When $1\le|x-2|$

$\therefore x-2\ge 1$ and $(x-2)\le -1$

$\Rightarrow x\ge 3$ and $x\le 1$

$\Rightarrow x\in(-\infty,1]\cup[3,\infty)$ ...(i)

Case II When $|x-2|\le 3$

$\Rightarrow -3\le x-2\le 3$

$\Rightarrow -1\le x\le 5$

$\Rightarrow x\in[-1,5]$ ...(ii)

From Eqs. (i) and (ii),

$x\in[-1,1]\cup[3,5]$

24. Since, $3^{x-1}+3^{-x-1}=\frac{1}{3}(3^x+3^{-x})$

$\ge\frac{1}{3}\cdot 2\sqrt{3^x\cdot 3^{-x}}$ $(\because AM\ge GM)$

$\Rightarrow 3^{x-1}+3^{-x-1}\ge\frac{2}{3}$

25. Let $b+c-a=x, c+a-b=y$ and $a+b-c=z$,

then $2a=y+z, 2b=z+x$ and $2c=x+y$

$\because \frac{y+z}{2}\ge\sqrt{yz}\Rightarrow y+z\ge 2\sqrt{yz}$

Similarly, $z+x\ge 2\sqrt{zx}$ and $x+y\ge 2\sqrt{xy}$

$\therefore (y+z)(z+x)(x+y)\ge 8xyz$

$\Rightarrow abc\ge(b+c-a)(c+a-b)(a+b-c)$

26. Put $\log_5 x=a$, then

$a^2+a-2<0 \Rightarrow (a+2)(a-1)<0$

$\Rightarrow -2<a<1 \Rightarrow -2<\log_5 x<1$

$\Rightarrow 5^{-2}<x<5 \Rightarrow \frac{1}{25}<x<5$

27. By definition of $\log x>0$ and $\left(\frac{x+5}{1-3x}\right)>0$, multiplying $\left(\frac{x+5}{1-3x}\right)$ by $(1-3x)$ in numerator and denominator both.

$\therefore \frac{(x+5)(1-3x)}{(1-3x)^2}>0$

$\Rightarrow (x+5)(1-3x)>0 \Rightarrow (x+5)(3x-1)<0$

$\Rightarrow -5<x<1/3$

As $x>0$, $0<x<1/3$

$\therefore \frac{x+5}{1-3x}<1 \Rightarrow x<-1$

This does not satisfy $0<x<1/3$.

Hence, there is no solution.

28. Since, $2\log_5(2^x - 5) = \log_5 2 + \log_5(2^x - 7/2)$

$\Rightarrow (2^x - 5)^2 = 2(2^x - 7/2)$

$\Rightarrow (2^x)^2 - 12(2^x) + 32 = 0$

$\Rightarrow (2^x - 4)(2^x - 8) = 0$

$\Rightarrow 2^x = 2^2, 2^3$

$\therefore x = 3 \quad (\because x \neq 2)$

29. Since, $x - 1 > 0 \Rightarrow x > 1$

and $\log_{0.3}(x-1) > \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{0.3}(x-1) > \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow \log_{0.3}(x-1) > 0$

$\Rightarrow x < 2$

Hence, $x \in (1, 2)$

30. I. When $x \geq -2$, $|x+2| = (x+2)$

$\therefore x^2 - x - 2 + x > 0 \Rightarrow x^2 - 2 > 0$

$\Rightarrow x \in [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ ...(i)

II. When $x \leq -2$, $|x+2| = -(x+2)$

$\therefore x^2 + x + 2 + x > 0$

$\Rightarrow (x+1)^2 + 1 > 0 \Rightarrow x \in R$

or $x \in (-\infty, -2)$ ...(ii)

From Eqs. (i) and (ii), $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

31. Now, $(\beta - \alpha) = [(\beta + \delta) - (\alpha + \delta)]$

and $(\beta + \alpha)^2 - 4\alpha\beta = [(\beta + \delta) + (\alpha + \delta)]^2 - 4(\beta + \delta)(\alpha + \delta)$

$\Rightarrow (-b_1)^2 - 4c_1 = (-b_2)^2 - 4c_2$

$\Rightarrow D_1 = D_2$ ...(i)

Since, least value of $f(x)$ is $-\frac{D_1}{4} = \frac{-1}{4} \Rightarrow D_1 = D_2 = 1$

Hence, least value of $g(x)$ is $-D_2 = -\frac{1}{4}$.

32. Least value of $g(x)$ occurs at $x = \frac{-b_2}{2} = \frac{7}{2}$ (given)

$\Rightarrow b_2 = -7$

33. Since, $x = -2$ is a root of $f(x)$.

$\therefore f(x) = (x+2)(ax+b)$

But $f(0) + f(1) = 0$ (given)

$\therefore 2b + 3a + 3b = 0$

$\Rightarrow -\frac{b}{a} = \frac{3}{5}$

34. Here, coefficient of $x^2 = 2 > 0$ is for the given condition $D \geq 0$, $f(1) > 0$ and $f(2) > 0$.

$\Rightarrow 1 - 8a \geq 0, 2 - 1 + a > 0$ and $8 - 2 + a > 0$

$\Rightarrow a \leq \frac{1}{8}, a > -1$ and $a > -6$

$\Rightarrow -1 < a \leq \frac{1}{8}$

35. $\cos\frac{\pi}{4} = 2\cos^2\left(\frac{\pi}{8}\right) - 1 \Rightarrow \cos^2\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}} + 1\right)\frac{1}{2}$

$\Rightarrow \cos^4\frac{\pi}{8} = \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{2}{\sqrt{2}}\right) = \left(\frac{3}{2} + \sqrt{2}\right)\frac{1}{4}$

Given equation becomes

$\frac{1}{4}\left(\frac{3}{2} + \sqrt{2}\right) + \frac{a}{2}\left(\frac{1}{\sqrt{2}} + 1\right) + b = 0$

$\Rightarrow \left(\frac{3}{8} + \frac{a}{2} + b\right) + \sqrt{2}\left(\frac{1}{4} + \frac{a}{4}\right) = 0$

Since, a and b are rational.

$\therefore \frac{1}{4} + \frac{a}{4} = 0, \frac{3}{8} + \frac{a}{2} + b = 0$

$\Rightarrow a = -1, b = \frac{1}{8}$

36. Since, $\log_{\frac{1}{2+|x|}}(5 + x^2) = \log_{(3+x^2)}(15 + \sqrt{x})$

$\Rightarrow -\log_{(2+x)}(5 + x^2) = \log_{(3+x^2)}(15 + \sqrt{x})$

Here, LHS < 0 and RHS > 0

Hence, no solution exist.

37. We know that, only even prime is 2, then $(2)^2 - \lambda(2) + 12 = 0$.

$\Rightarrow \lambda = 8$ ...(i)

and $x^2 + \lambda x + \mu = 0$ has equal roots.

$\therefore \lambda^2 - 4\mu = 0 \quad (\because D = 0)$

$\Rightarrow (8)^2 - 4\mu = 0$

$\Rightarrow \mu = 16$

38. $4x^2 - 16x + \frac{\lambda}{4} = 0$

$\therefore x = \frac{16 \pm \sqrt{(256 - 4\lambda)}}{8} = \frac{8 \pm \sqrt{(64 - \lambda)}}{4}$

$\Rightarrow \alpha, \beta = 2 \pm \frac{\sqrt{(64 - \lambda)}}{4}$

Here, $64 - \lambda > 0$

$\therefore \lambda < 64$

Also, $1 < \alpha < 2$ and $2 < \beta < 3$

$\Rightarrow 1 < 2 - \frac{\sqrt{64 - \lambda}}{4} < 2$ and $2 < 2 + \frac{\sqrt{64 - \lambda}}{4} < 3$

$\Rightarrow -1 < -\frac{\sqrt{64 - \lambda}}{4} < 0$ and $0 < \frac{\sqrt{(64 - \lambda)}}{4} < 1$

$\Rightarrow 1 > \frac{\sqrt{(64 - \lambda)}}{4} > 0$ and $0 < \frac{\sqrt{(64 - \lambda)}}{4} < 1$

i.e., $0 < \frac{\sqrt{(64 - \lambda)}}{4} < 1$

$\Rightarrow 0 < \sqrt{(64 - \lambda)} < 4$

$\Rightarrow 0 < 64 - \lambda < 16$

$\Rightarrow \lambda > 48$ or $48 < \lambda < 64$

$\therefore \lambda = \{49, 50, 51, 52, \ldots, 63\}$

39. $[\tan^2 x] =$ integer, $a =$ integer

So, $\tan x$ is also an integer.

Then, $\tan^2 x - \tan x - a = 0$

$\Rightarrow \quad a = \tan x(\tan x - 1)$

$= I(I-1)$

= Product of two consecutive integers

$\therefore \quad a = 2, 6, 12, 20, 30, 42, 56, 72, 90$

Hence, set S has 9 elements.

40. $2^{x+2} \cdot 3^{3x/(x-1)} = 9$

Taking log on both sides, we get

$$(x+2)\log 2 + \frac{3x}{(x-1)}\log 3 = 2\log 3$$

$\Rightarrow (x^2 + x - 2)\log 2 + 3x\log 3 = 2(x-1)\log 3$

$\Rightarrow x^2\log 2 + (\log 2 + \log 3)x - 2\log 2 + 2\log 3 = 0$

$$\therefore \; x = \frac{-(\log 2 + \log 3) \pm \sqrt{\{(\log 2 + \log 3)^2 - 4\log 2(-2\log 2 + 2\log 3)\}}}{2\log 2}$$

$$= \frac{-(\log 2 + \log 3) \pm \sqrt{\{(3\log 2)^2 - 6\log 2\log 3 + (\log 3)^2\}}}{2\log 2}$$

$$= \frac{-(\log 2 + \log 3) \pm (3\log 2 - \log 3)}{2\log 2}$$

$$\therefore \; x = -2, 1 - \frac{\log 3}{\log 2}$$

41. Since, α and β are the roots of $375x^2 - 25x - 2 = 0$.

$$\therefore \quad \alpha + \beta = \frac{25}{375} = \frac{1}{15} \text{ and } \alpha\beta = -\frac{2}{375}$$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n} S_r = \lim_{n\to\infty}\sum_{r=1}^{n}(\alpha^r + \beta^r)$$

$$= (\alpha + \alpha^2 + \alpha^3 + \dots \infty) + (\beta + \beta^2 + \beta^3 + \dots \infty)$$

$$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} = \frac{\alpha - \alpha\beta + \beta - \alpha\beta}{(1-\alpha)(1-\beta)}$$

$$= \frac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha+\beta) + \alpha\beta} = \frac{\frac{1}{15} + \frac{4}{375}}{1 - \frac{1}{15} - \frac{2}{375}}$$

$$= \frac{25 + 4}{375 - 25 - 2} = \frac{29}{348} = \frac{1}{12}$$

42. $\because a + b > c \Rightarrow ac + bc > c^2$

$b + c > a \Rightarrow ab + ca > a^2$

and $c + a > b \Rightarrow bc + ab > b^2$

$\therefore a^2 + b^2 + c^2 < 2(ab + bc + ca)$ ...(i)

Here $D \ge 0 \Rightarrow 4(a + b + c)^2 - 12\lambda(ab + bc + ca) \ge 0$

$$\Rightarrow \lambda \le \frac{(a+b+c)^2}{3(ab+bc+ca)} = \frac{2}{3} + \frac{a^2+b^2+c^2}{3(ab+bc+ca)} < \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

[from Eq. (i)]

43. If one root is less than α, then other root is greater than β.

Then, $D \ge 0$ and $f(\alpha) < 0$, $f(\beta) < 0$

Here, equation is

$$x^2 - (a+1)x + a^2 + a - 8 = 0$$

$$(a+1)^2 - 4(a^2 + a - 8) \ge 0$$

$\Rightarrow \quad a^2 + 1 + 2a - 4a^2 - 4a + 32 \ge 0$

$\Rightarrow \quad -3a^2 - 2a + 33 \ge 0$

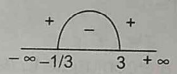
$\Rightarrow \quad 3a^2 + 2a - 33 \le 0$

$\Rightarrow \quad 3a^2 + 11a - 9a - 33 \le 0$

$\Rightarrow \quad a(3a + 11) - 3(3a + 11) \le 0$

$\Rightarrow \quad (3a + 11)(a - 3) \le 0$

$\therefore \quad a \in \left[\frac{-11}{3}, 3\right]$

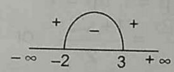


Again, $\quad f(2) < 0$

$\Rightarrow \quad 4 - (a+1)\cdot 2 + a^2 + a - 8 < 0$

$\Rightarrow \quad 4 - 2a - 2 + a^2 + a - 8 < 0$

$\Rightarrow \quad a^2 - a - 6 < 0$



$\Rightarrow \quad a^2 - 3a + 2a - 6 < 0$

$\Rightarrow \quad a(a-3) + 2(a-3) < 0$

$\Rightarrow \quad (a-3)(a+2) < 0$

$\therefore \quad a \in (-2, 3)$

44. $\because \; \alpha + \beta = -p$ and $\alpha\beta = \frac{3p}{4}$

Also, $\quad (\alpha - \beta)^2 = 10$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 10$

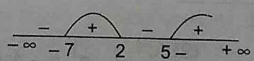
$\Rightarrow \quad p^2 - 3p = 10$

$\Rightarrow \quad (p+2)(p-5) = 0$

$\therefore \quad p = -2, 5$

45.
$$\frac{x-5}{x^2 + 5x - 14} > 0$$

$$\Rightarrow \quad \frac{(x+7)(x-2)(x-5)}{(x-2)^2(x+7)^2} > 0$$



$\Rightarrow \quad x \in (-7, 2) \cup (5, \infty)$

So, the least integral value α of x is -6, which satisfy the equation

$$\alpha^2 + 5\alpha - 6 = 0$$

46. $\therefore$ Sum $= \dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{-p}{q}$

$\therefore$ Product $= \dfrac{\alpha^2}{\beta} \times \dfrac{\beta^2}{\alpha} = \alpha\beta = q$

So, the required quadratic equation is

$$x^2 + \frac{p}{q}x + q = 0 \Rightarrow qx^2 + px + q^2 = 0$$

47. Given equations are

$$x^2 + 2x + 3 = 0 \quad \dots(i)$$

and $$ax^2 + bx + c = 0 \quad \dots(ii)$$

Since, Eq. (i) has imaginary roots.

So, Eq. (ii) will also have both roots same as Eq. (i).

Thus, $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$

Hence, $a : b : c$ is $1 : 2 : 3$.

48. Let $z = x + iy$, given $\text{Re}(z) = 1$

$\therefore \quad x = 1 \Rightarrow z = 1 + iy$

Since, the complex roots are conjugate to each other.

So, $z = 1 + iy$ and $1 - iy$ are two roots of $z^2 + \alpha z + \beta = 0$.

$\because$ Product of roots $= \beta \Rightarrow (1 + iy)(1 - iy) = \beta$

$\therefore \quad \beta = 1 + y^2 \geq 1 \Rightarrow \beta \in [1, \infty)$

49. Since, α and β are roots of the equation $x^2 - x + 1 = 0$.

Then, $\alpha + \beta = 1, \alpha\beta = 1$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{3}i}{2} = \dfrac{1+\sqrt{3}i}{2}$ or $\dfrac{1-\sqrt{3}i}{2}$

$\Rightarrow \quad x = -\omega$ or $-\omega^2$

Thus, $\alpha = -\omega^2$, then $\beta = -\omega$

$\Rightarrow \quad \alpha = -\omega$, then $\beta = -\omega^2 \quad (\because \omega^3 = 1)$

Hence, $\alpha^{2009} + \beta^{2009} = (-\omega)^{2009} + (-\omega^2)^{2009}$

$= -[(\omega^3)^{669} \cdot \omega^2 + (\omega^3)^{1339} \cdot \omega]$

$= -[\omega^2 + \omega] = -(-1) = 1$

50. Given, $bx^2 + cx + a = 0$ has imaginary roots.

$\therefore \quad c^2 - 4ab < 0$

$\Rightarrow \quad c^2 < 4ab$

$\Rightarrow \quad -c^2 > -4ab \quad \dots(i)$

Let $f(x) = 3b^2x^2 + 6bcx + 2c^2$

Here, $3b^2 > 0$

So, the given expression has a minimum value.

$\therefore$ Minimum value $= \dfrac{-D}{4a} = \dfrac{4AC - B^2}{4A}$

$= \dfrac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)}$

$= -\dfrac{12b^2c^2}{12b^2} = -c^2 > -4ab$ [from Eq. (i)]

51. Let the roots of $x^2 - 6x + a = 0$ be $\alpha, 4\beta$ and that of $x^2 - cx + 6 = 0$ be α and 3β.

$\therefore \alpha + 4\beta = 6$ and $4\alpha\beta = a$ and $\alpha + 3\beta = c$ and $3\alpha\beta = 6$

$\Rightarrow \quad \dfrac{a}{6} = \dfrac{4}{3} \Rightarrow a = 8$

$\therefore \quad x^2 - 6x + 8 = 0$

$\Rightarrow \quad (x - 4)(x - 2) = 0 \Rightarrow x = 2, 4$ and $x^2 - cx + 6 = 0$

$\Rightarrow \quad 2^2 - 2c + 6 = 0 \Rightarrow c = 5$

$\therefore \quad x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

Hence, the common root is 2.

52. Since, $\alpha + \beta = -a$ and $\alpha\beta = 1$

Now, $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \sqrt{a^2 - 4}$

Thus, $\sqrt{a^2 - 4} < \sqrt{5}$

$\Rightarrow \quad a^2 < 9$

$\Rightarrow \quad |a| < 3$

$\Rightarrow \quad -3 < a < 3$

53. Conditions are $D \geq 0, -2 < -\dfrac{b}{2a} < 4, f(4) > 0$ and $f(-2) > 0$.

$\Rightarrow 4m^2 - 4m^2 + 4 \geq 0, -2 < \left(\dfrac{2m}{2 \cdot 1}\right) < 4,$

$16 - 8m + m^2 - 1 > 0$ and $4 + 4m + m^2 - 1 > 0$

$\Rightarrow \quad 4 > 0, \forall m \in R, -2 < m < 4, (m - 3)(m - 5) > 0$

and $(m + 3)(m + 1) > 0$

$\Rightarrow \quad m \in R, -2 < m < 4, -\infty < m < 3 \cup 5 < m < \infty$

and $-\infty < m < -3 \cup -1 < m < \infty$

$\Rightarrow \quad -1 < m < 3$

54. Here, $\alpha + \beta = a - 2, \alpha\beta = -(a + 1)$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a - 2)^2 + 2(a + 1)$

$= a^2 - 2a + 6 = (a - 1)^2 + 5$

Hence, $\alpha^2 + \beta^2$ is least for $a = 1$.

55. Given, $x^2 - 2kx + k^2 = 5 - k$

$\Rightarrow \quad x - k = \pm\sqrt{5 - k}$

$\Rightarrow \quad x = k \pm \sqrt{5 - k} < 5$ (given)

$\Rightarrow \quad \sqrt{5 - k} < 5 - k$

But $\sqrt{x} < x$ for $x > 1$

$\therefore \quad 5 - k > 1$

$\Rightarrow \quad k < 4$

56. Since, $1 - p$ is a root of $x^2 + px + 1 - p = 0$

$\therefore \quad (1 - p)^2 + p(1 - p) + 1 - p = 0$

$\Rightarrow \quad -2p + 2 = 0$

$\Rightarrow \quad p = 1$

So, the equation is $x^2 + x = 0$.

Hence, the roots are 0 and -1.

57. Let α and 2α are the roots.

Then, $3\alpha = -\dfrac{(3a - 1)}{(a^2 - 5a + 3)}, 2\alpha^2 = \dfrac{2}{a^2 - 5a + 3}$

Eliminating α, we get

$$\frac{9}{a^2 - 5a + 3} = \frac{(3a - 1)^2}{(a^2 - 5a + 3)^2} \Rightarrow a = \frac{2}{3}$$