Day 24

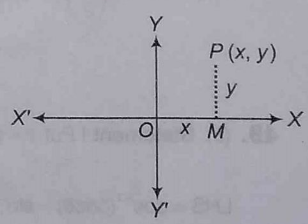
Cartesian System of Rectangular Coordinates

Day 24 Outlines ...

- Rectangular Coordinates
- Distance Formula
- Section Formulae
- Slope of a Line
- Parallel and Perpendicular Lines on the Coordinate Axes.
- Coordinates of Different Points of a Triangle

Rectangular Coordinates

Let XOX' and YOY' be two perpendicular axes in the plane intersecting at O. Let P be any point in the plane. Draw PM perpendicular to OX.



Then, the lengths OM and PM are called the **rectangular cartesian**. The ordered pair (x, y) is called the **coordinates of point** P.

Distance Formula

When coordinates of two points are given in cartesian form.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points.

Then, $\quad PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ or $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Distance between the points $(0, 0)$ and (x, y) is $\sqrt{x^2 + y^2}$.

Distance formula can also be given as $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

where, r is the distance of the point (x, y) on the line from the point (x_1, y_1) and θ be the angle making the line with the positive direction of X-axis.

Application of Distance Formulae

1. Collinearity of Three Given Points

The three given points A, B and C are collinear *i.e.*, lie on the same straight line, if any of three points (say B) lie on the straight line joining the other two points.

i.e., $AB + BC = AC$

2. The Other Conditions for Collinearity

(i) Area of ΔABC is zero. It means

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

(ii) Slope of AB = Slope of BC

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, if $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$.

3. Identifying the Types of Triangles

If A, B and C are vertices of triangle, then it would be

(i) equilateral triangle, when $AB = BC = CA$.

(ii) isosceles triangle, when any two sides are equal.

(iii) right angle triangle, when sum of the square of any two sides is equal to square of the third side.

- *If (x_1, y_1) and (x_2, y_2) are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by*
$$\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \pm (x_1 - x_2)}{2}\right)$$
- *Given two vertices (x_1, y_1) and (x_2, y_2) of an equilateral triangle, then third vertex is given by*
$$\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_1 - x_2)}{2}\right)$$
- *Triangle having integral coordinates can never be equilateral.*

4. Position of Four Points

Let A, B, C and D be the four given points in a plane. By joining these points following figure are formed

(i) Square, if $AB = BC = CD = DA$ and $AC = BD$.

(ii) Rhombus, if $AB = BC = CD = DA$ and $AC \neq BD$.

(iii) Parallelogram, if $AB = DC$, $BC = AD$, $AC \neq BD$.

(iv) Rectangle, if $AB = DC$, $BC = DA$, $AC = BD$.

Conditions of Diagonals in Quadrilaterals

Quadrilateral	Diagonals	Angles between Diagonals
Parallelogram	Not equal	$\theta \neq \frac{\pi}{2}$
Rectangle	Equal	$\theta \neq \frac{\pi}{2}$
Rhombus	Not equal	$\theta = \frac{\pi}{2}$
Square	Equal	$\theta = \frac{\pi}{2}$

Section Formulae

Coordinates of a point which divide the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ are

(i) $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$ (internal division)

(ii) $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$ (external division)

When m_1 and m_2 are of opposite sign, then division is ***external***.

- Mid-point of the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Coordinates of any point on one line segment which divide the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda : 1$ are given by
$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}\right), (\lambda \neq -1)$$
- Lines formed by joining (x_1, y_1) and (x_2, y_2) is divided by
  (i) X-axis in the ratio $-\frac{y_1}{y_2}$.
  (ii) Y-axis in the ratio $-\frac{x_1}{x_2}$.
  If the ratio is positive, then the axis divides it internally and if ratio is negative, then the axis divides externally.
- Line $Ax + By + C = 0$ divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $\lambda : 1$, then
$$\lambda = -\left(\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C}\right)$$

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If λ is positive, then it divides internally and if λ is negative, then it divides externally.

Locus and its Equation

It is the path or curve traced by a moving point satisfying the given condition.

Equation to the Locus of a Point

The equation to the locus of a point is the algebraic relation which is satisfied by the coordinates of every point on the locus of the point.

Steps to Find the of Locus of a Point

- *Step* I Assumes the coordinates of the point say (h, k) whose locus is to be find.
- *Step* II Write the given condition involving (h, k).
- *Step* III Eliminate the variable(s), if any.
- *Step* IV Replace $h \to x$ and $k \to y$. The equation so obtained is the locus of the point which moves under some definite conditions.

Translation of Axes

1. To Alter the Origin of Coordinates Without Altering the Direction of the Axes

Let origin $O(0,0)$ be shifted to a point (a, b) by moving the X and Y-axes parallel to themselves. If the coordinates of point P with reference to old axis are (x_1, y_1), then coordinates of this point with respect to new axis will be $(x_1 - a, y_1 - b)$.

2. To Change the Direction of the Axes of Coordinates without Changing Origin

Let OX and OY be the old axes and OX' and OY' be the new axes obtained by rotating the old OX and OY through an angle θ, then the coordinates of $P(x, y)$ with respect to new coordinate axes will be given by

	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

(i) x and y are old coordinates, x', y' are new coordinates.

(ii) The axes rotation in anti-clockwise is positive and clockwise rotation of axes is negative.

3. To Change the Direction of the Axes of Coordinates by Changing the Origin

If $P(x, y)$ and the axes are shifted parallel to the original axis, so that new origin is (α, β) and then the axes are rotated about the new origin (α, β) by angle ϕ in the anti-clockwise (x', y'), then the coordinates of P will be given by

$$x = \alpha + x'\cos\phi - y'\sin\phi$$
$$y = \beta + x'\sin\phi + y\cos\phi$$

Reflection (Image) of a Point

Let (x, y) be any point, then its image with respect to

X-axis is $(x, -y)$

Y-axis is $(-x, y)$

origin is $(-x, -y)$

line $y = x$ is (y, x)

To remove the term of xy in the equation $ax^2 + 2hxy + by^2 = 0$. The angel θ through which the axis must be turned $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$.

Slope of a Line

The trigonometrical tangent of the angle that a line makes with the positive direction of the X-axis in anti-clockwise sense is called the ***slope*** *or* ***gradient*** *of the line. The slope of a line is generally denoted by m. Thus,* $m = \tan\theta$.

Slope of a Line in Terms of Coordinates of any Two Points on it

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line making an angle θ with the positive direction of X-axis. Then, its slope m is given by

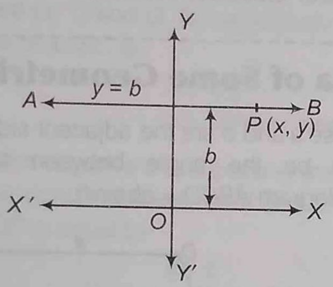
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

Parallel and Perpendicular Lines on the Coordinate Axes

A line parallel to X-axis makes an angle of $0°$ with X-axis. Therefore, its slope is $\tan 0° = 0$. A line parallel to Y-axis *i.e.*, perpendicular to X-axis makes an angle of $90°$ with X-axis, so its slope is $\tan \frac{\pi}{2} = \infty$. Also, the slope of a line equally inclined with axes is 1 or -1 as it makes an angle of $45°$ or $135°$ with X-axis.

Line Parallel to X-axis

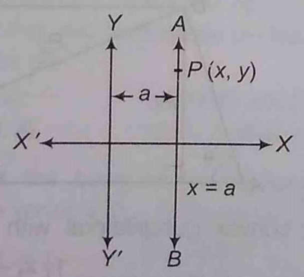
Let AB be a straight line parallel to X-axis at a distance b from it. The clearly ordinate of each point on AB is b.



Thus, AB can be considered at the locus of a point at a distance b from X-axis. Thus, if $P(x, y)$ is any point on AB, then $y = b$.

Line Parallel to Y-axis

Let AB be a line parallel to Y-axis and at a distance a from it. Then, the abscissa of every point on AB is a. So, it can be treated as the locus of a point at a distance a from Y-axis. Thus, if $P(x, y)$ is any point on AB, then $x = a$.



Intercepts of a Line on the Coordinate Axes

The equation of a line which cuts off intercepts a and b respectively from the X and Y-axes is $\frac{x}{a} + \frac{y}{b} = 1$. The intercept made by a line on X-axis can also be obtained by putting $y = 0$ in its equation. Similarly, y-intercept is the value of y obtained from the line when x is replaced by zero.

Image of a Point with Respect to a Line

Let the image of a point (x_1, y_1) with respect to $ax + by + c = 0$ be (x_2, y_2), then

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}.$$

- The image of the point $P(x_1, y_1)$ with respect to X-axis is $Q(x_1, -y_1)$.
- The image of the point $P(x_1, y_1)$ with respect to Y-axis is $Q(-x_1, y_1)$.
- The image of the point $P(x_1, y_1)$ with respect to mirror $y = x$ is $Q(y_1, x_1)$.
- The image of the point $P(x_1, y_1)$ with respect to the line mirror $y = x \tan\theta$ is
$$x = x_1 \cos 2\theta + y_1 \sin 2\theta;$$
$$y = x_1 \sin 2\theta - y_1 \cos 2\theta$$
- The image of the point $P(x_1, y_1)$ with respect to the origin is the point $(-x_1, -y_1)$.
- The length of perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

Coordinates of Different Points of a Triangle

The different coordinate system of a triangle at some particular points are given below.

1. Centroid

The centroid of a triangle is the point of intersection of its medians. It divides the medians in the ratio 2 : 1. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertries of ΔABC, then the coordinates of its centroid G are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

2. **Orthocentre**

The orthocentre of a triangle is the point of intersection of its altitudes. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC, then the coordinates of its orthocentre O are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

3. **Circumcentre**

The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of its sides. It is the centre of the circle passing through the vertices of a triangle and so it is equidistant from the vertices of the triangle.

Here, $OA = OB = OC$, where O is the centre of circle and A, B and C are the vertices of a triangle. The coordinates of the circumcentre are also given by

$$S\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

- *Circumcentre of the right angled ΔABC, right angled at A is $\frac{B+C}{2}$.*
- *Orthocentre of the right angled ΔABC, right angled at A is A.*
- *Orthocentre, centroid, circumcentre of a triangle are collinear.*
- *Centroid divides the line joining the orthocentre and circumcentre in the ratio 2 : 1.*
- *The circumcentre of right angled triangle is the mid-point of the hypotenuse.*
- *A triangle is isosceles, if any two of its medians are equal.*

Coordinates of Incentre and Excentre

1. **Incentre**

The point of intersection of the internal bisectors of the angles of a triangle is called its incentre. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC such that $BC = a, CA = b$ and $AB = c$, then the coordinates of the incentre are

$$I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

2. **Excentre**

Coordinate of excentre opposite of $\angle A$ is given by

$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

and similarly for excentres (I_2 and I_3) opposite to $\angle B$ and $\angle C$ are given by

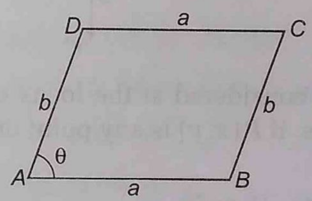
$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

and $$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right).$$

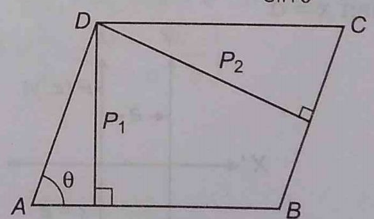
In an equilateral triangle, orthocentre, centroid, circumcentre, incentre, coincide.

Area of Some Geometrical Figures

- Suppose a and b are the adjacent sides of a parallelogram and θ be the angle between them, then area of parallelogram $ABCD = ab \sin\theta$.



- If length of perpendicular from one vertices to the opposite sides are P_1 and P_2 and angle between sides is θ, is given by area of parallelogram $ABCD = \frac{P_1 P_2}{\sin\theta}$.



- Area of convex quadrilateral with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ in that order is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$.
- A triangle having vertices $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$, then area of triangle $= a^2[(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)]$
- Area of triangle formed by coordinate axes and the lines $ax + by + c$ is $= \frac{c^2}{2ab}$.
- Area of rhombus formed by $|ax| + |by| + |c| = 0$ is $\frac{2c^2}{ab}$.

Practice Zone DAY 24

1. Let equation of the side BC of a ΔABC be $x + y + 2 = 0$. If the coordinates of its orthocentre and circumcentre are (1, 1) and (2, 0) respectively, then radius of the circumcircle of ΔABC is

(a) 3 (b) $\sqrt{10}$
(c) $2\sqrt{2}$ (d) None of these

2. A straight line with negative slope passing through the point (1, 4) meets the coordinate axes at A and B. The minimum value of $OA + OB$ is equal to

(a) 5 (b) 6
(c) 9 (d) 8

3. If $a > 0, b > 0$. Then, the sum of the distances of the point (a, b) from the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ is

(a) $\frac{a+b}{2}$ (b) $\sqrt{ab}$
(c) $\frac{2ab}{a+b}$ (d) $\sqrt{a^2 + b^2}$

4. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$ remaining vertices are

(a) (2, 0) and (4, 4) (b) (2, 0) and (– 4, – 4)
(c) (2, 0) and (– 4, 4) (d) (– 2, 0) and (4, 4)

5. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2, – 1). The area of triangle is

(a) $2\sqrt{3}$ (b) $\sqrt{3}/6$
(c) $1/\sqrt{3}$ (d) $2/\sqrt{3}$

6. If the points are $A(0, 4)$ and $B(0, -4)$, then find the locus of $P(x, y)$ such that $|AP - BP| = 6$.

(a) $9x^2 - 7y^2 + 63 = 0$
(b) $9x^2 + 7y^2 - 63 = 0$
(c) $9x^2 + 7y^2 + 63 = 0$
(d) None of the above

7. The area of the region bounded by the lines $y = |x - 2|$, $x = 1$, $x = 3$ and the X-axis is

(a) 1 (b) 2
(c) 3 (d) 4

8. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. Find the equation of the bisector of $\angle ABC$.

(a) $x + 7y - 2 = 0$ (b) $x - 7y - 2 = 0$
(c) $x - 7y + 2 = 0$ (d) None of these

9. The orthocentre of the triangle whose vertices are $\{at_1t_2, a(t_1 + t_2)\}$, $\{at_2t_3, a(t_2 + t_3)\}$, $\{at_3t_1, a(t_3 + t_1)\}$ is

(a) $\{-a, a(t_1 + t_2 + t_3 + t_1 + t_2t_3)\}$
(b) $\{-a, a(t_1 + t_2 + t_3 + t_1t_2t_3)\}$
(c) $\{-a, a(t_1 - t_2 - t_3 - t_1t_2t_3)\}$
(d) $\{-a, a(t_1 + t_2 - t_3 - t_1t_2t_3)\}$

10. Without change of axes the origin is shifted to (h, k), then from the equation $x^2 + y^2 - 4x + 6y - 7 = 0$ the terms containing linear powers are missing. The point (h, k) is

(a) (3, 2) (b) (–3, 2)
(c) (2, –3) (d) (–2, –3)

11. The orthocentre of the triangle formed by the points (0, 0), (4, 0) and (3, 4) is

(a) (2, 0) (b) $\left(\frac{3}{2}, 2\right)$ (c) $\left(\frac{3}{4}, 3\right)$ (d) $\left(3, \frac{3}{4}\right)$

12. If the coordinates of the vertices of a triangle are integers, then the triangle cannot be

(a) equilateral (b) isosceles
(c) scalene (d) None of these

13. If the coordinates of two points A and B are (3, 4) and (5, –2), respectively. Then, the coordinates of any point P, if $PA = PB$ and area of $\Delta PAB = 10$, are

(a) (7, 5), (1, 0) (b) (7, 2), (1, 0)
(c) (7, 2), (–1, 0), (d) None of these

14. The line $x + y = 1$ meets X-axis at A and Y-axis at B. P is the mid-point of AB, P_1 is the foot of the perpendicular from P of OA, M_1 is that of P_1 from OP, P_2 is that of M_1 from OA, M_2 is that of P_2 from OP, P_3 is that of M_2 from OP and so on. If P_n denotes the nth root of the perpendicular on OP from M_{n+1}, then OP_n is equal to

(a) $\frac{1}{2^{n-1}}$ (b) $\frac{1}{2^{n-2}}$
(c) $\frac{1}{2^n}$ (d) None of these

15. The point A divides the join of $P \equiv (-5, 1)$ and $Q \equiv (3, 5)$ in the ratio $k:1$. The two values of k for which the area of ΔABC, where $B \equiv (1, 5)$, $C \equiv (7, -2)$ is equal to 2 sq units are

(a) $\left(7, \frac{30}{9}\right)$ (b) $\left(7, \frac{31}{9}\right)$
(c) $\left(4, \frac{31}{9}\right)$ (d) $\left(7, \frac{31}{3}\right)$

16. If Δ_1 is the area of the triangle with vertices $(0, 0)$, $(a\tan\alpha, b\cot\alpha)$, $(a\sin\alpha, b\cos\alpha)$, Δ_2 is the area of the triangle with vertices (a, b), $(a\sec^2\alpha, b\operatorname{cosec}^2\alpha)$, $(a + a\sin^2\alpha, b + b\cos^2\alpha)$ and Δ_3 is the area of the triangle with vertices $(0, 0)$, $(a\tan\alpha, -b\cot\alpha)$, $(a\sin\alpha, b\cos\alpha)$. Then,

(a) $\Delta_1, \Delta_2, \Delta_3$ are in GP (b) $\Delta_1, \Delta_2, \Delta_3$ are not in GP
(c) Cannot be discussed (d) None of these

17. The line $x + y = a$ meets the axes of x and y at A and B, respectively. A ΔAMN is inscribed in the ΔOAB, O being the origin, with right angle at N. M and N lie respectively on OB and AB. If the area of the ΔAMN is $\frac{3}{8}$ of the area of the ΔOAB, then $\frac{AN}{BN}$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{3}, 3$ (c) $\frac{2}{3}, 3$ (d) 3

18. A line L intersects the three sides BC, CA and AB of a ΔABC at P, Q and R, respectively. Then, $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

19. The middle point of the line segment joining $(3, -1)$ and $(1, 1)$ is shifted by two units (in the sense of increasing y) perpendicular to the line segment. Then, the coordinates of the point in the new position are

(a) $(2 - \sqrt{2}, 2)$ (b) $(2, 2 - \sqrt{3})$
(c) $(2 + \sqrt{2}, \sqrt{2})$ (d) None of these

20. ABC is a variable triangle with the fixed vertex $C(1, 2)$ and A, B having the coordinates $(\cos t, \sin t)$, $(\sin t, -\cos t)$ respectively, where t is a parameter. The locus of the centroid of the ΔABC is

(a) $3(x^2 + y^2) - 2x - 4y - 1 = 0$ (b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
(c) $3(x^2 + y^2) + 2x + 4y - 1 = 0$ (d) $3(x^2 + y^2) + 2x + 4y + 1 = 0$

21. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in GP with the same common ratio, then the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

(a) lie on a straight line
(b) lie on an ellipse
(c) lie on one vertex of a triangle
(d) lie on a circle

22. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$, then Q is obtained from P by

(a) clockwise rotation around the origin through angles α
(b) anti-clockwise rotation around origin through angle α
(c) reflection in the line through the origin with slope $\tan\alpha$
(d) reflection in the line through the origin with slope $\tan\alpha/2$

23. A line joining $A(2, 0)$ and $B(3, 1)$ is rotated about A in anti-clockwise direction through $15°$. Find the equation of the line in the new position. If B goes to C in the new position, then coordinates of C are

(a) $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$ (b) $\left(2 - \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$
(c) $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)$ (d) None of these

24. Two points $P(a, 0)$ and $Q(-a, 0)$ are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is 2α, then

(a) locus of R is $x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$
(b) locus of R is $x^2 + y^2 + 2xy\cot\alpha - a^2 = 0$
(c) locus of R is a hyperbola, if $\alpha = \pi/4$
(d) locus of R is a circle, if $\alpha = \pi/4$

Directions (Q. Nos. 25 to 27) *Suppose we define the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as $d(P, Q) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$.*

25. For a point P along the line $y = \sqrt{3}x$, $d(O, P)$ is equal to

(a) x (b) y (c) x, if $x < 0$ (d) y, if $y \geq 0$

26. The area of the region bounded by the locus of a point P satisfying $d(P, A) = 4$, where A is $(1, 2)$, is

(a) 64 sq units (b) 54 sq units
(c) 16π sq units (d) None of these

27. Suppose that points A and B have coordinates $(1, 0)$ and $(-1, 0)$ respectively, then for a variable point P on this plane the equation $d(P, A) + d(P, B) = 2$ represents

(a) a line segment joining A and B
(b) an ellipse with foci at A and B
(c) region lying inside a square of area 2
(d) region inside a semi-circle with AB as diameter

Directions (Q. Nos. 28 to 31) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

28. Statement I If $(2a, 4a)$ and $B(2a, 6a)$ are two vertices of a equilateral ΔABC and the vertex C is given by $(2a + a\sqrt{3}, 5a)$.

Statement II An equilateral triangle all the coordinates of three vertices can be rational.

29. Statement I If the circumcentre of a triangle lies at the origin and centroid is the middle point of the line joining the points (2, 3) and (4, 7), then its orthocentre lies on the line $5x - 3y = 0$.

Statement II The circumcentre, centroid and the orthocentre of a triangle lie on the same line.

30. Statement I If the origin is shifted to the centroid of the triangle with vertices (0, 0), (3, 3) and (3, 6) without rotation of axes, then the vertices of the triangle in the new system of coordinates are (–2, 0), (1, 3) and (1, – 3).

Statement II If the origin is shifted to the point (2, 3) without rotation of the axes, then the coordinates of the point $P(\alpha - 1, \alpha + 1)$ in the new system of coordinates are $(\alpha - 3, \alpha - 2)$.

31. Let the equation of the line $ax + by + c = 0$.

Statement I If a, b and c are in AP, then $ax + by + c = 0$ pass through a fixed point $(1, -2)$.

Statement II Any family of lines always pass through a fixed point.

32. Consider three points

$$P = \{-\sin(\beta - \alpha), -\cos\beta\}$$
$$Q = \{\cos(\beta - \alpha), \sin\beta\} \text{ and}$$
$$R = \{\cos(\beta - \alpha + \theta), \sin\beta - \theta\}$$

where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$, then

(a) P lies on the line segment RQ
(b) Q lies on the line segment PR
(c) R lies on the line segment QP
(d) P, Q and R are non-collinear

33. Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of the ΔOPQ. The point R inside the ΔOPQ is such that ΔOPR, ΔPQR and ΔOQR are of equal area. Then, R is equal to

(a) $\left(\frac{4}{3}, 3\right)$ (b) $\left(3, \frac{2}{3}\right)$
(c) $\left(3, \frac{4}{3}\right)$ (d) $\frac{4}{3}$

34. The incentre of the triangle with vertices $A(1, \sqrt{3})$ $B(0, 0)$, $C(2, 0)$ is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

35. If by rotating the coordinate axes without translating the origin, the expression $a_1x^2 + 2h_1xy + b_1y^2$ becomes $a_2X^2 + 2h_2XY + b_2Y^2$, then

(a) $a_1 - b_1 = a_2 - b_2$ (b) $a_1 + b_1 = a_2 + b_2$
(c) $a_1 + b_2 = a_2 + b_1$ (d) $a_1 - b_1 = a_2 + b_2$

36. The point (4, 1) undergoes the following transformations

(i) Reflection in the line $x - y = 0$
(ii) Translation through a distance of 2 units along positive direction of X-axis.
(iii) Projection on X-axis.

The coordinates of the point in its final position are

(a) (3, 4) (b) (3, 0)
(c) (1, 0) (d) (4, 3)

37. If the areas be turned through an angle $\tan^{-1}2$, then what does the equation $4xy - 3x^2 = a^2$ become?

(a) $X^2 - 4Y^2 = a^2$ (b) $X^2 + 4Y^2 = a^2$
(c) $X^2 + 4Y^2 = -a^2$ (d) None of these

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38. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle, in square units, is **[JEE Main 2013]**

(a) $\frac{5}{4}a^2$ (b) $\frac{5}{2}a$
(c) $\frac{25a^2}{4}$ (d) $5a^2$

39. If the image of point $P(2, 3)$ in a line L is $Q(4, 5)$, then the image of point $R(0, 0)$ in the same line is **[JEE Main 2013]**

(a) (2, 2) (b) (4, 5) (c) (3, 4) (d) (7, 7)

40. Let $A(-3, 2)$ and $B(-2, 1)$ be the vertices of a ΔABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line **[JEE Main 2013]**

(a) $4x + 3y + 5 = 0$ (b) $3x + 4y + 2 = 0$
(c) $4x + 3y + 3 = 0$ (d) $3x + 4y + 5 = 0$

41. The x-coordinate of the incentre of the triangle that has coordinates of mid-points of its sides as (0, 1), (1, 1) and (1, 0) is **[JEE Main 2013]**

(a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$
(c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$

42. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k is equal to **[AIEEE 2012]**

(a) $\frac{29}{5}$ (b) 5 (c) 6 (d) $\frac{11}{5}$

43. The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement I The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement II In any triangle, bisector of an angle divides the triangle into two similar triangles. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false; Statement II is true.

44. If $A(2,-3)$ and $B(-2, 1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is **[AIEEE 2011]**

(a) $2x - 3y = 1$ (b) $x - y = 1$
(c) $2x + 3y = 1$ (d) $2x + 3y = 3$

45. Three distinct points A, B and C given in the two dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then, the circumcentre of the ΔABC is at the point **[AIEEE 2009]**

(a) $\left(\frac{5}{4}, 0\right)$ (b) $\left(\frac{5}{2}, 0\right)$
(c) $\left(\frac{5}{3}, 0\right)$ (d) (0, 0)

46. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which k can take is given by **[AIEEE 2007]**

(a) {1, 3} (b) {0, 2}
(c) {− 1, 3} (d) {− 3, − 2}

47. If a vertex of a triangle is (1, 1) and the mid-point of two sides of a triangle through this vertex are $(-1, 2)$ and $(3, 2)$, then the centroid of the triangle is **[AIEEE 2005]**

(a) $\left(-\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(-1, \frac{7}{3}\right)$
(c) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (d) $\left(1, \frac{7}{3}\right)$

48. Let $A(2, -3)$ and $B(-2, 1)$ be the vertices of a ΔABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line **[AIEEE 2004]**

(a) $2x + 3y = 9$ (b) $2x - 3y = 7$
(c) $3x + 2y = 5$ (d) $3x - 2y = 3$

Answers

1. (b)	2. (c)	3. (d)	4. (a)	5. (b)	6. (a)	7. (a)	8. (c)	9. (b)	10. (c)
11. (d)	12. (a)	13. (b)	14. (c)	15. (b)	16. (b)	17. (b)	18. (c)	19. (c)	20. (b)
21. (a)	22. (d)	23. (c)	24. (a)	25. (d)	26. (a)	27. (c)	28. (c)	29. (c)	30. (d)
31. (c)	32. (d)	33. (c)	34. (d)	35. (b)	36. (b)	37. (a)	38. (d)	39. (d)	40. (b)
41. (b)	42. (c)	43. (c)	44. (c)	45. (a)	46. (c)	47. (d)	48. (a)		

Hints & Solutions

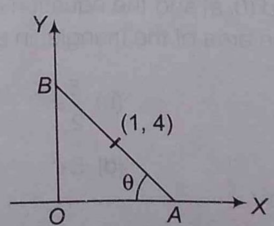
1. Since, reflection of the orthocentre of ΔABC in base BC will always lie on the circumcircle of the ΔABC, therefore coordinates of a point lying on the circumcircle are $\left(1 - \frac{1 \times 4}{2}, 1 - \frac{1 \times 4}{2}\right)$

i.e., $(-1, -1)$ and coordinates of the circumcentre are (2, 0).

$\therefore$ Radius of the circumcircle of ΔABC

$$= \sqrt{(2+1)^2 + (1)^2}$$
$$= \sqrt{10}$$

2. Let $\angle OAB = \theta$



Then, $OA + OB = 1 + 4\cot\theta + 4 + \tan\theta$
$= 5 + 4\cot\theta + \tan\theta \geq 5 + 4 = 9$ (using AM ≥ GM)

$$= \frac{\frac{a}{a}+\frac{b}{b}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}} + \frac{\frac{a}{b}+\frac{b}{a}-1}{\sqrt{\left(\frac{1}{b}\right)^2+\left(\frac{1}{a}\right)^2}}$$

$$= \left(\frac{a}{b}+\frac{b}{a}\right)\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \sqrt{a^2+b^2}$$

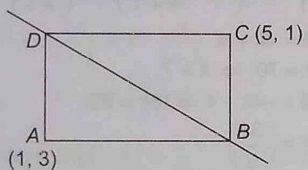
4. The diagonals meet at the mid-point of AC, *i.e.*, at (3, 2) which lies on $y = 2x + c$.

$\therefore \quad c = -4$

Let $\quad B = (\alpha, 2\alpha - 4)$

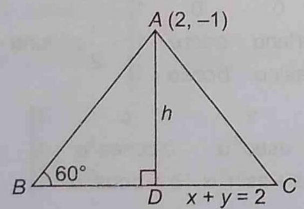
$\because \quad AB \perp BC \Rightarrow \left(\frac{2\alpha-7}{\alpha-1}\right)\left(\frac{2\alpha-5}{\alpha-5}\right) = -1$

$\therefore \quad \alpha^2 - 6\alpha + 8 = 0 \Rightarrow \alpha = 2, 4$



The other two vertices are (2, 0) and (4, 4).

5. The altitude h is $\frac{|2-1-2|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$



In ΔABC, $\quad BC = 2h \cot 60° = \sqrt{\frac{2}{3}}$

$\therefore \quad$ Area of triangle $= \frac{h}{2} BC = \frac{1}{2\sqrt{2}}\sqrt{\frac{2}{3}} = \frac{\sqrt{3}}{6}$

6. $BP - AP = \pm 6$ or $BP = AP \pm 6$

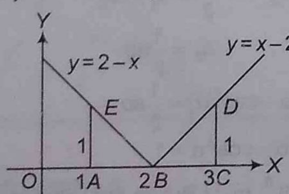
$\Rightarrow \quad \sqrt{x^2+(y+4)^2} = \sqrt{x^2+(y-4)^2} \pm 6$

On squaring and simplifying, we get

$4y - 9 = \pm 3\sqrt{x^2+(y-4)^2}$

Again squaring, we get $9x^2 - 7y^2 + 63 = 0$

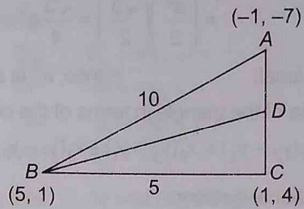
7. $y = |x-2| \Rightarrow y = x - 2$ and $y = 2 - x$



Area of shaded region $= 2\,ABC = 2 \cdot \frac{1}{2} \cdot 1 \cdot 1 = 1$

8. $BC = 5$, $BA = 10$

Let D divides AC in the ratio 2:1.



So, the coordinate of D is $\left(\frac{1}{3}, \frac{1}{3}\right)$.

Slope BD, $\quad \frac{\frac{1}{3}-1}{\frac{1}{3}-5} = \frac{1}{7}$

The equation of the line joining B and D is

$\frac{y-1}{x-5} = \frac{1}{7} \Rightarrow x - 7y + 2 = 0$

9. Let the vertices be C, A and B, respectively. The altitude from A is

$\frac{y - a(t_2+t_3)}{x - at_2t_3} = -t_1$

$\Rightarrow \quad xt_1 + y = at_1t_2t_3 + a(t_2+t_3) \quad$ ...(i)

The altitude from B is $xt_2 + y = at_1t_2t_3 + a(t_3+t_1) \quad$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), $x = -a$

Hence, $y = a(t_1+t_2+t_3+t_1t_2t_3)$

So, the orthocentre is $\{-a, a(t_1+t_2+t_3+t_1t_2t_3)\}$.

10. Let the new coordinates be $P(x', y')$ after shifting origin to $P(x', y')$

i.e., $x = x' + h$ and $y = y' + k$

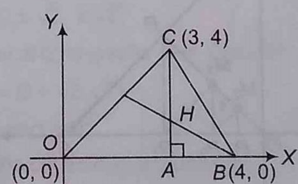
$\therefore (x'+h)^2 + (y'+k)^2 - 4(x'+h) + 6(y'+k) - 7 = 0$

$\Rightarrow \quad (x')^2 + (y')^2 + 2(h-2)x' + 2(k+3)y' + (h^2+k^2-4h+6k-7) = 0$

According to the question, $h - 2 = 0$ and $k + 3 = 0$

$\Rightarrow \quad (h, k) = (2, -3)$

11. Let $H(3, \alpha)$ is orthocentre.



$\therefore$ Slope of $BH \times$ Slope of $OC = -1$

$-\alpha \cdot \frac{4}{3} = -1 \Rightarrow \alpha = \frac{3}{4}$

Hence, the orthocentre of a triangle is (3, 3/4).

12. Let $A = (x_1 y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ be the vertices of a triangle and x_1, x_2, x_3 and y_1, y_2, y_3 be integers.

So, $BC^2 = (x_2-x_3)^2 + (y_2-y_3)^2$ is a positive integers.

If the triangle is equilateral, then $AB = BC = CA = a$ (say) and $\angle A = \angle B = \angle C = 60°$.

$\therefore$ Area of the triangle $= \left(\frac{1}{2}\right)\sin A\cdot bc = \left(\frac{1}{2}\right)a^2\sin 60^\circ$

$$= \left(\frac{a^2}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}a^2$$

which is irrational. (since, a^2 is a positive integer)

Now, the area of the triangle in terms of the coordinates

$$= \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

which is a rational number.

This contradicts that the area is an irrational number, if the triangle is equilateral.

13. Let the coordinates of P be (x, y).

Then, $PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow (x-3)^2+(y-4)^2 = (x-5)^2+(y+2)^2$

$\Rightarrow x-3y-1=0$ ...(i)

Now, area of $\Delta PAB = 10 \Rightarrow \frac{1}{2}\begin{vmatrix} x & y & 1\\ 3 & 4 & 1\\ 5 & -2 & 1\end{vmatrix} = \pm 10$

$\Rightarrow 6x+2y-26 = \pm 20$

$\Rightarrow 6x+2y-46=0$ or $6x+2y-6=0$

$\Rightarrow 3x+y-23=0$ or $3x+y-3=0$ ...(ii)

On solving, $x-3y-1=0$ and $3x+y-23=0$, we get

$x=7, y=2$

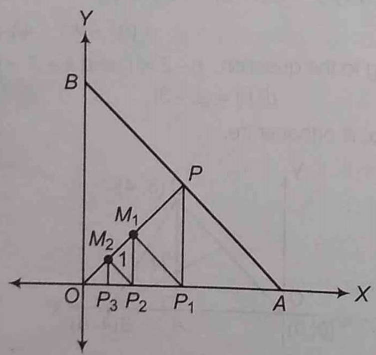
On solving $x-3y-1=0$ and $3x+y-3=0$, we get

$x=1, y=0$

Thus, the coordinates of P are (7,2) or (1, 0).

14. Let $x+y=1$ meets X-axis at $A(1, 0)$ and Y-axis at $B(1, 0)$.

The coordinates of P are $\left(\frac{1}{2},\frac{1}{2}\right)$ and PP_1 is perpendicular to OA



$\Rightarrow OP_1 = P_1P = \frac{1}{2}$

So, the equation of line OP is $y = x$.

We have, $OM_{n-1}^2 = OPn^2 + P_nM_{n-1}^2 = 20Pn^2 = 2P_n^2$ (say)

Also, $OP_{n-1}^2 = OM_{n-1}^2 + (P_{n-1}M_{n-1})^2$

$$= 2P_n^2 + \frac{1}{2}P_{n-1}^2$$

$\Rightarrow P_n^2 = \frac{1}{4}P_{n-1}^2$

$\Rightarrow P_n = \frac{1}{2}P_{n-1}$

$\therefore OP_n = P_n = \frac{1}{2}P_{n-1} = \dots = \frac{1}{2^{n-1}}P = \frac{1}{2^n}$

15. Coordinates of A, dividing the join of $P \equiv (-5,1)$ and $Q \equiv (3,5)$ in the ratio $k : 1$ are given by $A\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$.

Also, area of ΔABC is given by

$$\Delta = \left|\frac{1}{2}\Sigma x_1(y_2-y_3)\right|$$

$$= \frac{1}{2}|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]|$$

$\Rightarrow \left|\frac{1}{2}\left\{\frac{3k-5}{k+1}(7)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right\}\right| = 2$

$\Rightarrow \frac{1}{2}\left\{\frac{3k-5}{k+1}(7)+\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right\} = \pm 2$

$\Rightarrow 14k-66 = 4k+4$

$\Rightarrow 10k = 70 \Rightarrow k = 7$

or $14k-66 = -4k-4 \Rightarrow 18k = 62$

$\Rightarrow k = \left(\frac{31}{9}\right)$

Therefore, the values of k are 7 and $\frac{31}{9}$.

16. We have,

$$\Delta_1 = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1\\ a\tan\alpha & b\cot\alpha & 1\\ a\sin\alpha & b\cos\alpha & 1\end{Vmatrix} = \frac{1}{2}ab|\sin\alpha-\cos\alpha|$$

and $\Delta_2 = \frac{1}{2}\begin{Vmatrix} a & b & 1\\ a\sec^2\alpha & b\,\mathrm{cosec}^2\alpha & 1\\ a+a\sin^2\alpha & b+b\cos^2\alpha & 1\end{Vmatrix}$

Applying $C_1 \to C_1 - aC_3$ and $C_2 \to C_2 - bC_3$, we get

$$\Delta_2 = \frac{1}{2}ab\begin{Vmatrix} 0 & 0 & 1\\ \tan^2\alpha & \cot^2\alpha & 1\\ \sin^2\alpha & \cos^2\alpha & 1\end{Vmatrix} = \frac{1}{2}ab\,|\sin^2\alpha-\cos^2\alpha|$$

and $\Delta_3 = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 1\\ a\tan\alpha & -b\cot\alpha & 1\\ a\sin\alpha & b\cos\alpha & 1\end{Vmatrix} = \frac{1}{2}ab\,|\sin\alpha+\cos\alpha|$

So that, $\Delta_1\Delta_3 = \frac{1}{2}ab\Delta_2$

Suppose, Δ_1, Δ_2 and Δ_3 are in GP.

Then, $\Delta_1\Delta_3 = \Delta_2^2 \Rightarrow \frac{1}{2}ab\Delta_2 = \Delta_2^2$

$\Rightarrow \Delta_2 = \frac{1}{2}ab$

$\Rightarrow \frac{1}{2}ab(\sin^2\alpha-\cos^2\alpha) = \frac{1}{2}ab$

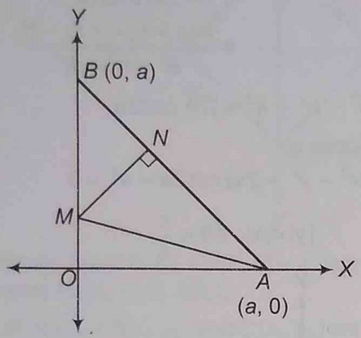
$\Rightarrow \sin^2\alpha-\cos^2\alpha = 1$

i.e., $\alpha = (2m+1)\frac{\pi}{2}, m \in I$. But for this value of α, the vertices of the given triangles are not defined.

Hence, Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α.

17. Let $\frac{AN}{BN} = \lambda$. Then, the coordinates of N are $\left(\frac{a}{1+\lambda}, \frac{\lambda a}{1+\lambda}\right)$.

where $(a, 0)$ and $(0, a)$ are the coordinates of A and B, respectively.



Now, equation of MN perpendicular to AB is

$$y - \frac{\lambda a}{1+\lambda} = x - \frac{a}{1+\lambda}$$

$$\Rightarrow \quad x - y = \frac{1-\lambda}{1+\lambda}.a$$

So, the coordinates of M are $\left(0, \frac{\lambda - 1}{\lambda + 1} \cdot a\right)$, therefore area of the

$$\Delta AMN = \frac{1}{2}\left|\left[a\left(\frac{-a}{\lambda+1}\right) + \frac{1-\lambda}{(1+\lambda)^2} \cdot a^2\right]\right| = \frac{\lambda a^2}{(1+\lambda)^2}$$

Also, area of the $\Delta AOB = \frac{a^2}{2}$

So that according to the given condition,

$$\frac{\lambda a^2}{(1+\lambda)^2} = \frac{3}{8} \cdot \frac{1}{2} a^2$$

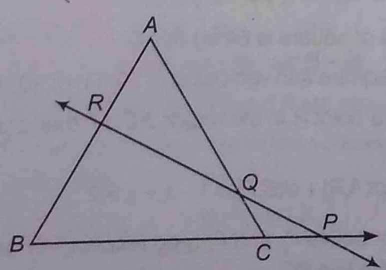
$$\Rightarrow \quad 3\lambda^2 - 10\lambda + 3 = 0 \Rightarrow \lambda = 3$$

$$\therefore \quad \lambda = \frac{1}{3}$$

For $\lambda = \frac{1}{3}$, M lies outside the segment OB and hence the required value of λ is 3.

18. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and let $lx + my + n = 0$ be the equation of the line. If P divides BC in the ratio $\lambda : 1$, then the coordinates of P are

$$\left(\frac{\lambda x_3 + x_2}{\lambda + 1}, \frac{\lambda y_3 + y_2}{\lambda + 1}\right)$$



Also, as P lies on L, we have

$$l\left(\frac{\lambda x_3 + x_2}{\lambda + 1}\right) + m\left(\frac{\lambda y_3 + y_2}{\lambda + 1}\right) + n = 0$$

$$\Rightarrow \quad -\frac{lx_2 + my_2 + n}{lx_3 + my_3 + n} = \frac{BP}{PC} = \lambda \quad \text{...(i)}$$

Similarly, we obtain

$$\frac{CQ}{QA} = -\frac{lx_3 + my_3 + n}{lx_1 + my_1 + n} \quad \text{...(ii)}$$

$$\text{and} \quad \frac{AR}{RB} = -\frac{lx_1 + my_1 + n}{lx_2 + my_2 + n} \quad \text{...(iii)}$$

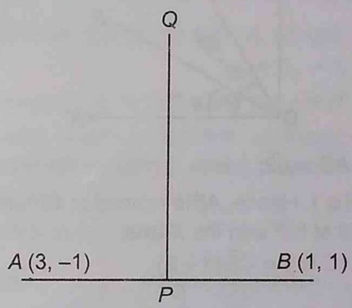
On multiplying Eqs. (i), (ii) and (iii), we get

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{PB} = -1$$

19. Let P be the middle point of the line segment joining $A(3, -1)$ and $B(1, 1)$.

Then, $\quad P = \left(\frac{3+1}{2}, \frac{-1+1}{2}\right) = (2, 0)$

Let P be shifted to Q, where $PQ = 2$ and y-coordinate of Q is greater than that of P (from question).



Now, slope m of $AB = \frac{1-(-1)}{1-3} = \frac{2}{-2} = -1$

$\therefore$ m of $PQ = +1$

Coordinates of Q by distance formula are $(2 \pm 2\cos\theta, 0 \pm 2\sin\theta)$, where $\tan\theta = 1$

$$= \left(2 \pm 2 \cdot \frac{1}{\sqrt{2}}, 0 \pm 2 \cdot \frac{1}{\sqrt{2}}\right)$$

$$= (2 \pm \sqrt{2}, \pm\sqrt{2})$$

As y-coordinate of Q, is greater than that of P.

$$\therefore \quad Q = (2 + \sqrt{2}, \sqrt{2})$$

This is the required point.

20. Let $G(\alpha, \beta)$ be the centroid in any position.

Then,

$$(\alpha, \beta) = \left(\frac{1 + \cos t + \sin t}{3}, \frac{2 + \sin t - \cos t}{3}\right)$$

$$\therefore \quad \alpha = \frac{1 + \cos t + \sin t}{3}$$

$$\text{and} \quad \beta = \frac{2 + \sin t - \cos t}{3}$$

$$\Rightarrow \quad 3\alpha - 1 = \cos t + \sin t \quad \text{...(i)}$$

$$\text{and} \quad 3\beta - 2 = \sin t - \cos t \quad \text{...(ii)}$$

On squaring and adding Eqs. (i) and (ii), we get

$$(3\alpha - 1)^2 + (3\beta - 2)^2 = (\cos t + \sin t)^2 + (\sin t - \cos t)^2$$

$$= 2(\cos^2 t + \sin^2 t) = 2$$

$\therefore$ The equation of the locus of the centroid is

$$(3x-1)^2 + (3y-2)^2 = 2$$

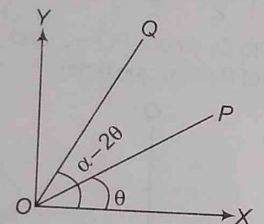
$$\Rightarrow \quad 9(x^2+y^2) - 6x - 12y + 3 = 0$$

$$\Rightarrow \quad 3(x^2+y^2) - 2x - 4y + 1 = 0$$

21. Area of triangle $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix} = 0$

So, points are collinear.

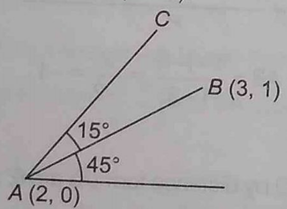
22. OP is inclined at angle θ with X-axis OQ is inclined at angle $\alpha - 2\theta$ with X-axis. The bisector of angle POQ is inclined at angle $\dfrac{\alpha - 2\theta}{2} + \theta = \dfrac{\alpha}{2}$ with X-axis.



23. Since, $AB = AC = \sqrt{2}$

The slope AB is 1. Hence, AB is inclined at 45° with the X-axis and AC is inclined at 60° with the X-axis.

Equation of AC is $y = \sqrt{3}(x-2)$.



The coordinates of C is $(2 + \sqrt{2}\cos 60°, 0 + \sqrt{2}\sin 60°)$

$$\Rightarrow \left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$$

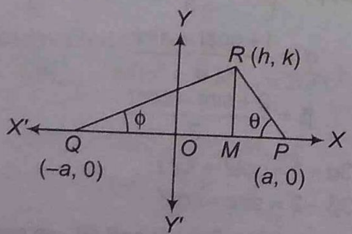
24. Let $\angle RPQ = \theta$ and $\angle RQP = \phi$

$$\therefore \quad \theta - \phi = 2\alpha$$

Let $RM \perp PQ$, so that $RM = k$, $MP = a - h$ and $MQ = a + h$

Then, $\tan\theta = \dfrac{RM}{MP} = \dfrac{k}{a-h}$

and $\tan\phi = \dfrac{RM}{MQ} = \dfrac{k}{a+h}$



Again, now $\quad 2\alpha = \theta - \phi$

$$\therefore \quad \tan 2\alpha = \tan(\theta - \phi)$$

$$= \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

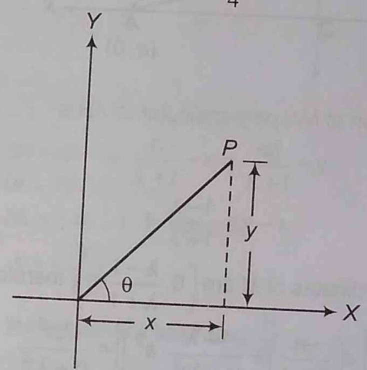
$$= \frac{k(a+h) - k(a-h)}{a^2 - h^2 + k^2}$$

$$\Rightarrow \quad a^2 - h^2 + k^2 = 2hk\cot 2\alpha$$

Hence, the locus is

$$x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$$

25. Clearly, $\quad |y| < |x|$, if $\theta < \dfrac{\pi}{4}$



$$|y| > |x|, \text{ if } \theta > \frac{\pi}{4}$$

and $\quad |y| = |x|$, if $\theta = \dfrac{\pi}{4}$

$$\Rightarrow \quad d(O,P) = |y| \quad \left(\because \theta > \frac{\pi}{4}\right)$$

$$\Rightarrow \quad d(O,P) = y,\ y \ge 0$$

26. We have, $\max\{|x-1|, |y-2|\} = 4$

If $\{|x-1| \ge |y-2|\}$, then $|x-1| = 4$,

i.e., if $(x+y-3)(x-y+1) \ge 0$, then $x = -3$ or 5,

If $|y-2| \ge |x-1|$, then $|y-2| = 4$

i.e., $(x+y-3)(x-y+1) \le 0$,

then $y = -2$ or 6

So, the locus of P bounds a square, the equation of whose sides are $x = -3, x = 5, y = -2, y = 6$.

So, the area of square is 64 sq units.

27. Consider a square with vertices at $(-1, 0)$, $(1, 0)$, $(0, 1)$ and $(0, -1)$.

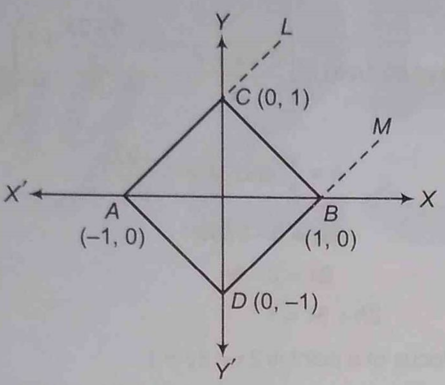
If we select a point P_1 in the region $BCLM$, then $d(A, P) = 1 + x$ $d(B, P) = y$

$$\therefore \quad d(A,P) + d(B,P) = 1 + x + y > 2$$

However, for points P_2 and P_3 lying respectively on the line BC and below the line BC,

$$d(P,A) + d(P,B) = 2$$

The same argument holds for other quadrant also.



Hence, the $d(P, A) + d(P, B) = 2$, represent the regions lying inside the square $ABCD$.

28. Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are all rational coordinates.

$\therefore$ Area $(\Delta ABC) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{\sqrt{3}}{4}[(x_1 - x_2)^2 + (y_1 - y_2)^2]$

LHS = rational, RHS = irrational

Hence, $(x_1, y_1)(x_2, y_2)$ and (x_3, y_3) cannot be all rational.

29. The orthocentre lies on the line joining the points (0, 0) and (3, 5) *i.e.*, $5x - 3y = 0$.

Also, Statement II is true.

30. Statement II is true as the coordinates of the point P in new system are $(\alpha - 1 - 2, \alpha + 1 - 3)$.

In Statement I, the centroid is (2, 3), so the coordinates of the vertices in the new system of coordinates are $(-2, -3)$, (1, 0), (1, 3).

31. Statement II is false as $L_1 + \lambda L_2 = 0$

$\Rightarrow$ Family of concurrent lines, if L_1 and L_2 are intersect.

$\Rightarrow$ Family of parallel lines, if L_1 and L_2 are parallel.

$\Rightarrow$ Family of coincident lines, if L_1 and L_2 are coincident.

As a, b and c are in AP.

$\Rightarrow \quad 2b = a + c$

$\Rightarrow \quad a - 2b + c = 0$

On comparing with $ax + by + c = 0$, it passes thorough fixed points $(1, -2)$.

32. Now, $\Delta = \begin{vmatrix} -\sin(\beta - \alpha) & -\cos\beta & 1 \\ \cos(\beta - \alpha) & \sin\beta & 1 \\ \cos(\beta - \alpha + \theta) & \sin(\beta - \theta) & 1 \end{vmatrix} \neq 0, \forall\, \alpha, \beta$ and θ

Hence, the points P, Q and R are non-collinear.

33. If the centroid is joined to the vertices, we get three triangles of equal area.

$\therefore \quad R = G = \left(3, \frac{4}{3}\right)$

34. $AB^2 = 4 = BC^2 = CA^2$

So, the triangle is equilateral.

$\therefore$ Incentre = Centroid $= \left(1, \frac{1}{\sqrt{3}}\right)$

35. Let the axes be rotated through an angle θ in anti-clockwise direction, then

$x = X\cos\theta - Y\sin\theta$ and $y = X\sin\theta + Y\cos\theta$

$\therefore a_1x^2 + 2h_1xy + b_1y^2 = a_1(X\cos\theta - Y\sin\theta)^2$
$+ 2h_1(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$
$+ b_1(X\sin\theta + Y\cos\theta)^2$

$= (a_1\cos^2\theta + b_1\sin^2\theta + h_1\sin 2\theta)X^2$
$+ XY(2h_1\cos 2\theta - a_1\sin 2\theta + b_1\sin 2\theta)$
$+ (a_1\sin^2\theta - h_1\sin 2\theta + b_1\cos^2\theta)Y^2 \quad \ldots(i)$

It is given that the expression $a_1x^2 + 2h_1xy + b_1y^2$ transforms to

$a_2X^2 + 2h_2XY + b_2Y^2 \quad \ldots(ii)$

by rotating the axes. Therefore, Eqs. (i) and (ii) are the same

$\therefore \quad a_2 = a_1\cos^2\theta + b_1\sin^2\theta + h_1\sin 2\theta \quad \ldots(iii)$

$2h_2 = 2h_1\cos 2\theta - a_1\sin 2\theta + b_1\sin 2\theta \quad \ldots(iv)$

and $\quad b_2 = a_1\sin^2\theta - h_1\sin 2\theta + b_1\cos^2\theta \quad \ldots(v)$

On adding Eqs. (iii) and (v), we get

$a_2 + b_2 = a_1 + b_1$

36. Image of (4, 1) in the line $x = y$ is (1, 4) on translating this point along positive direction of X-axis by 2 units, this point is transformed into (3, 4) and projection of the point (3, 4) on x-axis is (3, 0).

37. Here, $\tan\theta = 2$

So, $\quad \cos\theta = \frac{1}{\sqrt{5}}, \sin\theta = \frac{2}{\sqrt{5}}$

For x and y, we have $x = X\cos\theta - Y\sin\theta = \frac{X - 2Y}{\sqrt{5}}$

and $\quad y = X\sin\theta + Y\cos\theta = \frac{2X + Y}{\sqrt{5}}$

The equation $4xy - 3x^2 = a^2$ reduces to

$\frac{4(X - 2Y)}{\sqrt{5}} \cdot \frac{(2X + Y)}{\sqrt{5}} - 3\left(\frac{X - 2Y}{\sqrt{5}}\right)^2 = a^2$

$\Rightarrow \quad 4(2X^2 - 2Y^2 - 3XY) - 3(X^2 - 4XY + 4Y^2) = 5a^2$

$\Rightarrow \quad 5X^2 - 20Y^2 = 5a^2$

$\therefore \quad X^2 - 4Y^2 = a^2$

38. Area of required triangle $= 5a^2$

39. Mid-point of P and Q is (3, 4). Slope of P and Q is $\frac{5 - 3}{4 - 2} = 1$.

$\therefore$ Equation of line l is $y - 4 = -1(x - 3) \Rightarrow x + y = 7$

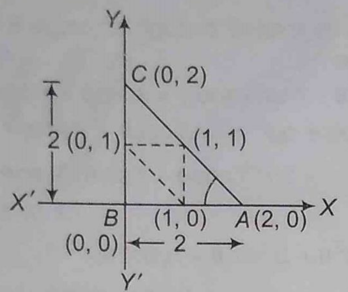
40. Let third vertex be $C(x_1, y_1)$.

$\therefore$ Centroid $\left(\frac{-3 - 2 + x_1}{3}, \frac{2 + 1 + y_1}{3}\right)$ lies on line

$3x + 4y + 2 = 0$

41. Given mid-points of a triangle are (0, 1), (1, 1) and (1, 0). Plotting these points on a graph paper and make a triangle.

So, the sides of a triangle will be 2, 2 and $\sqrt{2^2 + 2^2}$ *i.e.*, $2\sqrt{2}$



$$x\text{-coordinate of incentre} = \frac{2\times 0 + 2\sqrt{2}\cdot 0 + 2\cdot 2}{2+2+2\sqrt{2}}$$

$$= \frac{2}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}} = 2-\sqrt{2}$$

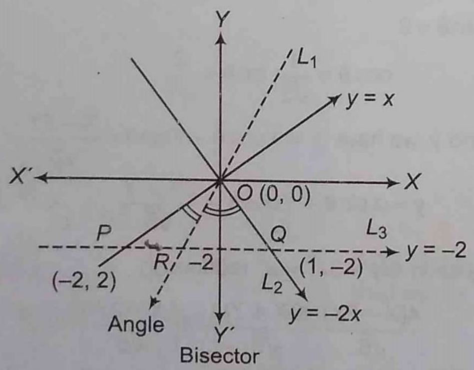
42. Using section formula, the coordinates of the point P, which divides AB internally in the ratio 3 : 2 are

$$P\left(\frac{3\times 2+2\times 1}{3+2}, \frac{3\times 4+2\times 1}{3+2}\right) \equiv P\left(\frac{8}{5}, \frac{14}{5}\right)$$

Also, since the line L passes through P, hence substituting the coordinates of $P\left(\frac{8}{5}, \frac{14}{5}\right)$ in the equation of $L: 2x+y=k$,

we get $2\left(\frac{8}{5}\right) + \left(\frac{14}{5}\right) = k \Rightarrow k = 6$

43. Here, $L_1: y-x=0$ and $L_2: 2x+y=0$ and $L_3: y+2=0$ shown as



$$|PO| = \sqrt{4+4} = 2\sqrt{2}; |OQ| = \sqrt{1+4} = \sqrt{5}$$

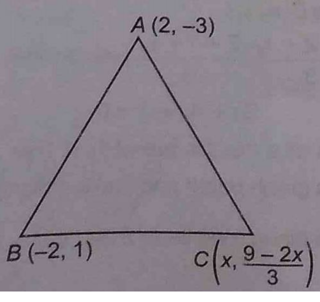
Since, OR is angle bisector $\frac{OP}{OQ} = \frac{PR}{RQ} \Rightarrow \frac{PR}{RQ} = \frac{2\sqrt{2}}{\sqrt{5}}$

Hence, Statement I is true.

But, it does not divide the triangle in two similar triangles.

Hence, Statement II is false.

44. The third vertex lies on $2x+3y=9$. *i.e.*, $\left(x, \frac{9-2x}{3}\right)$



$$\therefore \text{ Locus of centroid is } \left(\frac{2-2+x}{3}, \frac{-3+\frac{9-2x}{3}+1}{3}\right) = (h, k)$$

$$\therefore \quad h = \frac{x}{3} \text{ and } k = \frac{3-2x}{9}$$

$$\Rightarrow \quad 9k = 3-2(3h)$$

$$\Rightarrow \quad 9k = 3-6h$$

$$\Rightarrow \quad 2h+3k = 1$$

Hence, locus of a point is $2x+3y=1$.

45. Let (x, y) denotes the coordinates in A, B and C plane.

Then, $\frac{(x-1)^2+y^2}{(x+1)^2+y^2} = \frac{1}{9}$

$$\Rightarrow \quad 9x^2+9y^2-18x+9 = x^2+y^2+2x+1$$

$$\Rightarrow \quad 8x^2+8y^2-20x+8=0$$

$$\Rightarrow \quad x^2+y^2-\frac{5}{2}x+1=0$$

Hence, A, B and C lie on a circle with $C(5/4, 0)$.

46. Now, $AB = \sqrt{(1-h)^2+(1-k)^2}$; $BC = \sqrt{(2-1)^2+(1-1)^2} = 1$

and $CA = \sqrt{(h-2)^2+(k-1)^2}$

Since, $AC^2 = AB^2 + BC^2$

$$\Rightarrow \quad h^2+4-4h+k^2+1-2k = 1+h^2-2h+k^2+1-2k+1$$

$$\Rightarrow \quad h = 1 \quad \ldots\text{(i)}$$

$$\therefore \text{ Area of } \Delta ABC = \frac{1}{2}\times AB \times BC$$

$$\Rightarrow \quad 1 = \frac{1}{2}\times\sqrt{(1-h)^2+(1-k)^2}\times 1$$

$$\Rightarrow \quad 2 = \sqrt{(k-1)^2} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 4 = k^2+1-2k$$

$$\Rightarrow (k-3)(k+1)=0$$

$$\therefore \quad k = -1, 3$$

47. Let ABC be the triangle with $A(1, 1)$. The mid-point of AB is $(-1, 2)$.

Then, $B(-3, 3)$

The mid-point of AC is $(3, 2) \Rightarrow C(5, 3)$

Hence, the centroid is $\left(\frac{1-3+5}{3}, \frac{1+3+3}{3}\right) = \left(1, \frac{7}{3}\right)$.

48. Let the third vertex be $C(x_1, y_1)$.

The centroid is $\left(\frac{2-2+x_1}{3}, \frac{-3+1+y_1}{3}\right)$

i.e., $\left(\frac{x_1}{3}, \frac{y_1-2}{3}\right)$

It lies on the line $2x+3y=1$.

$$\therefore \quad \frac{2}{3}(x_1) + 3\left(\frac{y_1-2}{3}\right) = 1$$

Hence, the locus of C is $2x+3y=9$.

Day 25

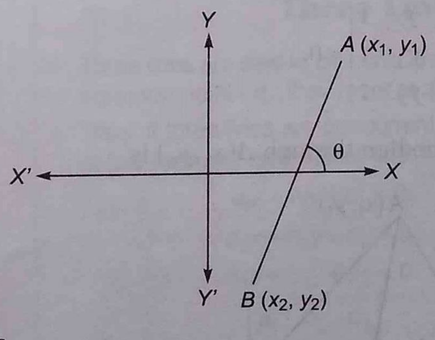
Straight Lines

Day 25 Outlines ...

- Concept of Straight Line
- Various Forms of Equations of a Line
- Angle between Two Lines
- Point of Intersection of Two Lines
- Distance of a Point from a Line

Concept of Straight Line

Any curve is said to be a **straight line**, if any two points are taken on the curve, such that every point on the line segment joining these two points lies on the curve.



The slope of a line AB is

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Various Forms of Equations of a Line

The equation of a line in the general form can be written as $ax + by + c = 0$

1. Slope Intercept Form

The equation of a line with slope m and making an intercept c on Y-axis is

$$y = mx + c$$

2. General Form to Slope Intercept Form

The general form of the equation of a line is

$$Ax + By + C = 0 \Rightarrow By = -Ax - C \quad \Rightarrow$$

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

This is of the form, $y = mx + c$, where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$ .Thus, for the straight line $Ax + By + C = 0$ having slope $m = -\frac{A}{B}$ and intercept on Y-axis $= -\frac{C}{B}$.

3. Point Slope Form

The equation of a line which passes through the point (x_1, y_1) and has the slope m is $y - y_1 = m(x - x_1)$.

4. Two Points Form

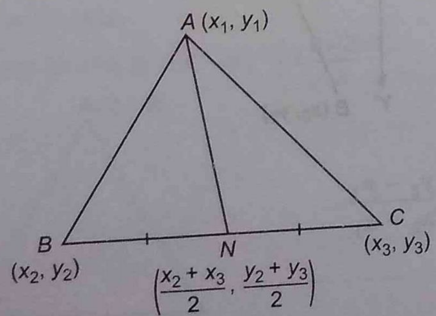
The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$.

5. Determinant Form

(i) The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) can also be written in the determinant form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

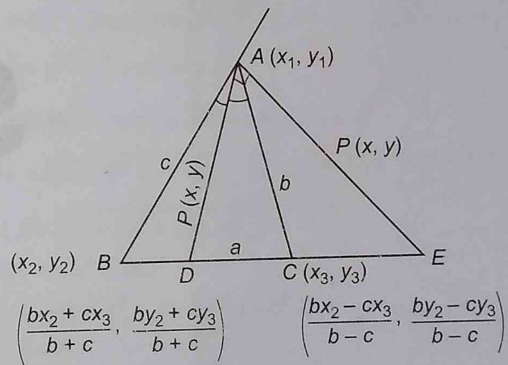
(ii) Equation of the median through $A(x_1, y_1)$ is



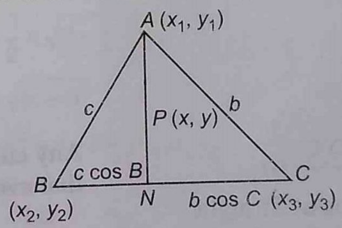
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(iii) Equation of internal and external angle bisector of A are

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \pm c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$



(iv) Equation of the altitude through A is



$$b\cos C\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\cos B\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

6. Intercept Form of a Line

The equation of a line which cuts off intercepts a and b respectively from the X and Y-axes is $\frac{x}{a} + \frac{y}{b} = 1$.

7. General Equation of a Line to Intercept Form

The general equation of a line $Ax + By + C = 0$ is

$$\frac{x}{-\left(\frac{C}{A}\right)} + \frac{y}{-\left(\frac{C}{B}\right)} = 1.$$

8. **Normal or Perpendicular Form**

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with X-axis is

$$x\cos\alpha + y\sin\alpha = p, \text{ where } 0 \le \alpha \le \pi.$$

9. **General Equation of a Line to the Normal Form**

The general equation of a line is

$$Ax + By + C = 0.$$

Now, to reduce the general equation of a line to normal form, we first shift the constant term on the RHS and make it positive, if it is not so and then divide both sides by

$$\sqrt{(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2}$$

$$\Rightarrow \left(\frac{A}{\sqrt{A^2+B^2}}\right)x + \left(\frac{B}{\sqrt{A^2+B^2}}\right)y = \left(\frac{-C}{\sqrt{A^2+B^2}}\right)$$

10. **Distance Form**

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of X-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where r is the distance of the point (x, y) on the line from the point (x_1, y_1).

- *The area of triangle formed by the lines*

$$y = m_1x + c_1,\ y = m_2x + c_2,$$

$$y = m_3x + c_3 \text{ is } \frac{1}{2}\left|\sum \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|$$

- *The foot of the perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by*

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

- *Area of the parallelogram formed by the lines*

$$a_1x + b_1y + c_1 = 0;\ a_2x + b_2y + c_2 = 0$$

$$a_1x + b_1y + d_1 = 0;\ a_2x + b_2y + d_2 = 0 \text{ is}$$

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}\right|$$

- *Foot of perpendicular from (a, b) on*

$$x - y = 0 \text{ is } \left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$

- *Foot of perpendicular from (a, b) on*

$$x + y = 0 \text{ is } \left(\frac{a-b}{2}, \frac{b-a}{2}\right).$$

Angle between Two Lines

The angle θ between the lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|.$$

1. **Parallelism of Lines**

If two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is $0°$.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1m_2} = 0$$

$$\Rightarrow \quad m_1 = m_2$$

Thus, when two lines are parallel, their slopes are equal.

2. **Perpendicularity of Two Lines**

If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is $90°$.

$$\therefore \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{1 + m_1m_2}{m_1 - m_2} = 0$$

$$\Rightarrow \quad m_1 \cdot m_2 = -1$$

Thus, when two lines are perpendicular, the product of their slopes is -1.

If m is the slope of a line, then the slope of a line perpendicular to it is $\left(-\frac{1}{m}\right)$.

Conditions for Concurrence of Three Lines

- Three lines are said to be concurrent, if they pass through a common point *i.e.*, they meet at a point.
- Thus, if three lines are concurrent, the point of intersection of two lines lies on the third line.

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots(iii)$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrence of three lines.

Point of Intersection of Two Lines

Let the equations of two lines be

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

and $$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

Suppose these two lines intersect at a point $P(x_1, y_1)$. Then, (x_1, y_1) satisfies each of the given equations.

$\therefore$ $$a_1x_1 + b_1y_1 + c_1 = 0$$

and $$a_2x_1 + b_2y_1 + c_2 = 0$$

On solving these two equations by cross multiplication method, we get

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$\Rightarrow$ $$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

and $$y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1},$$

which is the required intersection point.

Distance of a Point from a Line

The length of the perpendicular from a point (x_1, y_1) to a line

$$ax + by + c = 0 \text{ is } \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|.$$

(i) The length of the perpendicular from the origin to the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$.

(ii) Distance between parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$.

Equation of Internal and External Bisectors of Angles between Two Lines

The bisectors of the angles between two straight lines are the locus of a point which is equidistant from the two lines.

The equation of the bisector of the angles between the lines

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

and $$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

are given by, $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

where, if

$a_1a_2 + b_1b_2 > 0 \Rightarrow$ the positive sign for obtuse and negative sign for acute.

$a_1a_2 + b_1b_2 < 0 \Rightarrow$ negative sign for obtuse and positive sign for acute.

Equation of Family of Lines Through the Intersection of Two Given Lines

The equation of the family of lines passing through the intersection of the lines

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0 \text{ is}$$

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0,$$

where λ is a parameter.

Important Results

- The image of the line $a_1x + b_1y + c_1 = 0$ in the line $ax + by + c = 0$ is
  $$2(aa_1 + bb_1)(ax + by + c) = (a^2 + b^2)(a_1x + b_1y + c_1)$$
- The position of a point (x_1, y_1) and (x_2, y_2) relative to the line $ax + by + c = 0$
  (a) If $\frac{(ax_1 + by_1 + c)}{ax_2 + by_2 + c} > 0$, then points lie on the same side.
  (b) If $\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$, then the points lie on opposite side.
- The slope m of a line which is equally inclined with two lines of slopes m_1 and m_2 is given by $\frac{m - m_1}{1 + mm_1} = \frac{m_2 - m}{1 + mm_2}$.
- A point (x_1, y_1) will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the same sign.
- A point (x_1, y_1) will lie on the opposite side of the origin relative to the line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the opposite sign.
- The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.
- The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.
- The equations of the straight line which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are $(y - y_1) = \frac{m \pm \tan\alpha}{1 \mp \tan\alpha}(x - x_1)$.

Practice Zone DAY 25

1. The line parallel to the X-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is
(a) above the X-axis at a distance of (2/3) from it
(b) above the X-axis at a distance of (3/2) from it
(c) below the X-axis at a distance of (2/3) from it
(d) below the X-axis at a distance of (3/2) from it

2. If the two rays $x + y = |a|$ and $ax - y = 1$ intersect in the first quadrant, then $a \in (a_0, \infty)$, where a_0 is equal to
(a) 0 (b) -1
(c) 1 (d) 2

3. The image of the lines $2x - y = 1$ in the line $x + y = 0$ is
(a) $x + 2y = 1$ (b) $x - 2y = 1$
(c) $x + 2y = -1$ (d) $2x + y = 1$

4. The image of the curve $x^2 + y^2 = 1$ in the line $x + y = 1$ is
(a) $x^2 + y^2 + 2x + 2y + 1 = 0$
(b) $x^2 + y^2 - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 + x + y + 1 = 0$
(d) $x^2 + y^2 + x + y + 2 = 0$

5. The sides BC, CA, AB of ΔABC are respectively $x + 2y = 1$, $3x + y + 5 = 0$, $x - y + 2 = 0$. The altitude through B is
(a) $x - 3y + 1 = 0$ (b) $x - 3y + 4 = 0$
(c) $3x - y + 4 = 0$ (d) $x - y + 2 = 0$

6. The equations of the straight lines through $(-2, -7)$ and having intercept of length 3 between the lines $4x + 3y = 12$ and $4x + 3y = 3$ is
(a) $7x - 24y - 182 = 0$ (b) $7x + 24y + 182 = 0$
(c) $7x + 24y - 182 = 0$ (d) None of these

7. A straight line segment of length t moves with its ends on two mutually perpendicular lines. The locus of the point which divides the line in the ratio 1 : 2 is
(a) $9(x^2 - 4y^2) = 4t^2$ (b) $9(x^2 + 4y^2) = -4t^2$
(c) $9(x^2 - 4y^2) = -4t^2$ (d) $9(x^2 + 4y^2) = 4t^2$

8. Find the equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3, 2) is $\frac{7}{5}$. **[NCERT Exemplar]**
(a) $3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$
(b) $3x + 4y + 6 = 0$ and $4x + 3y + 1 = 0$
(c) $3x - 4y - 6 = 0$ and $4x + 3y + 1 = 0$
(d) None of the above

9. If P is a point (x, y) on the line $y = -3x$ such that P and the point (3, 4) are on the opposite sides of the line $3x - 4y = 8$, then
(a) $x > \frac{8}{15}, y < -\frac{8}{5}$ (b) $x > \frac{8}{5}, y < \frac{8}{15}$
(c) $x = \frac{8}{15}, y = -\frac{8}{5}$ (d) None of these

10. A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$. The equation to the line AC so that $AB = AC$ is
(a) $52x - 89y - 519 = 0$ (b) $52x + 89y - 519 = 0$
(c) $52x - 89y + 519 = 0$ (d) $52x + 89y + 519 = 0$

11. For all real values of a and b lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ and $mx + 2y + 6 = 0$ are concurrent, then m is equal to
(a) -2 (b) -3 (c) -4 (d) -5

12. Consider the family of lines
$$(x + y - 1) + \lambda(2x + 3y - 5) = 0$$
and $(3x + 2y - 4) + \mu(x + 2y - 6) = 0$,
equation of a straight line that belongs to both the families is
(a) $x - 2y - 8 = 0$ (b) $x - 2y + 8 = 0$
(c) $2x + y - 8 = 0$ (d) $2x - y - 8 = 0$

13. If $P\left(1 + \frac{\alpha}{\sqrt{2}}, 2 + \frac{\alpha}{\sqrt{2}}\right)$ be any point on a line, then the range of values of t for which the point P lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$ is
(a) $-\frac{4\sqrt{2}}{3} < \alpha < \frac{5\sqrt{2}}{6}$ (b) $0 < \alpha < \frac{5\sqrt{2}}{6}$
(c) $\frac{-4\sqrt{2}}{3} < \alpha < 0$ (d) None of these

14. A ray of light coming along the line $3x + 4y - 5 = 0$ gets reflected from the line $ax + by - 1 = 0$ and goes along the line $5x - 12y - 10 = 0$, then

(a) $a = \frac{64}{115}, b = \frac{112}{15}$ (b) $a = -\frac{64}{115}, b = \frac{8}{115}$
(c) $a = \frac{64}{115}, b = -\frac{8}{115}$ (d) $a = -\frac{64}{115}, b = -\frac{8}{115}$

15. Given n straight lines and a fixed point O, a straight line is drawn through O meeting lines in the points $R_1, R_2, R_3, \ldots, R_n$ and on it a point R is taken such that $\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \ldots + \frac{1}{OR_n}$, then the locus of R is

(a) a straight line (b) a circle
(c) a parabola (d) None of these

16. One side of a square of length a makes an angle α with X-axis and one vertex of the square is at the origin. Then, the equations of its diagonals are

(a) $y(\cos\alpha + \sin\alpha) = x(\sin\alpha + \cos\alpha)$
and $(\sin\alpha + \cos\alpha)y + x(\cos\alpha + \sin\alpha) = a$
(b) $y(\cos\alpha - \sin\alpha) = x(\sin\alpha + \cos\alpha)$
and $(\sin\alpha + \cos\alpha)y + x(\cos\alpha - \sin\alpha) = a$
(c) $y(\sin\alpha - \cos\alpha) = x(\sin\alpha + \cos\alpha)$
and $(\sin\alpha + \cos\alpha)y + x(\cos\alpha - \sin\alpha) = a$
(d) $y(\cos\alpha - \sin\alpha) = x(\sin\alpha + \cos\alpha)$
and $y(\sin\alpha - \cos\alpha) + x(\cos\alpha + \sin\alpha) = a$

17. If $u = a_1x + b_1y + c_1 = 0$, $v = a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then $u + \lambda v = 0$ is

(a) family of concurrent lines (b) family of parallel lines
(c) $u = 0$ or $v = 0$ (d) None of these

18. If (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Then, the minimum area of the ΔOPQ; where O being the origin, is

(a) $4hk$ sq units (b) $2hk$ sq units
(c) $3hk$ sq units (d) None of these

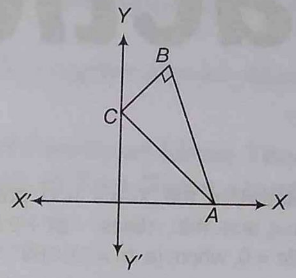
19. Abscissae and ordinates of n given points are in AP with first term a and common difference 1 and 2, respectively. If algebraic sum of perpendiculars from these given points on a variable line which always passes through the point $\left(\frac{13}{2}, 11\right)$ is zero, then the values of a and n is

(a) 2, 10 (b) 4, 20 (c) 3, 9 (d) 0, 1

20. Consider the point $A \equiv (3, 4)$ and $B \equiv (7, 13)$. If P be a point of the line $y = x$ such that $PA + PB$ is minimum, then coordinates of P is

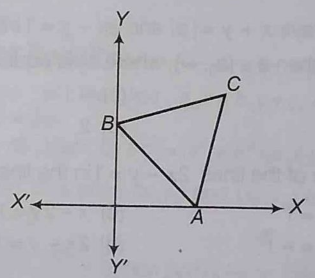
(a) $\left(\frac{13}{7}, \frac{13}{7}\right)$ (b) $\left(\frac{23}{7}, \frac{23}{7}\right)$ (c) $\left(\frac{31}{7}, \frac{31}{7}\right)$ (d) $\left(\frac{33}{7}, \frac{33}{7}\right)$

21. In the adjacent figure, ΔABC is right angle at B. If $AB = 4$ and $BC = 3$ and side AC slides along the coordinate axes in such a way that B always remains in the first quadrant, then B always lies on the straight line



(a) $y = x$ (b) $3y = 4x$
(c) $4y = 3x$ (d) $x + y = 0$

22. Adjacent figure represents a equilateral ΔABC of side length 2 units.



Locus of vertex C as the side AB slides along the coordinates axes is

(a) $x^2 + y^2 - xy + 1 = 0$ (b) $x^2 + y^2 + xy\sqrt{3} = 1$
(c) $x^2 + y^2 = 1 + xy\sqrt{3}$ (d) $x^2 + y^2 - xy\sqrt{3} + 1 = 0$

23. If the four lines with equations $x + 2y - 3 = 0$, $3x + 4y - 7 = 0$, $2x + 3y - 4 = 0$, $4x + 5y - 6 = 0$, then

(a) they are all concurrent
(b) they are the sides of a non-cyclic quadrilateral
(c) they form a cyclic quadrilateral
(d) None of the above

24. A variable straight line drawn through the point of intersection of the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ meets the coordinates axes at A and B, the locus of the mid-point of AB is

(a) $2xy(a + b) = ab(x + y)$ (b) $2xy(a - b) = ab(x - y)$
(c) $2xy(a + b) = ab(x - y)$ (d) None of these

25. Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Find the equation of the third side.

(a) $x - 3y = -31$ (b) $x - 3y = 31$
(c) $x + 3y = 31$ (d) $x + 3y = -31$

26. The equation of the line passing through the point (2, 3) and making an intercept of length 2 between the lines $y + 2x = 3$ and $y + 2x = 5$

(a) $x = -2, 3x + 4y = 18$ (b) $x = 2, 3x - 4y = 18$
(c) $x = 2, 3x + 4y = 18$ (d) None of these

27. The point Q is the image of point $P(a, b)$ in the line $x - y = 0$. Then, the foot of perpendicular from Q on the line $x + y = 0$ is

(a) $(a - b, b - a)$ (b) $(b - a, a - b)$
(c) $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$ (d) $\left[\frac{b-a}{2}, \frac{a-b}{2}\right]$

28. Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2). **[NCERT Exemplar]**

(a) $x + 7y + 12 = 0,\ 7x + y - 16 = 0$
(b) $x - 7y + 6 = 0,\ 7x + y - 16 = 0$
(c) $x - 7y + 12 = 0,\ 7x + y + 16 = 0$
(d) None of the above

29. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$. **[NCERT]**

(a) $3x + y + 7 = 0$ (b) $3x - y - 7 = 0$
(c) $3x + 2y - 7 = 0$ (d) None of these

30. A ray of light passing through the point (1, 2) reflected on the X-axis at point A and the reflected ray passes through the point (5, 3). Find coordinates of A. **[NCERT]**

(a) $\left(\frac{13}{5}, 0\right)$ (b) $\left(-\frac{13}{5}, 0\right)$ (c) $\left(0, \frac{13}{5}\right)$ (d) None of these

31. For all values of θ, the lines represented by the equation

$$(2\cos\theta + 3\sin\theta)\,x + (3\cos\theta - 5\sin\theta)\,y - (5\cos\theta - 2\sin\theta) = 0$$

(a) pass through the point (1, 2)
(b) pass through the point (0, 1)
(c) pass through a fixed point, where reflection in the line $x + y = \sqrt{2}$ is $(\sqrt{2} - 1, \sqrt{2} - 1)$
(d) pass through the origin

Directions (Q. Nos. 32 to 34) *Let the lines are*

$$L_1 : 3x + 4y - 8 = 0 \text{ and } L_2 : 2x + 7y - 1 = 0.$$

32. If L_1, L_2 represent the sides AB and AC of the isosceles ΔABC with $AB = AC = 2$, then the coordinates of

(a) B are $(28/5, -11/5)$
(b) B are $(28/5, 1/5)$
(c) C are $\left(\frac{14 + 4\sqrt{53}}{\sqrt{53}}, \frac{-4 - \sqrt{5}}{\sqrt{5}}\right)$
(d) C are $\left(\frac{14 + 4\sqrt{53}}{\sqrt{53}}, \frac{4 - \sqrt{5}}{\sqrt{5}}\right)$

33. Equation of the line through B parallel to AC is

(a) $2x + 7y + 21 = 0$ (b) $10x + 35y = 63$
(c) $10x + 35y + 21 = 0$ (d) $2x + 7y = 63$

34. If D is the mid-point of BC and E is the mid-point of CA, then DE is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

Directions (Q. Nos. 35 to 38) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

35. Statement I Each point on the line $y - x + 12 = 0$ is equidistant from the lines $4y + 3x - 12 = 0$, $3y + 4x - 24 = 0$.

Statement II The locus of a point which is equidistant from two given lines is the angular bisector of the two lines.

36. Let $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$ and there exist a line through the point (a, n) in the cartesian plane.

Statement I If the line through (a, n) cuts the circle $x^2 + y^2 = 4$ in A and B, then $PA \cdot PB = 16$.

Statement II The point (a, n) lies outside the circle.

37. Statement I Consider the points $A(0, 1)$ and $B(2, 0)$ and P be a point on the line $4x + 3y + 9 = 0$, then coordinates of P such that $|PA - PB|$ is maximum, is $\left(\frac{-12}{5}, \frac{17}{5}\right)$.

Statement II $|PA - PB| \le |AB|$

38. Statement I If point of intersection of the lines $4x + 3y = \lambda$ and $3x - 4y = \mu, \forall \lambda, \mu \in R$ is (x_1, y_1), then the locus of (λ, μ) is $x + 7y = 0, \forall x_1 = y_1$.

Statement II If $4\lambda + 3\mu > 0$ and $3\lambda - 4\mu > 0$, then (x_1, y_1) is in first quadrant.

39. A straight line through the origin meets the parallel lines $4x + 2y = 9$ and $2x + y = -6$ at points P and Q, respectively. Then, the point O divides the segment PQ in the ratio

(a) 1 : 2 (b) 3 : 4 (c) 2 : 1 (d) 4 : 3

40. Area of the parallelogram formed by the lines $y = mx, y = mx + 1, y = nx, y = nx + 1$ is equal to

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$
(c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

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41. Equation of the line passing through the points of intersection of the of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is **[JEE Main 2013]**

(a) $y - 3 = 0$ (b) $y + 3 = 0$
(c) $3y + 1 = 0$ (d) $3y - 1 = 0$

42. A light ray emerging from the point source placed at $P(1, 3)$ is reflected at a point Q in the axis of x. If the reflected ray passes through the point $R(6, 7)$, then the abscissa of Q is **[JEE Main 2013]**

(a) 1 (b) 3
(c) $\frac{7}{2}$ (d) $\frac{5}{2}$

43. If the three lines $x - 3y = p$, $ax + 2y = q$ and $ax = y = r$ from a right-angled triangle then **[JEE Main 2013]**

(a) $a^2 - 9a + 18 = 0$ (b) $a^2 - 6a - 12 = 0$
(c) $a^2 - 6a - 18 = 0$ (d) $a^2 - 9a + 12 = 0$

44. If the x-intercept of some line L is double as that of the line, $3x + 4y = 12$ and the y-intercept of L is half as that of the same line, then the slope of L is **[JEE Main 2013]**

(a) -3 (b) $-\frac{3}{8}$ (c) $-\frac{3}{2}$ (d) $-\frac{3}{16}$

45. Let θ_1 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_2 = 0$ and θ_2 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$, where c_1, c_2, c_3 are any real numbers.

Statement I If c_2 and c_3 are proportional, then $\theta_1 = \theta_2$.

Statement II $\theta_1 = \theta_2$ for all c_2 and c_3. **[JEE Main 2013]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

46. If two lines L_1 and L_2 in space are defined by
$L_1 = \{x = \sqrt{\lambda}\, y + (\sqrt{\lambda} - 1),\ z = (\sqrt{\lambda} - 1)\, y + \sqrt{\lambda}\}$ and
$L_2 = \{x = \sqrt{\mu}\, y + (1 - \sqrt{\mu}),\ z = (1 - \sqrt{\mu})\, y + \sqrt{\mu}\}$, then L_1 is perpendicular to L_2, for all non-negative reals λ and μ, such that **[JEE Main 2013]**

(a) $\sqrt{\lambda} + \sqrt{\mu} = 1$ (b) $\lambda \neq \mu$
(c) $\lambda + \mu = 0$ (d) $\lambda = \mu$

47. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching X-axis, the equation of the reflected ray is

(a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$
(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$

48. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then, the set of all possible values of a in the interval **[AIEEE 2011]**

(a) $(-1, 1]$ (b) $(0, \infty)$
(c) $[1, \infty)$ (d) $(-1, \infty)$

49. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point $(13, 32)$. The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then, the distance between L and K is **[AIEEE 2010]**

(a) $\frac{23}{\sqrt{15}}$ (b) $\sqrt{17}$ (c) $\frac{17}{\sqrt{15}}$ (d) $\frac{23}{\sqrt{17}}$

50. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1) y + 2q = 0$ are perpendicular to a common line for **[AIEEE 2009]**

(a) exactly one value of p (b) exactly two values of p
(c) more than two values of p (d) no values of p

51. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept -4. Then, a possible value of k is **[AIEEE 2008]**

(a) 4 (b) 1
(c) 2 (d) -2

52. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. The equation of the bisector of the $\angle PQR$ is **[AIEEE 2007]**

(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$
(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

53. A straight line through the point $A(3, 4)$ is such that is intercept between the axis is bisected at A. Its equation is **[AIEEE 2006]**

(a) $4x + 3y = 24$ (b) $3x + 4y = 25$
(c) $x + y = 7$ (d) $3x - 4y + 7 = 0$

54. If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}, x > 0$ and $y = 3x, x > 0$, then a belong to **[AIEEE 2006]**

(a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(-3, -\frac{1}{2}\right)$
(c) $\left(0, \frac{1}{2}\right)$ (d) $(3, \infty)$

55. If non-zero numbers a, b and c are in HP, then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point. That point is **[AIEEE 2005]**

(a) $\left(1, -\frac{1}{2}\right)$ (b) $(1, -2)$
(c) $(-1, -2)$ (d) $(-1, 2)$

56. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is – 1, is **[AIEEE 2004]**

(a) $\frac{x}{2}+\frac{y}{3}=-1, \frac{x}{-2}+\frac{y}{1}=-1$

(b) $\frac{x}{2}-\frac{y}{3}=-1, \frac{x}{-2}+\frac{y}{1}=-1$

(c) $\frac{x}{2}+\frac{y}{3}=1, \frac{x}{2}+\frac{y}{1}=1$

(d) $\frac{x}{2}-\frac{y}{3}=1, \frac{x}{-2}+\frac{y}{1}=1$

57. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle $\alpha\left(\text{where, } 0<\alpha<\frac{\pi}{4}\right)$ with the positive direction of X-axis. The equation of its diagonal not passing through the origin is **[AIEEE 2003]**

(a) $y(\cos\alpha-\sin\alpha)-x(\sin\alpha-\cos\alpha)=a$

(b) $y(\cos\alpha+\sin\alpha)-x(\sin\alpha-\cos\alpha)=a$

(c) $y(\cos\alpha+\sin\alpha)+x(\sin\alpha+\cos\alpha)=a$

(d) $y(\cos\alpha+\sin\alpha)+x(\sin\alpha-\cos\alpha)=a$

Answers

1. (d)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (a)	**10.** (d)
11. (a)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (a)	**16.** (b)	**17.** (c)	**18.** (b)	**19.** (a)	**20.** (c)
21. (b)	**22.** (c)	**23.** (d)	**24.** (a)	**25.** (b)	**26.** (c)	**27.** (d)	**28.** (d)	**29.** (b)	**30.** (a)
31. (c)	**32.** (a)	**33.** (c)	**34.** (c)	**35.** (a)	**36.** (b)	**37.** (d)	**38.** (b)	**39.** (b)	**40.** (d)
41. (d)	**42.** (d)	**43.** (a)	**44.** (d)	**45.** (a)	**46.** (d)	**47.** (b)	**48.** (c)	**49.** (d)	**50.** (a)
51. (a)	**52.** (c)	**53.** (a)	**54.** (a)	**55.** (b)	**56.** (d)	**57.** (b)			

Hints & Solutions

1. The intersection of lines is

$$ax+2by+3b+\lambda(bx-2ay-3a)=0 \quad \ldots(i)$$

or $\quad (a+\lambda b)x+(2b-2a\lambda)y+3b-3a\lambda=0$

which is parallel to the X-axis.

$\Rightarrow \quad a+\lambda b=0 \quad \text{or} \quad \lambda=-\frac{a}{b}$

$\therefore$ Eq. (i) becomes, $\quad y=-\frac{3}{2}$

2. Since, $x+y=|a|, ax-y=1$

$\Rightarrow \quad x=\frac{|a|+1}{a+1}$ and $y=\frac{a|a|-1}{a+1}$

But $x>0, y>0$

$\Rightarrow \quad a+1>0$ or $a>-1$ and $a|a|-1>0, a>0$

$\Rightarrow \quad a^2-1>0$

$\Rightarrow \quad a>1$ *i.e.*, $a_0=1$

3. The image of the point $(\alpha, 2\alpha-1)$ lying on the first line is given by

$$x=1-2\alpha, y=-\alpha$$

$\therefore \quad x-2y=1$

4. Any point of the curve is $x=\cos\theta$, $y=\sin\theta$. Its image is given by

$$\frac{x-\cos\theta}{1}=\frac{y-\sin\theta}{1}=\frac{-2(\cos\theta+\sin\theta-1)}{2}$$

$\therefore \quad x=1-\sin\theta, y=1-\cos\theta$

On eliminating θ, we get

$$(x-1)^2+(y-1)^2=1$$

$\Rightarrow \quad x^2+y^2-2x-2y+1=0$

5. The required line is

$$x+2y-1+\lambda(x-y+2)=0 \quad \ldots(i)$$

It is perpendicular to $3x+y+5=0$

$\therefore \quad 3(1+\lambda)+2-\lambda=0$

$\Rightarrow \quad \lambda=-\frac{5}{2}$

From Eq. (i),

$$x+2y-1-\frac{5}{2}(x-y+2)=0$$

$\Rightarrow \quad x-3y+4=0$

6. Let m be the slope of the line and angle θ it makes with the parallel line.

$\therefore \quad \sin\theta=\frac{3}{5}$ or $\tan\theta=\frac{3}{4}$

Hence, slope of the parallel lines is $-\frac{4}{3}$.

$\therefore \quad \left|\frac{m+\frac{4}{3}}{1-\frac{4m}{3}}\right|=\tan\theta=\frac{3}{4}$

$$\Rightarrow \quad \frac{3m+4}{3-4m} = \pm\frac{3}{4} \quad \Rightarrow \quad m = -\frac{7}{24}$$

The line is $y + 7 = -\frac{7}{24}(x+2)$ or $7x + 24y + 182 = 0$.

7. Taking the perpendicular lines as the coordinate axes, let the ends be $A(a, 0)$ and $B(0, b)$, we have

$$a^2 + b^2 = t^2 \quad \text{...(i)}$$

If $P(x, y)$ divides AB in the ratio $1 : 2$, we have

$$3x = 2a, 3y = b \quad \text{...(ii)}$$

On eliminating a, b from Eqs. (i) and (ii), we get the locus of P as

$$9(x^2 + 4y^2) = 4t^2$$

8. Equation of a line passing through the point of intersection of lines is

$$x - y + 1 + \lambda(2x - 3y + 5) = 0$$

$$\Rightarrow \quad x(1+2\lambda) + y(-1-3\lambda) + 1 + 5\lambda = 0 \quad \text{...(i)}$$

Its distance from point $(3, 2) = \frac{7}{5}$

$$\Rightarrow \quad \frac{|3(1+2\lambda)+2(-1-3\lambda)+1+5\lambda|}{\sqrt{(1+2\lambda)^2+(-1-3\lambda)^2}} = \frac{7}{5}$$

$$\Rightarrow \quad \frac{|2+5\lambda|}{\sqrt{13\lambda^2+10\lambda+2}} = \frac{7}{5}$$

On squaring, we get $25(4 + 25\lambda^2 + 20\lambda) = 49(13\lambda^2 + 10\lambda + 2)$

$$\Rightarrow \quad 6\lambda^2 - 5\lambda - 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{6}$$

On putting $\lambda = 1, -\frac{1}{6}$ in Eq. (i) respectively, we get

$$3x - 4y + 6 = 0 \quad \text{and} \quad 4x - 3y + 1 = 0$$

9. Let $L_1 = 3x - 4y - 8$

At $(3, 4)$, $L_1 = 9 - 16 - 8 = -15 < 0$

For the point $P(x, y)$, we should have $L_1 > 0$.

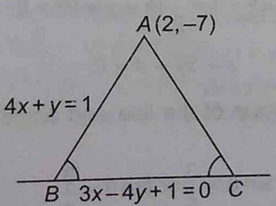
$$\Rightarrow \quad 3x - 4y - 8 > 0 \quad (\because y = -3x)$$

$$\Rightarrow \quad 3x - 4(-3x) - 8 > 0$$

$$[\because P(x, y) \text{ lies on } y = -3x]$$

$$\Rightarrow \quad x > 8/15 \text{ and } -y - 4y - 8 > 0 \Rightarrow y < -8/15$$

10. Let m be the slope of AC, then



$$\tan B = \tan C$$

$$\Rightarrow \quad \frac{\frac{3}{4}+4}{1-3} = \frac{m-\frac{3}{4}}{1+\frac{3m}{4}}$$

$$\Rightarrow \quad -\frac{19}{8} = \frac{4m-3}{4+3m}$$

$$\Rightarrow \quad m = -\frac{52}{89}$$

$\therefore$ Equation of AC is $y + 7 = -\frac{52}{89}(x-2)$

$$\Rightarrow \quad 52x + 89y + 519 = 0$$

11. Given equations,

$$(2a+b)x + (a+3b)y + (b-3a) = 0$$

and $mx + 2y + 6 = 0$ are concurrent for all real values of a and b, so they must represent the same line for same values of a and b. Then,

$$\frac{2a+b}{m} = \frac{(a+3b)}{2} = \frac{b-3a}{6}$$

Taking last two ratios,

$$\frac{a+3b}{2} = \frac{-3a+b}{6} \quad \Rightarrow \quad b = -\frac{3}{4}a$$

Taking first two ratios,

$$m = \frac{2(2a+b)}{a+3b} = \frac{2\{2a-(3/4)a\}}{a+3(-3/4)a} = -\frac{10}{5} = -2$$

12. The family of lines

$$(x + y - 1) + \lambda(2x + 3y - 5) = 0$$

passes through a point such that

$$x + y - 1 = 0$$
$$2x + 3y - 5 = 0$$

i.e., $(-2, 3)$ and family of lines

$$(3x + 2y - 4) + \mu(x + 2y - 6) = 0$$

passes through a point such that

$$3x + 2y - 4 = 0$$

and $x + 2y - 6 = 0$ *i.e.*, $\left(-1, \frac{7}{2}\right)$

$\therefore$ Equation of the straight line that belongs to both the families passes through $(-2, 3)$ and $(-1, 7/2)$ is

$$y - 3 = \frac{\frac{7}{2}-3}{-1+2}(x+2)$$

$$\Rightarrow \quad y - 3 = \frac{x+2}{2} \quad \Rightarrow \quad x - 2y + 8 = 0$$

13. $\because P\left(1+\frac{\alpha}{\sqrt{2}}, 2+\frac{\alpha}{\sqrt{2}}\right)$ lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$, then

$$\frac{\left(1+\frac{\alpha}{\sqrt{2}}\right)+2\left(2+\frac{\alpha}{\sqrt{2}}\right)-1}{2\left(1+\frac{\alpha}{\sqrt{2}}\right)+4\left(2+\frac{\alpha}{\sqrt{2}}\right)-15} < 0$$

$$\Rightarrow \quad \frac{4+\frac{3\alpha}{\sqrt{2}}}{-5+\frac{6\alpha}{\sqrt{2}}} < 0 \Rightarrow \frac{\left(\alpha+\frac{4\sqrt{2}}{3}\right)}{\left(\alpha-\frac{5\sqrt{2}}{6}\right)} < 0$$

$$\therefore \quad \frac{-4\sqrt{2}}{3} < \alpha < \frac{5\sqrt{2}}{6}$$

14. Equation of bisectors of the given lines are

$$\left(\frac{3x+4y-5}{\sqrt{3^2+4^2}}\right)=\pm\left(\frac{5x-12y-10}{\sqrt{5^2+(-12)^2}}\right)$$

$\therefore$ $(39x+52y-65)=\pm(25x-60y-50)$

$\Rightarrow$ $14x+112y-15=0$

$\Rightarrow$ $64x-8y-115=0$

$\Rightarrow$ $\frac{14}{15}x+\frac{112}{15}y-1=0$

$\Rightarrow$ $\frac{64}{115}x-\frac{8}{115}y-1=0$

$\therefore$ $a=\frac{14}{15}, b=\frac{112}{15}$

$\Rightarrow$ $a=\frac{64}{115}, b=-\frac{8}{115}$

15. Let equation of the given lines be $a_ix+b_iy+c_i=0$, $i=1,2,\ldots,n$ and the point O be the origin (0, 0). Then, equation of the line through O can be written as

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r$$

where, θ is the angle made by the line with the positive direction of x-axis and r is the distance of any point on the line from the origin O.

Let $r, r_1, r_2, \ldots, r_n$ be the distance of the points $R, R_1, R_2, \ldots, R_n$ from $O \Rightarrow OR=r$ and $OR_1=r_1, \ldots,$

Then, coordinates of R are $(r\cos\theta, r\sin\theta)$ and of R_i are $(r_i\cos\theta, r_i\sin\theta)$; $i=1,2,\ldots,n$.

Since, R_i lies on $a_ix+b_iy+c_i=0$

$\Rightarrow$ $a_ir_i\cos\theta+b_ir_i\sin\theta+c_i=0$, for $i=1,2,\ldots,n$.

$\Rightarrow$ $-\frac{a_i}{c_i}\cos\theta-\frac{b_i}{c_i}\sin\theta=\frac{1}{r_i}, i=1,2,\ldots,n$

$\Rightarrow$ $\sum_{i=1}^{n}\frac{1}{r_i}=-\left[\sum_{i=1}^{n}\frac{a_i}{c_i}\right]\cos\theta-\left[\sum_{i=1}^{n}\frac{b_i}{c_i}\right]\sin\theta$

$\Rightarrow$ $\frac{n}{r}=-\left[\sum_{i=1}^{n}\frac{a_i}{c_i}\right]\cos\theta-\left[\sum_{i=1}^{n}\frac{b_i}{c_i}\right]\sin\theta$

Hence, the locus of R is

$$\left(\sum_{i=1}^{n}\frac{a_i}{c_i}\right)x+\left(\sum_{i=1}^{n}\frac{b_i}{c_i}\right)y+n=0,$$

which is a straight line.

16. Let the side OQ make an angle α with X-axis and since the side of the square $OABC$ is a, therefore coordinates of point A are $(a\cos\alpha, a\sin\alpha)$.

Now, the diagonal OB will make an angle of $(45°+\alpha)$ with X-axis and pass through origin O.

Hence, its equation is

$$y=\tan(45°+\alpha)x$$

or $$y=\frac{1+\tan\alpha}{1-\tan\alpha}x$$

$\Rightarrow$ $y(\cos\alpha-\sin\alpha)-x(\cos\alpha+\sin\alpha)=0$ ...(i)

The other diagonal AC will be perpendicular to Eq. (i) and pass through the point A.

Hence, its equation is

$y(\cos\alpha+\sin\alpha)+x(\cos\alpha-\sin\alpha)=k$, where $k=a$

17. The given lines are

$u=a_1x+b_1y+c_1=0$ ...(i)

and $v=a_2x+b_2y+c_2=0$ ...(ii)

Also, given that

$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\lambda$ (say)

$\therefore$ $a_1=a_2\lambda, b_1=b_2\lambda, c_1=c_2\lambda$

Now, $u=a_1x+b_1y+c_1$

$=(a_2x+b_2y+c_2)\lambda=v\lambda$

$\therefore$ Line $u=0$ becomes $\lambda v=0$

or $v=0$ ...(iii)

If condition (iii) is satisfied, then $u=0$ and $v=0$ represent the same straight line.

Also, the curve $u+kv=0$ is

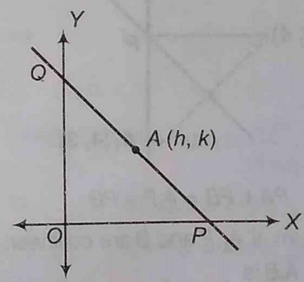
$\lambda v+kv=0 \Rightarrow (\lambda+k)v=0$

$\Rightarrow$ $v=0$

Hence, the curve $u+kv=0$ is nothing but any of the given straight line $u=0$ or $v=0$.

18. Let the equation of any line passing through $A(h,k)$ be

$$y-k=m(x-h)$$



Let this line cut the X and Y-axes at P and Q.

Then, $P\equiv(h-\frac{k}{m}, 0)$ and $Q\equiv(0, k-mh)$

Let S be the area of ΔOPQ, then

$S=\frac{1}{2}OP\times OQ=\frac{1}{2}\left(h-\frac{k}{m}\right)(k-mh)$

$=\frac{1}{2}\frac{(mh-k)(k-mh)}{m}$

$\Rightarrow$ $2mS=hkm-k^2-h^2m^2+khm$

$\Rightarrow$ $h^2m^2-2(hk-S)m+k^2=0$

Since, m is real.

So, its discriminant $D\geq0$.

$\therefore$ $4(hk-S)^2-4h^2k^2\geq0$

$\Rightarrow$ $S-2hk\geq0 \Rightarrow S\geq2hk$

Hence, minimum value of S is $2hk$ sq units.

19. Let the n given points be (a,a), $(a+1, a+2)$, $(a+2, a+4)$, ...

i.e., $\{a+i-1, a+2(i-1)\}$; $i=1,2,3,\ldots,n$

Let the variable line be

$px+qy+r=0$ ...(i)

Since, the algebraic sum of perpendiculars drawn from these n points on the variable line (i) is always zero.

$$\therefore \quad \sum \frac{p(a+i-1)+q(a+2i-2)+r}{\sqrt{p^2+q^2}} = 0$$

$$\Rightarrow \quad \sum [p(a+i-1)+q(a+2i-2)+r] = 0$$

$$\Rightarrow \quad p\sum (a+i-1)+q\sum (a+2i-2)+rn = 0$$

$$\Rightarrow \quad p\sum \frac{a+i-1}{n}+q\sum \frac{a+2i-2}{n}+r = 0$$

Hence, the line (i) always passes through a fixed point $\left(\sum \frac{a+i-1}{n}, \sum \frac{a+2i-2}{n}\right)$.

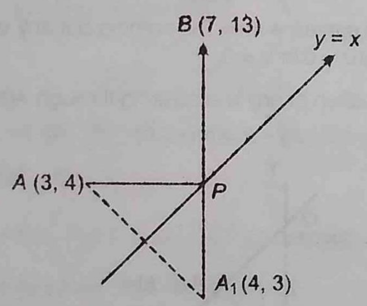
But it is given that this passes through the point $\left(\frac{13}{2}, 11\right)$.

$$\therefore \quad \sum \frac{a+i-1}{n} = \frac{13}{2} \text{ and } \sum \frac{a+2i-2}{n} = 11$$

On solving the two equations, we get $a = 2$ and $n = 10$

20. Let A_1 is the reflection of A in $y = x$.

$$\Rightarrow \quad A_1 = (4, 3)$$



Now, $\quad PA + PB = A_1P + PB$

which is minimum, if A_1, P and B are collinear.

$\therefore$ Equation of A_1B is

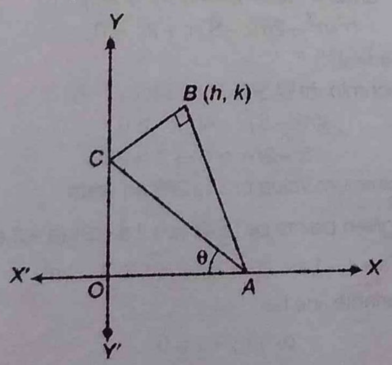
$$(y-3) = \frac{13-3}{7-4}(x-4) \Rightarrow 3y = 10x - 31$$

On solving it with $y = x$, we get $P = \left(\frac{31}{7}, \frac{31}{7}\right)$

21. $AB = 4, BC = 3 \Rightarrow AC = 5$

Let $\angle CAO = \theta$

$A \equiv (5\cos\theta, 0)$, $C \equiv (0, 5\sin\theta)$



If BC and AB make the angle θ_1 and θ_2 with X-axis, then

$$\theta_1 = C - \theta, \theta_2 = \pi - (A + \theta)$$

$$= \pi - \left(\frac{\pi}{2} - C + \theta\right) = \frac{\pi}{2} + (C - \theta)$$

Using parametric equation of line for BC, we get

$$\frac{h}{\cos(C-\theta)} = \frac{K - 5\sin\theta}{\sin(C-\theta)} = 3 \quad \text{...(i)}$$

Similarly, using parametric equation of line for AB, we get

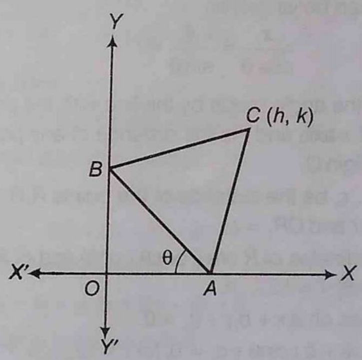
$$\frac{h}{-\sin(C-\theta)} = \frac{k}{\cos(C-\theta)} = 4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\cos(C-\theta) = \frac{h}{3} = \frac{k}{4}$$

So, the locus of B is $4x = 3y$.

22. $AB = BC = CA = 2$



Let $\quad \angle BAO = \theta$

$$\Rightarrow \quad A \equiv (2\cos\theta, 0)$$

$$B \equiv (0, 2\sin\theta)$$

BC makes an angle $\left(\frac{\pi}{3} - \theta\right)$ with X-axis and AC makes an angle $\left(\pi - \left(\frac{\pi}{3} + \theta\right)\right)$ with X-axis.

If $C \equiv (h, k)$, then

$$\frac{h}{\cos\left(\frac{\pi}{3}-\theta\right)} = \frac{k - 2\sin\theta}{\sin\left(\frac{\pi}{3}-\theta\right)} = 2$$

and $$\frac{h - 2\cos\theta}{-\cos\left(\frac{\pi}{3}+\theta\right)} = \frac{k}{\sin\left(\frac{\pi}{3}+\theta\right)} = 2$$

$$\Rightarrow \quad 2\cos\left(\frac{\pi}{3}-\theta\right) = h, 2\sin\left(\frac{\pi}{3}+\theta\right) = k$$

$$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = h, \sqrt{3}\cos\theta + \sin\theta = k$$

$$\Rightarrow \quad 2\cos\theta = (\sqrt{3}k - h), 2\sin\theta = (h\sqrt{3} - k)$$

Thus, locus of (h, k) is

$$1 = x^2 + y^2 - xy\sqrt{3}$$

or $$x^2 + y^2 = 1 + xy\sqrt{3}$$

23. Given lines are $x+2y=3$, $2x+3y=4$, $4x+5y-6=0$.

Now,

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & -4 \\ 4 & 5 & -6 \end{vmatrix} = 1(-18+20)-2(-12+16)-3(10-12)$$

$$=0$$

They are concurrent at $(-1, 2)$ and line $3x+4y-7=0$ does not pass through it.

24. The intersection of given lines is $\frac{x}{a}+\frac{y}{b}-1+\lambda\left(\frac{x}{b}+\frac{y}{a}-1\right)=0$

meets the coordinate axes at

$$A\left[\frac{1+\lambda}{\frac{1}{a}+\frac{\lambda}{b}}, 0\right] \text{ and } B\left[0, \frac{1+\lambda}{\frac{1}{b}+\frac{\lambda}{a}}\right]$$

The mid-point of AB is given by

$$2x=\frac{1+\lambda}{\frac{1}{a}+\frac{\lambda}{b}}, 2y=\frac{1+\lambda}{\frac{1}{b}+\frac{\lambda}{a}}$$

$$\Rightarrow (1+\lambda)\left[\frac{1}{x}+\frac{1}{y}\right]=2\left[\frac{1}{a}+\frac{\lambda}{b}\right]+2\left[\frac{1}{b}+\frac{\lambda}{a}\right]$$

$$=2(1+\lambda)\left[\frac{1}{a}+\frac{1}{b}\right]$$

$$\therefore \quad (x+y)ab=2xy(a+b)$$

25. The third side is parallel to a bisector of the angle between the equal sides.

The bisectors are

$$7x-y+3=\pm 5(x+y-3)$$

$$\Rightarrow \quad 2x-6y+18=0$$

or $\quad 12x+4y-12=0$

$$\Rightarrow \quad x-3y+9=0$$

or $\quad 3x+y-3=0$

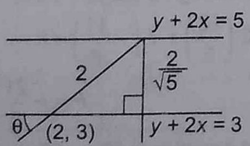
Let the third side be $x-3y=k$ or $3x+y=L$

It passes through $(1, -10)$.

$$\therefore \quad k=31, L=-7$$

Hence, required lines are $x-3y=31$, $3x+y=-7$.

26. The distance between the parallel lines is $\frac{2}{\sqrt{5}}$. If θ is the angle between the desired line and the line $y+2x=3$, then $\sin\theta=\frac{1}{\sqrt{5}}$ or $\tan\theta=\frac{1}{2}$.



If m is the slope of required line, then $\frac{m+2}{1-2m}=\pm\frac{1}{2}$

or $\quad 2m+4=\pm(1-2m)$

$$\Rightarrow \quad m=\infty \text{ or } m=-\frac{3}{4}.$$

The desired lines passing through $(2, 3)$ are $x=2$, $3x+4y=18$.

27. The image of $P(a, b)$ in the line $x-y=0$ is $Q(b, a)$. The image of $Q(b, a)$ in the line $x+y=0$ is $R(-a, -b)$. The mid-point of QR is $\left[\frac{b-a}{2}, \frac{a-b}{2}\right]$.

28. Let ABC be an isosceles right angled triangle at A with $AB=AC$.

$$\Rightarrow \quad \angle ABC=\angle ACB=45°$$

Now, we have to determine the angle between AC and BC.

The slope of line BC i.e., $3x+4y=4$ is

$$m_2=-\frac{3}{4}$$

Let the slope of the line AC is $m_1=m$.

$\because$ Angle between two lines is

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right| \Rightarrow \tan 45°=\left|\frac{m-\left(\frac{-3}{4}\right)}{1+m\left(\frac{-3}{4}\right)}\right|$$

$$\Rightarrow \quad 1=\left|\frac{m+\frac{3}{4}}{1-\frac{3m}{4}}\right| \Rightarrow \pm 1=\frac{4m+3}{4-3m}$$

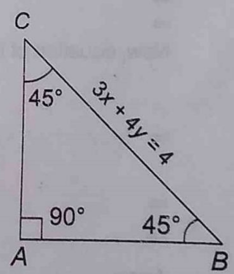
Taking positive sign, $1=\frac{4m+3}{4-3m}$.

$$\Rightarrow \quad 4-3m=4m+3$$

$$\Rightarrow \quad 4-3=4m+3m$$

$$\Rightarrow \quad 7m=1$$

$$\Rightarrow \quad m=\frac{1}{7}$$



Taking negative sign, $\quad -1=\frac{4m+3}{4-3m}$

$$\Rightarrow \quad -4+3m=4m+3$$

$$\Rightarrow \quad -4-3=m \Rightarrow m=-7$$

Hence, equation of line AB and line AC are

$$y-2=\frac{1}{7}(x-2) \text{ and } y-2=-7(x-2)$$

$$\Rightarrow \quad 7y-14=x-2 \text{ and } y-2=-7x+14$$

$$\Rightarrow \quad x-7y+12=0 \text{ and } 7x+y-16=0$$

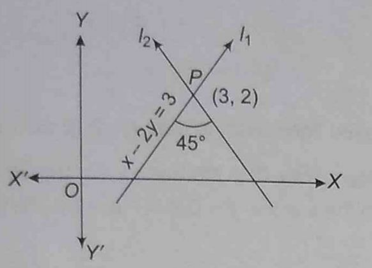
29. Equation of line l_1 is

$$x-2y=3 \quad \ldots(i)$$

Slope of Eq. (i),

$$m_1=-\left(\frac{1}{-2}\right)$$

$$\Rightarrow \quad m_1=\frac{1}{2}$$



Angle between two lines is

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

Given, $\theta=45°$, $m_1=\frac{1}{2}$, $m_2=m$ (let)

$$\therefore \quad \tan 45°=\left|\frac{\frac{1}{2}-m}{1+\frac{1}{2}m}\right| \Rightarrow 1=\left|\frac{1-2m}{2+m}\right| \Rightarrow \frac{1-2m}{2+m}=\pm 1$$

Taking positive sign,

$$\frac{1-2m}{2+m}=1$$

$\Rightarrow \quad 1-2m=2+m \Rightarrow 1-2=3m$

$\Rightarrow \quad m=-\frac{-1}{3}$

Taking negative sign,

$$\frac{1-2m}{2+m}=-1$$

$\Rightarrow \quad 1-2m=-2-m$

$\Rightarrow \quad 2m-m=1+2$

$\Rightarrow \quad m=3$

Now, equation of line l_2 by using

$$y-y_1=m(x-x_1)$$

$\Rightarrow \quad y-2=-\frac{1}{3}(x-3) \quad \left(\text{when } m=-\frac{1}{3}, x_1=3 \text{ and } y_1=2\right)$

$\Rightarrow \quad 3y-6=-x+3$

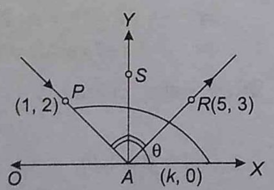
$\Rightarrow \quad x+3y-9=0$

Again, $\quad y-2=3(x-3) \quad \left(\text{when } m=3, x_1=3 \text{ and } y_1=2\right)$

$\Rightarrow \quad y-2=3x-9 \Rightarrow 3x-y-9+2=0$

$\Rightarrow \quad 3x-y-7=0$

30. In a figure, PA is the incident ray and AR is the reflected ray which makes an angle θ from the X-axis.



It is clear from the figure that

$AS \perp OX$

It means AS bisect the $\angle PAR$.

Then, $\quad \angle PAS=\angle RAS$

$\Rightarrow \quad \angle RAX=\angle PAO=\theta$ (let)

$\Rightarrow \quad \angle XAP=180°-\theta$

Slope of $\quad AR=\tan\theta=\frac{y_2-y_1}{x_2-x_1}=\frac{3-0}{5-k}$ ...(i)

[where, point A is $(k, 0)$]

Slope of $AP=\tan(180°-\theta)$

$$=-\tan\theta=\frac{-y_2-y_1}{x_2-x_1}=\frac{-2-0}{1-k} \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\frac{3}{5-k}=-\frac{2}{1-k}$$

$\Rightarrow \quad 3-3k=-10+2k$

$\Rightarrow \quad 5k=13 \Rightarrow k=\frac{13}{5}$

Hence, the coordinate of A is $\left(\frac{13}{5},0\right)$.

31. The given equation can be rewritten as

$$(2x+3y-5)\cos\theta+(3x-5y+2)\sin\theta=0$$

or $\quad (2x+3y-5)+\tan\theta\,(3x-5y+2)=0$

This passes through the point of intersection of the lines $2x+3y-5=0$ and $3x-5y+2=0$ for all values of θ. The point of intersection is $P(1, 1)$. Let $Q(h, k)$ be the reflection of $P(1, 1)$ in the line

$$x+y=\sqrt{2} \quad \text{...(i)}$$

Then, PQ is perpendicular to Eq. (i) and the mid-point of PQ lies on Eq. (i), we get

$$\frac{k-1}{h-1}=1 \Rightarrow k=h$$

and $\quad \frac{h+1}{2}+\frac{k+1}{2}=\sqrt{2}$

$\Rightarrow \quad h=k=\sqrt{2}-1$

32. The intersection point of L_1 and L_2 is $A(4, -1)$.

Equation of a line through A is

$$\frac{x-4}{\cos\theta}=\frac{y+1}{\sin\theta} \quad \text{...(i)}$$

Coordinates of any point on this at 2 units distance from A is

$$(2\cos\theta+4, 2\sin\theta-1) \quad \text{...(ii)}$$

From Eq. (i), L_1 represents AB, then

$$\tan\theta=-\frac{3}{4}$$

$\Rightarrow \quad \sin\theta=\pm\frac{3}{5}, \cos\theta=\pm\frac{4}{5}$

On taking $\cos\theta=\frac{4}{5}$, $\sin\theta=-\frac{3}{5}$ in Eq. (ii), we get the coordinates of B are $\left(\frac{28}{5}, -\frac{11}{5}\right)$.

33. Let the equation parallel to AC is $2x+7y=k$.

Since, it passes through $B\,(28/5, -11/5)$.

$$\therefore \quad 2\left(\frac{28}{5}\right)+7\left(-\frac{11}{5}\right)=k$$

$\Rightarrow \quad k=-\frac{21}{5}$

So, the required equation is $10x+35y+21=0$.

34. Since, $DE=\frac{AB}{2}=1$

35. Equation of bisector of

$4y + 3x - 12 = 0$ and $3y + 4x - 24 = 0$ is

$$\frac{4y + 3x - 12}{\sqrt{16 + 9}} = \pm \frac{3y + 4x - 24}{\sqrt{9 + 16}}$$

$\Rightarrow$ $y - x + 12 = 0$ and $7y + 7x - 36 = 0$

So, the line $y - x + 12 = 0$ is the angular bisector.

36. From Statement I, $na = 8$

$$\frac{n(n-1)}{2} a^2 = 24$$

$$a^2 = \frac{64}{n^2} \Rightarrow a = 2 \text{ and } n = 4$$

From Statement II, $PA \cdot PB = (\sqrt{S_1})^2 = 2^2 + 4^2 - 4 = 16$

37. Equation of line AB is $y - 1 = \frac{0 - 1}{2 - 0}(x - 0)$

$\Rightarrow$ $x + 2y - 2 = 0$

Here, $|PA - PB| \le |AB|$

Thus, $|PA - PB|$ to be maximum, then A, B and P must be collinear.

38. The point of intersection of lines $4x + 3y = \lambda$ and $3x - 4y = \mu$ is

$$x_1 = \frac{4\lambda + 3\mu}{25} \text{ and } y_1 = \frac{3\lambda - 4\mu}{25}$$

$\because$ $x_1 = y_1 \Rightarrow \frac{4\lambda + 3\mu}{25} = \frac{3\lambda - 4\mu}{25}$

$\Rightarrow$ $\lambda + 7\mu = 0$

Hence, locus of a point (λ, μ) is $x + 7y = 0$.

39. Now, distance of origin from $4x + 2y - 9 = 0$ is

$$\frac{|-9|}{\sqrt{4^2 + 2^2}} = \frac{9}{\sqrt{20}}$$

and distance of origin from $2x + y + 6 = 0$ is

$$\frac{|6|}{\sqrt{2^2 + 1^2}} = \frac{6}{\sqrt{5}}$$

Hence, the required ratio $= \frac{9/\sqrt{20}}{6/\sqrt{5}} = \frac{3}{4}$

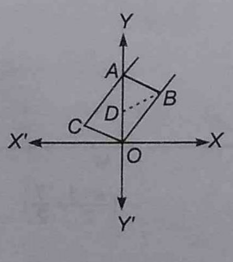
40. Let lines $OB: y = mx$

$CA: y = mx + 1$

$BA: y = nx + 1$

and $OC: y = nx$

So, the point of intersection B of OB and AB has x-coordinate $\frac{1}{m-n}$.



Now, area of a parallelogram $OBAC = 2 \times$ Area of ΔOBA

$$= 2 \times \frac{1}{2} \times OA \times DB$$

$$= 2 \times \frac{1}{2} \times \frac{1}{m-n}$$

$$= \frac{1}{m-n} = \frac{1}{|m-n|},$$

depending upon whether $m > n$ or $m < n$.

41. On solving both the equations, we get

$$\frac{8y}{3} + y^2 = 1$$

$\Rightarrow$ $3y^2 + 8y - 3 = 0$

$\Rightarrow$ $(3y - 1)(y + 3) = 0$

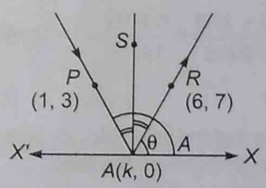
$\Rightarrow$ $y = -3, \frac{1}{3}$ here $y \ne -3$

At $y = \frac{1}{3}$, $x = \pm 2\sqrt{\frac{2}{3}}$

So, the point of intersection is $\left(2\sqrt{\frac{2}{3}}, \frac{1}{3}\right)$ and $\left(-2\sqrt{\frac{2}{3}}, \frac{1}{3}\right)$.

From option (d); $3y - 1 = 0$ is the required equation which satisfied the intersection points.

42. Here, $AS \perp OX$



It means AS bisect the angle PAR.

Then, $PAS = RAS$

$\Rightarrow$ $RAX = PAO = \theta$ (let)

$\Rightarrow$ $XAP = 180° - \theta$

Slope of $AR = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 0}{6 - k}$ ...(i)

Slope of $AP = \tan(180° - \theta) = -\tan\theta$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{1 - k} \quad \text{...(ii)}$$

$\therefore$ From Eqs. (i) and (ii),

$$\frac{7}{6-k} = -\frac{3}{1-k} \Rightarrow 7 - 7k = -18 + 3k$$

$$10k = 25 \Rightarrow k = \frac{5}{2}$$

Hence, the coordinate of A is $\left(\frac{5}{2}, 0\right)$.

43. ***Case* I** Let line $l_1 \equiv x - 3y = p$ and $l_2 \equiv ax + 2y = p$ are perpendicular, then $\frac{1}{3} x - \frac{a}{2} = -1 \Rightarrow a = 6$

***Case* II** Let line $l_2 \equiv ax + 2y = p$ and $l_3 \equiv ax + y = r$ are perpendicular, then $\frac{-a}{2} \times -a = -1 \Rightarrow a^2 = -2$ (not possible)

Case III Let line $l_3 \equiv ax + y = r$ and $l_1 \equiv x - 3y = p$ are perpendicular, then $-a \times \frac{1}{3} = -1 \Rightarrow a = 3$

So, formation of quadratic equation in a, whose roots are 3 and 6.

$\therefore \quad a^2 - (6+3)a + (6 \cdot 3) = 0 \Rightarrow a^2 - 9a + 18 = 0$

44. $\frac{x}{4} + \frac{y}{3} = 1$

For line L, x-intercept $= 2 \times 4 = 8$

$$y\text{-intercept} = \frac{1}{2} \times 3 = \frac{3}{2}$$

$\therefore$ Line L is $\frac{x}{8} + \frac{y}{3/2} = 1$.

Slope, $m = -\frac{3}{16}$

45. Here, angle between the lines $2x + 3y + c_1 = 0$ and $-x + 5 + c_2 = 0$ is θ_1.

$$\therefore \quad \tan\theta_1 = \left|\frac{1/5 + 2/3}{1 - 2/15}\right| = \left|\frac{13/15}{13/15}\right| = 1 = \tan 45^\circ$$

$\Rightarrow \quad \theta_1 = 45^\circ$

Also, the angle between the lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$ is θ_2.

$$\therefore \quad \tan\theta_2 = \left|\frac{1/5 + 2/3}{1 - 2/15}\right| = \left|\frac{13/15}{13/15}\right| = 1 = \tan 45^\circ$$

$\Rightarrow \quad \theta_2 = 45^\circ$

Here, we observe that the value of c_1, c_2 and c_3 is not depend to measuring the angle between the lines.

So, c_2 and c_3 are proportional or for all c_2 and c_3

$$\theta_1 = \theta_2$$

46. We can written the given lines in symmetric form

$$L_1 \equiv \frac{x - (\sqrt{\lambda} - 1)}{\sqrt{\lambda}} = \frac{y - 0}{1} = \frac{z - \sqrt{\lambda}}{\sqrt{\lambda} - 1},$$

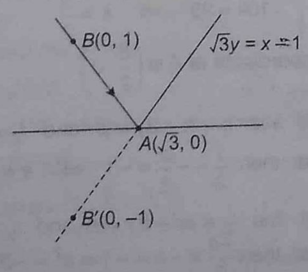
$$L_2 \equiv \frac{x - (1 - \sqrt{\mu})}{\sqrt{\mu}} = \frac{y - 0}{1} = \frac{z - \sqrt{\mu}}{1 - \sqrt{\mu}}$$

Since, L and L_2 are perpendicular.

$\therefore \quad \sqrt{\lambda} \cdot \sqrt{\mu} + 1 \cdot 1 + (\sqrt{\lambda} - 1)(1 - \sqrt{\mu}) = 0$

$\Rightarrow \quad \sqrt{\lambda} + \sqrt{\mu} = 0 \Rightarrow \lambda = \mu$

47. Take any point $B(0,1)$ on given line.



Equation of AB' is

$$y - 0 = \frac{-1-0}{0-\sqrt{3}}(x - \sqrt{3})$$

$\Rightarrow \quad -\sqrt{3}y = -x + \sqrt{3}$

$\Rightarrow \quad x - \sqrt{3}y = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}y = x - \sqrt{3}$

48. As $x + y = |a|$ and $ax - y = 1$. Intersect in first quadrant.

So, x and y-intercepts are positive.

$\therefore \quad x = \frac{1 + |a|}{1 + a} \geq 0$

and $\quad y = \frac{a|a| - 1}{a + 1} \geq 0$

$\Rightarrow \quad 1 + a \geq 0$ and $a|a| - 1 \geq 0$

$\Rightarrow \quad a \geq -1$ and $a|a| \geq 1 \quad \ldots(\text{i})$

If $\quad -1 \leq a < 0$

$\Rightarrow \quad -a^2 > 1 \quad$ (not possible)

If $\quad a \geq 0$

$\Rightarrow \quad a^2 \geq 1$

$\Rightarrow \quad a \geq 1$

$\therefore \quad a \geq 1$ or $a \in [1, \infty)$

49. Since, the line L is passing through the point $(13, 32)$.

Therefore, $\frac{13}{5} + \frac{32}{b} = 1 \Rightarrow \frac{32}{b} = -\frac{8}{5} \Rightarrow b = -20$

The line K is parallel to the line L, its equation must be

$$\frac{x}{5} - \frac{y}{20} = a \quad \text{or} \quad \frac{x}{5a} - \frac{y}{20a} = 1$$

On comparing with $\frac{x}{c} + \frac{y}{3} = 1$, we get

$$20a = -3, c = 5a = -\frac{3}{4}$$

Hence, the distance between lines

$$= \frac{|a - 1|}{\sqrt{\frac{1}{25} + \frac{1}{400}}} = \frac{\left|\frac{-3}{20} - 1\right|}{\sqrt{\frac{17}{400}}} = \frac{23}{\sqrt{17}}$$

50. Line perpendicular to same line are parallel to each other.

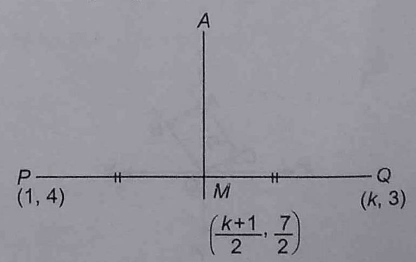
$\therefore \quad -p(p^2 + 1) = p^2 + 1$

$\Rightarrow \quad p = -1$

So, there is exactly one value of p.

51. Since, slope of $PQ = \frac{4-3}{1-k} = \frac{1}{1-k}$

Slope of $AM = (k - 1)$



$\therefore$ Equation of AM is

$$y - \frac{7}{2} = (k-1)\left[x - \left(\frac{k+1}{2}\right)\right]$$

For y-intercept, $x = 0$, $y = -4$

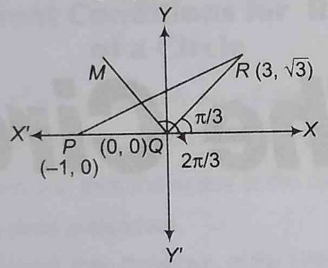
$$-4 - \frac{7}{2} = -(k-1)\left(\frac{k+1}{2}\right)$$

$$\Rightarrow \quad \frac{15}{2} = \frac{k^2 - 1}{2}$$

$$\Rightarrow \quad k^2 - 1 = 15 \Rightarrow k^2 = 16$$

$$\Rightarrow \quad k = \pm 4$$

52. Now, slope of $QR = \frac{3\sqrt{3} - 0}{3 - 0} = \sqrt{3} = \tan\theta \Rightarrow \theta = \frac{\pi}{3}$



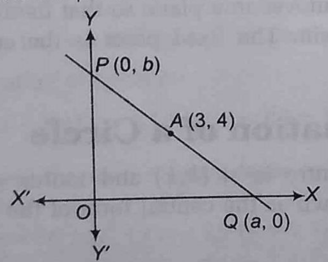
$\therefore$ The angle between $\angle PQR$ is $\frac{2\pi}{3}$, so the line QM makes an angle $\frac{2\pi}{3}$ from positive direction of X-axis.

Slope of the line $QM = \tan\frac{2\pi}{3} = -\sqrt{3}$

Hence, equation of line QM is

$$y = -\sqrt{3}x \Rightarrow \sqrt{3}x + y = 0$$

53. Since, A is the mid-point of line PQ.



$$\therefore \quad 3 = \frac{a+0}{2} \Rightarrow a = 6$$

$$\text{and} \quad 4 = \frac{0+b}{2} \Rightarrow b = 8$$

Hence, the equation of line is

$$\frac{x}{6} + \frac{y}{8} = 1 \quad \text{or } 4x + 3y = 24.$$

54. The point (a, a^2) lies in the sector bounded by the lines $x - 2y = 0$ and $3x - y = 0$.

$$\therefore \quad (a - 2a^2)(3a - a^2) < 0 \Rightarrow \left[a - \frac{1}{2}\right](a - 3) < 0$$

$$\therefore \quad a \in \left(\frac{1}{2}, 3\right)$$

55. Since, a, b and c are in HP. Then, $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP.

$$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

Hence, straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ is always passes through a fixed point $(1, -2)$.

56. Let the line intercept the X-axis at a and Y-axis at b distance.

Since, $a + b = -1$

$$\Rightarrow \quad b = -(a+1)$$

$\therefore$ Equation of line is $\frac{x}{a} - \frac{y}{a+1} = 1$

$$\therefore \quad \frac{4}{a} - \frac{3}{a+1} = 1$$

$$\Rightarrow \quad \frac{4a + 4 - 3a}{a(a+1)} = 1$$

$$\Rightarrow \quad a + 4 = a^2 + a$$

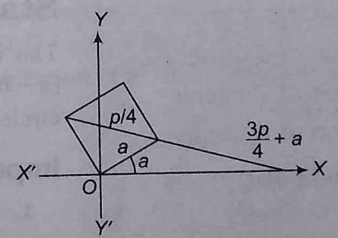
$$\Rightarrow \quad a = \pm 2$$

Hence, equation of lines are

$$\frac{x}{2} - \frac{y}{3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1.$$

57. Slope of the diagonal $= \tan\left(\frac{3\pi}{4} + \alpha\right)$

$$= \frac{-1 + \tan\alpha}{1 + \tan\alpha} = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$$



The equation is

$$\frac{y - a\sin\alpha}{x - a\cos\alpha} = \frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$$

$$\Rightarrow \quad y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$$

Day 26

The Circle

Day 26

Outlines ...

- Concept of Circle
- Standard Form of Equation of a Circle
- Equation of Tangents
- Equation of Normals
- Pole and polar
- Family of Circles

Concept of Circle

Circle is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is a constant. The fixed point is the **centre** and the constant distance is the **radius**.

Standard Form of Equation of a Circle

The equation of a circle whose centre is at (h,k) and radius r is given, is $(x-h)^2+(y-k)^2=r^2$. It is also known as the central form of the equation of a circle.

Important Results

1. When the circle is at the origin and radius is r, then the equation of the circle is $x^2+y^2=r^2$. When the circle touches both the axes then in this case, $h=k=r$

$\therefore$ Equation of the circle is $(x-r)^2+(y-r)^2=r^2$

$\Rightarrow \quad x^2+y^2-2rx-2ry+r^2=0$

2. When the circle passes through the origin and centre lies on X-axis. In this case, $k=0$ and $h=r$.

$\therefore$ Equation of the circle is $(x-r)^2+(y-0)^2=r^2 \Rightarrow x^2+y^2-2rx=0$

3. When the circle passes through the origin and centre lies on Y-axis. In this case, $h = 0$ and $k = r$

$\therefore$ Equation of the circle is

$$(x-0)^2 + (y-r)^2 = r^2 \Rightarrow x^2 + y^2 - 2ry = 0.$$

4. The equation of the circle passing through the three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Different Conditions for Radius of a Circle

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

- If $g^2 + f^2 - c > 0$, then the radius of the circle is **real** and hence, the circle is also **real**.
- If $g^2 + f^2 - c = 0$, then the radius of the circle is **zero.** Such a circle is known a **point circle.**
- If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2 + f^2 - c}$ of circle is imaginary but the centre is real. Such a circle is called an **imaginary circle** as it is not possible to draw such a circle.

Equation of Circle when the end Points of a Diameter are Given

If A and B are end points of a diameter of a circle whose coordinates are (x_1, y_1) and (x_2, y_2), respectively.

Then, the equation of circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

» *The general equation of second degree represent a general equation of a circle $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, if $a = b$ and $h = 0$,*

» *Parametric equations of a circle $(x-h)^2 + (y-k)^2 = r^2$, where $x = h + r\cos\theta$, $y = k + r\sin\theta$, $0 \le \theta \le 2\pi$.*

Intercept on Axes

The length of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with X and Y-axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$, respectively.

(i) If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches x-axis and the intercept on X-axis is zero.

(ii) If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle meets the X-axis in two real and distinct points and the length of the intercept on X-axis is $2\sqrt{g^2 - c}$.

(iii) If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the circle does not meet X-axis in real points.

Similarly, the circle cuts the Y-axis in real and distinct points, touches or does not meet in real points according as $f^2 > 0$, $=$ or $< c$.

Position of a Point with Respect to a Circle

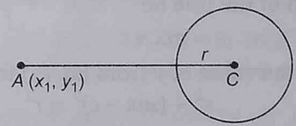
A point (x_1, y_1) lies outside, on or inside a circle

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

according as $S_1 >, =,$ or < 0,

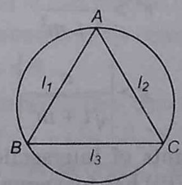
where $\quad S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Greatest and **least distance** of a point $A(x_1, y_1)$ from a circle with centre C and radius r is $|AC + r|$ and $|AC - r|$.



Equation of a Circle Circumscribing Triangle *and* Quadrilateral

1. Equation of circle circumscribing a triangle whose sides are given by $l_1 = 0; l_2 = 0$ and $l_3 = 0$ is given by $l_1 l_2 + \lambda l_2 l_3 + \mu l_3 l_1 = 0$. Provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



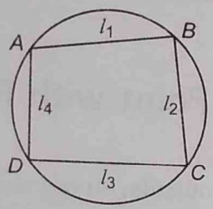
2. Circumcircle of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given ahead.

$$\frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)} = \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}$$

3. Equation of a circle circumscribing a quadrilateral whose sides in order are represented by the lines $l_1 = 0, l_2 = 0, l_3 = 0; l_4 = 0$ is given by

$$l_1 l_3 + \lambda l_2 l_4 = 0 \quad \text{...(i)}$$

represents a second degree curve and also A, B, C and D satisfy Eq. (i) provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



Points of Intersection of a Line and a Circle with the Centre at the Origin

Let the equation of the circle be

$$x^2 + y^2 = r^2 \quad \text{...(i)}$$

and the equation of the line be

$$y = mx + c \quad \text{...(ii)}$$

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$x^2 + (mx + c)^2 = r^2$$

$$\Rightarrow \quad x^2(1 + m^2) + 2mcx + (c^2 - r^2) = 0 \quad \text{...(iii)}$$

(i) When points of intersection are real and distinct. In this case, Eq. (iii) has two distinct roots.

$\therefore \quad D > 0$

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0$

$\Rightarrow \quad 4r^2(1 + m^2) - 4c^2 > 0$

$\Rightarrow \quad r^2(1 + m^2) > c^2$

$\Rightarrow \quad r^2 > \dfrac{c^2}{1 + m^2}$

$\Rightarrow \quad r > \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

(ii) When the points of intersection are coincident. In this case, Eq. (iii) has two equal roots.

$\therefore \quad D = 0$

$\Rightarrow \quad r = \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

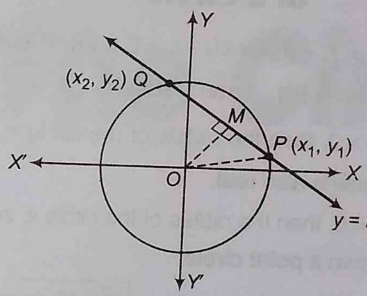
(iii) When the points of intersection are imaginary. In this case, Eq. (iii) has imaginary roots.

$\therefore \quad D < 0 \Rightarrow r < \left|\dfrac{c}{\sqrt{1 + m^2}}\right|$

- *Lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ cut the axes at concyclic points, if $a_1a_2 = b_1b_2$.*
- *Circle through the concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = (a_1b_2 + a_2b_1)xy$*

Length of the Intercept Cut off from a Line by a Circle

The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = r^2$ is



$$PQ = 2\sqrt{\frac{r^2(1 + m^2) - c^2}{1 + m^2}}$$

Condition for a Line to be Tangent to a Circle

The line $y = mx + c$ touches the circle $x^2 + y^2 = r^2$, if the length of the intercept is zero.

i.e., $\quad PQ = 0 \Rightarrow 2\sqrt{\dfrac{r^2(1+m^2) - c^2}{(1+m^2)}} = 0$

$\Rightarrow \quad r^2(1+m^2) - c^2 = 0 \Rightarrow c = \pm r\sqrt{1+m^2}$

which is the required condition.

Equation of Tangents

A line which touch only one point of a circle is called its tangent. *This tangent may be in slope or point form as given below.*

1. **Slope form**

(i) The equation of a tangent of slope m to the circle $x^2 + y^2 = r^2$ is $y = mx \pm r\sqrt{1 + m^2}$.

The coordinates of the point of contact are

$$\left(\pm \frac{rm}{\sqrt{1 + m^2}}, \mp \frac{r}{\sqrt{1 + m^2}}\right).$$

(ii) The equation of tangents of slope m to the circle $(x-a)^2+(y-b)^2=r^2$ are given by

$$y-b=m(x-a)\pm r\sqrt{1+m^2}.$$

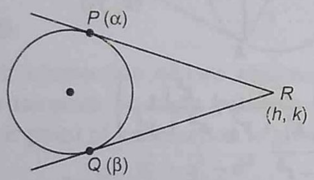
and the coordinates of the points of contact are

$$\left(a\pm\frac{mr}{\sqrt{1+m^2}}, b\mp\frac{r}{\sqrt{1+m^2}}\right).$$

(iii) The equation of the tangents of slope m to the circle $x^2+y^2+2gx+2fy+c=0$ are $y+f=m(x+g)\pm\sqrt{(g^2+f^2-c)(1+m^2)}$

(iv) Point of intersection of the tangent drawn to the circle $x^2+y^2=a^2$ at the point $P(\alpha)$ and $Q(\beta)$ is

$$h=\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \text{ and } k=\frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$



2. Point Form

(i) The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2+y^2+2gx+2fy+c=0$ is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

(ii) Equation of tangent for

$$(x-a)^2+(y-b)^2=r^2$$

at $(a+r\cos\theta, b+r\sin\theta)$ is

$$(x-a)\cos\theta+(y-b)\sin\theta=r$$

(iii) Equation of tangent for $x^2+y^2=r^2$ at (x_1, y_1) is $xx_1+yy_1=r^2$.

Equation of Normals

The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at that point and it is always passes through origin.

1. Point Form

The equation of normal to the circle $x^2+y^2+2gx+2fy+c=0$ or $x^2+y^2=a^2$ at any point (x_1, y_1) is

$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f} \text{ or } \frac{x}{x_1}=\frac{y}{y_1}.$$

2. Parametric Form

The equation of normal to the circle $x^2+y^2=a^2$ at point $(a\cos\theta, a\sin\theta)$ is $\dfrac{x}{\cos\theta}=\dfrac{y}{\sin\theta}$.

Pair of Tangents

From a given point, two tangents can be drawn to a circle which are real and distinct, coincident or imaginary according as the given point lies outside, on or inside the circle.

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is $SS_1=T^2$.

where, $S=x^2+y^2-a^2, S_1=x_1^2+y_1^2-a^2$

and $T=xx_1+yy_1-a^2$

Length of the Tangents

The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is equal to

$$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}=\sqrt{S_1}$$

Coaxial System of Circles

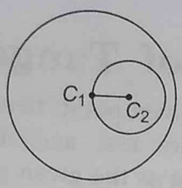
A system of circles is said to be coaxial system of circles, if every pair of the circles in the system has the same radical axis.

- Since, the lines joining the centres of two circles is perpendicular to their radical axis. Therefore, the centres of all circles of a coaxial system lie on a straight line which is perpendicular to the common radical axis.
- Circles passing through two fixed points P and Q form a coaxial system because every pair of circles has the same common chord PQ and therefore the same radical axis which is perpendicular bisector of PQ.
- If the equation of a member of a system of coaxial circles is $S=0$ and the equation of the common radical axis is $L=0$, then the equation representing the coaxial system of circle is $S+\lambda L=0$, where $\lambda\in R$.
- If $S_1=0$ and $S_2=0$ are two circles, then $S_1+\lambda S_2=0$
  $S_1+\lambda(S_1-S_2)=0$ or $S_2+\lambda(S_1-S_2)=0, \lambda\in R$
  represent a family of coaxial circles having $S_1-S_2=0$ as the common radical axis.
- Limiting points of a coaxial system of circles are the members of the system which are of zero radius.

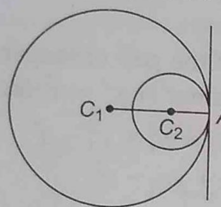
Common Tangents of Two Circles

Let the centres and radii of two circles are C_1, C_2 and r_1, r_2, respectively.

- When one circle contains other, no common tangent is possible. Condition $C_1C_2 < r_1 - r_2$.

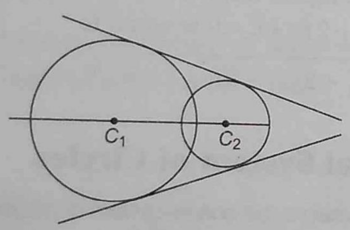


- When two circles touch internally, one common tangent is possible.



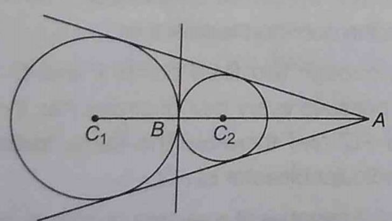
Condition, $C_1C_2 = r_1 - r_2$

- When two circles intersect, two common tangents are possible.



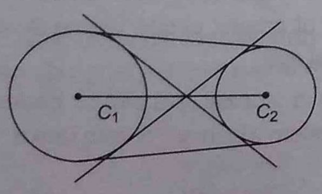
Condition, $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

- When two circles touch externally, three common tangents are possible.



Condition, $C_1C_2 = r_1 + r_2$
A divides C_1C_2 externally in the ratio $r_1 : r_2$. B divides C_1C_2 internally in the ratio $r_1 : r_2$.

- When two circles are separately, four common tangents are possible.



Condition, $C_1C_2 > r_1 + r_2$

Director Circle

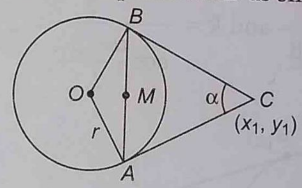
The locus of the point of intersection of two perpendicular tangents to a given conic is known as its director circle.

The equation of the director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

Chord of Contact

The chord joining the points of contact of the two tangents from a point is called the chord of contact of tangents. The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $\quad xx_1 + yy_1 = a^2$ or $T = 0$.

If AB is a chord of contact of tangents from C to the circle $x^2 + y^2 = r^2$ and M is the mid-point of AB as shown in figure. Then,



(i) Coordinates of $M\left(\dfrac{r^2 x_1}{x_1^2 + y_1^2}, \dfrac{r^2 y_1}{x_1^2 + y_1^2}\right)$

(ii) $AB = 2r \dfrac{\sqrt{x_1^2 + y_1^2 - r^2}}{\sqrt{x_1^2 + y_1^2}}$

(iii) $BC = \sqrt{x_1^2 + y_1^2 - r^2}$

(iv) Area of quadrilateral $OACB = r\sqrt{x_1^2 + y_1^2 - r^2}$

(v) Area of $\Delta ABC = \dfrac{r}{x_1^2 + y_1^2}(x_1^2 + y_1^2 - r^2)^{3/2}$

(vi) Area of $\Delta OAB = \dfrac{r^3}{x_1^2 + y_1^2}\sqrt{x_1^2 + y_1^2 - r^2}$

(vii) Angle between two tangents $\angle ACB$ is $2\tan^{-1}\dfrac{r}{\sqrt{S_1}}$.

Chord Bisected at a Given Point

The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at the point (x_1, y_1) is given by

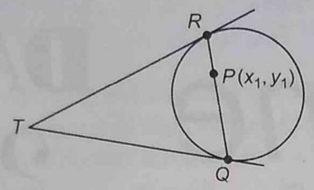
$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2 \text{ or } T = S_1$$

Of all the chords which passes through a given point $M(a,b)$ inside the circle the shortest chord is one whose middle point is (a,b).

Pole and Polar

If through a point $P(x_1, y_1)$ (inside or outside a circle) there be drawn any straight line to meet the given circle at Q and$R$, the locus of the point of intersection of the tangents at Q and R is called the **polar** of P and P is called the **pole** of the polar.

The polar of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$ or $T = 0$.



1. **Conjugate Points**
   Two points A and B are conjugate points with respect to a given circle, if each lies on the polar of the other with respect to the circle.
2. **Conjugate Lines**
   If two lines be such that the pole of one line lies on the other, then they are called conjugate lines with respect to the given circle.

Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents or angle between the normals to the two circles at their point of intersection is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$$

where, d is distance between centres of the circles.

Orthogonal Circles

Two circles are said to be intersect orthogonally, if their angle of intersection is a right angle.

(Radius of Ist circle)^2 + (Radius of IInd circle)^2 = (Distance between centres)^2 $\Rightarrow 2(g_1g_2 + f_1f_2) = c_1 + c_2$

which is the condition of orthogonality of two circles.

The circles having radii r_1 and r_2 intersect orthogonally.

Length of their common chord is $\dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from it to the two circles are equal.

The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by,

$$S_1 - S_2 = 0.$$

(i) The equations of radical axis and the common chord of two circles are identical.

(ii) The radical axis of two circles is always perpendicular to the line joining the centres of the circles.

(iii) The radical axis of three circles, whose centres are non-collinear, taken in pairs of concurrent.

(iv) The centre of the circle cutting two given circles orthogonally, lies on their radical axis.

(v) **Radical centre** The point of intersection of radical axis of three circles whose centres are non-collinear, taken in pairs is called their radical centre.

(vi) The circle with centre at the radical centre and radius equal to the length of the tangents from it to one of the circles intersects all the three circles orthogonally.

Family of Circles

- The equation of a family of circles passing through the intersection of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and line $L \equiv lx + my + n = 0$ is
  $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$
  or $S + \lambda L = 0$
  where, λ is any real number.
- The equation of the family of circles passing through the point $A(x_1, y_1)$ and $B(x_2, y_2)$ is
  $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$
  $\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda L = 0$
  where, $L = 0$ represents the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ and $\lambda \in R$.
- The equation of the family circles touching the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is
  $$x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$$
  or $S + \lambda L = 0$
  where, $L = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.
- The equation of a family of circles passing through the intersection of the circles
  $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
  and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is
  $S_1 + \lambda S_2 = 0$, where $(\lambda \neq -1)$ is an arbitrary real number.

Practice Zone DAY 26

1. A circle passes through the points of intersection of the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ with the coordinate axes, then λ is
(a) 0 (b) 1 (c) 2 (d) 1/2

2. The number of tangents that can be drawn from the point $\left(\frac{5}{2}, 1\right)$ to the circle passing through the points $(1, \sqrt{3})$, $(1, -\sqrt{3})$ and $(3, -\sqrt{3})$ is
(a) 1 (b) 0
(c) 2 (d) None of these

3. From the origin chords are drawn to the circle $(x-1)^2 + y^2 = 1$. The equation of the locus of the mid-points of these chords
(a) $x^2 + y^2 - x = 0$ (b) $-x^2 - y^2 + x = 0$
(c) $x^2 + y - x = 0$ (d) None of these

4. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5).
(a) $x^2 + y^2 - 18x - 16y + 120 = 0$
(b) $x^2 + y^2 - 27x - 19y + 130 = 0$
(c) $x^2 - y^2 - 18x - 16y + 120 = 0$
(d) None of the above

5. The equation of circle which passes through the points (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$ is
(a) $25(x^2 + y^2) - 20x + 2y + 60 = 0$
(b) $25(x^2 + y^2) - 20x + 2y - 60 = 0$
(c) $25(x^2 - y^2) - 20x - 2y - 60 = 0$
(d) None of the above

6. AB is chord of the circle $x^2 + y^2 = 25$. The tangents of A and B intersect at C. If (2, 3) is the mid-point of AB, then area of the quadrilateral $OACB$ is
(a) $50\sqrt{\frac{13}{3}}$ (b) $50\sqrt{\frac{3}{13}}$
(c) $50\sqrt{3}$ (d) $\frac{50}{\sqrt{3}}$

7. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. If the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at C, then find the area of the quadrilateral $ABCD$.
(a) 78 (b) 75 (c) 79 (d) 85

8. Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.
(a) $x^2 + y^2 - 6x + 12y - 15 = 0$
(b) $x^2 + y^2 - 6x - 12y + 15 = 0$
(c) $x^2 + y^2 - 6x + 12y + 15 = 0$
(d) None of the above

9. The abscissae of two points A and B are the roots of $x^2 + 2ax - b^2 = 0$ and their coordinates are the roots of $y^2 + 2py - q^2 = 0$. Find the equation and the radius of the circle on AB as diameter.
(a) $\sqrt{a^2 + p^2 + b^2 + q^2}$ (b) $\sqrt{a^2 - p^2 + b^2 - q^2}$
(c) $-a^2 + b^2 - p^2 - q^2$ (d) None of these

10. The tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$, also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find its point of contact.
(a) $x = 2, y = 1$ (b) $x = 3, y = -1$
(c) $x = 5, y = 7$ (d) None of these

11. Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and the third vertex lies above the x-axis. Find the equation of its circumcircle.
(a) $x^2 - y^2 + \frac{2y}{\sqrt{3}} + 1 = 0$ (b) $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$
(c) $x^2 - y^2 - \frac{y}{\sqrt{3}} = 0$ (d) None of these

12. The ΔPQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (–4, 3), respectively, then $\angle QPR$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

13. The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is **[NCERT Exemplar]**

(a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$

14. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents from the point $P(6, 8)$ to the circle and the chord of contact is maximum.

(a) $r = 4$ (b) $r = 5$ (c) $r = 3$ (d) $r = 1$

15. The lines $3x - y + 3 = 0$ and $x - 3y - 6 = 0$ cut the coordinate axes at concyclic points. The equation of the circle through these points is

(a) $x^2 + y^2 - 5x - y - 6 = 0$ (b) $x^2 + y^2 + 5x + y + 6 = 0$
(c) $x^2 + y^2 + 2xy = 0$ (d) None of these

16. Equation of a circle through the origin and belonging to the coaxial system of which the limiting points are (1, 2) and (4, 3) is

(a) $x^2 + y^2 - 8x - 6y = 0$ (b) $x^2 + y^2 - 2x + 4y = 0$
(c) $2x^2 + 2y^2 - x - 7y = 0$ (d) None of these

17. Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

(a) $x^2 + y^2 - 4x - 10y + 25 = 0$
(b) $x^2 + y^2 - 4x - 10y - 25 = 0$
(c) $x^2 + y^2 - 4x + 10y - 25 = 0$
(d) None of the above

18. Let L_1 be a line passing through the origin and L_2 be the line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 is

(a) $x + y = 0$ (b) $x + y = 2$
(c) $x + 7y = 0$ (d) $x - 7y = 0$

19. Circles are drawn through the points (a, b) and $(b, -a)$ such that the chord joining the two points subtends an angle of 45° at any point of the circumference. Then, the distance between the centres is

(a) $\sqrt{3}$ times the radius of either circle
(b) 2 times the radius of either circle
(c) $\frac{1}{\sqrt{2}}$ times the radius of either circle
(d) $\sqrt{2}$ times the radius of either circle

20. If $a_n, n = 1, 2, 3, 4$ represent four distinct positive real numbers other than unit such that each pair of the logarithm of a_n and the reciprocal of logarithm denotes a point on a circle, whose centre lies on Y-axis. Then, the product of these four member is

(a) 0 (b) 1 (c) 2 (d) 3

21. The set of values of a for which the point $(2a, a + 1)$ is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord $x - y + 1 = 0$, is

(a) $\left(\frac{5}{9}, \frac{9}{5}\right)$ (b) $\left(0, \frac{5}{9}\right)$
(c) $\left(0, \frac{9}{5}\right)$ (d) $\left(1, \frac{9}{5}\right)$

22. Three concentric circles of which biggest circle is $x^2 + y^2 = 1$, have their radii in AP. If the line $y = x + 1$ cuts all the circles in real and distinct points, then the interval in which the common difference of AP will lie, is

(a) $\left[0, \left(1 - \frac{1}{\sqrt{2}}\right)\right]$ (b) $\left(0, \frac{1}{2}\right)$
(c) (1, 1) (d) $\left[0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right]$

23. Let a circle be $2x(x - a) + y(2y - b) = 0$, $a \neq 0, b \neq 0$. Then, the condition on a and b, if two chords each bisected by the X-axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$, is

(a) $a^2 > 2b^2$ (b) $a^2 < 2b^2$
(c) $a^2 = 2b^2$ (d) None of these

24. The length of the common chord of two circles $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is

(a) $\sqrt{4c^2 + 2(a - b)^2}$ (b) $\sqrt{4c^2 - (a - b)^2}$
(c) $\sqrt{4c^2 - 2(a - b)^2}$ (d) $\sqrt{2c^2 - 2(a - b)^2}$

25. A circle touches the hypotenuse of a right angle triangle at its middle point and passes through the mid-point of the shorter side. If a and b $(a < b)$ be the length of the sides, then the radius is

(a) $\frac{b}{a}\sqrt{a^2 + b^2}$ (b) $\frac{b}{2a}\sqrt{a^2 - b^2}$
(c) $\frac{b}{4a}\sqrt{a^2 + b^2}$ (d) None of these

26. Let P, Q and R be the centres and r_1, r_2 and r_3 be the corresponding radii of the three circles form a system of coaxial circle, then $r_1^2 \cdot QR + r_2^2 \cdot RP + r_3^2 \cdot PQ$ is equal to

(a) $PQ \cdot QR \cdot RP$ (b) $-PQ \cdot QR \cdot RP$
(c) $PQ + QR + RP$ (d) $\frac{PQ}{QR} \times RP$

27. The limiting points of the coaxial system of circles given by $x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$. Subtend a right angle at the origin, if

(a) $-\frac{c}{g^2} - \frac{k}{f^2} = 2$ (b) $\frac{c}{g^2} + \frac{k}{f^2} = -2$
(c) $\frac{c}{g^2} - \frac{k}{f^2} = 2$ (d) $\frac{c}{g^2} + \frac{k}{f^2} = 2$

28. The equation of the circle of minimum radius which contains the three circles

$$x^2+y^2-4y-5=0,$$
$$x^2+y^2+12x+4y+31=0$$
and $$x^2+y^2+6x+12y+36=0 \text{ is}$$

(a) $\left(x-\frac{31}{18}\right)^2+\left(y-\frac{23}{12}\right)^2=\left(3-\frac{5}{36}\sqrt{949}\right)^2$

(b) $\left(x+\frac{23}{12}\right)^2+\left(y-\frac{31}{18}\right)^2=\left(3+\frac{5}{36}\sqrt{949}\right)^2$

(c) $\left(x+\frac{31}{18}\right)^2+\left(y+\frac{23}{12}\right)^2=\left(3+\frac{5}{36}\sqrt{949}\right)^2$

(d) None of the above

29. The equation of the circle circumscribing the triangle formed by the lines $x+y=6$, $2x+y=4$ and $x+2y=5$ is

(a) $x^2+y^2+17x+19y+50=0$

(b) $x^2+y^2-17x-19y+50=0$

(c) $x^2+y^2-19x-17y+50=0$

(d) None of the above

30. The equation of the circle circumscribing the quadrilateral formed by the lines in order are $5x+3y-9=0$, $x-3y=0$, $2x-y=0$, $x+4y-2=0$, is

(a) $2x^2+2y^2-\frac{16}{9}x+\frac{2}{3}y=0$

(b) $2x^2+y^2+\frac{16}{9}x+\frac{2}{3}y=0$

(c) $x^2+y^2+\frac{16}{9}x+\frac{1}{3}y=0$

(d) $x^2+y^2-\frac{16}{9}x+\frac{1}{3}y=0$

Directions (Q. Nos. 31 and 32) *P is a variable point on the line $L=0$. Tangents are drawn to the circle $x^2+y^2=4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.*

31. If $L\equiv 2x+y=6$, then the locus of circumcentre of ΔPQR is

(a) $2x-y=4$
(b) $2x+y=3$
(c) $x-2y=4$
(d) $x+2y=3$

32. If $P\equiv(2,3)$, then the centre of circumcircle of ΔQRS is

(a) $\left(\frac{2}{13},\frac{7}{26}\right)$ (b) $\left(\frac{2}{13},\frac{3}{26}\right)$ (c) $\left(\frac{3}{13},\frac{9}{26}\right)$ (d) $\left(\frac{3}{13},\frac{2}{13}\right)$

Directions (Q. Nos. 33 and 34) *Let the point to $A(3,7)$ and $B(6,5)$ and equation of circle, $C\equiv x^2+y^2-4x-6y-3=0$.*

33. The chords in which the circle C cuts the members of the family S of circles through A and B are concurrent at

(a) $(2,3)$
(b) $\left(2,\frac{23}{3}\right)$
(c) $\left(3,\frac{23}{2}\right)$
(d) $(3,2)$

34. Equation of the member of the family S which bisects the circumference of C is

(a) $x^2+y^2-5x-1=0$
(b) $x^2+y^2-5x+6y-1=0$
(c) $x^2+y^2-5x-6y-1=0$
(d) $x^2+y^2+5x-6y-1=0$

Directions (Q. Nos. 35 to 38) *Consider the two circles $C_1: x^2+y^2=r_1^2$ and $C_2: x^2+y^2=r_2^2$ $(r_2<r_1)$. Let A be a fixed point on the circle C_1, say $A(r_1,0)$ and B be a, variable point on the circle C_2. The line BA meets the circle C_2 again at C.*

35. The maximum value of BC^2 is

(a) $4r_1^2$
(b) $4r_2^2$
(c) $4r_2^2-4r_1^2$
(d) None of these

36. The minimum value of BC^2 is

(a) $4r_1^2$
(b) $4r_2^2$
(c) $4r_2^2-4r_1^2$
(d) None of these

37. The set of values of $OA^2+OB^2+BC^2$ is

(a) $[5r_2^2-3r_1^2, 5r_2^2+r_1^2]$
(b) $[4r_2^2-4r_1^2, -4r_1^2]$
(c) $[4r_1^2, 4r_2^2]$
(d) $[5r_2^2-3r_1^2, 5r_2^2+3r_1^2]$

38. The locus of the mid-point of AB, O being the origin

(a) $\left(x-\frac{r_1}{2}\right)^2+y^2=\frac{r_2^2}{2}$
(b) $\left(x-\frac{r_1}{2}\right)^2+y^2=\frac{r_2^2}{4}$
(c) $\left(x-\frac{r_2}{2}\right)^2+y^2=\frac{r_1^2}{2}$
(d) $\left(x-\frac{r_2}{2}\right)^2+y^2=\frac{r_1^2}{4}$

Directions (Q. Nos. 39 to 44) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

39. Tangents are drawn from the point (17, 7) to the circle $x^2+y^2=169$.

Statement I The tangents are mutually perpendicular.

Statement II The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$.

40. Consider the radius should be zero in limiting points.

Statement I Equation of a circle through the origin and belonging to the coaxial system, of which the limiting points are (1, 1) and (3, 3) is $2x^2+2y^2-3x-3y=0$.

Statement II Equation of a circle passing through the point, (1, 1) and (3, 3) is $x^2+y^2-2x-6y+6=0$.

41. **Statement I** A ray of light incident at the point $(-3, -1)$ gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. If the reflected ray touches the circle, then equation of the reflected ray is $4y - 3x = 5$.

Statement II The angle of incidence = angle of reflection i.e., $\angle i = \angle r$.

42. In three collinear points, no circle can be drawn.

Statement I Number of circles passing through $(-2, 1), (-1, 0), (-4, 3)$ is 1.

Statement II Through three non-collinear points in a plane only one circle can be drawn.

43. **Statement I** The circle of smallest radius passing through two given points A and B must be of radius $\frac{1}{2}AB$.

Statement II A straight line is a shortest distance between two points.

44. Consider $L_1 \equiv 2x + 3y + p - 3 = 0$, $L_2 \equiv 2x + 3y + p + 3 = 0$, where p is a real number and $C \equiv x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement I If line L_1 is a chord of circle C, then L_2 is not always a diameter of circle C.

Statement II If line L_1 is a diameter of circle C, then L_2 is not a chord of circle C.

45. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 (b) 2 (c) 3/2 (d) 1

46. The centre of the circle which circumscribes the square formed by $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is

(a) (3, 7) (b) (4, 7) (c) (2, 5) (d) (6, 9)

47. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and QR intersect at a point X on the circumference of the circle, then $2r$ is equal to

(a) $\sqrt{PQ \cdot RS}$ (b) $\frac{PQ + RS}{2}$

(c) $\frac{2PQ \cdot RS}{PQ + RS}$ (d) $\sqrt{\frac{PQ^2 + RS^2}{2}}$

48. If two circles $x^2 + y^2 + 4x + 6y = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then

(a) $3g' = 2f'$ (b) $3f' = 2g'$ (c) $f' + g' = 6$ (d) $f' - g' = 1$

49. The point of contact of $4x + 5y + 6 = 0$ and $x^2 + y^2 - 2x - 4y - 8 = 0$ is

(a) $\left(\frac{2}{3}, \frac{2}{5}\right)$ (b) $\left(\frac{2}{5}, \frac{5}{4}\right)$ (c) $(3, -2)$ (d) None of these

50. The equation of the circle described on the common chord of the circles, $x^2 + y^2 - 12x + 2y - 10 = 0$ and $x^2 + y^2 - 8x + 5y - 37 = 0$ as a diameter is

(a) $25(x^2 + y^2) - 348x + 14y - 74 = 0$

(b) $25(x^2 + y^2) - 348x + 140y - 74 = 0$

(c) $25(x^2 + y^2) - 300x + 14y + 70 = 0$

(d) None of the above

51. A point P moves such that the sum of the squares of its distances from the sides of a square of side unity is equal to 9. The locus of P is a circle whose centre coincides with the centre of the square. Find its radius, also.

(a) $C\left(\frac{1}{3}, \frac{1}{2}\right), r = 2$ (b) $C\left(\frac{1}{2}, \frac{1}{2}\right), r = 2$

(c) $C\left(\frac{1}{5}, \frac{7}{3}\right), r = 2$ (d) None of these

52. If the straight line $ax + by = 2$; $a, b \neq 0$, touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are

(a) $a = 1, b = 2$ (b) $a = 1, b = -1$

(c) $a = -\frac{4}{3}, b = 1$ (d) None of these

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53. If each of the lines $5x + 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then **[JEE Main 2013]**

(a) $a = 5$ and $b \notin (-1, 1)$ (b) $a = 1$ and $b \notin (-1, 1)$

(c) $a = 2$ and $b \in (-\infty, 1)$ (d) $a = 5$ and $b \in (-\infty, 1)$

54. If a circle C passing through $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$, externally at a point $(1, -1)$, then the radius of the circle C is **[JEE Main 2013]**

(a) 5 (b) $2\sqrt{5}$ (c) 4 (d) $\sqrt{57}$

55. If two vertices of an equilateral triangle are $A(-a, 0)$ and $B(a, 0)$, where $a > 0$ and the third vertex C lies above X-axis, then the equation of the circumcircle of ΔABC is **[JEE Main 2013]**

(a) $3x^2 + 3y^2 - 2\sqrt{3}ay = 3a^2$ (b) $3x^2 + 3y^2 - 2ay = 3a^2$

(c) $x^2 + y^2 - 2ay = a^2$ (d) $x^2 + y^2 - \sqrt{3}ay = a^2$

56. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then a equals to **[JEE Main 2013]**

(a) 0 (b) ± 4 (c) ± 2 (d) ± 3

57. **Statement I** The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$.

Statement II $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$. **[JEE Main 2013]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

58. The circle passing through $(1, -2)$ and touching the x axis to at $(3, 0)$ also passes through the point **[JEE Main 2013]**

(a) $(-5, 2)$ (b) $(2, -5)$
(c) $(5, -2)$ (d) $(-2, 5)$

59. The length of the diameter of the circle which touches the X-axis at the point $(1, 0)$ and passes through the point $(2, 3)$ is **[AIEEE 2012]**

(a) $\frac{10}{3}$ (b) $\frac{3}{5}$ (c) $\frac{6}{5}$ (d) $\frac{5}{3}$

60. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$, $(c > 0)$ touch each other, if **[AIEEE 2011]**

(a) $|a| = c$ (b) $a = 2c$
(c) $|a| = 2c$ (d) $2|a| = c$

61. The equation of the circle passing through the point $(1, 0)$ and $(0, 1)$ and having the smallest radius is **[AIEEE 2011]**

(a) $x^2 + y^2 + x + y - 2 = 0$ (b) $x^2 + y^2 - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 - x - y = 0$ (d) $x^2 + y^2 + 2x + 2y - 7 = 0$

62. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points, if **[AIEEE 2010]**

(a) $-85 < m < -35$ (b) $-35 < m < 15$
(c) $15 < m < 65$ (d) $35 < m < 85$

63. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and $(1, 1)$ and **[AIEEE 2009]**

(a) all values of p (b) all except one value of p
(c) all except two values of p (d) exactly one value of p

64. The point diameterically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is **[AIEEE 2008]**

(a) $(3, 4)$ (b) $(3, -4)$
(c) $(-3, 4)$ (d) $(-3, -4)$

65. Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) is the centre of circle, then **[AIEEE 2007]**

(a) $k \geq 1/2$
(b) $-1/2 \leq k \leq 1/2$
(c) $k \leq 1/2$
(d) $0 < k < 1/2$

66. Any chord of the circle $x^2 + y^2 = 25$ subtends a right angle at the centre. Then, the locus of the centroid of the triangle made by the chord and a moving point P on the circle is **[AIEEE 2007]**

(a) parabola (b) circle
(c) rectangular hyperbola (d) ellipse

67. Let C be the circle with centre $(0, 0)$ and radius 3. The equation of the locus of the mid-points of the chords of the circle C that subtend an angle $2\pi/3$ at its centre is **[AIEEE 2006]**

(a) $x^2 + y^2 = \frac{27}{4}$ (b) $x^2 + y^2 = \frac{9}{4}$
(c) $x^2 + y^2 = \frac{3}{2}$ (d) $x^2 + y^2 = 1$

68. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q, then the line $5x + by - a = 0$ passes through P and Q for **[AIEEE 2005]**

(a) no value of a
(b) exactly one value of a
(c) exactly two values of a
(d) infinitely many values of a

69. If the two circles $(x-1)^2 + (y-3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then **[AIEEE 2003]**

(a) $2 < r < 8$ (b) $r < 2$
(c) $r = 2$ (d) $r > 2$

Answers

1. (c)	2. (b)	3. (a)	4. (a)	5. (b)	6. (b)	7. (b)	8. (a)	9. (a)	10. (b)
11. (b)	12. (c)	13. (c)	14. (b)	15. (a)	16. (c)	17. (a)	18. (c)	19. (d)	20. (b)
21. (c)	22. (d)	23. (a)	24. (c)	25. (c)	26. (b)	27. (d)	28. (c)	29. (b)	30. (d)
31. (b)	32. (c)	33. (b)	34. (c)	35. (b)	36. (c)	37. (a)	38. (b)	39. (a)	40. (b)
41. (b)	42. (d)	43. (b)	44. (c)	45. (b)	46. (b)	47. (a)	48. (a)	49. (b)	50. (a)
51. (b)	52. (c)	53. (d)	54. (a)	55. (a)	56. (b)	57. (d)	58. (c)	59. (a)	60. (a)
61. (c)	62. (b)	63. (c)	64. (d)	65. (a)	66. (b)	67. (b)	68. (a)	69. (a)	

Hints & Solutions

1. Using the condition, if the circle passing through the intersection of two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ with coordinate axes, then

$$a_1 \cdot a_2 = b_1 \cdot b_2$$

$\therefore \quad \lambda \cdot 1 = (-1)(-2)$

$\Rightarrow \quad \lambda = 2$

2. The triangle is right angled. Its circumcircle is $x^2+y^2-4x=0$.

At point $\left(\frac{5}{2}, 1\right), \left(\frac{5}{2}\right)^2 + 1 - 4 \cdot \frac{5}{2} < 0$

Hence, point is inside the circle.

3. Circle is $x^2+y^2-2x=0$

Let (x_1, y_1) be mid-point of the chord, $T = S_1$

$\therefore \quad xx_1 + yy_1 - (x + x_1) = x_1^2 + y_1^2 - 2x_1$

It passes through (0, 0), whose locus is $x^2+y^2-x=0$.

4. The centre of the required circle is such that (5, 5) is the mid-point of the two centres. Hence, required equation of circle having centre (9, 8) is $(x-9)^2+(y-8)^2=25$

i.e., $\quad x^2+y^2-18x-16y+120=0$

5. Given lines are $\quad 3x+5y=1 \quad$ ...(i)

and $\quad (2+c)x+5c^2y=1 \quad$ ...(ii)

From Eqs. (i) and (ii),

$$(1-c)x+5(1-c^2)y=0$$

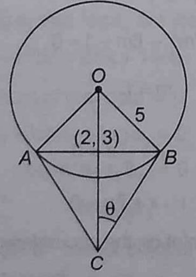
$$c=1, x+10y=0$$

$\therefore \quad$ Centre $= \left(\frac{2}{5}, -\frac{1}{25}\right)$

$\therefore$ Equation of circle is $\left(x-\frac{2}{5}\right)^2+\left(y+\frac{1}{25}\right)^2=\left(2-\frac{2}{5}\right)^2+\frac{1}{25^2}$

$\Rightarrow \quad 25(x^2+y^2)-20x+2y-60=0$

6. Area of quadrilateral $OACB$,



$$A = OB \cdot BC = 5^2 \cot\theta$$

$$= 50\sqrt{\frac{3}{13}} \qquad \left(\because \sin\theta = \frac{\sqrt{13}}{5}\right)$$

7. The tangent at $B(1, 7)$ is $y=7$

and $D(4, -2)$ is $3x-4y-20=0$.

Then, meet at $C(16, 7)$.

Now, $\quad AB=5, BC=15$

Area of quadrilateral $ABCD = AB \cdot BC = 75$

8. Centre of given circle $x^2+y^2-6x+12y+15=0$ is $(3, -6)$.

$\therefore \quad$ Radius $= \sqrt{(3)^2+(-6)^2-15} = \sqrt{30}$

Area of circle $= \pi r^2 = \pi(\sqrt{30})^2 = 30\pi$

Area of require circle = 2 (Area of given circle)

$\therefore \quad \pi R^2 = 2 \times 30\pi = 60\pi$

$\Rightarrow \quad R^2 = 60 \Rightarrow R = 2\sqrt{15}$

$\therefore$ Equation of required circle is

$$(x-3)^2+(y+6)^2=(2\sqrt{15})^2$$

$\Rightarrow \quad x^2+9-6x+y^2+36+12y=60$

$\Rightarrow \quad x^2+y^2-6x+12y-15=0$

9. Let x_1 and x_2 be the roots of $x^2+2ax-b^2=0$ and y_1 and y_2 be the roots of $y^2+2py-q^2=0$. Then, coordinates are $A(x_1, y_1)$ and $B(x_2, y_2)$.

Now, $\quad AB = (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\therefore x^2 - x(x_1+x_2)+x_1x_2+y^2-y(y_1+y_2)+y_1y_2=0$

i.e., $\quad x^2+y^2+2ax+2py-b^2-q^2=0$

$\therefore \quad r = \sqrt{a^2+p^2+b^2+q^2}$

10. Equation of tangent at $(1, -2)$ is $x-2y-5=0$.

For IInd circle centre $(4, -3)$ and $r = \sqrt{5}$.

Point of contact is $\frac{x-4}{1} = \frac{y+3}{-2}$

$$= -\frac{(4+6-5)}{1^2+2^2} = -1$$

$\therefore \quad x=3, y=-1$

11. Length of side of triangle is 2.

The equation of circle passing through $(-1, 0)$ and $(1, 0)$ is $(x+1)(x-1)+y^2+k\lambda=0$.

The coordinates of third vertex will be $(0, \sqrt{3})$, which is passing by the circle.

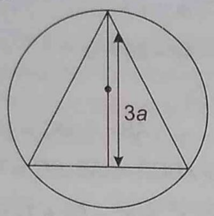
$\therefore \quad \lambda = -\frac{2}{\sqrt{3}}$

So, the equation of circle is $x^2+y^2-\frac{2y}{\sqrt{3}}-1=0$.

12. QR subtend $\frac{\pi}{2}$ at the centre and QR subtends $\frac{\pi}{4}$ at P or $\angle QPR = \frac{\pi}{4}$.

13. Let equation of circle be $x^2 + y^2 = r^2$

Since, in an equilateral triangle, the centroid coincides with the centre of the circle.



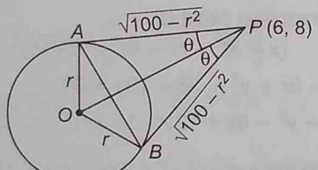
$\therefore$ Radius of circle, $r = \frac{2}{3}(3a) = 2a$

On putting $r = 2a$ in Eq. (i), we get

$$x^2 + y^2 = (2a)^2$$

$$\Rightarrow \quad x^2 + y^2 = 4a^2$$

14. Now, $OP = \sqrt{6^2 + 8^2} = 10$



$$PA = \sqrt{S_1} = \sqrt{100 - r^2}$$

Let $f(r) = \Delta PAB = \frac{1}{2} PA \cdot PB \cdot \sin 2\theta$

$$= (100 - r^2)\sin\theta \cdot \cos\theta$$

$$= \frac{r}{100}(100 - r^2)^{3/2}$$

Put $f'(r) = 0$

$$\Rightarrow \quad \frac{3}{2}(100 - r^2)^{1/2}(-2r^2) + (100 - r^2)^{3/2}$$

$$\Rightarrow \quad \sqrt{100 - r^2}\,(-3r^2 + 100 - r^2) = 0$$

$$\Rightarrow \quad r = \pm 10 \text{ or } r = 5$$

Hence, $r = 5$

15. Here, $3 \times 1 = (-1)(-3) = 3$

Hence, the points are concyclic.

$$\therefore \quad L_1L_2 + \lambda xy = 0$$

$$\Rightarrow \quad (3x - y + 3)(x - 3y - 6) + \lambda xy = 0$$

Now, Coefficient of $xy = 0$

$$\lambda - 10 = 0 \Rightarrow \lambda = 10$$

$$\therefore \quad 3(x^2 + y^2) - 15x - 3y - 18 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 5x - y - 6 = 0$$

16. Point circles of the coaxial system are

$$(x - 1)^2 + (y - 2)^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 5 = 0$$

and $(x - 4)^2 + (y - 3)^2 = 0$

$$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 25 = 0$$

$\therefore$ Equation of coaxial system is

$$(x^2 + y^2 - 2x - 4y + 5) + \lambda(x^2 + y^2 - 8x - 6y + 25) = 0 \quad \ldots(i)$$

Since, its passes through (0, 0).

$$\therefore \quad 5 + 25\lambda = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{5}$$

From Eq. (i), $2x^2 + 2y^2 - x - 7y = 0$

17. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(i)$$

Since, it is passing through the points (2, 3) and (4, 5).

$$\therefore \quad 2^2 + 3^2 + 2g(2) + 2f(3) + c = 0$$

$$\Rightarrow \quad 4g + 6f + c + 13 = 0 \quad \ldots(ii)$$

and $4^2 + 5^2 + 2g(4) + 2f(5) + c = 0 \quad \ldots(iii)$

$$\Rightarrow \quad 8g + 10f + c + 41 = 0$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$4g + 4f + 28 = 0$$

$$\Rightarrow \quad g + f + 7 = 0 \quad \ldots(iv)$$

Also, centre $(-g, -f)$ lies on line $y - 4x + 3 = 0$

$$\therefore \quad -f + 4g + 3 = 0 \quad \ldots(v)$$

On solving Eqs. (iv) and (v), we get

$$g = -2, f = -5$$

On putting $g = -2, f = -5$ in Eq. (ii), we get

$$4(-2) + 6(-5) + c + 13 = 0 \Rightarrow c = 25$$

From Eq. (i), $x^2 + y^2 - 4x - 10y + 25 = 0$

18. The chords are equal length, then the distances of the centre from the lines are equal. Let L_1 be $y - mx = 0$ centre is $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

$$\therefore \quad \frac{\left|-\frac{3}{2} - \frac{m}{2}\right|}{\sqrt{m^2 + 1}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}}$$

$$\Rightarrow \quad 7m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad m = 1, -\frac{1}{7}$$

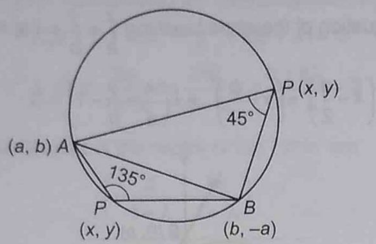
Hence, L_1 be $y + \frac{1}{7}x = 0$

$$\Rightarrow \quad x + 7y = 0$$

19. Let $P(x, y)$ be any point on the circumference of the circle.

Then, $m_1 = \text{Slope of } PA = \frac{b - y}{a - x}$

and $m_2 = \text{Slope of } PB = \frac{-a - y}{b - x}$



We have, $\angle APB = 45°$ or $135°$

$$\Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 45° \text{ or } \tan 135°$$

$$\Rightarrow \quad \frac{\frac{b-y}{a-x} - \frac{-a-y}{b-x}}{1 + \frac{b-y}{a-x} \times \frac{-a-y}{b-x}} = \pm 1$$

$$\Rightarrow \quad \frac{(b-y)(b-x) + (a+y)(a-x)}{(a-x)(b-x) - (b-y)(a+y)} = \pm 1$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

$$\Rightarrow \quad \{x - (a+b)\}^2 + \{y - (b-a)\}^2 = a^2 + b^2$$

The centres of these circles are $O(0,0)$ and $C(a+b, b-a)$.

$\therefore$ Distance between the centre

$$= \sqrt{(a+b)^2 + (a-b)^2} = \sqrt{2}\sqrt{a^2 + b^2}$$

$$= \sqrt{2} \text{ (Radius of either circle)}$$

20. Let $(0, b)$ be the centre and r be the radius of the given circle, then its equation is

$$(x-0)^2 + (y-b)^2 = r^2$$

$$\Rightarrow \quad x^2 + y^2 - 2yb + b^2 - r^2 = 0 \quad \ldots \text{(i)}$$

It is given that the point $P_n\left(\log a_n, \frac{1}{\log a_n}\right)$; $n = 1, 2, 3, 4$ lie on the circle given by Eq. (i).

Therefore,

$$(\log a_n)^2 + \frac{1}{(\log a_n)^2} - \frac{2b}{(\log a_n)} + b^2 - r^2 = 0, n = 1, 2, 3, 4$$

Since, $\log a_1, \log a_2, \log a_3$ and $\log a_4$ are roots of the equation.

then, $\quad \lambda^4 + (b^2 - r^2)\lambda^2 - 2b\lambda + 1 = 0$

$\therefore$ Sum of the roots $= 0$

$$\Rightarrow \quad \log a_1 + \log a_2 + \log a_3 + \log a_4 = 0$$

$$\Rightarrow \quad \log (a_1 a_2 a_3 a_4) = 0$$

$$\Rightarrow \quad a_1 a_2 a_3 a_4 = 1$$

21. The point $(2a, a+1)$ will be an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$

(i) The point $(2a, a+1)$ is an interior point.

(ii) The point $(2a, a+1)$ and the centre $(1, 1)$ are on the same side of the chord $x - y + 1 = 0$.

$$\therefore \quad (2a)^2 + (a+1)^2 - 2(2a) - 2(a+1) - 8 < 0$$

and $\quad (2a - a - 1 + 1)(1 - 1 + 1) > 0$

$$\Rightarrow \quad 5a^2 - 4a - 9 < 0 \text{ and } a > 0$$

$$\Rightarrow \quad (5a - 9)(a + 1) < 0 \text{ and } a > 0$$

$$\Rightarrow \quad -1 < a < \frac{9}{5} \text{ and } a > 0$$

$$\Rightarrow \quad a \in \left(0, \frac{9}{5}\right)$$

22. The equation of the biggest circle is

$$x^2 + y^2 = 1^2$$

Clearly, it is centred at $O(0, 0)$ and has radius 1. Let the radii of the other two circles be $1 - r, 1 - 2r$, where $r > 0$.

Thus, the equations of the concentric circles are

$$x^2 + y^2 = 1 \quad \ldots \text{(i)}$$

$$x^2 + y^2 = (1 - r)^2 \quad \ldots \text{(ii)}$$

$$x^2 + y^2 = (1 - 2r)^2 \quad \ldots \text{(iii)}$$

Clearly, $y = x + 1$ cuts the circle (i) at $(1, 0)$ and $(0, 1)$. This line will cut circles (ii) and (iii) in real and distinct points, if

$$\left|\frac{1}{\sqrt{2}}\right| < 1 - r \text{ and } \left|\frac{1}{\sqrt{2}}\right| < 1 - 2r$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} < 1 - r \text{ and } \frac{1}{\sqrt{2}} < 1 - 2r$$

$$\Rightarrow \quad r < 1 - \frac{1}{\sqrt{2}} \text{ and } r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

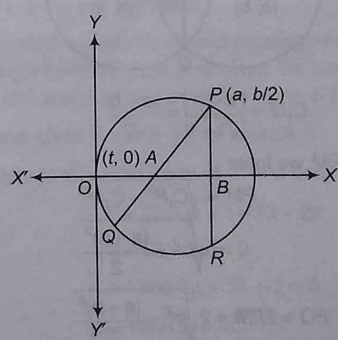
$$\Rightarrow \quad r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad r \in \left[0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right] \quad (\because r > 0)$$

23. The equation of the circle is

$$2x(x - a) + y(2y - b) = 0$$

$$\Rightarrow \quad x^2 + y^2 - ax - \frac{b}{2}y = 0$$



Let PQ and PR be two chords drawn from $P\left(a, \frac{b}{2}\right)$ such that they are bisected by X-axis.

Let $A(t, 0)$ be the mid-point of PQ.

Then, its equation is

$$tx + 0y - \frac{a}{2}(x+t) - \frac{b}{4}(y+0) = t^2 - at \qquad \text{(using } T = S'\text{)}$$

$$\Rightarrow \quad \left(t - \frac{a}{2}\right)x - \frac{b}{4}y = t^2 - \frac{a}{2}t$$

This passes through $P\left(a, \frac{b}{2}\right)$.

Therefore, $\left(t - \frac{a}{2}\right)a - \frac{b^2}{8} = t^2 - \frac{a}{2}t$

$$\Rightarrow \quad t^2 - \frac{3}{2}at + \left(\frac{a^2}{2} + \frac{b^2}{8}\right) = 0$$

This should give two distinct values of t for points A and B.

$$\therefore \quad \frac{9}{4}a^2 - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\Rightarrow \quad \frac{a^2}{4} - \frac{b^2}{2} > 0$$

$$\Rightarrow \quad a^2 > 2b^2$$

24. The equations of two circles are

$$S_1 \equiv (x-a)^2 + (y-b)^2 = c^2 \qquad \ldots\text{(i)}$$

and

$$S_2 \equiv (x-b)^2 + (y-a)^2 = c^2 \qquad \ldots\text{(ii)}$$

The equation of the common chord of these circles is

$$S_1 - S_2 = 0$$

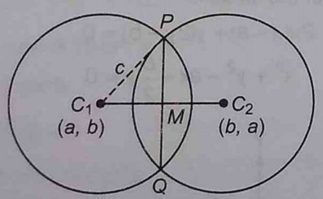
$$\Rightarrow \quad (x-a)^2 - (x-b)^2 + (y-b)^2 - (y-a)^2 = 0$$

$$\Rightarrow \quad (2x-a-b)(b-a) + (2y-b-a)(a-b) = 0$$

$$\Rightarrow \quad 2x - a - b - 2y + b + a = 0$$

$$\Rightarrow \quad x - y = 0$$

The centre coordinates of circles S_1 and S_2 are $C_1(a, b)$ and $C_2\ (b, a)$, respectively.



Now, $\quad C_1M = \left|\frac{a-b}{\sqrt{1+1}}\right| = \frac{|a-b|}{\sqrt{2}}$

In right ΔC_1PM, we have

$$PM = \sqrt{C_1P^2 - C_1M^2}$$

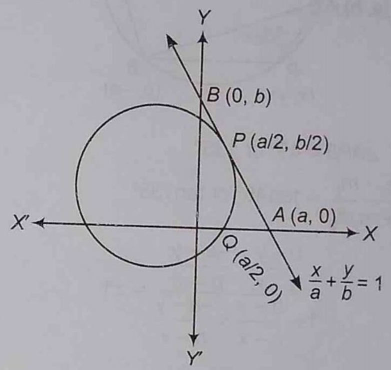
$$= \sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$$\therefore \quad PQ = 2PM = 2\sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$$= \sqrt{4c^2 - 2(a-b)^2}$$

25. The equation of the circle touching $\frac{x}{a} + \frac{y}{b} = 1$ at $P\left(\frac{a}{2}, \frac{b}{2}\right)$ is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \lambda\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0 \qquad \ldots\text{(i)}$$



Since, it passes through $Q\left(\frac{a}{2}, 0\right)$.

Therefore, $\quad \lambda = \frac{b^2}{2} \qquad$ [from Eq. (i)]

On putting the value of λ in Eq. (i), we get

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \frac{b^2}{2}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$$

$$\Rightarrow \quad x^2 + y^2 - \left(a - \frac{b^2}{2a}\right)x - \frac{by}{2} + \frac{a^2 - b^2}{4} = 0$$

Let r be the radius of this circle.

Then, $r^2 = \frac{1}{4}\left(a - \frac{b^2}{2a}\right)^2 + \frac{b^2}{16} - \left(\frac{a^2 - b^2}{4}\right) = \frac{b^2(a^2 + b^2)}{16a^2}$

$$\Rightarrow \quad r = \frac{b}{4a}\sqrt{a^2 + b^2}$$

26. Let equation of circle be $x^2 + y^2 + 2gx + c = 0$.

where, g is a variable and c is a constant, be a coaxial system of circle having common radical axis as x-axis.

Let $x^2 + y^2 + 2g_i\, x + c = 0;\ i = 1, 2, 3$

be three members of the given coaxial system of circles.

Then, the coordinates of their centres and radii are

$$P(-g_1, 0),\ Q(-g_2, 0),\ R(-g_3, 0)$$

and $\quad r_1^2 = g_1^2 - c,\ r_2^2 = g_2^2 - c,\ r_3^2 = g_3^2 - c$

Now, $r_1^2 \cdot QR + r_2^2 \cdot RP + r_3^2 \cdot PQ$

$$= (g_1^2 - c)(g_2 - g_3) + (g_2^2 - c)(g_3 - g_1) + (g_3^2 - c)(g_1 - g_2)$$

$$= g_1^2(g_2 - g_3) + g_2^2(g_3 - g_1) + g_3^2(g_1 - g_2) - c\{(g_2 - g_3) + (g_3 - g_1) + (g_1 - g_2)\}$$

$$= -(g_1 - g_2)\cdot(g_2 - g_3)\cdot(g_3 - g_1)$$

$$= -PQ \cdot QR \cdot RP$$

27. The equation representing the coaxial system of circle is

$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$

$\Rightarrow \quad x^2 + y^2 + \frac{2g}{1+\lambda}x + \frac{2f\lambda}{1+\lambda}y + \frac{c+k\lambda}{1+\lambda} = 0 \quad \ldots(i)$

The coordinates of the centre of this circle are

$\left(-\frac{g}{1+\lambda}, -\frac{f\lambda}{1+\lambda}\right) \quad \ldots(ii)$

and radius $= \sqrt{\frac{g^2 + f^2\lambda^2 - (c+k\lambda)(1+\lambda)}{(1+\lambda)^2}}$

For the limiting points, we must have

Radius = 0

$\Rightarrow \quad g^2 + f^2\lambda^2 - (c+k\lambda)(1+\lambda) = 0$

$\Rightarrow \quad \lambda^2(f^2 - k) - \lambda(c+k) + (g^2 - c) = 0 \quad \ldots(iii)$

Let λ_1 and λ_2 be the roots of this equation.

Then, $\quad \lambda_1 + \lambda_2 = \frac{c+k}{f^2-k}$

and $\quad \lambda_1\lambda_2 = \frac{g^2-c}{f^2-k} \quad \ldots(iv)$

Thus, the coordinates of limiting points L_1 and L_2 are,

$L_1\left(\frac{-g}{1+\lambda_1}, \frac{-f\lambda_1}{1+\lambda_1}\right)$

and $\quad L_2\left(\frac{-g}{1+\lambda_2}, \frac{-f\lambda_2}{1+\lambda_2}\right) \quad$ [from Eq. (iv)]

Now, L_1L_2 will subtend a right angle at the origin.

If slope of $OL_1 \times$ slope of $OL_2 = -1$

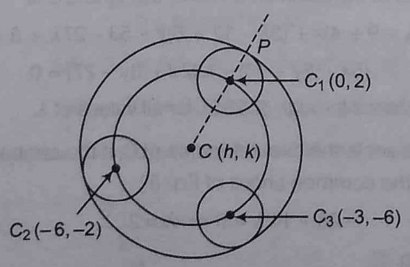
$\Rightarrow \quad \frac{f\lambda_1}{g} \times \frac{f\lambda_2}{g} = -1 \Rightarrow f^2\lambda_1\lambda_2 = -g^2$

$\Rightarrow \quad f^2\left(\frac{g^2-c}{f^2-k}\right) = -g^2$

$\Rightarrow \quad f^2(g^2-c) + g^2(f^2-k) = 0$

$\Rightarrow \quad \frac{c}{g^2} + \frac{k}{f^2} = 2$

28. The coordinates of the centres and radii of three given circles are as given



Circle	Centre	Radius
Ist Circle	$C_1(0,2)$	$r_1 = 3$
IInd Circle	$C_2(-6,-2)$	$r_2 = 3$
IIIrd Circle	$C_3(-3,-6)$	$r_3 = 3$

Let $C(h,k)$ be the centre of the circle passing through the centres of the circles Ist, IInd and IIIrd.

Then, $\quad CC_1 = CC_2 = CC_3$

$\Rightarrow \quad CC_1^2 = CC_2^2 = CC_3^2$

$\Rightarrow \quad (h-0)^2 + (k-2)^2 = (h+6)^2 + (k+2)^2$

$= (h+3)^2 + (k+6)^2$

$\Rightarrow \quad -4k + 4 = 12h + 4k + 40 = 6h + 12k + 45$

$\Rightarrow 12h + 8k + 36 = 0$ and $6h - 8k - 5 = 0$

$\Rightarrow \quad 3h + 2k + 9 = 0$ and $6h - 8k - 5 = 0$

$\Rightarrow \quad h = \frac{-31}{18}, k = \frac{-23}{12}$

$\therefore \quad CC_1 = \sqrt{\left(0+\frac{31}{18}\right)^2 + \left(2+\frac{23}{12}\right)^2}$

$= \frac{5}{36}\sqrt{949}$

Now, $\quad CP = CC_1 + C_1P$

$\Rightarrow \quad CP = \left(\frac{5}{36}\sqrt{949} + 3\right)$

Thus, required circle has its centre at $\left(\frac{-31}{18}, \frac{-23}{12}\right)$ and radius

$= CP = \left(\frac{5}{36}\sqrt{949} + 3\right).$

Hence, its equation is

$\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$

29. Consider the equation

$(x+y-6)(2x+y-4) + \lambda(2x+y-4)(x+2y-5)$

$+\mu(x+2y-5)(x+y-6) = 0 \ldots(i)$

where, λ and μ are constants.

This equation represents a curve passing through the vertices of the triangle formed by the given lines. We have to determine the values of λ and μ so that the curve given in Eq. (i) is a circle.

The curve given in Eq. (i) will be a circle, if

coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$\Rightarrow \quad 2 + 2\lambda + \mu = 1 + 2\lambda + 2\mu$

and $\quad 3 + 5\lambda + 3\mu = 0$

$\Rightarrow \quad \mu = 1$ and $3\mu + 5\lambda + 3 = 0$

$\Rightarrow \quad \mu = 1$ and $\lambda = -\frac{6}{5}$

On substituting the values of λ and μ in Eq. (i), we get

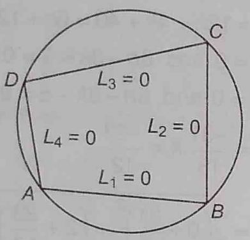
$$(x+y-6)(2x+y-4)-\left(\frac{6}{5}\right)(2x+y-4)(x+2y-5)$$
$$+(x+2y-5)(x+y-6)=0$$
$$\Rightarrow \quad x^2+y^2-17x-19y+50=0$$

This is the equation of the required circle.

30. The equation of the sides of the quadrilateral are

$$L_1 \equiv 5x+3y-9=0$$
$$L_2 \equiv x-3y=0$$
$$L_3 \equiv 2x-y=0$$

and $\quad L_4 \equiv x+4y-2=0$



The equation of the curve passing through the vertices of the quadrilateral is $L_1L_3+\lambda L_2L_4=0$

$$\Rightarrow \quad (5x+3y-9)(2x-y)+\lambda(x-3y)(x+4y-2)=0 \quad \ldots(i)$$

This will represent a circle, if

coefficient of x^2 = coefficient of y^2 and coefficient of $xy=0$

$$\Rightarrow \quad 10+\lambda=-3-12\lambda$$
$$\Rightarrow \quad \lambda=-1$$

On substituting the value of λ in Eq. (i), we get

$$(5x+3y-9)(2x-y)-(x-3y)(x+4y-2)=0$$
$$\Rightarrow \quad x^2+y^2-\frac{16}{9}x+\frac{1}{3}y=0$$

which is the equation of the required circle

31. Length of tangents from a point to the circle are equal.

$$PQ=PR$$

Then, parallelogram $PQSR$ is rhombus.

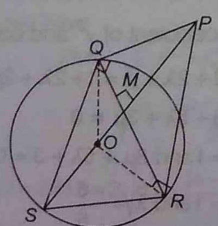
$\therefore$ Mid-point of QR = Mid-point of PS and $QR \perp PS$

So, S is the mirror image of P with respect to QR.

$\because \quad L \equiv 2x+y=6$

Let $\quad P \equiv (\lambda, 6-2\lambda)$

$\because \quad \angle PQO = \angle PRO = \pi/2$



Since, OP is diameter of circumcircle PQR, then centre is $\left(\frac{\lambda}{2}, 3-\lambda\right)$.

$$\therefore \quad x=\frac{\lambda}{2} \Rightarrow \lambda=2x$$

and $\quad y=3-\lambda$

Then, $\quad 2x+y=3$

32. Since, $P \equiv (2, 3)$

So, the equation of QR is $2x+3y=4$.

Let $\quad S \equiv (\alpha, \beta)$

$$\Rightarrow \quad \frac{\alpha-2}{2}=\frac{\beta-3}{3}=\frac{-2(4+9-4)}{(4+9)}$$
$$=-\frac{18}{13}$$
$$\therefore \quad \alpha=-\frac{10}{13}$$
$$\beta=-\frac{15}{13}$$

Now, equation of circumcircle of ΔQRS is

$$(x^2+y^2-4)+\lambda(2x+3y-4)=0.$$

Since, its passes through $S \equiv \left(-\frac{10}{13}, -\frac{15}{13}\right)$, then

$$\left(\frac{100}{169}+\frac{225}{169}-4\right)-\lambda\left(-\frac{20}{13}-\frac{45}{13}-4\right)=0$$
$$\Rightarrow \quad \left(-\frac{351}{169}\right)+\lambda(-9)=0$$
$$\Rightarrow \quad \lambda=-\frac{39}{169}=-\frac{3}{13}$$

Hence, circumcentre is $\left(-\lambda, -\frac{3\lambda}{2}\right)$ *i.e.*, $\left(\frac{3}{13}, \frac{9}{26}\right)$.

33. Equation of a member of the family S with AB as diameter is $(x-3)(x-6)+(y-7)(y-5)=0$

Equation of line AB is $2x+3y-27=0$.

Equation of the family S is

$$(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)=0$$
$$\Rightarrow \quad x^2+y^2+(2\lambda-9)x+(3\lambda-12)y+53-27\lambda=0 \quad \ldots(i)$$

Equation of the common chord of Eq. (i) and C is

$$(2\lambda-9+4)x+(3\lambda-12+6)y+53-27\lambda+3=0$$
$$\Rightarrow \quad (5x+6y-56)-\lambda(2x+3y-27)=0. \quad \ldots(ii)$$

which passes through (2, 23/3), for all values of λ.

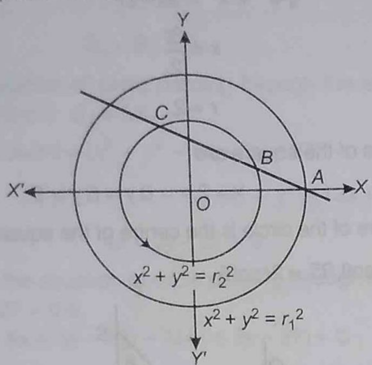
34. Eq. (i) bisects the circumference of C, if the centre (2, 3) of C lies on the common chord of Eq. (ii).

$$\therefore \quad -28+14\lambda=0 \Rightarrow \lambda=2$$

From Eq. (i),

$$x^2+y^2-5x-6y-1=0$$

35. Let the equation of AB be,



$$\frac{x - r_1}{\cos\theta} = \frac{y_1 - 0}{\sin\theta} = r$$

$\therefore$ Coordinates of any point on this line are

$$(r_1 + r\cos\theta, r\sin\theta)$$

If it lies on $x^2 + y^2 = r_2^2$. Then, we have

$$(r_1 + r\cos\theta)^2 + (r\sin\theta)^2 = r_2^2$$

$\Rightarrow$ $r^2 + 2rr_1\cos\theta + r_1^2 - r_2^2 = 0$ ...(i)

Let $AB = r_B$ and $AC = r_C$.

Then, r_B and r_C are the roots of Eq. (i), we get

$$r_B - r_C = -2r_1\cos\theta$$

and $r_B r_C = r_1^2 - r_2^2$

$\Rightarrow$ $(BC)^2 = (r_C - r_B)^2 = (r_C + r_B)^2 - 4r_B r_C$

$$= 4r_1^2\cos^2\theta - 4r_1^2 + 4r_2^2$$

which is maximum, when $\cos^2\theta = 1$.

$\therefore$ $(BC)^2_{\max} = 4r_2^2$

36. Again, $(BC)^2$ is minimum, when $\cos^2\theta = 0$

$\therefore$ $(BC)_{\min} = 4r_2^2 - 4r_1^2$

37. Now, $OA^2 + OB^2 + BC^2$

$\Rightarrow$ $r_1^2 + r_2^2 + 4r_1^2\cos^2\theta - 4r_1^2 + 4r_2^2$

$\Rightarrow$ $5r_2^2 - 3r_1^2 + 4r_1^2\cos^2\theta$

Now, $0 \le \cos^2\theta \le 1$

$\Rightarrow$ $0 \le 4r_1^2\cos^2\theta \le 4r_1^2$

$\Rightarrow$ $5r_2^2 - 3r_1^2 \le 5r_2^2 - 3r_1^2 + 4r_1^2\cos^2\alpha \le 5r_2^2 + r_1^2$

$\Rightarrow$ $OA^2 + OB^2 + BC^2 \in [5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$

38. Let (h,k) be the mid-point of AB and let (α,β) be the coordinate of B.

Then, $\frac{\alpha + r_1}{2} = h$

and $\frac{\beta}{2} = k$

$\Rightarrow$ $\alpha = 2h - r_1$

and $\beta = 2k$

where, (α, β) lies on $x^2 + y^2 = r_2^2$

$\therefore$ $\alpha^2 + \beta^2 = r_2^2$

$\Rightarrow$ $(2h - r_1)^2 + (2k)^2 = r_2^2$

$\therefore$ Locus of mid-point of AB is

$$\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$$

39. Since, the tangents are perpendicular. So, locus of perpendicular tangents to the circle $x^2 + y^2 = 169$ is a director circle having equation $x^2 + y^2 = 338$.

40. Equation of circle, when the limiting points are (1, 1) and (3, 3) is

$$(x - 1)^2 + (y - 1)^2 = 0$$

and $(x - 3)^2 + (y - 3)^2 = 0$

$\Rightarrow$ $x^2 + y^2 - 2x - 2y + 2 = 0$

and $x^2 + y^2 - 6x - 6y + 18 = 0$

Equation of the coaxial system of circle is

$$x^2 + y^2 - 6x - 6y + 18 + \lambda(x^2 + y^2 - 2x - 2y + 2) = 0$$

It passes through origin, therefore

$$\lambda = -9$$

Hence, required circle is $2x^2 + 2y^2 - 3x - 3y = 0$.

Statement II is also true but it is not a correct explanation for Statement I.

41. Equation of tangent to the given circle is

$$y = mx \pm \sqrt{1 + m^2}$$

If it passes through $(-3, -1)$, then we have

$$-1 = -3m \pm \sqrt{1 + m^2}$$

$\Rightarrow$ $1 + m^2 = (3m - 1)^2$

$\Rightarrow$ $8m^2 - 6m = 0$

$\therefore$ $m = 0, m = \frac{3}{4}$

Thus, reflected ray is $y + 1 = \frac{3}{4}(x + 3)$

$\Rightarrow$ $4y - 3x = 5$

42. Let $P \equiv (-2, 1)$, $Q \equiv (-1, 0)$ and $R \equiv (-4, 3)$

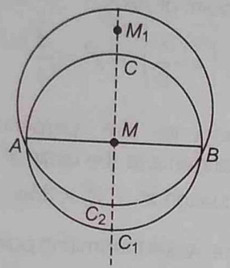
$\therefore$ Slope of $PQ = \frac{0 - 1}{-1 + 2} = -1$

and slope of $QR = \frac{3 - 0}{-4 + 1} = -1$

$\because$ Slope of PQ = Slope of QR

Hence, P, Q and R are collinear *i.e.*, no circle is drawn.

43. Let C_1 be a circle which passes through A and B and C whose diameter is AB and C_2 be another circle which passes through A and B, then centres of C_1 and C_2 must lie on perpendicular bisector of AB. Indeed centre of C_1 is mid-point M of AB and centre of any other circle lies somewhere else on bisector.



Then, $M_1A > AM$ (hypotenuse of right angled ΔAMM_1)

$\Rightarrow$ Radius of $C_2 > \frac{1}{2}AB$

So, C_1 is the circle whose radius is least.

Thus, Statement I is true but does not actually follow from Statement II which is certainly true.

44. Equation of circle C is

$$(x+3)^2 + (y-5)^2 = 9 + 25 - 30$$

i.e., $(x+3)^2 + (y-5)^2 = 4$

$\therefore$ Centre is $(3, -5)$.

If L_1 is a diameter of a circle, then

$$2(3) + 3(-5) + p - 3 = 0$$

$\Rightarrow$ $p = 12$

$\therefore$ L_1 is $2x + 3y + 9 = 0$ and L_2 is $2x + 3y + 15 = 0$.

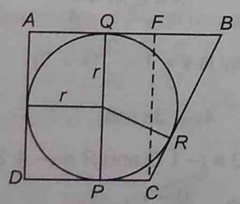
Distance of centre of circle C from L_2

$$= \left|\frac{2(3) + 3(-5) + 15}{\sqrt{2^2 + 3^2}}\right|$$

$$= \frac{6}{\sqrt{13}} < 2$$

Hence, L_2 is a chord of circle C.

45. Let $CD = x$, $AQ = r$



Area of quadrilateral $= \frac{1}{2}(x + 2x) \times 2r = 18$

$\Rightarrow$ $2rx = 12$

$\Rightarrow$ $rx = 6$ ...(i)

$\Rightarrow$ $CB = CR + RB = PC + QB$

$\Rightarrow$ $x - r + 2x - r = 3x - 2r$

$\Rightarrow$ $\sqrt{4r^2 + x^2} = 3x - 2r$

$\Rightarrow$ $x = \frac{3r}{2}$ ...(ii)

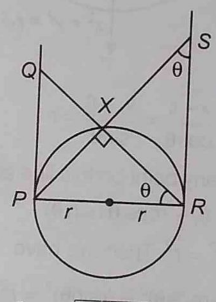
$\therefore$ $r = 2$

46. The sides of the square are

$$x = 2, x = 6, y = 5, y = 9$$

The centre of the circle is the centre of the square (4, 7).

47. $PQ = 2r\tan\theta, RS = 2r\cot\theta$



$\therefore$ $2r = \sqrt{PQ \cdot RS}$

48. The centre of the first circle is $(-2, -3)$ and radius $r_1 = \sqrt{13}$ and centre of second circle is $(-g', -f')$ and radius $r_2 = \sqrt{g'^2 + f'^2}$

$\therefore$ $C_1C_2 = \sqrt{(g'+2)^2 + (f'+3)^2}$

$\because$ Circles touch each other $C_1C_2 = r_1 \pm r_2$

i.e., $\sqrt{g'^2 + f'^2 + 4 + 9 + 4g' + 6f'} = \sqrt{13} \pm \sqrt{(g')^2 + (f')^2}$

On squaring both sides, we get

$$(2g' + 3f')^2 = 13(g'^2 + f'^2)$$

$\Rightarrow$ $12g'f' = 4f'^2 + 9g'^2$

$\Rightarrow$ $(3g' - 2f')^2 = 0$

$\Rightarrow$ $3g' = 2f'$

49. Let the point of contact be (x_1, y_1).

$\therefore$ The equation of tangent to the circle

$x^2 + y^2 - 2x - 4y - 8 = 0$ is

$$xx_1 + yy_1 - (x + x_1) - 2(y + y_1) - 8 = 0$$

or $x(x_1 - 1) + y(y_1 - 2) - (x_1 + 2y_1 + 8) = 0$

But $4x + 5y + 6 = 0$, also represents the same line

$\therefore$ $\frac{(x_1 - 1)}{4} = \frac{(y_1 - 2)}{5} = \frac{-(x_1 + 2y_1 + 8)}{6}$

$5x_1 - 4y_1 + 3 = 0$ ...(i)

and $5x_1 + 16y_1 + 28 = 0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x_1 = \frac{2}{5} \text{ and } y_1 = \frac{5}{4}$$

So, the point of contact is $\left(\frac{2}{5}, \frac{5}{4}\right)$.

50. $\because$ The equation of common chord of two given circles S_1 and S_2 is

$$S_1 - S_2 = 0 \quad \ldots(\text{i})$$

and the equation of circle passing through the end points of common chord is $S_1 + \lambda S_2 = 0$ ...(ii)

$\therefore$ Common chord $= (x^2 + y^2 - 8x + 5y - 37) - (x^2 + y^2 - 12x + 2y - 10) = 0$

$$\Rightarrow \quad 4x + 3y - 27 = 0 \quad \ldots(\text{iii})$$

Therefore, the equation of circle passing through end points of $4x + 3y - 27 = 0$ is

$(x^2 + y^2 - 8x + 5y - 37) + \lambda(4x + 3y - 27) = 0.$

$$\Rightarrow \quad x^2 + y^2 + (4\lambda - 8)x + (5 + 3\lambda)y - (37 + 27\lambda) = 0 \quad \ldots(\text{iv})$$

The centre of the circle (iv) is $\left[-\frac{(4\lambda - 8)}{2}, -\frac{(5 + 3\lambda)}{2}\right]$ and it lies on the common chord.

$$\therefore \quad -4(2\lambda - 4) - 3\left(\frac{5 + 3\lambda}{2}\right) - 27 = 0$$

$$\Rightarrow \quad -8\lambda + 16 - \frac{15}{2} - \frac{9\lambda}{2} - 27 = 0$$

$$\Rightarrow \quad \frac{-25\lambda}{2} + \frac{17}{2} - 27 = 0$$

$$\Rightarrow \quad \frac{-25\lambda}{2} - \frac{37}{2} = 0$$

$$\Rightarrow \quad \lambda = -\frac{37}{25}$$

Therefore, the required circle is

$$x^2 + y^2 + \left[4 \times \left(-\frac{37}{25}\right) - 8\right]x + \left[5 + 3\left(-\frac{37}{25}\right)\right]y - \left[37 + 27\left(-\frac{37}{25}\right)\right] = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{348x}{25} + \frac{14y}{25} - \frac{74}{25} = 0$$

$$\Rightarrow \quad 25(x^2 + y^2) - 348x + 14y - 74 = 0$$

51. Let the sides of the squares be $x = 0$, $y = 0$, $x = 1$, $y = 1$

Locus of P is $x^2 + y^2 + (x - 1)^2 + (y - 1)^2 = 9$

$$\Rightarrow \quad x^2 + y^2 - x - y - \frac{7}{2} = 0$$

Centre of square $C\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $r = 2$.

52. For circle $x^2 + y^2 - 2x = 3$ ...(i)

Centre $\equiv (1, 0)$ and radius $= \sqrt{1^2 + 0^2 + 3} = 2$

and for circle $x^2 + y^2 - 4y = 6$ ...(ii)

Centre $\equiv (0, 2)$ and radius

$$= \sqrt{0^2 + 2^2 + 6} = \sqrt{10}$$

Since, line $ax + by = 2$ ...(iii)

Touches the circle (i), then

$$\frac{a(1) + b(0) - 2}{\sqrt{a^2 + b^2}} = 2$$

$$\Rightarrow \quad a - 2 = 2\sqrt{a^2 + b^2} \quad \ldots(\text{iv})$$

Also, line (iii) is normal to the circle (ii), so it will pass through the centre $(0, 2)$ of the circle (ii).

$$\therefore \quad a - 2 = 2\sqrt{a^2 + 1}$$

$$\Rightarrow \quad (a - 2)^2 = 4(a^2 + 1)$$

$$\Rightarrow \quad 3a^2 + 4a = 0$$

$$\Rightarrow \quad a = 0, -4/3$$

Thus, $a = -4/3$ and $b = 1$ according to the given choices.

53. On solving the equation of diameters, we get

$$x = 1 \text{ and } y = 1$$

i.e., centre of circle is $(1, 1)$.

Given, $\quad x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$ ...(i)

So, centre of the circle is $\{a^2 - 7a + 11, a^2 - 6a + 6\}$.

$$\therefore \quad a^2 - 7a + 11 = 1$$

and $\quad a^2 - 6a + 6 = 1$

$$\Rightarrow \quad a^2 - 7a + 10 = 0$$

and $\quad a^2 - 6a + 5 = 0$

$$\therefore \quad a = 2, 5 \quad \text{and} \quad a = 1, 5$$

On putting $a = 5$ in Eq. (i), we get

$$x^2 + y^2 - 2x - 2y + b^3 + 1 = 0$$

$$\Rightarrow \quad (x - 1)^2 + (y - 1)^3 = (\sqrt{1 - b^3})^2$$

Here, radius $= \sqrt{1 - b^3} > 0$

$$\Rightarrow \quad b^3 - 1 < 0$$

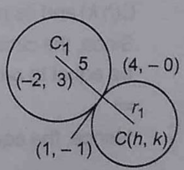
$$\Rightarrow \quad (b - 1)(b^2 + 1 + b) < 0$$

$$\Rightarrow \quad b < 1 \; i.e., b \in (-\infty, 1)$$

54. Let centre and radius of circle C is (h, k), and r_1 respectively.

$\therefore$ Centre if radius of circle

$(-2, 3)$ and $r = \sqrt{4 + 9 + 12} = 5$



55. In an equilateral triangle circumcentre lies on the centroid of a triangle.

$\therefore$ Centre of circle = Centroid of triangle

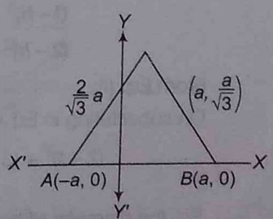
$$= \left(0, \frac{a}{\sqrt{2}}\right)$$

$$r = \frac{2a}{\sqrt{3}}$$

$$\therefore \quad (x - 0)^2 + \left(y - \frac{a}{\sqrt{3}}\right)^2 = \left(\frac{2a}{\sqrt{3}}\right)^2$$

$$\Rightarrow 3x^2 + 3y^2 + a^2 - 2a\sqrt{3}y = 4a^2$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 2\sqrt{3}ay = 3a^2$$



56. When the circle touches abscissa (X-axis), then the equation of circle is

$$(x-h)^2+(y-k)^2=k^2, \text{ where } (h,k)\rightarrow \text{centre}$$

$$\Rightarrow \quad x^2+y^2-2hx-2ky+h^2=0 \quad \ldots(\text{i})$$

Compair with, $x^2+y^2-6x-8y+(25-a^2)=0$

$$\Rightarrow \quad h^2=25-a^2 \quad (\because h=-3)$$

$$\Rightarrow \quad a^2=25-9=16$$

$$\therefore \quad a=\pm 4$$

57. **Statement I** Centre of circle = (3, – 1)

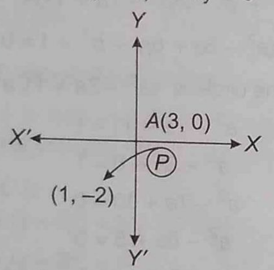
Now, $2(3)+(-1)=5=5$, (true)

Statement II Centre = (3, – 1), which lies on given line.

Simplify it and get the result.

58. Let the equation of circle be

$$(x-3)^2+(y-0)^2+\lambda y=0$$



As it passes through (1, – 2).

$$\therefore \quad (1-3)^2+(-2)^2+\lambda(-2)=0$$

$$\Rightarrow \quad 4+4-2\lambda=0$$

$$\Rightarrow \quad \lambda=4$$

$\therefore$ Equation of circle is $(x-3)^2+y^2+4y=0$

Now, by hit and trial method, we see that point (5, – 2) satisfies equation of circle.

59. Let us assume that the coordinates of the centre of the circle are $C(h,k)$ and its radius is r.

Since, the circle touches x-axis at (1, 0), hence its radius should be equal to ordinate of centre.

$$\Rightarrow \quad r=k$$

Hence, the equation of the circle is

$$(x-h)^2+(y-k)^2=k^2$$

Also, given that the circle passes through the points (1, 0) and (2, 3). Hence, substituting them in the equation of the circle, we get

$$(1-h)^2+(0-k)^2=k^2 \quad \ldots(\text{i})$$

$$(2-h)^2+(3-k)^2=k^2 \quad \ldots(\text{ii})$$

From Eq. (i), $h=1$

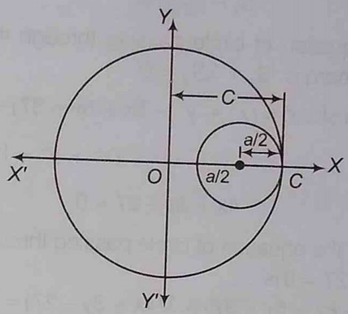
On substituting in Eq. (ii), we get

$$(2-1)^2+(3-k)^2=k^2 \Rightarrow k=\frac{5}{3}$$

So, the diameter of the circle is $2k=\dfrac{10}{3}$.

60. $x^2+y^2-ax=0$ and $x^2+y^2=c^2$ touch each other.

(i) If circles touch internally,



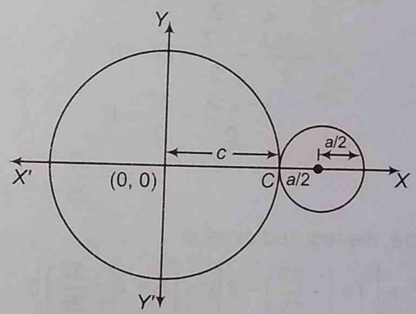
$$\left|c-\frac{a}{2}\right|=\frac{a}{2}$$

$$\Rightarrow \quad c-\frac{a}{2}=\frac{a}{2}$$

$$\Rightarrow \quad c=a, c>0$$

$$\therefore \quad |a|=c$$

(ii) If circles touches externally,



$$\left|c+\frac{a}{2}\right|=\frac{a}{2}$$

$$\Rightarrow \quad c+\frac{a}{2}=\frac{a}{2}$$

$\therefore c=0$, *i.e.*, not possible as $c>0$

Hence, the circles should touch internally, when $|a|=c$.

61. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.

$$\Rightarrow \quad (x-1)(x-0)+(y-0)(y-1)=0 \Rightarrow x^2+y^2-x-y=0$$

62. Since, the coordinates of the centre of the circle are(2, 4).

Also, $\quad r^2=4+16+5=25$

The line will intersect the circle at two distinct points, if the distance of (2, 4) from $3x-4y=m$ is less than radius of the circle.

$$\therefore \quad \frac{|6-16-m|}{5}<5$$

$$\Rightarrow \quad -25<10+m<25$$

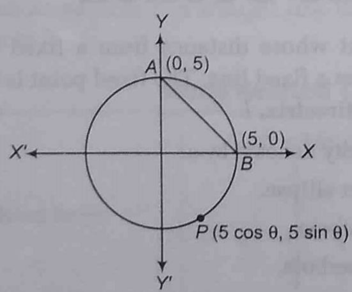
$$\therefore \quad -35<m<15$$

63. Let $S \equiv x^2 + y^2 + 3x + 7y + 2p - 5 = 0$
and $S' \equiv x^2 + y^2 + 2x + 2y - p^2 = 0$
So, equation of the required circle is $S + \lambda S' = 0$.
As it passes through (1, 1), the value of
$$\lambda = -(7 + 2p)/(6 - p^2)$$
Here, λ is not defined at $p = \pm\sqrt{6}$
Hence, it is true for all except two values of p.

64. Given equation can be rewritten as
$$(x + 1)^2 + (y + 2)^2 = (2\sqrt{2})^2$$
Let required point be $Q(\alpha, \beta)$.
Then, mid-point of $P(1, 0)$ and $Q(\alpha, \beta)$ is the centre of the circle.
i.e., $\frac{\alpha + 1}{2} = -1$ and $\frac{\beta + 0}{2} = -2$
$\Rightarrow \alpha = -3$ and $\beta = -4$
So, the required point is $(-3, -4)$.

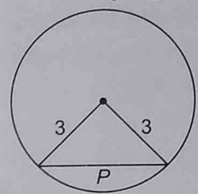
65. Circle with centre (h, k) and touching x-axis is
$$x^2 + y^2 - 2hx - 2ky + h^2 = 0$$
Since, $(-1, 1)$ lies on it.
$\therefore \ 2 + 2h - 2k + h^2 = 0 \Rightarrow k = \frac{(h + 1)^2 + 1}{2} \geq \frac{1}{2}$

66. Let the centroid of the triangle be (h, k). Then,
$h = \frac{0 + 5\cos\theta + 5}{3}$ and $k = \frac{5 + 5\sin\theta + 0}{3}$
$\Rightarrow 3h = 5(1 + \cos\theta)$ and $3k = 5(1 + \sin\theta)$
$\Rightarrow \frac{(3h - 5)}{5} = \cos\theta$ and $\frac{(3k - 5)}{5} = \sin\theta$



$\Rightarrow \frac{9h^2 + 25 - 30h}{25} + \frac{9k^2 + 25 - 30k}{25} = 1$
$\Rightarrow 9h^2 + 9k^2 + 50 - 30h - 30k - 25 = 0$
$\Rightarrow 9(x^2 + y^2) - 30x - 30y + 25 = 0$
Hence, it represents a circle.

67. The locus is the circle with centre (0, 0) and radius
$$3\cos\frac{\pi}{3} = \frac{3}{2}$$



Hence, its equation is $x^2 + y^2 = \frac{9}{4}$.

68. The common chord of the circles is
$$S - S' = 0$$
$\Rightarrow 5ax + (c - d)y + a + 1 = 0$
But it is same as $5x + by - a = 0$
$\therefore \frac{5a}{5} = \frac{c - d}{b} = \frac{a + 1}{-a}$
$\Rightarrow a^2 + a + 1 = 0$
has no real value of a.

69. Centres and radii of given circles are
$C_1(1, 3), r_1 = r, C_2(4, -1)$
and $r_2 = \sqrt{4^2 + 1^2 - 8} = 3$
Since, $r_1 - r_2 < C_1C_2 < r_1 + r_2$
$\Rightarrow r - 3 < \sqrt{(4 - 1)^2 + (-1 - 3)^2} < r + 3$
$\Rightarrow r - 3 < 5 < r + 3$
$\Rightarrow r - 3 < 5$ and $5 < r + 3$
$\Rightarrow r < 8$ and $2 < r$
$\Rightarrow 2 < r < 8$

Day 27

Parabola

Day 27

Outlines ...

Conic Section

A conic is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is a **focus** S. The fixed line is the corresponding **directrix**, l.

The constant ratio is the **eccentricity** denoted by e.

(i) If $0 < e < 1$, then conic is an ellipse.

(ii) If $e = 1$, then conic is a parabola

(iii) If $e > 1$, then conic is a hyperbola.

(iv) If fixed point of a curve is (x_1, y_1) and fixed line is $(ax + by + c)^2$, then equation of conic is

$$(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = e^2 (ax + by + c)^2$$

General Equation of Conic Section

A second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents*

Case I *When the focus lies on the directrix*

(i) Pair of straight lines $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If $e > 1$, then the lines will be real and distinct intersecting at fixed point.

(iii) If $e = 1$, then the lines will coincident passing through a fixed point.

(iv) If $e < 1$, then the lines will be imaginary.

Case II *When the focus does not lie on the directrix*

(i) Circle : $a = b, h = 0, e = 0$ and $\Delta \neq 0$

(ii) Parabola : $h^2 = ab, \Delta \neq 0, e = 1$

(iii) Ellipse : $h^2 < ab, \Delta \neq 0, 0 < e < 1$

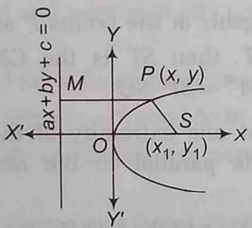
(iv) Hyperbola : $h^2 > ab, 0\ \Delta \neq 0, e > 1$

(v) Rectangular hyperbola : $a + b = 0, \Delta \neq 0, e > 1, h^2 > ab$

Concept of Parabola

Parabola is the locus of a point which moves in a plane such that its distance from a fixed point (focus, S) is equal to its distance from a fixed straight line (directrix, L).

Let $S \equiv (x_1, y_1)$ and $L \equiv ax + by + c = 0$.
Then, equation of parabola is
$(a^2 + b^2)\,[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$.

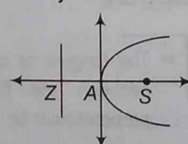
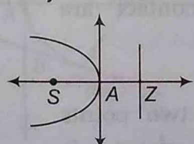
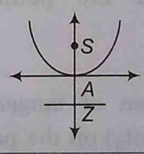
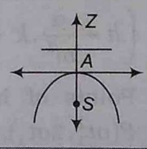


If S lies on L, parabola reduces to a straight line through S and perpendicular to L.

Definitions Related to Parabola

1. **Vertex** The intersection point of parabola and axis.
2. **Centre** The point which bisects every chord of the conic passing through it.
3. **Focal Chord** Any chord passing through the focus.
4. **Double Ordinate** A chord perpendicular to the axis of a conic.
5. **Latusrectum** A double ordinate passing through the focus of the parabola.
6. **Focal Distance** The distance of a point $P(x, y)$ from the focus S is called the focal distance of the point P.

Some Related Terms of Parabolas (In Standard Form)

S. No.	Related Terms	$y^2 = 4ax$ 	$y^2 = -4ax$ 	$x^2 = 4ay$ 	$x^2 = -4ay$ 
1.	Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
2.	Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
3.	Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
4.	Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
5.	Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
6.	Extremities of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
7.	Length of latusrectum	$4a$	$4a$	$4a$	$4a$
8.	Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
9.	Parametric equation	$\begin{cases} x = at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = -at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}$	$\begin{cases} x = 2at \\ y = -at^2 \end{cases}$
10.	Focal distance of any point $P(h, k)$ on the parabola	$h + a$	$h - a$	$k + a$	$k - a$
11.	Equation of latusrectum	$x = a$	$x + a = 0$	$y = a$	$y + a = 0$

Results on Parabola $y^2 = 4ax$

1. The general equation of a parabola whose axis is parallel to X-axis, is $x = ay^2 + by + c$ and parabola whose axis is parallel to Y-axis, is $y = ax^2 + bx + c$.
2. The parametric equations of $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$.
3. Length of latusrectum = 2 (Harmonic mean of focal segment)
4. If y_1, y_2 and y_3 are the ordinates of the vertices of triangle inscribed in the parabola $y^2 = 4ax$, then its area $= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$
5. For the ends of latusrectum of the parabola $y^2 = 4ax$, the values of the perimeter are ± 1.
6. Vertex is equidistant from focus and directrix.

Position of a Point

The position of a point (h, k) with respect to the parabola S lie inside, on or outside the parabola, if $S_1 < 0, S_1 = 0$ or $S_1 > 0$.

Equation of Tangent

A line which intersect the parabola at only one point is called the tangent to the parabola. The different forms of the equation of the parabola for different sections are given below.

(i) In point (x_1, y_1) form, $yy_1 = 2a(x + x_1)$

(ii) In slope m form, $y = mx + \frac{a}{m}$

(iii) In parametric t form, $ty = x + at^2$

(iv) The line $y = mx + c$ touches a parabola iff $c = \frac{a}{m}$ and the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Results on Tangent

1. The equation of a tangent of slope m to the parabola $(y-k)^2 = 4a(x-h)$ is given by $y - k = m(x-h) + \frac{a}{m}$.
   The coordinates of the point of contact are $\left(h - \frac{a}{m^2}, k + \frac{2a}{m}\right)$.
2. Points of intersection of tangents at two points $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{at_1t_2, a(t_1 + t_2)\}$ (where, R is GM of x-coordinates of P,Q and AM of y-coordinates of P,Q).
3. Angle θ between tangents at two points $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1 t_2}\right|$.
4. Locus of the point of intersection of perpendicular tangents to the parabola is directrix.
5. If the tangents at the points P and Q on a parabola meet in T, then ST is the GM between SP and SQ i.e., $ST^2 = SP \cdot SQ$
6. Tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.
7. If the tangent and normal at any point P of the parabola intersect the axis at T and G, then $ST = SG = SP$, where S is the focus.
8. Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.

- *The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.*
- *The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.*
- *The circumcircle formed by the intersection points of tangents at any three points on a parabola passes through the focus of the parabola.*
- *The length of the subtangent at any point on a parabola is equal to twice the abscissa of the point.*
- *Two tangents can be drawn from a point to a parabola. Two tangents are real and distinct or coincident or imaginary according as given point lies outside, on or inside the parabola.*

Equation of Normal

A line which is perpendicular to the tangent of the parabola is called the normal to the parabola.

The different forms of the equation of the parabola for different sections are given below

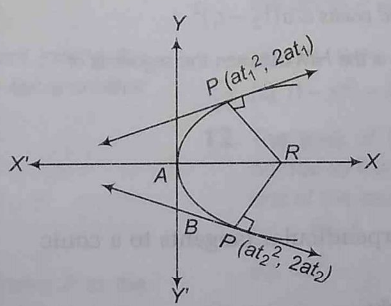
(i) In point (x_1, y_1) form, $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$.

(ii) In slope m form, $y = mx - 2am - am^3$.

(iii) In parametric t form, $y + tx = 2at + at^3$.

Results on Normal

1. Normals at t_1 and t_2 meet at the point $\{2a + a(t_1^2 + t_2^2 + t_1t_2), - at_1t_2(t_1 + t_2)\}$.
2. If the normals at t_1 and t_2 meet at the parabola, then $t_1t_2 = 2$.
3. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersects at a third point R on the curve, then the product of the ordinates of P and Q is $8a^2$.
4. The normal chord at a point t on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, if $t^2 = 2$.
5. Normal at the ends of latusrectum of the parabola $y^2 = 4ax$ meet at right angles on the axis of the parabola.
6. If the tangent and normal at any point P of the parabola intersect the axis at T and G, then $ST = SG = SP$, where S is the focus.
7. Tangents and normals at the extremities of the latusrectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ and $(3a, 0)$.
8. The normal at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.
9. The normal drawn at a point $P(at_1^2, 2at_1)$ to the parabola $y^2 = 4ax$ meets again the parabola at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.



PQ is normal to the curve at P and not at Q.

- *The tangent at one extremity of the focal chord of a parabola is parallel to the normal at other extremity.*
- *The normal chord of a parabola at a point whose ordinate is equal to the abscissa, subtends a right angle at the focus.*
- *Three normals can be drawn from a point to a parabola.*

Equation of a Pair of Tangents

The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the parabola is $SS_1 = T^2$.

where, $S = y^2 - 4ax$

$S_1 = y_1^2 - 4ax$

and $T = yy_1 - 2a(x + x_1)$

Equations of Chord of Contact

1. The equation of chord of contact is
$$yy_1 - 2a(x + x_1) = 0$$
2. The equation of chord of parabola, whose mid-point (x_1, y_1) is $T = S_1$.
i.e., $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
3. Length of the chord of contact is
$$l = \frac{\sqrt{(y_1^2 + 4ax_1)(y_1^2 + 4a^2)}}{a}.$$
4. Area of the ΔPAB formed by the pair of tangents and their chord of contact is
$$A = \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}.$$

- *Equation of the chord joining points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ is $(t_1 + t_2)y = 2x + 2at_1t_2$.*
- *For PQ to be focal chord $t_1 t_2 = -1$.*
- *Length of the focal chord having t_1, t_2 as end points is $a(t_2 - t_1)^2$.*
- *Semi-latusrectum of the parabola $y^2 = 4ax$ is the HM between the segments of any focal chord of the parabola.*

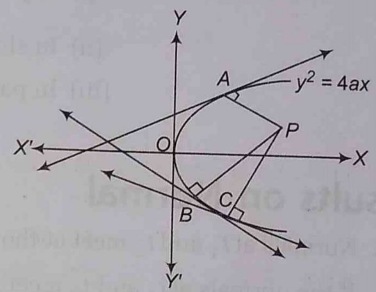
Director Circle

The locus of the point of intersection of perpendicular tangents to a conic is known as **director circle**.

The director circle of a parabola is its directrix.

Conormal Points

The points on the parabola at which the normals pass through a common point are called conormal points. The conormal points are called the feet of the normals.



Points A, B and C are called conormal points.

- The algebraic sum of the slopes of the normals at conormals point is 0.
- The sum of the ordinates of the conormal points is 0.
- The centroid of the triangle formed by the conormal points on a parabola lies on its axis.

Diameter

Diameter is the locus of mid-points of a system of parallel chords of parabola.

- The tangent at the extremities of a focal chord intersect at right angles on the directrix and hence a circle on any focal chord as diameter touches the directrix.
- A circle on any focal radii of point a $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.
- The diameter bisecting chords of slope m to the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

Practice Zone DAY 27

1. If the normals at the end points of variable chord PQ of the parabola $y^2 - 4y - 2x = 0$ are perpendicular, then the tangents at P and Q will intersect on the line
   (a) $x + y = 3$
   (b) $3x - 7 = 0$
   (c) $y + 3 = 0$
   (d) $2x + 5 = 0$

2. If two distinct chords of a parabola $y^2 = 4ax$ passing through the point $(a, 2a)$ are bisected by the line $x + y = 1$, then the length of the latusrectum can be
   (a) 2 (b) 7 (c) 4 (d) 5

3. The parabola $y = x^2 - 8x + 15$ cuts the x-axis at P and Q. A circle is drawn through P and Q, so that the origin is outside it. The length at a tangent to the circle from O is
   (a) 15 (b) $\sqrt{8}$ (c) $\sqrt{15}$ (d) $\dot{8}$

4. The curve described parametrically by $x = t^2 + t + 1$; $y = t^2 - t + 1$ represents
   (a) a pair of line
   (b) an ellipse
   (c) a parabola
   (d) a hyperbola

5. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix
   (a) $x = -a$
   (b) $x = -\dfrac{a}{2}$
   (c) $x = 0$
   (d) $x = \dfrac{a}{2}$

6. If the chord of contact of tangents from a point P to the parabola $y^2 = 12x$ touches the parabola $x^2 = 24y$, then the locus of P is
   (a) straight line
   (b) circle
   (c) parabola
   (d) hyperbola

7. A parabola of latusrectum l touches a fixed equal parabola. The axes of two parabolas are parallel. Then, the locus of the vertex of the moving parabola is
   (a) A parabola whose latusrectum is $2l$
   (b) An ellipse
   (c) A hyperbola
   (d) A circle whose radius is $2l$

8. Set of values of h for which the number of distinct common normals of $(x-2)^2 = 4(y-3)$ and $x^2 + y^2 - 2x - hy - c = 0$ where, $(c > 0)$ is 3, is
   (a) $(2, \infty)$
   (b) $(4, \infty)$
   (c) $(2, 4)$
   (d) $(10, \infty)$

9. The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y = 4x$ above the x-axis is
   (a) $\sqrt{3}y = 3x + 1$
   (b) $\sqrt{3}y = -(x + 3)$
   (c) $\sqrt{3}y = x + 3$
   (d) $\sqrt{3}y = -3(3x + 1)$

10. The locus of trisection point of any double ordinate of the parabola $y^2 = 4ax$ is
    (a) $y^2 = 9ax$
    (b) $y^2 = ax$
    (c) $9y^2 = 4ax$
    (d) None of these

11. At any points P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q the locus of the points P which divides QP externally in the ratio $\dfrac{1}{2} : 1$, is
    (a) $(x + 1)(1 - y)^2 + 4 = 0$
    (b) $x + 1 = 0$
    (c) $(1 - y)^2 - 4 = 0$
    (d) None of these

12. The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$, both drawn at the same end of the latusrectum and the axis of the parabola is
    (a) $2\sqrt{2}\,a^2$
    (b) $2a^2$
    (c) $4a^2$
    (d) None of these

13. Find the lengths of the normals drawn from the point on the axis of the parabola $y^2 = 8x$ whose distance from the focus is 8.
    (a) 10
    (b) 8
    (c) 9
    (d) None of these

14. P is a point on the parabola $y^2 = 4x$ and Q is a point on the line $2x + y + 4 = 0$. If the line $x - y + 1 = 0$ is the perpendicular bisector of PQ, then the coordinates of P is
    (a) $(8, 9), (10, 11)$
    (b) $(1, -2), (9, -6)$
    (c) $(7, 8), (9, 8)$
    (d) None of these

15. If $x+y=k$ is a normal to the parabola $y^2=12x$, p is the length of the perpendicular from the focus of the parabola on this normal, then $3k^3+2p^2$ is equal to

(a) 2223
(b) 2224
(c) 2222
(d) None of these

16. The line $x-b+\lambda y=0$ cuts the parabola $y^2=4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$. If $b\in[2a, 4a]$ and $\lambda\in R$, then t_1t_2 belongs to

(a) $[-4,-2]$
(b) $[4,-3]$
(c) $[-3,-2]$
(d) None of these

17. The parabola $y^2=\lambda x$ and $25[(x-3)^2+(y+2)^2]=(3x-4y-2)^2$ are equal, if λ is equal to

(a) 1 (b) 2 (c) 3 (d) 6

18. The centre of the circle passing through the point (0, 1) and touching the curve $y=x^2$ at (2, 4) is

(a) $\left(\frac{-16}{5}, \frac{27}{10}\right)$
(b) $\left(\frac{-16}{7}, \frac{53}{10}\right)$
(c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$
(d) None of these

19. Consider two concentric circles $C_1: x^2+y^2-1=0$ and $C_2: x^2+y^2-4=0$. A parabola is drawn through the points, where C_1 meets the x-axis and having arbitrary tangent of C_2 as it's directrix. Then, the locus of the focus of drawn parabola is

(a) $\frac{4}{3}x^2-y^2=3$
(b) $\frac{3}{4}x^2-y^2=3$
(c) $\frac{4}{3}x^2+y^2=3$
(d) $\frac{3}{4}x^2+y^2=3$

20. A line is drawn from $A(-2, 0)$ to intersect the curve $y^2=4x$ in P and Q in the first quadrant such the $\frac{1}{AP}+\frac{1}{AQ}<\frac{1}{4}$, then slope of the line is always

(a) $<\sqrt{3}$
(b) $>\sqrt{3}$
(c) $\geq\sqrt{3}$
(d) None of these

21. Vertex A of a parabola $y^2=4ax$ is joined to any point P on it and line PQ is drawn at right angle to AP to meet the axis at Q. Then, the projection of PQ on the axis is always equal to

(a) $3a$
(b) $2a$
(c) $\sqrt{3}a$
(d) $4a$

22. If on a given base triangle be described such that the sum of the tangents of the base angles is constant (k), then the locus of third vertex is

(a) a hyperbola
(b) an ellipse
(c) a circle
(d) a parabola

Directions *(Q. Nos. 23 to 25) $y=f(x)$ is a parabola of the form $y=x^2+ax+1$, its tangent at the point of intersection of Y-axis and parabola also touches the circle $x^2+y^2=r^2$. It is known that no point of the parabola is below X-axis.*

23. The radius of circle when a attains its maximum value is

(a) $\frac{1}{\sqrt{10}}$
(b) $\frac{1}{\sqrt{5}}$
(c) 1
(d) $\sqrt{5}$

24. The slope of the tangent when radius of the circle is maximum

(a) 0
(b) 1
(c) -1
(d) Not defined

25. The minimum area bounded by the tangent and the coordinate axes

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 1

Directions *(Q. Nos. 26 and 27) Consider the equations*
$C: y=x^2-3, D: y=kx^2, L_1: x=a, L_2: x=1$ (where, $a\neq 0$).

26. If the parabolas C and D intersect at a point A on the line L_1, then equation of the tangent line L at A to the parabola D is

(a) $2(a^2-3)x-ay+a^3-3a=0$
(b) $2(a^2-3)x-ay-a^3+3a=0$
(c) $(a^2-3)x-2ay-2a^3+6a=0$
(d) None of the above

27. If $a>0$ then the angle subtended by the chord AB at the vertex of the parabola C is

(a) $\tan^{-1}(5/7)$
(b) $\tan^{-1}(1/2)$
(c) $\tan^{-1}(2)$
(d) $\tan^{-1}(1/8)$

Directions *(Q. Nos. 28 to 31) Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

28. Statement I The perpendicular bisector of the line segment joining the points $(-a, 2at)$ and $(a, 0)$ is tangent to the parabola $y^2=-4ax$, where $t\in R$.

Statement II Number of parabolas with a given point as vertex and length of latusrectum equal to 4 is 2.

29. Consider the equation of the parabola is $y^2 = 4ax$.

Statement I Length of focal chord of a parabola having focus (2, 0) making an angle of 60° with X-axis is 32.

Statement II Length of focal chord of a parabola $y^2 = 4ax$ making an angle α with X-axis is $4a\operatorname{cosec}^2\alpha$.

30. Consider the equation of the parabola is $y^2 = 4ax$.

Statement I Area of triangle formed by pair of tangents drawn from a point (12, 8) to the parabola having focus (1,0) and their corresponding chord of contact is 32 sq units.

Statement II If from a point $P(x_1, y_1)$ tangents are drawn to a parabola then area of triangle formed by these tangents and their corresponding chord of contact is $\dfrac{(y_1^2 - 4ax_1)^{3/2}}{4\,|a|}$ sq units.

31. Statement I The latusrectum of a parabola is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$, then the equation of directrix of the parabola is $4x - 3y + 8 = 0$.

Statement II If P be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then $(PM)^2 =$ (latusrectum) (PN).

32. Consider the two curves

$$C_1 : y^2 = 4x,\ C_2 : x^2 + y^2 - 6x + 1 = 0, \text{ then}$$

(a) C_1 and C_2 touch each other only at one point
(b) C_1 and C_2 touch each other exactly at two points
(c) C_1 and C_2 intersect (but do not touch) at exactly two points
(d) C_1 and C_2 neither intersect nor touch each other

33. The axis of a parabola is along the line $y = x$ and the distance of its vertex from the orgin is $\sqrt{2}$ and that of its focus from the origin is $2\sqrt{2}$. If the vertex and focus lie in the first quadrant, then equation of the parabola is

(a) $(x + y)^2 = x - y - 2$
(b) $(x - y)^2 = x + y - 2$
(c) $(x - y)^2 = 4(x + y - 2)$
(d) $(x - y)^2 = 8(x + y - 2)$

34. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord are

(a) $\{-1, 1\}$
(b) $\{-2, 2\}$
(c) $\left\{-2, \frac{1}{2}\right\}$
(d) $\left\{2, \frac{1}{2}\right\}$

35. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latusrectum. **[NCERT]**

(a) 19 sq units
(b) 18 sq units
(c) 20 sq units
(d) 17 sq units

36. If the area of the triangle inscribed in the parabola $y^2 = 8x$, then the ordinate of whose vertices are 1, 3 and 4 is

(a) $\frac{3}{4}$ (b) $\frac{3}{8}$ (c) $\frac{4}{3}$ (d) $\frac{5}{4}$

37. Find the length of the line segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola, where the line segment makes an angle θ to the X-axis. **[NCERT Exemplar]**

(a) $\dfrac{2a\cos\theta}{\sin^2\theta}$
(b) $\dfrac{4a\cos\theta}{\sin^2\theta}$
(c) $\dfrac{4a\cos\theta}{3\sin^2\theta}$
(d) None of these

38. If the normals at three points P, Q and R of the parabola $y^2 = 36x$ meet at (x, y), then the centroid of the ΔPQR lies on the

(a) latusrectum
(b) tangent at vertex
(c) axis
(d) None of these

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39. Statement I The line $x - 2y = 2$ meets the parabola, $y^2 + 2x = 0$ only at the point $(-2, -2)$.

Statement II The line $y = mx - \dfrac{1}{2m}\ (m \neq 0)$ is tangent to the parabola, $y^2 = -2x$ at the point $\left(-\dfrac{1}{2m^2}, -\dfrac{1}{m}\right)$. **[JEE Main 2013]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

40. The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latusrectum is **[JEE Main 2013]**

(a) (0, 2)
(b) (3, 0)
(c) (0, 3)
(d) (2, 0)

41. Given A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement I An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement II If the line, $y = mx + \frac{\sqrt{5}}{m}$ where, $m \neq 0$ is the common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

[JEE Main 2013]

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statemen II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

42. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is [AIEEE 2010]

(a) $x = 1$ (b) $2x + 1 = 0$
(c) $x = -1$ (d) $2x - 1 = 0$

43. If a parabola has the origin as its focus and the line $x = 2$ as the directrix. Then, the vertex of the parabola is at

[AIEEE 2008]

(a) (2, 0) (b) (0, 2) (c) (1, 0) (d) (0, 1)

44. The equation of the tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent, is

[AIEEE 2007]

(a) (0, 2) (b) (2, 4) (c) (–2, 0) (d) (–1, 1)

45. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is

[AIEEE 2006]

(a) $xy = \frac{35}{36}$ (b) $xy = \frac{64}{105}$
(c) $xy = \frac{105}{64}$ (d) $xy = \frac{3}{4}$

46. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then [AIEEE 2004]

(a) $d^2 + (2b + 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b - 3c)^2 = 0$
(d) $d^2 + (3b + 2c)^2 = 0$

47. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again at the points $(bt_2^2, 2bt_2)$, then [AIEEE 2003]

(a) $t_2 = -t_1 - \frac{2}{t_1}$ (b) $t_2 = -t_1 + \frac{2}{t_1}$
(c) $t_2 = t_1 - \frac{2}{t_1}$ (d) $t_2 = t_1 + \frac{2}{t_1}$

Answers

1. (d)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (c)	**19.** (d)	**20.** (b)
21. (d)	**22.** (d)	**23.** (b)	**24.** (a)	**25.** (a)	**26.** (b)	**27.** (b)	**28.** (c)	**29.** (d)	**30.** (c)
31. (d)	**32.** (b)	**33.** (d)	**34.** (a)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (b)
41. (b)	**42.** (c)	**43.** (c)	**44.** (c)	**45.** (c)	**46.** (a)	**47.** (a)			

Hints & Solutions

1. The tangents and normals form a rectangle.

Hence, tangents meet on the directrix.

Now, $(y - 2)^2 = 2(x + 2)$

Vertex = (–2, 2) and directrix, $x = -\frac{5}{2}$

$\Rightarrow$ $2x + 5 = 0$

2. The chord whose mid-point $(\alpha, 1 - \alpha)$ is $S_1 = S_{11}$.

$\therefore$ $y(1 - \alpha) - 2a(x + \alpha) = (1 - \alpha)^2 - 4a\alpha$

Since, it passes through $(a, 2a)$.

$\therefore$ $(1 - \alpha)^2 = 2a(1 - a) > 0$

$\Rightarrow$ $a \in (0, 1) \Rightarrow 4a \in (0, 4)$

3. Since, the points P and Q are (3, 0) and (5, 0).

Equation of circle is $(x - 3)(x - 5) + y^2 + \lambda y = 0$.

So, the length of tangent from (0, 0) is $\sqrt{S_{11}} = \sqrt{15}$.

4. $\frac{x + y}{2} = t^2 + 1$, $\frac{x - y}{2} = t$

$\therefore$ $\frac{x + y}{2} = \frac{(x - y)^2}{4} + 1$

$\Rightarrow$ $(x - y)^2 = 2(x + y - 2)$,

Hence, it represents a parabola.

5. Let $P(x, y)$ be the mid-point of $(a, 0)$ and $(at^2, 2at)$.

Then, $2x = a(1 + t^2)$ and $y = at$

$\therefore \quad 2x = a\left(1+\frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$

$\Rightarrow \quad y^2 = 2a\left(x - \frac{a}{2}\right)$

$\therefore \quad \text{Vertex} = \left(\frac{a}{2}, 0\right)$

and directrix, $x = \frac{a}{2} - \frac{a}{2} = 0$

6. Let there be a point $P(x_1, y_1)$.

So, the chord of contact is $yy_1 = 6(x + x_1)$

$\Rightarrow \quad x = \frac{yy_1}{6} - x_1 \quad \ldots(\text{i})$

On comparing Eq. (i) by $x = my + c$, we get

$$m = \frac{y_1}{6}, c = -x_1$$

Now, $x = my + c$ touches

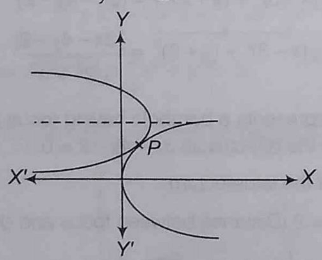
$$x^2 = 24y, \text{ if } c = \frac{6}{m}$$

$\Rightarrow \quad -x_1 = \frac{6}{\frac{y_1}{6}} \Rightarrow -x_1y_1 = 36$

Therefore, the locus of (x_1, y_1) is hyperbola $xy = -36$.

7. Let the fixed parabola is

$$y^2 = 4ax \quad \ldots(\text{i})$$



and moving parabola is

$$(y-k)^2 = -4a(x-h) \quad \ldots(\text{ii})$$

On solving Eqs. (i) and (ii), we get

$$(y-k)^2 = -4a\left(\frac{y^2}{4a} - h\right)$$

$\Rightarrow \quad y^2 - 2ky + k^2 = -y^2 + 4ah$

$\Rightarrow \quad 2y^2 - 2ky + k^2 - 4ah = 0$

Since, the two parabolas touch each other.

So, $\quad D = 0$

$\Rightarrow \quad 4k^2 - 8(k^2 - 4ah) = 0$

$\Rightarrow \quad -4k^2 + 32ah = 0$

$\Rightarrow \quad k^2 = 8ah$

Hence, locus of the vertex of the moving parabola is $y^2 = 8ax$ whose latusrectum $= 8a = 2(4a) = 2l$.

8. The equation of any normal of $(x-2)^2 = 4(y-3)$ is $(x-2) = m(y-3) - 2m - m^3$.

If it passes through $\left(1, \frac{h}{2}\right)$, then

$$1 - 2 = m\left(\frac{h}{2} - 3\right) - 2m - m^3$$

$\Rightarrow \quad 2m^3 + m(10-h) - 2 = 0 = f(m) \quad \text{(say)}$

This equation will given three distinct values of m.

If $f'(m) = 0$ has two distinct roots, where

$$f(m) = 2m^3 + m(10-h) - 2$$

Now, $\quad f'(m) = 6m^2 + (10-h)$

Put $\quad f'(m) = 0 \Rightarrow m = \pm\sqrt{\frac{h-10}{6}}$

So, the values of m are real and distinct, if $h > 10$ *i.e.*, $h \in (10, \infty)$.

9. Equation of tangent to the parabola is $y = mx + \frac{1}{m}$, which touches the circle.

$\therefore \quad \frac{3m + \frac{1}{m}}{\sqrt{m^2+1}} = \pm 3$

$\Rightarrow \quad (3m^2 + 1)^2 = 9m^2(m^2 + 1)$

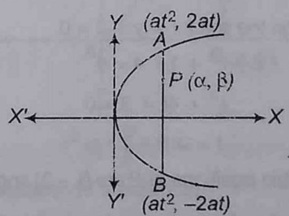
$\Rightarrow \quad 3m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$

$\therefore \quad y = \frac{x}{\sqrt{3}} + \sqrt{3}$

$\Rightarrow \quad \sqrt{3}y = x + 3$

10. Let $P(\alpha, \beta)$ be the trisection point.

$\therefore \quad \alpha = at^2, \quad \beta = \frac{2(2at) + 1(-2at)}{3}$



$\Rightarrow \quad \beta = \frac{2}{3}at \Rightarrow t = \frac{3\beta}{2a}$

$\therefore \quad \alpha = a\left(\frac{3\beta}{2a}\right)^2 \Rightarrow 9\beta^2 = 4a\alpha$

Hence, the locus of P is $9y^2 = 4ax$.

11. Given, $(y-1)^2 = 4(x-1)$. P has coordinates $x = 1 + t^2, y = 1 + 2t$.

Tangent at P is $(x-1) - (y-1)t + t^2 = 0$.

So, the directrix is $x = 0$.

$\therefore \quad Q = \left[0, t + 1 - \frac{1}{t}\right]$

If $R(x, y)$ divides QP externally in the ratio 1 : 2.

$\therefore \quad x = -(1+t^2)$ and $y = 1 - \dfrac{2}{t}$

$\Rightarrow \quad t = \dfrac{2}{1-y}$

$\therefore \quad x + 1 + \dfrac{4}{(1-y)^2} = 0$

$\Rightarrow \quad (x+1)(1-y)^2 + 4 = 0$

12. The coordinate of end of the latusrectum is $(a, 2a)$. The equation of the tangent at $(a, 2a)$ is $y \cdot 2a = 2a(x+a)$, *i.e.*, $y = x + a$. The normal at $(a, 2a)$ is $y + x = 2a + a$, *i.e.*, $y + x = 3a$.

On solving $y = 0$ and $y = x + a$, we get

$$x = -a, y = 0$$

On solving $y = 0$ and $y + x = 3a$, we get

$$x = 3a, y = 0$$

The area of the triangle with vertices $(a, 2a), (-a, 0), (3a, 0)$

$$= \frac{1}{2} \times 4a \times 2a = 4a^2$$

13. Here, $a = 2$ normal at t is $xt + y = 2t^3 + 4t$. Focus $= (2, 0)$.

So, the point on the axis is $(10, 0)$.

normal passes through $(10, 0)$

$\therefore \quad 10 = 2t^2 + 4 \Rightarrow t^2 = 3$

So, the normal is at the point $(6, 4\sqrt{3})$.

So, the required length is

$$\sqrt{(10-6)^2 + (4\sqrt{3})^2} = \sqrt{16+48} = 8$$

14. Any point on the parabola is $P = (t^2, 2t)$.

Q is its image of the line $x - y + 1 = 0$.

$\therefore \quad \dfrac{x - t^2}{1} = \dfrac{y - 2t}{-1} = -(t^2 - 2t + 1)$

$\Rightarrow \quad Q = (2t - 1, t^2 + 1)$

Since, it lies on the line $2x + y + 4 = 0$

$\therefore \quad 4t - 2 + t^2 + 1 + 4 = 0$

$\Rightarrow \quad t^2 + 4t + 3 = 0$

$\Rightarrow \quad t = -1, -3$

So, the possible positions of P are $(1, -2)$ and $(9, -6)$.

15. The equation of normal to the parabola $y^2 = 12x$ with slope -1 is

$$y = -x - 2(3)(-1) - 3(-1)^3$$

$\Rightarrow \quad y = -x + 9$

$\Rightarrow \quad x + y = 9$

$\therefore \quad k = 9$

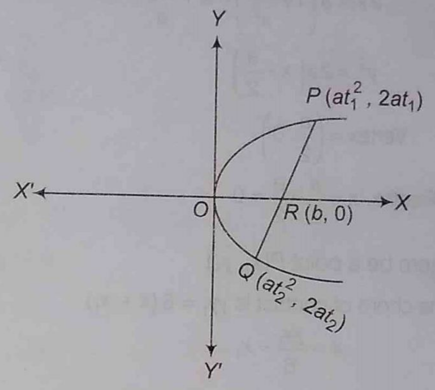
Hence, the focus of the parabola is $(3, 0)$.

$\therefore \quad p = \left|\dfrac{3-9}{\sqrt{2}}\right|$

$\Rightarrow \quad 2p^2 = 36$

$\therefore \quad 3k^3 + 2p^2 = 3(9)^3 + 36 = 2223$

16. Line $x - b + \lambda y = 0$ always passes through $(b, 0)$.



Slope of PR = Slope of RQ

$\Rightarrow \quad t_1 t_2 = -\dfrac{b}{a}$

$\therefore$ Minimum value of $t_1 t_2 = -4$

and maximum value of $t_1 t_2 = -2$

17. Let us recall that two parabolas are equal, if the length of their latusrectum are equal.

Length of the latusrectum of $y^2 = \lambda x$ is λ.

The equation of the second parabola is

$$25\{(x-3)^2 + (y+2)^2\} = (3x - 4y - 2)^2$$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y+2)^2} = \dfrac{|3x - 4y - 2|}{\sqrt{3^2 + 4^2}}$

Clearly, it represents a parabola having focus at $(3, -2)$ and equation of the directrix as $3x - 4y - 2 = 0$.

$\therefore$ Length of the latusrectum

= 2 (Distance between focus and directrix)

$$= 2\left|\frac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}}\right|$$

$$= 6$$

Thus, the two parabolas are equal, if $\lambda = 6$.

18. The slope of the tangent to $y = x^2$ at $(2, 4)$ is 4 and the equation of the tangent is

$$4x - y - 4 = 0$$

Equation of the circle is

$$(x-2)^2 + (y-4)^2 + \lambda(4x - y - 4) = 0$$

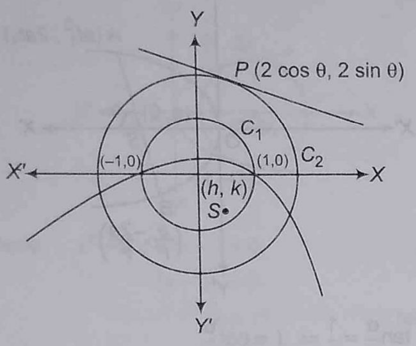
Since, it passes through $(0, 1)$.

Hence, $\quad \lambda = \dfrac{13}{5}$

So, the centre of the circle is $\left(\dfrac{-16}{5}, \dfrac{53}{10}\right)$.

19. Clearly, the parabola should pass through (1, 0) and (−1 0).

Let directrix of this parabola be $x\cos\theta + y\sin\theta = 2$.



If $S(h, k)$ be the focus of this parabola, then distance of $(\pm 1, 0)$ from S and from the directrix should be same.

Then, $(h-1)^2 + k^2 = (\cos\theta - 2)^2$ ...(i)

and $(h+1)^2 + k^2 = (\cos\theta + 2)^2$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$$4h = 8\cos\theta$$

$\Rightarrow$ $\cos\theta = \frac{h}{2}$

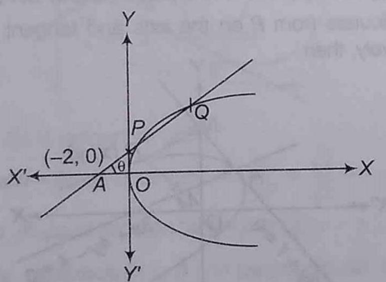
On adding Eqs. (i) and (ii), we get

$$2(h^2 + k^2 + 1) = 2(\cos^2\theta + 4)$$

$\Rightarrow$ $h^2 + k^2 + 1 = 4 + \frac{h^2}{4} \Rightarrow \frac{3}{4}h^2 + k^2 = 3$

Hence, locus of focus is $\frac{3}{4}x^2 + y^2 = 3$.

20. Let $P(-2 + r\cos\theta, r\sin\theta)$ and P lies on parabola.



$\Rightarrow$ $r^2\sin^2\theta - 4(-2 + r\cos\theta) = 0$

$\Rightarrow$ $r_1 + r_2 = \frac{4\cos\theta}{\sin^2\theta} \Rightarrow r_1r_2 = \frac{8}{\sin^2\theta}$

$\therefore$ $\frac{r_1 + r_2}{r_1r_2} = \frac{1}{AP} + \frac{1}{AQ}$

$\Rightarrow$ $\cos\theta < \frac{1}{2}$

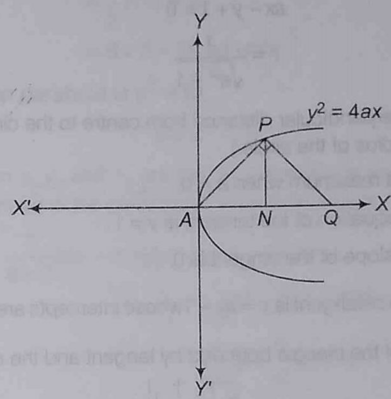
$\Rightarrow$ $\tan\theta > \sqrt{3}$

[because $\cos\theta$ is decreasing and $\tan\theta$ is increasing in $\left(0, \frac{\pi}{2}\right)$]

$\Rightarrow$ $m > \sqrt{3}$

21. Let $P \equiv (at^2, 2at)$

Equation of the line PQ is $y - 2at = -\frac{t}{2}(x - at^2)$.



On putting $y = 0$, we get $x = 4a + at^2$

So, the coordinates of Q and N are $(4a + at^2, 0)$ and $(at^2, 0)$, respectively.

So, length of projection $= 4a + at^2 - at^2 = 4a$

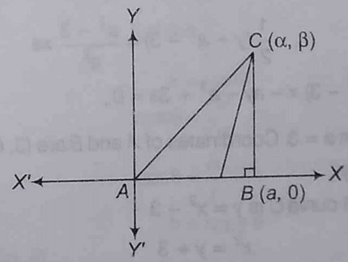
22. Let AB is the given base.

Then, $\tan A + \tan B = k$

$$\tan A = \frac{\beta}{\alpha}, \tan B = \frac{\beta}{\alpha - a}$$

Now, $\frac{\beta}{\alpha} + \frac{\beta}{\alpha - \alpha} = k$

$\Rightarrow$ $-\alpha\beta = k(\alpha^2 - \alpha a)$



$\Rightarrow$ $x^2 = -\frac{a}{k}y + ax$

which is an equation of a parabola.

23. Since, no point of the parabola is below x-axis.

$\therefore$ $a^2 - 4 \le 0$

So, the maximum value of a is 2.

Equation of the parabola when $a = 2$ is

$$y = x^2 + 2x + 1$$

Since, it intersects y-axis at (0, 1).

Equation of the tangent at (0,1) is $y = 2x + 1$.

Since, $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$.

$\therefore$ $r = \frac{1}{\sqrt{5}}$

24. Equation of the tangent at (0, 1) to the parabola

$y = x^2 + ax + 1$ is $\frac{y+1}{2} = \frac{a}{2}(x+0) + 1$

$\Rightarrow \quad ax - y + 1 = 0$

$\therefore \quad r = \frac{1}{\sqrt{a^2+1}}$

(since, perpendicular distance from centre to the circle is equal to the radius of the circle.)

Radius is maximum when $a = 0$.

Hence, equation of the tangent is $y = 1$.

So, the slope of the tangent is 0.

25. Equation of tangent is $y = ax + 1$ whose intercepts are $-\frac{1}{a}$ and 1.

$\therefore$ Area of the triangle bounded by tangent and the axes

$$= \frac{1}{2}\left|-\frac{1}{a}\cdot 1\right|$$

$$= \frac{1}{2|a|}$$

Since, it is minimum when $a = 2$.

$\therefore \quad$ Minimum area $= \frac{1}{4}$

26. C and D intersect at the points for which $x^2 - 3 = kx^2$.

But $x = a$ (given), therefore $k = \frac{a^2-3}{a^2}$.

So, the coordinates of A are $(a, a^2 - 3)$.

Equation of the tangent L at A to $D : y = kx^2$ is

$$\frac{1}{2}(y + a^2 - 3) = \frac{a^2-3}{a^2}xa$$

$\Rightarrow \quad 2(a^2 - 3)x - ay - a^3 + 3a = 0$

27. If $a > 0$, then $a = 3$. Coordinates of A and B are (3, 6) and $(1, -2)$, respectively.

Equation of curve C is $y = x^2 - 3$

or $\quad x^2 = y + 3$

So, coordinates of the vertex O of the parabola C are $(0, -3)$.

Slope of $OA = 3$, slope of $OB = 1$

$\therefore$ Angle between OA and OB is

$$\tan^{-1}\left(\frac{3-1}{1+3}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

28. Image of $(a, 0)$ with respect to tangent $yt = x + at^2$ is $(-a, 2at)$.

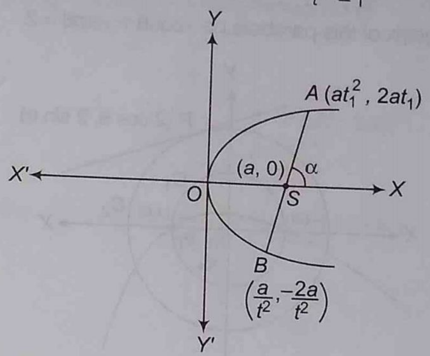
So, perpendicular bisector of $(a, 0)$ and $(-a, 2at)$ is the tangent line $yt = x + at^2$ to the parabola.

Hence, statement I is true.

Statement II Infinitely many parabolas are possible.

Hence, Statement II is false.

29. Let AB be a focal chord slope of $AB = \frac{2t}{t^2-1} = \tan\alpha$



$\Rightarrow \quad \tan\frac{\alpha}{2} = \frac{1}{t} \Rightarrow t = \cot\frac{\alpha}{2}$

Length of $AB = a\left(t + \frac{1}{t}\right)^2 = 4a\text{cosec}^2\alpha$

When $a = 2, \alpha = 60°$

$\therefore$ Length of $AB = 4(2)\text{cosec}^2(60°) = \frac{32}{3}$

30. **Statement II** Area of triangle formed by these tangents and their corresponding chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2|a|}$.

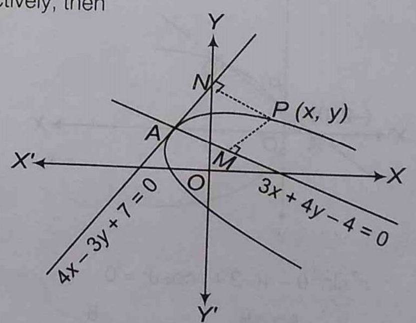
Hence, Statement II is false.

Statement I $x_1 = 12, y_1 = 8$

$\therefore \quad \text{Area} = \frac{(y_1^2 - 4ax_1)^{3/2}}{2} = \frac{(64-48)^{3/2}}{2} = 32$

Hence, Statement I is true.

31. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively, then



$$(PM)^2 = (\text{latusrectum})(PN)$$

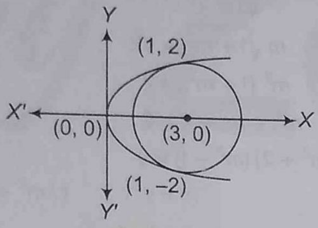
$\Rightarrow \quad \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x+3y+7}{\sqrt{4^2+(-3)^2}}\right) \Rightarrow Y^2 = 4AX$

$\therefore \quad A = 1, Y = \frac{3x+4y-4}{5}, X = \frac{4x-3y+7}{5}$

So, the directrix is $X + A = 0$.

$\Rightarrow \quad \frac{4x-3y+7}{5} + 1 = 0 \Rightarrow 4x - 3y + 12 = 0$

32. For the points of intersection of the two given curves,
$C_1 : y^2 = 4x$ and $C_2 : x^2 + y^2 - 6x + 1 = 0$



We have, $x^2 + 4x - 6x + 1 = 0$

$\Rightarrow \quad (x-1)^2 = 0$

$\Rightarrow \quad x = 1, 1$ (equal real roots)

$\Rightarrow \quad y = 2, -2$

Thus, the given curves touch each other at exactly two points (1, 2) and (1, −2).

33. Vertex = (1, 1), Focus = (2, 2). So, the directrix is $x + y = 0$.

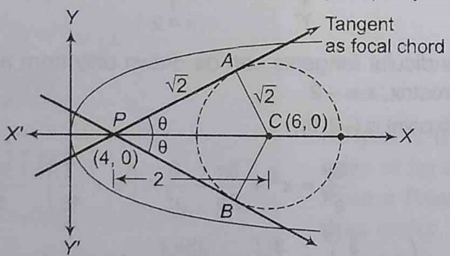
Then, equation is $\frac{x+y}{\sqrt{2}} = \sqrt{(x-2)^2 + (y-2)^2}$.

$\Rightarrow \quad (x-y)^2 = 8(x + y - 2)$

34. Here, the focal chord to $y^2 = 16x$ is tangent to circle $(x-6)^2 + y^2 = 2$.

So, the focus of parabola is $(a, 0)$ *i.e.*, (4, 0).

Now, tangents are drawn from (4, 0) to $(x-6)^2 + y^2 = 2$



Since, PA is tangent to circle.

$\therefore \quad \tan\theta = \text{Slope of tangent} \Rightarrow \frac{AC}{AP} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \Rightarrow \frac{BC}{BP} = -1$

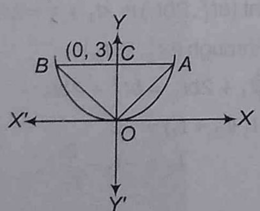
Hence, slope of focal as tangent to circle = ± 1.

35. Clearly, latusrectum is a line perpendicular to the axes and passing through focus whose length is $4a$.

Given, $x^2 = 12y$, which is of the form $x^2 = 4ay$.

$\Rightarrow \quad 4a = 12 = \text{Length of } ABC$

$\therefore \quad$ Focus, $C = (0, 3)$



Area of $\Delta OAB = \frac{1}{2} \times AB \times OC$

$= \frac{1}{2} \times 12 \times 3$

$= 6 \times 3 = 18$ sq units

36. Given parabola is $y^2 = 8x$.

$\Rightarrow \quad a = 2$

When y_1, y_2 and y_3 are the ordinates of the vertices of triangle inscribed in the parabola $y^2 = 8x$, then area is

$$\frac{1}{8 \times 2}[(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)]$$

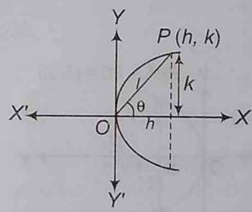
$$= \frac{1}{16}[(1-3)(3-4)(4-1)]$$

$$= \frac{1}{16}(6) = \frac{3}{8} \text{ sq unit}$$

37. Let any point (h, k) will satisfy

$y^2 = 4ax$ *i.e.*, $k^2 = 4ah$ ...(i)

Let a line OP makes an angle θ from the the x-axis.



In ΔOPA, $\quad \sin\theta = \frac{PA}{OP}$

$\Rightarrow \quad \sin\theta = \frac{k}{l}$

$\Rightarrow \quad k = l\sin\theta$

and $\quad \cos\theta = \frac{OA}{OP}$

$\Rightarrow \quad \cos\theta = \frac{h}{l}$

$\Rightarrow \quad h = l\cos\theta$

From Eq. (i),

$l^2\sin^2\theta = 4a \times l\cos\theta$

(put $k = l\sin\theta$, $h = l\cos\theta$)

$\Rightarrow \quad l = \frac{4a\cos\theta}{\sin^2\theta}$

38. Any normal to the parabola $y^2 = 36x$ is

$y = mx - 18m - 9m^3$

$\Rightarrow \quad 9m^3 + (18 - x)m + y = 0$

Here, $\quad m_1 + m_2 + m_3 = 0$

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{18-x}{9}$

$m_1m_2m_3 = -\frac{y}{9}$

So, the feet of normals are $(9m_1^2, -18m_1), (9m_2^2, -18m_2)$ and $(9m_3^2, -18m_3)$ and if $(\bar{x}, \bar{y})$ be the centroid of the triangle, then

$$\bar{x} = \frac{9(m_1^2 + m_2^2 + m_3^2)}{3}$$

$$\bar{y} = -\frac{18(m_1 + m_2 + m_3)}{3}$$

$$\Rightarrow \quad \bar{x} = \frac{2(18 - x)}{9}, \ \bar{y} = 0$$

So, $y = 0$ is the axis of the parabola.

39. **Statement I** Intersection point of $x - 2y = 2$ and $y^2 + 2x = 0$ is $(-2, -2)$.

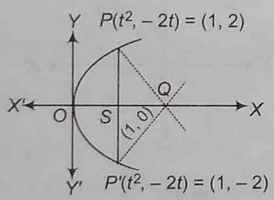
Statement II Here, $a = -\frac{1}{2}$

Equation of tangent $y = mx + \frac{a}{m}$

$$y = mx - \frac{1}{2m}$$

And point of contact is $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$.

40. Given, $y^2 = 4x$



$$\frac{dy}{dx} = \frac{2}{y} \quad \text{...(i)}$$

$\because S$ is the mid-point of PP'.

$$\therefore \quad t^2 = 1 \Rightarrow t = \pm 1$$

Equation of normal PQ is $(y - 2) = \left(\frac{-2}{2}\right)(x - 1)$

$$\Rightarrow \quad x + y = 3 \quad \text{...(ii)}$$

Equation of normal $P'Q\,(y + 2) = 1(x - 1) \Rightarrow x - y = 3$ ...(iii)

On solving Eqs. (ii) and (iii), we get

$$2x = 6 \Rightarrow x = 3$$

and $\quad y = 0$

So, the required intersection point is $(3, 0)$.

41. Equation of circle can be rewritten as

$$x^2 + y^2 = \frac{5}{2}$$

Let common tangent be

$$y = mx + \frac{\sqrt{5}}{m}$$

So, the perpendicular from centre to the tangent is equal to radius.

$$\therefore \quad \frac{\frac{\sqrt{5}}{m}}{\sqrt{1 + m^2}} = \sqrt{\frac{5}{2}}$$

$$\Rightarrow \quad m\sqrt{1 + m^2} = \sqrt{2}$$

$$\Rightarrow \quad m^2(1 + m^2) = 2$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad m = \pm 1 \quad (\because m^2 + 2 \neq 0, \text{ as } m \in R)$$

$$\therefore \quad y = \pm (x\sqrt{5}),$$

Both statements are correct as

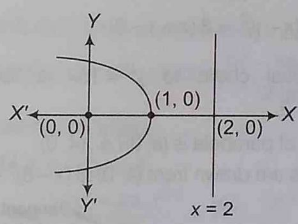
$$m = \pm 1$$

satisfies the given equation of Statement II.

42. We know that, the locus of point P from which two perpendicular tangents are drawn to the parabola, is the directrix of the parabola.

Hence, the required locus is $x = -1$.

43. From the figure, it is clear that vertex of the parabola at $(1, 0)$.



44. Perpendicular tangents can be drawn only from a point on the directrix, $x = -2$.

So, the point is $(-2, 0)$.

45. $$\frac{3y}{a^3} = x^2 + \frac{3x}{2a} - \frac{6}{a^2} = \left(x + \frac{3}{4a}\right)^2 - \frac{105}{16a^2}$$

$$\Rightarrow \quad \left(x + \frac{3}{4a}\right)^2 = \frac{3}{a^3}\left(y + \frac{35a}{16}\right)$$

If (α, β) is the vertex, then $\alpha = -\frac{3}{4a}, \beta = -\frac{35a}{16}$

Hence, the locus of (α, β) is $xy = \frac{105}{64}$

46. The common chord is the line $x - y = 0$.

But given line is $2bx + 3cy + 4d = 0$

$$\therefore \quad d = 0, 2b + 3c = 0.$$

47. Normal at a point $(bt_1^2, 2bt_1)$ is $xt_1 + y = bt_1^3 + 2bt_1$.

Since, it passes through $(bt_2^2, 2bt_2)$.

$$\therefore \quad bt_2^2 t_1 + 2bt_2 = bt_1^3 + 2bt_1$$

$$\Rightarrow \quad t_1(t_2 + t_1) = -2$$

$$\Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1}$$

Day 28

Ellipse

Day 28

Outlines ...

Concept of Ellipse

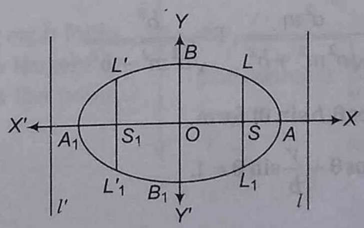
Ellipse is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (focus) in the same plane to its distance from a fixed straight line (directrix) is always constant, which is always less than unity.

Equations of Ellipse in Standard Form

Different forms of an ellipse and their equations have been given below

1. Ellipse of the Form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a$

If the coefficient of x^2 has the larger denominator, then its major axis lies along the X-axis, then it is said to be horizontal ellipse as shown below.



(i) Centre : $O(0,0)$

(ii) Coordinates of the vertices : $A(a, 0)$ and $A_1(-a, 0)$

(iii) Equation of the major-axis, $y = 0$

(iv) Equation of the minor-axis, $x = 0$

(v) Focal distance of a point (x, y) is $a \pm ex$.

(vi) Major-axis, $AA_1 = 2a$, Minor-axis, $BB_1 = 2b$

(vii) Foci are $S(ae, 0)$ and $S_1(-ae, 0)$.

(viii) Equations of directrices are $l : x = \frac{a}{e}, l' : x = -\frac{a}{e}$

(ix) Length of latusrectum : $LL_1 = L'L_1' = \frac{2b^2}{a}$

(x) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

(xi) Sum of focal distances of a point (x, y) is $2a$.

2. Ellipse of the Form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a$

If the coefficient of x^2 has the smaller denominator, then its major axis lies along the Y-axis, then it is said to be vertical ellipse as shown below.

(i) Centre : $O(0,0)$

(ii) Coordinates of the vertices : $B(0, b)$ and $B_1(0, -b)$

(iii) Equation of the major-axis, $x = 0$

(iv) Equation of the minor-axis, $y = 0$.

(v) Focal distance of a point (x, y) is $b \pm ey$.

(vi) Major-axis, $BB_1 = 2b$, Minor-axis, $AA_1 = 2a$

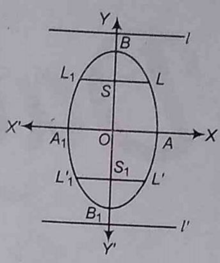
(vii) Foci are $S(0, ae)$ and $S_1(0, -ae)$.

(viii) Equation of directrices are $l : y = \frac{b}{e}, l' : y = -\frac{b}{e}$.

(ix) Length of latusrectum : $LL_1 = L'L_1' = \frac{2a^2}{b}$

(x) Eccentricity : $e = \sqrt{1 - \frac{a^2}{b^2}}$

(xi) Sum of focal distance of a point (x, y) is $2b$.



- *The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.*
- *The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.*

Results on Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b\right)$

- The parametric equation of an ellipse be $x = a\cos\theta$, $y = b\sin\theta$ and hence the coordinates $(a\cos\theta, b\sin\theta)$, where θ is the parameter.
- The location of a point (h, k) with respect to ellipse 'S' lie inside, on or outside the ellipse, if

$$S_1 < 0, S_1 = 0 \quad \text{or} \quad S_1 > 0$$

where, $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

- Locus of mid-points of focal chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a^2}$.

Tangent to an Ellipse

The equation of tangent to an ellipse for different forms at the particular coordinate system are given below.

1. In point (x_1, y_1) form, $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

2. In slope 'm' form, $y = mx \pm \sqrt{a^2m^2 + b^2}$.

   at the points $\pm \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 + b^2}}$

3. In parametric $(a\cos\theta, b\sin\theta)$ form,

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

Results on Tangent to an Ellipse

1. The line $y = mx + c$ touches an ellipse, iff $c^2 = a^2m^2 + b^2$ and the point of contact is $\left(\pm\frac{a^2m}{c}, \mp\frac{b^2}{c}\right)$.

2. The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the ellipse is $SS_1 = T^2$,

   where $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

   and $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

3. The equation of chord of contact of tangents is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$.

4. The equation of chord of an ellipse, whose mid-point is (x_1, y_1), is $T=S_1$ i.e., $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

5. The chord joining the points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ is $\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$

6. If the chord passes through the focus, then $\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2}=\frac{e+1}{e-1}$

7. The locus of the point of intersection of perpendicular tangents to the ellipse is a director circle i.e., $x^2+y^2=a^2+b^2$.

Normal to an Ellipse

The different equation of normal to an ellipse at the particular coordinate system are given below

(i) In point (x_1, y_1) form, $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

(ii) In slope 'm' form, $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$ at the points $\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{b^2m}{\sqrt{a^2+b^2m^2}}\right)$.

(iii) In parametric $(a\cos\theta, b\sin\theta)$ form, $ax\sec\theta-by\,\text{cosec}\,\theta=a^2-b^2$.

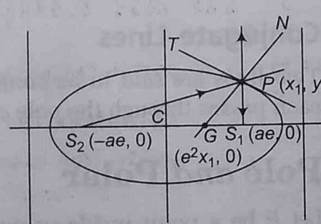
(iv) The point of intersection of normals to the ellipse at two points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ are (λ,μ), where

$$\lambda=\frac{(a^2-b^2)}{2}\cdot\cos\theta_1\cdot\cos\theta_2\cdot\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}$$

and $$\mu=-\frac{(a^2-b^2)}{2}\cdot\sin\theta_1\cdot\sin\theta_2\cdot\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}$$

Reflection Property

The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus and *vice-versa*. Hence, we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it, where G is the point, where normal at P meets the major-axis.



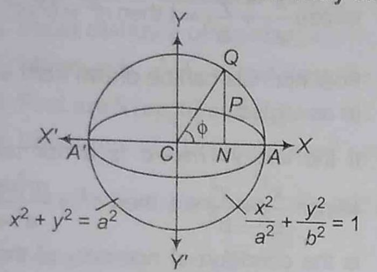
$$\frac{S_2G}{GS_1}=\frac{e^2x_1+ae}{ae-e^2x_1}=\frac{a+ex_1}{a-ex_1} \text{ or } \frac{PS_2}{PS_1}=\frac{e\left(\frac{a}{e}+x_1\right)}{e\left(\frac{a}{e}-x_1\right)}=\frac{a+ex_1}{a-ex_1}$$

Results on Normal to an Ellipse

- The line $lx+my+n=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $n^2=b^2m^2+a^2l^2$
- Four normals can be drawn from a point P to an ellipse.
- If the line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=\frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$ is the condition of normality of the line to the ellipse.
- If the normal at any point P on the ellipse with centre C meet the major and minor-axes in G and g respectively and if CF be perpendicular upon this normal, then
  - (a) $PF\cdot PG=b^2$
  - (b) $PF\cdot Pg=a^2$
  - (c) $PG\cdot Pg=SP\cdot S'P$
  - (d) $CG\cdot CT=(CS)^2$
  - (e) Locus of the mid-point of Gg is another ellipse having the same eccentricity as that of the original ellipse.
- The points on the ellipse, the normals at which to the ellipse pass through a given point are called **conormal** points.
- If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$
- Tangent at an end of a latusrectum (Ist quadrant) is $\frac{ex}{a}+\frac{\sqrt{1-e^2}}{b}y=1$ or $ex+y=a$ and normal is $\frac{ax}{e}-\frac{by}{\sqrt{1-e^2}}=a^2-b^2$ or $x-ye=ae^3$

Auxiliary Circle and Eccentric Angle

The circle described on the major-axis of an ellipse as diameter is called an **auxiliary circle**.



Here, the equation of auxiliary circle is $x^2 + y^2 = a^2$

So, the coordinates of Q and P are $(a\cos\phi, a\sin\phi)$ and $(a\cos\phi, b\sin\phi)$, where ϕ is an eccentric angle.

Results on Eccentric Angles

- Eccentric angles of the extremities of latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.
- A circle cut an ellipse in four points real or imaginary. The sum of the eccentric angles of these four concyclic points on the ellipse is an even multiple of π.
- The sum of the eccentric angles of the conormal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.
- If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the ellipse the normals at which are concurrent, then
  (a) $\Sigma\cos(\theta_1+\theta_2)=0$
  (b) $\Sigma\sin(\theta_1+\theta_2)=0$
  (c) $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$
- If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$, then the normals at these points are concurrent.

Diameter and Conjugate Diameters

The locus of the middle points of a system of parallel chords is called a **diameter.** If $y = mx + c$ represents a system of parallel chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the line $y=-\frac{b^2}{a^2m}x$ is the equation of the diameter.

The two diameters are said to be **conjugate diameters,** when each bisects all chords parallel to the other.

If $y = mx$ and $y = m_1x$ be two conjugate diameters of an ellipse, then $mm_1=-\frac{b^2}{a^2}$

Results on Conjugate Diameters

1. The area of a parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.
2. The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse *i.e.*, $CP^2 + CD^2 = a^2 + b^2$.
3. The product of the focal distance of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

- *The eccentric angles of the ends of a pair of conjugate diameter of an ellipse differ by a right angle.*
- *The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.*

Conjugate Points

Two points P and Q are **conjugate points** with respect to an ellipse, if the polar of P passes through Q and the polar of Q passes through P.

Conjugate Lines

Two lines are said to be **conjugate lines** with respect to an ellipse, if each passes through the pole of the polar.

Pole and Polar

Let P be a point inside or outside an ellipse. Then, the locus of the point of intersection of tangents to the ellipse at the point, where secants drawn through 'P' intersect the ellipse is called the **polar** of point P with respect to the ellipse and the point P is called the **pole** of the polar.

The polar of a point (x_1, y_1) with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

Practice Zone DAY 28

1. If equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{25}=1$, then coordinate of the foci, eccentricity and the length of the latusrectum are respectively **[NCERT]**
(a) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{5}, \frac{7}{5}$ (b) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{5}, \frac{8}{5}$
(c) $(0, \pm\sqrt{21}), \frac{\sqrt{21}}{7}, \frac{8}{5}$ (d) None of these

2. If vertices and foci of an ellipse are $(0, \pm 13)$ and $(0, \pm 5)$ respectively, then the equation of an ellipse is **[NCERT]**
(a) $\frac{x^2}{144}+\frac{y^2}{169}=1$ (b) $\frac{x^2}{169}+\frac{y^2}{144}=1$
(c) $\frac{x^2}{12}+\frac{y^2}{13}=1$ (d) None of these

3. Find the equation of an ellipse, if major axis on the x-axis and passes through the points (4, 3) and (6, 2). **[NCERT]**
(a) $\frac{x^2}{13}+\frac{y^2}{52}=1$ (b) $\frac{x^2}{40}+\frac{y^2}{10}=1$
(c) $\frac{x^2}{52}+\frac{y^2}{13}=1$ (d) None of these

4. If CP and CD are semi-conjugate diameters of an ellipse $\frac{x^2}{14}+\frac{y^2}{8}=1$, then CP^2+CD^2 is equal to
(a) 20 (b) 22 (c) 24 (d) 26

5. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{(5)^2}+\frac{y^2}{(6)^2}=1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to
(a) $-\left(\frac{5}{6}\right)^4$ (b) $\left(\frac{6}{5}\right)^4$ (c) $-\left(\frac{6}{5}\right)^4$ (d) $\left(\frac{5}{6}\right)^4$

6. If θ and ϕ are eccentric angles of the end of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\theta-\phi$ is equal to
(a) $\pm\frac{\pi}{2}$ (b) $\pm\pi$ (c) 0 (d) None of these

7. If tangent at any point P on the ellipse $7x^2+16y^2=12$ cuts the tangent at the end points of the major-axis at the points A and B, then the circle with AB as diameter passes through a fixed point whose coordinates are
(a) $(\pm\sqrt{a^2-b^2}, 0)$ (b) $(\pm\sqrt{a^2+b^2}, 0)$
(c) $(0, \pm\sqrt{a^2-b^2})$ (d) $(0, \pm\sqrt{a^2+b^2})$

8. The length of the common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ intercepted by the coordinate axes is
(a) 5 (b) $2\sqrt{7}$
(c) $\frac{7}{\sqrt{3}}$ (d) $\frac{14}{\sqrt{3}}$

9. The minimum area of triangle formed by the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes is
(a) ab (b) $\frac{a^2+b^2}{2}$
(c) $\frac{(a+b)^2}{2}$ (d) $\frac{a^2+ab+b^2}{3}$

10. PQ is a chord of the ellipse through the centre. If the square of its length is the HM of the squares of major and minor-axes, find its inclination with X-axis.
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{2\pi}{3}$ (d) None of these

11. The distance of the centre of ellipse $x^2+2y^2-2=0$ to those tangents of the ellipse which are equally inclined from both the axes, is
(a) $\frac{3}{\sqrt{2}}$ (b) $\sqrt{3/2}$
(c) $\sqrt{2}/3$ (d) $\frac{\sqrt{3}}{2}$

12. Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. If P and Q be the points $(1, 2)$ and $(2, 1)$, respectively. Then,

(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) None of the above

13. Image of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ in the line $x+y=10$ is

(a) $\frac{(x-10)^2}{16}+\frac{(y-10)^2}{25}=1$
(b) $\frac{(x-10)^2}{25}+\frac{(y-10)^2}{16}=1$
(c) $\frac{(x-5)^2}{16}+\frac{(y-5)^2}{25}=1$
(d) $\frac{(x-5)^2}{25}+\frac{(y-5)^2}{16}=1$

14. The eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ whose distance from the centre of the ellipse is 2, is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{3}$ (d) $\frac{7\pi}{6}$

15. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) 2/3 (b) $-$ 2/3
(c) 1/3 (d) $-$ 1/3

16. If any tangent to the ellipse is cut by the tangents at the end points of the major-axis in T and T', then the circle whose diameter is TT' will pass through the

(a) centre of ellipse
(b) foci of the ellipse
(c) other end point of minor-axis
(d) None of the above

17. If the focal distance of an end of the minor-axis of any ellipse (its axes as x and y-axes respectively) is k and the distance between the foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$ (b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$
(c) $\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}=1$ (d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

18. From any point P on the ellipse PN is drawn perpendicular to the major-axis and produced at Q. So that, NQ equals to PS, where S is a focus. Then, the locus of Q is the

(a) $y-ex-a=0$ (b) $y\pm ex+a=0$
(c) $y+ex-a=0$ (d) None of these

19. The coordinates of all the points P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for which the area of the ΔPON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P, is

(a) $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$
(b) $\left(\pm\frac{a^2}{\sqrt{a^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2-b^2}}\right)$
(c) $\left(\pm\frac{2a^2}{\sqrt{a^2+b^2}}, \pm\frac{2b^2}{\sqrt{a^2+b^2}}\right)$
(d) $\left(\frac{2a^2}{\sqrt{a^2-b^2}}, \frac{2b^2}{\sqrt{a^2-b^2}}\right)$

20. If two points are taken on the minor-axis of an ellipse at the same distance from the centre as the foci, then the sum of the squares of the perpendicular distances from these points on any tangent to the ellipse is

(a) $2a^2$ (b) $3a^2$
(c) $4a^2$ (d) None of these

21. In an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on

(a) tangent at vertex (b) corresponding directrix
(c) tangent at $(0, b)$ (d) None of these

22. A parabola is drawn whose focus is one of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where, $a>b$) and whose directrix passes through the other focus and perpendicular to the major-axis of the ellipse. Then, the eccentricity of the ellipse for which the latusrectum of the ellipse and the parabola are same, is

(a) $\sqrt{2}-1$ (b) $2\sqrt{2}+1$
(c) $\sqrt{2}+1$ (d) $2\sqrt{2}-1$

23. At a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tangent PQ is drawn. If the point Q be at a distance $\frac{1}{p}$ from the point P, where p is distance of the tangent from the origin, then the locus of the point Q is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{1}{a^2b^2}$
(b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1-\frac{1}{a^2b^2}$
(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{a^2b^2}$
(d) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{1}{a^2b^2}$

24. Given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ with foci at S and S' and vertices at A and A'. A tangent is drawn at any point P on the ellipse and let R, R', B, B' respectively be the feet of the perpendiculars drawn from S, S', A, A' on the tangent at P. Then, the ratio of the areas of the quadrilaterals $S'R'RS$ and $A'B'BA$ is

(a) $e:2$ (b) $e:3$ (c) $e:1$ (d) $e:4$

25. A triangle is drawn such that it is right angled at the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where, $a>b$) and its other two vertices lie on the ellipse with eccentric angles α and β. Then, $\dfrac{1-e^2\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$ is equal to

(a) $\frac{a^2}{a^2+b^2}$ (b) $\frac{a^2+b^2}{a^2}$ (c) $\frac{a^2}{a^2-b^2}$ (d) $\frac{a^2-b^2}{a^2}$

26. If the tangent at a point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points, the chord joining them subtends a right angle at the centre, then the eccentricity of the ellipse is given by

(a) $(1+\cos^2\theta)^{-1/2}$ (b) $(1+\sin^2\theta)$
(c) $(1+\sin^2\theta)^{-1/2}$ (d) $(1+\cos^2\theta)$

27. Tangent is drawn to the ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where, $\theta\in(0,\pi/2)$). Then, the value of θ such that the sum of intercepts on axes made by this tangent is minimum, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

28. The equation of normal to the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the end of the latusrectum is

(a) $x-ey-2e^3=0$ (b) $x+ey+2e^3=0$
(c) $x+2ey+e^3=0$ (d) $x-2ey+e^3=0$

Directions (Q. Nos. 29 and 30) *Consider the standard equation of an ellipse whose focus and corresponding foot of directrix are* $(\sqrt{7},0)$ *and* $\left(\frac{16}{\sqrt{7}},0\right)$ *and a circle with equation* $x^2+y^2=r^2$. *If in the first quadrant, the common tangent to a circle of this family and the above ellipse meets the coordinate axes at A and B.*

29. The equation of the ellipse is

(a) $16x^2+9y^2=144$ (b) $9x^2+16y^2=144$
(c) $16x^2+y^2=144$ (d) $x^2+9y^2=144$

30. If mid-point of A and B is (x_1, y_1) and slope of common tangent be m, then

(a) $2mx_1+y_1=0$ (b) $2my_1+x_1=0$
(c) $my_1+x_1=0$ (d) $mx_1+y_1=0$

Directions (Q. Nos. 31 and 32)

$$C: x^2+y^2=9, E: \frac{x^2}{9}+\frac{y^2}{4}=1, L: y=2x$$

31. P is a point on the circle C, the perpendicular PQ to the major-axis of the ellipse E meets the ellipse at M, then $\frac{MQ}{PQ}$ is equal to

(a) 1/3 (b) 2/3
(c) 1/2 (d) None of these

32. If L represents the line joining the points P on C to its centre O, then equation of the tangent at M to the ellipse E is

(a) $x+3y=3\sqrt{5}$ (b) $4x+3y=\sqrt{5}$
(c) $x+3y+8\sqrt{5}=0$ (d) $4x+3y+\sqrt{5}=0$

Directions (Q. Nos. 33 to 35) *An ellipse E has its centre C(1, 3) focus at S(6, 4) and passing through the point P (4, 7).*

33. The product of the lengths of the perpendicular segments from the foci on tangent at point P is

(a) 20 (b) 45
(c) 40 (d) Cannot be determined

34. The point of intersection of the lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at point P, is

(a) $\left(\frac{5}{3},5\right)$ (b) $\left(\frac{4}{3},3\right)$ (c) $\left(\frac{8}{3},3\right)$ (d) $\left(\frac{10}{3},5\right)$

35. If the normal at a variable point on the ellipse (E) meets its axes in Q and R, then the locus of the mid-point of QR is a conic with an eccentricity (e'), then

(a) $e'=\frac{3}{\sqrt{10}}$ (b) $e'=\frac{\sqrt{5}}{3}$
(c) $e'=\frac{3}{\sqrt{5}}$ (d) $e'=\frac{\sqrt{10}}{3}$

Directions (Q. Nos. 36 to 39) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

36. Let $S_1 \equiv (x-1)^2 + (y-2)^2 = 0$ and $S_2 \equiv (x+2)^2 + (y-1)^2 = 0$ be the equations of two circles.

Statement I Locus of centre of a variable circle touching two circles S_1 and S_2 is an ellipse.

Statement II If a circle $S_1 = 0$ lies completely inside the circle $S_2 = 0$, then locus of centre of variable circle $S = 0$ which touches both the circles is an ellipse.

37. Statement I If $P\left(\frac{3\sqrt{3}}{2}, 1\right)$ is a point on the ellipse $4x^2 + 9y^2 = 36$. Circle drawn AP as diameter touches another circle $x^2 + y^2 = 9$, where $A \equiv (-\sqrt{5}, 0)$.

Statement II Circle drawn with focal radius as diameter touches the auxiliary circle.

38. Statement I The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

Statement II If $y = mx$ and $y = m_1x$ be two conjugate diameters of an ellipse, then $mm_1 = \frac{b^2}{a^2}$.

39. Statement I The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by the line $x + y = b$ is $a^2 + 6ab - 7b^2 \geq 0$.

Statement II Equation of chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid-point is (x_1, y_1), is $T = S_1$.

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40. If a and c are positive real numbers and the ellipse $\frac{x^2}{4c^2} + \frac{y^2}{c^2} = 1$ has four distinct points in common with the circle $x^2 + y^2 = 9a^2$, then **[JEE Main 2013]**

(a) $9ac - 9a^2 - 2c^2 < 0$ (b) $6ac - 9a^2 - 2c^2 < 0$

(c) $9ac - 9a^2 - 2c^2 > 0$ (d) $6ac + 9a^2 - 2c^2 > 0$

41. Let the equations of two ellipses be

$E_1 : \frac{x^2}{3} + \frac{y^2}{2} = 1$ and

$E_2 : \frac{x^2}{16} + \frac{y^2}{b^2} = 1$. If the product of their eccentricities is $\frac{1}{2}$, then the length of the minor axis of ellipse E_2 is **[JEE Main 2013]**

(a) 8 (b) 9

(c) 4 (d) 2

42. A point on the ellipse, $4x^2 + 9y^2 = 36$, where the normal is parallel to the line $4x - 2y - 5 = 0$, is **[JEE Main 2013]**

(a) $\left(\frac{9}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, \frac{-9}{5}\right)$

(c) $\left(-\frac{9}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{5}, \frac{9}{5}\right)$

43. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at (0, 3) is **[JEE main 2013]**

(a) $x^2 + y^2 - 6y - 7 = 0$

(b) $x^2 + y^2 - 6y + 7 = 0$

(c) $x^2 + y^2 - 6y - 5 = 0$

(d) $x^2 + y^2 - 6y + 5 = 0$

44. Statement I An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement II If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[AIEEE 2012]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

45. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor-axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as its semi-major-axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is **[AIEEE 2012]**

(a) $4x^2 + y^2 = 4$

(b) $x^2 + 4y^2 = 8$

(c) $4x^2 + y^2 = 8$

(d) $x^2 + 4y^2 = 16$

46. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{\frac{2}{5}}$ is **[AIEEE 2011]**

(a) $5x^2 + 3y^2 - 48 = 0$ (b) $3x^2 + 5y^2 - 15 = 0$

(c) $5x^2 + 3y^2 - 32 = 0$ (d) $3x^2 + 5y^2 - 32 = 0$

47. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which is turn in inscribed in another ellipse that passes through the point (4, 0). Then, the equation of the ellipse is **[AIEEE 2009]**

(a) $x^2 + 12y^2 = 16$
(b) $4x^2 + 48y^2 = 48$
(c) $4x^2 + 64y^2 = 48$
(d) $x^2 + 16y^2 = 16$

48. A focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is $\frac{1}{2}$, then length of semi-major-axis is **[AIEEE 2008]**

(a) $\frac{5}{3}$ (b) $\frac{8}{3}$
(c) $\frac{2}{3}$ (d) $\frac{4}{3}$

49. In an ellipse, the distance between its foci is 6 and minor-axis is 8. Then, its eccentricity is **[AIEEE 2006]**

(a) 1/2 (b) 4/5
(c) $1/\sqrt{5}$ (d) 3/5

50. If the angle between the lines joining the end points of minor-axis of an ellipse with its foci is $\frac{\pi}{2}$, then the eccentricity of the ellipse is **[AIEEE 2005]**

(a) 1/2 (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $1/2\sqrt{2}$

51. The eccentricity of an ellipse, with centre at the origin, is $\frac{1}{2}$. If one directrix is $x = 4$, the equation of the ellipse is **[AIEEE 2004]**

(a) $3x^2 + 4y^2 = 1$ (b) $3x^2 + 4y^2 = 12$
(c) $4x^2 + 3y^2 = 1$ (d) $4x^2 + 3y^2 = 12$

Answers

1. (b)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (a)
11. (d)	**12.** (d)	**13.** (a)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (b)	**19.** (a)	**20.** (a)
21. (b)	**22.** (a)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (b)	**28.** (a)	**29.** (b)	**30.** (d)
31. (b)	**32.** (a)	**33.** (a)	**34.** (d)	**35.** (b)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (c)
41. (c)	**42.** (a)	**43.** (a)	**44.** (b)	**45.** (d)	**46.** (d)	**47.** (a)	**48.** (b)	**49.** (d)	**50.** (b)
51. (b)									

Hints & Solutions

1. Given equation of ellipse is

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

Here, $a^2 = 4, b^2 = 25$

i.e., $a < b$

So, the major-axis is along Y-axis.

$\therefore$ $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$

Here, major-axis is along Y-axis.

$\therefore$ Foci $= (0, \pm e) = (0, \pm \sqrt{21})$

Length of latusrectum $= \frac{2a^2}{b} = \frac{2 \times 4}{5} = \frac{8}{5}$

2. Given vertices $= (0, \pm 13)$ and foci $= (0, \pm 5)$

Here, we see that x-coordinate is 0, it means major-axis is lies along Y-axis.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > a$

Here, $b = 13$ and $be = 5$

$\Rightarrow$ $13e = 5 \Rightarrow e = \frac{5}{13}$

$\because$ $a^2 = b^2(1 - e^2) = 169\left(1 - \frac{25}{169}\right) = 144$

$\Rightarrow$ $a^2 = 144$

So, the required equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{169} = 1.$

3. Since, major-axis is along X-axis.

Hence, equation of ellipse will be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Given that, Eq. (i) passes through the points (4, 3) and (6, 2) *i.e.,* they will satisfy it.

$\therefore$ $\frac{(4)^2}{a^2} + \frac{(3)^2}{b^2} = 1 \Rightarrow \frac{16}{a^2} + \frac{9}{b^2} = 1$ ...(ii)

and $\frac{(6)^2}{a^2} + \frac{(2)^2}{b^2} = 1 \Rightarrow \frac{36}{a^2} + \frac{4}{b^2} = 1$ ...(iii)

On multiplying Eq. (ii) by 4 and Eq. (iii) by 9, then subtracting, we get

$$\frac{64}{a^2} - \frac{324}{a^2} = 4 - 9 \Rightarrow -\frac{260}{a^2} = -5$$

$\Rightarrow$ $a^2 = \frac{260}{5} \Rightarrow a^2 = 52$

From Eq. (ii), $\frac{9}{b^2}=1-\frac{16}{52} \Rightarrow \frac{9}{b^2}=\frac{52-16}{52}$

$\Rightarrow \frac{9}{b^2}=\frac{36}{52} \Rightarrow b^2=\frac{9\times 52}{36}=13$

Put the values $a^2=52$ and $b^2=13$ in Eq. (i), we get

$$\frac{x^2}{52}+\frac{y^2}{13}=1$$

4. CP and CD are semi-conjugate diameters of an ellipse $\frac{x^2}{14}+\frac{y^2}{8}=1$ and let eccentric angle of P is ϕ, then eccentric angle of D is $\frac{\pi}{2}+\phi$, therefore the coordinates of P and D are

$(a\cos\phi, b\sin\phi)$ and $\left[\sqrt{14}\cos\left(\frac{\pi}{2}+\phi\right), \sqrt{8}\sin\left(\frac{\pi}{2}+\phi\right)\right]$

i.e., $CP^2+CD^2=(a^2\cos^2\phi+b^2\sin^2\phi)+(a^2\sin^2\phi+b^2\cos^2\phi)$

$=a^2+b^2$

$=14+8=22$

5. The chords of contact are

$$\frac{xx_1}{25}+\frac{yy_1}{36}=1 \text{ and } \frac{xx_2}{25}+\frac{yy_2}{36}=1.$$

Product of the slopes $=-1$

$\Rightarrow \left(-\frac{x_1/25}{y_1/36}\right)\left(-\frac{x_2/25}{y_2/36}\right)=-1$

$\Rightarrow \frac{x_1x_2}{625}\frac{(36)^2}{y_1y_2}=-1$

$\Rightarrow \frac{x_1x_2}{y_1y_2}=-\frac{(5)^4}{(6)^4}=-\left(\frac{5}{6}\right)^4$

6. Let $y=m_1x$ and $y=m_2x$ be a pair of conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be ends of these two diameters. Then,

$$m_1m_2=-\frac{b^2}{a^2}$$

$\Rightarrow \frac{b\sin\theta-0}{a\cos\theta-0}\times\frac{b\sin\phi-0}{a\cos\phi-0}=-\frac{b^2}{a^2}$

$\Rightarrow \sin\theta\sin\phi=-\cos\theta\cos\phi$

$\Rightarrow \cos(\theta-\phi)=0$

$\Rightarrow \theta-\phi=\pm\frac{\pi}{2}$

7. Equation of tangent at any point $P(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. The equation of tangents at the end points of the major-axis are $x=a, x=-a$.

$\therefore$ The intersection point of these tangents are

$$A=\left(a, b\tan\frac{\theta}{2}\right), B=\left(-a, b\cot\frac{\theta}{2}\right).$$

Equation of circle with AB as diameter.

$$(x-a)(x+a)+\left(y-b\tan\frac{\theta}{2}\right)\left(y-b\cot\frac{\theta}{2}\right)=0$$

$\Rightarrow x^2-a^2+y^2-by\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)+b^2=0$

$\Rightarrow x^2+y^2-a^2+b^2-2by\operatorname{cosec}\theta=0$

which is the equation of family of circles passing through the point of intersection of the circle

$$x^2+y^2-a^2+b^2=0 \text{ and } y=0$$

So, the fixed point is $(\pm\sqrt{a^2-b^2}, 0)$.

8. The tangent to $\frac{x^2}{25}+\frac{y^2}{4}=1$ is $\frac{x}{5}\cos\theta+\frac{y}{2}\sin\theta=1$. If it is also tangent to the circle, then

$$16=\frac{1}{\frac{\cos^2\theta}{25}+\frac{\sin^2\theta}{4}}=\frac{100}{4+21\sin^2\theta}$$

$\Rightarrow \sin^2\theta=\frac{3}{28}, \cos^2\theta=\frac{25}{28}$

If the tangents meets the axes at A and B, then

$$A=\left(\frac{5}{\cos\theta}, 0\right) \text{ and } B=\left(0, \frac{2}{\sin\theta}\right)$$

$\therefore AB^2=\frac{25}{\cos^2\theta}+\frac{4}{\sin^2\theta}=28+\frac{4}{3}\cdot 28=\frac{196}{3}$

$\Rightarrow AB=\frac{14}{\sqrt{3}}$

9. Equation of tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

Meets the axis at $A\left(\frac{a}{\cos\theta}, 0\right)$ and $B\left(0, \frac{b}{\sin\theta}\right)$.

$\therefore$ Area of $\Delta AOB=\frac{1}{2}OA\cdot OB=\frac{ab}{\sin 2\theta}\geq ab$

10. The straight line $x=r\cos\theta, y=r\sin\theta$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where r is given.

$\therefore \frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$ ...(i)

But $PQ^2=4r^2=$ HM of $4a^2, 4b^2$

$\therefore \frac{1}{r^2}=\frac{1}{2a^2}+\frac{1}{2b^2}$ ...(ii)

From Eqs. (i) and (ii), $\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{2a^2}+\frac{1}{2b^2}$

It is possible when $\theta=\frac{\pi}{4}$

11. Given equation of ellipse is $\frac{x^2}{2}+\frac{y^2}{1}=1$. General equation of tangent to the ellipse of slope m is $y=mx\pm\sqrt{2m^2+1}$

Since, this is equally inclined to axes, so $m=\pm 1$.

Then, tangents are $y=\pm x\pm\sqrt{2+1}=\pm x\pm\sqrt{3}$

Distance of any tangent from origin $=\frac{|0+0\pm\sqrt{3}|}{\sqrt{1^2+1^2}}=\frac{\sqrt{3}}{2}$

12. The position of the points (1, 2) and (2, 1) with respect to the circle $x^2+y^2=9$ is $1^2+2^2=5<9$ and $2^2+1^2=5<9$. Thus, both P and Q lie inside C.

The position of the points (1, 2) and (2, 1) with respect to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

$$\frac{1^2}{9}+\frac{2^2}{4}=\frac{1}{9}+1>1$$

and $$\frac{2^2}{9}+\frac{1}{4}=\frac{16+9}{36}=\frac{25}{36}<1$$

Since, P lies outside E and Q lies inside E. Thus, P lies inside C but outside E.

13. Given equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$ and equation of line is $x+y=10$. Replace x by $10-y$ and y by $10-x$ in the equation of ellipse, we get

$$\frac{(x-10)^2}{16}+\frac{(y-10)^2}{25}=1$$

which is the required equation of ellipse.

14. Let point is $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ and it's distance d from origin.

$\therefore \quad d=\sqrt{6\cos^2\theta+2\sin^2\theta}$

$\Rightarrow \quad 2=\sqrt{2+4\cos^2\theta}$

$\Rightarrow \quad 2+4\cos^2\theta=4$

$\Rightarrow \quad \cos^2\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{4}$

15. Given equation of ellipse is $\frac{x^2}{14}+\frac{y^2}{5}=1$.

Any point on the ellipse is $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$.

$\therefore$ Equation of normal at $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$ is

$$\sqrt{14}x\sec\theta-\sqrt{5}y\operatorname{cosec}\theta=9.$$

It passes through $(\sqrt{14}\cos 2\theta, \sqrt{5}\sin 2\theta)$.

$\therefore \quad \sqrt{14}\sqrt{14}\cos 2\theta\sec\theta-\sqrt{5}\sqrt{5}\sin 2\theta\operatorname{cosec}\theta=9$

$\Rightarrow \quad 14\frac{\cos 2\theta}{\cos\theta}-5\frac{\sin 2\theta}{\sin\theta}=9$

$\Rightarrow \quad 14(2\cos^2\theta-1)-10\cos^2\theta=9\cos\theta$

$\Rightarrow \quad 18\cos^2\theta-9\cos\theta-14=0$

$\Rightarrow \quad \cos\theta=-\frac{2}{3}, \cos\theta\neq-\frac{7}{6}$

16. Let the equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Any tangent to the ellipse be

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad \ldots\text{(i)}$$

Tangents at the vertices are

$$x=a \text{ and } x=-a.$$

On solving with Eq. (i), we get the points T and T' as

$$T\left[a, \frac{b(1-\cos\theta)}{\sin\theta}\right], T'\left[-a, \frac{b(1+\cos\theta)}{\sin\theta}\right]$$

i.e., $$T\left[a, b\tan\frac{\theta}{2}\right], T'\left[-a, b\cot\frac{\theta}{2}\right]$$

$\therefore$ Equation of circle on TT' as diameter is

$$(x-a)(x+a)+\left(y-b\tan\frac{\theta}{2}\right)\left(y-b\cot\frac{\theta}{2}\right)=0$$

$$\Rightarrow x^2-a^2+y^2+b^2-by\left[\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right]=0$$

Since, it will pass through the foci $(\pm ae, 0)$.

$\therefore \quad a^2e^2-a^2+b^2=0$

$\Rightarrow \quad b^2=a^2(1-e^2)$, which is true.

17. Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and e is eccentricity of ellipse.

Therefore, $\quad 2h=2ae$

$\Rightarrow \quad ae=h \quad \ldots$(i)

Focal distance of one end of minor-axis say $(0, b)$ is k.

Therefore, $\quad a+e\,(0)=k$

$\Rightarrow \quad a=k \quad \ldots$(ii)

Now, $\quad b^2=a^2(1-e^2) \quad$ [from Eqs. (i) and (ii)]

$$b^2=a^2-a^2e^2=k^2-h^2$$

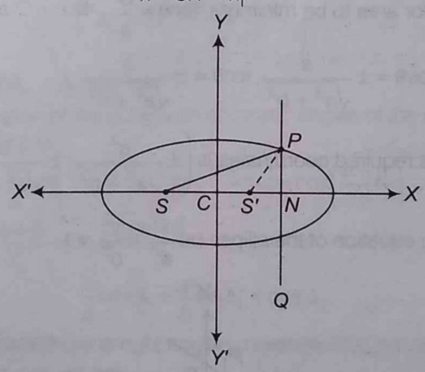
Therefore, equation of ellipse is

$$\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1.$$

18. Let any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be (x_1, y_1), then $NQ=PS=a+ex_1$.

Let the coordinates of Q be (x, y), then

$$x=CN=x_1 \quad \ldots\text{(i)}$$



and $y=NQ=-(a+ex_1) \quad \ldots$(ii)

On putting the value of x_1 from Eq. (i) in Eq. (ii), the required locus is

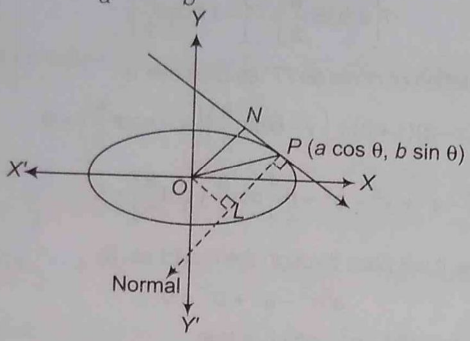
$$y=-(a+ex)$$

$\Rightarrow \quad y+a+ex=0$

Taking S' instead of S the locus of Q will be $y+a-ex=0$.

19. Equation of tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$



$$ON=\frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Equation of the normal at P is $ax\sec\theta-by\,\text{cosec}\,\theta=a^2-b^2$.

$$OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\text{cosec}^2\theta}}$$

$$=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

where, L is the foot of perpendicular from O on the normal.

Area of $\Delta PON=\frac{1}{2}\times ON\times OL$ $\quad(\because NP=OL)$

$$=\frac{(a^2-b^2)ab\tan\theta}{a^2\tan^2\theta+b^2}=\frac{(a^2-b^2)ab}{a^2\tan\theta+b^2\cot\theta}$$

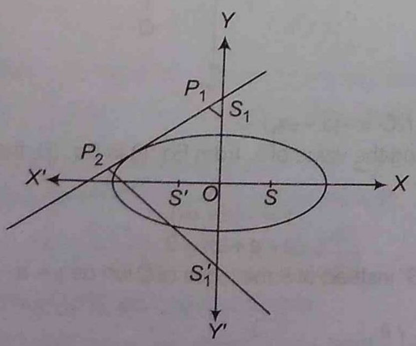
which is minimum when $a^2\tan\theta+b^2\cot\theta$ is maximum.

Thus, for area to be minimum, $\tan\theta=\frac{b}{a}$

$\therefore \quad \cos\theta=\pm\frac{a}{\sqrt{a^2+b^2}},\ \sin\theta=\pm\frac{b}{\sqrt{a^2+b^2}}$

So, the required coordinates is $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}},\pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

20. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Then, the distance of a focus from the centre $=ae$

$$=a\sqrt{1-\frac{b^2}{a^2}}=\sqrt{a^2-b^2}$$

So, the two points on the minor-axis are

$$S_1(0,\sqrt{a^2-b^2})\text{ and }S_1'(0,-\sqrt{a^2-b^2})$$

Now, any tangent to the ellipse is $y=mx+\sqrt{a^2m^2+b^2}$, where m is a parameter.

The sum of the squares of the perpendiculars on this tangent from the two points S_1 and S'_1

$$=\left(\frac{\sqrt{a^2-b^2}-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right)^2+\left(\frac{-\sqrt{a^2-b^2}-\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right)^2$$

$$=2\frac{(a^2-b^2+a^2m^2+b^2)}{1+m^2}=2a^2$$

21. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and O be the centre.

Tangent at $P(x_1,y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, whose slope $=-\frac{b^2x_1}{a^2y_1}$

Focus is $S(ae,0)$.

Equation of the line perpendicular to tangent at S is

$$y=\frac{a^2y_1}{b^2x_1}(x-ae) \quad \ldots\text{(i)}$$

Equation of OP is $y=\frac{y_1}{x_1}x$ $\quad \ldots$(ii)

Since, Eqs. (i) and (ii) intersect.

$\therefore \quad \frac{y_1}{x_1}\cdot x=\frac{a^2y_1}{b^2x_1}(x-ae)\Rightarrow x(a^2-b^2)=a^3e$

$\Rightarrow \quad x\cdot a^2e^2=a^3e\Rightarrow x=\frac{a}{e}$

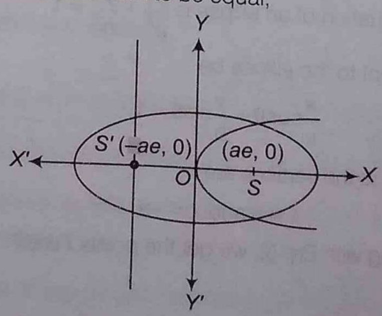
which is the corresponding directrix.

22. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Equation of the parabola with focus $S(ae,0)$ and directrix $x+ae=0$ is $y^2=4aex$.

Now, length of latusrectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

For the two latusrectum to be equal,



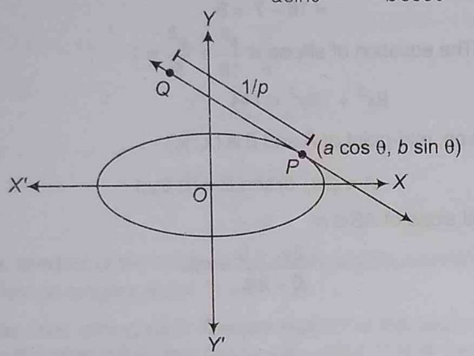
$$\frac{2b^2}{a} = 4ae \Rightarrow \frac{2a^2(1-e^2)}{a} = 4ae$$

$\Rightarrow \quad 1-e^2 = 2e \Rightarrow e^2 + 2e - 1 = 0$

Therefore, $e = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

Hence, $e = \sqrt{2} - 1$ as $0 < e < 1$ for ellipse.

23. Equation of the tangent at P is $\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta}$



The distance of the tangent from the origin is

$$p = \left|\frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$\Rightarrow \quad \frac{1}{p} = \frac{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}{ab}$$

Now, the coordinates of the point Q are given as follows,

$$\frac{\frac{x - a\cos\theta}{-a\sin\theta}}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{\frac{y - b\sin\theta}{b\cos\theta}}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

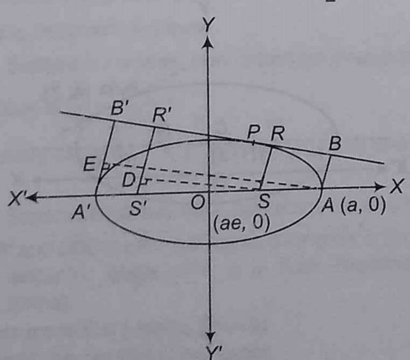
$$= \frac{1}{p} = \frac{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}{ab}$$

$$\Rightarrow \quad x = a\cos\theta - \frac{a\sin\theta}{ab} \text{ and } y = b\sin\theta + \frac{b\cos\theta}{ab}$$

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \frac{1}{a^2b^2} \text{ is the required locus.}$$

24. Equation of tangent at P is $y = mx + \sqrt{a^2m^2 + b^2}$ …(i)

$\therefore S'R'RS$ is a trapezium and its area $\Delta_1 = \frac{1}{2}(SR + S'R') \times SD$



Equation of the line $S'R'$ is

$$y = -\frac{1}{m}(x + ae)$$

$$\Rightarrow \quad x + my + ae = 0 \qquad \text{…(ii)}$$

Therefore, $SD = \left|\frac{ae + ae}{\sqrt{1 + m^2}}\right|$

$$SR = \left|\frac{aem + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right|$$

and $S'R' = \left|\frac{-aem + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right|$

Then, $\Delta_1 = \frac{1}{2}(S'R' + SR) \times SD$

$$\Rightarrow \quad \Delta_1 = 2ae\left(\frac{\sqrt{a^2m^2 + b^2}}{1 + m^2}\right)$$

Area of $A'B'BA$ is $\Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$

Equation of $A'B'$ is $y = -\frac{1}{m}(x + a)$

$$\Rightarrow \quad x + my + a = 0 \qquad \text{…(iii)}$$

Therefore, $AE = \left|\frac{a + a}{\sqrt{1 + m^2}}\right|$

and $AB = \left|\frac{ma + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right|$

and $A'B' = \left|\frac{-ma + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right|$

Then, $\Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$

$$\Rightarrow \quad \Delta_2 = 2a\left(\frac{\sqrt{a^2m^2 + b^2}}{1 + m^2}\right)$$

Hence, $\Delta_1 : \Delta_2 = e : 1$

25. Equation of the chord with eccentric angles of the extremities as α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha + \beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\alpha - \beta}{2}\right)$$

Let $\frac{\alpha + \beta}{2} = \Delta_1$ and $\frac{\alpha - \beta}{2} = \Delta_2$, so that

$$\frac{x}{a}\cos\Delta_1 + \frac{y}{b}\sin\Delta_1 = \cos\Delta_2$$

As the triangle is right angled, homogenizing the equation of the curve, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \left(\frac{x\cos\Delta_1}{a\cos\Delta_2} + \frac{y\sin\Delta_1}{b\cos\Delta_2}\right)^2 = 0$$

$$\Rightarrow \left(\frac{1}{a^2} - \frac{\cos^2\Delta_1}{a^2\cos^2\Delta_2}\right) + \left(\frac{1}{b^2} - \frac{\sin^2\Delta_1}{b^2\cos^2\Delta_2}\right) = 0$$

(as coefficient of x^2 + coefficient of $y^2 = 0$)

$\Rightarrow \quad b^2(\cos^2\Delta_2 - \cos^2\Delta_1) + a^2(\cos^2\Delta_2 - \sin^2\Delta_1) = 0$

$\Rightarrow \quad \cos^2\Delta_2(a^2+b^2) - b^2\cos^2\Delta_1 - a^2 + a^2\cos^2\Delta_1 = 0$

$\Rightarrow \quad \cos^2\Delta_2(a^2+b^2) = a^2(1 - e^2\cos^2\Delta_1)$

$$\Rightarrow \quad \frac{1-e^2\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} = \frac{a^2+b^2}{a^2}$$

26. Equation of tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \text{...(i)}$$

The joint equation of the lines joining the points of intersection of Eq. (i) and the auxiliary circle $x^2+y^2=a^2$ to the origin, which is the centre of the circle, is

$$x^2+y^2 = \left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2$$

Since, these lines are at right angles.

$$\therefore \quad 1 - a^2\left(\frac{\cos^2\theta}{a^2}\right) + 1 - a^2\left(\frac{\sin^2\theta}{b^2}\right) = 0$$

($\because$ coefficient of x^2 + coefficient of $y^2 = 0$)

$$\Rightarrow \quad \sin^2\theta\left(1 - \frac{a^2}{b^2}\right) + 1 = 0$$

$\Rightarrow \quad \sin^2\theta(b^2 - a^2) + b^2 = 0$

$\Rightarrow \quad (1+\sin^2\theta)(a^2e^2) = a^2$

$\Rightarrow \quad e = (1+\sin^2\theta)^{-1/2}$

27. The equation of tangent from the point $(3\sqrt{3}\cos\theta, \sin\theta)$ to the curve is

$$\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1.$$

Thus, sum of intercepts $= 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta = f(\theta)$ (say)

$$\Rightarrow \quad f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$$

Put $\quad f'(\theta) = 0$

$$\Rightarrow \quad \sin^3\theta = \frac{1}{3^{3/2}}\cos^3\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

Now, $f''(\theta) > 0$, for $\theta = \pi/6$ (*i.e.*, minimum)

Hence, value of θ is $\frac{\pi}{6}$.

28. Since, end of latusrectum is $\left(2e, \frac{3}{2}\right)$.

Normal at $\left(2e, \frac{3}{2}\right)$ is $\dfrac{x-2e}{2e/4} = \dfrac{\left(y-\frac{3}{2}\right)}{\frac{3}{2\times 3}}$.

$$\Rightarrow \quad \frac{2(x-2e)}{e} = 2(y) - 4(1-e^2) \quad [\because b^2 = a^2(1-e^2) \Rightarrow 3 = 4(1-e^2)]$$

$\Rightarrow \quad 2x - 4e = 2ey + 4e^3 - 4e$

$\Rightarrow \quad 2x - 2ey = 4e^3$

$\Rightarrow \quad (x - ey) = 2e^3 \Rightarrow x - ey - 2e^3 = 0$

29. Given, $ae = \sqrt{7}, \dfrac{a}{e} = \dfrac{16}{\sqrt{7}}$

$\Rightarrow \quad a^2 = 16 \Rightarrow a = 4$

Then, $\quad e = \dfrac{\sqrt{7}}{4}$

and $\quad b^2 = a^2(1-e^2) = 16\left(1 - \dfrac{7}{16}\right)$

$= 16 - 7 = 9$

$\therefore$ The equation of ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$\Rightarrow \quad 9x^2 + 16y^2 = 144.$

30. Given, mid-point of A and B is (x_1, y_1).

$\therefore \quad A \equiv (2x_1, 0)$ and $B \equiv (0, 2y_1)$

and slope of AB is m. (given)

$$\therefore \quad \frac{2y_1 - 0}{0 - 2x_1} = m$$

$\Rightarrow \quad y_1 + mx_1 = 0$

$\Rightarrow \quad mx_1 + y_1 = 0$

31. Let the coordinates of P be $(3\cos\theta, 3\sin\theta)$, then the eccentric angle at the point M, where the ordinate PQ through P meets the ellipse is θ and the coordinates of M are $(3\cos\theta, 2\sin\theta)$.

Here, $\quad PQ = 3\sin\theta$

$MQ = 2\sin\theta$, so $\dfrac{MQ}{PQ} = \dfrac{2}{3}$

32. Line $L: y = 2x$ meets the circle $C: x^2+y^2 = 9$ at points for which $x^2 + 4x^2 = 9 \Rightarrow x = \pm 3/\sqrt{5}$

$\therefore$ Coordinates of P are $(\pm 3/\sqrt{5}, \pm 6/\sqrt{5})$.

Coordinates of M are $(\pm 3/\sqrt{5}, \pm 4/\sqrt{5})$.

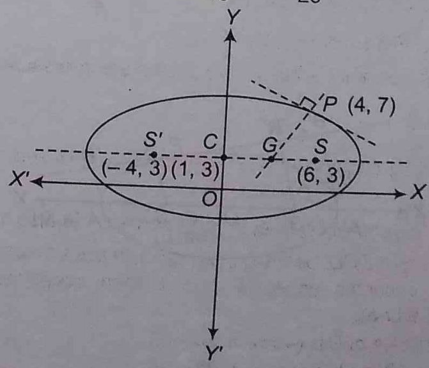
Equation of the tangent at M to the ellipse E is

$$\frac{x(\pm 3)}{9\sqrt{5}} + \frac{y(\pm 4)}{4\sqrt{5}} = 1$$

$\Rightarrow \quad x + 3y = \pm 3\sqrt{5}$

Solutions (Q. Nos. 33 to 35)

$\because$ Equation of ellipse is $\dfrac{(x-1)^2}{45} + \dfrac{(y-3)^2}{20} = 1 \quad \text{...(i)}$



$\because \quad PS + PS' = 2a$

$\therefore \quad PS' = \sqrt{64+16} = 4\sqrt{5}$

and $\quad PS' = \sqrt{4+16} = 2\sqrt{5}$

$\therefore \quad 2a = 6\sqrt{5} \Rightarrow a = 3\sqrt{5}$

and $\quad ae = 6 - 1 = 5$

$\Rightarrow \quad e \times 3\sqrt{5} = 5 \Rightarrow e = \frac{\sqrt{5}}{3}$

$\because \quad e^2 = 1 - \frac{b^2}{a^2}$

$\Rightarrow \quad \frac{b^2}{a^2} = 1 - \frac{5}{9} = \frac{4}{9}$

$\therefore \quad b^2 = \frac{4}{9} \times 9 \times 5$

$\Rightarrow \quad b = 2\sqrt{5}$

33. Now, product of the length of the perpendicular segments from the foci on tangent at $P(4, 7)$ is $b^2 = 20$.

34. Since, lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at $P(4, 7)$ meet the normal PG and bisects it.

Hence, required point is the mid-point of PG.

$\therefore$ Equation of normal at $P(4,7)$ is given by

$$3x - y - 5 = 0$$

$\because \quad G\left(\frac{8}{3}, 3\right)$

So, the required point is mid-point of PG *i.e.*, $\left(\frac{10}{3}, 5\right)$.

35. Locus of mid-point of QR is another ellipse having the same eccentricity as that of ellipse (e).

$\Rightarrow \quad e' = e = \frac{\sqrt{5}}{3}$

36. Let C_1 and C_2 be the centres and R_1 and R_2 be the radii of the two circles.

Let $S_1 = 0$ lie completely inside in the circle $S_2 = 0$.

Let 'C' and 'r' be the centre and radius of the variable centre.

Then, $\quad CC_2 = R_2 - r \quad$ and $\quad C_1C = R_1 + r$

$\therefore \quad C_1C + C_2C = R_1 + R_2 \quad$ (constant)

So, the locus of C is an ellipse.

Therefore, Statement II is true.

Hence, Statement I is false. (two circles are intersecting).

37. The ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

So, auxiliary circle is $x^2 + y^2 = 9$ and $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$ are focii.

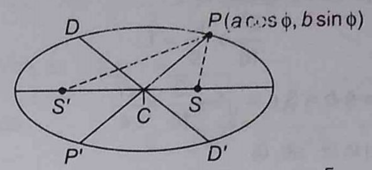
Hence, Statement I is true, then Statement II is also true.

38. Let PCP' and DCD' be the conjugate diameters of an ellipse and let the eccentric angle of P is ϕ, then coordinate of P is $(a\cos\phi, b\sin\phi)$.

$\therefore$ Coordinate of D is $(-a\sin\phi, b\cos\phi)$.

Let S and S' be two foci of the ellipse.

Then, $SP \cdot S'P = (a - ae\cos\phi) \cdot (a + ae\cos\phi)$



$= a^2 - a^2e^2\cos^2\phi \qquad \left[\begin{array}{l}\Rightarrow b^2 = a^2(1-e^2) \\ \Rightarrow a^2 - b^2 = a^2e^2\end{array}\right]$

$= a^2 - (a^2 - b^2)\cos^2\phi$

$= a^2\sin^2\phi + b^2\cos^2\phi$

$= CD^2$

39. Let $(t, b - t)$ be a point on the line $x + y = b$, then equation of chord whose mid-point is $(t, b - t)$, is

$$\frac{tx}{2a^2} + \frac{(b-t)y}{2b^2} - 1 = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2} - 1 \quad \ldots(i)$$

Since, point $(a, -b)$ lies on Eq. (i), then

$$\frac{ta}{2a^2} - \frac{b(b-t)}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$$

$\Rightarrow \quad t^2(a^2 + b^2) - ab(3a + b)t + 2a^2b^2 = 0$

Since, t is real.

$\therefore \quad B^2 - 4AC \ge 0$

$\Rightarrow \quad a^2b^2(3a+b)^2 - 4(a^2+b^2)2a^2b^2 \ge 0$

$\Rightarrow \quad 9a^2 + 6ab + b^2 - 8a^2 - 8b^2 \ge 0$

$\therefore \quad a^2 + 6ab - 7b^2 \ge 0$

41. E_1 : Eccentricity $e_1 = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$

E_2 : Eccentricity $e_2 = \sqrt{1 - \frac{b^2}{16}} = \frac{\sqrt{16 - b^2}}{4}$

Also, $\quad \frac{1}{\sqrt{3}} \times \frac{\sqrt{16-b^2}}{4} = \frac{1}{2}$

$\Rightarrow \quad b^2 = 4$

Minor-axis of ellipse, $E_2 = 2b = 2 \times 2 = 4$

42. Let (x_1, y_1) be a point on the ellipse.

$\therefore \quad 4x_1^2 + 9y_1^2 = 36 \quad \ldots(i)$

Now, $\quad 4(2x) + 18y\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{4x}{9y} = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -\frac{4x_1}{9y_1}$

$\therefore$ Slope of normal $= \frac{9y_1}{4x_1}$

which is parallel to $4x - 2y - 5 = 0$

$\therefore \quad \frac{9y_1}{4x_1} = 2$

$\Rightarrow \quad 9y_1 = 8x_1 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$$\left(\frac{9}{5}, \frac{8}{5}\right)$$

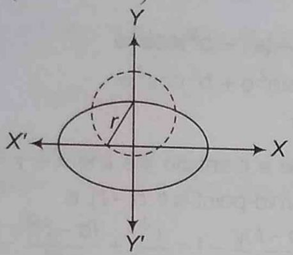
43. Given equation of ellipse is

$$\frac{x^2}{16}+\frac{y^2}{9}=1.$$

Here, $a=4, b=3, e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$

$\therefore$ Foci is $(\pm\ ae, 0)$

$$=\left(\pm\ 4\times\frac{\sqrt{7}}{4}, 0\right)=(\pm\ \sqrt{7}, 0)$$



$\therefore$ Radius of the circle,

$$r=\sqrt{(ae)^2+b^2}=\sqrt{7+9}=\sqrt{16}=4$$

Now, the equation of circle is

$$(x-0)^2+(y-3)^2=16$$

$\Rightarrow\quad x^2+y^2-6y-7=0$

44. Statement I Given, a parabola $y^2=16\sqrt{3}x$ and an ellipse $2x^2+y^2=4$. To find the equation of common tangent to the given parabola and the ellipse. This can be very easily done by comparing the standard equation of tangents. Standard equation of tangent with slope 'm' to the parabola $y^2=16\sqrt{3}x$ is

$$y=mx+\frac{4\sqrt{3}}{m} \quad \ldots\text{(i)}$$

Standard equation of tangent with slope 'm' to the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$ is $y=mx\pm\sqrt{2m^2+4}$ ...(ii)

If a line L is a common tangent to both parabola and ellipse, then L should be tangent to parabola *i.e.*, its equation should be like Eq. (i) and L should be tangent to ellipse *i.e.*, its equation should be like Eq. (ii) *i.e.*, L must be like both of the Eqs. (i) and (ii).

Hence, comparing Eqs. (i) and (ii), we get

$$\frac{4\sqrt{3}}{m}=\pm\sqrt{2m^2+4}$$

On squaring both sides, we get

$$m^2(2m^2+4)=48$$

$\Rightarrow\quad m^4+2m^2-24=0$

$\Rightarrow\quad (m^2+6)(m^2-4)=0$

$\Rightarrow\quad m^2=4 \quad (\because m^2\neq -6)$

$\Rightarrow\quad m=\pm 2$

On substituting $m=\pm 2$ in the Eq. (i), we get the required equation of the common tangent as

$$y=2x+2\sqrt{3} \text{ and } y=-2x-2\sqrt{3}$$

Hence, Statement I is correct.

Statement II In Statement II, we have already seen that, if the line $y=mx+\frac{4\sqrt{3}}{m}$ is a common tangent to the parabola $y^2=16\sqrt{3}x$ and the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$, then it satisfies the equation $m^4+2m^2-24=0$.

Hence, Statement II is also correct but is not able to explain the statement I.

It is an intermediate step in the final answer.

45. Given

(i) An ellipse whose semi-minor-axis coincides with one of the diameters of the circle $(x-1)^2+y^2=1$.

(ii) The semi-major-axis of the ellipse coincides with one of the diameters of circle $x^2+(y-2)^2=4$.

(iii) The centre of the ellipse is at origin.

(iv) The axes of the ellipse are coordinate axes.

Now, diameter of circle $(x-1)^2+y^2=1$ is 2 units and that of circle $x^2+(y-2)^2=4$ is 4 units.

Semi-minor-axis of ellipse, $b=2$ units and semi-major-axis of ellipse, $a=4$ units.

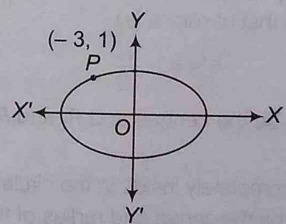
Hence, the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\Rightarrow\quad \frac{x^2}{16}+\frac{y^2}{4}=1$

$\Rightarrow\quad x^2+4y^2=16$

46. Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



It passes through $P(-3, 1)$ and $e=\sqrt{\frac{2}{5}}$.

$\therefore\quad \frac{9}{a^2}+\frac{1}{b^2}=1 \quad \ldots\text{(i)}$

and $\quad e^2=1-\frac{b^2}{a^2}$

$\Rightarrow\quad \frac{2}{5}=1-\frac{b^2}{a^2}$

$$\frac{b^2}{a^2}=\frac{3}{5}$$

From Eq. (i), $\frac{9}{a^2}+\frac{5}{3a^2}=1$

$\Rightarrow\quad \frac{27+5}{3a^2}=1$

$\Rightarrow\quad a^2=\frac{32}{3}$ and then $b^2=\frac{32}{5}$

$\therefore$ Equation of ellipse is

$$\frac{3x^2}{32} + \frac{5y^2}{32} = 1$$

$$\Rightarrow \quad 3x^2 + 5y^2 = 32$$

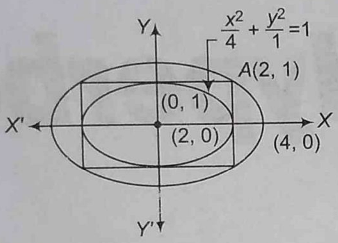
47. Let the equation of the required ellipse be $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$.

But the ellipse passes through the point (2, 1).

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{b^2} = 1$$

$$\Rightarrow \quad \frac{1}{b^2} = \frac{3}{4}$$

$$\Rightarrow \quad b^2 = \frac{4}{3}$$



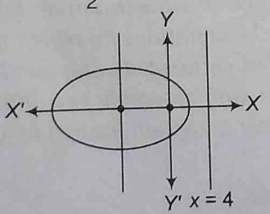
Hence, the equation is

$$\frac{x^2}{16} + \frac{3y^2}{4} = 1$$

$$\Rightarrow \quad x^2 + 12y^2 = 16$$

48. Given, $\frac{a}{e} - ae = 4$ and $e = \frac{1}{2}$

$$\therefore \quad 2a - \frac{a}{2} = 4$$



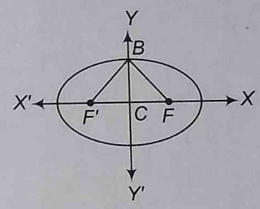
$$\Rightarrow \quad \frac{3a}{2} = 4 \Rightarrow a = \frac{8}{3}$$

49. Given, $ae = 3$, $b = 4$

Now, $\frac{16}{9} = \frac{b^2}{a^2e^2} = \frac{a^2(1-e^2)}{a^2e^2} = \frac{1-e^2}{e^2}$

$$\Rightarrow \quad e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$$

50. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$



$$\because \quad \angle FBF' = \frac{\pi}{2}$$

$$\therefore \quad \angle FBC = \frac{\pi}{4}$$

$\therefore \angle CFB$ is also an angle of $\frac{\pi}{4}$.

$$\Rightarrow \quad BC = CF \Rightarrow b = ae$$

$$\Rightarrow \quad b^2 = a^2e^2 \quad \ldots(i)$$

We know, that $b^2 = a^2(1-e^2)$

$$\therefore \quad a^2e^2 = a^2(1-e^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad e^2 = 1 - e^2$$

$$\Rightarrow \quad 2e^2 = 1$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

51. Given, $\frac{a}{e} = 4 \Rightarrow 2a = 4 \Rightarrow a = 2$ and $e = \frac{1}{2}$

Now, $b^2 = a^2(1-e^2) = 4\left(1 - \frac{1}{4}\right) = 3$

So, the equation is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ or $3x^2 + 4y^2 = 12$.

Day 29

Hyperbola

Day 29

Outlines ...

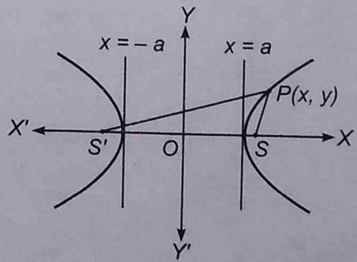
- Concept of Hyperbola
- Equations of Hyperbola in Standard Form
- Tangent to a Hyperbola
- Normal to a Hyperbola
- Rectangular Hyperbola
- Asymptotes

Concept of Hyperbola

In Mathematics, a hyperbola is a type of smooth curve, lying in a plane, defined by its geometric properties or by equations for which it is the solution set. A hyperbola has two pieces, called connected components or branches, that are mirror images of each other and resemble two infinite bows. The hyperbola is one of the four kinds of conic section formed by the intersection of a plane and a cone.

or

Hyperbola is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (focus) in the same plane to its distance from a fixed line (directrix) is always constant which is always greater than unity. Mathematically, $\frac{SP}{PM} = e$, where $e > 1$.

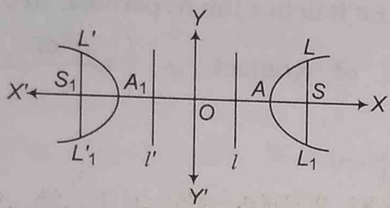


Equations of Hyperbola in Standard Form

If the centre of the hyperbola is at the origin and foci are on the X-axis or Y-axis, then that types of equation are called standard equation of an ellipse. Different forms of hyperbola and their equations have been given below

1. Hyperbola of the Form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

When the hyperbola is in the given form, then it is also called the the equation of auxiliary circle. The important parameters are



(i) Centre, $O(0, 0)$

(ii) Foci : $S(ae, 0), S_1(-ae, 0)$

(iii) Vertices : $A(a, 0), A_1(-a, 0)$

(iv) Directrices $l : x = \frac{a}{e}, l' : x = -\frac{a}{e}$

(v) Length of latusrectum, $LL_1 = L'L'_1 = \frac{2b^2}{a}$

(vi) Eccentricity, $e = \sqrt{1 + \left(\frac{b}{a}\right)^2}$

or $\quad b^2 = a^2(e^2 - 1)$

(vii) Length of transverse axis, $2a$

(viii) Length of conjugate axis, $2b$

(ix) Equation of transverse axis, $y = 0$

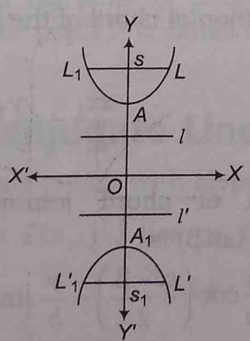
(x) Equation of conjugate axis, $x = 0$

(xi) Focal distances of a point on the hyperbola is $ex \pm a$.

(xii) Difference of the focal distances of a point on the hyperbola is $2a$.

2. Conjugate Hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.



(i) Centre, $O(0, 0)$

(ii) Foci, $S(0, be), S_1(0, -be)$

(iii) Vertices, $A(0, b), A_1(0, -b)$

(iv) Directrices

$$l : y = \frac{b}{e}, l' : y = -\frac{b}{e}$$

(v) Length of latusrectum

$$LL_1 = L'L_1' = \frac{2a^2}{b}$$

(vi) Eccentricity,

$$e = \sqrt{1 + \left(\frac{a}{b}\right)^2}$$

(vii) Length of transverse axis, $2b$

(viii) Length of conjugate axis, $2a$

(ix) Equation of transverse axis, $x = 0$

(x) Equation of conjugate axis, $y = 0$

(xi) Focal distances of a point on the hyperbola is $ey \pm b$.

(xii) Difference of the focal distances of a point on the hyperbola is $2b$.

Results on Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

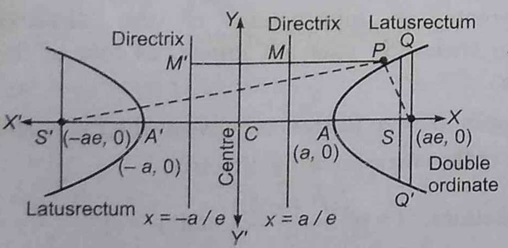
1. The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.
2. The points of intersection of the directrix with the transverse axis are known as foot of the directrix.
3. Latusrectum $(l) = 2e$ (distance between the focus and the foot of the corresponding directrix).
4. The equations, $x = a\left(\frac{e^\theta + e^{-\theta}}{2}\right)$ and $y = b\left(\frac{e^\theta - e^{-\theta}}{2}\right)$ are also known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
5. The parametric equation of a hyperbola be $x = a\sec\theta$ and $y = b\tan\theta$, where $\theta \in (0, 2\pi)$.
6. The position of a point (h, k) with respect to the hyperbola S lie inside, on or outside the hyperbola, if $S_1 > 0$, $S_1 = 0$ or $S_1 < 0$

   where, $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

Terms Related to Hyperbola

There are some important terms related to hyperbola are given as follow.

- **Vertices** The points A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.
- **Transverse and Conjugate Axes** Transverse axis is the one which lie along the line passing through the foci and perpendicular to the directrices and conjugate axis is the one which is perpendicular to the transverse axis and passes through the mid-point of the foci *i.e.*, centre.
- **Centre** The mid-point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.
- **Focal Chord** A chord of a hyperbola which is passing through the focus is called a focal chord of the hyperbola.
- **Directrix** A line which is perpendicular to the axis and it lies between centre and vertex. The equation of directrix is $x = \pm \frac{a}{e}$.
- **Double Ordinates** If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then, QQ' is called a double ordinate of Q.
- **Latusrectum** The double ordinate passing through focus is called latusrectum.



- *The vertex divides the join of focus and the point of intersection of directrix with axis internally and externally in the ratio e : 1.*
- *Domain and range of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x \le -a$ or $x \ge a$ and $y \in R$, respectively.*
- *The line through the foci of the hyperbola is called its transverse axis.*
- *The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.*

Tangent to a Hyperbola

A line which intersect the hyperbola at only one point is called the tangent to the hyperbola.

(i) In point (x_1, y_1) form, $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

(ii) In slope 'm' form, $y = mx \pm \sqrt{a^2m^2 - b^2}$

(iii) In parametric form, $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ at $(a\sec\theta,\ b\tan\theta)$.

(iv) The line $y = mx + c$ touches the hyperbola, iff $c^2 = a^2m^2 - b^2$ and the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$, where $c = \sqrt{a^2m^2 - b^2}$.

Results on Tangent

1. Two tangents can be drawn from a point to a hyperbola.
2. The point of intersection of tangents at t_1 and t_2 to the curve $xy = c^2$ is $\left(\frac{2c\,t_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.
3. The tangent at the point $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ intersect at the point
$$R\left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right).$$
4. The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the hyperbola is $SS_1 = T^2$
where $S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$,
$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$
and $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$
5. The equation of chord of contact is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$.
6. The equation of chord of the hyperbola, whose mid-point is (x_1, y_1) is $T = S_1$
i.e., $$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$
7. Equation of chord joining the points $(a\sec\alpha, b\tan\alpha)$ and $(a\sec\beta, b\tan\beta)$ is
$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right).$$

Normal to a Hyperbola

A line which is perpendicular to the tangent of the hyperbola is called the normal to the hyperbola.

(i) In point (x_1, y_1) form $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

(ii) In slope 'm' form

$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - m^2b^2}}$ and the point of intersection is

$\left(\pm \frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp \frac{b^2 m}{\sqrt{a^2 - b^2m^2}}\right)$

(iii) In parametric form,

$ax\cos\theta + by\cot\theta = a^2 + b^2$ at $(a\sec\theta, b\tan\theta)$

Results on Normals

1. If the straight line $lx + my + n = 0$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$
2. Four normals can be drawn from any point to a hyperbola.
3. The line $y = mx + c$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = \frac{m^2(a^2+b^2)^2}{a^2 - b^2m^2}$.
4. If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then

$$(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4.$$

Pole and Polar

Let P be a point inside or outside a hyperbola. Then, the locus of the point of intersection of two tangents to the hyperbola at the point where secants drawn through P intersect the hyperbola, is called the **polar** of point P with respect to the hyperbola and the point P is called the **pole** of the polar.

Conjugate Points and Conjugate Lines

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $P(x_1, y_1)$ lies on the polar of $Q(x_2, y_2)$

i.e., on $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$.

Therefore, $\frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} = 1$

Two lines are said to be conjugate lines with respect to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if each passes through the pole of the other.

Diameter

The locus of the mid-point of a system of parallel chords of a hyperbola is called a diameter.

(i) The equation of a diameter bisecting a system of parallel chords of slope 'm' to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2 m} \cdot x$.

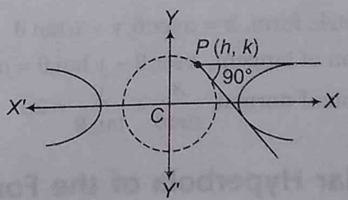
(ii) Two diameters of a hyperbola are said to be **conjugate diameters**, if each bisects the chord parallel to the other.

(iii) Length of chord cut off by hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the line $y = mx + c$ is

$$\frac{2ab\sqrt{[c^2 - (a^2m^2 - b^2)](1+m^2)}}{(b^2 - a^2m^2)}$$

Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other, is called a director circle. The equation of director circle is $x^2 + y^2 = a^2 - b^2$.



The circle $x^2 + y^2 = a^2$ is known the auxiliary circle of both hyperbola.

Conormal Points

Points on the hyperbola, the normals at which passes through a given point are called conormal points.

(i) The sum of the eccentric angles of conormal points is an odd multiple of π.

(ii) If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normals at which are concurrent, then (a) $\Sigma \cos(\theta_1 + \theta_2) = 0$, (b) $\Sigma \sin(\theta_1 + \theta_2) = 0$.

(iii) If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$, then the normals at these points are concurrent.

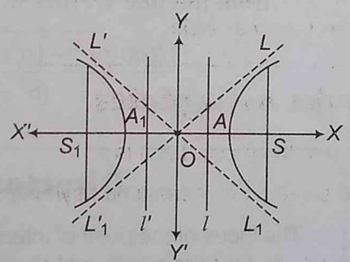
(iv) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then $(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$.

Rectangular Hyperbola

A hyperbola whose asymptotes include a right angle is said to be rectangular hyperbola or we can say that, if the lengths of transverse and conjugate axes of any hyperbola be equal, then it is said to be a rectangular hyperbola.

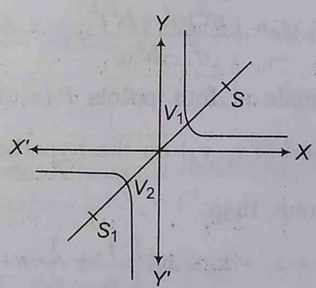
Rectangular Hyperbola of the Form $x^2 - y^2 = a^2$

1. Asymptotes are perpendicular lines *i.e.*, $x \pm y = 0$
2. Eccentricity, $e = \sqrt{2}$,
3. Centre, $O(0, 0)$
4. Foci, S and S_1 $(\pm\sqrt{2}a, 0)$
5. Directrices, $x = \pm\frac{a}{\sqrt{2}}$
6. Latusrectum $= 2a$
7. Point form, $x = x_1, y = y_1$
   Equation of tangent, $xx_1 - yy_1 = a^2$
   Equation of normal, $\frac{x_1}{x} + \frac{y_1}{y} = 2$
8. Parametric form, $x = a\sec\theta, y = a\tan\theta$
   Equation of tangent, $x\sec\theta - y\tan\theta = a$
   Equation of normal, $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$



Rectangular Hyperbola of the Form $xy = c^2$

1. Asymptotes are perpendicular lines *i.e.*, $x = 0$ and $y = 0$
2. Eccentricity, $e = \sqrt{2}$
3. Centre, $(0, 0)$
4. Foci $S(\sqrt{2}c, \sqrt{2}c)$, $S_1(-\sqrt{2}c, -\sqrt{2}c)$
5. Vertices, $V_1(c, c)$, $V_2(-c, -c)$



6. Directrices, $x + y = \pm\sqrt{2}c$
7. Latusrectum $= 2\sqrt{2}c$
8. Point form, $x = x_1, y = y_1$
   Equation of tangent, $xy_1 + yx_1 = 2c^2$
   $\Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$
   Equation of Normal, $xx_1 - yy_1 = x_1^2 - y_1^2$
9. Parametric form : $x = ct, y = \frac{c}{t}$
   Equation of tangent, $x + yt^2 = 2ct$
   Equation of normal, $t^2x - y = c\left(t^3 - \frac{1}{t}\right)$

- *Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.*
- *If a hyperbola is equilateral, then the conjugate hyperbola is also equilateral.*

Asymptotes

An asymptote to a curve is a straight line, at a finite distance from the ***origin****, to which the tangent to a curve tends as the point of contact goes to infinity.*

Results on Asymptotes

1. If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi-conjugate axis.
2. Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.
3. If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ, then $e = \sec\theta$.
4. A hyperbola and its conjugate hyperbola have the same asymptotes.
5. The angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
6. Asymptotes always passes through the centre of the hyperbola.
7. The bisectors of the angle between the asymptotes are the coordinate axes.
8. The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extrimities of each axis parallel to the other axis.

Step to Determine the Asymptotes

- ***Step* I** Determine the degree of the equation of the curve. Let it is n.
- ***Step* II** If x^n is present in the equation of the curve, then there is no asymptote parallel to X-axis.
- ***Step* III** If the term of x^n is missing, then equate to zero the coefficient of x^{n-1}, this will give the asymptote parallel to X-axis.

Similarly, we can find the asymptote parallel to Y-axis.

Practice Zone DAY 29

1. Length of the latusrectum of the hyperbola $xy - 3x - 4y + 8 = 0$ is
(a) 4 (b) $4\sqrt{2}$
(c) 8 (d) None of these

2. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, |r| < 1$ represents
(a) an ellipse (b) a hyperbola
(c) a circle (d) None of these

3. The angle between the rectangular hyperbolas $(y - mx)(my + x) = a^2$ and $(m^2 - 1)(y^2 - x^2) + 4mxy = b^2$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

4. The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci is **[NCERT Exemplar]**
(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$ (d) None of these

5. A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S. Then, $CP^2 + CQ^2 + CR^2 + CS^2$ is equal to
(a) r^2 (b) $2r^2$
(c) $3r^2$ (d) $4r^2$

6. Given the base BC of ΔABC and if $\angle B - \angle C = K$, a constant, then locus of the vertex A is a hyperbola.
(a) No (b) Yes
(c) Both (d) None of these

7. If a line $21x + 5y = 116$ is tangent to the hyperbola $7x^2 - 5y^2 = 232$, then point of contact is
(a) $(-6, 3)$ (b) $(6, -2)$
(c) $(8, 2)$ (d) None of these

8. If the equation of hyperbola is $9y^2 - 4x^2 = 36$, then vertices and length of latusrectum is **[NCERT]**
(a) $(0, +2), 0$ (b) $(0, \pm 2), 9$
(c) $(0, -2), 0$ (d) None of these

9. Equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is **[NCERT Exemplar]**
(a) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$ (b) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$
(c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (d) None of these

10. The product of the length of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes, is
(a) $\frac{1}{2}$ (b) $\frac{2}{3}$
(c) $\frac{3}{2}$ (d) 2

11. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then, its equation is
(a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$
(b) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$
(c) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$
(d) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

12. If e_1 is the eccentricity of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and e_2 is the eccentricity of the hyperbola passing through the foci of the ellipse and $e_1 e_2 = 1$, then equation of the hyperbola is
(a) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = -1$
(c) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (d) None of these

13. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is
(a) $(-2, \sqrt{6})$ (b) $(-5, 2\sqrt{6})$
(c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (d) $(4, -\sqrt{6})$

14. The common tangent to $9x^2 - 4y^2 = 36$ and $x^2 + y^2 = 3$ is
(a) $y - 2\sqrt{3}x - \sqrt{39} = 0$
(b) $y + 2\sqrt{3}x + \sqrt{39} = 0$
(c) $y - 2\sqrt{3}x + \sqrt{39} = 0$
(d) None of the above

15. The locus of the points of intersection of perpendicular tangents to $\frac{x^2}{16}-\frac{y^2}{9}=1$ is

(a) $x^2-y^2=7$ (b) $x^2-y^2=25$
(c) $x^2+y^2=25$ (d) $x^2+y^2=7$

16. If P is a point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ and N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT\cdot ON$ is equal to

(a) 9 (b) 4
(c) e^2 (d) None of these

17. Let $P(a\sec\theta,\ b\tan\theta)$ and $Q(a\sec\phi,\ b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If $(h,\ k)$ is the point of intersection of normals at P and Q, then k is

(a) $\frac{a^2+b^2}{a}$ (b) $-\frac{(a^2+b^2)}{a}$
(c) $\frac{a^2+b^2}{b}$ (d) $-\frac{(a^2+b^2)}{b}$

18. If $x=9$ is the chord of contact of tangents of $x^2-y^2=9$, then the equation of the corresponding tangents is

(a) $9x^2-8y^2+18x+9=0$ (b) $9x^2-8y^2-18x+9=0$
(c) $9x^2-8y^2-18x-9=0$ (d) $9x^2-8y^2+18x-9=0$

19. If the latusrectum subtends a right angle at the centre of a hyperbola, then its eccentricity is

(a) $\frac{\sqrt{3}+1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$
(c) $\frac{\sqrt{5}+\sqrt{2}}{2}$ (d) $\frac{\sqrt{3}+\sqrt{2}}{2}$

20. Tangents are drawn from points on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ to the circle $x^2+y^2=9$. The locus of the mid-point of the chord of contact is

(a) $x^2+y^2=\frac{x^2}{9}-\frac{y^2}{4}$
(b) $(x^2+y^2)^2=\frac{x^2}{9}-\frac{y^2}{4}$
(c) $(x^2+y^2)^2=81\left(\frac{x^2}{9}-\frac{y^2}{4}\right)$
(d) $(x^2+y^2)^2=9\left(\frac{x^2}{9}-\frac{y^2}{4}\right)$

21. Equation of a rectangular hyperbola whose asymptotes are $x=3$ and $y=5$ and passing through (7, 8) is

(a) $xy-3y+5x+3=0$ (b) $xy+3y+4x+3=0$
(c) $xy-3y+5x-3=0$ (d) $xy-3y-5x+3=0$

22. If two points P and Q on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, whose centre C be such that CP is perpendicular to CQ, $a<b$, then $\frac{1}{CP^2}+\frac{1}{CQ^2}$ is equal to

(a) $-\frac{1}{a^2}+\frac{1}{b^2}$ (b) $\frac{1}{a^2}+\frac{1}{b^2}$
(c) $-\frac{1}{a^2}-\frac{1}{b^2}$ (d) $\frac{1}{a^2}-\frac{1}{b^2}$

23. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Then, the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is

(a) $(y^2-x^2)^2=4x^2(x^2-a^2)$ (b) $(x^2-y^2)^2=x^2(x^2-a^2)$
(c) $(x^2-y^2)=4x^2(x^2-a^2)$ (d) None of these

24. If the lines $x=h$ and $y=k$ are conjugate with respect to the hyperbola $xy=c^2$, then the locus of (h,k) is

(a) $xy=2c^2$ (b) $xy=c^2$
(c) $2xy=c^2$ (d) $xy=4c^2$

25. A triangle is inscribed in the rectangular hyperbola $xy=c^2$, such that two of its sides are parallel to the lines $y=m_1x$ and $y=m_2x$. Then, the third side of the triangle touches the hyperbola

(a) $xy=\left\{\frac{c^2(m_1+m_2)^2}{4m_1m_2}\right\}$ (b) $xy=\left\{\frac{c^2(m_1-m_2)^2}{4m_1m_2}\right\}$
(c) $xy=\left\{\frac{c^2(m_1-m_2)^2}{m_1m_2}\right\}$ (d) $xy=\left\{\frac{c^2(m_1+m_2)^2}{m_1m_2}\right\}$

26. From the centre C of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ perpendicular CN is drawn on any tangent to it at the point $P(a\sec\theta, b\tan\theta)$ in the first quadrant. Then, the value of θ, so that area of ΔCPN is maximum, is

(a) $\sin^{-1}\left(\frac{a}{b}\right)$ (b) $\sin^{-1}\left(\frac{b}{a}\right)$ (c) $\cos^{-1}\left(\frac{a}{b}\right)$ (d) $\tan^{-1}\left(\frac{b}{a}\right)$

27. A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy=4$ is equal to the sum of ordinates of feet of normals. Then, the locus of P is

(a) a parabola (b) a hyperbola
(c) an ellipse (d) a circle

28. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent nearer to the point P_1 to the hyperbola $x^2-y^2=1$ and the circle $x^2+y^2=1$. Find the equation of the ellipse.

(a) $(3x+1)^2+12(y+1)^2=1$
(b) $(2x-1)^2+4(y-1)^2=(x-1)^2$
(c) $9x^2+6x+12y+1=0$
(d) $(3x+1)^2-12(y+1)^2=1$

29. A tangent to $x^2 = 4ay$ meets the curve $xy = c^2$ at two points P and Q. The locus of the mid-point of PQ is

(a) $ay - 2x^2 = 0$
(b) $ay + 2x^2 = 0$
(c) $-ax + by^2 = 0$
(d) None of these

30. If tangent and normal to a rectangular hyperbola $xy = c^2$ cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other, then

(a) $a_1 = b_1$
(b) $a_2 = b_2$
(c) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$
(d) $a_1a_2 + b_1b_2 = 0$

31. If a variable circle $x^2 + y^2 - 2ax + 4ay = 0$ intersects the hyperbola $xy = 4$ at the points (x_i, y_i), $= 1, 2, 3, 4$, then locus of the point $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right)$ is

(a) $y + 2x = 0$
(b) $y - 2x + 5 = 0$
(c) $y - 2x = 0$
(d) $y + 4x - 7 = 0$

32. The diameter of the hyperbola $9x^2 - 16y^2 = 144$ which is conjugate to $y = 2x$ is

(a) $y = \frac{8}{9}x$
(b) $y = \frac{9}{8}x$
(c) $y = \frac{7}{8}x$
(d) None of these

33. The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M, respectively. If locus of the mid-point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\left(\frac{e+1}{e-1}\right)$
(b) $\frac{e}{\sqrt{(e^2-1)}}$
(c) e
(d) None of these

34. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the ΔABC is

(a) $\left(1 - \sqrt{\frac{2}{3}}\right)$ sq unit
(b) $\left(\sqrt{\frac{3}{2}} - 1\right)$ sq unit
(c) $\left(1 + \sqrt{\frac{2}{3}}\right)$ sq units
(d) $\left(\sqrt{\frac{3}{2}} + 1\right)$ sq units

Directions (Q. Nos. 35 and 36) *Let the curves be*

$$H : x^2 - y^2 = 9, P : y^2 = 4(x - 5), L : x = 9.$$

35. If L is the chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$

36. If R is the point of intersection of the tangents of H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$
(b) $x = 9$
(c) $y = 7$
(d) $y = 9$

Directions (Q. Nos. 37 to 39) *For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normal at $P(\theta)$ meets the transverse axis AA' in G and the conjugate axis BB' in g and CF be perpendicular to the normal from the centre.*

37. $PF^2 . PG^2 = K \sec\theta \cdot \tan\theta \, CB^4$, then K is equal to

(a) $\frac{(a-b)}{ab}$
(b) $\frac{(b-a)}{ab}$
(c) $\frac{ab}{(a-b)}$
(d) $\frac{ab}{(b-a)}$

38. $PG : Pg$ is equal to

(a) $b^2 : a^2$
(b) $a^2 : b^2$
(c) $b : a$
(d) $a : b$

39. Locus of middle point of G and g is a hyperbola of eccentricity

(a) $\frac{1}{\sqrt{e^2-1}}$
(b) $\frac{e}{\sqrt{e^2-1}}$
(c) $2\sqrt{e^2-1}$
(d) $\frac{e}{2}$

Directions (Q. Nos. 40 to 43) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

40. Statement I A bullet is fired and hit a target. An observer in the same thunder plane heared two sounds the crack of the rifle and the thud of the bullet striking the target at the same instant, then locus of the observer is hyperbola where velocity of sound is smaller than velocity of bullet.

Statement II If difference of distances of a point P from the two fixed points is constant and less than the distance between the fixed points, then locus of P is a hyperbola.

41. Consider e and e' are the eccentricities of hyperbola and its conjugate hyperbola

Statement I If $e = 2$, then $e' = \frac{2}{\sqrt{3}}$.

Statement II $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

42. Consider the director circle to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$.

Statement I The equation of the director circle to the hyperbola $4x^2-3y^2=12$ is $x^2+y^2=1$.

Statement II Director circle is the locus of the point of intersection of perpendicular tangents to a hyperbola.

43. **Statement I** A hyperbola and its conjugate hyperbola have the same asymptotes.

Statement II The difference between the second degree curve and pair of asymptotes is constant.

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44. A tangent to the hyperbola $\frac{x^2}{4}-\frac{y^2}{2}=1$ meets X-axis at P and Y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where, O is the origin). Then, R lies on **[JEE Main 2013]**

(a) $\frac{4}{x^2}+\frac{2}{y^2}=1$ (b) $\frac{2}{x^2}-\frac{4}{y^2}=1$
(c) $\frac{2}{x^2}+\frac{4}{y^2}=1$ (d) $\frac{4}{x^2}-\frac{2}{y^2}=1$

45. A common tangent to the conics $x^2=6y$ and $2x^2-4y^2=9$ is **[JEE Main 2013]**

(a) $x-y=\frac{3}{2}$
(b) $x+y=1$
(c) $x+y=\frac{9}{2}$
(d) $x-y=1$

46. The equation of the hyperbola whose foci are $(-2, 0)$ and $(2, 0)$ and eccentricity 2 is given by **[AIEEE 2011]**

(a) $-3x^2+y^2=3$
(b) $x^2-3y^2=3$
(c) $3x^2-y^2=3$
(d) $-x^2+3y^2=3$

47. For the hyperbola $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$. Which of the following remains constant when α varies? **[AIEEE 2007]**

(a) Directrix (b) Abscissae of vertices
(c) Abscissae of foci (d) Eccentricities

48. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is **[AIEEE 2005]**

(a) circle (b) ellipse
(c) parabola (d) hyperbola

49. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Then, the value of b^2 is **[AIEEE 2003]**

(a) 1 (b) 5 (c) 7 (d) 9

50. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy=c^2$ is **[AIEEE 2002]**

(a) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ (b) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$
(c) $\frac{x}{y_1+y_2}+\frac{y}{x_1+x_2}=1$ (d) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

Answers

1. (b)	2. (b)	3. (a)	4. (c)	5. (d)	6. (b)	7. (b)	8. (b)	9. (a)	10. (b)
11. (a)	12. (b)	13. (d)	14. (a)	15. (d)	16. (a)	17. (d)	18. (b)	19. (b)	20. (c)
21. (d)	22. (d)	23. (a)	24. (a)	25. (a)	26. (b)	27. (a)	28. (b)	29. (b)	30. (d)
31. (a)	32. (d)	33. (b)	34. (b)	35. (b)	36. (b)	37. (b)	38. (a)	39. (b)	40. (a)
41. (a)	42. (d)	43. (b)	44. (d)	45. (a)	46. (c)	47. (c)	48. (d)	49. (c)	50. (a)

Hints & Solutions

1. Given equation can be rewritten as $(x-4)(y-3)=4$

which is a rectangular hyperbola of the type $xy = c^2$.

$\therefore \quad c = 2$

Then, $\quad a = b = c\sqrt{2} = 2\sqrt{2}$

$\therefore$ Length of latusrectum $= \dfrac{2b^2}{a} = 2a = 4\sqrt{2}$

2. As $|r| < 1$, then $1 - r > 0, 1 + r > 0$

Hence, it represents a hyperbola.

3. On differentiating first equation, we get

$$(y - mx)\left(m\frac{dy}{dx} + 1\right) + (my + x)\left(\frac{dy}{dx} - m\right) = 0$$

$$\Rightarrow \frac{dy}{dx}(my + x + my - m^2x) + y - mx - m^2y - mx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1 \quad \text{(say)}$$

On differentiating second equation, we get

$$(m^2 - 1)\left(2y\frac{dy}{dx} - 2x\right) + 4m\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx}(2y(m^2 - 1) + 4mx) = -4my + 2x(m^2 - 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2 \text{(say)}$$

$$\because \quad m_1 m_2 = -1$$

So, angle between the hyperbola $= \dfrac{\pi}{2}$.

4. Let equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Given, $\quad \dfrac{2b^2}{a} = 8$

$\Rightarrow \quad \dfrac{b^2}{a} = 4$ and $2b = \dfrac{1}{2}(2\,ae)$

$\Rightarrow \quad 2b = ae \Rightarrow 4b^2 = a^2e^2$

$\Rightarrow \quad 4\left(\dfrac{b^2}{a^2}\right) = e^2$

$\Rightarrow \quad 4(e^2 - 1) = e^2 \quad [\because b^2 = a^2(e^2 - 1)]$

$\Rightarrow \quad 3e^2 = 4$

$\Rightarrow \quad e^2 = \dfrac{4}{3} \Rightarrow e = \dfrac{2}{\sqrt{3}}$

5. Let equation of the rectangular hyperbola be

$$xy = c^2 \quad \text{...(i)}$$

and equation of circle be $x^2 + y^2 = r^2$. ...(ii)

From Eq. (i) and (ii) eliminating y, we get

$$x^4 - r^2x^2 + c^4 = 0 \quad \text{...(iii)}$$

Let x_1, x_2, x_3 and x_4 are the roots of Eq. (iii).

$\therefore \quad$ Sum of roots $= \sum_{i=1}^{4} x_i = 0$

Sum of products of the roots taken two at a time

$$= \Sigma\, x_1x_2 = -r^2$$

From Eq. (i) and (ii) eliminating x, we get

$$y^4 - r^2y^2 + c^4 = 0 \quad \text{...(iv)}$$

Let y_1, y_2, y_3 and y_4 are the roots of Eq. (iv).

$\therefore \quad \sum_{i=1}^{4} y_i = 0$ and $\Sigma\, y_1y_2 = -r^2$

Now, $\quad CP^2 + CQ^2 + CR^2 + CS^2$

$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + x_4^2 + y_4^2$

$= (x_1^2 + x_2^2 + x_3^2 + x_4^2 + (y_1^2 + y_2^2 + y_3^2 + y_4^2)$

$= \left[(\sum_{i=1}^{4} x_i)^2 - 2\,\Sigma\, x_1x_2\right] + \left[(\sum_{i=1}^{4} y_i)^2 - 2\,\Sigma\, y_1y_2\right]$

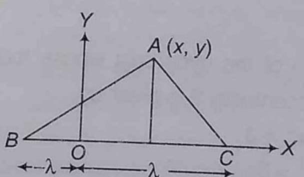
$= (0 + 2r^2) + (0 + 2r^2)$

$[\because (\sum_{i=1}^{4} xi) = 0, \Sigma\, x_1x_2 = -r^2, \sum_{i=1}^{4} y_i = 0, \Sigma\, y_1y_2 = -r^2]$

$= 4r^2$

6. Let $B(-\lambda, 0)$, $C(\lambda, 0)$ and $A(x, y)$

Given, $K = B - C$



$$\therefore \quad \tan K = \frac{\tan B - \tan C}{1 + \tan B \cdot \tan C} = \frac{\dfrac{y}{\lambda + x} - \dfrac{y}{\lambda - x}}{1 + \dfrac{y^2}{\lambda^2 - x^2}}$$

$\Rightarrow \quad \lambda^2 - x^2 + y^2 = -2\,xy\cot K$

$\Rightarrow \quad x^2 - 2\cot K \cdot xy - y^2 = \lambda^2$

which is a hyperbola.

7. Here, $a^2 = \dfrac{232}{7}$, $b^2 = \dfrac{232}{5}$ and $y = -\dfrac{21}{5}x + \dfrac{116}{5}$ with slope $-\dfrac{21}{5}$

Now, $a^2m^2 - b^2 = \left(\dfrac{116}{5}\right)^2$ (since line is tangent)

If (x_1, y_1) is the point of contact, then tangent is $S_1 = 0$

$\therefore \quad 7x_1x - 5y_1y = 232$

On comparing it with $21x + 5y = 116$, we get

$$x_1 = 6, y_1 = -2$$

So, the point of contact is $(6, -2)$.

8. Given equation is $9y^2 - 4x^2 = 36$.

Divide it by 36, we get

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36}$$

$$\Rightarrow \quad \frac{y^2}{4} - \frac{x^2}{9} = 1$$

Now, comparing $\frac{y^2}{4} - \frac{x^2}{9} = 1$ with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get

$$b^2 = 4 \text{ and } a^2 = 9$$

$$\Rightarrow \quad b = 2 \text{ and } a = 3$$

Here, in hyperbolic equation coefficient of y^2 is positive, so transverse axis is along Y-axis.

$$\text{Vertices} = (0, \pm b) = (0, \pm 2)$$

and $\quad \text{latusrectum} = \frac{2a^2}{b} = \frac{2 \times 9}{2} = 9$

9. Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \text{...(i)}$$

Given, $e = \frac{3}{2}$ and foci $= (\pm ae, 0) = (\pm 2, 0)$

$$\therefore \quad e = \frac{3}{2} \text{ and } ae = 2$$

$$\Rightarrow \quad a \times \frac{3}{2} = 2 \Rightarrow a^2 = \frac{16}{9}$$

Now, $\quad b^2 = a^2(e^2 - 1)$

$$\Rightarrow \quad b^2 = \frac{16}{9}\left(\frac{9}{4} - 1\right) = \frac{16}{9} \times \frac{5}{4} = \frac{20}{9}$$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{16/9} - \frac{y^2}{20/9} = 1$$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$$

10. Given, equation can be rewritten as

$$\frac{x^2}{2} - \frac{y^2}{1} = 1$$

Here, $\quad a^2 = 2, b^2 = 1$

We know that, the length of perpendicular drawn from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the asymptotes is

$$\frac{a^2 b^2}{a^2 + b^2} = \frac{2(1)}{2+1} = \frac{2}{3}$$

11. Here, $a = \sin\theta$

Since, foci of the ellipse are $(\pm 1, 0)$.

$$\therefore \quad \pm 1 = \sqrt{a^2 + b^2} \Rightarrow b^2 = \cos^2\theta$$

Then, equation is $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

$$\Rightarrow x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$$

12. Here, $e_1 = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

$$\because \quad e_1 e_2 = 1 \Rightarrow e_2 = \frac{5}{3}$$

Since, foci of ellipse are $(0, \pm 3)$.

Hence, equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = -1$.

13. As we know, equation of tangent at (x_1, y_1) is $xx_1 - 2yy_1 = 4$, which is same as $2x + \sqrt{6}y = 2$

$$\therefore \quad \frac{x_1}{2} = -\frac{2y_1}{\sqrt{6}} = \frac{4}{2}$$

$$\Rightarrow \quad x_1 = 4 \text{ and } y_1 = -\sqrt{6}$$

14. Suppose the common tangent is $y = mx + c$ to

$$9x^2 - 4y^2 = 36 \text{ and } x^2 + y^2 = 3$$

$$\therefore \quad c^2 = a^2m^2 - b^2 = 4m^2 - 9 \quad \text{...(i)}$$

and $\quad c^2 = 3 + 3m^2 \quad \text{...(ii)}$

From Eqs. (i) and (ii), we get

$$4m^2 - 9 = 3m^2 + 3$$

$$\Rightarrow \quad m^2 = 12 \Rightarrow m = 2\sqrt{3}$$

$$\therefore \quad c = \sqrt{3 + 3 \times 12} = \sqrt{39}$$

Hence, the common tangent is

$$y = 2\sqrt{3}x + \sqrt{39}$$

15. The equation of tangent in slope form to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is $y = mx + \sqrt{16m^2 - 9}$

Since, it passes through (h, k).

$$\therefore \quad k = mh + \sqrt{16m^2 - 9^2}$$

$$\Rightarrow \quad (k - mh)^2 = (16m^2 - 9^2)$$

$$\Rightarrow \quad k^2 + m^2h^2 - 2mkh - 16m^2 + 9^2 = 0$$

It is quadratic in m and let the slope of two tangents be m_1 and m_2, then

$$m_1 m_2 = \frac{k^2 + 9}{h^2 - 16}$$

$$\Rightarrow \quad -1 = \frac{k^2 + 9}{h^2 - 16}$$

$$\Rightarrow \quad h^2 + k^2 = 7$$

The required locus is $x^2 + y^2 = 7$

16. The point on the hyperbola is $P(x_1, y_1)$, then N is $(x, 0)$.

$\therefore$ Tangent at (x_1, y_1) is $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$

This meets x-axis at $T\left(\frac{9}{x_1}, 0\right)$

$$\therefore \quad OT \cdot ON = \frac{9}{x_1} \cdot x_1 = 9$$

17. Equation of normal at θ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$ and normal at $\phi=\frac{\pi}{2}-\theta$ is $\frac{ax}{\text{cosec}\theta}+\frac{by}{\cot\theta}=a^2+b^2$

Eliminating x, we get

$$by\left(\frac{1}{\sin\theta}-\frac{1}{\cos\theta}\right)=(a^2+b^2)\left(\frac{1}{\cos\theta}-\frac{1}{\sin\theta}\right)$$

$$\Rightarrow \quad by=-(a^2+b^2) \text{ or } k=-\frac{(a^2+b^2)}{b}$$

18. Let $P(x_1, y_1)$ be the point from which tangents are drawn.

The chord of contact is $S_1=-8$.

i.e., $xx_1-yy_1=9$.

It is same as $x=9$.

$\therefore \quad x_1=1, y_1=0$

Equation of tangents is $T_1^2=SS_1$.

$$\therefore \quad (x-9)^2=-8(x^2-y^2-9)$$

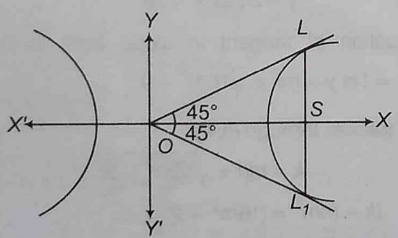
$$\Rightarrow \quad 9x^2-8y^2-18x+9=0$$

19. Since, $\angle LOS=45^\circ$

$$\because \quad LS=OS$$

$$\Rightarrow \quad a(e^2-1)=ae$$

$$\therefore \quad e^2-e-1=0$$



So, the required eccentricity, $e=\frac{1+\sqrt{5}}{2}$

20. Any point on the hyperbola is $(3\sec\theta, 2\tan\theta)$. The chord of contact to the circle is $3x\sec\theta+2y\tan\theta=9$ ...(i)

If (x_1, y_1) is the mid-point of the chord, then its equation is

$$xx_1+yy_1=x_1^2+y_1^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{3\sec\theta}{x_1}=\frac{2\tan\theta}{y_1}=\frac{9}{x_1^2+y_1^2}$$

Eliminating θ, $\frac{1}{81}(x_1^2+y_1^2)^2=\frac{x_1^2}{9}-\frac{y_1^2}{4}$

Hence, the locus is $(x^2+y^2)^2=81\left(\frac{x^2}{9}-\frac{y^2}{4}\right)$.

21. The equation of rectangular hyperbola is

$$(x-3)(y-5)+\lambda=0$$

which passes through (7, 8).

$$\therefore \quad 4.3+\lambda=0$$

$$\Rightarrow \quad \lambda=-12$$

Then, $\quad xy-5x-3y+15-12=0$

$$\Rightarrow \quad xy-3y-5x+3=0$$

22. Let $CP=r_1$ and $CQ=r_2$, CP be inclined to transverse axis at an angle θ, so that P is $(r_1\cos\theta, r_1\sin\theta)$ and P lies on the hyperbola.

$$\Rightarrow \quad r_1^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)=1$$

Replacing θ by $90^\circ+\theta$, we get $r_2^2\left(\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}\right)=1$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\cos^2\theta\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\sin^2\theta\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\therefore \quad \frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

23. The equation of a hyperbola of the series is $\frac{x^2}{a^2}-\frac{y^2}{\lambda^2}=1$ where, λ is a parameter.

The asymptotes of this hyperbola $\frac{x}{a}=\pm\frac{y}{\lambda}$.

Suppose (x', y') is a point P on the hyperbola which is equidistant from the transverse axis and asymptote. Then,

$$\frac{x'^2}{a^2}-\frac{y'^2}{\lambda^2}=1 \quad \text{...(i)}$$

and
$$y'=\frac{\frac{x'}{a}-\frac{y'}{\lambda}}{\sqrt{\frac{1}{a^2}+\frac{1}{\lambda^2}}} \quad \text{...(ii)}$$

i.e.,
$$\frac{y'^2}{\lambda^2}=\frac{x'^2}{a^2}-1 \quad \text{[from Eq. (i)] ...(iii)}$$

and
$$y'^2\left(\frac{1}{a^2}+\frac{1}{\lambda^2}\right)=\frac{x'^2}{a^2}+\frac{y'^2}{\lambda^2}-\frac{2x'y'}{a\lambda} \quad \text{[from Eq. (ii)] ...(iv)}$$

On simplification the second relation gives

$$(y'^2-x'^2)^2=\frac{4x'^2y'^2a^2}{\lambda^2}=4x'^2(x'^2-a^2) \quad \text{[using Eq. (iii)]}$$

So, the locus of P is $(y^2-x^2)^2=4x^2(x^2-a^2)$.

24. Let (x_1, y_1) be the pole of the line $x-h=0$ with respect to the hyperbola $xy=c^2$. Then, the equation of the polar is $xy_1+yx_1=2c^2$.

Thus, the equations $x-h=0$ and $xy_1+yx_1=2c^2$ represent the same line.

$$\therefore \quad \frac{y_1}{1}=\frac{x_1}{0}=\frac{2c^2}{h}$$

$$\Rightarrow \quad x_1=0 \text{ and } y_1=\frac{2c^2}{h}$$

Since, the lines $x - h = 0$ and $y - k = 0$ are conjugate lines. Therefore, the pole of the line $x - h = 0$ lies on the line $y - k = 0$.

$$\therefore \quad y_1 - k = 0 \Rightarrow \frac{2c^2}{h} - k = 0$$

$$\Rightarrow \quad hk = 2c^2$$

Hence, the locus of (h, k) is $xy = 2c^2$.

25. Let $P\left(ct_1, \frac{c}{t_1}\right), Q\left(ct_2, \frac{c}{t_2}\right), R\left(ct_3, \frac{c}{t_3}\right)$ be the vertices of a ΔPQR inscribed in the rectangular hyperbola $xy = c^2$ such that the sides PQ and QR are parallel to $y = m_1 x$ and $y = m_2 x$, respectively.

$$\therefore \quad m_1 = \text{slope of } PQ \quad \text{and} \quad m_2 = \text{slope of } QR$$

$$\Rightarrow \quad m_1 = -\frac{1}{t_1 t_2} \quad \text{and} \quad m_2 = -\frac{1}{t_2 t_3}$$

$$\therefore \quad \frac{m_1}{m_2} = \frac{t_3}{t_1} \Rightarrow t_3 = \left(\frac{m_1}{m_2}\right) t_1$$

The equation of PR is

$$x + y t_1 t_3 = c(t_1 + t_3)$$

$$\Rightarrow \quad x + y\left(\frac{m_1}{m_2}\right) t_1^2 = c\left(t_1 + \frac{m_1}{m_2} t_1\right)$$

$$\Rightarrow \quad x + y\left(\frac{m_1}{m_2}\right) t_1^2 = c\left(1 + \frac{m_1}{m_2}\right) t_1$$

$$\Rightarrow \quad x + y\left(\sqrt{\frac{m_1}{m_2}}\, t_1\right)^2 = 2\left\{\frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}} \sqrt{\frac{m_1}{m_2}} \cdot t_1\right\}$$

$$\Rightarrow \quad x + yt^2 = 2\lambda t,$$

where, $t = \sqrt{\frac{m_1}{m_2}} \cdot t_1$

and $\lambda = \frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}}$

Clearly, it touches the hyperbola,

$$xy = \left\{\frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}}\right\}^2 \text{ or } xy = \left\{\frac{c^2 (m_1 + m_2)^2}{4 m_1 m_2}\right\}$$

26. Equation of tangent is

$$b \sec\theta \cdot x - a \tan\theta \cdot y - ab = 0$$

$$\therefore \quad CN = \frac{ab}{\sqrt{b^2 \sec^2\theta + a^2 \tan^2\theta}}$$

Equation of normal at P is $ax\cos\theta + by\cot\theta = a^2 + b^2$

$$\therefore \quad CM = \frac{a^2 + b^2}{\sqrt{a^2 \cos^2\theta + b^2 \cot^2\theta}}$$

$$\text{area } A = \frac{1}{2} CM \times CN \qquad (\because CM = NP)$$

$$= \frac{ab(a^2 + b^2)}{\sqrt{a^2 b^2 + b^4 \operatorname{cosec}^2\theta + a^4 \sin^2\theta}}$$

So, A is maximum when $b^4 \operatorname{cosec}^2\theta + a^4 \sin^2\theta$ is minimum.

Now, $b^4 \operatorname{cosec}^2\theta + a^4 \sin^2\theta \ge 2a^2 b^2 \quad (\because \text{AM} \ge \text{GM})$

$$\therefore \quad A_{max} = \frac{ab(a^2 + b^2)}{2ab} = \frac{a^2 + b^2}{2}, \quad \text{where } \theta = \sin^{-1}\left(\frac{b}{a}\right)$$

27. Any point on the hyperbola $xy = 4$ is $\left(2t, \frac{2}{t}\right)$. Now, normal at $\left(2t, \frac{2}{t}\right)$ is $y - \frac{2}{t} = t^2(x - 2t)$. (its slope is t^2)

If the normal passes through $P(h, k)$, then

$$k - \frac{2}{t} = t^2(h - 2t)$$

$$\Rightarrow \quad 2t^4 - ht^3 + tk - 2 = 0 \qquad \ldots(\text{i})$$

Roots of Eq. (i) give parameters of feet of normals passing through (h, k). Let roots be t_1, t_2, t_3 and t_4, then

$$t_1 + t_2 + t_3 + t_4 = \frac{h}{2} \qquad \ldots(\text{ii})$$

$$t_1 t_2 + t_2 t_3 + t_1 t_3 + t_1 t_4 + t_2 t_4 + t_3 t_4 = 0 \qquad \ldots(\text{iii})$$

$$t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4 = -\frac{k}{2} \qquad \ldots(\text{iv})$$

$$\text{and} \quad t_1 t_2 t_3 t_4 = -1 \qquad \ldots(\text{v})$$

On dividing Eq. (iv) by Eq. (v), we get

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{2}$$

$$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = k \qquad \left(\because y = \frac{2}{t}\right)$$

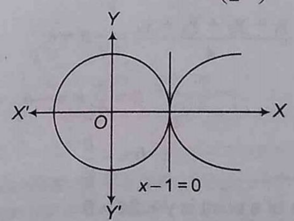
From Eqs. (ii) and (iii),

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{4}$$

Given that, $\frac{h^2}{4} = k$

Hence, locus of (h, k) is $x^2 = 4ay$, which is a parabola.

28. Ellipse has directrix, $x - 1 = 0$, focus $\left(\frac{1}{2}, 1\right)$



Equation of the ellipse is

$$\left(x - \frac{1}{2}\right)^2 + (y - 1)^2 = \frac{1}{4}(x - 1)^2$$

$$\Rightarrow \quad (2x - 1)^2 + 4(y - 1)^2 = (x - 1)^2$$

29. Let (x_1, y_1) be the mid-point of PQ. The equation of PR is $T = S_1$.

$$xy_1 + yx_1 = 2x_1 y_1 \qquad \ldots(\text{i})$$

Eliminating y between Eq. (i) and $x^2 = 4ay$, we get

$$x^2 = 4a\left(2 - \frac{x}{x_1}\right) y_1$$

$\Rightarrow \quad x^2 + 4a\dfrac{y_1}{x_1}\cdot x - 8ay_1 = 0$

Put, discriminant = 0

$\Rightarrow \quad 16a^2\left(\dfrac{y_1}{x_1}\right)^2 + 32ay_1 = 0$

So, the locus of the mid point is $\dfrac{a^2y^2}{x^2} + 2ay = 0.$

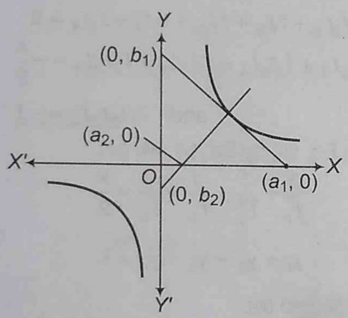
$\therefore \quad ay + 2x^2 = 0$

30. Since, tangent and normal are at perpendicular.

$\therefore$ Product of slopes is -1.

$\Rightarrow \quad \left(-\dfrac{b_1}{a_1}\right)\left(-\dfrac{b_2}{a_2}\right) = -1$

$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$



31. On Putting $y = \dfrac{4}{x}$ in the equation of the circle, we get $x^4 - 2ax^3 + 16ax + 16 = 0$

If x_1, x_2, x_3 and x_4 are the roots of the equation.

Then, $\quad \dfrac{x_1 + x_2 + x_3 + x_4}{4} = \dfrac{a}{2} = h \quad$ (say)

Similarly, $\quad \dfrac{y_1 + y_2 + y_3 + y_4}{4} = -a = k \quad$ (say)

$\Rightarrow \quad h = \dfrac{a}{2}, k = -a$

$\therefore \quad h = -\dfrac{k}{2}$

Hence, locus of a point is $y + 2x = 0$.

32. If $y = m_1x$ and $y = m_2x$ be two conjugate diameters of a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

Then, $\quad m_1m_2 = \dfrac{b^2}{a^2}$

Here, the hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ and diameter is $y = 2x$

$\therefore \quad m_1 = 2, a^2 = 16, b^2 = 9, m_2 = ?$

Then, $\quad 2m_2 = \dfrac{9}{16}$

$\Rightarrow \quad m_2 = \dfrac{9}{32}$

The required diameter is $y = \dfrac{9}{32}x.$

33. Equation of normal at $P(a\sec\phi, b\tan\phi)$ is

$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

Hence, coordinates of L and M are

$$L \equiv \left(\frac{a^2 + b^2}{a}\sec\phi, 0\right)$$

and $\quad M \equiv \left(0, \dfrac{a^2 + b^2}{b}\tan\phi\right)$, respectively.

Let mid-point of ML is $Q(h, k)$, then $h = \dfrac{(a^2 + b^2)}{2a}\sec\phi$

$\Rightarrow \quad \sec\phi = \dfrac{2ah}{(a^2 + b^2)} \quad \ldots$(i)

and $\quad k = \dfrac{(a^2 + b^2)}{2b}\tan\phi$

$\Rightarrow \quad \tan\phi = \dfrac{2bk}{(a^2 + b^2)} \quad \ldots$(ii)

From Eqs. (i) and (ii), we get

$$\sec^2\phi - \tan^2\phi = \frac{4a^2h^2}{(a^2 + b^2)^2} - \frac{4b^2k^2}{(a^2 + b^2)^2}$$

Hence, locus is $\dfrac{x^2}{\left(\dfrac{a^2 + b^2}{2a}\right)^2} - \dfrac{y^2}{\left(\dfrac{a^2 + b^2}{2b}\right)^2} = 1$

Let eccentricity is e_1,

$\therefore \quad \left(\dfrac{a^2 + b^2}{2b}\right)^2 = \left(\dfrac{a^2 + b^2}{2a}\right)^2 (e_1^2 - 1)$

$\Rightarrow \quad a^2 = b^2(e_1^2 - 1)$

$\Rightarrow \quad a^2 = a^2(e^2 - 1)(e_1^2 - 1) \quad [\because b^2 = a^2(e^2 - 1)]$

$\Rightarrow \quad e^2e_1^2 - e^2 - e_1^2 + 1 = 1$

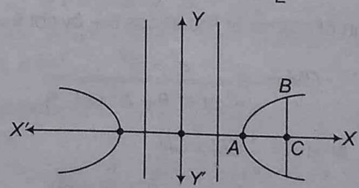
$\Rightarrow \quad e_1^2(e^2 - 1) = e^2$

$\therefore \quad e_1 = \dfrac{e}{\sqrt{(e^2 - 1)}}$

34. The given equation can be rewritten as

$$\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$$

For $A(x, y)$,



$$e = \sqrt{1 + \frac{2}{4}} = \sqrt{\frac{3}{2}}$$

$\therefore \quad x - \sqrt{2} = 2 \Rightarrow x = 2 + \sqrt{2}$

For $C(x, y)$, $x - \sqrt{2} = ae = \sqrt{6}$

$\therefore \quad x = \sqrt{6} + \sqrt{2}$

Now, $AC = \sqrt{6} + \sqrt{2} - 2 - \sqrt{2} = \sqrt{6} - 2$ and $BC = \dfrac{b^2}{a} = \dfrac{2}{2} = 1$

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2} \times AC \times BC$

$$= \frac{1}{2} \times (\sqrt{6} - 2) \times 1 = \left(\sqrt{\frac{3}{2}} - 1\right) \text{ sq unit}$$

35. Let $R(h, k)$ be the point of intersection of the tangents to the extremities of the chord $L : x = 9$ to the hyperbola, then equation of L is $hx - ky = 9 \Rightarrow h = 1$ and $k = 0$.

$\therefore$ Coordinates of R are $(1, 0)$.

Equation of the pair of tangents from R to the hyperbola is

$$(x^2 - y^2 - 9)(1 - 9) = (x - 9)^2 \qquad (\because SS_1 = T^2)$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

36. The equation of chord of contact at $(1, 0)$ to the parabola is

$$y \times 0 = 2(x + 1) - 20$$

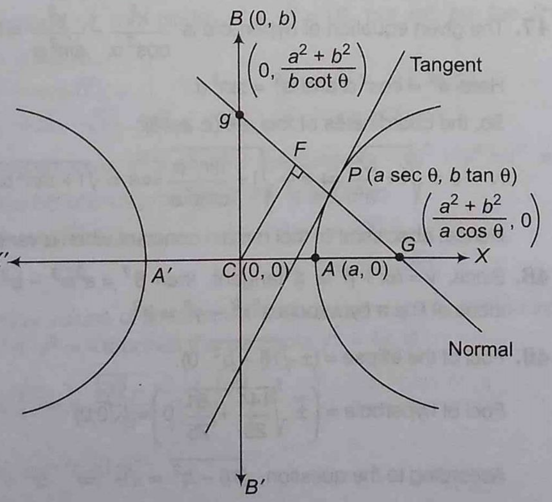
$$\therefore \quad x = 9$$

37. Equation of tangent of hyperbola at $P(\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \qquad \text{...(i)}$$

and equation of normal of hyperbola at $P(\theta)$ is

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \qquad \text{...(ii)}$$



By foot of perpendicular formula

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \left|\frac{(ax_1 + by_1 + c)}{a^2 + b^2}\right|$$

For point F; $\dfrac{x - 0}{a\cos\theta} = \dfrac{y - 0}{b\cot\theta} = \dfrac{a^2 + b^2}{a^2\cos^2\theta + b^2\cot^2\theta}$

$$\therefore \quad F \to \left\{\frac{(a^2 + b^2)a\cos\theta}{(a^2\cos^2\theta + b^2\cot^2\theta)}, \frac{(a^2 + b^2)b\cot\theta}{(a^2\cos^2\theta + b^2\cot^2\theta)}\right\}$$

$$\therefore \quad PF^2 = \left\{\frac{a}{\cos\theta} - \frac{(a^2 + b^2)a\cos\theta}{(a^2\cos^2\theta + b^2\cot^2\theta)}\right\}^2 + \left\{\frac{b}{\cot\theta} - \frac{(a^2 + b^2)b\cot\theta}{(a^2\cos^2\theta + b^2\cot^2\theta)}\right\}^2$$

$$= \frac{ab\,(b - a)\,(\cot^2\theta - \cos^2\theta)}{\cos\theta \cdot (a^2\tan^2\theta + b^2\sec^2\theta) \cdot \cot\theta}$$

$$\Rightarrow \quad PF^2\sec^2\theta \cdot \tan^2\theta = \frac{ab\,(b - a)}{\cos\theta \cdot \cot\theta\,(a^2\sec^2\theta + b^2\tan^2\theta)}$$

and $\quad PG^2 = \left\{\dfrac{a}{\cos\theta} - \dfrac{a^2 + b^2}{a\cos\theta}\right\}^2 + (b\tan\theta)^2$

$$= \frac{b^2}{a^2}(b^2\sec^2\theta + a^2\tan^2\theta)$$

$$Pg^2 = \{a^2\sec^2\theta\} + \left\{\frac{a^2 + b^2}{b\cot\theta} - b\tan\theta\right\}^2$$

$$= \frac{a^2}{b^2}(b^2\sec^2\theta + a^2\tan^2\theta)$$

So, $\quad PF^2 \cdot PG^2 = \dfrac{b^4(b - a)}{ab} \cdot \sec\theta \cdot \tan\theta \qquad \text{...(iii)}$

and $\quad PF^2 \cdot Pg^2 = \dfrac{a^4(b - a)}{ab} \cdot \sec\theta \cdot \tan\theta \qquad \text{...(iv)}$

$\therefore \quad PF^2 \cdot PG^2 = CB^4\left(\dfrac{b - a}{ab}\right) \cdot \sec\theta \cdot \tan\theta \qquad (\because CB^2 = b^2)$

$$k = \frac{b - a}{ab}$$

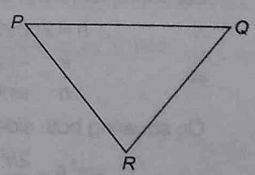
38. Now, from Eqs. (iii) and (iv), we get

$$\frac{PG^2}{Pg^2} = \frac{b^4}{a^4} \Rightarrow PG : Pg = b^2 : a^2$$

39. Locus of middle point is $\dfrac{x^2}{\dfrac{a^2e^4}{4}} - \dfrac{y^2}{\dfrac{a^2e^4}{4b^2}} = 1$

$$\therefore \quad e_1 = \sqrt{\frac{\dfrac{a^2e^4}{4} + \dfrac{a^2e^4}{4b^2}}{\dfrac{a^2e^4}{4}}} = \frac{e}{\sqrt{e^2 - 1}}$$

40. Let P be the position of the gun and Q be the position of the target. Let u be the velocity of sound, v be the velocity of bullet and R be the position of the man then, we have



$$\frac{PR}{u} = \frac{QR}{u} + \frac{PQ}{v}$$

$$\Rightarrow \quad \frac{PR}{u} - \frac{QR}{u} = \frac{PQ}{v}$$

$$\Rightarrow \quad PR - QR = \frac{u}{v}PQ$$

$PQ =$ Constant and $\dfrac{u}{v}PQ < PQ$. $\qquad (\because u < v)$

Hence, locus of R is a hyperbola.

41. Statement II is true.

Since, $\frac{1}{2^2}+\left(\frac{\sqrt{3}}{2}\right)^2=1$

Hence, Statement I is also true.

42. Now, $4x^2-3y^2=12$

$\Rightarrow \frac{x^2}{3}-\frac{y^2}{4}=1$

Then, director circle is $x^2+y^2=3-4=-1$

Does not exist for existence $a>b$.

43. Let hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\lambda=0$.

Then, $\Delta=0$

$\therefore \lambda=0$

$\therefore$ Pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

and equation of conjugate hyperbola is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then, pair of asymptotes is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\mu=0$

Then, $\Delta=0$

$\therefore \mu=0$

$\therefore$ Pair of asymptotes is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

44. Given hyperbola is $\frac{x^2}{4}-\frac{y^2}{2}=1$.

Here, $a^2=4$, $b^2=2 \Rightarrow a=2, b=\sqrt{2}$

The equation of tangent is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

$$\Rightarrow \frac{x}{2}\sec\theta-\frac{y}{\sqrt{2}}\tan\theta=1$$

So, the coordinates of P and Q are $P(2\cos\theta, 0)$ and $Q(0, -\sqrt{2}\cot\theta)$, respectively.

Let coordinates of R are (h, k).

$\therefore h=2\cos\theta, k=-\sqrt{2}\cot\theta$

$\Rightarrow \frac{k}{h}=\frac{-\sqrt{2}}{\sin\theta} \Rightarrow \sin\theta=\frac{-\sqrt{2}h}{2k}$

On squaring both sides, we get

$$\sin^2\theta=\frac{2h^2}{4k^2} \Rightarrow 1-\cos^2\theta=\frac{2h^2}{4k^2}$$

$$\Rightarrow 1-\frac{h^2}{4}=\frac{2h^2}{4k^2} \Rightarrow \frac{2h^2}{4k^2}+\frac{h^2}{4}=1$$

$$\Rightarrow \frac{h^2}{4}\left(\frac{2}{k^2}+1\right)=1 \Rightarrow \frac{2}{k^2}+1=\frac{4}{h^2} \Rightarrow \frac{4}{R^2}-\frac{2}{k^2}$$

Hence, R lies on $\frac{4}{x^2}-\frac{2}{y^2}=1$.

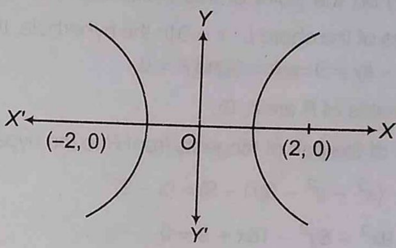
45. $\frac{x^2}{9/2}-\frac{y^2}{9/4}=1$

Equation of tengent in $y=mx\pm\sqrt{\frac{9}{2}m^2-\frac{9}{4}}$

This equation also intersect the conic $x^2=6y$. This will give us a quadratic equation, whose discriminant is zero.

Simplify it and get the result.

46. Let equation of hyperbola be



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where, $2ae=4$ and $e=2 \Rightarrow a=1$

Now, $a^2e^2=a^2+b^2$

$\Rightarrow 4=1+b^2 \Rightarrow b^2=3$

Thus, equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3}=1$

$\Rightarrow 3x^2-y^2=3$

47. The given equation of hyperbola is $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$

Here, $a^2=\cos^2\alpha$ and $b^2=\sin^2\alpha$

So, the coordinates of foci are $(\pm ae, 0)$.

$\therefore e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow e=\sqrt{1+\frac{\sin^2\alpha}{\cos^2\alpha}} \Rightarrow e=\sqrt{1+\tan^2\alpha}=\sec\alpha$

Hence, abscissae of foci remain constant when α varies.

48. Since, $y=\alpha x+\beta$ is a tangent, then $\beta^2=a^2\alpha^2-b^2$. Hence, locus of P is a hyperbola $a^2x^2-y^2=b^2$

49. Foci of the ellipse $=(\pm\sqrt{16-b^2}, 0)$.

Foci of hyperbola $=\left(\pm\sqrt{\frac{144}{25}+\frac{81}{25}}, 0\right)=(\sqrt{9}, 0)$

According to the question, $\sqrt{16-b^2}=\sqrt{9} \Rightarrow b^2=7$

50. The mid-point of the chord is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

So, the equation of chord when mid-point is given, is

$$x\left(\frac{y_1+y_2}{2}\right)+y\left(\frac{x_1+x_2}{2}\right)=2\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right)$$

$$\Rightarrow x(y_1+y_2)+y(x_1+x_2)=(x_1+x_2)(y_1+y_2)$$

$$\therefore \frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$$

Unit Test 4 (Coordinate Geometry)

DAY 30

1. If in a ΔABC (whose circumcentre is origin), $a \le \sin A$, then for any point (x, y) inside the circumcircle of ΔABC

(a) $|xy| < \frac{1}{8}$ (b) $|xy| > \frac{1}{8}$

(c) $\frac{1}{8} < xy < \frac{1}{2}$ (d) None of these

2. If $A(n, n^2)$ (where, $n \in N$) is any point in the interior of the quadrilateral formed by the lines $x = 0$, $y = 0, 3x + y - 4 = 0$ and $4x + y - 21 = 0$, then the possible number of positions of the point A is

(a) 0 (b) 1 (c) 2 (d) 3

3. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut off by the line $x + y = 2$, is

(a) $(-\infty, 5\sqrt{2})$ (b) $(4\sqrt{2} - \sqrt{14},\ 5\sqrt{2})$

(c) $(4\sqrt{2} - \sqrt{14},\ 4\sqrt{2} + \sqrt{14})$ (d) None of these

4. The parabola $y^2 = 4x$ and the circle $(x - 6)^2 + y^2 = r^2$ will have no common tangent, if r is equal to

(a) $r > \sqrt{20}$ (b) $r < \sqrt{20}$

(c) $r > \sqrt{18}$ (d) $r \in (\sqrt{20}, \sqrt{28})$

5. Set of values of m for which a chord of slope m of the circle $x^2 + y^2 = 4$ touches the parabola $y^2 = 4x$, is

(a) $\left(-\infty, -\sqrt{\frac{\sqrt{2}-1}{2}}\right) \cup \left(\sqrt{\frac{\sqrt{2}-1}{2}}, \infty\right)$

(b) $(-\infty, 1) \cup (1, \infty)$

(c) $(-1, 1)$

(d) R

6. AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to parabola at A and B meet Y-axis at A_1 and B_1, respectively. If the area of trapezium AA_1B_1B is equal to $24a^2$, then angle subtended by A_1B_1 at the focus of the parabola is equal to

(a) $2\tan^{-1}(3)$ (b) $\tan^{-1}(3)$ (c) $2\tan^{-1}(2)$ (d) $\tan^{-1}(2)$

7. The equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$, is

(a) $(x - 1)^2 + (y - 0)^2 = \frac{11}{3}$

(b) $(x - 1)^2 - (y - 0)^2 = \frac{11}{3}$

(c) $(x + 1)^2 - (y + 0)^2 = \frac{11}{3}$

(d) $(x + 1)^2 + (y + 0)^2 = \frac{11}{3}$

8. If ω is one of the angles between the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$, then $\frac{2\cot\omega}{\sin 2\theta}$ is equal to

(a) $\frac{e^2}{\sqrt{1-e^2}}$ (b) $\frac{e^2}{\sqrt{1+e^2}}$

(c) $\frac{e^2}{1-e^2}$ (d) $\frac{e^2}{1+e^2}$

9. If the latusrectum of a hyperbola through one focus subtends 60° angle at the other focus, then its eccentricity e is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) $\sqrt{5}$ (d) $\sqrt{6}$

10. If two tangents can be drawn to the different branches of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (α, α^2), then

(a) $\alpha \in (-2, 0)$

(b) $\alpha \in (-3, 0)$

(c) $\alpha \in (-\infty, -2)$

(d) $\alpha \in (-\infty, -3)$

11. The lines $lx + my + n = 0$, $mx + ny + l = 0$ and $nx + ly + m = 0$ are concurrent, if

(a) $l + m + n = 0$

(b) $l + m - n = 0$

(c) $l - m + n = 0$

(d) $l^2 + m^2 + n^2 = lm + mn + nl$

12. If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points, then the locus of S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is

(a) a straight line parallel to X-axis
(b) a circle through the origin
(c) a circle with centre at the origin
(d) a straight line parallel to Y-axis

13. The point $(a^2, a+1)$ lies in the angle between the lines $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin, if

(a) $a \in (-3, 0) \cup \left(\frac{1}{3}, 1\right)$ (b) $a \in (-\infty, -3) \cup \left(\frac{1}{3}, 1\right)$
(c) $a \in \left(-3, \frac{1}{3}\right)$ (d) $a \in \left(\frac{1}{3}, \infty\right)$

14. The locus of poles with respect to the parabola $y^2 = 12x$ of tangent to the hyperbola $x^2 - y^2 = 9$ is

(a) $4x^2 + y^2 = 36$ (b) $x^2 + 4y^2 = 9$
(c) $x^2 + 4y^2 = 36$ (d) $4x^2 + y^2 = 81$

15. A point moves such that the area of the triangle formed by it with the points (1, 5) and (3, 7) is 21 sq units. Then, locus of the point is

(a) $6x + y - 32 = 0$ (b) $6x - y + 32 = 0$
(c) $6x - y - 32 = 0$ (d) $x + 6y - 32 = 0$

16. If the straight lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$ and $ax + by - 1 = 0$ form a triangle with origin as orthocentre, then (a, b) is given by

(a) $(-3, 3)$ (b) $(6, 4)$ (c) $(-8, 8)$ (d) $(0, 7)$

17. If two vertices of an equilateral triangle are (0, 0) and $(3, 3\sqrt{3})$, then the third vertex is

(a) $(3, -3)$ (b) $(-3, 3)$
(c) $(-3, 3\sqrt{3})$ (d) None of these

18. Let ABC is a triangle with vertices $A(-1, 4)$, $B(6, -2)$ and $C(-2, 4)$. If D, E and F are the points which divide each AB, BC and CA respectively in the ratio 3 : 1 internally. Then, the centroid of the ΔDEF is

(a) (3, 6) (b) (1, 2)
(c) (4, 8) (d) None of these

19. The four distinct points (0, 0), (2, 0), (0, −2) and $(k, -2)$ are concyclic, if k is equal to

(a) 0 (b) −2
(c) 2 (d) 1

20. If a point $P(4, 3)$ is rotated through an angle 45° in anti-clockwise direction about origin, then coordinates of P in new position are

(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $\left(-\frac{7}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $\left(\frac{1}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$

21. The coordinates of the circumcentre of the triangle with vertices (8, 6), (8, −2) and (2, −2) are

(a) $\left(6, \frac{2}{3}\right)$ (b) (8, 2)
(c) (5, −2) (d) (5, 2)

22. The exhaustive range of values of 'a' such that the angle between the pair of tangents drawn from (a, a) to the circle $x^2 + y^2 - 2x - 2y - 6 = 0$ lies in the range $\left(\frac{\pi}{3}, \pi\right)$ is

(a) $(0, \infty)$ (b) $(-3, -1) \cup (3, 5)$
(c) $(-2, -1) \cup (2, 3)$ (d) $-(3, 0) \cup (1, 2)$

23. A variable circle through the fixed point $A(p, q)$ touches the x-axis. The locus of the outer end of the diameter through A is

(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(x - p)^2 = 4qx$ (d) $(x - q)^2 = 4px$

24. The diameter of $16x^2 - 9y^2 = 144$ which is conjugate to $x = 2y$ is

(a) $y = \frac{16x}{9}$ (b) $y = \frac{32x}{9}$
(c) $x = \frac{16y}{9}$ (d) $x = \frac{32y}{9}$

25. The number of integral values of λ for which the equation $x^2 + y^2 - 2\lambda x + 2\lambda y + 14 = 0$ represents a circle whose radius cannot exceed 6, is

(a) 9 (b) 10
(c) 11 (d) 12

26. The slopes of tangents to the circle $(x - 6)^2 + y^2 = 2$ which passes through the focus of the parabola $y^2 = 16x$ are

(a) ± 2 (b) 1/2, −2
(c) −1/2, 2 (d) ± 1

27. The range of values of n for which $(n, -1)$ is exterior to both the parabolas $y^2 = |x|$ is

(a) (0, 1) (b) (−1, 1)
(c) (−1, 0) (d) None of these

28. The length of the latusrectum of the parabola $169[(x-1)^2 + (y-3)^2] = (5x - 12y + 17)^2$ is

(a) $\frac{14}{13}$ (b) $\frac{28}{13}$
(c) $\frac{12}{13}$ (d) $\frac{16}{13}$

29. The parameters t and t' of two points on the parabola $y^2 = 4ax$, are connected by the relation $t = k^2t'$. The tangents at their points intersect on the curve

(a) $y^2 = ax$ (b) $y^2 = k^2x$
(c) $y^2 = ax\left(k + \frac{1}{k}\right)^2$ (d) None of these

30. The angle of intersection between the curves $x^2 = 4(y+1)$ and $x^2 = -4(y+1)$, is

(a) $\pi/6$ (b) $\pi/4$
(c) 0° (d) $\pi/2$

31. The equation of common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x+3)$
(c) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x+1)$

32. The equation of the ellipse having vertices at $(\pm 5, 0)$ and foci at $(\pm 4, 0)$ is

(a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (b) $4x^2 + 5y^2 = 20$
(c) $9x^2 + 25y^2 = 225$ (d) None of these

33. If the tangent at the point 'ϕ' on the ellipse $\frac{x^2}{16} + \frac{11y^2}{256} = 1$ touches the circle $x^2 + y^2 - 2x - 15 = 0$, then ϕ is equal to

(a) $\pm\frac{\pi}{2}$ (b) $\pm\frac{\pi}{4}$ (c) $\pm\frac{\pi}{3}$ (d) $\pm\frac{\pi}{6}$

34. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at (0, 3) is

(a) 3 (b) 4 (c) $\sqrt{12}$ (d) $\frac{7}{2}$

35. The foci of an ellipse are $(0, \pm 1)$ and minor-axis is of unit length. The equation of the ellipse is

(a) $2x^2 + y^2 = 2$ (b) $x^2 + 2y^2 = 2$
(c) $4x^2 + 20y^2 = 5$ (d) $20x^2 + 4y^2 = 5$

36. The locus of a point which moves so that the chord of contact of the tangent from the point to two fixed given circles are perpendicular to each other is

(a) circle (b) parabola
(c) ellipse (d) None of these

37. Tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ $\left[\text{where } \theta \in \left(0, \frac{\pi}{2}\right)\right]$. Then, the value of θ such that sum of intercept on axes made by this tangent is minimum, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

38. The condition for the line $px + qy + r = 0$ to be tangent to the rectangular hyperbola $x = ct$, $y = \frac{c}{t}$ is

(a) $p < 0, q > 0$ (b) $p > 0, q > 0$
(c) $p > 0, q < 0$ (d) None of these

39. If the line $x + 3y + 2 = 0$ and its perpendicular line are conjugate w.r.t. $3x^2 - 5y^2 = 15$, then equation to conjugate line is

(a) $3x - y = 15$ (b) $3x - y + 12 = 0$
(c) $3x - y + 10 = 0$ (d) $3x - y = 4$

40. A hyperbola has the asymptotes $x + 2y = 3$ and $x - y = 0$ and passes through (2, 1). Its centre is

(a) (1, 2) (b) (2, 2) (c) (1, 1) (d) (2, 1)

41. The product of the lengths of perpendicular drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes, is

(a) 1/2 (b) 2/3 (c) 3/2 (d) 2

42. Tangents at any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the axes at A and B respectively, if the rectangle $OAPB$, where O is the origin is completed, then locus of the point P is given by

(a) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ (b) $\frac{a^2}{k^2} + \frac{b^2}{y^2} = 1$
(c) $\frac{a^2}{y^2} - \frac{b^2}{x^2} = 1$ (d) None of these

Directions (Q. Nos. 43 to 45) *P is a variable point on the line $L = 0$. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.*

43. If $L \equiv 2x + y - 6 = 0$, then the locus of circumcentre of ΔPQR is

(a) $2x - y = 4$ (b) $2x + y = 3$
(c) $x - 2y = 4$ (d) $x + 2y = 3$

44. If $P \equiv (6, 8)$, then the area of ΔQRS is

(a) $\frac{196\sqrt{5}}{25}$ sq units (b) $\frac{196\sqrt{6}}{52}$ sq units
(c) $\frac{192\sqrt{6}}{25}$ sq units (d) $\frac{196\sqrt{6}}{25}$ sq units

45. If $P \equiv (3, 4)$, then the coordinate of S is

(a) $\left(-\frac{46}{25}, -\frac{63}{25}\right)$ (b) $\left(-\frac{51}{25}, -\frac{68}{25}\right)$
(c) $\left(-\frac{46}{25}, -\frac{68}{25}\right)$ (d) $\left(-\frac{68}{25}, -\frac{51}{25}\right)$

Directions (Q. Nos. 46 to 48) *A ΔABC is given where vertex A is (1, 1) and the orthocentre is (2, 4). Also, side AB and BC are members of the family of lines $ax + by + c = 0$, where a, b and c are in AP.*

46. The vertex B is

(a) (2, 1) (b) (1, −2)
(c) (−1, 2) (d) None of these

47. The vertex C is

(a) (4, 16) (b) (17, − 4)
(c) (4, − 17) (d) (− 17, 4)

48. ΔABC is a/an

(a) obtuse angled triangle
(b) right angled triangle
(c) acute angled triangle
(d) equilateral triangle

Directions (Q. Nos. 49 to 55) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

49. If p, x_1, x_2, x_3 and q, y_1, y_2, y_3 form two arithmetic progression with common differences a and b.

Statement I The centroid of triangle formed by points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), lies on a straight line.

Statement II The point (h, k) given by $h = \frac{x_1 + x_2 + \ldots + x_n}{n}$ and $k = \frac{y_1 + y_2 + \ldots + y_n}{n}$ always lies on the line $b(x - p) = a(y - q)$ for all values of n.

50. Statement I A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola, then slope of AB is $\frac{1}{\sqrt{2}}$.

Statement II If normal at $(at_1^2, 2at_1)$ cuts again the parabola at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

51. Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle.

Statement I If radius is one unit and $AB \cdot BC \cdot CD \cdot DA \geq 4$, then $ABCD$ is a square.

Statement II A cyclic quadrilateral is a square, if its diagonals are the diameters of the circle.

52. If a circle $S = 0$ intersects a hyperbola $xy = c^2$ at four points.

Statement I If $c = 2$ and three of the intersection points are (2, 2), (4, 1) and $\left(6, \frac{2}{3}\right)$, then coordinates of the fourth point are $\left(\frac{1}{4}, 16\right)$.

Statement II If a circle intersects a hyperbola at t_1, t_2, t_3, t_4, then $t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 1$.

53. The auxiliary circle of an ellipse is described on the major-axis of an ellipse.

Statement I The circle $x^2 + y^2 = 4$ is auxiliary circle of an ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$ (where, $b < 2$).

Statement II A given circle is auxiliary circle of exactly one ellipse.

54. The tangent at a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is not an extremity of major-axis meets a directrix at T.

Statement I The circle on PT as diameter passes through the focus of the ellipse corresponding to the directrix on which T lies.

Statement II PT subtends a right angle at the focus of the ellipse corresponding to the directrix on which T lies.

55. Statement I If the perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept − 4, then $k^2 - 16 = 0$.

Statement II Locus of a point equidistant from two given points is the perpendicular bisector of the line joining the given points.

56. If two circles that passes through the points (0, 4) and (0, − 4) and touch the straight line $y = 2x + c$, cut orthogonally, then c is equal to

(a) $\pm 2\sqrt{7}$ (b) $\pm 3\sqrt{5}$
(c) $2\sqrt{9}$ (d) $\pm 4\sqrt{6}$

57. If one of the diagonal of a square is along the line $x = 2y$ and one of its vertices is (3, 0), then its sides through this vertex are given by the equations

(a) $y - 3x + 9 = 0, 3y + x - 3 = 0$
(b) $y + 3x + 9 = 0, 3y + x - 3 = 0$
(c) $y - 3x + 9 = 0, 3y - x + 3 = 0$
(d) $y - 3x + 3 = 0, 3y + x + 9 = 0$

Answer *with* Solutions

1. (*a*) Given, $a \le \sin A \Rightarrow \frac{a}{\sin A} \le 1$

$\Rightarrow \quad 2R \le 1$

$\Rightarrow \quad R \le \frac{1}{2}$

So, for any point (x, y) inside the circumcircle,

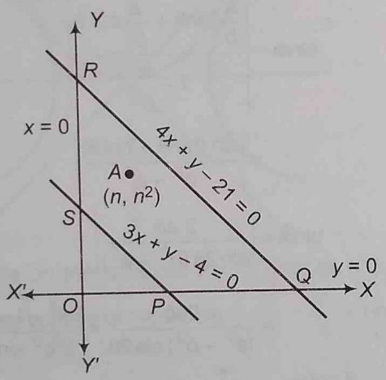
$$x^2 + y^2 < \frac{1}{4}$$

Using AM ≥ GM, $\left(\frac{x^2 + y^2}{2} \ge |xy|\right)$

$\Rightarrow \quad |xy| < \frac{1}{8}$

2. (*b*) Origin is on the left of PS.

$\because \quad 0 + 0 - 4 < 0$



At point $A(n, n^2)$,

$3n + n^2 - 4 > 0$

$\Rightarrow \quad n^2 + 3n - 4 > 0$

$\Rightarrow \quad (n + 4)(n - 1) > 0$

$\Rightarrow \quad n > 1$

or $\quad n - 1 > 0 \quad \ldots$(i)

Now, A and O lies on the same sides of QR.

$\therefore \quad 4x + y - 21 = 0 + 0 - 21 < 0$

At point $A(n, n^2)$,

$4n + n^2 - 21 < 0$

$\Rightarrow \quad n^2 + 4n - 21 < 0$

$\Rightarrow \quad (n + 7)(n - 3) < 0$

$\Rightarrow \quad 0 < n < 3 \because n \in N \quad \ldots$(ii)

From Eqs. (i) and (ii)

$1 < n < 3$

$\Rightarrow \quad n = 2$

Hence, $A(2, 4)$ is only one point.

3. (*b*) The given point is an interior point.

$\therefore \quad \left(-5 + \frac{r}{\sqrt{2}}\right)^2 + \left(-3 + \frac{r}{\sqrt{2}}\right)^2 - 16 < 0$

$\Rightarrow \quad r^2 - 8\sqrt{2}\, r + 18 < 0$

$\Rightarrow \quad 4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14}$

So, the point is on the major segment.

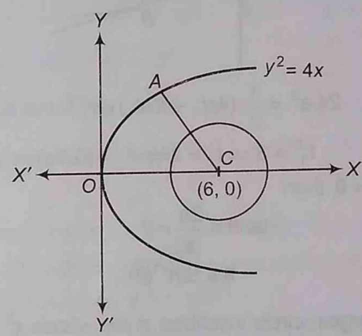
Since, the centre and the point are on the same side of the line $x + y = 2$.

$\therefore \quad -5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2}} - 2 < 0$

$\Rightarrow \quad r < 5\sqrt{2}$

So, $\quad 4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$

4. (*b*) Any normal of parabola is $y = -tx + 2t + t^3$.



If it pass through (6,0), then $-6t + 2t + t^3 = 0$

$\Rightarrow \quad t = 0, t^2 = 4$

Thus, $\quad A \equiv (4, 4)$

Thus, for no common tangent,

$$AC = \sqrt{4 + 16} > r \Rightarrow r < \sqrt{20}$$

5. (*a*) The equation of tangent of slope m to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

This will be a chord of the circle $x^2 + y^2 = 4$, if length of the perpendicular from the centre (0, 0) is less than the radius.

i.e., $\quad \left|\frac{1}{m\sqrt{m^2 + 1}}\right| < 2$

$\Rightarrow \quad 4m^4 + m^2 \cdot 4 - 1 > 0$

$\Rightarrow \quad \left(m^2 - \frac{\sqrt{2} - 1}{2}\right)\left(m^2 + \frac{1 + \sqrt{2}}{2}\right) > 0$

$\Rightarrow \quad \left(m^2 - \frac{\sqrt{2} - 1}{2}\right) > 0$

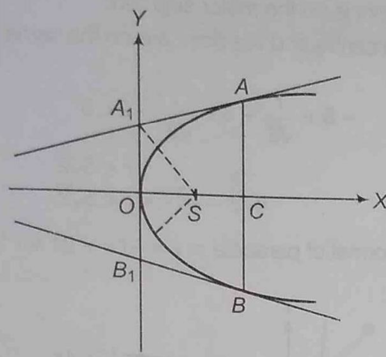
$\Rightarrow \left(m - \sqrt{\frac{\sqrt{2} - 1}{2}}\right)\left(m + \sqrt{\frac{\sqrt{2} - 1}{2}}\right) > 0$

$\Rightarrow m \in \left(-\infty, -\sqrt{\frac{\sqrt{2} - 1}{2}}\right) \cup \left(\sqrt{\frac{\sqrt{2} - 1}{2}}, \infty\right)$

6. (d) Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_1^2, -2at_1)$. Equation of tangents at A and B are $yt_1 = x + at_1^2$ and $yt_1 = x - at_1^2$, respectively.

Now, $A_1 \equiv (0, at_1)$, $B_1 = (0, -at_1)$

Area of trapezium $AA_1B_1B = \frac{1}{2}(AB + A_1B_1) \cdot OC$



$$\Rightarrow \quad 24a^2 = \frac{1}{2}\cdot(4at_1 + 2at_1)(at_1^2)$$

$$\Rightarrow \quad t_1^3 = 8 \Rightarrow t_1 = 2 \Rightarrow A_1 = (0, 2a)$$

If $\angle OSA_1 = \theta$, then

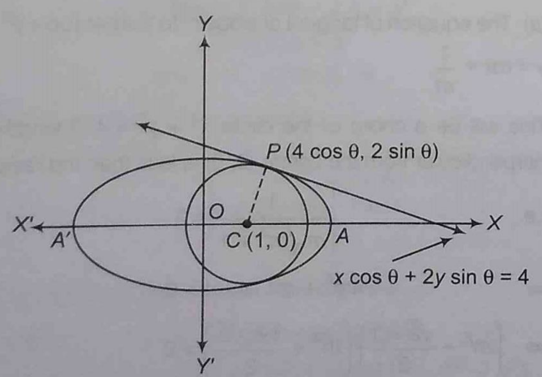
$$\tan\theta = \frac{2a}{a} = 2$$

$$\Rightarrow \quad \theta = \tan^{-1}(2)$$

7. (a) The largest circle inscribed in the ellipse $x^2 + 4y^2 = 16$ will touch the ellipse at some point. So, let r be the radius of the largest circle centred at (1, 0) and inscribed in the ellipse $x^2 + 4y^2 = 16$. Suppose it touches the ellipse at $P(4\cos\theta, 2\sin\theta)$. Then, the equation of the tangent to the ellipse at P is

$$4x\cos\theta + 8y\sin\theta = 16$$

$$\Rightarrow \quad x\cos\theta + 2y\sin\theta = 4 \quad \ldots\text{(i)}$$



Clearly, CP is perpendicular to Eq. (i).

Therefore, $\frac{2\sin\theta - 0}{4\cos\theta - 1} \times \frac{-\cos\theta}{2\sin\theta} = -1$

$$\Rightarrow \quad -\cos\theta = -4\cos\theta + 1$$

$$\Rightarrow \quad \cos\theta = \frac{1}{3}$$

$$\therefore \quad CP = r$$

$$= \sqrt{(4\cos\theta - 1)^2 + (2\sin\theta - 0)^2}$$

$$= \sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(2 \times \frac{2\sqrt{2}}{3} - 0\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{32}{9}} = \sqrt{\frac{11}{3}}$$

Hence, the equation of the circle is

$$(x-1)^2 + (y-0)^2 = \frac{11}{3}.$$

8. (a) The equations of the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$ are

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$$

and $-ax\,\text{cosec}\,\theta - by\sec\theta = a^2 - b^2$, respectively.

Since, ω is the angle between these two normals.

Therefore,

$$\tan\omega = \left|\frac{\frac{a}{b}\tan\theta + \frac{a}{b}\cot\theta}{1 - \frac{a^2}{b^2}}\right|$$

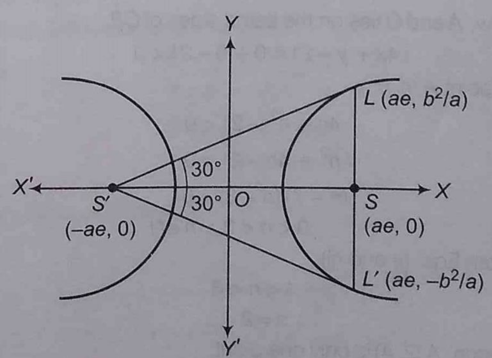
$$= \left|\frac{ab(\tan\theta + \cot\theta)}{b^2 - a^2}\right|$$

$$\Rightarrow \quad \tan\omega = \left|\frac{2ab}{\sin 2\theta\,(b^2 - a^2)}\right|$$

$$= \frac{2ab}{(a^2 - b^2)\sin 2\theta} = \frac{2a^2\sqrt{1-e^2}}{a^2e^2\sin 2\theta}$$

$$\therefore \quad \frac{2\cot\omega}{\sin 2\theta} = \frac{e^2}{\sqrt{1-e^2}}$$

9. (b) Let LSL' be a latusrectum through the focus $S(ae, 0)$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It subtends angle 60° at the other focus $S'(-ae, 0)$.



We have, $\angle LS'L' = 60°$

$\therefore \quad \angle LS'S = 30°$

In $\Delta LS'S$, we have

$$\tan 30^\circ = \frac{LS}{S'S}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{b^2/a}{2ae}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{b^2}{2a^2e} = \frac{e^2-1}{2e}$$

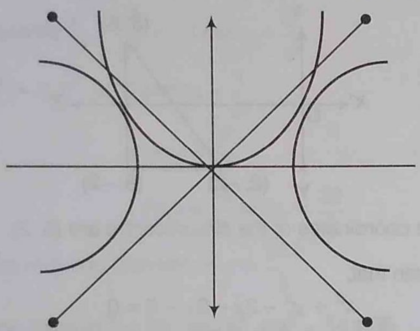
$$\Rightarrow \quad \sqrt{3}e^2 - 2e - \sqrt{3} = 0$$

$$\Rightarrow \quad (e-\sqrt{3})(\sqrt{3}e+1) = 0$$

$$\therefore \quad e = \sqrt{3}$$

10. (c) Given that,

$$\frac{x^2}{1} - \frac{y^2}{4} = 1$$



Since, (α, α^2) lie on the parabola $y = x^2$, then (α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ in 1st and 2nd quadrant.

So, the asymptotes are $y = \pm 2x$.

$$\therefore \quad 2\alpha < \alpha^2$$

$$\Rightarrow \quad \alpha < 0 \text{ or } \alpha > 2 \text{ and } -2\alpha < \alpha^2$$

$$\alpha < -2 \text{ or } \alpha > 0$$

$$\therefore \quad \alpha \in (-\infty, -2) \text{ or } (2, \infty)$$

11. (a) Since, lines are concurrent.

$$\therefore \quad 1(lx + my + n) + 1(mx + ny + l) + 1(nx + ly + m) = 0$$

$$\Rightarrow \quad x(l + m + n) + y(l + m + n) + (l + m + n) = 0$$

$$\therefore \quad l + m + n = 0$$

12. (d) Let the coordinates of a point S be (x, y).

Since, $SQ^2 + SR^2 = 2SP^2$

$$\Rightarrow \quad (x+1)^2 + y^2 + (x-2)^2 + y^2 = 2[(x-1)^2 + y^2]$$

$$\Rightarrow \quad 2x + 3 = 0$$

Hence, it is a straight line parallel to Y-axis.

13. (a) Since, origin and the point $(a^2, a+1)$ lie on the same side of both the lines.

$\therefore$ $3a^2 - (a+1) + 1 > 0$ and $a^2 + 2(a+1) - 5 < 0$

i.e., $a(3a-1) > 0$ and $a^2 + 2a - 3 < 0$

i.e., $a \in (-\infty, 0) \cup \left(\frac{1}{3}, \infty\right)$

and $(a-1)(a+3) < 0 \Rightarrow a \in (-3, 1)$

Then, $a \in (-3, 0) \cup \left(\frac{1}{3}, 1\right)$

14. (a) Let the pole be (h, k), so that polar is

$$ky = 6(x+h) \Rightarrow y = \frac{6x}{k} + \frac{6h}{k}$$

Since, it is tangent to the hyperbola,

$x^2 - y^2 = 9$ then

$$\therefore \quad c^2 = 9m^2 - 9$$

$$\Rightarrow \quad \frac{36h^2}{k^2} = \frac{324}{k^2} - 9 \left(\because c = \frac{6h}{k}, m = \frac{6}{k}\right)$$

$$\Rightarrow \quad 4h^2 + k^2 = 36$$

Hence, the locus is $4x^2 + y^2 = 36$.

15. (a) Let (x, y) be the required point.

Then, $$\frac{1}{2}\begin{vmatrix} x & y \\ 1 & 5 \\ 3 & -7 \\ x & y \end{vmatrix} = 21$$

$$\Rightarrow \quad 5x - y - 7 - 15 + 3y + 7x = 42$$

$$\Rightarrow \quad 6x + y = 32$$

16. (c) Equation of AO is $2x + 3y - 1 + \lambda(x + 2y - 1) = 0$.

Since, it passes through $(0, 0)$, then $\lambda = -1$.

$$\therefore \quad x + y = 0$$

Since, AO is perpendicular to BC.

$$\therefore \quad (-1)\left(-\frac{a}{b}\right) = -1 \Rightarrow a = -b$$

Similarly, $(2x + 3y - 1) + \mu(ax - ay - 1) = 0$ will be equation of BO for $\mu = -1$.

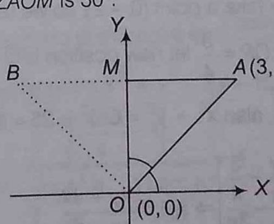
Thus, BO is perpendicular to AC.

$$\Rightarrow \quad -\frac{2-a}{3+a} \cdot \left(\frac{-1}{2}\right) = -1$$

$$\Rightarrow \quad 2 - a = -6 - 2a$$

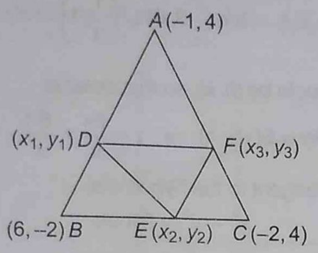
$$\Rightarrow \quad a = -8 \text{ and } b = 8$$

17. (c) Since, $\angle AOM$ is 30°.



Hence, the required point B is $(-3, 3\sqrt{3})$.

18. (*b*) Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) are coordinates of the points *D*, *E* and *F* which divide each *AB*, *BC* and *CA* respectively in the ratio 3 : 1 (internally).



$\therefore \quad x_1 = \frac{3 \times 6 - 1 \times 1}{4} = \frac{17}{4}$

$y_1 = \frac{-2 \times 3 + 4 \times 1}{4}$

$= -\frac{2}{4} = -\frac{1}{2}$

Similarly, $\quad x_2 = 0, \; y_2 = \frac{5}{2}$

and $\quad x_3 = -\frac{5}{4}, \; y_3 = 4$

Let (x, y) be the coordinates of centroid of ΔDEF.

$\therefore \quad x = \frac{1}{3}\left(\frac{17}{4} + 0 - \frac{5}{4}\right) = 1$

and $\quad y = \frac{1}{3}\left(-\frac{1}{2} + \frac{5}{2} + 4\right) = 2$

So, the coordinates of centroid are (1, 2).

19. (*c*) Let the general equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$

The equation of circle passing through (0, 0), (2, 0) and (0, – 2).

Then, $\quad c = 0 \quad$ ...(i)

$4 + 4g + c = 0 \quad$ ...(ii)

and $\quad 4 - 4f + c = 0 \quad$ ...(iii)

On solving Eqs. (i), (ii) and (iii), we get

$c = 0, g = -1, f = 1$

$\therefore$ The equation of circle becomes

$x^2 + y^2 - 2x + 2y = 0$

Since, it is passes through $(k, -2)$,

$k^2 + 4 - 2k - 4 = 0$

$\Rightarrow \quad k^2 - 2k = 0$

$\Rightarrow \quad k = 0, 2$

We have already take a point (0, – 2), so we take only $k = 2$.

20. (*a*) Slope of line $OP = \frac{3}{4}$, let new position is $Q(x, y)$.

Slope of $OQ = \frac{y}{x}$, also $x^2 + y^2 = OQ^2 = 25 = (OP^2)$

$\therefore \quad \tan 45^\circ = \left|\frac{\frac{y}{x} - \frac{3}{4}}{1 + \frac{3y}{4x}}\right| \Rightarrow \pm 1 = \frac{4y - 3x}{4x + 3y}$

$\Rightarrow \quad 4x + 3y = 4y - 3x$

or $\quad -4x - 3y = 4y - 3x$

$\Rightarrow \quad x = \frac{1}{7}y \quad$ ...(i)

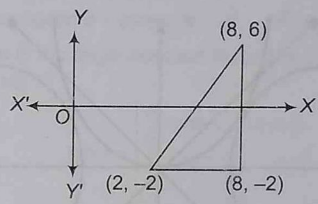
or $\quad -x = 7y \quad$ ...(ii)

Correct relation is $x = \frac{1}{7}y$ as new point must lies in Ist quadrant.

$\therefore \quad x^2 + 49x^2 = 25$

$\Rightarrow \quad x = +\frac{1}{\sqrt{2}}, \; y = \frac{7}{\sqrt{2}}$

21. (*d*) Given triangle is a right angled triangle. In right angled triangle mid-point of hypotenuse is circumcentre.



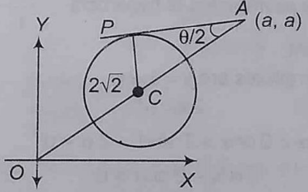
So, the coordinates of the circumcentre are (5, 2).

22. (*b*) Given that,

$x^2 + y^2 - 2x - 2y - 6 = 0$

Centre = C (1, 1), radius = $2\sqrt{2}$

Since, point (a, a) must lie out side the circle.



So, $\quad 2a^2 - 4a - 6 > 0$

$\Rightarrow \quad a < -1 \;$ or $\; a > 3$

Now, in ΔPAC,

$\tan\frac{\theta}{2} = \frac{2\sqrt{2}}{\sqrt{2a^2 - 4a - 6}}$

As given that, $\frac{\pi}{3} < \theta < \pi$

$\Rightarrow \quad \frac{\pi}{6} < \frac{\theta}{2} < \frac{\pi}{2}$

$\therefore \quad \frac{2\sqrt{2}}{\sqrt{2a^2 - 4a - 6}} > \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \sqrt{a^2 - 2a - 3} < 2\sqrt{3}$

$\therefore \quad a^2 - 2a - 3 < 12$

$\Rightarrow \quad a^2 - 2a - 15 < 0$

$\Rightarrow \quad -3 < a < 5$

$\therefore \quad a \in (-3, -1) \cup (3, 5)$

23. (a) The circle touching the X-axis is

$$x^2 + y^2 + 2gx + 2fy + g^2 = 0.$$

Since, it passes through (p, q).

$$\therefore \quad p^2 + q^2 + 2gp + 2fq + g^2 = 0 \quad \text{(i)}$$

If (x, y) is the other end of the diameter, then

$$p + x = -2g, q + y = -2f$$

Now, Eq. (i) gives

$$p^2 + q^2 - p(p + x) - q(q + y) + \frac{(p + x)^2}{4} = 0$$

$$\Rightarrow \quad (x + p)^2 = 4px + 4qy$$

$$\Rightarrow \quad (x - p)^2 = 4qy$$

24. (b) Diameters $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $m_1m_2 = \frac{b^2}{a^2}$.

Here, $a^2 = 9$, $b^2 = 16$ and $m_1 = \frac{1}{2}$

$$\because \quad m_1m_2 = \frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{1}{2}(m_2) = \frac{16}{9} \Rightarrow m_2 = \frac{32}{9}$$

Thus, the required diameter is $y = \frac{32x}{9}$.

25. (c) Since, $(\text{radius})^2 \le 36 \Rightarrow \lambda^2 + \lambda^2 - 14 \le 36$

$$\Rightarrow \quad \lambda^2 \le 25 \Rightarrow -5 \le \lambda \le 5$$

$$\Rightarrow \quad \lambda = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$$

Hence, number of integer values of λ is 11.

26. (d) Tangent to the circle with slope m is

$$y = m(x - 6) \pm \sqrt{2(1 + m^2)}.$$

Since, it passes through (4, 0).

$$\therefore \quad 4m^2 = 2 + 2m^2 \Rightarrow m = \pm 1$$

27. (b) Since, $1 - |n| > 0 \Rightarrow |n| < 1$ or $n \in (-1, 1)$

28. (b) Given parabola is $(x - 1)^2 + (y - 3)^2 = \left(\frac{5x - 12y + 17}{13}\right)^2$

Focus = (1, 3), directrix is $5x - 12y + 17 = 0$

$\therefore$ Length of latusrectum $= 2\left|\frac{5 - 36 + 17}{13}\right| = \frac{28}{13}$

29. (c) Tangents at t and t' meet on the point (x, y) given by

$$x = att' = ak^2t'^2 \quad \text{...(i)}$$

and $\quad y = a(t + t') = a(k^2t' + t') = at'(k^2 + 1) \quad \text{...(ii)}$

From Eqs. (i) and (ii)

$$x = \frac{ak^2 \cdot y^2}{a^2(k^2 + 1)^2} = \frac{k^2y^2}{a(k^2 + 1)^2}$$

$$\Rightarrow \quad y^2 = \frac{ax(k^2 + 1)^2}{k^2} = ax\left(k + \frac{1}{k}\right)^2$$

30. (c) The point of intersection between the curves $x^2 = 4(y + 1)$ and $x^2 = -4(y + 1)$ is $(0, -1)$.

Since, the slopes of curve first and curve second at the point $(0, -1)$ are respectively, then

$$m_1 = \frac{2x}{4} = 0 \text{ and } m_2 = -\frac{2x}{4} = 0$$

$$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = 0 \Rightarrow \theta = 0°$$

31. (c) Any tangent to $y^2 = 4x$ is of the form $y = mx + \frac{1}{m}$, $(a = 1)$, this touches the circle $(x - 3)^2 + y^2 = 9$.

So, the centre of the circle is (3, 0) and radius is r.

If $$\left|\frac{m(3) + \frac{1}{m} - 0}{\sqrt{m^2 + 1}}\right| = 3$$

$$3m^2 + 1 = \pm 3m\sqrt{m^2 + 1}$$

$$\Rightarrow \quad 9m^4 + 1 + 6m^2 = 9m^2(m^2 + 1) \text{ (squaring on both sides)}$$

$$\Rightarrow \quad 3m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

If the tangent touches the parabola and circle above x-axis, then slope m should be positive.

$\therefore \quad m = \frac{1}{\sqrt{3}}$ and the equation is $y = \frac{x}{\sqrt{3}} + \sqrt{3}$

$$\Rightarrow \quad \sqrt{3}y = x + 3$$

which is the required equation of tangent.

32. (c) The line joining foci and vertices is x-axis and the centre is (0, 0). So, axes of the ellipse coincide with coordinate axes.

Here, $a = 5$ and $ae = 4 \Rightarrow e = \frac{4}{5}$

Now, $\quad b^2 = a^2(1 - e^2) = 25\left[1 - \left(\frac{4}{5}\right)^2\right] = 9$

Hence, the equation of the ellipse is $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$

or $$9x^2 + 25y^2 = 225$$

33. (c) Given equation is $\frac{x^2}{16} + \frac{y^2}{(16/\sqrt{11})^2} = 1.$

Thus, the parametric coordinates are $\left(4\cos\phi, \frac{16}{\sqrt{11}}\sin\phi\right)$. The equation of tangent at this point is $\frac{x\cos\phi}{4} + \frac{\sqrt{11}y\sin\phi}{16} = 1.$

This touches the circle $x^2 + y^2 - 2x - 15 = 0$

$$\therefore \quad \frac{\left|\frac{\cos\phi}{4} - 1\right|}{\sqrt{\frac{\cos^2\phi}{16} + \frac{11\sin^2\phi}{256}}} = 4$$

$$\Rightarrow \cos^2\phi + 16 - 8\cos\phi = 256\left(\frac{\cos^2\phi}{16} + \frac{11\sin^2\phi}{256}\right)$$

$$\Rightarrow 15\cos^2\phi + 11(1-\cos^2\phi) + 8\cos\phi - 16 = 0$$

$$\Rightarrow \quad 4\cos^2\phi + 8\cos\phi - 5 = 0$$

$$\Rightarrow \quad \cos\phi = \frac{1}{2} \quad \left(\because \cos\phi \neq \frac{5}{2}\right)$$

$$\Rightarrow \quad \phi = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$$

34. (*b*) Since, $a^2(1-e^2) = 9 \Rightarrow 16 - a^2e^2 = 9 \Rightarrow ae = \sqrt{7}$

So, foci are at $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$.

$\therefore$ Required radius $= \sqrt{(\sqrt{7}-0)^2 + (3-0)^2} = 4$

35. (*d*) Given, $2b = 1 \Rightarrow b = \frac{1}{2}$ and $a \cdot e = 1$

Since, $a^2(1-e^2) = \frac{1}{4} \Rightarrow a^2 - 1^2 = \frac{1}{4} \Rightarrow a^2 = \frac{5}{4}$

Hence, the equation of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

is $\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1$ or $20x^2 + 4y^2 = 5$.

36. (*a*) Let the equations of two given circles

are $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

Now, the equations of the chords of contacts from $P(h, k)$ to Eqs. (i) and (ii) are

$$x(h+g_1) + y(k+f_1) + g_1h + f_1k + c_1 = 0$$

and $x(h+g_2) + y(k+f_2) + g_2h + f_2k + c_2 = 0$.

According to given condition,

$$\frac{(h+g_1)}{(k+f_1)} \times \frac{(h+g_2)}{(k+f_2)} = -1$$

$$\Rightarrow h^2 + (g_1+g_2)h + g_1g_2 + k^2 + k(f_1+f_2) + f_1f_2 = 0$$

Hence, the locus of point is

$$x^2 + y^2 + (g_1+g_2)x + (f_1+f_2)y + g_1g_2 + f_1f_2 = 0$$

which is the equation of a circle.

37. (*b*) Equation of tangent at $(3\sqrt{3}\cos\theta, \sin\theta)$ to the ellipse $\frac{x^2}{27} + y^2 = 1$ is $\frac{x\cos\theta}{3\sqrt{3}} + y\sin\theta = 1$.

This intercepts on the coordinate axes.

$\therefore$ Sum of intercepts on axes is

$$3\sqrt{3}\sec\theta + \operatorname{cosec}\theta = f(\theta) \quad \text{(say)}$$

On differentiating w.r.t. θ, we get

$$f'(\theta) = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$$

$$= \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$$

For maxima and minima, put $f'(\theta) = 0$

$$3\sqrt{3}\sin^3\theta - \cos^3\theta = 0$$

$$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

At $\theta = \frac{\pi}{3}$, $f''(\theta) > 0$

So, $f(\theta)$ is minimum at $\theta = \frac{\pi}{6}$.

38. (*b*) Given, $x = ct$, $y = c/t$

Then, $\frac{dy}{dt} = \frac{-c}{t^2}$ and $\frac{dx}{dt} = c$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{t^2}$$

But equation of tangent is $px + qy + r = 0$.

$$\therefore \quad -\frac{p}{q} = -\frac{1}{t^2}$$

$$\Rightarrow \quad \frac{p}{q} = \frac{1}{t^2} > 0$$

$$\Rightarrow \quad \frac{p}{q} > 0$$

$$\Rightarrow \quad p > 0, q > 0 \text{ or } p < 0, q < 0$$

39. (*b*) Since, the lines $x + 3y + 2 = 0$ and $3x - y + k = 0$ are conjugate w.r.t. $\frac{x^2}{5} - \frac{y^2}{3} = 1$.

$$\therefore \quad 5(1)(3) - 3(3)(-1) = 2k$$

$$\Rightarrow \quad k = 12$$

Hence, equation of conjugate line is

$$3x - y + 12 = 0.$$

40. (*c*) Given equations of as asymptotes are

$x + 2y = 3$ ...(i)

and $x - y = 0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = 1, y = 1$$

So, the centre of hyperbola is (1, 1).

41. (*b*) Given, equation can be rewritten as $\frac{x^2}{2} - \frac{y^2}{1} = 1$

Here, $a^2 = 2, b^2 = 1$

The product of length of perpendicular drawn from any point on the hyperbola to the asymptotes is

$$\frac{a^2b^2}{a^2+b^2} = \frac{2(1)}{2+1} = \frac{2}{3}.$$

42. (*a*) The equation of tangent is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

So, the coordinates of A and B are $(a\cos\theta, 0)$ and $(0, -b\cot\theta)$, respectively.

Let coordinates of P are (h, k).

$\therefore \quad h = a\cos\theta, k = -b\cot\theta$

$\Rightarrow \quad \frac{k}{h} = -\frac{b}{a\sin\theta} \Rightarrow \frac{b^2h^2}{a^2k^2} = \sin^2\theta$

$\Rightarrow \quad \frac{b^2h^2}{a^2k^2} + \frac{h^2}{a^2} = 1 \qquad (\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow \quad \frac{a^2}{h^2} - \frac{b^2}{k^2} = 1$

Hence, the locus of P is $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$.

43. (*b*) $\because PQ = PR$, so parallelogram $PQSR$ is a rhombus.

$\therefore$ Mid-point of QR = Mid-point of PS and $QR \perp PS$

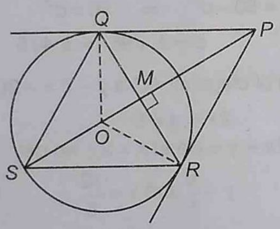
So, S is the mirror image of P w.r.t. QR.

$\because \quad L \equiv 2x + y = 6$

Let $\quad P \equiv (k, 6-2k)$

$\because \quad \angle PQO = \angle PRO = \frac{\pi}{2}$

So, OP is diameter of circumcircle of ΔPQR, then centre is $\left(\frac{k}{2}, 3-k\right)$.



$\therefore \quad x = \frac{k}{2} \Rightarrow k = 2x$

and $\quad y = 3 - k \Rightarrow y = 3 - 2x \quad \Rightarrow \quad 2x + y = 3$

44. (*c*) Given, $P(6, 8)$

So, the equation of QR is $6x + 8y = 4$.

$\Rightarrow \quad 3x + 4y - 2 = 0$

$\therefore \quad PM = \frac{48}{5}$ and $PQ = \sqrt{96}$

Then, $\quad QM = \sqrt{96 - \frac{(48)^2}{25}} = \sqrt{\frac{96}{25}}$

$\therefore \quad QR = 2\sqrt{\frac{96}{25}}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2} \cdot PM \cdot QR = \frac{192 \cdot \sqrt{6}}{25}$

Since, $PQRS$ is a rhombus.

$\therefore$ Area of ΔQRS = Area of $\Delta PQR = \frac{192 \cdot \sqrt{6}}{25}$ sq units

45. (*b*) Given, $\quad P \equiv (3, 4)$

So, the equation of QR is $3x + 4y = 4$. $\quad$ ...(i)

Let $\quad S \equiv (x_1, y_1)$

Since, S is the mirror image of P w.r.t. Eq. (i), then

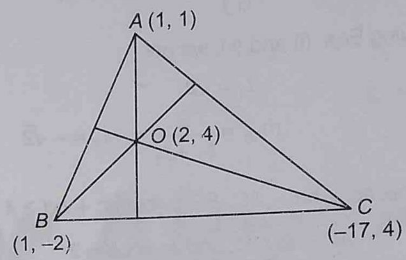
$$\frac{x_1 - 3}{3} = \frac{y_1 - 4}{4} = \frac{-2(3 \times 3 + 4 \times 4 - 4)}{3^2 + 4^2} = -\frac{42}{25}$$

$\therefore \quad x_1 = -\frac{51}{25}, y_1 = -\frac{68}{25}$

So, the coordinate of s is $S\left(-\frac{51}{25}, -\frac{68}{25}\right)$.

46. (*b*) Slope of $AO = \frac{4-1}{2-1} = 3$ $\quad$ ...(i)

Also, $\left(-\frac{a}{b}\right)3 = -1 \Rightarrow 3a = b \Rightarrow b - 3a = 0$



Also, a, b and c are in AP. So, $a + c = 2b \Rightarrow c = 2b - a$

If vertex $B = (1, -2)$, then $a(1) + b(-2) + c = a - 2b + c$

$= a - 2b + 2b - a = 0$

47. (*d*) From option (d), if $c = (-17, 4)$

then $-17a + 4b + c = -17a + 4b + 2b - a = -18a + 6b$

$= 6(b - 3a) = 0 \qquad$ [from Eq.]

48. (*a*) Slope of $AC\ (m_1) = \frac{1-4}{1+17} = -\frac{3}{18} = -\frac{1}{6}$

and slope of $BC\ (m_2) = \frac{-2-4}{1+17} = -\frac{6}{18} = -\frac{1}{3}$

$\therefore \quad \tan A = \frac{m_2 - m_1}{1 + m_1m_2} = \frac{-\frac{1}{3} + \frac{1}{6}}{1 + \left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)} < 0$

Hence, ΔABC is an obtuse angled triangle.

49. (*a*) Since, $p, x_1, x_2, \ldots$ and $q, y_1, y_2, \ldots$ are in AP with common differences a and b, respectively.

$\Rightarrow \quad x_i = p + ai$ and $y_i = q + ib$

$\therefore \quad h = \frac{x_1 + x_2 + \ldots + x_n}{n}$ and $k = \frac{y_1 + y_2 + \ldots + y_n}{n}$

$\Rightarrow \quad nh = \sum_{i=1}^{n} x_i$ and $nk = \sum_{i=1}^{n} y_i$

$\Rightarrow \quad nh = \sum_{i=1}^{n} (p + ia)$ and $nk = \sum_{i=1}^{n} (q + bi)$

$\Rightarrow \quad nh = np + \frac{n(n+1)}{2}a$ and $nk = nq + \frac{n(n+1)}{2}b$

$\Rightarrow \quad \frac{h-p}{a} = \frac{n+1}{2}$ and $\frac{k-q}{b} = \frac{n+1}{2}$

$\therefore \quad \frac{h-p}{a} = \frac{k-q}{b}$

Hence, locus of (h, k) is $b(x-p) = a(k-q)$.
Hence, Statement II is true and since for Statement I, $n = 3$
So, Statement I is true and Statement II is a correct explanation of Statement I.

50. (*d*) If t_1 and t_2 are parameters of a and b, then

$$t_1 t_2 = -4 \quad \ldots\text{(i)}$$

Also, $t_1\left(-t_1 - \frac{2}{t_1}\right) = -4 \quad \ldots\text{(ii)}$

On solving Eqs. (i) and (ii), we get

$$t_1 = \sqrt{2}$$

Also, $m_{AB} = \frac{2}{t_2 + t_1} = -t_1 = -\sqrt{2}$

51. (*c*) Since, $ac + bd = AC \cdot BD \le 4$ but $ac + bd \ge 4$ ($\because$ AM $\ge$ GM)

$\Rightarrow \quad AC = BD = 2$ and $ac = bd = 2$

52. (*d*) Statement II is true.
For the point (2, 2), $t_1 = 1$
For the point (4, 1), $t_2 = 2$
For the point (6, 2/3), $t_3 = 3$
For the point $\left(\frac{1}{4}, 16\right)$, $t_4 = \frac{1}{8}$

Now, $t_1 \cdot t_2 \cdot t_3 \cdot t_4 = \frac{3}{4} \ne 1$

Hence, Statement I is false.

53. (*c*) The auxiliary circle of an ellipse

$\frac{x^2}{4} + \frac{y^2}{b^2} = 1, b < 2$ is $x^2 + y^2 = 4$

54. (*a*) The equation of tangent to the ellipse is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ and meets the directrix $x = \frac{a}{e}$ at $T\left[\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right]$.

Since, focus is $S(ae, 0)$.

Slope of $SP = \frac{b\sin\theta}{a(\cos\theta - e)}$

and slope of $ST = \frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$

$\therefore$ Product of slopes $= -\frac{b^2}{a^2(1-e^2)}$

55. (*a*) Statement II is true, using in Statement I,

$(x-1)^2 + (y-4)^2 = (x-k)^2 + (y-3)^2$

$\Rightarrow \quad 2(k-1)x - 2y = k^2 - 8$

y-intercept $= -\frac{k^2-8}{2} = -4$ (given)

$\Rightarrow \quad k^2 = 16 = k^2 - 16 = 0$

56. (*d*) Since, the given points lie on the y-axis.
So, the centre of two circles lie on the x-axis.
Let $(g, 0)$ be the coordinate of centre of one of the circles.
Then, its radius is $\sqrt{g^2 + 16}$.
$\because$ It touches the given straight line.

$\therefore \quad \frac{2g+c}{\sqrt{1+m^2}} = \frac{2g+c}{\sqrt{5}} = \pm\sqrt{g^2+16}$

$\Rightarrow \quad (2g+c)^2 = (5)(g^2+16)$

$\Rightarrow \quad 4g^2 + c^2 + 4gc = 5g^2 + 80$

$\Rightarrow \quad g^2 - 4gc + 80 - c^2 = 0 \quad \ldots\text{(i)}$

Let g_1 and g_2 be two roots of Eq. (i).
Then, equations of two circles are

$$x^2 + y^2 - 2g_1 x - 16 = 0$$

and $x^2 + y^2 - 2g_2 x - 16 = 0$

Since, it cut orthogonally, then

$$2g_1 g_2 = -32 \quad \ldots\text{(ii)}$$

From Eq. (i),

$$g_1 g_2 = 16 \times 5 - c^2$$

Again from Eq. (ii),

$-16 = 80 - c^2 \Rightarrow 96 = c^2$

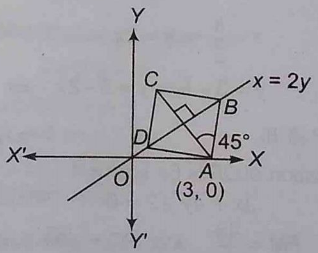
$\Rightarrow \quad c = \pm\sqrt{96} = \pm 4\sqrt{6}$

57. (*a*) Equation of diagonal AC is $y - 0 = -2(x-3)$

$\Rightarrow \quad 2x + y = 6$

On solving $2x + y = 6$ and $x = 2y$, we get

$$y = \frac{6}{5} \text{ and } x = \frac{12}{5}$$



So, the centre of square is $\left(\frac{12}{5}, \frac{6}{5}\right)$.

Let slope of side AB or AD is m, then

$$\left|\frac{m-(-2)}{1+m(-2)}\right| = 1$$

$\Rightarrow \quad (m+2) = \pm(1-2m)$

$\Rightarrow \quad m = -\frac{1}{3}$ and $m = 3$

Hence, slopes of AB and AD are 3 and $-\frac{1}{3}$, respectively.

$\therefore$ Equations of sides AB and AD are

$$y - 0 = 3(x-3)$$

and $y - 0 = -\frac{1}{3}(x-3)$

or $y - 3x + 9 = 0$ and $3y + x - 3 = 0$, respectively.