Day 31

Vector Algebra

Day 31
Outlines ...

- Physical Quantities
- Types of Vectors
- Position of Vectors
- Components of a Vector
- Scalar (Dot) Product
- Vector (Cross) Product
- Product of Three Vectors
- Scalar Triple Product
- Application of Vectors in Geometry

Physical Quantities

Physical quantities are divided into two categories **Scalar Quantities** and **Vector Quantities**. Those quantities which have only magnitude and which are not related to any fixed direction in space are called **scalar quantities** or briefly scalars.

Examples of scalars are mass, volume, density, work, temperature etc.,

Vectors are those quantities which have both magnitude and direction. Displacement, velocity, acceleration, momentum, weight, force etc., are examples of vector quantities.

Magnitude of a vector **a** is denoted by $|\mathbf{a}|$ or a. It is **non-negative scalar.**

Types of Vectors

1. **Zero or Null Vector**
   A vector whose initial and terminal points are coincident is called the zero or the null vector.

2. **Unit Vector**
   A vector whose modulus is unity, is called a unit vector.

3. **Like and Unlike Vectors**

Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.

4. **Equality of Vectors**

Two vectors **a** and **b** are said to be equal, written as $\mathbf{a} = \mathbf{b}$, if they have same length, same or parallel support and same sense.

5. **Collinear or Parallel Vectors**

Vectors having the same or parallel supports are called collinear vectors.

6. **Coinitial Vectors**

Vectors having the same initial point are called coinitial vectors.

7. **Coplanar Vectors**

A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

8. **Coterminous Vectors**

Vectors having the same terminal point are called coterminous vectors.

9. **Negative of a Vector**

The vector which has the same magnitude as that of a given vector **a** but opposite direction, is called the negative of **a** and is denoted by $-\mathbf{a}$.

10. **Reciprocal of a Vector**

A vector having the same direction as that of a given vector **a** but magnitude equal to the reciprocal of the given vector is known as the reciprocal of **a** and is denoted by $\mathbf{a}^{-1}$.

11. **Localized and Free Vectors**

A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. Force acting on a rigid body is a localized vector as its effect depends on the line of action of the force.

12. **Displacement Vector**

When a partical is displaced from point A to other point B, then the displacement AB is a vector *i.e.*, **AB** is called displacement vector of the particle.

13. **Orthogonal Vector**

Two vectors are called orthogonal, if angle between the two is a right angle.

Position of Vectors

Every point $P(x, y, z)$ is associated with a vector, called position vector *i.e.*, PV of P as $\mathbf{OP} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

O is the origin, its PV is the zero vector.

(i) If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ have PV **a** and **b**, then

$$\mathbf{AB} = \mathbf{b} - \mathbf{a}$$

and $$AB = |\mathbf{b} - \mathbf{a}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

(ii) If the point C divides AB in the ratio $m:n$, then its PV is

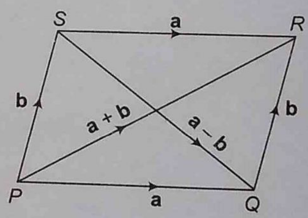
$$\mathbf{C} = \frac{m\,\mathbf{b} + n\,\mathbf{a}}{m + n}$$

Addition and Subtraction of Vectors

Let $PQRS$ is a parallelogram and $\mathbf{PQ} = \mathbf{a} = \mathbf{SR}$, $\mathbf{PS} = \mathbf{b} = \mathbf{QR}$. Then,

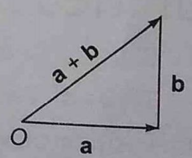
Diagonal of $\mathbf{PR} = \mathbf{a} + \mathbf{b}$ (addition of vectors)

and Diagonal of $\mathbf{SQ} = \mathbf{a} - \mathbf{b}$ (subtraction of vectors)

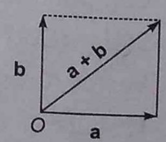


Three Methods of Vector Addition

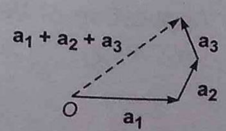
- **Triangle Law** If two vectors **a** and **b** lie along the two sides of a triangle in consecutive order (as shown in the adjoining figure), then third side represents the sum (resultant) $\mathbf{a} + \mathbf{b}$.



- **Parallelogram Law** If two vectors lie along two adjacent sides of a parallelogram (as shown in the adjoining figure), then diagonal of the parallelogram through the common vertex represents their sum.



- **Polygon Law** If $(n-1)$ sides of a polygon represents vector $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$ in consecutive order, then nth side represents their sum (as shown in the adjoining figure).



Components of a Vector

The process of splitting a vector is called resolution of a vector. In simpler language, it would mean, determining the effect of a vector in a particular direction. The parts of the vector obtained after splitting the vectors are known as the components of the vector.

Components of a Vector in Two Dimensional Space

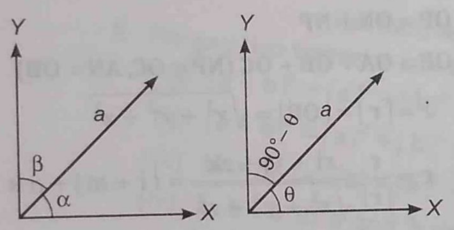
The planar components of a vector lies in the plane of vector. Since, there are two perpendicular axes involved with a plane, a vector is resolved in two components which lie in the same plane as that of vector. Clearly, a vector is composed by components in only two directions

$$\mathbf{a} = a\cos\alpha\mathbf{i} + a\cos\beta\mathbf{j}$$

From the figure depicting a planar coordinate, it is clear that angle 'β' is compliment of angle 'α'. If $\alpha = \theta$, then

$$\alpha = \theta \text{ and } \beta = 90° - \theta$$

Planar vector



The direction of a planar vector with respect to rectangular axes can be described by a single angle.

Putting in the expression for the vector,

$$\mathbf{a} = a_X\mathbf{i} + a_Y\mathbf{j}$$
$$\mathbf{a} = a\cos\theta\mathbf{i} + a\cos(90° - \theta)\mathbf{j}$$
$$\mathbf{a} = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j}$$

From graphical representation, the tangent of the angle that vector makes with X-axis is $\tan\alpha = \tan\theta = \dfrac{a\sin\theta}{a\cos\theta} = \dfrac{a_Y}{a_X}$

Similarly, the tangent of the angle that vector makes with Y-axis is $\tan\beta = \tan(90° - \theta) = \cot\theta = \dfrac{a\cos\theta}{a\sin\theta} = \dfrac{a_X}{a_Y}$

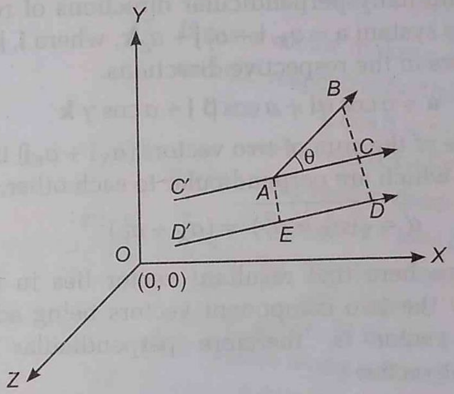
Components of a Vector in Three Dimensional Space

A scalar component, also known as projection, of a vector **AB** in the positive direction of a straight line $C'C$ is defined as $\quad AC = |\mathbf{AB}|\cos\theta = AB\cos\theta$

where, θ is the angle that vector **AB** makes with the specified direction $C'C$ as shown in the figure.

Analytical Method

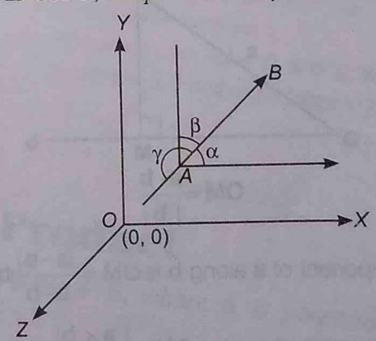
It is clear that **scalar component** can either be **positive** or **negative** depending on the value of angle that the vector makes with the referred direction. The angle lies between the range given by $0 \le \theta \le 180°$. This interval means that we should consider the smaller angle between two vectors.



In accordance with the above definition, we resolve a given vector in three components in **three mutually perpendicular directions of rectangular coordinate system.** Note, here that we measure angle with respect to parallel lines to the axes. By convention, we denote components by using the non-bold type face of the vector symbol with a suffix representing direction (*X* or *Y* or *Z*).

Components of a Vector

$$AB_X = AB\cos\alpha,\ AB_Y = AB\cos\beta \text{ and } AB_Z = AB\cos\gamma.$$



where α, β and γ are the angles that vector *AB* makes with the positive directions of *X*, *Y* and *Z*, respectively.

As a vector can be resolved in a set of components, the reverse processing of components in a vector is also expected. A vector is composed from vector components in three directions along the axes of rectangular coordinate system by combining three component vectors. The vector component is the vector counterpart of the scalar component, which is obtained by multiplying the scalar component with the unit vector in axial direction. The vector components of a vector **a** are $a_X\mathbf{i}$, $a_Y\,\mathbf{j}$ and $a_Z\mathbf{k}$.

- *If two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal, then their resolved parts will also equal i. e., $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.*
- *The resolved parts of a resultant vector of addition of two vectors are equal to the sum of resolved parts of these vectors.*

A vector **a** is equal to the vector sum of component vectors in three mutually perpendicular directions of rectangular coordinate system $\mathbf{a} = a_X\ \mathbf{i} + a_Y\mathbf{j} + a_Z\mathbf{k}$, where **i**, **j** and **k** are unit vectors in the respective directions.

$$\mathbf{a} = a\cos\alpha\mathbf{i} + a\cos\beta\ \mathbf{j} + a\cos\gamma\ \mathbf{k}$$

Magnitude of the sum of two vectors $(a_X\mathbf{i} + a_Y\mathbf{j})$ in X and Y direction, which are perpendicular to each other, is

$$a' = \sqrt{(a_X^2 + a_Y^2)}\ = (a_X^2 + a_Y^2)^{1/2}$$

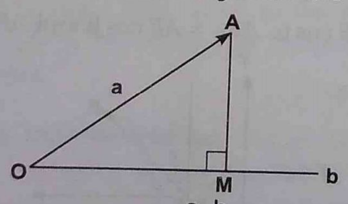
We observe here that resultant vector lies in the plane formed by the two component vectors being added. The resultant vector is, therefore perpendicular to third component vector.

Thus, the magnitude of the sum of vector $(a_X\mathbf{i} + a_Y\mathbf{j})$ and third vector $a_Z\mathbf{k}$, which are perpendicular to each other, is

$$\Rightarrow\ a = \sqrt{[\{(a_X^2 + a_Y^2)^{1/2}\}^2 + a_Z^2]} = \sqrt{(a_X^2 + a_Y^2 + a_Z^2)}$$

Results on Components of a Vector

- The scalar component of **a** along a non-zero vector **b** is



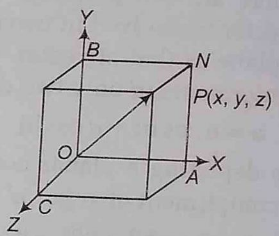
$$\mathrm{OM} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

- Vector component of **a** along **b** is $\mathrm{OM} = \dfrac{(\mathbf{a}\cdot\mathbf{b})}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$.
- Scalar component of **a** $\perp$ **b** is $\mathrm{MA} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$.
- Vector component of **a** $\perp$ **b** is $\mathrm{MA} = \mathbf{a} - \dfrac{(\mathbf{a}\cdot\mathbf{b})}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \dfrac{\mathbf{b}\times\mathbf{a}\times\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}$

Orthogonal System of Unit Vectors

Let OX, OY and OZ be three mutually perpendicular straight lines. Given any point $P(x, y, z)$ in space, we can construct the rectangular parallelopiped of which OP is a diagonal and $OA = x, OB = y, OC = z$. Here, A, B and C are $(x, 0, 0), (0, y, 0)$ and $(0, 0, z)$ respectively and L, M and N are $(0, y, z), (x, 0, z)$ and $(x, y, 0)$, respectively.

Let **i**, **j** and **k** denote unit vectors along OX, OY and OZ respectively. We have, $\mathbf{r} = \mathbf{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as $\mathbf{OA} = x\mathbf{i}$, $\mathbf{OB} = y\mathbf{j}$ and $\mathbf{OC} = z\mathbf{k}$.



$$\mathbf{ON} = \mathbf{OA} + \mathbf{AN}$$

$$\mathbf{OP} = \mathbf{ON} + \mathbf{NP}$$

So, $$\mathbf{OP} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}\ (\mathbf{NP} = \mathbf{OC}, \mathbf{AN} = \mathbf{OB})$$

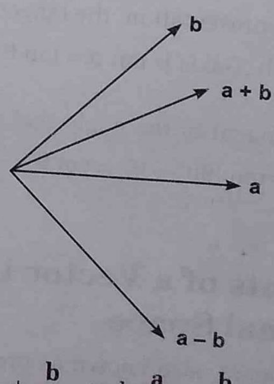
$$r = |\mathbf{r}| = |\mathbf{OP}| = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow\quad \mathbf{r} = \frac{\mathbf{r}}{|\mathbf{r}|}\ \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$$

$$\Rightarrow\quad \mathbf{r} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$$

Angular Bisectors

Let **a** and **b** are unit vectors. The internal bisector of angle between **a** and **b** is along **a** + **b**. External bisector of angle is along **a** − **b**. If **a** and **b** are not unit vectors, then above angle bisectors are



$$\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\ \text{and}\ \frac{\mathbf{a}}{|\mathbf{a}|} - \frac{\mathbf{b}}{|\mathbf{b}|}$$

These bisectors are perpendicular to each other.

Scalar (Dot) Product

The scalar product of two vectors **a** and **b** *is given by* $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta, 0<\theta\le\pi$.

Properties of Scalar Product

1. $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ (commutative)
2. $\mathbf{a}\cdot\mathbf{b}=0$, if $\mathbf{a}\perp\mathbf{b}$
3. $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$
4. $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$
   and $\mathbf{i}\cdot\mathbf{j}=\mathbf{i}\cdot\mathbf{k}=\mathbf{j}\cdot\mathbf{k}=\mathbf{j}\cdot\mathbf{i}=\mathbf{k}\cdot\mathbf{i}=\mathbf{k}\cdot\mathbf{j}=0$
5. If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$,
   then $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$
6. $\mathbf{a}\cdot(\alpha\cdot\mathbf{b})=\alpha(\mathbf{a}\cdot\mathbf{b})$
7. $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c})=\mathbf{a}\cdot\mathbf{b}\pm\mathbf{a}\cdot\mathbf{c}$
   (distributive)
8. For any two vectors **a** and **b**, we have
   (i) $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2+2(\mathbf{a}\cdot\mathbf{b})$
   (ii) $|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2(\mathbf{a}\cdot\mathbf{b})$
   (iii) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2-|\mathbf{b}|^2$
   (iv) $\mathbf{a}\cdot\mathbf{b}<0$ iff **a** and **b** are inclined at an obtuse angle.
   (v) $\mathbf{a}\cdot\mathbf{b}>0$ iff **a** and **b** are inclined at an acute angle.
9. If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ inclined at an angle θ, then
   $$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$
10. $|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2=(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})$
    $=|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2(\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a})$
11. If **r** is a vector making angles α,β and γ with OX, OY and OZ respectively, then
    $\cos\alpha=\mathbf{r}\cdot\mathbf{i}, \cos\beta=\mathbf{r}\cdot\mathbf{j}, \cos\gamma=\mathbf{r}\cdot\mathbf{k}$
    $\mathbf{r}=|\mathbf{r}|\cos\alpha\,\mathbf{i}+|\mathbf{r}|\cos\beta\,\mathbf{j}+|\mathbf{r}|\cos\gamma\,\mathbf{k}$
    If **r** is a unit vector, then
    $\mathbf{r}=(\cos\alpha)\,\mathbf{i}+(\cos\beta)\,\mathbf{j}+(\cos\gamma)\,\mathbf{k}$
12. If **a**, **b** and **c** are non-coplanar vectors in space, any vector **r** in space can be written as $\mathbf{r}=(\mathbf{r}\cdot\mathbf{a})\,\mathbf{a}+(\mathbf{r}\cdot\mathbf{b})\,\mathbf{b}+(\mathbf{r}\cdot\mathbf{c})\,\mathbf{c}$, where **a**, **b** and **c** are unit vectors along **a**, **b** and **c**, respectively.
13. If **r** is a non-zero vector in space and **a**, **b** and **c** are three vectors, such that $\mathbf{r}\cdot\mathbf{a}=\mathbf{r}\cdot\mathbf{b}=\mathbf{r}\cdot\mathbf{c}=0$
    $\Rightarrow$ **a**, **b** and **c** are coplanar.
14. If **r** is a non-zero coplanar to two given vectors **a** and **b**, then $\mathbf{r}\cdot\mathbf{a}=\mathbf{r}\cdot\mathbf{b}=0 \Rightarrow$ **a** and **b** are collinear.
15. **Work done** If a particle acted on by a force **F** has displacement **d**, then Work done $=\mathbf{F}\cdot\mathbf{d}$

- *The angle between two vectors* **a** *and* **b** *is defined as the smaller angle* θ *between them, when they are drawn with the same initial point.*
- *Usually, we take* $0\le\theta\le\pi$. *Angle between two like vectors is 0 and angle between two unlike vectors is* π.

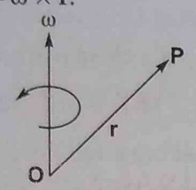
Vector (Cross) Product

The vector product of two vectors **a** and **b** *is given by* $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\cdot\hat{\mathbf{n}}$, *where* $\hat{\mathbf{n}}$ *is perpendicular to* **a** and **b** *and* $\theta\,(0\le\theta\le\pi)$ *is the angle between* **a** and **b**.

Properties of Vector Product

1. $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
2. $(\mathbf{a}\times\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2$
3. The vector perpendicular to both **a** and **b** is given by $\mathbf{a}\times\mathbf{b}$.
4. $m\mathbf{a}\times\mathbf{b}=m(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times m\mathbf{b}$
5. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=\mathbf{a}\times\mathbf{b}\pm\mathbf{a}\times\mathbf{c}$
   and $(\mathbf{b}\pm\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}\pm\mathbf{c}\times\mathbf{a}$
6. $\mathbf{a}\times\mathbf{b}=\mathbf{0}\Leftrightarrow\mathbf{a}\,||\,\mathbf{b}$,
   where **a** and **b** are non-null vectors.
7. If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$
   then, $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$
8. The unit vectors perpendicular to the plane of **a** and **b** are $\pm\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ and a vector of magnitude λ perpendicular to the plane of (**a** and **b** or **b** and **a**) is $\dfrac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.

9. **Rotating Body** A rigid body is spinning with angular velocity ω about an axis through θ. The velocity of a point P in the body is $\mathbf{v} = \omega \times \mathbf{OP} = \omega \times \mathbf{r}$.



10. If $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are three unit vectors along three mutually perpendicular lines, then
$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

11. **Moment of a Force** A force $\mathbf{F}$ is acting along a line through B. Its moment about the point A is $\mathbf{AB} \times \mathbf{F}$ *i.e.*, $\mathbf{r} \times \mathbf{F}$.

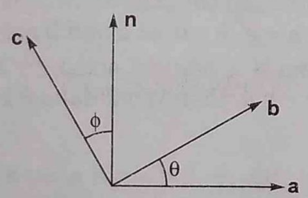
12. Let $\mathbf{a}$ and $\mathbf{b}$ be the two adjacent sides of a parallelogram $PQRS$.
   (i) Area of ΔPQR is $\frac{1}{2} |\mathbf{a} \times \mathbf{b}|$.
   (ii) Area of parallelogram $PQRS$ is $|\mathbf{a} \times \mathbf{b}|$.
   (iii) Area of quadrilateral $PQRS$ is $\frac{1}{2} |\mathbf{PR} \times \mathbf{SQ}|$, where PR and SQ are diagonals.
   (iv) Area of $\Delta PQR = \frac{1}{2} |\mathbf{PQ} \times \mathbf{PR}|$

Product of Three Vectors

There are two methods of product of three vectors

Scalar Triple Product

Let $\mathbf{a} \cdot \mathbf{b}$ are inclined at angle θ and $\mathbf{c}$ is inclined at an angle ϕ with the vector $\mathbf{a} \times \mathbf{b}$. Then,



$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = |\mathbf{a}||\mathbf{b}|(\sin\theta)\, \mathbf{n} \cdot \mathbf{c}$$
$$= |\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin\theta \cos\phi$$

It is the volume of the parallelopiped with concurrent edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and is denoted by the box product $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Properties of Scalar Triple Product

1. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$
2. If λ is a scalar, then $[\lambda\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \lambda\, [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
3. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}_1 + \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}_1] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}_2]$
4. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$
5. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$
6. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
7. The scalar triple product of three vectors is zero, if any two of them are equal or parallel or collinear.
8. If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$, then any two of the vectors are parallel or $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar.
   or $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$
9. Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ will be coplanar, if $[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
10. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
   and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
11. $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$
12. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \cdot [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{a}\cdot\mathbf{v} & \mathbf{a}\cdot\mathbf{w} \\ \mathbf{b}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{w} \\ \mathbf{c}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$

Vector Triple Product

If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three vector quantities, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ represents the vector triple product and is given by

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$$
$$\Rightarrow \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$

Properties of Vector Triple Product

1. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if and only if $\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}$
2. $\mathbf{a} \times \mathbf{b} \times \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a})$
3. Vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

4. **a** and **b** are two vectors as adjacent sides of a parallelogram, then the vector which is the altitude of the parallelogram and is perpendicular to **a** is

$$\frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{a}|}\mathbf{a}-\mathbf{b} \text{ or } \frac{1}{|\mathbf{a}|^2}\{|\mathbf{a}|^2\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}\}.$$

Linearly Dependent Vectors

1. If two non-zero vectors **a** and **b** are linearly dependent, then it means
   (i) there are non-zero scalar α and β such that $\alpha\,\mathbf{a}+\beta\mathbf{b}=\mathbf{0}$.
   (ii) **a** and **b** are parallel.
   (iii) **a** and **b** are collinear.
   (iv) $\mathbf{a}\times\mathbf{b}=\mathbf{0}$

   Otherwise, **a** and **b** are linearly independent.
2. If three non-zero vectors **a**, **b** and **c** are linearly dependent, then it means
   (i) there are non-zero scalars α,β and γ such that $\alpha\,\mathbf{a}+\beta\,\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$
   (ii) **a**, **b** and **c** are parallel to plane.
   (iii) **a**, **b** and **c** are coplanar.
   (iv) $\mathbf{a}=\alpha_1\mathbf{b}+\alpha_2\mathbf{c}$ etc.
   (v) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$

   Otherwise **a**, **b** and **c** are linearly independent.

A vector **r** *is said to be a linear combination of vectors* **a**, **b**, **c**, ... *etc., if there exist scalars x, y, z, ... etc., such that* $\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+\dots$

Reciprocal System of Vector

Let **a**, **b** and **c** be three non-coplanar vectors, so that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\neq 0$.

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

These three vectors are said to form a reciprocal system of vectors for the vectors **a**, **b** and **c**

- $(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}'+(\mathbf{b}+\mathbf{c})\cdot\mathbf{b}'+(\mathbf{c}+\mathbf{a})\cdot\mathbf{c}'=3$
- $[\mathbf{a}'\,\mathbf{b}'\,\mathbf{c}']=\frac{1}{[\mathbf{abc}]}$

Applications of Vectors in Geometry

1. 'The points A, B and C are collinear' means
   (i) Area of $\Delta\,ABC$ is zero.
   (ii) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are collinear vectors.
   (iii) $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are parallel.
   (iv) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$
   (v) There exist α,β and γ not all zero such that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$ and $\alpha+\beta+\gamma=0$.

   Otherwise A, B and C are not collinear.
2. 'A, B, C and D are coplanar' means.
   (i) Volume of tetrahedron $ABCD$ is zero.
   (ii) $\mathbf{b}-\mathbf{a},\mathbf{c}-\mathbf{a}$ and $\mathbf{d}-\mathbf{a}$ are coplanar.
   (iii) $[\mathbf{b}-\mathbf{a},\mathbf{c}-\mathbf{a},\mathbf{d}-\mathbf{a}]=0$
   (iv) There exist α,β,γ and δ not all zero such that $\alpha\,\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0$ and $\alpha+\beta+\gamma+\delta=0$

   Otherwise A, B, C and D are not coplanar.
3. If **a** and **b** are the PV of A and B and **r** be the PV of the point P which divides the join of A and B in the ratio $m:n$, then

$$\mathbf{r}=\frac{m\mathbf{b}\pm n\mathbf{a}}{m\pm n}$$

Here, '+' sign takes for internal ratio and '–' sign takes for external ratio.

4. If **a**, **b** and **c** be the PV of three vertices of ΔABC and **r** be the PV of the centroid of ΔABC, then

$$\mathbf{r}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$$

5. (i) Vector equation of the straight line passing through origin and parallel to $\mathbf{b}$ is given by $\mathbf{r} = t\mathbf{b}$, where t is scalar.
   (ii) Vector equation of the straight line passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is scalar.
   (iii) Vector equation of the straight line passing through $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{r} = \mathbf{a} + t\,(\mathbf{b} - \mathbf{a})$, where t is scalar.
6. (i) Vector equation of the plane through origin and parallel $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{c}$ is given by $\mathbf{r} = s\mathbf{b} + t\mathbf{c}$, where s and t are scalars.
   (ii) Vector equation of the plane passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ is given by $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ where, s and t are scalars.
   (iii) Vector equation of the plane passing through $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is
   $$\mathbf{r} = (1 - s - t)\,\mathbf{a} + s\mathbf{b} + t\mathbf{c},$$
   where s and t are scalars.
   (iv) Plane through the non-collinear points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is
   $$[\mathbf{r} - \mathbf{a}\;\mathbf{b} - \mathbf{a}\;\mathbf{c} - \mathbf{a}] = 0.$$
   (v) The distance of the point $\mathbf{a}$ from the plane $\mathbf{r} \cdot \mathbf{n} = p$ is
   $$\frac{|\mathbf{a} \cdot \mathbf{n} - p|}{|\mathbf{n}|}$$
   (vi) The foot of the perpendicular from the point $\mathbf{a}$ to the plane $\mathbf{r} \cdot \mathbf{n} = p$ is $\mathbf{a} - \dfrac{(\mathbf{a} \cdot \mathbf{n} - p)\,\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}$.
   (vii) The image of the point $\mathbf{a}$ in the plane $\mathbf{r} \cdot \mathbf{n} = p$ is
   $$\mathbf{a} - \frac{2\,(\mathbf{r} \cdot \mathbf{n} - p)\,\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}$$
7. Let the position vectors of quadrilateral $ABCD$ be a, b, c and d. Then,
   (i) Area of $\Delta ABC = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$
   $$= \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$
   (ii) Area of quadrilateral $ABCD$ is
   $$= \frac{1}{2}|(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{d})|$$
   (iii) Volume of a parallelopiped with edges DA, DB and DC is $|[\mathbf{a} - \mathbf{d}, \mathbf{b} - \mathbf{d}, \mathbf{c} - \mathbf{d}]|$

Tetrahedron and Its Properties

A tetrahedron is a three dimensional figure formed by four triangles.

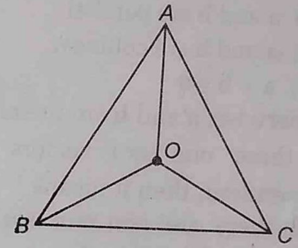
In figure, $OABC \to$ Tetrahedron

$\Delta ABC \to$ Base

$OAB, OBC, OCA \to$ Faces

OA, OB, OC, AB, BC and $CA \to$ Edges

OA, BC, OB, CA, OC and $AB \to$ Pair of opposite edges.



- A tetrahedron in which all edges are equal is called a regular tetrahedron.
- If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair is also perpendicular to each other.
- The sum of the squares of two opposite edges is the same for each pair of opposite edges.
- Any two opposite edges in a regular tetrahedron are perpendicular.
- Volume of a tetrahedron $ABCD$ is $\frac{1}{6}|[\mathbf{a} - \mathbf{d}, \mathbf{b} - \mathbf{d}, \mathbf{c} - \mathbf{d}]|$.
- Volume of a tetrahedron whose three coterminous edges are in the right handed system are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given $\frac{1}{6}[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$
- Centroid of tetrahedron is $\dfrac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$

Practice Zone

DAY 31

1. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$, then a vector $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$ is **[NCERT Exemplar]**

(a) $\frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ (b) $\frac{2}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{5}{3}\mathbf{k}$

(c) $\frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ (d) None of these

2. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors inclined at an angle α, $\alpha \in [0, \pi]$ to each other and $|\mathbf{a} + \mathbf{b}| < 1$. Then, α belong to

(a) $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ (b) $\left(\frac{2\pi}{3}, \pi\right)$

(c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

3. A unit vector $\mathbf{d}$ is equally inclined at an angle α with the vectors $\mathbf{a} = \cos\theta \cdot \mathbf{i} + \sin\theta \cdot \mathbf{j}$, $\mathbf{b} = -\sin\theta \cdot \mathbf{i} + \cos\theta \cdot \mathbf{j}$ and $\mathbf{c} = \mathbf{k}$. Then, α is equal to

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\cos^{-1}\frac{1}{3}$ (d) $\frac{\pi}{2}$

4. Vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $|\mathbf{a}| = 1$, $|\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 2$. If $\mathbf{c} = 2\mathbf{a} \times \mathbf{b} - 3\mathbf{b}$, then the angle between $\mathbf{b}$ and $\mathbf{c}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

5. Let $\mathbf{p} = 3ax^2\,\mathbf{i} - 2(x-1)\,\mathbf{j}$, $\mathbf{q} = b(x-1)\,\mathbf{i} + x\mathbf{j}$ and $ab < 0$. Then, $\mathbf{p}$ and $\mathbf{q}$ are parallel for atleast one x in

(a) $(0, 1)$ (b) $(-1, 0)$

(c) $(1, 2)$ (d) None of these

6. Let $\mathbf{b}$ and $\mathbf{c}$ be non-collinear vectors. If $\mathbf{a}$ is a vector such that $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 4$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (x^2 - 2x + 6)\,\mathbf{b} + \sin y \cdot \mathbf{c}$ then (x, y) lies on the line

(a) $x + y = 0$ (b) $x - y = 0$

(c) $x = 1$ (d) $y = \pi$

7. The distance of the point $3\mathbf{i} + 5\mathbf{k}$ from the line parallel to $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and passing through the point $8\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ is

(a) 1 (b) 2 (c) 3 (d) 4

8. Let $|\mathbf{a}| = 2\sqrt{2}$, $|\mathbf{b}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$. If a parallelogram is constructed with adjacent sides $2\mathbf{a} - 3\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$, then its longer diagonal is of length

(a) 10 (b) 8 (c) $2\sqrt{26}$ (d) 6

9. If $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \cos\theta$, then the maximum value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{2\pi}{5}$

10. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and a unit vector $\mathbf{c}$ be coplanar. If $\mathbf{c}$ is perpendicular to $\mathbf{a}$, then $\mathbf{c}$ is equal to **[NCERT Exemplar]**

(a) $(-\mathbf{j} + \mathbf{k})$ (b) $\pm\frac{1}{\sqrt{2}}(-\mathbf{j} + \mathbf{k})$

(c) $\pm\frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k})$ (d) None of these

11. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$, then λ is equal to **[NCERT Exemplar]**

(a) $\frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]}$ (b) $\frac{[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]}{[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]}$

(c) $\frac{[\mathbf{b}\ \mathbf{d}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ (d) None of these

12. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $|\mathbf{c}| = 1$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

(a) 0 (b) 1 (c) $|\mathbf{a}|^2\,|\mathbf{b}|^2$ (d) $|\mathbf{a} \times \mathbf{b}|^2$

13. The distance of the point B with position vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ from the line passing through the point A whose position vector is $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ is

(a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $\sqrt{6}$ (d) $\sqrt{8}$

14. Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of points A and B with respect to origin and $|\mathbf{a}| = a$, $|\mathbf{b}| = b$. The points C and D divide AB internally and externally in the ratio $2:3$. If $\mathbf{OC}$ and $\mathbf{OD}$ are perpendicular, then

(a) $9a^2 = 4b^2$ (b) $4a^2 = 9b^2$

(c) $9a = 4b$ (d) $4a = 9b$

15. If the positive numbers a, b and c are the pth, qth and rth terms of GP, then the vectors $\log a \cdot \mathbf{i} + \log b \cdot \mathbf{j} + \log c \cdot \mathbf{k}$ and $(q-r)\mathbf{i} + (r-p)\mathbf{j} + (p-q)\mathbf{k}$ are

(a) equal
(b) parallel
(c) perpendicular
(d) None of these

16. A vector of magnitude $\sqrt{2}$ coplanar with $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is

(a) $-\mathbf{j}+\mathbf{k}$ (b) $\mathbf{i}-\mathbf{k}$ (c) $\mathbf{i}-\mathbf{j}$ (d) $\mathbf{i}-2\mathbf{j}+\mathbf{k}$

17. If $\mathbf{a}$ is a unit vector and projection of $\mathbf{x}$ along $\mathbf{a}$ is 2 and $\mathbf{a}\times\mathbf{r}+\mathbf{b}=\mathbf{r}$, then $\mathbf{r}$ is equal to

(a) $\frac{1}{2}[\mathbf{a}-\mathbf{b}+\mathbf{a}\times\mathbf{b}]$
(b) $\frac{1}{2}[2\mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}]$
(c) $\mathbf{a}+\mathbf{a}\times\mathbf{b}$
(d) $\mathbf{a}-\mathbf{a}\times\mathbf{b}$

18. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$, then $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|^2$ is equal to

(a) 1/3 (b) 1/2 (c) 1 (d) 2

19. A line passes through $(2, -1, 3)$ and is perpendicular to the line $\mathbf{r}=(\mathbf{i}+\mathbf{j}-\mathbf{k})+\lambda(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r}=(2\mathbf{i}-\mathbf{j}-3\mathbf{k})+\mu(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$. Then, its equation is

(a) $\mathbf{r}=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})+\mu(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$
(b) $\mathbf{r}=(2\mathbf{i}+\mathbf{j}-3\mathbf{k})+\mu(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$
(c) $\mathbf{r}=(\mathbf{i}+\mathbf{j}+3\mathbf{k})+\mu(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$
(d) None of the above
(where, $\mu=-2\lambda$)

20. If a vector $\mathbf{r}$ of magnitude $3\sqrt{6}$ is directed along the bisector of the angle between the vectors $\mathbf{a}=7\mathbf{i}-4\mathbf{j}-4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$, then $\mathbf{r}$ is equal to

(a) $\mathbf{i}-7\mathbf{j}+2\mathbf{k}$
(b) $\mathbf{i}+7\mathbf{j}-2\mathbf{k}$
(c) $\mathbf{i}+7\mathbf{j}+2\mathbf{k}$
(d) $\mathbf{i}-7\mathbf{j}-2\mathbf{k}$

21. If $\mathbf{b}=4\mathbf{i}+3\mathbf{j}$ and $\mathbf{c}$ be two vectors perpendicular to each other in the xy-plane. Then, a vector in the same plane having projections 1 and 2 along $\mathbf{b}$ and $\mathbf{c}$ respectively, is

(a) $\mathbf{i}+2\mathbf{j}$
(b) $2\mathbf{i}-\mathbf{j}$
(c) $2\mathbf{i}+\mathbf{j}$
(d) None of these

22. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. Which one of the following is correct?

(a) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}=\mathbf{0}$
(b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}\neq\mathbf{0}$
(c) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{c}=\mathbf{0}$
(d) $\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ are mutually perpendicular.

23. The unit vector which is orthogonal to the vector $3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ and is coplanar with the vectors $2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$, is

(a) $\frac{2\mathbf{i}-6\mathbf{j}+\mathbf{k}}{\sqrt{41}}$
(b) $\frac{2\mathbf{i}-3\mathbf{j}}{\sqrt{13}}$
(c) $\frac{3\mathbf{j}-\mathbf{k}}{\sqrt{10}}$
(d) $\frac{4\mathbf{i}+3\mathbf{j}-3\mathbf{k}}{\sqrt{34}}$

24. Let $\mathbf{v}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{w}=\mathbf{i}+3\mathbf{k}$. If $\mathbf{u}$ is a unit vector, then the maximum value of $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is

(a) -1
(b) $\sqrt{10}+\sqrt{6}$
(c) $\sqrt{59}$
(d) $\sqrt{60}$

25. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors, then $|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{c}-\mathbf{a}|^2$ does not exceed to

(a) 4
(b) 9
(c) 8
(d) 6

26. If the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form the sides BC, CA and AB respectively of ΔABC, then

(a) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=0$
(b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$
(c) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$
(d) $\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=0$

27. The projection of the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ on the line whose vector equation is $\mathbf{r}=(2+t)\mathbf{i}+(t-1)\mathbf{j}+3t\,\mathbf{k}$, where t being a scalar is

(a) $\frac{5}{\sqrt{11}}$ (b) $\frac{5}{\sqrt{13}}$ (c) $\frac{7}{\sqrt{11}}$ (d) $\frac{7}{\sqrt{13}}$

28. If $\mathbf{a}=\mathbf{i}-\mathbf{j}$, $\mathbf{b}=\mathbf{i}-2\mathbf{k}$, then the point of intersection of the lines $\mathbf{r}\times\mathbf{a}=\mathbf{b}\times\mathbf{a}$ and $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ is

(a) $\mathbf{i}+\mathbf{j}-\mathbf{k}$
(b) $\mathbf{i}-\mathbf{j}+\mathbf{k}$
(c) $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$
(d) None of these

29. The vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. The value of x, for which the vectors, $\mathbf{c}=(2x+3)\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=(2x+3)\mathbf{a}-\mathbf{b}$ are collinear is

(a) 1/2
(b) 1/3
(c) $-\frac{3}{2}$
(d) None of these

30. The values of x for which the angle between the vectors $2x^2\mathbf{i}+4x\mathbf{j}+\mathbf{k}$ and $7\mathbf{i}-2\mathbf{j}+x\mathbf{k}$ is obtuse and the angle between the z-axis and $7\mathbf{i}-2\mathbf{j}+x\mathbf{k}$ is acute and less than $\frac{\pi}{6}$ is given by

(a) $0<x<\frac{1}{2}$
(b) $x>\frac{1}{2}$ or $x<0$
(c) $\frac{1}{2}<x<15$
(d) No such value for x

31. Let G_1, G_2 and G_3 be the centroids of the triangular faces OBC, OCA and OAB of a tetrahedron $OABC$. If V_1 denote the volume of the tetrahedron $OABC$ and V_2 that of the parallelopiped with OG_1, OG_2 and OG_3 as three concurrent edges, then

(a) $4V_1=9V_2$
(b) $9V_1=4V_2$
(c) $3V_1=2V_2$
(d) $3V_2=2V_1$

32. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are position vectors of three non-collinear points A, B and C respectively, the shortest distance of A from BC is

(a) $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})$
(b) $\sqrt{|\mathbf{b}-\mathbf{a}|^2-\left\{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{c}|}\right\}^2}$
(c) $|\mathbf{b}-\mathbf{a}|$
(d) None of these

33. Let $\vec{\alpha} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, $\vec{\beta} = b\mathbf{i} + c\mathbf{j} + a\mathbf{k}$ and $\vec{\gamma} = c\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ be three coplanar vectors with $a \neq b$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then, $\mathbf{v}$ is perpendicular to

(a) $\vec{\alpha}$ (b) $\vec{\beta}$
(c) $\vec{\gamma}$ (d) All of these

34. $\mathbf{a}$ and $\mathbf{c}$ are unit collinear vectors and $|\mathbf{b}| = 6$, then $\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$, if λ is

(a) $-9, 3$ (b) $9, 3$
(c) $3, -3$ (d) None of these

35. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors, then the projection of the vector $l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$ along the angle bisector of the vector $\mathbf{a}$ and $\mathbf{b}$ is

(a) $\frac{l^2+m^2}{\sqrt{l^2+m^2\ n^2}}$ (b) $\sqrt{l^2+m^2+n^2}$
(c) $\frac{\sqrt{l^2+m^2}}{\sqrt{l^2+m^2+n^2}}$ (d) $\frac{l+m}{\sqrt{2}}$

36. Let us define the length of a vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ as $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|$. This definition coincides with the usual definition of length of a vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ iff

(a) $\mathbf{a} = \mathbf{b} = \mathbf{c} = 0$
(b) any two of a, b and c are zero
(c) any one of a, b and c are zero
(d) $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

37. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. If $\hat{\mathbf{n}}$ is a unit vector such that $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ and $\bar{\mathbf{v}}\cdot\hat{\mathbf{n}} = 0$, then $|\mathbf{w}\cdot\hat{\mathbf{n}}|$ is equal to

(a) 3 (b) 0
(c) 1 (d) 2

38. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then the greatest value of $|\mathbf{a}+\mathbf{b}| + |\mathbf{a}-\mathbf{b}|$ is

(a) 2 (b) 4
(c) $2\sqrt{2}$ (d) $\sqrt{2}$

39. If V is the volume of the parallelopiped having three coterminus edges, as $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, then the volume of the parallelopiped having three coterminus edges as

$$\alpha = (\mathbf{a}\cdot\mathbf{a})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b} + (\mathbf{a}\cdot\mathbf{c})\mathbf{c}$$
$$\beta = (\mathbf{a}\cdot\mathbf{b})\mathbf{a} + (\mathbf{b}\cdot\mathbf{b})\mathbf{b} + (\mathbf{b}\cdot\mathbf{c})\mathbf{c}$$
$$\gamma = (\mathbf{a}\cdot\mathbf{c})\mathbf{a} + (\mathbf{b}\cdot\mathbf{c})\mathbf{b} + (\mathbf{c}\cdot\mathbf{c})\mathbf{c}$$

(a) V^3 (b) $3V$
(c) V^2 (d) $2V$

40. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and r is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for

(a) no value of λ
(b) all except one value of λ
(c) all except two values of λ
(d) all values of λ

41. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors such that

$$|\mathbf{u}| = 1, |\mathbf{v}| = 2, |\mathbf{w}| = 3$$

If the projection of $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other, then $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ is equal to

(a) 4 (b) $\sqrt{7}$ (c) $\sqrt{14}$ (d) 2

42. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that no two are collinear and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ is equal to

(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

43. Resolved part of vector a along the vector $\mathbf{b}$ is $\mathbf{a}_1$ and that perpendicular to $\mathbf{b}$ is $\mathbf{a}_2$, then $\mathbf{a}_1 \times \mathbf{a}_2$ is equal to

(a) $\frac{(\mathbf{a}\times\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$ (b) $\frac{(\mathbf{a}\times\mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}$
(c) $\frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}|^2}$ (d) $\frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}\times\mathbf{a}|}$

Directions (Q. Nos. 44 to 46) *P is a point of intersection of*

$$x^2 + y^2 = 20 \quad \ldots(i)$$

and $$y = 2 \quad \ldots(ii)$$

in the first quadrant. Q is a point on Eq. (ii), in the first quadrant such that it is an interior point of Eq. (i) and whose abscissae is an integer.

44. The sum of the projection of $\mathbf{OQ}$ on $\mathbf{OP}$ (O is the origin) for possible of Q is

(a) $\frac{18\sqrt{5}}{5}$ (b) $3\sqrt{5}$
(c) $\frac{12\sqrt{5}}{5}$ (d) $\sqrt{5}$

45. The sum of $|\mathbf{OP}\times\mathbf{OQ}|$ for possible positions of Q is

(a) 6 (b) 8 (c) 12 (d) 10

46. Let R be $(-5, 2)$, then ratio of the maximum of $|\mathbf{OR}|$ and the sum of the projectionis of $\mathbf{OQ}$ on y-axis for possible positions of Q is

(a) 1 : 2 (b) 1 : 1
(c) 4 : 3 (d) 5 : 4

Directions (Q. Nos. 47 and 48) *Let A be the given point whose position vector relative to an origin O be* $\mathbf{a}$ *and* $\mathbf{ON} = \mathbf{n}$. *Let* $\mathbf{r}$ *be the position vector of any point P which lies on the plane and passing through* $\mathbf{a}$ *and perpendicular to* $\mathbf{ON}$. *Then, for any point P on the plane*

$$\mathbf{AP}\cdot\mathbf{n} = 0$$
$$\Rightarrow \quad (\mathbf{r} - \mathbf{a})\cdot\mathbf{n} = 0 \Rightarrow \mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$$
$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{n} = p$$

where, P is perpendicular distance of the plane from origin.

47. The equation of the plane through the point $2i - j - 4k$ and parallel to the plane $r \cdot (4i - 12j - 3k) - 7 = 0$ is

(a) $r \cdot (4i - 12j - 3k) = 0$ (b) $r \cdot (4i - 12i - 3k) = 16$
(c) $r \cdot (4i - 12j - 3k) = 24$ (d) $r \cdot (4i - 12j - 3k) = 32$

48. The equation of the plane through the point $2i + j - k$ and passing through the line of intersection of the planes $r \cdot (i + 3j - k) = 0$ and $r \cdot (j + 2k) = 0$ is

(a) $r \cdot (4i + 9j + 11k) = 6$ (b) $r \cdot (3i + 9j - 3k) = 18$
(c) $r \cdot (i + 9j + 11k) = 0$ (d) $r \cdot (i + 9j - 11k) = 22$

Directions (Q. Nos. 49 to 53) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

49. Statement I If u and v are unit vectors inclined at an angle α and x is a unit vector bisecting the angle between them, then $x = \dfrac{u + v}{2\cos\dfrac{\alpha}{2}}$.

Statement II If ΔABC is an isosceles triangle with $AB = AC = 1$, then vector representing bisector of angle A is given by $AD = \dfrac{AB + AC}{2}$.

50. Statement I A relation between the vectors $r \cdot a$ and b is $r \times a = b \Rightarrow r = \dfrac{a \times b}{a \cdot a}$.

Statement II $r \cdot a = 0$.

51. Statement I For $a = -\dfrac{1}{\sqrt{3}}$ the volume of the parallelopiped formed by vectors $i + aj$, $ai + j + k$ and $j + ak$ is maximum.

Statement II The volume of the parallelopiped having three coterminous edges a, b and c is $|[a\ b\ c]|$.

52. Statement I If a, b and c are coplanar, then $a \times b, b \times c$ and $c \times a$ are also coplanar.

Statement II $[a \times b\ b \times c\ c \times a] = 2\,[a\ b\ c]^2$.

53. Statement I $|a| = |b|$ does not implies that $a = b$.

Statement II If $a = b$, then $a \cdot b = |a|^2 = |b|^2$.

54. Let $a = 2i - j + k, b = i + 2j - k$ and $c = i + j - 2k$ be three vectors. A vector of the type $b + \lambda c$ for some scalar λ, whose projection on a is of magnitude $\sqrt{\dfrac{2}{3}}$, is

(a) $2i + j + 5k$
(b) $2i + 3j - 3k$
(c) $2i - j + 5k$
(d) $2i + 3j + 3k$

55. The vector $(i \times a \cdot b)\, i + (j \times a \cdot b)\, j + (k \times a \cdot b)\, k$ is equal to

(a) $b \times a$ (b) a
(c) $a \times b$ (d) b

56. If a, b and c are unit vectors satisfying $a - \sqrt{3}\, b + c = 0$, then the angle between the vectors a and c is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$
(c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{2}$

57. If a and b are non-collinear vectors, then the value of α for which the vectors $u = (\alpha - 2)a + b$ and $v = (2 + 3\alpha)a - 3b$ are collinear is

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$
(c) $\dfrac{-3}{2}$ (d) $\dfrac{-2}{3}$

58. Let $a = 2i + j - 2k, b = i + j$. If c is a vector such that $a \cdot c = |c|, |c - a| = 2\sqrt{2}$ and the angle between $a \times b$ and c is $30°$, then $|(a \times b) \times c|$ is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{3\sqrt{3}}{2}$
(c) 3 (d) $\dfrac{3}{2}$

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59. If the vectors $AB = 3i + 4k$ and $AC = 5i - 2j + 4k$ are the sides of a ΔABC, then the length of the median through A is **[JEE Main 2013]**

(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{45}$

60. Let a and b be two unit vectors. If the vectors $c = a + 2b$ and $d = 5a - 4b$ are perpendicular to each other, then the angle between a and b is **[AIEEE 2012]**

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{4}$

61. Let $ABCD$ be a parallelogram such that $AB = q$, $AB = p$, and $\angle BAD$ be an acute angle. If r is the vector that coincides with the altitude directed from the vertex B to the side AD, then r is given by **[AIEEE 2012]**

(a) $r = 3q - \dfrac{3(p \cdot q)}{(p \cdot q)}p$ (b) $r = -q + \left(\dfrac{p \cdot q}{q \cdot p}\right)p$
(c) $r = q - \left(\dfrac{p \cdot q}{p \cdot q}\right)p$ (d) $r = -3q + \dfrac{3(p \cdot q)}{(p \cdot q)}p$

62. If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{k})$ and $\mathbf{b} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$, then the value of $(2\mathbf{a} - \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is **[AIEEE 2011]**

(a) -3 (b) 5 (c) 3 (d) -5

63. Vectors **a** and **b** are not perpendicular and **c** and **d** are two vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \cdot \mathbf{d} = 0$. Then, the vector **d** is equal to **[AIEEE 2011]**

(a) $\mathbf{c} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} + \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} - \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

64. Let **a**, **b** and **c** be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with **c** and $\mathbf{b} + 2\mathbf{c}$ is collinear with **a**, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is equal to **[AIEEE 2011]**

(a) $\mathbf{a} + \mathbf{c}$ (b) $\mathbf{a}$

(c) $\mathbf{c}$ (d) 0

65. If the vectors $p\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + q\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + r\mathbf{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is **[AIEEE 2011]**

(a) -2 (b) 2

(c) 0 (d) -1

66. If $\mathbf{a} = \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Then, the vector **b** satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 3$, is **[AIEEE 2010]**

(a) $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (b) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

(c) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (d) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

67. If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal, then (λ, μ) is equal to **[AIEEE 2010]**

(a) $(-3, 2)$ (b) $(2, -3)$ (c) $(-2, 3)$ (d) $(3, -2)$

68. If **u**, **v** and **w** are non-coplanar vectors and p and q are real numbers, then the equality $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$ holds for **[AIEEE 2009]**

(a) exactly two values of (p, q)

(b) more than two but not all values of (p, q)

(c) all values of (p, q)

(d) exactly one value of (p, q)

69. The vector $\mathbf{a} = \alpha\mathbf{i} + 2\mathbf{j} + \beta\mathbf{k}$ lies in the plane of the vectors $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$ and bisects the angle between **b** and **c**. Then, which one of the following gives possible values of α and β? **[AIEEE 2008]**

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = 2$

(c) $\alpha = 1, \beta = 2$ (d) $\alpha = 2, \beta = 1$

70. The non-zero vectors **a**, **b** and **c** are related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then, the angle between **a** and **c** is **[AIEEE 2008]**

(a) π (b) 0

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

71. If the edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors **a**, **b** and **c** such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 1/2$. Then, the volume of the parallelopiped is

(a) $\frac{1}{\sqrt{2}}$ cu unit (b) $\frac{1}{2\sqrt{2}}$ cu unit

(c) $\frac{\sqrt{3}}{2}$ cu unit (d) $\frac{1}{\sqrt{3}}$ cu unit

72. If **u** and **v** are unit vectors and θ is the acute angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for **[AIEEE 2007]**

(a) exactly two values of θ

(b) more than two values of θ

(c) no value of θ

(d) exactly one value of θ

73. ABC is triangle, right angled at A. The resultant of the forces acting along **AB** and **AC** with magnitudes $\frac{1}{AB}$ and $\frac{1}{AC}$ respectively is the force along **AD**, where D is the foot of the perpendicular from A onto BC. The magnitude of the resultant is **[AIEEE 2006]**

(a) $\frac{(AB)(AC)}{AB + AC}$ (b) $\frac{1}{AB} + \frac{1}{AC}$

(c) $\frac{1}{AD}$ (d) $\frac{AB^2 + AC^2}{(AB)^2 (AC)^2}$

74. The distance between the line $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ is **[AIEEE 2005]**

(a) $\frac{10}{3\sqrt{3}}$ (b) $\frac{10}{9}$

(c) $\frac{10}{3}$ (d) $\frac{3}{10}$

75. Let **a**, **b** and **c** are non-zero vectors such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is acute angle between the vectors **b** and **c**, then $\sin\theta$ is equal to **[AIEEE 2004]**

(a) $\frac{1}{3}$ (b) $\frac{\sqrt{2}}{3}$

(c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

76. If **u**, **v** and **w** are three non-coplanar vectors, then $(\mathbf{u} + \mathbf{v} - \mathbf{w}) \cdot (\mathbf{u} - \mathbf{v}) \times (\mathbf{v} - \mathbf{w})$ is equal to **[AIEEE 2003]**

(a) **0** (b) $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$

(c) $\mathbf{u} \cdot \mathbf{w} \times \mathbf{v}$ (d) $3\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$

77. If the vectors **c**, $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$ are such that **a**, **c** and **b** form a right handed system, then **c** is equal to **[AIEEE 2002]**

(a) $z\mathbf{i} - x\mathbf{k}$ (b) **0**

(c) $y\mathbf{j}$ (d) $-z\mathbf{i} + x\mathbf{k}$

Answers

1. (a)	2. (b)	3. (b)	4. (b)	5. (d)	6. (c)	7. (c)	8. (c)	9. (c)	10. (b)
11. (b)	12. (d)	13. (a)	14. (a)	15. (c)	16. (a)	17. (b)	18. (c)	19. (a)	20. (a)
21. (b)	22. (b)	23. (c)	24. (c)	25. (b)	26. (b)	27. (a)	28. (c)	29. (c)	30. (d)
31. (a)	32. (d)	33. (d)	34. (a)	35. (d)	36. (b)	37. (a)	38. (c)	39. (a)	40. (c)
41. (c)	42. (a)	43. (c)	44. (a)	45. (c)	46. (c)	47. (d)	48. (c)	49. (a)	50. (b)
51. (d)	52. (c)	53. (b)	54. (b)	55. (c)	56. (b)	57. (b)	58. (d)	59. (c)	60. (c)
61. (b)	62. (d)	63. (c)	64. (d)	65. (a)	66. (a)	67. (a)	68. (d)	69. (a)	70. (a)
71. (a)	72. (d)	73. (c)	74. (a)	75. (d)	76. (b)	77. (a)			

Hints & Solutions

1. Given, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$

Let $\quad \mathbf{c} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$

Now, $\quad \mathbf{a} \times \mathbf{c} = \mathbf{b} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\Rightarrow \quad (z - y)\mathbf{i} - \mathbf{j}(z - x) + \mathbf{k}(y - x) = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$

On comparing, we get

$$z - y = 0 \Rightarrow y = z \qquad \ldots\text{(i)}$$

$$-z + x = 1 \Rightarrow x = 1 + z \qquad \ldots\text{(ii)}$$

and $\quad y - x = -1 \qquad \ldots$(iii)

Also, $\quad \mathbf{a} \cdot \mathbf{c} = 3 \Rightarrow (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3$

$\Rightarrow \quad x + y + z = 3 \qquad \ldots$(iv)

On putting the values of x and y from Eqs. (i) and (ii) in Eq. (iv), we get

$$(1 + z) + z + z = 3$$

$$\Rightarrow \quad 3z = 2 \Rightarrow z = \frac{2}{3}$$

On putting the value of z in Eqs. (i) and (ii), we get

$$y = \frac{2}{3} \text{ and } x = \frac{5}{3}$$

These values of x and y also satisfy Eq. (iii), we get

$$x = \frac{5}{3}, y = \frac{2}{3}, z = \frac{2}{3}$$

Hence, $\mathbf{c} = \frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, which is the required vector.

2. Since, $\quad |\mathbf{a} + \mathbf{b}|^2 < 1$

$\Rightarrow \quad 2 + 2\cos\alpha < 1$

$\Rightarrow \quad 4\cos^2\frac{\alpha}{2} < 1$

$\Rightarrow \quad \cos\frac{\alpha}{2} < \frac{1}{2}$

$\Rightarrow \quad \alpha \in \left(\frac{2\pi}{3}, \pi\right)$

3. Let $\mathbf{d} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \cos\alpha$

$\Rightarrow \quad \mathbf{d} \cdot (\mathbf{a} - \mathbf{k}) = 0$

$\mathbf{d}$ is parallel to $(\mathbf{a} - \mathbf{k}) \times (\mathbf{b} - \mathbf{k})$

$\therefore \quad d = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -\sin\theta & \cos\theta & -1 \end{vmatrix}$

$\Rightarrow \quad \mathbf{d} = \dfrac{(\cos\theta - \sin\theta)\,\mathbf{i} + (\cos\theta + \sin\theta)\,\mathbf{j} + \mathbf{k}}{\sqrt{3}}$

$$\cos\alpha = \mathbf{d} \cdot \mathbf{k} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

4. Now, $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = 16 - 4 = 12$

and $\quad |\mathbf{c}|^2 = (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b}) \cdot (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b})$

$= 4|\mathbf{a} \times \mathbf{b}|^2 + 9|\mathbf{b}|^2 = 4 \cdot 12 + 9 \cdot 16$

$= 192$

$\Rightarrow \quad |\mathbf{c}| = 8\sqrt{3}$

Now, $\quad \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b})$

$= -3|\mathbf{b}|^2 = -48$

$\because \quad \cos\theta = \dfrac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = -\dfrac{48}{4 \cdot 8\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \theta = \dfrac{5\pi}{6}$

5. Hence, $\mathbf{p} \times \mathbf{q} = \{3ax^3 + 2b(x - 1)^2\}k = f(x)\,k$,

where, $f(0)\,f(1) = 6ab < 0$

$\therefore$ By intermediate value theorem there exists, x in $(1, 0)$ such that $f(x) = 0$.

6. By expanding $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, we get

$$\mathbf{a} \cdot \mathbf{c} = x^2 - 2x + 6, \mathbf{a} \cdot \mathbf{b} = -\sin y$$

Given, $\quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 4 \Rightarrow x^2 - 2x + 2 = \sin y$

$\Rightarrow \quad \sin y = x^2 - 2x + 2 = (x - 1)^2 + 1 \ge 1$

But $\quad \sin y \le 1$

So, both sides are equal only for $x = 1$.

7. We know that the distance of the point $\mathbf{c}$ from the line $\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$ is $\dfrac{|(\mathbf{c} - \mathbf{a}) \times \mathbf{b}|}{|\mathbf{b}|}$.

Hence, $c = 3\mathbf{i} + 5\mathbf{k}$

$a = 8\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$b = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Now, $(\mathbf{c}-\mathbf{a})\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -5 & -3 & 4\\ 6 & 1 & -2\end{vmatrix}=2\mathbf{i}+14\mathbf{j}+13\mathbf{k}$

$\therefore$ Distance $=\dfrac{\sqrt{4+196+169}}{\sqrt{36+1+4}}=\sqrt{9}=3$

8. Now, $\mathbf{a}\cdot\mathbf{b}=2\sqrt{2}\cdot3\cdot\dfrac{1}{\sqrt{2}}=6$

The diagonals are $2\mathbf{a}-3\mathbf{b}\pm(\mathbf{a}+\mathbf{b})$.

$\therefore$ Length of diagonals are

$|3\mathbf{a}-2\mathbf{b}|^2=9\cdot8+4\cdot9-12\cdot6=36$

and $|\mathbf{a}-4\mathbf{b}|^2=8+16\cdot9-8\cdot6=104$

So, the length of the longer diagonal is $\sqrt{104}$ *i.e.*, $2\sqrt{26}$.

9. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c}\end{vmatrix}$

$=\begin{vmatrix}1 & \cos\theta & \cos\theta\\ \cos\theta & 1 & \cos\theta\\ \cos\theta & \cos\theta & 1\end{vmatrix}$

$=1-3\cos^2\theta+2\cos^3\theta$

$=(1-\cos\theta)^2\,(1+2\cos\theta)$

$\Rightarrow\quad 1+2\cos\theta\ge0$

$\Rightarrow\quad \theta\le\dfrac{2\pi}{3}$

10. $\mathbf{a}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{a}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\mathbf{b}$

$=3(2\mathbf{i}+\mathbf{j}+\mathbf{k})-6(\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$=-9\mathbf{j}+9\mathbf{k}$

$\therefore$ Required unit vector $=\dfrac{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times(\mathbf{a}\times\mathbf{b})|}$

$=\pm\dfrac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$

11. Given $\quad\mathbf{d}=\lambda\mathbf{a}+\mu\mathbf{b}+v\mathbf{c}$

$\therefore\quad \mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})=\lambda[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+0+0$

$=\lambda[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow\quad \lambda=\dfrac{[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]}{[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]}$

12. If $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{0}$, then $\mathbf{c}=\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$,

$\therefore\quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=|\mathbf{a}\times\mathbf{b}|$

$\Rightarrow\quad \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2=|\mathbf{a}\times\mathbf{b}|^2$

13. Here, $\mathbf{AB}=-3\mathbf{i}+\mathbf{k}$

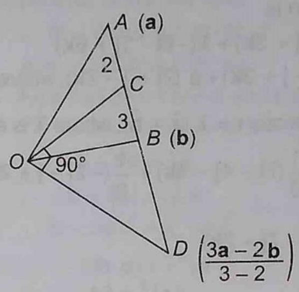
Now, $\mathbf{AB}\cdot(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})=-6+6=0$

Hence, **AB** is perpendicular to the given line.

Thus, the required distance

$=|\mathbf{AB}|=\sqrt{9+1}=\sqrt{10}$

14. $\left(\dfrac{3\mathbf{a}+2\mathbf{b}}{5}\right)\cdot(3\mathbf{a}-2\mathbf{b})=0$



$\Rightarrow\quad 9|\mathbf{a}|^2-4|\mathbf{b}|^2=0$

$\therefore\quad 9a^2=4b^2$

15. Let first term and common ratio of a GP be α and β. Then, $a=\alpha\cdot\beta^{p-1}$, $b=\alpha\cdot\beta^{q-1}$, $c=\alpha\cdot\beta^{r-1}$

$\therefore \log a=\log\alpha+(p-1)\log\beta$, etc.

The dot product of the given two vectors is

$\sum\{\log\alpha+(p-1)\log\beta\}\,(q-r)$

$=(\log\alpha-\log\beta)\sum(q-r)+\log\beta\sum p\,(q-r)$

$=0$

Hence, given vectors are prependicular

16. A vector coplanar with $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is

$\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+\mathbf{k})=(1+\lambda)\mathbf{i}+(1+2\lambda)\mathbf{j}+(2+\lambda)\mathbf{k}$

It is perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$\therefore\quad 1+\lambda+1+2\lambda+2+\lambda=0,\Rightarrow\lambda=-1$

So, the required vector is $-\mathbf{j}+\mathbf{k}$

17. Here, $\mathbf{a}\cdot\mathbf{x}=2$ and $\mathbf{a}\times\mathbf{r}+\mathbf{b}=\mathbf{r}$ ...(i)

Dot product of Eq. (i) with **a** gives, $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{r}=2$

Cross product of Eq. (i) with **a** gives

$\mathbf{a}\times(\mathbf{a}\times\mathbf{r})+\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{r}=\mathbf{r}-\mathbf{b}$ [from Eq. (i)]

$\Rightarrow\quad 2\mathbf{a}-\mathbf{r}+\mathbf{a}\times\mathbf{b}=\mathbf{r}-\mathbf{b}$

$\therefore\quad \mathbf{r}=\dfrac{1}{2}[2\mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}]$

18. Since, $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=0$ and $\mathbf{b}\cdot\mathbf{c}=\dfrac{1}{2}$

$\therefore\quad |\mathbf{a}\times\mathbf{b}|=|\mathbf{a}\times\mathbf{c}|=1$

Now, $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|^2=|\mathbf{a}\times\mathbf{b}|^2+|\mathbf{a}\times\mathbf{c}|^2-2(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$

$=1+1-2\begin{vmatrix}1 & 0\\ 0 & 1/2\end{vmatrix}=1$

19. The required line is perpendicular to the line which are parallel to vectors $\mathbf{b}_1=2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{b}_2=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, respectively. So, it is parallel to the vector $\mathbf{b}=\mathbf{b}_1\times\mathbf{b}_2$.

Now, $\quad\mathbf{b}=\mathbf{b}_1\times\mathbf{b}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -2 & 1\\ 1 & 2 & 2\end{vmatrix}$

$=-6\mathbf{i}-3\mathbf{j}+6\mathbf{k}$

Thus, the required line passes through the point (2, − 1, 3) and is parallel to the vector $\mathbf{b} = -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

So, its equation is

$$\mathbf{r} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \lambda(-6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$
$$\Rightarrow \quad \mathbf{r} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \mu\,(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \text{ where } \mu = -2\lambda$$

20. The required vector $\mathbf{r} = \lambda\,(\hat{\mathbf{a}} + \hat{\mathbf{b}})$, where λ is a scalar.

$$\Rightarrow \quad \mathbf{r} = \lambda\left(\frac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \frac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right)$$
$$\Rightarrow \quad \mathbf{r} = \frac{\lambda}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$$

Given, $|\mathbf{r}|^2 = 54$

$$\Rightarrow \quad \frac{\lambda^2}{81}(1 + 49 + 4) = 54$$
$$\Rightarrow \quad \lambda = \pm 9$$

Thus, the required vector is

$$\mathbf{r} = \pm(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}).$$

21. Let $\mathbf{c} = x\mathbf{i} + y\mathbf{j}$

Then, $\mathbf{b} \perp \mathbf{c} \Rightarrow \mathbf{b} \cdot \mathbf{c} = 0 \Rightarrow 4x + 3y = 0$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{-4} = \lambda \quad \text{(say)}$$
$$\Rightarrow \quad x = 3\lambda,\ y = -4\lambda$$
$$\therefore \quad \mathbf{c} = \lambda\,(3\mathbf{i} - 4\mathbf{j})$$

Let the required vector be $\alpha = p\mathbf{i} + q\mathbf{j}$. Then, the projections of α on $\mathbf{b}$ and $\mathbf{c}$ are $\frac{\alpha \cdot \mathbf{b}}{|\mathbf{b}|}$ and $\frac{\alpha \cdot \mathbf{c}}{|\mathbf{c}|}$, respectively.

$$\therefore \quad \alpha \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = 1 \text{ and } \frac{\alpha \cdot \mathbf{c}}{|\mathbf{c}|} = 2 \quad \text{(given)}$$
$$\Rightarrow \quad 4p + 3q = 5 \text{ and } 3p - 4q = 10$$
$$\Rightarrow \quad p = 2, q = -1$$
$$\alpha = 2\mathbf{i} - \mathbf{j}$$

22. Since, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, therefore $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represents an equilateral triangle.

$$\therefore \quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq 0$$

23. A vector coplanar to $(2\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and orthogonal to $(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$$= \lambda\,[\{(2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k})\} \times (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})] = \lambda\,(21\mathbf{j} - 7\mathbf{k})$$

$\therefore$ Required unit vector is $\pm \dfrac{(21\mathbf{j} - 7\mathbf{k})}{\sqrt{(21)^2 + (7)^2}} = \pm \dfrac{(3\mathbf{j} - \mathbf{k})}{\sqrt{10}}$

24. Here, $|\mathbf{u}| = 1$ and $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 7\mathbf{j} - \mathbf{k}$

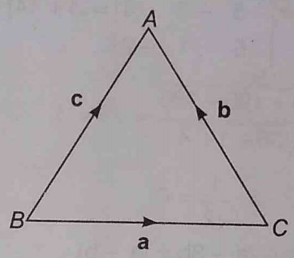
$$|\mathbf{v} \times \mathbf{w}| = \sqrt{9 + 49 + 1} = \sqrt{59}$$
$$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \leq |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}| \leq \sqrt{59}$$

25. Since, $(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 \geq 0 \Rightarrow 3 + 2\Sigma\mathbf{a} \cdot \mathbf{b} \geq 0$

or $-2\,\Sigma\,\mathbf{a} \cdot \mathbf{b} \leq 3$

$$\therefore \quad |\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = 6 - 2\,\Sigma\,\mathbf{a} \cdot \mathbf{b} \leq 9$$

26. Since, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$



Taking cross product with $\mathbf{a}$, we get

$$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$$

or $\quad \mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

Similarly, $\quad \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

$$\therefore \quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

27. The vector parallel to the given vector is $\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The unit vector along the line is

$$\mathbf{b} = \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{1 + 1 + 9}} = \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{11}}$$

$$\therefore \text{Projection} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\left(\frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{11}}\right) = \frac{5}{\sqrt{11}}$$

28. Since, $\quad \mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$$\therefore \quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0 \Rightarrow (\mathbf{r} - \mathbf{a}) \text{ parallel to } \mathbf{b}$$
$$\therefore \quad \mathbf{r} - \mathbf{a} = m\mathbf{b}$$
$$\Rightarrow \quad \mathbf{r} = \mathbf{a} + m\mathbf{b} \quad \ldots\text{(i)}$$

Similarly, for another condition we have equation.

$$\mathbf{r} = \mathbf{b} + n\mathbf{a} \quad \ldots\text{(ii)}$$

For point of intersection,

$$\mathbf{a} + m\mathbf{b} = \mathbf{b} + n\mathbf{a}$$
$$n = 1 \text{ and } m = 1$$

$\therefore$ The point of intersection is equal to

$$\mathbf{a} + \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

29. Since, $\mathbf{d}$ is collinear to vector $\mathbf{c}$.

$$\therefore \quad \mathbf{c} = \lambda\mathbf{d}$$
$$\Rightarrow \quad (2x + 3)\mathbf{a} + \mathbf{b} = \lambda[(2x + 3)\mathbf{a} - \mathbf{b}]$$
$$\Rightarrow \quad 2x\mathbf{a} + 3\mathbf{a} + \mathbf{b} = 2\lambda x\mathbf{a} + 3\lambda\mathbf{a} - \mathbf{b}\lambda$$
$$\Rightarrow \quad (2x - 2\lambda x\mathbf{a} + 3\mathbf{a} - 3\lambda\mathbf{a}) + (\mathbf{b}\lambda + \mathbf{b}) = 0$$
$$\Rightarrow \quad (2x - 2\lambda x + 3 - 3\lambda)\mathbf{a} + (\lambda + 1)\mathbf{b} = 0$$
$$\therefore \quad (2x - 2\lambda x + 3 - 3\lambda) = 0 \text{ and } (\lambda + 1) = 0$$
$$\Rightarrow \quad (2x + 3) - \lambda(2x + 3) = 0 \text{ and } \lambda = -1$$
$$\Rightarrow \quad (2x + 3)(1 - \lambda) = 0$$
$$\therefore \quad x = -\frac{3}{2} \text{ and } \lambda = 1$$

30. Let $\mathbf{a} = 2x^2\mathbf{i} + 4x\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$.

The angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} < 0$$
$$\Rightarrow \quad 14x^2 - 8x + x < 0$$
$$\Rightarrow \quad 7x(2x - 1) < 0$$
$$\therefore \quad x \in \left(0, \frac{1}{2}\right) \quad \ldots\text{(i)}$$

Also, it is given, $\mathbf{b}\cdot\mathbf{k} = x$ and $\dfrac{\mathbf{b}\cdot\mathbf{k}}{|\mathbf{b}|} < \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad 2x > \sqrt{3}\sqrt{53 + x^2}$

$\therefore \quad x^2 > 159 \quad \ldots\text{(ii)}$

Hence, there is no common value for Eqs. (i) and (ii).

31. Taking O as the origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively.

Then, the position vectors G_1, G_2 and G_3 are $\dfrac{\mathbf{b}+\mathbf{c}}{3}$, $\dfrac{\mathbf{c}+\mathbf{a}}{3}$ and $\dfrac{\mathbf{a}+\mathbf{b}}{3}$, respectively.

$\therefore \quad V_1 = \dfrac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

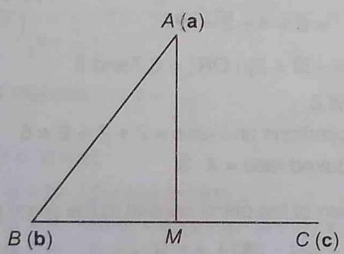
and $\quad V_2 = [\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3]$

Now, $\quad V_2 = [\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3] = \left[\dfrac{\mathbf{b}+\mathbf{c}}{3}\ \dfrac{\mathbf{c}+\mathbf{a}}{3}\ \dfrac{\mathbf{a}+\mathbf{b}}{3}\right]$

$= \dfrac{1}{27}[\mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}\ \ \mathbf{a}+\mathbf{b}] = \dfrac{2}{27}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= \dfrac{2}{27} \times 6V_1 \Rightarrow 9V_2 = 4V_1$

32. The vector equation of the line BC is



$\mathbf{r} = \mathbf{b} + \lambda(\mathbf{c} - \mathbf{b})$

Clearly, BM = Projection of $\mathbf{BA}$ on $\mathbf{BC}$

$\Rightarrow \quad BM = \dfrac{\mathbf{BA}\cdot\mathbf{BC}}{|\mathbf{BC}|} = \dfrac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{c}-\mathbf{b}|}$

Required distance $= AM = \sqrt{AB^2 - BM^2}$

$\therefore$ Required distance $= \sqrt{|\mathbf{b}-\mathbf{a}|^2 - \left\{\dfrac{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{c}-\mathbf{b}|}\right\}^2}$

33. It is given that α, β and γ are coplanar vectors

$\therefore \quad [\alpha\ \beta\ \gamma] = 0$

$\Rightarrow \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$\Rightarrow \quad 3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad a + b + c = 0$

$(\because a^2 + b^2 + c^2 - ab - bc - ca \neq 0)$

$\Rightarrow \quad \mathbf{v}\cdot\alpha = \mathbf{v}\cdot\beta = \mathbf{v}\cdot\gamma = 0$

Hence, $\mathbf{v}$ is perpendicular to α, β and γ.

34. We have, $\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

Taking scalar product with $\mathbf{c}$, we have

$(\mathbf{b} - 3\mathbf{c})\cdot\mathbf{c} = \lambda(\mathbf{a}\cdot\mathbf{c})$

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} - 3(\mathbf{c}\cdot\mathbf{c}) = \lambda(\mathbf{a}\cdot\mathbf{a})$

$(\because |\mathbf{a}| = |\mathbf{c}| = 1$ and $\mathbf{a}$ and $\mathbf{c}$ are collinear vectors)

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} - 3 = \lambda \Rightarrow \mathbf{b}\cdot\mathbf{c} = 3 + \lambda \quad \ldots\text{(i)}$

Again, $\quad \mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}| = |\lambda\mathbf{a}|$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}|^2 = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad |\mathbf{b}|^2 + 9|\mathbf{c}|^2 - 6(\mathbf{b}\cdot\mathbf{c}) = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad 36 + 9 - 6(3 + \lambda) = \lambda^2 \quad$ [from Eq. (i)]

$\Rightarrow \quad 27 - 6\lambda = \lambda^2$

$\Rightarrow \quad \lambda^2 + 6\lambda - 27 = 0$

$\therefore \quad \lambda = -9, 3$

35. A vector parallel to the bisector of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{a} + \mathbf{b}$.

$\therefore$ Unit vector along the bisector $= \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} = \dfrac{1}{\sqrt{2}}(\mathbf{a}+\mathbf{b})$

$\left(\begin{array}{l}\because |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} \\ \Rightarrow |\mathbf{a}+\mathbf{b}|^2 = 1 + 1 + 0 = 2\end{array}\right)$

$\therefore$ Required projection

$= \left\{l\cdot\dfrac{\mathbf{a}}{|\mathbf{a}|} + m\dfrac{\mathbf{b}}{|\mathbf{a}|} + n\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}\right\}\cdot\dfrac{1}{\sqrt{2}}(\mathbf{a}+\mathbf{b}) = \dfrac{1}{\sqrt{2}}(l + m)$

$[\because |\mathbf{a}| = |\mathbf{b}| = 1$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0]$

36. We have, $\quad |a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = |a| + |b| + |c|$

$\Rightarrow \quad \sqrt{a^2 + b^2 + c^2} = |a| + |b| + |c|$

$\Rightarrow \quad a^2 + b^2 + c^2 = a^2 + b^2 + c^2 + 2$

$(|a||b| + |b||c| + |c||a|)$

$\Rightarrow \quad |a||b| + |b||c| + |c||a| = 0$

$\Rightarrow \quad ab = bc = ca = 0$

Hence, any two of a, b and c are zero.

37. We have, $\mathbf{u}\cdot\mathbf{n} = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$

$\Rightarrow \quad \mathbf{n} \perp \mathbf{u}$ and $\mathbf{n} \perp \mathbf{v} \Rightarrow \mathbf{n} = \pm\dfrac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$

Now, $\quad \mathbf{u}\times\mathbf{v} = (\mathbf{i} + \mathbf{j})\times(\mathbf{i} - \mathbf{j}) = -2\mathbf{k}$

$\therefore \quad \mathbf{n} = \pm\mathbf{k}$

Hence, $\quad |\mathbf{w}\cdot\mathbf{n}| = |(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\cdot(\pm\mathbf{k})| = 3$

38. Let θ be an angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$.

Then, $\mathbf{a}\cdot\mathbf{b} = \cos\theta$

Now, $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b} = 2 + 2\cos\theta = 4\cos^2\dfrac{\theta}{2}$

$\Rightarrow \quad |\mathbf{a}+\mathbf{b}| = 2\cos\dfrac{\theta}{2}, |\mathbf{a}-\mathbf{b}| = 2\sin\dfrac{\theta}{2}$

$\Rightarrow \quad |\mathbf{a}+\mathbf{b}| + |\mathbf{a}-\mathbf{b}| = 2\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right) \le 2\sqrt{2}$

39. We have, $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]| = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ.

Then, $V_1 = |[\alpha\ \beta\ \gamma]|$

Now, $[\alpha\ \beta\ \gamma] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{abc}]^2\ [\mathbf{abc}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{abc}]^3$

$\therefore \quad V_1 = |[\alpha\ \beta\ \gamma]| = |[\mathbf{a}\,\mathbf{bc}]^3| = V^3$

40. Let $\alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\beta = \lambda\mathbf{b} + 4\mathbf{c}$ and $\gamma = (2\lambda - 1)\mathbf{c}$

Then, $[\alpha\ \beta\ \gamma] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda - 1) \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = \lambda\,(2\lambda - 1)\,[\mathbf{abc}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = 0$, if $\lambda = 0, \frac{1}{2}$ $\qquad (\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0)$

Hence, α, β and γ are non-coplanar for all values of λ except two values 0 and $\frac{1}{2}$.

41. We have, projection of $\mathbf{v}$ along $\mathbf{u}$ = Projection of $\mathbf{w}$ along $\mathbf{u}$

$\Rightarrow \quad \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{u}|}$

$\Rightarrow \quad \mathbf{v}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u}$ $\qquad \ldots$(i)

Also, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other.

$\therefore \quad \mathbf{v}\cdot\mathbf{w} = 0$ $\qquad \ldots$(i)

Now, $|\mathbf{u} - \mathbf{v} + \mathbf{w}| = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\,(\mathbf{u}\cdot\mathbf{v}) - 2\,(\mathbf{v}\cdot\mathbf{w}) + 2\,(\mathbf{u}\cdot\mathbf{w})$

$\Rightarrow \quad |\mathbf{u} - \mathbf{v} + \mathbf{w}|^2 = 1 + 4 + 9$ $\qquad$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad |\mathbf{u} - \mathbf{v} + \mathbf{w}| = \sqrt{14}$

42. We have, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$

$\Rightarrow (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - \{(\mathbf{b}\cdot\mathbf{c}) + \frac{1}{3}|\mathbf{b}||\mathbf{c}|\}\mathbf{a} = \mathbf{0}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{c}) = 0$ and $\mathbf{b}\cdot\mathbf{c} + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$

(since, $\mathbf{a}$ and $\mathbf{b}$ are non-collinear)

$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\cos\theta + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$

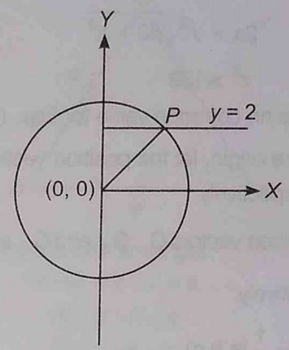
$\Rightarrow \quad \cos\theta = -\frac{1}{3}$

$\therefore \quad \sin\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

43. $\mathbf{a}_1 = \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} \Rightarrow \mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$

Thus, $\mathbf{a}_1 \times \mathbf{a}_2 = \frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2} \times \left(\mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}\right) = \frac{(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}|^2}$

44. On solving given Eqs. (i) and (ii) we get the point P is (4, 2) and Q can be (1, 2), (2, 2) and (3, 2).



$\therefore \mathbf{OP} = 4\mathbf{i} + 2\mathbf{j}$

and $\quad \mathbf{OQ} = \mathbf{i} + 2\mathbf{j}$ or $2\mathbf{i} + 2\mathbf{j}$ or $3\mathbf{i} + 2\mathbf{j}$

Sum of $\mathbf{OQ}$ for three positions $6\mathbf{i} + 6\mathbf{j}$.

So, the required projection

$= \frac{(4\mathbf{i} + 2\mathbf{j})\cdot(6\mathbf{i} + 6\mathbf{j})}{\sqrt{16 + 4}} = \frac{36}{2\sqrt{5}} = \frac{18\sqrt{5}}{5}$

45. $|\mathbf{OP} \times \mathbf{OQ}| = |(4\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + 2\mathbf{j})| + |(4\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 2\mathbf{j})| + |(4\mathbf{i} + 2\mathbf{j}) \times (3\mathbf{i} + 2\mathbf{j})|$

$= 6 + 4 + 2 = 12$

46. Here, $\mathbf{OR} = -5\mathbf{i} + 2\mathbf{j}$; $|\mathbf{OR}| = 6, 7$ and 8

Maximum is 8.

Sum of projections on Y-axis $= 2 + 2 + 2 = 6$

So, the required ratio = 4 : 3

47. The equation of the plane parallel to the given plane is

$\mathbf{r}\cdot(4\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}) + \lambda = 0$ $\qquad \ldots$(i)

Since, this plane passes through $2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$.

Therefore, $(2\mathbf{i} - \mathbf{j} - 4\mathbf{k})\cdot(4\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}) + \lambda = 0$

$\Rightarrow \quad 8 + 12 + 12 + \lambda = 0$

$\Rightarrow \quad \lambda = -32$

Hence, the required plane is $\mathbf{r}\cdot(4\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}) = 32$.

48. The equation of a plane through the line of intersection of the planes

$\mathbf{r}\cdot(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 0$ and $\mathbf{r}\cdot(\mathbf{j} + 2\mathbf{k}) = 0$ is

$\{\mathbf{r}\cdot(\mathbf{i} + 3\mathbf{j} - \mathbf{k})\} + \lambda\,\{\mathbf{r}\cdot(\mathbf{j} + 2\mathbf{k})\} = 0$ $\qquad \ldots$(i)

This passes through $(2\mathbf{i} + \mathbf{j} - \mathbf{k})$.

Therefore, $\{(2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(\mathbf{i} + 3\mathbf{j} - \mathbf{k})\} + \lambda\,\{(2\mathbf{i} + \mathbf{j} - \mathbf{k})\cdot(\mathbf{j} + 2\mathbf{k})\} = 0$

$\Rightarrow \quad (2 + 3 + 1) + \lambda\,(0 + 1 - 2) = 0$

$\therefore \quad \lambda = 6$

From Eq. (i), required plane is

$\mathbf{r}\cdot(\mathbf{i} + 9\mathbf{j} + 11\mathbf{k}) = 0$

49. In an isosceles ΔABC in which $AB = AC$, the median and bisector from A must be same line.

Statement II is true.

Now, $\quad \mathbf{AD} = \frac{\mathbf{u} + \mathbf{v}}{2}$

and $\quad |AD|^2 = \frac{1}{2} \cdot 2\cos^2 \frac{\alpha}{2}$

So, $\quad |AD| = \cos \frac{\alpha}{2}$

Unit vector along *AD*, *i.e.*, *x* is given by

$$\mathbf{x} = \frac{\mathbf{AD}}{|\mathbf{AD}|} = \frac{\mathbf{u}+\mathbf{v}}{2\cos\frac{\alpha}{2}}.$$

50. Since, $\mathbf{b} = \mathbf{r} \times \mathbf{a}$

We have, $\quad \mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{r} \times \mathbf{a})$

$= (\mathbf{a}\cdot\mathbf{a})\mathbf{r} - (\mathbf{a}\cdot\mathbf{r})\mathbf{a}$

$= (\mathbf{a}\cdot\mathbf{a})\mathbf{r}$

$\because \quad \mathbf{a}\cdot\mathbf{r} = 0$

$\therefore \quad \mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$

51. $V = \begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 0 & 1 & a \end{vmatrix} = a - 1 - a^3$

$\therefore \quad \frac{dV}{da} = 1 - 3a^2 = 0 \quad$ (say)

Now, $\quad a = \pm\frac{1}{\sqrt{3}} \quad$ and $\quad \frac{d^2V}{da^2} = -6a$

$\Rightarrow \quad \left(\frac{d^2V}{da^2}\right)_{\left(a = \frac{1}{\sqrt{3}}\right)} = -\frac{6}{\sqrt{3}}$ (– ve)

Hence, *V* is maximum at $a = \frac{1}{\sqrt{3}}$.

52. $\because [\mathbf{a}\times\mathbf{b} \ \ \mathbf{b}\times\mathbf{c} \ \ \mathbf{c}\times\mathbf{a}]$

$= (\mathbf{a}\times\mathbf{b}) \cdot \{(\mathbf{b}\times\mathbf{c}) \times (\mathbf{c}\times\mathbf{a})\}$

$= (\mathbf{a}\times\mathbf{b}) \cdot \{[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})]\mathbf{c} - [\mathbf{c}\cdot(\mathbf{c}\times\mathbf{a})]\mathbf{b}\}$

$= (\mathbf{a}\times\mathbf{b}) \cdot \{[\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\mathbf{c} - 0\}$

$= (\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})[\mathbf{b}\,\mathbf{c}\,\mathbf{a}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$

Since, **a**, **b** and **c** are coplanar.

$\therefore [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 0$ and then $[\mathbf{a}\times\mathbf{b} \ \mathbf{b}\times\mathbf{c} \ \mathbf{c}\times\mathbf{a}] = 0$

Hence, $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ are also coplanar.

53. If $\mathbf{a} = \mathbf{b}$, then $|\mathbf{a}| = |\mathbf{b}|$

Now, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

and $\quad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2$

$\therefore \quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2 = |\mathbf{b}|^2$

But it is true that, if $|\mathbf{a}| = |\mathbf{b}|$ does not implies that $\mathbf{a} = \mathbf{b}$.

54. Given, $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Now, we have; $\mathbf{b} + \lambda\mathbf{c} = (1+\lambda)\mathbf{i} + (2+\lambda)\mathbf{j} + (-1-2\lambda)\mathbf{k}$

$\therefore$ Projection of $(\mathbf{b}+\lambda\mathbf{c})$ on $\mathbf{a} = \left|\frac{(\mathbf{b}+\lambda\mathbf{c})\cdot\mathbf{a}}{|\mathbf{a}|}\right| = \sqrt{\frac{2}{3}}$ (given)

$\Rightarrow \quad \left|\frac{2(1-\lambda) - (2+\lambda) + (-1-2\lambda)}{\sqrt{4+1+1}}\right| = \sqrt{\frac{2}{3}}$

$\Rightarrow \quad \left|\frac{-\lambda - 1}{\sqrt{6}}\right| = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \lambda + 1 = 2 \Rightarrow \lambda = 1$

$\therefore \quad \mathbf{b} + \lambda\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$

55. $(\mathbf{i}\times\mathbf{a}\cdot\mathbf{b})\mathbf{i} + (\mathbf{j}\times\mathbf{a}\cdot\mathbf{b})\mathbf{j} + (\mathbf{k}\times\mathbf{a}\cdot\mathbf{b})\mathbf{k}$

$= [\mathbf{i}\,\mathbf{a}\,\mathbf{b}]\mathbf{i} + [\mathbf{j}\,\mathbf{a}\,\mathbf{b}]\mathbf{j} + [\mathbf{k}\,\mathbf{a}\,\mathbf{b}]\mathbf{k}$

Let $\quad \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$\therefore \quad [\mathbf{i}\cdot\mathbf{ab}] = \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)$

$[\mathbf{j}\,\mathbf{ab}] = \begin{vmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_1a_3 - a_1b_3)$

and $\quad [\mathbf{k}\mathbf{ab}] = \begin{vmatrix} 0 & 0 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_1b_2 - a_2b_1)$

$\therefore \quad [\mathbf{i}\,\mathbf{a}\,\mathbf{b}]\mathbf{i} + [\mathbf{j}\mathbf{a}\,\mathbf{b}]\mathbf{j} + [\mathbf{k}\,\mathbf{a}\,\mathbf{b}]\mathbf{k}$

$= (a_2b_3 - b_2a_3)\mathbf{i} + (b_1a_3 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$= \mathbf{a}\times\mathbf{b}$

56. $\sqrt{3}\mathbf{b} = (\mathbf{a} + \mathbf{c})$

$\Rightarrow \quad 3|\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{c}|^2 + 2\mathbf{a}\cdot\mathbf{c}$

$\Rightarrow \quad 3(1) = 1 + 1 + 2\mathbf{a}\cdot\mathbf{c} \Rightarrow 2\mathbf{a}\cdot\mathbf{c} = 1$

$\Rightarrow \quad |\mathbf{a}||\mathbf{c}|\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

57. Since vectors **u** and **v** are collinear

$\therefore \quad \mathbf{u} = \lambda\mathbf{v}$

$\Rightarrow \quad \{(\alpha - 2)\mathbf{a} + \mathbf{b}\} = \lambda(2 + 3\alpha)\mathbf{a} - 3\mathbf{b}\}$

On comparing, we get

$(2 + 3\alpha)\lambda = \alpha - 2 \quad$ ...(i)

and $\quad 1 = -3\lambda \Rightarrow \lambda = \frac{-1}{3}$

$\therefore \quad (2 + 3\alpha)\left(\frac{-1}{3}\right) = \alpha - 2 \Rightarrow \alpha = \frac{2}{3}$

58. $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2} \Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{c} = 8$

$\Rightarrow \quad |\mathbf{c}|^2 + (\sqrt{9})^2 - 2|\mathbf{c}| = 8$

$\Rightarrow \quad |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0 \Rightarrow (|\mathbf{c}| - 1)^2 = 0$

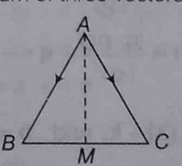
$\Rightarrow \quad |\mathbf{c}| = 1$

Now, $\quad \mathbf{a}\times\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\therefore \quad |\mathbf{a}\times\mathbf{b}| = \sqrt{4+4+1} = 3$

$\therefore \quad |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\sin 30^\circ = 3\times 1\times\frac{1}{2} = \frac{3}{2}$

59. We know that, the sum of three vectors of a triangle is zero.



$\therefore \quad \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$

$\Rightarrow \quad \mathbf{BC} = \mathbf{AC} - \mathbf{AB}$

$\Rightarrow \quad BM = \dfrac{AC - AB}{2}$

(since, *M* is a mid-point of *BC*)

Also, $\quad AB + BM + MA = 0$ (by properties of a triangle)

$\Rightarrow \quad AB + \dfrac{AC - AB}{2} = AM$

$\Rightarrow \quad AM = \dfrac{AB + AC}{2}$

$= \dfrac{3i + 4k + 5i - 2j + 4k}{2} = 4i - j + 4k$

$\therefore \quad |AM| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

60. Given that,

(i) **a** and **b** are unit vectors, *i.e.*, $|\mathbf{a}| = |\mathbf{b}| = 1$

(ii) $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{b} = 5\mathbf{a} - 4\mathbf{b}$

(iii) **c** and **d** are perpendicular to each other, *i.e.*, $\mathbf{c} \cdot \mathbf{d} = 0$

Now, $\mathbf{c} \cdot \mathbf{d} = 0 \Rightarrow (\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow \quad 5\mathbf{a} \cdot \mathbf{a} - 4\mathbf{a} \cdot \mathbf{b} + 10\mathbf{b} \cdot \mathbf{a} - 8\mathbf{b} \cdot \mathbf{b} = 0$

$\Rightarrow \quad 6\mathbf{a} \cdot \mathbf{b} = 3$

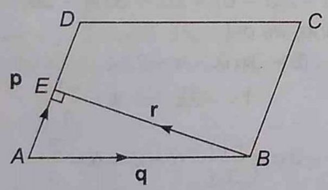
$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \dfrac{1}{2}$

So, the angle between **a** and **b** is $\dfrac{\pi}{3}$.

61. Given,

(i) A parallelogram *ABCD* such that $\mathbf{AB} = \mathbf{q}$ and $\mathbf{AD} = \mathbf{p}$.

(ii) The altitude from vertex *B* to side *AD* coincides with a vector **r**.



To find The vector **r** in terms of **p** and **q**.

Let *E* be the foot of perpendicular from *B* to side *AD*.

AE = Projection of vector **q**

$\mathbf{AE}$ = Vector along AE of length AE

$= |\mathbf{AE}|\,\mathbf{AE} = \dfrac{(\mathbf{q} \cdot \mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}$

Now, applying triangles law in ΔABE, we get

$\mathbf{AB} + \mathbf{BE} = \mathbf{AE}$

$\Rightarrow \quad \mathbf{q} + \mathbf{r} = \dfrac{(\mathbf{q} \cdot \mathbf{p})\mathbf{p}}{|\mathbf{p}|^2}$

$\Rightarrow \quad \mathbf{r} = \dfrac{(\mathbf{q} \cdot \mathbf{p})\mathbf{p}}{|\mathbf{p}|^2} - \mathbf{q} = -\mathbf{q} + \left(\dfrac{\mathbf{q} \cdot \mathbf{p}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$

62. Given, $\mathbf{a} = \dfrac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{k})$ and $\mathbf{b} = \dfrac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$

$\therefore \quad (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})\}$

$= (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \times 2\mathbf{b}\}$

$= (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \cdot \mathbf{a})\,\mathbf{b} + (\mathbf{b} \cdot \mathbf{a})\,\mathbf{a} + 2\,(\mathbf{a} \cdot \mathbf{b})\,\mathbf{b} - 2\,(\mathbf{b} \cdot \mathbf{b})\,\mathbf{a}\}$

$= (2\mathbf{a} - \mathbf{b}) \cdot \{1\,(\mathbf{b}) - (0)\,\mathbf{a} + 2\,(0)\,\mathbf{b} - 2\,(1)\,\mathbf{a}\}$

$\{\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1\}$

$= (2\mathbf{a} - \mathbf{b})\,(\mathbf{b} - 2\mathbf{a}) = -(4|\mathbf{a}|^2 + 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2)$

$= -\{4 + 0 + 1\} = -5$

63. Given, $\mathbf{a} \cdot \mathbf{b} \neq 0$, $\mathbf{a} \cdot \mathbf{d} = 0$ ...(i)

and $\quad \mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow \quad \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = 0$

$\therefore \quad \mathbf{b} \,||\, (\mathbf{c} - \mathbf{d})$

$\Rightarrow \quad \mathbf{c} - \mathbf{d} = \lambda \mathbf{b}$

$\Rightarrow \quad \mathbf{d} = \mathbf{c} - \lambda \mathbf{d}$ ...(ii)

Taking dot product with **a**, we get

$\mathbf{a} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{c} - \lambda \mathbf{a} \cdot \mathbf{b}$

$\Rightarrow \quad 0 = \mathbf{a} \cdot \mathbf{c} - \lambda\,(\mathbf{a} \cdot \mathbf{b})$ [from Eq. (i)]

$\Rightarrow \quad \lambda = \dfrac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}$ ...(iii)

$\therefore \quad \mathbf{d} = \mathbf{c} - \dfrac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{a} \cdot \mathbf{b})}\mathbf{b}$

64. As $\mathbf{a} + 3\mathbf{b}$ is collinear with **c**.

$\Rightarrow \quad \mathbf{a} + 3\mathbf{b} = \lambda \mathbf{c}$ ...(i)

Also, $\mathbf{b} + 2\mathbf{c}$ is collinear with **a**.

$\Rightarrow \quad \mathbf{b} + 2\mathbf{c} = \mu \mathbf{a}$ ...(ii)

From Eq. (i), $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (\lambda + 6)\,\mathbf{c}$ ...(iii)

From Eq. (ii), $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\mu)\,\mathbf{a}$ ...(iv)

From Eqs. (iii) and (v), we get

$(\lambda + 6)\,\mathbf{c} = (1 + 3\mu)\,\mathbf{a}$

Since, **a** is not collinear with **c**.

$\Rightarrow \quad \lambda + 6 = 1 + 3\mu = 0$

From Eq. (iv), $\quad \mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = 0$

65. Given, $\mathbf{a} = p\,\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + q\,\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + r\,\mathbf{k}$ are coplanar and $p \neq q \neq r \neq 1$.

Since, **a**, **b** and **c** are coplanar.

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$\Rightarrow \quad \begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$

$\Rightarrow \quad p\,(q\,r - 1) - 1\,(r - 1) + 1\,(1 - q) = 0$

$\Rightarrow \quad pqr - p - r + 1 + 1 - q = 0$

$\therefore \quad pqr - (p + q + r) = -2$

66. We have, $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{c} = 0$

$\Rightarrow \quad (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$

$\Rightarrow \quad 3\mathbf{a} - 2\mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$

$\Rightarrow \quad 2\mathbf{b} = 3\mathbf{a} + \mathbf{a} \times \mathbf{c}$

$\Rightarrow \quad 2\mathbf{b} = 3\mathbf{j} - 3\mathbf{k} - 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$= -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\therefore \quad \mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

67. Since, the given vectors are mutually orthogonal, therefore

$$\mathbf{a}\cdot\mathbf{b} = 2 - 4 + 2 = 0$$
$$\mathbf{a}\cdot\mathbf{c} = \lambda - 1 + 2\mu = 0 \quad \text{...(i)}$$
$$\mathbf{b}\cdot\mathbf{c} = 2\lambda + 4 + \mu = 0 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$\mu = 2 \text{ and } \lambda = -3$$

Hence, $(\lambda, \mu) = (-3, 2)$

68. Since, $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$

$\therefore\ 3p^2\ [\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] - pq\ [\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})] - 2q^2\ [\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})] = 0$

$\Rightarrow\ (3p^2 - pq + 2q^2)\,[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = 0$

But $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$

$\Rightarrow\ 3p^2 - pq + 2q^2 = 0$

$\therefore\ p = q = 0$

69. Given that, $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$

The equation of bisector of **b** and **c** is

$$\mathbf{r} = \lambda(\mathbf{b}+\mathbf{c}) = \lambda\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} + \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\right) = \frac{\lambda}{\sqrt{2}}(\mathbf{i}+2\mathbf{j}+\mathbf{k}) \quad \text{...(i)}$$

Since, point a lies on **BC**.

$$\therefore\ \frac{\lambda}{\sqrt{2}}(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = (\mathbf{i}+\mathbf{j}) + \mu(\mathbf{j}+\mathbf{k})$$

On equating the coefficient of $\hat{\mathbf{i}}$ both sides, we get

$$\therefore\ \frac{\lambda}{\sqrt{2}} = 1$$
$$\Rightarrow\ \lambda = \sqrt{2}$$

On putting $\lambda = \sqrt{2}$ in Eq. (i), we get

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Since, the given vector **a** represents the same bisector equation **r**.

$\therefore\ \alpha = 1$ and $\beta = 1$

Alternatively

Since, **a**, **b** and **c** are coplanar.

$$\Rightarrow \begin{vmatrix} \alpha & 2 & \beta \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow\ \alpha(1-0) - 2(1-0) + \beta(1-0) = 0$

$\Rightarrow \alpha + \beta = 2$ which is possible for $\alpha = 1, \beta = 1$.

70. Since, $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$

So, **a** is parallel to **b** and **c** is anti-parallel to **b**.

Since, **a** and **c** are anti-parallel.

So, the angle between **a** and **c** is π.

71. The volume of the parallelopiped with coterminus edges as **a**, **b** and **c** is given by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$$\text{Now, } [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\Rightarrow\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \frac{1}{\sqrt{2}} \text{ cu unit}$$

72. Let $(2\mathbf{u}\times 3\mathbf{v})$ is a unit vector.

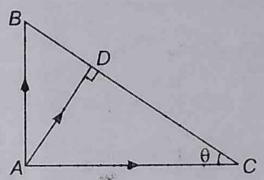
$$\therefore\ |2\mathbf{u}\times 3\mathbf{v}| = 1$$
$$\Rightarrow\ 6|\,|\mathbf{u}|\,|\mathbf{v}|\sin\theta\,| = 1$$
$$\therefore\ \sin\theta = \frac{1}{6}$$

Hence, there is exactly one value of θ for which $(2\mathbf{u}\times 3\mathbf{v})$ is a unit vector.

73. Let $|\mathbf{BC}| = l$

In ΔABC, $l = \sqrt{AB^2 + AC^2}$

$$\therefore\ \tan\theta = \frac{AB}{AC}$$



$$\Rightarrow\ \sin\theta = \frac{AB}{l} \text{ and } \cos\theta = \frac{AC}{l}$$

$$\therefore\ \text{Resultant vector} = \frac{1}{AB}\mathbf{i} + \frac{1}{AC}\mathbf{j} = \left(\frac{1}{l\sin\theta}\mathbf{i} + \frac{1}{l\cos\theta}\mathbf{j}\right)$$

In ΔADC, $AD = AC\sin\theta = l\sin\theta\cos\theta = \dfrac{AB\cdot AC}{l}$

$\therefore$ Magnitude of resultant vector

$$= \sqrt{\frac{1}{l^2}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right)} = \frac{l}{(AB)(AC)} = \frac{1}{AD}$$

74. The line is parallel to the plane. Since, $(\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = 0$ $(\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+5\mathbf{j}+\mathbf{k}) = 0$ Now, the distance between the line and the plane is the distance of $(2, -2, 3)$ from the plane $x + 5y + z - 5 = 0$

i.e., $\dfrac{|2 - 10 + 3 - 5|}{\sqrt{1+25+1}} = \dfrac{10}{3\sqrt{3}}$

75. Given $\dfrac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a} = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

$$\therefore\ \mathbf{b}\cdot\mathbf{c} = -\frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|$$

$$\Rightarrow\ \cos\theta = -\frac{1}{3} \Rightarrow \sin\theta = \frac{2\sqrt{2}}{3}$$

76. Now, $(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{w} + \mathbf{w}\times\mathbf{u}$

$\therefore\ (\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v})\times(\mathbf{v}-\mathbf{w})$

$$= (\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{w} + \mathbf{w}\times\mathbf{u})$$
$$= \mathbf{u}\cdot\mathbf{v}\times\mathbf{w} + \mathbf{v}\cdot\mathbf{w}\times\mathbf{u} - \mathbf{w}\cdot\mathbf{u}\times\mathbf{v}$$
$$= \mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$$

77. Here, $\mathbf{c} = \mathbf{b}\times\mathbf{a}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = z\mathbf{i} - x\mathbf{k}$$

Day 32

Three Dimensional Geometry

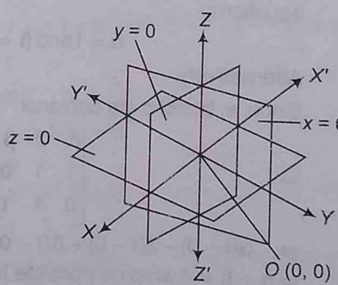
Day 32

Outlines ...

- Coordinates of a Point in a Space
- Section Formula
- Direction Cosines and Ratios
- Plane
- Line

Coordinates of a Point in a Space

The three mutually perpendicular lines in a space which divides the space into eight parts are called coordinates axes. The coordinates of a point are the distances from the origin to the feet of the perpendiculars from the point on the respective coordinate axes. The coordinates of any point on the X, Y and Z-axes will be as $(x,0,0)$ $(0,y,0)$ and $(0,0,z)$ respectively and the coordinates of any point will be as (x,y,z).



Sign Convention of Coordinates of a Point in Space

Octant Coordinate	x	y	z
OXYZ	+	+	+
OX′YZ	−	+	+
OXY′Z	+	−	+
OXYZ′	+	+	−
OX′Y′Z	−	−	+
OX′YZ′	−	+	−
OXY′Z′	+	−	−
OX′Y′Z′	−	−	−

Distance between Two Points

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formula

If $M(x, y, z)$ divides the line joining of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m:n$, then

For Internal Division

$$x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}$$

and $z = \frac{mz_2 + nz_1}{m+n}$

For External Division

$$x = \frac{mx_2 - nx_1}{m-n}, y = \frac{my_2 - ny_1}{m-n}$$

and $z = \frac{mz_2 - nz_1}{m-n}$

The coordinates of the mid-point of the line joining of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Direction Cosines and Ratios

If a vector makes angles α, β and γ with the positive directions of X-axis, Y-axis and Z-axis respectively, then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the **direction cosines** and it is denoted by l, m, n i.e., $l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$.

If numbers a, b and c are proportional to l, m and n respectively, then a, b and c are called **direction ratios.**

Thus, a, b and c are the direction ratios of a vector, provided $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

where, $l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}$,

$$m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(i) If a vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ having direction cosines l, m and n, then $l = \frac{a}{|\mathbf{r}|}, m = \frac{b}{|\mathbf{r}|}$ and $n = \frac{c}{|\mathbf{r}|}$

where, a, b and c are direction ratios of $\mathbf{r}$.

(ii) Direction ratios of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its direction cosines are

$$\frac{x_2 - x_1}{|\mathbf{PQ}|}, \frac{y_2 - y_1}{|\mathbf{PQ}|}, \frac{z_2 - z_1}{|\mathbf{PQ}|}$$

(iii) If $P(x, y, z)$ is a point in space, then
 (a) $x = l|\mathbf{r}|, y = m|\mathbf{r}|, z = n|\mathbf{r}|$
 (b) $l|\mathbf{r}|, m|\mathbf{r}|$ and $n|\mathbf{r}|$ are projections of $\mathbf{r}$ on OX, OY and OZ, respectively.
 (c) $\mathbf{r} = |\mathbf{r}|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and $\mathbf{r} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$

(iv) The sum of squares of direction cosines is always unity i.e., $l^2 + m^2 + n^2 = 1$

(v) DC's of X-axis are (1, 0, 0), Y-axis are (0, 1, 0) and Z-axis are (0, 0, 1).

(vi) Direction cosines of a line are unique but direction ratio of a line are not unique and can be infinite.

(vii) The DC's of a line which is equally inclined to the coordinate axes are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.

(viii) If $G(\alpha, \beta, \gamma)$ is the centroid of ΔABC where, A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2), then coordinates of C is $(3\alpha - x_1 - x_2, 3\beta - y_1 - y_2, 3\gamma - z_1 - z_2)$.

(ix) If l, m and n are the DC's of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Plane

A plane is a surface such that line joining any two points of the plane totally lies in it.

Equation of a Plane in Different Forms

1. The general equation of a plane is $ax + by + cz + d = 0$ and $a^2 + b^2 + c^2 \neq 0$, where, a, b and c are the DR's of the normal to the plane.
 (i) Plane through the origin is $ax + by + cz = 0$.
 (ii) Planes parallel to the coordinate planes (perpendicular to coordinate axes) $x = k$ parallel to YOZ plane $y = k$ parallel to ZOX plane $z = k$ parallel to XOY plane

(iii) Planes parallel to coordinate axes

$$by + cz + d = 0 \text{ parallel to } X\text{-axis}$$
$$ax + cz + d = 0 \text{ parallel to } Y\text{-axis}$$
$$ax + by + d = 0 \text{ parallel to } Z\text{-axis}$$

2. If a, b and c are the intercepts of the coordinate axes, then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

   It meets the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
3. If l, m and n are DC's of normal to the plane, p is the distance of the origin from the plane, then $lx + my + nz = p \geq 0$
4. If a plane meets the coordinate axes in A, B and C such that the centroid of the ΔABC is the point (p, q, r), then the equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3s$. (where, s = semi-perimeter of triangle)

Results on Plane

1. Plane through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
2. Plane through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

   where, $a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$
3. Plane through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
4. Plane $ax + by + cz + d = 0$ intersecting a line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ divides in the ratio

$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$$

   (i) If this ratio is positive, then A and B are on opposite sides of the plane.

   (ii) If this ratio is negative, then A and B are on the same side of the plane.
5. If θ be the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\theta = \cos^{-1}\left(\left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|\right)$$

6. Distance of a point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

   Distance of the origin is $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.
7. The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$
8. Any plane passing through the line of intersection of the planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ is $(ax + by + cz + d) + \lambda(a_1x + b_1y + c_1z + d_1) = 0$
9. The equation of the plane passing through the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) such that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = x_3 + y_3 + z_3 = k$ (say) is $x + y + z = k$.
10. A variable plane passes through a fixed point (α, β, γ) and meets the coordinate axes in A, B and C. Then, locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 1$.

Line

A line is the locus of the intersection of any two planes. In other words, two intersecting planes determine a straight line.

Equations of a Line in Different Forms

1. If $ax + by + cz + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ be the equations of any two planes, then taken together $ax + bx + cz + d_1 = 0 = a_1x + b_1y + c_1z + d_2$ gives the equation of straight line.
2. The symmetric form of equation of line is $\frac{x - p_1}{q_1} = \frac{y - p_2}{q_2} = \frac{z - p_3}{q_3}$, where (p_1, p_2, p_3) is a point and (q_1, q_2, q_3) is DR's.
3. The parametric equations of a line through (a_1, a_2, a_3) with DC's l, m and n are $x = a_1 + lr$, $y = a_2 + mr$ and $z = a_3 + nr$
4. Line through two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Its vector form is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{r} - \mathbf{a})$.

Angle between Two Intersecting Lines

1. If DR's of two lines are a_1, b_1, c_1 and a_2, b_2, c_2, then $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$

   (i) For parallel lines, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

   (ii) For perpendicular lines $a_1a_2 + b_1b_2 + c_1c_2 = 0$

2. Angle between two lines with DC's l_1, m_1, n_1 and l_2, m_2, n_2 is $\cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$ or $\sin^{-1}(\sqrt{\Sigma(m_1n_2 - m_2n_1)^2})$

   (i) For parallel lines, $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

   (ii) For perpendicular lines, $l_1l_2 + m_1m_2 + n_1n_2 = 0$

- *The angle between any two diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.
- *The angle between a diagonal of a cube and a faces is* $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.
- *The angle between the diagonal of a cube and edge of cube is* $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.
- *If a straight line makes angles* α, β, γ *and* δ *with the diagonals of a cube, then*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Skew-Lines

Two straight lines in a space which are neither parallel nor intersecting are called **skew-lines**. *Thus, skew-lines are those lines which do not lie in the same plane.*

Shortest Distance between Two Skew-Lines

If l_1 and l_2 are two skew-lines, then there is one and only one line perpendicular to each of the line l_1 and l_2, which is known as the line of shortest distance.

The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q, where the line of shortest distance intersects the two given lines

1. Vector Form

The shortest distance between two skew-lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\,\mathbf{d}$ is given by $SD = \left|\frac{\mathbf{c} - \mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})}{|\mathbf{b} \times \mathbf{d}|}\right|$

2. Cartesian Form

The shortest distance between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ is}$$

$$SD = |(x_1 - x_2)l + (y_1 - y_2)m + (z_1 - z_2)n|$$

Angle between a Line *and* a Plane

If angle between the line, $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

and the plane $a_1x + b_1y + c_1z + d = 0$ is θ, then $(90° - \theta)$ is the angle between normal and the line *i.e.*,

$$\cos(90° - \theta) = \left|\frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2 + b^2 + c^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}\right|$$

Coplanar Line

A line which is in the same plane as another line. Any two intersecting lines must lie in the same plane and therefore will be coplanar.

Condition for Coplanarity of Two Lines

1. Vector Form

Two lines $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\,\mathbf{d}$ are coplanar or intersecting, if $(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) = 0 \Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = [\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$

2. Cartesian Form

The lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$

and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$

Some Important Results

1. If $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of a ΔABC, then the
   (i) Centroid of triangle $= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$
   (ii) Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$
   (iii) If area of $\Delta ABC = 0$, then these points are collinear.
2. Centroid G, Incentre I, Excentres I_1, I_2 and I_3, Orthocentre O, Circumcentre S of a triangle are given by

$\left(\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}, \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}, \frac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3}\right)$,

where $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are the vertices of ΔABC.

	m_1	m_2	m_3
G	1	1	1
I	sin A	sin B	sin C
I_1	– sin A	sin B	sin C
I_2	sin A	–sin B	sin C
I_3	sin A	sin B	– sin C
O	tan A	tan B	tan C
S	sin 2A	sin 2B	sin 2C

3. Four non-coplanar points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ forms a tetrahedron with vertices A, B, C and D, edges AB, AC, AD, BC, BD and CD, faces ABC, ABD, ACD and BCD, then

(i) Centroid $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$

(ii) Volume

$$=\frac{1}{6}|AB \times AC \cdot AD| = \frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}$$

(iii) If two pairs of opposite edges are perpendicular, then the third pair is also perpendicular.

4. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. The locus of the centroid of the ΔABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$.

5. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. The locus of the centroid of the tetrahedron $OABC$ is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

Important Points Related to Line and Plane

- Foot of the perpendicular from a point (x_1, y_1, z_1) on the plane $ax+by+cz+d=0$ is (x, y, z),
  where $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$.
- Image of the point (x_1, y_1, z_1) in the plane $ax+by+cz+d=0$ is (x, y, z),
  where $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=-\frac{2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$.
- Four points (x_i, y_i, z_i), where $i=1, 2, 3$ and 4 are coplanar, if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}=0$.
- Two planes $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$ are **Coincident**, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{d_1}{d_2}$
- Parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\neq\frac{d_1}{d_2}$ Perpendicular , if $a_1a_2+b_1b_2+c_1c_2=0$
- A point $A(x_1, y_1, z_1)$ lies or doest not lie between the parallel planes $ax+by+cz+d_i=0$, where $i=1, 2$. according as
  $$\frac{ax_1+by_1+cz_1+d_1}{ax_2+by_2+cz_2+d_2}<0 \text{ or } >0$$
- Planes bisecting the angle between two intersecting planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are given by $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{(a_2x+b_2y+c_2z+d_2)}{\sqrt{a_2^2+b_2^2+c_2^2}}$

  (i) If $a_1a_2+b_1b_2+c_1c_2<0$, then origin is in acute angle and the acute angle bisector is obtained by taking positive sign in the above equation. The obtuse angle bisector is obtained by taking negative sign in the above equation.

  (ii) If $a_1a_2+b_1b_2+c_1c_2>0$, then origin lies in obtuse angle and the obtuse angle bisector is obtained by taking positive sign in above equation. Acute angle bisector is obtained by taking negative sign.

Practice Zone DAY 32

1. The angle between a diagonal of a cube and an edge of the cube intersecting the diagonal is

(a) $\cos^{-1}\frac{1}{3}$ (b) $\cos^{-1}\sqrt{\frac{2}{3}}$
(c) $\tan^{-1}\sqrt{2}$ (d) None of these

2. The equation of the plane through $(3, 1, -3)$ and $(1, -2, 2)$ and parallel to the line with DR's $1, 1, -2$ is

(a) $x - y + z + 1 = 0$ (b) $x + y - z + 1 = 0$
(c) $x - y - z - 1 = 0$ (d) $x + y + z - 1 = 0$

3. The equation of the plane through the line of intersection of the planes $x + y + z - 1 = 0$ and $2x + y - 3z + 2 = 0$ passing through the point $(1, 1, 1)$ is

(a) $x - 4z + 3 = 0$ (b) $x - y + z = 1$
(c) $x + y + z = 3$ (d) $2x - y + z = 2$

4. The angle between the lines whose direction cosines are given by $2l - m + 2n = 0, lm + mn + nl = 0$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

5. A line makes an angle θ with X and Y-axes both. A possible value of θ is in

(a) $\left[0, \frac{\pi}{4}\right]$ (b) $\left[0, \frac{\pi}{2}\right]$ (c) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{3}, \frac{\pi}{6}\right]$

6. A plane passes through the point $(1, -2, 3)$ and is parallel to the plane $2x - 2y + z = 0$. The distance of the point $(-1, 2, 0)$ from the plane, is

(a) 2 (b) 3
(c) 4 (d) 5

7. The angle between the lines whose direction cosine are given by $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

8. The shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$, is **[NCERT]**

(a) $\sqrt{29}$ units (b) 29 units
(c) $\frac{29}{2}$ units (d) $2\sqrt{29}$ units

9. The volume of the tetrahedron formed by coordinate planes and $2x + 3y + z = 6$, is

(a) 5 (b) 4
(c) 6 (d) 0

10. The foot of perpendicular from $(0, 2, 3)$ to the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$, is

(a) $(-2, 3, 4)$ (b) $(2, -1, 3)$
(c) $(2, 3, -1)$ (d) $(3, 2, -1)$

11. If the points $(1, 2, 3)$ and $(2, -1, 0)$ lie on the opposite sides of the plane $2x + 3y - 2z = k$, then

(a) $k < 1$ (b) $k > 2$
(c) $k < 1$ or $k > 2$ (d) $1 < k < 2$

12. Find the distance of the plane $x + 2y - z = 2$ from the point $(2, -1, 3)$ as measured in the direction with DR's $(2, 2, 1)$.

(a) 2 (b) -3
(c) -2 (d) 3

13. Find the planes bisecting the acute angle between the planes $x - y + 2z + 1 = 0$ and $2x + y + z + 2 = 0$.

(a) $x + z - 1 = 0$
(b) $x + z + 1 = 0$
(c) $x - z - 1 = 0$
(d) None of these

14. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive X-axis, then $\cos\alpha$ is equal to

(a) $1/2$ (b) 1
(c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

15. If the orthocentre and centroid of a triangle are $(-3, 5, 1)$ and $(3, 3, -1)$ respectively, then its circumcentre is

(a) $(6, 2, -2)$ (b) $(1, 2, 0)$
(c) $(6, 2, 2)$ (d) $(6, -2, 2)$

16. A vector **r** is inclined at equal angles to OX, OY and OZ. If the magnitude of **r** is 6 units, then **r** is equal to

(a) $\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (b) $-\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$
(c) $-2\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (d) None of these

17. The coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane passing through three points $(2, 2, 1)$, $(3, 0, 1)$ and $(4, -1, 0)$, is [NCERT Exemplar]

(a) $(1, 2, 7)$ (b) $(-1, 2, -7)$
(c) $(1, -2, 7)$ (d) None of these

18. The perpendicular distance of $P(1, 2, 3)$ from the lines $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(a) 7 (b) 5
(c) 8 (d) 0

19. The shortest distance between the diagonals of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it, are

(a) $\frac{bc}{\sqrt{b^2-c^2}}, \frac{ca}{\sqrt{c^2-a^2}}, \frac{ab}{\sqrt{a^2-b^2}}$

(b) $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

(c) $\frac{2bc}{\sqrt{b^2-c^2}}, \frac{2ca}{\sqrt{c^2-a^2}}, \frac{2ab}{\sqrt{a^2-b^2}}$

(d) None of the above

20. The points $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ are quidistant from the plane $\mathbf{r} \cdot (5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}) + 9 = 0$, then they are

(a) on the same sides of the plane
(b) parallel of the plane
(c) on the opposite sides of the plane
(d) None of the above

21. If $|x_1| > |y_1| + |z_1|, |y_2| > |x_2| + |z_2|$ and $|z_3| > |x_3| + |y_3|$, then $x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and $x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$ are

(a) perpendicular
(b) collinear
(c) coplanar
(d) non-coplanar

22. A line L_1 passes through the point $3\mathbf{i}$ and is parallel to the vector $-\mathbf{i} + \mathbf{j} + \mathbf{k}$ and another line L_2 passes through the point $\mathbf{i} + \mathbf{j}$ and is parallel to the vector $\mathbf{i} + \mathbf{k}$, then point of intersection of the lines is

(a) $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
(b) $2\mathbf{i} + \mathbf{j} + \mathbf{k}$
(c) $\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
(d) $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

23. The ΔABC is such that the mid-points of the sides BC, CA and AB are $(l, 0, 0)$, $(0, m, 0)$, $(0, 0, n)$, respectively. Then, $\frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ is equal to

(a) 2 (b) 4
(c) 8 (d) 16

24. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, then the direction cosines of the line perpendicular to both of these are [NCERT]

(a) $m_1n_2 + m_2n_1, n_1l_2 + n_2l_1, l_1m_2 + l_2m_1$
(b) $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$
(c) $m_1m_2 - n_1n_2, n_1n_2 - l_1l_2, l_1l_2 - m_1m_2$
(d) $m_1m_2 + n_1n_2, n_1n_2 + l_1l_2, l_1l_2 + m_1m_2$

25. A plane is such that the foot of perpendicular drawn from the origin to it is $(2, -1, 1)$. The distance of $(1, 2, 3)$ from the plane is

(a) $3/2$ (b) $\sqrt{3/2}$
(c) 2 (d) None of these

26. A and B are two given points. If C divides AB internally and D divides AB externally in the same ratio. Then, AC, AB and AD are in

(a) AP (b) GP
(c) HP (d) None of these

27. The projection of the line segment joining (2, 5, 6) and (3, 2, 7) on the line with direction ratios 2, 1, –2, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 1

28. The plane passing through the point $(-2, -2, 2)$ and containing the line joining the points $(1, -1, 2)$ and $(1, 1, 1)$ makes intercepts on the coordinate axes and the sum of whose length is

(a) 3 (b) 6 (c) 12 (d) 20

29. If the plane $x + y + z = 1$ is rotated through an angle $90°$ about its line of intersection with the plane $x - 2y + 3z = 0$, the new position of the plane is

(a) $x - 5y + 4z = 1$ (b) $x - 5y + 4z = -1$
(c) $x - 8y + 7z = 2$ (d) $x - 8y + 7z = -2$

30. $OABC$ is a tetrahedron such that $OA = OB = OC = k$ and each of the edges OA, OB and OC is inclined at an angle θ with the other two, then range of θ is

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) $\left[0, \frac{2\pi}{3}\right]$

31. $OABC$ is a regular tetrahedron of unit edge. Its volume is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{6}}$
(c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{6\sqrt{2}}$

32. A plane passes through $(1, -2, 1)$ and is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$. The distance of the plane from the point $(1, 2, 2)$ is

(a) 0 (b) 1
(c) $\sqrt{2}$ (d) $2\sqrt{2}$

33. A variable plane at a distance of 1 unit from the origin cut the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of ΔABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then k is equal to

(a) 3 (b) 1
(c) $\frac{1}{3}$ (d) 9

34. If α, β, γ and δ are the angles between a straight line with the diagonals of a cube, then $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{8}{3}$
(c) $\frac{7}{4}$ (d) None of these

35. A plane meets the coordinate axes in A, B and C and (α, β, y) is the centroid of the ΔABC. Then, the equation of the plane is

(a) $\frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma}=1$ (b) $\frac{3x}{\alpha}+\frac{3y}{\beta}+\frac{3z}{\gamma}=1$
(c) $\alpha x+\beta y+\gamma z=1$ (d) None of these

36. The direction cosines to two lines at right angles are $(1, 2, 3)$ and $\left(-2, \frac{1}{2}, \frac{1}{3}\right)$, then the direction cosine perpendicular to both the given lines are

(a) $\sqrt{\frac{25}{2198}}, \frac{19}{\sqrt{2198}}, \sqrt{\frac{729}{2198}}$ (b) $\sqrt{\frac{24}{2198}}, \sqrt{\frac{38}{2198}}, \sqrt{\frac{730}{2198}}$
(c) $\frac{1}{3}, -2, \frac{-7}{2}$ (d) None of these

37. The equation of the plane passing through $(2, 1, 5)$ and parallel to the plane $3x-4y+5z=4$ is

(a) $3x-4y+5z-27=0$ (b) $3x-4y+5z+21=0$
(c) $3x-4y+5z+26=0$ (d) $3x-4y+5z+17=0$

38. The equation of the line passing through the points $(3, 0, 1)$ and parallel to the planes $x+2y=0$ and $3y-z=0$, is **[NCERT Exemplar]**

(a) $\frac{x-3}{-2}=\frac{y-0}{1}=\frac{z-1}{3}$ (b) $\frac{x-3}{1}=\frac{y-0}{-2}=\frac{z-1}{3}$
(c) $\frac{x-3}{3}=\frac{y-0}{1}=\frac{z-1}{-2}$ (d) None of these

Directions (Q. Nos. 39 to 41) *Two lines whose equations are* $\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{\lambda}$ *and* $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z-2}{3}$ *lie in the same plane.*

39. The value of $\sin^{-1}\sin\lambda$ is equal to

(a) 3 (b) $\pi-3$
(c) 4 (d) $\pi-4$

40. The point of intersection of the lines lies on

(a) $3x+y+z=20$ (b) $2x+y+z=25$
(c) $3x+2y+z=24$ (d) $x=y=z$

41. Angle between the plane containing both lines and the plane $4x+y+2z=0$ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{6}$ (d) $\cos^{-1}\frac{2}{\sqrt{186}}$

Directions (Q. Nos. 42 and 43) *Consider the lines*

$$L_1: \frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2} \text{ and } L_2: \frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}$$

42. The unit vector perpendicular to both L_1 and L_2, is

(a) $\frac{-\mathbf{i}+7\mathbf{j}+7\mathbf{k}}{\sqrt{99}}$ (b) $\frac{-\mathbf{i}-7\mathbf{j}+5\mathbf{k}}{5\sqrt{3}}$
(c) $\frac{-\mathbf{i}+7\mathbf{j}+5\mathbf{k}}{5\sqrt{3}}$ (d) $\frac{7\mathbf{i}-7\mathbf{j}-\mathbf{k}}{\sqrt{99}}$

43. The distance of the point $(1, 2, -1)$ from the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the lines L_1 and L_2, is

(a) $\frac{2}{\sqrt{75}}$ (b) $\frac{7}{\sqrt{75}}$ (c) $2\sqrt{3}$ (d) $\frac{23}{\sqrt{75}}$

Directions (Q. Nos. 44 to 48) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

44. Statement I A point on the straight line $2x+3y-4z=5$ and $3x-2y+4z=7$ can be determined by taking $x=k$ and then solving the two equations for y and z, where k is any real number.

Statement II If $c'\neq kc$, then the straight line $ax+by+cz+d=0$, $kax+kby+c'z+d'=0$, does not intersect the plane $z=\alpha$, where α is any real number.

45. Consider a line is perpendicular to the plane, then DR's of plane is proportional to the line.

Statement I The lines $\frac{x-1}{1}=\frac{y}{-1}=\frac{z+1}{1}$ and $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ are coplanar and equation of the plane containing them is $5x+2y-3z-8=0$

Statement II The line $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z}{3}$ is perpendicular to the plane $3x+6y+9z-8=0$ and parallel to the plane $x+y-z=0$

46. Consider the lines

$$L_1:\frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2},\ L_2:\frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}.$$

Statement I The distance of the point (1, 1, 1) from the plane passing through the point (−1, −2, −1) and whose normal is perpendicular to both the lines L_1 and L_2 is $\frac{13}{5\sqrt{3}}$.

Statement II The unit vector perpendicular to both the lines L_1 and L_2 is $\frac{-\mathbf{i}-7\mathbf{j}+5\mathbf{k}}{5\sqrt{3}}$.

47. Consider the intersection of two planes, P_1 and P_2 is $P_1+\lambda P_2=0$

Statement I The plane $5x+2z-8=0$ contains the line $2x-y+z-3=0$ and $3x+y+z=5$ and is perpendicular to $2x-y-5z-3=0$

Statement II The plane $3x+y+z=5$ meets the line $x-1=y+1=z-1$ at the point (1, 1, 1).

48. Consider the points $A(2, 9, 12)$, $B(1, 8, 8)$, $C(-2, 11, 8)$ and $D(-1, 12, 12)$ are the vertices of a quadrilateral.

Statement I Vertices A, B, C and D are the vertices of a rhombus.

Statement II $AB=BC=CD=DA$ and $AC\neq BD$.

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49. If the lines $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+1}{3}$ and $\frac{x+2}{2}=\frac{y-k}{3}=\frac{z}{4}$ are coplanar, then the value of k is **[JEE Main 2013]**

(a) $\frac{11}{2}$ (b) $-\frac{11}{2}$ (c) $\frac{9}{2}$ (d) $-\frac{9}{2}$

50. A vector **n** is inclined to X-axis at 45°, to Y-axis at 60° and at an acute angle to Z-axis. If **n** is a normal to a plane passing through the point $(\sqrt{2}, -1, 1)$, then the equation of the plane is **[JEE Main 2013]**

(a) $4\sqrt{2}x+7y+z=2$ (b) $\sqrt{2}\,x+y+z=2$
(c) $3\sqrt{2}\,x-4y-3z=7$ (d) $\sqrt{2}x-y-z=2$

51. Let Q be the foot of perpendicular from the origin to the plane $4x-3y+z+13=0$ and R be a point $(-1, 1, -6)$ on the plane. Then, length QR is **[JEE Main 2013]**

(a) $\sqrt{14}$ (b) $\sqrt{\frac{19}{2}}$ (c) $3\sqrt{\frac{7}{2}}$ (d) $\frac{3}{\sqrt{2}}$

52. The acute angle between two lines such that the direction cosines l, m, n of each of them satisfy the equations $l+m+n=0$ and $l^2+m^2-n^2=0$ is

(a) 15° (b) 30°
(c) 60° (d) 45°

53. If the projections of a line segment on the X, Y and Z-axes in 3-dimensional space are 2, 3 and 6 respectively, then the length of the line segment is **[JEE Main 2013]**

(a) 12 (b) 7 (c) 9 (d) 6

54. The equation of a plane through the line of intersection of the planes $x+2y=3$, $y-2z+1=0$ and perpendicular to the first plane is **[JEE Main 2013]**

(a) $2x-y-10z=9$ (b) $2x-y+7z=11$
(c) $2x-y+10z=11$ (d) $2x-y-9z=10$

55. Let ABC be a triangle with vertices at points $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then (λ, μ) is equal to **[JEE Main 2013]**

(a) (10, 7) (b) (7, 5)
(c) (7, 10) (d) (5, 7)

56. Distance between two parallel planes $2x+y+2z=8$ and $4x+2y+4z+5=0$ is **[AIEEE 2013]**

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$

57. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, then k can have **[AIEEE 2013]**

(a) any value (b) exactly one value
(c) exactly two values (d) exactly three values

58. If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to **[AIEEE 2012]**

(a) −1 (b) $\frac{2}{9}$ (c) $\frac{9}{2}$ (d) 0

59. An equation of a plane parallel to the plane $x-2y+2z-5=0$ and at a unit distance from the origin is **[AIEEE 2012]**

(a) $x-2y+2z\pm 3=0$ (b) $x-2y+2z+1=0$
(c) $x-2y+2z-1=0$ (d) $x-2y+2z+5=0$

60. If the angle between the line $x=\frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x+2y+3z=4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ is equal to **[AIEEE 2011]**

(a) $\frac{3}{2}$ (b) $\frac{2}{5}$ (c) $\frac{5}{3}$ (d) $\frac{2}{3}$

61. The length of the perpendicular drawn from the point $(3, -1, 11)$ to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is **[AIEEE 2011]**

(a) $\sqrt{66}$ (b) $\sqrt{29}$ (c) $\sqrt{33}$ (d) $\sqrt{53}$

62. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along a straight line $x = y = z$ is **[AIEEE 2011]**

(a) $3\sqrt{5}$ (b) $10\sqrt{3}$ (c) $5\sqrt{3}$ (d) $3\sqrt{10}$

63. Statement I The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} \frac{z-2}{3}$.

Statement II The line $\frac{x}{1} = \frac{y-1}{2} \frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

64. A line AB in three dimensional space makes angles 45° and 120° with the positive x-axis and the positive y-axis, respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals **[AIEEE 2010]**

(a) 30° (b) 45° (c) 60° (d) 75°

65. Statement I The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.

Statement II The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$. **[AIEEE 2010]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

66. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then, (α, β) is equal to **[AIEEE 2009]**

(a) $(6, -17)$ (b) $(-6, 7)$
(c) $(5, -15)$ (d) $(-5, 15)$

67. The projections of a vector on the three coordinate axes are 6, -3 and 2, respectively. The direction cosines of the vector, are **[AIEEE 2009]**

(a) $6, -3, 2$ (b) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$
(c) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (d) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

68. The line passing through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at the point $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$. Then,

(a) $a = 8, b = 2$ (b) $a = 2, b = 8$ **[AIEEE 2008]**
(c) $a = 4, b = 6$ (d) $a = 6, b = 4$

69. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to **[AIEEE 2008]**

(a) -2 (b) -5 (c) 5 (d) 2

70. The image of the point $(-1, 3, 4)$ in the plane $x - 2y = 0$ is **[AIEEE 2006]**

(a) $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$ (b) $\left(-\frac{17}{3}, -\frac{19}{3}, 1\right)$
(c) $(8, 4, 4)$ (d) $\left(-\frac{17}{3}, -\frac{19}{3}, 4\right)$

71. If the angle θ between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}\, z + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$, the value of λ is **[AIEEE 2005]**

(a) $-\frac{3}{5}$ (b) $\frac{5}{3}$
(c) $-\frac{4}{3}$ (d) $\frac{3}{4}$

72. The distance between parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is **[AIEEE 2004]**

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$
(c) $\frac{7}{2}$ (d) $\frac{9}{2}$

73. If the lines $x = 1 + s, y = -3 - \lambda s, z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$ are coplanar, then λ is equal to **[AIEEE 2004]**

(a) -2 (b) -1
(c) $-\frac{1}{2}$ (d) 0

74. The two lines $x = ay + b, z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular if and only if **[AIEEE 2003]**

(a) $aa' + bb' + cc' + 1 = 0$
(b) $aa' + bb' + cc' = 0$
(c) $(a + a')(b + b')(c + c') = 0$
(d) $aa' + cc' + 1 = 0$

75. The direction ratios of normal to the plane through $(1, 0, 0), (0, 1, 0)$ which makes an angle $\frac{\pi}{4}$ with the plane $x + y = 3$ are **[AIEEE 2002]**

(a) $1, \sqrt{2}, 1$ (b) $1, 1, \sqrt{2}$
(c) $1, 1, 2$ (d) $\sqrt{2}, 1, 1$

Answers

1. (c)	2. (d)	3. (a)	4. (d)	5. (c)	6. (d)	7. (c)	8. (d)	9. (c)	10. (c)
11. (d)	12. (d)	13. (b)	14. (d)	15. (a)	16. (c)	17. (c)	18. (a)	19. (b)	20. (c)
21. (d)	22. (b)	23. (c)	24. (b)	25. (b)	26. (c)	27. (d)	28. (c)	29. (d)	30. (d)
31. (d)	32. (d)	33. (d)	34. (b)	35. (a)	36. (a)	37. (a)	38. (a)	39. (d)	40. (d)
41. (b)	42. (b)	43. (c)	44. (b)	45. (b)	46. (a)	47. (c)	48. (c)	49. (a)	50. (b)
51. (c)	52. (c)	53. (b)	54. (c)	55. (c)	56. (c)	57. (c)	58. (c)	59. (a)	60. (d)
61. (d)	62. (b)	63. (a)	64. (c)	65. (a)	66. (b)	67. (c)	68. (d)	69. (b)	70. (a)
71. (b)	72. (c)	73. (a)	74. (b)	75. (b)					

Hints & Solutions

1. If three edges of the cube are along x, y and z, then diagonal has DR's 1, 1, 1 and edge along x-axis has DR's 1, 0, 0. The angle between them is

$$\cos^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\sqrt{2}$$

2. The DR's of the line joining $(3, 1, -3)$ and $(1, -2, 2)$ are $-2, -3, 5$.

Equation of plane passing through $(3, 1, -3)$ is

$$a(x-3)+b(y-1)+c(z+3)=0$$

It passes through $(1, -2, 2)$, then

$$-2a-3b+5c=0$$

Also, $\quad a+b-2c=0$

On solving, we get $a=1, b=1$ and $c=1$

$\therefore$ Equation of plane is

$$(x-3)\cdot 1+(y-1)\cdot 1+(z+3)\cdot 1=0$$

$\Rightarrow \quad x+y+z-1=0$

3. The required plane is $2x+y-3z+2+\lambda(x+y+z-1)=0$

It passes through $(1, 1, 1)$, then $2+2\lambda=0$

$\therefore \quad \lambda=-1$

Hence, plane is $x-4z+3=0$

4. On eliminating m from given equations, we get

$$2(l+n)^2+nl=0 \qquad (\because \text{ put } m=2l+2n)$$

$\Rightarrow \quad (2l+n)(l+2n)=0$

$\Rightarrow \quad n=-2l \Rightarrow m=-2l$

or $\quad l=-2n \Rightarrow m=-2n$

The DR's are $1, -2, -2$ and $-2, -2, 1$.

Now, $\quad 1(-2)-2(-2)-2(1)=0$

Hence, lines are perpendicular.

So, angle between them is $\pi/2$.

5. We know that, $\cos^2\theta+\cos^2\theta+\cos^2\gamma=1$

$\Rightarrow \quad \cos^2\gamma=-\cos 2\theta$

$\Rightarrow \quad \cos 2\theta \le 0$

$\therefore \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

6. Let parallel plane be $2x-2y+z+\lambda=0$. It passes through $(1, -2, 3)$.

$\therefore \quad \lambda=-9$

The distance of $(-1, 2, 0)$ from the plane

$2x-2y+z-9=0$ is $\left|\frac{-2-4-9}{3}\right|=5$

7. $l^2+m^2-n^2=0 \Rightarrow 1-2n^2=0 \Rightarrow n=\pm\frac{1}{\sqrt{2}}$

Eliminating n from the given relations, we get $lm=0$

The two lines having direction cosines are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

Then, angle between them is $\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$.

8. The given lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

For line Ist DR's $=(7, -6, 1)$ and it passes through $(-1, -1, -1)$, then equation of given lines (in vector form) is

$$\mathbf{r}_1=-\mathbf{i}-\mathbf{j}-\mathbf{k}+\lambda(7\mathbf{i}-6\mathbf{j}+\mathbf{k})$$

Similarly, $\quad \mathbf{r}_2=3\mathbf{i}+5\mathbf{j}+7\mathbf{k}+\mu(\mathbf{i}-2\mathbf{j}+\mathbf{k})$

which are of the form $\mathbf{r}_1=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}_2=\mathbf{a}_2+\mu\mathbf{b}_2$

where, $\quad \mathbf{a}_1=-\mathbf{i}-\mathbf{j}-\mathbf{k}, \mathbf{b}_1=7\mathbf{i}-6\mathbf{j}+\mathbf{k}$

and $\quad \mathbf{a}_2=3\mathbf{i}+5\mathbf{j}+7\mathbf{k}, \mathbf{b}_2=\mathbf{i}-2\mathbf{j}+\mathbf{k}$

Now, $\mathbf{a}_2-\mathbf{a}_1=(3\mathbf{i}+5\mathbf{j}+7\mathbf{k})-(-\mathbf{i}-\mathbf{j}-\mathbf{k})=4\mathbf{i}+6\mathbf{j}+8\mathbf{k}$

and $\quad \mathbf{b}_1\times\mathbf{b}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 7 & -6 & 1\\ 1 & -2 & 1\end{vmatrix}$

$$=\mathbf{i}(-6+2)-\mathbf{j}(7-1)+\mathbf{k}(-14+6)$$
$$=-4\mathbf{i}-6\mathbf{j}-8\mathbf{k}$$

$$|\mathbf{b}_1\times\mathbf{b}_2|=\sqrt{(-4)^2+(-6)^2+(-8)^2}$$
$$=\sqrt{16+36+64}$$
$$=\sqrt{116}=2\sqrt{29}$$

So, the shortest distance between the given lines

$$d = \left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$$

$$= \frac{|(-4\mathbf{i}-6\mathbf{j}-8\mathbf{k})\cdot(4\mathbf{i}+6\mathbf{j}+8\mathbf{k})|}{2\sqrt{29}}$$

$$= \frac{|(-4)\times 4+(-6)\times 6+(-8)\times 8|}{2\sqrt{29}}$$

$$= \frac{|-16-36-64|}{2\sqrt{29}}$$

$$= \frac{116}{2\sqrt{29}}$$

$$= \frac{58}{\sqrt{29}} = 2\sqrt{29} \text{ units}$$

Note The two lines should be parallel and non-intercepting, then we can only determined the shortest distance.

9. Since, the vertices of the tetrahedron are (0, 0, 0), (3, 0, 0), (0, 2, 0) and (0, 0, 6).

$$\therefore \text{Volume of tetrahedron} = \frac{1}{6}\begin{vmatrix}3 & 0 & 0\\0 & 2 & 0\\0 & 0 & 6\end{vmatrix} = 6$$

10. Let L be foot of perpendicular from $P(0, 2, 3)$ on the line

$$\frac{x-(-3)}{5} = \frac{y-1}{2} = \frac{z-(-4)}{3} = t \quad \ldots(i)$$

Any point on Eq. (i) is $L(-3+5t, 1+2t, -4+3t)$.

Then, DR's of PL are

$(-3+5t-0, 1+2t-2, -4+3t-3)$

or $(5t-3, 2t-1, 3t-7)$.

Since, PL is perpendicular to Eq. (i), therefore

$5(5t-3)+2(2t-1)+3(3t-7)=0 \Rightarrow t=1$

So, the coordinate of L is (2, 3,–1).

11. On substituting the coordinates of the points in the equation $2x+3y-2z-k=0$, we get

$$(2+6-6-k)(4-3-k)<0$$

$$\Rightarrow \quad (k-1)(k-2)<0$$

$$\therefore \quad 1<k<2$$

12. Consider the line through (2, – 1, 3) with DC's $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ is

$$\frac{x-2}{2/3} = \frac{y+1}{2/3} = \frac{z-3}{1/3} = r \quad \text{(say)}$$

$$\therefore \quad x = 2+\frac{2r}{3}, y = -1+\frac{2r}{3}, z = 3+\frac{r}{3}$$

Since, it lies on the plane $x+2y-z=2$.

$$\therefore \quad 2+\frac{2r}{3}-2+\frac{4r}{3}-3-\frac{r}{3} = 2 \Rightarrow r=3$$

13. Now, $a_1a_2+b_1b_2+c_1c_2 = 2-1+2>0$.

The acute angle bisecting plane is

$$x-y+2z+1 = -(2x+y+z+2)$$

i.e., $$x+z+1=0$$

14. The two normal vectors are $\mathbf{m} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{n} = \mathbf{i}+3\mathbf{j}+2\mathbf{k}$

The line L is along, $\mathbf{m}\times\mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 3 & 1\\1 & 3 & 2\end{vmatrix} = 3(\mathbf{i}-\mathbf{j}+\mathbf{k})$

and DC's of x-axis are (1, 0, 0).

$$\therefore \quad \cos\alpha = \frac{3(\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(\mathbf{i})}{\sqrt{3^2(1+1+1)}\sqrt{1}} = \frac{1}{\sqrt{3}}$$

15. Since, S divides OG in the ratio 3 : – 1.

Then, $S = \left(\frac{9+3}{2}, \frac{-5+9}{2}, \frac{-3-1}{2}\right) = (6, 2, -2)$

16. Let $\mathbf{r}$ be inclined at an angle α to each axis, then

$$l = m = n = \cos\alpha$$

Since, $l^2+m^2+n^2 = 1 \Rightarrow 3\cos^2\alpha = 1$

If α is acute, then $l = m = n = \frac{1}{\sqrt{3}}$ and $|\mathbf{r}| = 6$

$$\therefore \quad \mathbf{r} = |\mathbf{r}|(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$$

$$= 6\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right)$$

$$= 2\sqrt{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

If α is obtuse, then $l = m = n = -\frac{1}{\sqrt{3}}$ and $|\mathbf{r}| = 6$

$$\therefore \quad \mathbf{r} = |\mathbf{r}|(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$$

$$= 6\left(-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}\right)$$

$$= -2\sqrt{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

17. Equation of plane through three points (2, 2, 1), (3, 0, 1) and (4, – 1, 0) is $[(\mathbf{r}-\mathbf{i}+2\mathbf{j}+\mathbf{k})]\cdot[(\mathbf{i}-2\mathbf{j})\times(\mathbf{i}-\mathbf{j}-\mathbf{k})] = 0$

i.e., $\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k}) = 7$ or $2x+y+z-7=0 \quad \ldots(i)$

Equation of line through (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \quad \ldots(ii)$$

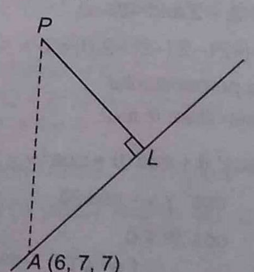
Any point on line (ii) is $(-\lambda+3, \lambda-4, 6\lambda-5)$. This point lies on plane (i).

Therefore, $2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7=0$

$$\Rightarrow \quad \lambda = 2$$

Hence, the required point is (1, – 2, 7).

18. The point A (6, 7, 7) lie on the line. Let the perpendicular from P meet the line at L. Then,



$AP^2 = (6-1)^2 + (7-2)^2 + (7-3)^2 = 66$

Also, AL = Projection of AP on line

$\left[\text{since, actual DC's are } \left(\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)\right]$

$$= (6-1)\cdot\frac{3}{\sqrt{17}} + (7-2)\cdot\frac{2}{\sqrt{17}} + (7-3)\left(\frac{-2}{\sqrt{17}}\right) = \sqrt{17}$$

$\therefore$ Perpendicular distance d of P from the line is given by

$d^2 = AP^2 - AL^2 \Rightarrow d^2 = 66 - 17 = 49 \Rightarrow d = 7$

19. Let one vertex of the parallelopiped be at the origin O and three coterminus edges OA, OB and OC be along OX, OY and OZ and respectively. The coordinates of the vertices of the parallelopiped are marked in figure.

The edges which do not meet the diagonal OF are AH, AD and BD and their parallels are BE, CE and CH, respectively.

The vector equation of the diagonal OF is

$$\mathbf{r} = 0 + \lambda(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \quad \ldots\text{(i)}$$

The vector equation of the edge BD is

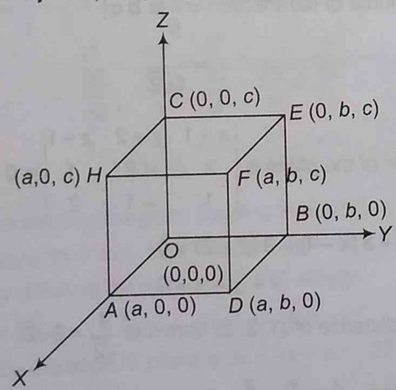
$$\mathbf{r} = b\,\mathbf{j} + \mu\, a\mathbf{i} \quad \ldots\text{(ii)}$$

We have, $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times a\mathbf{i} = ba(\mathbf{j}\times\mathbf{i}) + ca(\mathbf{k}\times\mathbf{i})$

$= -ba\,\mathbf{k} + ca\,\mathbf{j}$

$\therefore \quad |(a\mathbf{i} \times b\mathbf{j} \times c\mathbf{k}) \times a\mathbf{i}| = \sqrt{b^2a^2 + c^2a^2}$

and $\{(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times a\mathbf{i}\}\cdot(b\mathbf{j} - \mathbf{0}) = (-ba\,\mathbf{k} + ca\,\mathbf{j})\cdot b\mathbf{j} = abc$



Thus, the shortest distance between Eqs. (i) and (ii) is given by

$$SD = \frac{|\{(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times a\mathbf{i}\}\cdot(b\mathbf{j} - \mathbf{0})|}{|(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times a\mathbf{i}|}$$

$$= \frac{abc}{\sqrt{b^2a^2 + c^2a^2}} = \frac{bc}{\sqrt{b^2 + c^2}}$$

Similarly, it can be shown that the shortest distance between OF and AD is $\frac{ca}{\sqrt{a^2 + c^2}}$ and that between OF and AH is $\frac{ab}{\sqrt{a^2 + b^2}}$.

20. The given plane is $\mathbf{r}\cdot(5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}) = -9$.

Length of the perpendicular from $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ to the above plane is

$$\frac{-9 - (\mathbf{i} - \mathbf{j} + 3\mathbf{k})\cdot(5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k})}{\sqrt{25 + 4 + 49}} = \frac{-9 - 5 + 2 + 21}{\sqrt{78}} = \frac{9}{\sqrt{78}}$$

Length of the perpendicular from $3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ to the above plane

$$= \frac{-9 - (3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})\cdot(5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k})}{\sqrt{78}}$$

$$= \frac{-9 - 15 - 6 + 21}{\sqrt{78}} = \frac{-9}{\sqrt{78}}$$

Thus, the length of the two perpendiculars are equal in magnitude but opposite in sign. Hence, they are located on opposite sides of the plane.

21. If the given vectors are coplanar, then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

The set of equations,

$$x_1x + y_1y + z_1z = 0$$
$$x_2x + y_2y + z_2z = 0$$
and $$x_3x + y_3y + z_3z = 0$$

has a non-trivial solution.

Let the given set has a non-trivial solution x, y and z without loss of generality, we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have

$$x_1x = -y_1y - z_1z$$

$\Rightarrow \quad |x_1\,x| = |y_1\,y + z_1\,z| \leq |y_1\,y| + |z_1\,z|$

$\Rightarrow \quad |x_1\,x| \leq |y_1\,x| + |z_1\,x|$

$\Rightarrow \quad |x_1| < |y_1| + |z_1|$

which is contradiction to the given inequality *i.e.*,

$$|x_1| > |y_1| + |z_1|$$

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Therefore, the given equations have only a trivial solution.

So, the given vectors are non-coplanar.

22. Equation of L_1 is

$$\mathbf{r} = 3\mathbf{i} + \lambda(-\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= (3 - \lambda)\mathbf{i} + \lambda\mathbf{j} + \lambda\mathbf{k}$$

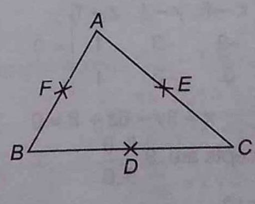
Equation of L_2 is

$$\mathbf{r} = (\mathbf{i} + \mathbf{j}) + \mu(\mathbf{i} + \mathbf{k})$$
$$= (1 + \mu)\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$$

For point of intersection, we get

$$\lambda = \mu = 1 \Rightarrow \mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

23. Given, mid-points of sides are $D(l, 0, 0)$, $E(0, m, 0)$ and $F(0, 0, n)$



Also, $EF^2 = \frac{BC^2}{4}$ (by mid-point theorem)

$\Rightarrow$ $BC^2 = 4(m^2 + n^2)$

Similarly, $AB^2 = 4(l^2 + m^2)$ and $CA^2 = 4(l^2 + n^2)$

$\therefore$ $\frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2} = 8$

24. Given lines are respectively parallel to unit vector

$$\mathbf{b}_1 = l_1\,\mathbf{i} + m_1\,\mathbf{j} + n_1\,\mathbf{k} \quad \ldots\text{(i)}$$

and $$\mathbf{b}_1 = l_2\,\mathbf{i} + m_2\,\mathbf{j} + n_2\,\mathbf{k} \quad \ldots\text{(ii)}$$

Now, $\mathbf{b}_1 \times \mathbf{b}_2$ is a vector which is at right angles to both $\mathbf{b}_1$ and $\mathbf{b}_2$ and is of magnitude unity.

Hence, components of $\mathbf{b}_1 \times \mathbf{b}_2$ are direction cosines of a line which is at right angles to both $\mathbf{b}_1$ and $\mathbf{b}_2$. So, we compute $\mathbf{b}_1 \times \mathbf{b}_2$.

$$\mathbf{b}_1 + \mathbf{b}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1n_2 - m_2n_1)\,\mathbf{i} - (n_2l_1 - n_1l_2)\,\mathbf{j} + (l_1m_2 - l_2m_1\,\mathbf{k})$$

$$= (m_1n_2 - m_2n_1)\mathbf{i} + (n_1l_2 - n_2l_1)\,\mathbf{j} + (l_1m_2 - l_2m_1)\,\mathbf{k}$$

Thus, the direction cosines of the required line are

$m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1$.

25. Given, plane is $2x - y + z = k$

It passes through $(2, -1, 1)$.

$\therefore$ $k = 6$

The distance of $(1, 2, 3)$ from

$2x - y + z = 6$ is $\frac{|2 - 2 + 3 - 6|}{\sqrt{2^2 + 1 + 1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$

26. Let C and D divide AB internally and externally in the ratio $\lambda : 1$.

Then, $AC = \frac{\lambda}{\lambda + 1} \cdot AB$

$AD = \frac{\lambda}{\lambda - 1}\,AB$

$\therefore$ $\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$

Hence, AC, AB and AD are in HP.

27. The vector joining the points is $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Its projection along the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$= \frac{|(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{|2 - 3 - 2|}{3} = 1$$

28. Required equation of plane is

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ -3 & -3 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $x - 3y - 6z + 8 = 0$

Since, the intercepts are $8, \frac{8}{3}, \frac{8}{6}$.

So, their sum is 12.

29. The new position of plane is

$$x - 2y + 3z + \lambda(x + y + z - 1) = 0$$

$$\Rightarrow (1 + \lambda)x + (\lambda - 2)y + (\lambda + 3)z - \lambda = 0$$

Since, it is perpendicular to $x + y + z - 1 = 0$.

$\therefore$ $1 + \lambda + \lambda - 2 + \lambda + 3 = 0 \Rightarrow \lambda = -\frac{2}{3}$

Hence, required plane is $x - 8y + 7z = -2$.

30. Let $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}, \mathbf{OC} = \mathbf{c}$. Then, the centroid G of ΔABC is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$

$$\mathbf{OG}^2 = \frac{1}{9}(3k^2 + 6k^2\cos\theta) = \frac{k^2}{3}(1 + 2\cos\theta) > 0$$

$$\theta \in \left[0, \frac{2\pi}{3}\right]$$

31. Let $\mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b}, \mathbf{OC} = \mathbf{c}$

Then, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1; \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \frac{1}{2}$

Now, $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}$$

$\because$ Volume of tetrahedron $= \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= \frac{1}{6\sqrt{2}}$

32. Equation of the plane is $\begin{vmatrix} x-1 & y+2 & z-1 \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow$ $-3(x - 1) - 3(y + 2) = 0$

$\Rightarrow$ $x + y + 1 = 0$

So, the distance of $(1, 2, 2)$ from it is $\frac{4}{\sqrt{2}} = 2\sqrt{2}$.

33. Let plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Centroid of plane ABC is $D\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

Distance of the plane from the origin

$$d = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 1 \quad \text{(given)}$$

$\Rightarrow$ $1 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

$\therefore D(x, y, z) \Rightarrow a = 3x, b = 3y, c = 3z$

$\Rightarrow$ $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9$

Hence, the value of k is 9.

34. $\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

and $\cos^2\alpha = 1 - \sin^2\alpha$, similarly for all other angles.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$
$= 4 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta)$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \left(4 - \frac{4}{3}\right) = \frac{8}{3}$

35. The intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, where a, b and c are the intercepts made by the plane on the axes.

$\therefore \quad \alpha = \frac{0+0+a}{3}, \beta = \frac{0+0+b}{3}, \gamma = \frac{0+0+c}{3}$

$\Rightarrow \quad a = 3\alpha, b = 3\beta$ and $c = 3\gamma$

So, the equation of plane is $\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1$

36. Let the direction cosine of the line perpendicular to two given lines is (l, m, n), then $l + 2m + 3n = 0$ and $-2l + \frac{m}{2} + \frac{n}{3} = 0$

From the above equation,

$$\frac{l}{2\times\frac{1}{3} - \frac{1}{2}\times 3} = -\frac{m}{-3\times(-2) - 1\times\frac{1}{3}} = \frac{n}{1\times\frac{1}{2} - 2(-2)}$$

$$\Rightarrow \quad \frac{l^2}{\frac{25}{36}} = \frac{m^2}{\frac{361}{9}} = \frac{n^2}{\frac{81}{4}} = \frac{1}{\frac{25}{36} + \frac{361}{9} + \frac{81}{4}}$$

$$\therefore \quad l = \sqrt{\frac{25}{2198}}, m = \frac{19}{\sqrt{2198}}, n = \sqrt{\frac{729}{2198}}$$

37. The equation of the plane passing through (2, 1, 5) and parallel to the plane $3x - 4y + 5z = 4$ is $3x - 4y + 5z + k = 0$

On substituting coordinates (2, 1, 5), we get

$3\times 2 - 4\times 1 + 5\times 5 + k = 0 \Rightarrow \quad k = -27$

So, the equation of plane is $3x - 4y + 5z - 27 = 0$.

38. Let a, b and c be the direction ratios of the required line.

Then, its equation is $\frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c}$ ...(i)

Since, Eq. (i) is parallel to the planes $x + 2y + 0z = 0$ and $0x + 3y - z = 0$. Therefore, normal to the plane is perpendicular to the line.

$\therefore \quad a(1) + b(2) + c(0) = 0$ and $a(0) + b(3) + c(-1) = 0$

On solving these two equations by cross-multiplication, we get

$$\frac{a}{(2)(-1) - (0)(3)} = \frac{b}{(0)(0) - (1)(-1)} = \frac{c}{(1)(3) - (0)(2)}$$

$$\Rightarrow \quad \frac{a}{-2} = \frac{b}{1} = \frac{c}{3} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = -2\lambda, b = \lambda \text{ and } c = 3\lambda$$

On substituting the values of a, b and c in Eq. (i), we get the equation of the required line as

$$\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$$

39. Both lines are coplanar.

$$\therefore \quad \begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-2+3) + 3(3+3) + \lambda(-3-2) = 0$

$\Rightarrow \quad \lambda = 4$

$\therefore \quad \sin^{-1}\sin 4 = \sin^{-1}\sin(\pi - 4) = \pi - 4$

40. Let $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{4} = r_1$

$\Rightarrow \quad x = 3 + 2r_1, y = 2 + 3r_1, z = 1 + 4r_1$

It will lie on, $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$

$$\Rightarrow \quad \frac{3 + 2r_1 - 2}{3} = \frac{2 + 3r_1 - 3}{2}$$

$\Rightarrow \quad r_1 = 1$

So, the point of intersection is (5, 5, 5), which lies on $x = y = z$.

41. Equation of the plane contains both lines is

$$\begin{vmatrix} x-3 & y-2 & z-1 \\ 2 & 3 & 4 \\ 3 & 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-3)(1) + (y-2)(12-6) + (z-1)(4-9) = 0$

$\Rightarrow \quad x + 6y - 5z = 10$

$\because \quad (1)(4) + (6)(1) + (-5)(2) = 4 + 6 - 10 = 0$

Hence, the required angle is $\frac{\pi}{2}$.

42. The equations of given lines in vector form may be written as

$L_1 : \mathbf{r} \cdot (-\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

and $L_2 : \mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$\therefore$ The vector perpendicular to both L_1 and L_2 is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$$

$\therefore$ Required unit vector $= \frac{(-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})}{\sqrt{(-1)^2 + (-7)^2 + (5)^2}}$

$= \frac{1}{5\sqrt{3}}(-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$

43. The equation of the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as $(x+1) + 7(y+2) - 5(z+1) = 0$

$\Rightarrow \quad x + 7y - 5z + 10 = 0$

The distance of the point $(1, 2, -1)$ from the plane

$$= \left|\frac{1 + 7(2) - 5(-1) + 10}{\sqrt{1 + 49 + 25}}\right|$$

$$= \frac{30}{\sqrt{75}} = 2\sqrt{3}$$

44. **Statement I** $3y-4z=5-2k$

$-2y+4z=7-3k$

$\therefore x=k,\ y=12-5k$ and $z=\dfrac{31-13k}{4}$ is a point on the line for all real values of k.

Statement I is true.

Statement II Direction ratios of the straight line are $< bc'-kbc, kac-ac', 0 >$ and direction ratios of normal the plane are $< 0, 0, 1 >$.

Now, $0\times(bc'-kbc)+0\times(kac-ac')+1\times 0=0$

$\therefore$ The straight line is parallel to the plane.

45. The equation of the plane containing them is

$$\begin{vmatrix} x-1 & y & z+1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -(5x+2y-3z-8)=0$$

Statement II Here, $\dfrac{1}{3}=\dfrac{2}{6}=\dfrac{3}{9}\Rightarrow \dfrac{1}{3}=\dfrac{1}{3}=\dfrac{1}{3}$

and $1(1)+2(1)+3(-1)=0$

$\therefore$ Statement II is true.

46. **Statement II** Lines L_1 and L_2 are parallel to the vectors $\mathbf{a}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, respectively. The unit vector perpendicular to both L_1 and L_2 is

$$\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\frac{-\mathbf{i}-7\mathbf{j}+5\mathbf{k}}{\sqrt{1+49+25}}=\frac{-\mathbf{i}-7\mathbf{j}+5\mathbf{k}}{5\sqrt{3}}$$

Statement I Plane is $-(x+1)-7(y+2)+5(z+1)=0$ whose distance from (1, 1, 1) is $\dfrac{13}{5\sqrt{3}}$.

Hence, Statement II is true.

47. Equation of the intersection plane is

$2x-y+z-3+\lambda(3x+y+z-5)=0$

For $\lambda=1$, we get

$5x+2z-8=0$, which is perpendicular to

$2x-y-5z-3=0$ as $5\times 2+0(-1)+2(-5)=0$

48. $\because$ $AB=\sqrt{(2-1)^2+(9-8)^2+(12-8)^2}$

$=\sqrt{18}=3\sqrt{2}$

$BC=\sqrt{(1+2)^2+(8-11)^2+(8-8)^2}$

$=\sqrt{18}=3\sqrt{2}$

$CD=\sqrt{(-2+1)^2+(11-12)^2+(8-12)^2}$

$=\sqrt{18}=3\sqrt{2}$

$DA=\sqrt{(-1-2)^2+(12-9)^2+(12-12)^2}$

$=\sqrt{18}=3\sqrt{2}$

$AC=\sqrt{(2+2)^2+(9-11)^2+(12-8)^2}$

$=\sqrt{36}=6$

and $BD=\sqrt{(1+1)^2+(8-12)^2+(8-12)^2}=\sqrt{36}=6$

Hence, $AB=BC=CD=DA$ and $AC=BD$

49. Since, the lines $\dfrac{x+1}{2}=\dfrac{y-1}{1}=\dfrac{z+1}{3}$

and $\dfrac{x+2}{2}=\dfrac{y-k}{3}=\dfrac{z}{4}$ are coplanar.

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -2+1 & k-1 & 0+1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} -1 & k-1 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = -(4-0)-(k-1)(8-6)+1(6-2)=0$$

$\Rightarrow 5-2k+2+4=0$

$\Rightarrow 2k=11 \Rightarrow k=\dfrac{11}{2}$

50. $\because$ $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow \cos^2 45^\circ+\cos^2 60^\circ+\cos^2\gamma=1$

$\Rightarrow \cos^2\gamma=1-\dfrac{1}{2}-\dfrac{1}{4}=1-\dfrac{3}{4}=\dfrac{1}{4}\Rightarrow \cos\gamma=\dfrac{1}{2}$

$\therefore$ Direction Ratio's of normal to the plane is

$\angle\cos 45^\circ; \cos 60^\circ, \dfrac{1}{2}> = <\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}, \dfrac{1}{2}>$

$\therefore$ Equation of plane passing through $(\sqrt{2}, -1, 1)$

$\dfrac{1}{\sqrt{2}}(x-\sqrt{2})+\dfrac{1}{2}(y+1)+\dfrac{1}{2}(z-1)=0$

$\Rightarrow 2(x-\sqrt{2})+\sqrt{2}(y+1)+\sqrt{2}(z-1)=0$

$\Rightarrow \sqrt{2}(x-\sqrt{2})+(y+1)+d(z-1)=0$

$\Rightarrow \sqrt{2}x-2+y+1+z-1=0$

$\Rightarrow \sqrt{2}x+y+z=2$

51. Let foot of perpendicular $Q(x, y, z)$ from $O(0, 0, 0)$

$\dfrac{x-0}{4}=\dfrac{y-0}{-3}=\dfrac{z-0}{1}=-\dfrac{\{4(0)-3(0)+1(0)+13\}}{4^2+3^2+1^2}$

$\Rightarrow \dfrac{x}{4}=\dfrac{y}{-3}=\dfrac{z}{1}=\dfrac{-13}{26}=-\dfrac{1}{2}$

$x=-2,\ y=\dfrac{3}{2},\ z=-\dfrac{1}{2}$

$\therefore Q\left(-2, \dfrac{3}{2}, -\dfrac{1}{2}\right)$

$\therefore PQ=\sqrt{(-1+2)^2+\left(1-\dfrac{3}{2}\right)^2+\left(-6+\dfrac{1}{2}\right)^2}$

$=\sqrt{1+\dfrac{1}{4}+\dfrac{121}{4}}=\dfrac{\sqrt{126}}{2}=3\sqrt{\dfrac{7}{2}}$

52. $\because l+m+n=0,\ l^2+m^2-n^2=0$ and $l^2+m^2+n^2=1$

On solving, we get $m=\pm\dfrac{1}{\sqrt{2}}, n=\pm\dfrac{1}{\sqrt{2}}$ and $l=0$

$\therefore \theta=\dfrac{\pi}{3}$ or $\dfrac{\pi}{2}$

53. Given that, the projections of a line segment on the X, Y and Z-axes in 3D-space are, $lr=2$, $mr=3$ and $nr=6$

$\because (lr)^2+(mr)^2+(nr)^2=(2)^2+(3)^2+(6)^2$

$\Rightarrow (l^2+m^2+n^2)r^2=4+9+36$

$\Rightarrow r^2=49 \Rightarrow r=7$

54. Intersection of two planes is

$$(x+2y-3)+\lambda(y-2z+1)$$
$$\Rightarrow \quad x+(2+\lambda)y-2\lambda z+\lambda-3=0$$
$$\therefore \quad 1(1)+2(2+\lambda)+0(-2\lambda)=0$$
$$\Rightarrow \quad \lambda=-\frac{5}{2}$$

Simplify it and get the result.

55. Centroid of ΔABC, $G=\left(\frac{2-1+\lambda}{3},\frac{3+3+5}{3},\frac{5+2+\mu}{3}\right)$

$$=\left(\frac{1+\lambda}{3},\frac{11}{3},\frac{7+\mu}{3}\right)$$

Since, median is always passes through centroid.
Since, they are equally inclined

$$\therefore \quad \frac{1+\lambda}{3}-2=\frac{11}{3}-3=\frac{7+\mu}{3}-5$$
$$\Rightarrow \quad \frac{\lambda-5}{3}=\frac{2}{3}=\frac{\mu-8}{3}$$
$$\Rightarrow \quad \lambda=7,\mu=10$$

56. Given planes are,

$$2x+y+2z-8=0$$
and
$$2x+y+2z+\frac{5}{2}=0$$

Distance between two planes

$$=\frac{|c_1-c_2|}{\sqrt{a^2+b^2+c^2}}=\left|\frac{-8-\frac{5}{2}}{\sqrt{2^2+1^2+2^2}}\right|$$
$$=\frac{\frac{21}{2}}{3}=\frac{7}{2}$$

57. Condition for two lines are coplanar.

$$[a-c,b,d]=0$$

where, a and c are the points on a line and b and d are the direction ratios of the lines.

$$\therefore \quad \begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0$$
$$\Rightarrow \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0$$
$$\Rightarrow \quad 1(1+2k)+(1+k^2)-(2-k)=0$$
$$\Rightarrow \quad k^2+2k+k=0$$
$$\Rightarrow \quad k^2+3k=0 \Rightarrow k=0,-3$$

Note If 0 appears in the denominator, then the correct way of representing the equation of straight line is

$$\frac{x-2}{1}=\frac{y-3}{1};\ z=4$$

58. Let $L_1:\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=p$

and $L_2:\frac{x-3}{1}=\frac{y-k}{2}=\frac{z-0}{1}=q$

$\Rightarrow$ Any point P on line L_1 is of type $P(2p+1,3p-1,4p+1)$ and any point Q on line L_2 is of type $Q(q+3,2q+k,q)$

Since, L_1 and L_2 are intersecting each other, hence both point P and Q should coincide at the point of intersection, *i.e.*, corresponding coordinates of P and Q should be same.

$2p+1=q+3$, $4p+1=q$ and $3p-1=2q+k$

On solving $2p+1=q+3$ and $4p+1=q$, we get the values of p and q as

$$p=\frac{-3}{2}$$
and
$$q=-5$$

On substituting the values of p and q in the third equation $3p-1=2q+k$, we get

$$\therefore \quad 3\left(\frac{-3}{2}\right)-1=2(-5)+k$$
$$\Rightarrow \quad k=\frac{9}{2}$$

59. Given, a plane $P: x-2y+2z-5=0$

Equation of family of planes parallel to the given plane P is

$$Q: x-2y+2z+d=0$$

Also, perpendicular distance of Q from origin is 1 unit.

$$\Rightarrow \quad \left|\frac{0-2(0)+2(0)+d}{\sqrt{1^2+2^2+2^2}}\right|=1$$
$$\Rightarrow \quad \left|\frac{d}{3}\right|=1 \Rightarrow d=\pm 3$$

Hence, the required equation of the plane parallel to P and at unit distance from origin is

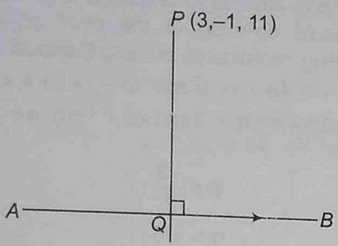
$$x-2y+2z\pm 3=0$$

60. Angle between straight line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and plane $\mathbf{r}\cdot\mathbf{n}=\mathbf{d}$,

$$\sin\theta=\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$
$$\therefore \quad \sin\theta=\frac{(\mathbf{i}+2\mathbf{j}+\lambda\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}}$$
$$\Rightarrow \quad \sin\theta=\frac{5+3\lambda}{\sqrt{\lambda^2+5}\cdot\sqrt{14}}$$

Given,
$$\cos\theta=\sqrt{\frac{5}{14}}$$
$$\therefore \quad \sin\theta=\frac{3}{\sqrt{14}}$$
$$\Rightarrow \quad \frac{3}{\sqrt{14}}=\frac{5+3\lambda}{\sqrt{\lambda^2+5}\cdot\sqrt{14}}$$
$$\Rightarrow \quad 9(\lambda^2+5)=9\lambda^2+30\lambda+25$$
$$\Rightarrow \quad 9\lambda^2+45=9\lambda^2+30\lambda+25$$
$$\Rightarrow \quad 30\lambda=20$$
$$\therefore \quad \lambda=\frac{2}{3}$$

61. Let the coordinate of Q be $(2\lambda, 3\lambda + 2, 4\lambda + 3)$, which is any point on the straight line AB.



So, the DR's of PQ, is $(2\lambda - 3, 3\lambda + 3, 4\lambda - 8)$ and it is perpendicular to straight line AB.

i.e., $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$ having DR's (2, 3, 4).

$\Rightarrow 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$

$\Rightarrow 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$\Rightarrow \quad 29\lambda - 29\lambda = 0$

$\Rightarrow \quad \lambda = 1$

$\therefore$ The coordinates of Q are (2, 5, 7).

$\therefore \quad |PQ| = \sqrt{(3-2)^2 + (-1-5)^2 + (7-11)^2}$

$= \sqrt{1 + 36 + 16}$

$= \sqrt{53}$

62. Equation of PQ is $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$

So, $x = \lambda + 1$, $y = \lambda - 5$ and $z = \lambda + 9$ lies on the plane $x - y + z = 5$.

$\Rightarrow \lambda + 1 - \lambda + 5 + \lambda + 9 = 5$

$\therefore \quad \lambda = -10$

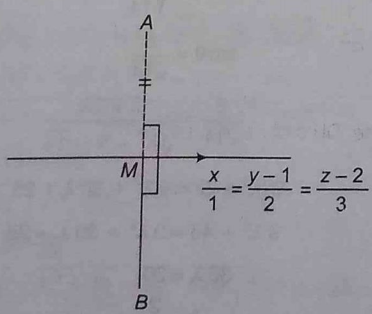
So, the coordinate of Q is $(-9, -15, -1)$ and coordinate of P is $(1, -5, 9)$.

$\therefore \quad |PQ| = \sqrt{(10)^2 + (10)^2 + (10)^2}$

$= 10\sqrt{3}$

63. Since, mid-point on AB is M (1, 3, 5).

which lies on $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.



$\therefore \quad \frac{1}{1} = \frac{3-1}{2} = \frac{5-2}{3} \Rightarrow 1 = 1 = 1$

Hence, Statement II is true.

Also, direction ratio of AB is

$(1-1, 6-0, 3-7) = (0, 6, -4)$ ...(i)

and direction ratio of straight line is

$(1, 2, 3)$ ... (ii)

These two lines are perpendicular, if

$0(1) + 6(2) - 4(3) = 12 - 12 = 0$

Hence, Statement I is true.

64. We know that, $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$

$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$

$\therefore \quad \theta = 60°$ or $120°$

65. The image of the point (3, 1, 6) with respect to the plane $x - y + z = 5$ is

$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = \frac{-2(3-1+6-5)}{1+1+1}$

$\Rightarrow \quad \frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = -2$

$\Rightarrow \quad x = 3 - 2 = 1$

$y = 1 + 2 = 3$

$z = 6 - 2 = 4$

which shows that Statement I is true.

We observe that the line segment joining the points $A(3, 1, 6)$ and $B(1, 3, 4)$ has direction ratios 2, – 2, 2 which are proportional to 1, – 1, 1 the direction ratios of the normal to the plane. Hence, Statement II is true.

Thus, the Statements I and II are true and Statement II is correct explanation of Statement I.

66. DR's of given line are (3, – 5, 2).

DR's of normal to the plane $= (1, 3, -\alpha)$

$\therefore$ Line is perpendicular to the normal

$\Rightarrow \quad 3(1) - 5(3) + 2(-\alpha) = 0$

$\Rightarrow \quad 3 - 15 - 2\alpha = 0$

$\Rightarrow \quad 2\alpha = -12$

$\Rightarrow \quad \alpha = -6$

Also, point (2, 1, – 2) lies on the plane

$2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$

$\therefore \quad (\alpha, \beta) = (-6, 7)$

67. Projection of a vector on coordinate axes are

$x_2 - x_1, y_2 - y_1, z_2 - z_1$

$x_2 - x_1 = 6, y_2 - y_1 = -3$ and $z_2 - z_1 = 2$

$\therefore \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{36 + 9 + 4} = 7$

Hence, DC's of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$.

68. Equation of the line passing through (5, 1, a) and (3, b, 1) is

$$\frac{x-3}{5-3}=\frac{y-b}{1-b}=\frac{z-1}{a-1} \quad \ldots(i)$$

Point $\left(0, \frac{17}{2}, \frac{13}{2}\right)$ satisfies Eq. (i), we get

$$-\frac{3}{2}=\frac{\frac{17}{2}-b}{1-b}=\frac{\frac{13}{2}-1}{a-1}$$

$$\Rightarrow \quad (a-1)=\frac{\left(-\frac{15}{2}\right)}{\left(-\frac{3}{2}\right)}=5 \Rightarrow a=6$$

Also, $$-3(1-b)=2\left(\frac{17}{2}-b\right)$$

$$\Rightarrow \quad 3b-3=17-2b \Rightarrow 5b=20$$

$$\therefore \quad b=4$$

69. Given, $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ ...(i)

and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$ ...(ii)

Since, lines intersect at a point. Then, shortest distance between them is zero.

$$\therefore \quad \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix}=0$$

$$\Rightarrow \quad k(-2k-2)-2(-6-2)+3(3-k)=0$$

$$\Rightarrow \quad -2k^2-5k+25=0$$

$$\Rightarrow \quad 2k^2+5k-25=0$$

$$\Rightarrow \quad 2k^2+10k-5k-25=0$$

$$\Rightarrow \quad 2k(k+5)-5(k+5)=0$$

$$\Rightarrow \quad k=\frac{5}{2}, -5$$

Hence, the integer value of k is -5.

70. The image of point $(-1, 3, 4)$ in a plane $x-2y=0$ is given by

$$\frac{x+1}{1}=\frac{y-3}{-2}=\frac{z-4}{0}=-\frac{2(1\times(-1)+3\times(-2)+0)}{1+4}$$

$$\Rightarrow \quad \frac{x+1}{1}=\frac{y-3}{-2}=\frac{z-4}{0}=\frac{-2(-7)}{5}$$

$$\therefore \quad x=\frac{14}{5}-1=\frac{9}{5}, y=-\frac{28}{5}+3=\frac{-13}{5} \text{ and } z=4$$

Hence, the image of the given point is $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$.

71. The direction ratios of a line are 1, 2, 2. The direction ratios of the normal to the plane are $2, -1, \sqrt{\lambda}$.

$$\therefore \quad \frac{1}{3}=\sin\theta=\frac{2(1)-1(2)+2\sqrt{\lambda}}{3(\sqrt{\lambda+5})}$$

$$\Rightarrow \quad \lambda+5=4\lambda \Rightarrow \lambda=\frac{5}{3}$$

72. Given, plane are $2x+y+2z-8=0$

and $2x+y+2z+\frac{5}{2}=0$

$$\therefore \quad \text{Distance between planes}=\frac{\left|8+\frac{5}{2}\right|}{\sqrt{4+1+4}}=\frac{7}{2}$$

73. Given, lines are $\frac{x-1}{1}=\frac{y+3}{-\lambda}=\frac{z-1}{\lambda}$

and $\frac{x}{1/2}=\frac{y-1}{1}=\frac{z-2}{-1}$

If lines are coplanar, then

$$\begin{vmatrix} 1-0 & -3-1 & 1-2 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -4 & -1 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix}=0$$

$$\Rightarrow \quad \lambda=-2$$

74. The lines are $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$, $\frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'}$

The lines are perpendicular, if $aa'+bb'+cc'=0$

75. Let the equation of plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Since, it passes through the point (1, 0, 0) and (0, 1, 0).

$\therefore$ Equation of plane is $\frac{x}{1}+\frac{y}{1}+\frac{z}{c}=1$

DR's of normal are $1, 1, \frac{1}{c}$ and of given plane are 1, 1, 0.

$$\text{Now,} \quad \cos\frac{\pi}{4}=\frac{1\cdot 1+1\cdot 1+\frac{1}{c}\cdot 0}{\left(\sqrt{\frac{1}{c^2}+2}\right)\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}}=\frac{2}{\left(\sqrt{\frac{1}{c^2}+2}\right)\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{c^2}+2=4 \Rightarrow c^2=\frac{1}{2}$$

$$\therefore \quad c=\frac{1}{\sqrt{2}}$$

So, the DR's of normal are $1, 1, \sqrt{2}$.

Unit Test 5 (Vectors & 3D Geometry)

DAY 33

1. The vector **B** satisfying the vector equation $\mathbf{A}+\mathbf{B}=\mathbf{a}$, $\mathbf{A}\times\mathbf{B}=\mathbf{b}$ and $\mathbf{A}\cdot\mathbf{a}=1$, where **a** and **b** are given vectors is

(a) $\frac{(\mathbf{b}\times\mathbf{a})+\mathbf{a}\,(a^2-1)}{a^2}$ (b) $\frac{(\mathbf{a}\times\mathbf{b})+\mathbf{a}}{\mathbf{a}}$
(c) $\frac{\mathbf{a}\,(a^2-1)+\mathbf{b}(b^2-1)}{a^2}$ (d) None of these

2. If $(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}\cdot\mathbf{b})^2=144$ and $|\mathbf{a}|=4$, then $|\mathbf{b}|$ is equal to

(a) 3 (b) 8
(c) 12 (d) 16

3. The values of x for which the angle between $\mathbf{a}=2x^2\mathbf{i}+4x\mathbf{j}+\mathbf{k}$, $\mathbf{b}=7\mathbf{i}-2\mathbf{j}+x\mathbf{k}$ is obtuse, is

(a) $x>1/2$ or $x<0$ (b) $0<x<1/2$
(c) $1/2<x<15$ (d) None of these

4. The vector **c**, directed along the bisectors of the angle between the vectors $\mathbf{a}=7\mathbf{i}-4\mathbf{j}-4\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $|\mathbf{c}|=5\sqrt{6}$ is

(a) $\pm\frac{5}{3}(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$ (b) $\pm\frac{5}{3}(5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$
(c) $\pm\frac{5}{3}(\mathbf{i}+7\mathbf{j}+2\mathbf{k})$ (d) $\pm\frac{5}{3}(-5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

5. Vectors **a** and **b** are inclined at an angle $\theta=120°$. If $|\mathbf{a}|=1$, $|\mathbf{b}|=2$, then $[(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})]^2$ is equal to

(a) 300 (b) 325 (c) 275 (d) 225

6. Let **a**, **b** and **c** be three non-zero and non-coplanar vectors and **p**, **q** and **r** be three vectors given by $\mathbf{p}=\mathbf{a}+\mathbf{b}-2\mathbf{c}$, $\mathbf{q}=3\mathbf{a}-2\mathbf{b}+\mathbf{c}$ and $\mathbf{r}=\mathbf{a}-4\mathbf{b}+2\mathbf{c}$. If the volume of the parallelopiped determined by **a**, **b** and **c** is V_1 and the volume of the parallelopiped determined by **p**, **q** and **r** is V_2, then $V_2:V_1$ is equal to

(a) 7 : 1 (b) 3 : 1
(c) 11 : 1 (d) None of these

7. For a non-zero vector **a**, the set of real numbers satisfying the inequality $|(5-x)\mathbf{a}|<|2\mathbf{a}|$ consists of all x such that

(a) $0<x<3$ (b) $3<x<7$
(c) $-7<x<-3$ (d) $-7<x<3$

8. A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors $\mathbf{i}, \mathbf{i}+\mathbf{j}$ and the plane determined by the vectors $\mathbf{i}-\mathbf{j}, \mathbf{i}+\mathbf{k}$. Then, the angle between **a** and the vector $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

9. Let **a**, **b** and **c** be three vectors having magnitudes 1, 1 and 2, respectively. If $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})+\mathbf{b}=\mathbf{0}$, then the acute angle between **a** and **c** is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) None of these

10. A point moves so that the sum of the squares of its distances from the six faces of a cube given by $x=\pm1$, $y=\pm1$, $z=\pm1$ is 10 units. The locus of the point is

(a) $x^2+y^2+z^2=1$ (b) $x^2+y^2+z^2=2$
(c) $x+y+z=1$ (d) $x+y+z=2$

11. If **a**, **b** and **c** are unit coplanar vectors, then the scalar triple product $[2\mathbf{a}-\mathbf{b}\;\;2\mathbf{b}-\mathbf{c}\;\;2\mathbf{c}-\mathbf{a}]$ is equal to

(a) 0 (b) 1
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

12. Point (α,β,γ) lies on the plane $x+y+z=2$. Let $\mathbf{a}=\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}$, $\mathbf{k}\times(\mathbf{k}\times\mathbf{a})=\mathbf{0}$. Then, γ is equal to

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

13. Let $\mathbf{a}=3\mathbf{i}+2\mathbf{k}$ and $\mathbf{b}=2\mathbf{j}+\mathbf{k}$. If **c** is a unit vector, then the maximum value of the vector triple product $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$, is

(a) $\sqrt{61}$ (b) $\sqrt{59}$
(c) $\sqrt{3}\cdot\sqrt{36}$ (d) None of these

14. If the four plane faces of a tetrahedron are represented by the equation $\mathbf{r}\cdot(l\,\mathbf{i}+m\,\mathbf{j})=0$, $\mathbf{r}\cdot(m\,\mathbf{j}+n\,\mathbf{k})=0$, $\mathbf{r}\cdot(n\mathbf{k}+p\,\mathbf{i})=0$ and $\mathbf{r}\cdot(l\,\mathbf{i}+m\,\mathbf{j}+n\mathbf{k})=p$, then the volume of the tetrahedron is

(a) $\frac{p^3}{6\,lmn}$ (b) $\frac{2p^3}{3\,lmn}$ (c) $\frac{3\,p^3}{lmn}$ (d) $\frac{6p^3}{lmn}$

15. If a variable plane forms a tetrahedron of constant volume $64k^3$ with the coordinate planes, then the locus of the centroid of the tetrahedron is

(a) $xyz = k^3$ (b) $xyz = 2k^3$
(c) $xyz = 12k^3$ (d) $xyz = 6k^3$

16. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\alpha\mathbf{d}$ and $\mathbf{b}+\mathbf{c}+\mathbf{d}=\beta\mathbf{a}$, then $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}$ is equal to

(a) $\mathbf{0}$ (b) $\alpha\mathbf{a}$
(c) β (d) $(\alpha+\beta)\mathbf{c}$

17. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the unit vectors such that $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular and $\mathbf{c}$ is equally inclined at $\mathbf{a}$ and $\mathbf{b}$ at an angle θ. If $\mathbf{c} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b})$, then

(a) $z^2 = 1-2x^2$ (b) $z^2 = 1-x^2+y^2$
(c) $z^2 = 1+2y^2$ (d) None of these

18. The ratio of lengths of diagonals of the parallelogram constructed on the vectors $\mathbf{a} = 3\mathbf{p}-\mathbf{q}, \mathbf{b} = \mathbf{p}+3\mathbf{q}$ is (given that $|\mathbf{p}| = |\mathbf{q}| = 2$, and the angle between $\mathbf{p}$ and $\mathbf{q}$ is $\frac{\pi}{3}$)

(a) $\sqrt{7}:\sqrt{3}$ (b) $\sqrt{3}:\sqrt{7}$
(c) $\sqrt{5}:\sqrt{7}$ (d) $\sqrt{5}:\sqrt{3}$

19. Let $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector x satisfies equation

$$\mathbf{p}\times\{(\mathbf{x}-\mathbf{q})\times\mathbf{p}\}+\mathbf{q}\times\{(\mathbf{x}-\mathbf{r})\times\mathbf{q}\}+\mathbf{r}\times\{(\mathbf{x}-\mathbf{p})\times\mathbf{r}\}=\mathbf{0}$$

Then, $\mathbf{x}$ is given by

(a) $\frac{1}{2}(\mathbf{p}+\mathbf{q}-2\mathbf{r})$ (b) $\frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r})$
(c) $\frac{1}{3}(2\mathbf{p}+\mathbf{q}-\mathbf{r})$ (d) None of these

20. A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equation $x = y+2 = z$; $x+2 = 2y = 2z$.
The coordinates of the point of intersection are

(a) (6, 4, 6), (2, 4, 2) (b) (6, 6, 6), (2, 6, 2)
(c) (6, 4, 6), (2, 2, 0) (d) None of these

21. The equation of plane perpendicular to $2x+6y+6z=1$ and passing through the points (2, 2, 1) and (9, 3, 6), is

(a) $3x+4y+5z-9=0$
(b) $3x+4y-5z+9=0$
(c) $3x+4y-5z-9=0$
(d) $3x+4y+5z+9=0$

22. The line through $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ and perpendicular to the line $\mathbf{r} = (\mathbf{i}+2\mathbf{j}-\mathbf{k})+\lambda(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = (2\mathbf{i}+6\mathbf{j}+\mathbf{k})+\mu(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ is

(a) $\mathbf{r} = (\mathbf{i}+2\mathbf{j}-\mathbf{k})+\lambda(-\mathbf{i}+5\mathbf{j}-3\mathbf{k})$
(b) $\mathbf{r} = (\mathbf{i}+3\mathbf{j}+2\mathbf{k})+\lambda(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$
(c) $\mathbf{r} = (\mathbf{i}+3\mathbf{j}+2\mathbf{k})+\lambda(\mathbf{i}+5\mathbf{j}+3\mathbf{k})$
(d) None of the above

23. The orthogonal projection A' of the point A with position vector (1, 2, 3) on the plane $3x-y+4z=0$, is

(a) $(-1, 3, -1)$ (b) $\left(-\frac{1}{2}, \frac{5}{2}, 1\right)$
(c) $\left(\frac{1}{2}, -\frac{5}{2}, -1\right)$ (d) $(6, -7, -5)$

24. Let $A(3, 2, 0), B(5, 3, 2), C(-9, 6, -3)$ are three points forming a triangle. If AD, the bisector of $\angle BAC$ meets BC in D, then coordinates of D are

(a) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$
(c) $\left(\frac{19}{8}, \frac{57}{16}, -\frac{17}{16}\right)$ (d) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

25. The equation of the plane containing the points $A(1, 0, 1)$ and $B(3, 1, 2)$ and parallel to the line joining the origin to the point $C(1, -1, 2)$ is

(a) $x+y-z=0$ (b) $x+y+z=0$
(c) $x-y+z=0$ (d) $x-y-z=0$

26. If the axes are rectangular, the distance from the point (3, 4, 5) to the point, where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=17$ is

(a) 1 (b) 2
(c) 3 (d) None of these

27. The plane $ax+by=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Then, the equation to the plane in new position is

(a) $ax-by\pm z\sqrt{a^2+b^2}\cot\alpha=0$
(b) $ax+by\pm z\sqrt{a^2+b^2}\cot\alpha=0$
(c) $ax-by\pm z\sqrt{a^2+b^2}\tan\alpha=0$
(d) $ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0$

28. The planes $3x-y+z+1=0$ and $5x+y+3z=0$ intersect in the line PQ. The equation of the plane through the point (2, 1, 4) and perpendicular to PQ is

(a) $x+y-2z=5$ (b) $x+y-2z=-5$
(c) $x+y+2z=5$ (d) $x+y+2z=-5$

29. The line of intersection of the planes $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$ and $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$ is parallel to the vector

(a) $-2\mathbf{i}+7\mathbf{j}+13\mathbf{k}$ (b) $-2\mathbf{i}-7\mathbf{j}+13\mathbf{k}$
(c) $2\mathbf{i}+7\mathbf{j}-13\mathbf{k}$ (d) None of these

30. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x=y+a=z$ and $x+a=2y=2z$. The coordinates of each of the points of intersection are given by

(a) $(3a, 2a, 3a), (a, a, 2a)$ (b) $(3a, 3a, 3a), (a, a, a)$
(c) $(3a, 2a, 3a), (a, a, a)$ (d) None of these

31. The equation of the plane containing the line $2x-y+z-3=0$ and $3x+y+z=5$ at a distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$ is

(a) $x+y+z-3=0$
(b) $2x-y-z-3=0$
(c) $2x-y+z+3=0$
(d) $62x+29y+19z-105=0$

32. The equation of the plane through the point $(2, -1, -3)$ and parallel to the lines $\frac{x-1}{3}=\frac{y+2}{2}=\frac{z}{-4}$ and $\frac{x}{2}=\frac{y-1}{-3}=\frac{z-2}{2}$ is

(a) $8x+14y+13z+37=0$ (b) $8x-14y+13y+37=0$
(c) $8x+14y-13z+37=0$ (d) None of these

33. The shortest distance between the lines

$\mathbf{r}=-(\mathbf{i}+\mathbf{j}+\mathbf{k})+\lambda(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$

and $\mathbf{r}=-\mathbf{i}+\mu(3\mathbf{i}+4\mathbf{j}+5\mathbf{k})$ is

(a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

34. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. The angle between the faces OAB and ABC is

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$
(c) $30°$ (d) $90°$

35. The sides of a parallelogram are $3\mathbf{i}+4\mathbf{j}-6\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$. The unit vector parallel to one of the diagonals is

(a) $\frac{5\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{27}}$ (b) $\frac{4\mathbf{i}+3\mathbf{j}-2\mathbf{k}}{\sqrt{29}}$
(c) $\frac{\mathbf{i}-2\mathbf{j}+4\mathbf{k}}{\sqrt{21}}$ (d) None of these

36. Let $\mathbf{p}=8\mathbf{i}+6\mathbf{j}$ and $\mathbf{q}$ be two vectors perpendicular to each other in the xy-plane. Then, the vector in the same plane having projections 2 and 4 along $\mathbf{p}$ and $\mathbf{q}$ respectively is

(a) $\pm 3(\mathbf{i}-2\mathbf{j})$ (b) $\pm(\mathbf{i}+2\mathbf{j})$
(c) $\pm 2(2\mathbf{i}-\mathbf{j})$ (d) None of these

37. If $\mathbf{p}=2\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ and $\mathbf{q}=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ be two vectors and $\mathbf{r}$ is a vector perpendicular to $\mathbf{p}$ and $\mathbf{q}$ and satisfying the condition. $\mathbf{r}(2\mathbf{i}-4\mathbf{j}+2\mathbf{k})=-12$, then $\mathbf{r}$ is equal to

(a) $2\mathbf{i}-\frac{20}{3}\mathbf{j}+16\mathbf{k}$ (b) $\frac{2}{3}(3\mathbf{i}+10\mathbf{j}+8\mathbf{k})$
(c) $\frac{1}{3}(3\mathbf{i}-10\mathbf{j}+8\mathbf{k})$ (d) None of these

38. The direction ratios of a normal to the plane through $(2, 0, 0)$ and $(0, 2, 0)$ that makes an angle $\frac{\pi}{3}$ with the plane $2x+3y=5$ is

(a) $1:1:2$ (b) $1:1:\sqrt{3}$
(c) $\sqrt{2}:1:3$ (d) $1:1:\sqrt{5.7}$

39. The shortest distance between the lines $\frac{x-2}{3}=\frac{y-1}{2}=\frac{z-3}{2}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-4}{4}$ is equal to

(a) 1.14 units (b) 2.01 units
(c) 3.16 units (d) None of these

Directions (Q. Nos. 40 and 41)

$$L_1=\frac{x+1}{-3}=\frac{y-3}{2}=\frac{y+2}{1}$$
$$L_2=\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$$

40. The lines L_1 and L_2 are

(a) perpendicular (b) parallel
(c) coplanar (d) None of these

41. The lines L_1 and L_2 intersect at the point

(a) $(-3, 2, 1)$ (b) $(2, 1, -3)$
(c) $(1, -3, 2)$ (d) None of these

Directions (Q. Nos. 42 and 43) *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be three vectors such that* $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=4$ *and angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is* $\frac{\pi}{3}$, *angle between* $\mathbf{b}$ *and* $\mathbf{c}$ *is* $\frac{\pi}{3}$ *and angle between* $\mathbf{c}$ *and* $\mathbf{a}$ *is* $\frac{\pi}{3}$.

42. The volume of the parallelopiped whose adjacent edges are represented by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

(a) $24\sqrt{2}$ (b) $24\sqrt{3}$
(c) $32\sqrt{2}$ (d) $32\sqrt{3}$

43. The volume of the tetrahedron whose adjacent edges are represented by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

(a) $\frac{4\sqrt{3}}{2}$ (b) $\frac{8\sqrt{2}}{3}$
(c) $\frac{16}{\sqrt{3}}$ (d) $\frac{16\sqrt{2}}{3}$

Directions (Q. Nos. 44 to 47) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

44. Consider vectors $\mathbf{a}$ and $\mathbf{c}$ are non-collinear, then

Statement I The lines $\mathbf{r}=6\mathbf{a}-\mathbf{c}+\lambda(2\mathbf{c}-\mathbf{a})$ and $\mathbf{r}=\mathbf{a}-\mathbf{c}+\mu(\mathbf{a}+3\mathbf{c})$ are coplanar.

Statement II There exist λ and μ such that the two values of r become same.

45. Consider **u** and **v** are unit vectors inclined at an angle α and **a** is a unit vector bisecting the angle between them,

Statement I Then, $\mathbf{a} = \frac{\mathbf{u}+\mathbf{v}}{2\cos(\alpha/2)}$.

Statement II If *ABC* is an isosceles triangle with $AB = AC = 1$, then vector representing bisector of $\angle A$ is $\frac{\mathbf{AB}+\mathbf{AC}}{2}$.

46. Suppose $\pi : x + y - 2z = 3, P : (2, 1, 6), Q : (6, 5, -2)$

Statement I The line joining *PQ* is perpendicular to the normal to the plane π.

Statement II *Q* is the image of *P* in the plane π.

47. Consider the equation of planes $P_1 = x + y + z - 6 = 0$ and $P_2 = 2x + 3y + 4z + 5 = 0$.

Statement I The equation of the plane through the intersection of the planes P_1 and P_2 and the point (4, 4, 4) is $29x + 23y + 17y = 276$.

Statement II Equation of the plane through the line of intersection of the planes $P_1 = 0$ and $P_2 = 0$ is $P_1 + \lambda P_2 = 0$, $\lambda \neq 0$.

Answer *with* Solutions

1. (*d*) Given,
$$\mathbf{A} + \mathbf{B} = \mathbf{a} \quad \text{...(i)}$$
$$\Rightarrow \mathbf{A}\cdot\mathbf{a} + \mathbf{B}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}$$
$$\Rightarrow 1 + \mathbf{B}\cdot\mathbf{a} = a^2$$
$$\Rightarrow \mathbf{B}\cdot\mathbf{a} = a^2 - 1 \quad \text{...(ii)}$$
Also,
$$\mathbf{A}\times\mathbf{B} = \mathbf{b}$$
$$\Rightarrow \mathbf{a}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{a}\times\mathbf{b}$$
$$\Rightarrow (\mathbf{a}\cdot\mathbf{B})\,\mathbf{A} - (\mathbf{a}\cdot\mathbf{A})\,\mathbf{B} = \mathbf{a}\times\mathbf{b}$$
$$\Rightarrow (a^2 - 1)\,\mathbf{A} - \mathbf{B} = \mathbf{a}\times\mathbf{b} \quad \text{[from Eq. (ii)] ...(iii)}$$
From Eqs. (i) and (iii), we get
$$\mathbf{A} = \frac{(\mathbf{a}\times\mathbf{b}) + \mathbf{a}}{a^2}$$
and
$$\mathbf{B} = \mathbf{a} - \left\{\frac{(\mathbf{a}\times\mathbf{b}) + \mathbf{a}}{a^2}\right\}$$
$$\therefore \mathbf{B} = \frac{(\mathbf{b}\times\mathbf{a}) + \mathbf{a}\,(a^2 - 1)}{a^2}$$

2. (*a*)
$$(\mathbf{a}\times\mathbf{b})^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta + |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$$
$$= |\mathbf{a}|^2|\mathbf{b}|^2$$
$$\Rightarrow 144 = (4)^2|\mathbf{b}|^2$$
$$\Rightarrow |\mathbf{b}| = 3$$

3. (*b*) Since,
$$\mathbf{a}\cdot\mathbf{b} < 0$$
$$\Rightarrow 14x^2 - 8x + x < 0$$
$$\Rightarrow 14x^2 - 7x < 0$$
$$\Rightarrow 7x(2x - 1) < 0$$
$$\therefore 0 < x < \frac{1}{2}$$

4. (*a*) The required vector **c** is given by
$$\mathbf{c} = \lambda(\mathbf{a} + \mathbf{b})$$
$$= \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$$
$$= \lambda\left\{\frac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \frac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right\}$$
$$\Rightarrow |\mathbf{c}| = \pm\frac{\lambda}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$$
$$\Rightarrow 5\sqrt{6} = \pm\frac{\lambda}{9}\sqrt{1 + 49 + 4} = \pm\frac{\lambda}{9}\sqrt{54}$$
$$\Rightarrow \lambda = \pm 15$$
$$\therefore \mathbf{c} = \pm\frac{15}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$$
$$= \pm\frac{5}{3}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$$

5. (*a*) $[(\mathbf{a} + 3\mathbf{b})\times(3\mathbf{a} - \mathbf{b})]^2 = [10\,(\mathbf{b}\times\mathbf{a})]^2$
$$= 100\,[|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2]$$
$$= 100\,[1\times 4 - (1\times 2\times\cos 120°)^2]$$
$$= 100\,(4 - 1) = 300$$

6. (*d*) Given, $V_1 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ and $V_2 = [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]$

Then,
$$V_2 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 1 & -4 & 2 \end{vmatrix}[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$$
$$= 15\,[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$$
$$\therefore V_2 : V_1 = 15 : 1$$

7. (*b*) Given,
$$|(5 - x)\,\mathbf{a}|^2 < |2\mathbf{a}|^2$$
$$\Rightarrow (5 - x)^2 < 4$$
$$\Rightarrow 5 - x < 2$$
or
$$5 - x > -2$$
$$\Rightarrow 3 < x$$
or
$$x < 7$$
$$\therefore 3 < x < 7$$

8. (*b*) The normal to the first plane is along $\mathbf{i}\times(\mathbf{i} + \mathbf{j}) = \mathbf{k}$ and the normal to the second plane is along
$$(\mathbf{i} - \mathbf{j})\times(\mathbf{i} + \mathbf{k}) = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$
Since, **a** is perpendicular to the two normals.

So, **a** is along $\mathbf{k}\times(-\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{j}$

Hence the angle between **a** and the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is

$$\cos^{-1}\frac{\mathbf{a}\cdot(\mathbf{i}-2\mathbf{j}+2\mathbf{k})}{|\mathbf{a}||\mathbf{i}-2\mathbf{j}+2\mathbf{k}|}=\cos^{-1}\frac{(\mathbf{i}-\mathbf{j})}{\sqrt{2}}\cdot\frac{(\mathbf{i}-2\mathbf{j}+2\mathbf{k})}{3}$$

$$=\cos^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$$

9. (*a*) Since, $\mathbf{b}=(\mathbf{a}\times\mathbf{c})\times\mathbf{a}$

$\Rightarrow$ $|\mathbf{b}|=|\mathbf{a}\times\mathbf{c}||\mathbf{a}| \Rightarrow 1=2\sin\theta$

$\therefore$ $\theta=\frac{\pi}{6}$

10. (*b*) Let $P(x, y, z)$ be any point on the locus, then the distances from the six faces are

$|x+1|, |x-1|, |y+1|, |y-1|, |z+1|, |z-1|$

According to the given condition,

$$|x+1|^2+|x-1|^2+|y+1|^2+|y-1|^2+|z+1|^2+|z-1|^2=10$$

$\Rightarrow$ $2(x^2+y^2+z^2)=10-6=4$

$\therefore$ $x^2+y^2+z^2=2$

11. (*a*) If **a**, **b** and **c** lie in a plane, then $2\mathbf{a}-\mathbf{b}$, $2\mathbf{b}-\mathbf{c}$ and $2\mathbf{c}-\mathbf{a}$, also lie in the same plane. So, their scalar triple product is '0'.

12. (*c*) Since, $\mathbf{k}\times(\mathbf{k}\times\mathbf{a})=\mathbf{0}$

$\Rightarrow$ $(\alpha\mathbf{i}+\beta\mathbf{j})=0$

$\Rightarrow$ $\alpha=\beta=0$

$\because$ $\alpha+\beta+\gamma=2$

$\therefore$ $\gamma=2$

13. (*a*) Here, $\mathbf{a}\times\mathbf{b}=-4\mathbf{i}-3\mathbf{j}+6\mathbf{k}$

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

$=(-4\mathbf{i}-3\mathbf{j}+6\mathbf{k})\cdot\mathbf{c}$

$\therefore$ The maximum value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$=|-4\mathbf{i}-3\mathbf{j}+6\mathbf{k}||\mathbf{c}|$

$=\sqrt{61}$

14. (*b*) The first three planes meet at the point whose position vector is (0, 0, 0). The first two and the fourth planes meet at the point whose position vector is $\left(\frac{p}{l},\frac{-p}{m},\frac{p}{n}\right)$. Similarly, the other two vertices of the tetrahedron have position vectors

$$\left(\frac{-p}{l},\frac{p}{m},\frac{p}{n}\right) \text{ and } \left(\frac{p}{l},\frac{p}{m},\frac{-p}{n}\right)$$

$\therefore$ Volume of the tetrahedron

$$=\frac{1}{6}\begin{Vmatrix} p/l & -p/m & p/n \\ -p/l & p/m & p/n \\ p/l & p/m & -p/n \end{Vmatrix}$$

$$=\frac{p^3}{6\,lmn}\begin{Vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{Vmatrix}$$

$$=\frac{p^3}{6\,lmn}\begin{Vmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{Vmatrix} \quad \begin{pmatrix} R_1\to R_1+R_2, \\ R_2\to R_2+R_3 \end{pmatrix}$$

$$=\frac{|-4|\,p^3}{6\,lmn}=\frac{2p^3}{3\,lmn}$$

15. (*d*) Let the variable plane intersects the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then, the equation of the plane will be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \ldots(i)$$

Let $P(\alpha, \beta, \gamma)$ be the centroid of tetrahendron $OABC$, then

$$\alpha=\frac{a}{4}, \beta=\frac{b}{4} \text{ and } \gamma=\frac{c}{4}$$

$\therefore$ $a=4\alpha, b=4\beta, c=4\gamma$

Now, volume of tetrahedron = (Area of ΔAOB) OC

$\Rightarrow$ $64k^3=\frac{1}{3}\left(\frac{1}{2}ab\right)c=\frac{abc}{6}$

$\Rightarrow$ $64k^3=\frac{(4\alpha)(4\beta)\times(4\gamma)}{3\times 2}$

$\therefore$ $k^3=\frac{\alpha\beta\gamma}{6}$

Hence, locus of $P(\alpha, \beta, \gamma)$ is $xyz=6k^3$.

16. (*a*) Given, $\mathbf{a}+\mathbf{b}+\mathbf{c}=\alpha\mathbf{d}$ and $\mathbf{b}+\mathbf{c}+\mathbf{d}=\beta\mathbf{a}$

$\therefore$ $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=(\alpha+1)\mathbf{d}$

and $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=(\beta+1)\mathbf{a}$

$\Rightarrow$ $(\alpha+1)\mathbf{d}=(\beta+1)\mathbf{a}$

If $\alpha=-1$, then $(\alpha+1)\mathbf{d}=(\beta+1)\mathbf{a}$

$\Rightarrow$ $\mathbf{d}=\frac{\beta+1}{\alpha+1}\mathbf{a}$

$\therefore$ $\mathbf{a}+\mathbf{b}+\mathbf{c}=\alpha\,\mathbf{d}$

$\Rightarrow$ $\mathbf{a}+\mathbf{b}+\mathbf{c}=\alpha\left(\frac{\beta+1}{\alpha+1}\right)\alpha$

$\Rightarrow$ $\left(1-\frac{\alpha(\beta+1)}{\alpha+1}\right)\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

Hence, **a**, **b** and **c** are coplanar, which is a contradiction to the given condition.

$\therefore$ $\alpha=-1$

$\Rightarrow$ $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$

17. (*a*) Now, $\mathbf{a}\cdot\mathbf{c}=x(\mathbf{a}\cdot\mathbf{a})+y(\mathbf{a}\cdot\mathbf{b})+z\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$

$\Rightarrow$ $x=\cos\theta$

Similarly, $y=\cos\theta$

Now, $|\mathbf{c}|^2=x^2|\mathbf{a}|^2+y^2|\mathbf{b}|^2+z^2|\mathbf{a}\times\mathbf{b}|^2$

$\Rightarrow$ $1-2\cos^2\theta=z^2$

$\Rightarrow$ $1-2x^2=z^2$

where, $z=|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|$

18. (a) Now, $\mathbf{d}_1 = \mathbf{a} + \mathbf{b} = 4\mathbf{p} + 2\mathbf{q}$

and $\mathbf{d}_2 = \mathbf{a} - \mathbf{b} = 2\mathbf{p} - 4\mathbf{q}$

$\Rightarrow \quad \mathbf{d}_1^2 = 16\mathbf{p}^2 + 4\mathbf{q}^2 + 16\mathbf{p}\cdot\mathbf{q}$

$$= 16(4) + 4(4) + 16\left(2 \times 2 \times \cos\frac{\pi}{3}\right) = 112$$

$\Rightarrow \quad |\mathbf{d}_1| = 4\sqrt{7}$

Similarly, $|\mathbf{d}_2| = 4\sqrt{3}$

$\therefore \quad \mathbf{d}_1 : \mathbf{d}_2 = \sqrt{7} : \sqrt{3}$

19. (b) $|\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = c$ (say)

and $\mathbf{p}\cdot\mathbf{q} = 0 = \mathbf{p}\cdot\mathbf{r} = \mathbf{q}\cdot\mathbf{r}$

Given that,

$\mathbf{p} \times \{(\mathbf{x} - \mathbf{q}) \times \mathbf{p}\} + \mathbf{q} \times \{(\mathbf{x} - \mathbf{r}) \times \mathbf{q}\} + \mathbf{r} \times \{(\mathbf{x} - \mathbf{p}) \times \mathbf{r}\} = \mathbf{0}$

$\Rightarrow (\mathbf{p}\cdot\mathbf{p})(\mathbf{x} - \mathbf{q}) - \{\mathbf{p}\cdot(\mathbf{x} - \mathbf{q})\}\mathbf{p} + \ldots = \mathbf{0}$

$\Rightarrow c^2(\mathbf{x} - \mathbf{q} + \mathbf{x} - \mathbf{r} + \mathbf{x} - \mathbf{p}) - (\mathbf{p}\cdot\mathbf{x})\mathbf{p} - (\mathbf{q}\cdot\mathbf{x})\mathbf{q} - (\mathbf{r}\cdot\mathbf{x})\mathbf{r} = 0$

$\Rightarrow c^2\{3\mathbf{x} - (\mathbf{p} + \mathbf{q} + \mathbf{r})\} - \{(\mathbf{p}\cdot\mathbf{x})\mathbf{p} + (\mathbf{q}\cdot\mathbf{x})\mathbf{q} + (\mathbf{r}\cdot\mathbf{x})\mathbf{r}\} = 0$

which is satisfied by $\mathbf{x} = \frac{1}{2}(\mathbf{p} + \mathbf{q} + \mathbf{r})$

20. (d) Let $P(r, r-2, r)$ and $Q(2k-2, k, k)$ are the general coordinates of points on the two given lines.

$\therefore$ DR's of PQ are

$(r - 2k + 2, r - k - 2, r - k)$

$\therefore \quad \frac{r - 2k + 2}{2} = \frac{r - k - 2}{1} = \frac{r - k}{2}$

$\Rightarrow \quad r = 6, k = 2$

So, the points of intersection are (6, 4, 6) and (2, 2, 2).

21. (c) The plane passing through (2, 2, 1), is

$$a(x-2) + b(y-2) + c(z-1) = 0$$

Since, it passes through (9, 3, 6).

$\therefore \quad 7a + b + 5c = 0 \quad \ldots(i)$

Since, it is perpendicular to $2x + 6y + 6z - 1 = 0$.

$\therefore \quad 2a + 6b + 6c = 0 \quad \ldots(ii)$

$\Rightarrow \quad \frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$

The required plane is

$3(x-2) + 4(y-2) - 5(z-1) = 0$

$\Rightarrow \quad 3x + 4y - 5z - 9 = 0$

22. (b) The required line passes through the point $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and is perpendicular to the lines

$\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

and $\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

So, it is parallel to the vector.

$\therefore \quad \mathbf{b} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$

The required equation is

$$\mathbf{r} = (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$$

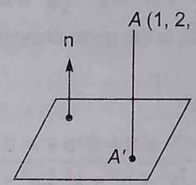
23. (b) Let $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

Line through A and parallel to $\mathbf{n}$ is

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$= (3\lambda + 1)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + 4\lambda)\mathbf{k} \quad \ldots(i)$

From Eq. (i) must satisfy the plane $3x - y + 4z = 0$.



$\therefore \quad 3(3\lambda + 1) - (2 - \lambda) + 4(3 + 4\lambda) = 0$

$\Rightarrow \quad 26\lambda + 13 = 0$

$\Rightarrow \quad \lambda = -\frac{1}{2}$

Hence, A' is $\left(-\frac{1}{2}, \frac{5}{2}, 1\right)$ which is the foot of the perpendicular from A on the given plane.

24. (d) Here, $AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} = 3$

and $AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2} = 13$

Since, AD is the bisector of $\angle BAC$.

$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{3}{13}$

Since, D divides BC in the ratio 3 : 13.

$\therefore$ The coordinates of D are

$$\left[\frac{3(-9) + 13(5)}{3 + 13}, \frac{3(6) + 13(3)}{3 + 13}, \frac{3(-3) + 13(2)}{3 + 13}\right] = \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

25. (d) DR's of OC are $(1, -1, 2)$.

Let the equation of plane passing through (1, 0, 1) is

$a(x-1) + b(y-0) + c(z-1) = 0 \quad \ldots(i)$

Since, its normal is perpendicular to OC

$\therefore \quad 1\cdot a + (-1)b + 2c = 0$

$\Rightarrow \quad a - b + 2c = 0 \quad \ldots(ii)$

As Eq. (i) passes through (3, 1, 2).

$\therefore \quad 2a + b + c = 0 \quad \ldots(iii)$

$\Rightarrow \quad \frac{a}{-1} = \frac{b}{1} = \frac{c}{1}$ [from Eqs. (ii) and (iii)]

Hence, required equation of plane be $x - y - z = 0$.

26. (c) Any point on the line is $(r + 3, 2r + 4, 2r + 5)$.

which lies on the plane $x + y + z = 17$.

$\because \quad (r+3) + (2r+4) + (2r+5) = 17$

$\therefore \quad r = 1$

Thus, the point of intersection is (4, 6, 7).

So, the required distance $= \sqrt{(4-3)^2 + (6-4)^2 + (7-5)^2}$

$= \sqrt{1+4+4} = 3$

27. (*d*) Equation of any plane passing through the line of intersection of given plane, is

$$ax + by + kz = 0 \quad \ldots(i)$$

$\therefore$ DC's of Eq. (i) are

$$\frac{a}{\sqrt{a^2+b^2+k^2}}, \frac{b}{\sqrt{a^2+b^2+k^2}}, \frac{k}{\sqrt{a^2+b^2+k^2}}$$

The DC's of a normal to the given plane is

$$\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0.$$

$$\therefore \quad \cos\alpha = \frac{a\cdot a + b\cdot b + k\cdot 0}{\sqrt{a^2+b^2+k^2}\sqrt{a^2+b^2}}$$

$$= \sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

$$\Rightarrow \quad k^2\cos^2\alpha = a^2(1-\cos^2\alpha) + b^2(1-\cos^2\alpha)$$

$$\Rightarrow \quad k^2 = \frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}$$

$$\therefore \quad k = \pm\sqrt{a^2+b^2}\tan\alpha$$

From Eq. (i), $ax + by \pm z\sqrt{a^2+b^2}\tan\alpha = 0$

28. (*b*) Let DC's of PQ be l, m and n.

$$\therefore \quad 3l - m + n = 0$$

and $\quad 5l + m + 3n = 0$

$$\therefore \quad \frac{l}{-3-1} = \frac{m}{5-9} = \frac{n}{3+5}$$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$$

Thus, the equation of plane perpendicular to PQ will have $x + y - 2z = \lambda$.

It passes through (2, 1, 4), therefore $\lambda = -5$.

Hence, the required equation of plane be

$$x + y - 2z = -5$$

29. (*a*) The line of intersection of the planes $\mathbf{r}\cdot(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and $\mathbf{r}\cdot(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is perpendicular to each of the normal vectors.

Here, $\quad \mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

and $\quad \mathbf{n}_2 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\therefore$ It is parallel to the vector $\mathbf{n}_1 \times \mathbf{n}_2$

$$= (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$= -2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$$

30. (*c*) Since, lines are

$$\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1} \quad \text{and} \quad \frac{x+a}{2} = \frac{y}{1} = \frac{z}{1}$$

Let $P \equiv (r, r-a, r)$ and $Q \equiv (2\lambda - a, \lambda, \lambda)$ be the points of Ist and IInd lines.

So, DR's of PQ are $r - 2\lambda + a, r - \lambda - a, r - \lambda$.

According to the given question,

$$\frac{r - 2\lambda + a}{2} = \frac{r - \lambda - a}{1} = \frac{r - \lambda}{2}$$

From Ist and IInd terms, $r - a = 2a$

$$\Rightarrow \quad r = 3a$$

From IInd and IIIrd terms, $\lambda = a$

$\therefore P \equiv (3a, 2a, 3a)$ and $Q \equiv (a, a, a)$

31. (*d*) The plane is

$$(2 + 3\lambda)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$$

Its distance from (2, 1, −1) is $\frac{1}{\sqrt{6}}$.

$$\therefore \quad \frac{(4 + 6\lambda + \lambda - 1 - \lambda - 1 - 5\lambda - 3)^2}{(2+3\lambda)^2 + (\lambda-1)^2 + (\lambda+1)^2} = \frac{1}{6}$$

$$\Rightarrow \quad (5\lambda + 24)\lambda = 0 \Rightarrow \lambda = \frac{-24}{5} \text{ or } 0$$

The planes are $2x - y + z - 3 = 0$

and $\quad 62x + 29y + 19z - 105 = 0$

32. (*a*) Equation of a plane passing through the point (2, −1, −3) and parallel to the given line is

$$\begin{vmatrix} x-2 & y+1 & z+3 \\ 3 & 2 & -4 \\ 2 & -3 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(4-12) - (y+1)(6+8) + (z+3)(-9-4) = 0$$

$$\Rightarrow \quad 8x + 14y + 13z + 37 = 0$$

33. (*d*) The common normal is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$$\mathbf{r} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$\therefore$ Shortest distance $= (\mathbf{j} + \mathbf{k})\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{\sqrt{6}}$

34. (*a*) Normal to OAB is $\mathbf{OA} \times \mathbf{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

Normal to ABC is $\mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix}$

$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

If θ is angle between the planes, then

$$\cos\theta = \frac{5+5+9}{\sqrt{35}\cdot\sqrt{35}} = \frac{19}{35}$$

35. (*a*) Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$\because$ Diagonals of a parallelogram in terms of its sides are

$$\mathbf{p} = \mathbf{a} + \mathbf{b} \text{ and } \mathbf{p} = \mathbf{b} - \mathbf{a}$$

$$\Rightarrow \mathbf{p} = 5\mathbf{i} + \mathbf{j} - \mathbf{k} \text{ and } \mathbf{q} = -\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}$$

The unit vectors along the diagonals are

$$\frac{5\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{25+1+1}} \text{ and } \frac{-\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}}{\sqrt{(-1)^2 + 49 + 121}}$$

$$\Rightarrow \frac{5\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{27}} \text{ and } \frac{-\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}}{\sqrt{171}}$$

36. (*c*) Since, $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

$$\therefore \quad \mathbf{p} \cdot \mathbf{q} = 0$$

Let $\mathbf{q} = x\mathbf{i} + y\mathbf{j}$, then

$$(8\mathbf{i} + 6\mathbf{j})(x\mathbf{i} + y\mathbf{j}) = 0$$

$$\Rightarrow \quad 8x + 6y = 0$$

$$\therefore \quad y = -\frac{8x}{6} = -\frac{4x}{3}$$

$$\mathbf{q} = x\mathbf{i} + \left(\frac{-4x}{3}\right)\mathbf{j} = \frac{3x\mathbf{i} - 4x\mathbf{j}}{3} = \frac{x}{3}(3\mathbf{i} - 4\mathbf{j})$$

Again, the projection of vector $\mathbf{r} = \pm(x_1\mathbf{i} + x_2\mathbf{j})$ on vector $\mathbf{p}$ is 2 and on $\mathbf{q}$ is 4.

$$\therefore \quad 2 = \left|\frac{8x_1 + 6x_2}{10}\right|$$

$$\text{and} \quad 4 = \left|\frac{6x_1 - 8x_2}{10}\right|$$

$$\Rightarrow \quad 8x_1 + 6x_2 = 20$$

$$\text{and} \quad 6x_1 - 8x_2 = 40$$

$$\Rightarrow \quad 4x_1 + 3x_2 = 10$$

$$\text{and} \quad 3x_1 - 4x_2 = 20$$

$$\Rightarrow \quad x_1 = 4 \text{ and } x_2 = -2$$

$$\therefore \quad \mathbf{r} = \pm(4\mathbf{i} - 2\mathbf{j}) = \pm 2(2\mathbf{i} - \mathbf{j})$$

37. (*d*) Let $\mathbf{r} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$$\text{Since,} \quad \mathbf{r} \perp \mathbf{p} \text{ and } \mathbf{r} \perp \mathbf{q}$$

$$\therefore \quad \mathbf{r} \cdot \mathbf{p} = 0 \text{ and } \mathbf{r} \cdot \mathbf{q} = 0$$

$$\Rightarrow \quad 2a_1 - 3a_2 + 3a_3 = 0$$

$$\text{and} \quad 4a_1 - 2a_2 + a_3 = 0$$

$$\Rightarrow \quad 2a_1 - 3a_2 + 3a_3 = 0$$

$$\text{and} \quad 4a_1 - 2a_2 + a_3 = 0$$

$$\therefore \quad \frac{a_1}{-3-(-6)} = \frac{-a_2}{12-2} = \frac{a_3}{-4-(-12)} = k \text{ (say)}$$

$$\Rightarrow \quad \frac{a_1}{3} = \frac{a_2}{10} = \frac{a_3}{8} = k$$

$$\therefore \quad a_1 = 3k, a_2 = 10k, a_3 = 8k \quad \text{...(i)}$$

Again, $(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})(2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -12$

$$\Rightarrow \quad 2a_1 - 4a_2 + 2a_3 = -12 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$6k - 4(10k) + 2(8k) = -12$$

$$6k - 40k + 16k = -12$$

$$-18k = -12 \Rightarrow k = \frac{12}{18} = \frac{2}{3}$$

$$\therefore \quad a_1 = 2, a_2 = \frac{20}{3}, a_3 = \frac{16}{3}$$

$$\therefore \quad \mathbf{r} = 2\mathbf{i} + \frac{20}{3}\mathbf{j} + \frac{16}{3}\mathbf{k}$$

38. (*d*) Plane passing through a point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

$\therefore$ Plane through (2, 0, 0) is

$$a(x-2) + b(y-0) + c(z-0) = 0 \quad \text{...(i)}$$

It contains (0, 2, 0), if

$$-2a + 2b = 0 \Rightarrow -a + b = 0 \quad \text{...(ii)}$$

Since, plane $a(x-2) + b(y-0) + c(z-0) = 0$ makes an angle $\frac{\pi}{3}$ with the plane $2x + 3y = 5$.

$$\therefore \quad \cos\frac{\pi}{3} = \frac{2a + 3b}{\sqrt{a^2 + b^2 + c^2}\sqrt{4+9}}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{2a + 3b}{\sqrt{(a^2 + b^2 + c^2)(13)}}$$

$$\Rightarrow \quad \frac{1}{4} = \frac{(2a + 3b)^2}{[\sqrt{(a^2 + b^2 + c^2)(13)}]^2}$$

$$\Rightarrow \quad 13(a^2 + b^2 + c^2) = 100a^2$$

$$\because \quad a = b$$

$$\therefore \quad 13(2a^2 + c^2) = 100a^2$$

$$\Rightarrow \quad 26a^2 + 13c^2 = 100a^2$$

$$\Rightarrow \quad 13c^2 = 74a^2$$

$$\Rightarrow \quad c = \sqrt{\frac{74}{13}}a$$

$$\therefore \quad a : b : c = a : a : \sqrt{\frac{74}{13}}a$$

$$= a : a : \sqrt{5.7}\,a = 1 : 1 : \sqrt{5.7}$$

39. (*d*) Shortest distance

$$= \frac{\begin{Vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}}{\begin{Vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}}$$

$$= \frac{\begin{Vmatrix} -1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 4 \end{Vmatrix}}{\begin{Vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 2 & 3 & 4 \end{Vmatrix}}$$

$$= \left|\frac{-1(8-6)+1(4-12)+1(9-4)}{|\mathbf{i}(8-6)+\mathbf{j}(4-12)+\mathbf{k}(9-4)|}\right|$$

$$= \left|\frac{-2-8+5}{\sqrt{93}}\right|$$

$$= \frac{5}{\sqrt{93}} = 0.52 \text{ unit}$$

40. (*c*) Now, $a_1a_2 + b_1b_2 + c_1c_2 = -3\times 1 + 2\times -3 + 1\times 2$
$= -7$

Therefore, the lines are neither perpendicular nor parallel.

Now, $\begin{vmatrix} -1-0 & 3-7 & -2+7 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -4 & 5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$

$= -1(4+3) + 4(-6-1) + 5(9-2)$
$= -7 - 28 + 35 = 0$

41. (*b*) Any point on the lines L_1 and L_2 are $(-3r_1 - 1, 2r_1 + 3, r_1 - 2)$ and $(r_2, -3r_2 + 7, 2r_2 - 7)$

Since, they intersect each other, therefore

$-3r_1 - 1 = r_2, 2r_1 + 3 = -3r_2 + 7$

and $r_1 - 2 = 2r_2 - 7$

On solving, we get $r_2 = 2$ and $r_1 = -1$

Hence, the required point is $(2, 1, -3)$.

42. (*c*) Here, $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = 16, \mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2 = 16, \mathbf{c}\cdot\mathbf{c} = |\mathbf{c}|^2 = 16$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = |\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{3} = 4\cdot 4\cdot\frac{1}{2} = 8$$

$$\mathbf{a}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = |\mathbf{a}||\mathbf{c}|\cos\frac{\pi}{3} = 4\cdot 4\cdot\frac{1}{2} = 8$$

Similarly, $\mathbf{b}\cdot\mathbf{c} = 8$

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$

$$= \begin{vmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{vmatrix}$$

$$= 8^3 \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 8^3\cdot 4 = 64\times 32$$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 32\sqrt{2}$

$\therefore$ Volume of parallelopiped $= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 32\sqrt{2}$

43. (*d*) Volume of the tetrahedron $= \frac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$= \frac{1}{6}\times 32\sqrt{2}$$

$$= \frac{16\sqrt{2}}{3}$$

44. (*a*) If the lines have a common point, then there exists λ and μ such that

$6 - \lambda = 1 + \mu$

and $-1 + 2\lambda = -1 + 3\mu$

$\Rightarrow \quad \lambda = 3, \mu = 2$

$\therefore \quad \mathbf{r} = 3\mathbf{a} + 5\mathbf{c}$

45. (*a*) In an isosceles ΔABC in which $AB = AC$ the median and bisector from A must be same line, so Statement II is true.

Now, $\mathbf{AD} = \frac{\mathbf{u}+\mathbf{v}}{2}$

$\Rightarrow \quad |\mathbf{AD}|^2 = \frac{1}{4}[|\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}\cdot\mathbf{v}]$

$$= \frac{1}{4}(2 + 2\cos\alpha)$$

$$= \cos^2\frac{\alpha}{2}$$

$\therefore$ Unit vector along $\mathbf{AD} = \frac{1}{2\cos\frac{\alpha}{2}}(\mathbf{u}+\mathbf{v})$

46. (*d*) Direction ratios of PQ are $6-2, 5-1, -2-6$ *i.e.*, $4, 4, -8$ which are proportional to the direction ratios of the normal to the plane π, so PQ is perpendicular to π.

Hence, Statement I is false and Statement II is true.

47. (*a*) The equation of plane through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ is

$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$ ...(i)

Since, it passes through $(4, 4, 4)$, then

$(4 + 4 + 4 - 6) + \lambda(8 + 12 + 16 + 5) = 0$

$\Rightarrow \quad 6 + 41\lambda = 0$

$\Rightarrow \quad \lambda = -\frac{6}{41}$

From Eq. (i), we get

$41(x + y + z - 6) - 6(2x + 3y + 4z + 5) = 0$

$\therefore \quad 29x + 23y + 17z = 276$

Day 34

Statistics

Day 34
Outlines ...

- Measure of Central Tendency
- Mean (Arithmetic Mean)
- Geometric Mean (GM)
- Harmonic Mean (HM)
- Median
- Mode
- Measure of Dispersion

Statistics is concerned with scientific method for collecting, organising, summarising, presenting and analysing data as well as with drawing valid conclusions and making reasonable decision on the basis of such analysis.

Measure *of* Central Tendency

An average or a central value of the statistical (series) data is the value of the variable which describes the characteristics of entire distribution.

There are five types of central tendency.

1. Mean (Arithmetic Mean)

The sum of all the observations is divided by the number of observation is called **mean** and it is denoted by $\bar{x}$. The most stable measure of central tendency is mean.

(i) Mean of Ungrouped Data

If $x_1, x_2, \ldots, x_n$ be the n observations, then mean (arithmetic mean) is given by

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum x_i}{n}$$

or $\bar{x} = A + \frac{\sum_{i=1}^{n} d_i}{n}$, where $d_i = x_i - A$ and A is assumed mean.

(ii) Mean of Grouped Data

If corresponding frequencies of n observations are $f_1, f_2, \ldots, f_n$, then

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

or $$\bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{N}, \text{ where } d_i = x_i - A \text{ and } N = \sum_{i=1}^{n} f_i$$

(iii) Weighted Mean

If corresponding weights are $w_1, w_2, \ldots, w_n$, then weighted mean, $\bar{x}_w = \frac{\sum wx}{\sum w}$.

The weighted mean of first n natural numbers whose weights are equal to the corresponding numbers is equal to $\frac{(2n+1)}{3}$.

(iv) Combined Mean

If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ be the mean of k groups of observations with corresponding frequencies of size $n_1, n_2, \ldots, n_k$, then combined mean, $\bar{x}_{1k} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots + n_k}$

$\bar{x}_1$ = mean of first set of observations

x_1 = number of observations in first set

$\bar{x}_2$ = mean of second set of observations

x_2 = number of observations in second set

2. Geometric Mean (GM)

If $x_1, x_2, \ldots, x_n$ are the n observations, then nth root of the product of all observations is called **geometric mean**.

(i) GM for Ungrouped Data

If $x_1, x_2, \ldots, x_n$ be n non-zero observations, then

$$GM = (x_1 \cdot x_2 \ldots x_n)^{1/n} = \text{antilog}\left(\frac{1}{n}\sum_{i=1}^{n} \log x_i\right)$$

(ii) GM for Grouped Data

If corresponding frequencies of each observation are $f_1, f_2, \ldots, f_n$, then $GM = [x_1^{f_1} \cdot x_2^{f_2} \ldots x_n^{f_n}]^{\frac{1}{N}}$

$$= \text{antilog}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right] \text{ where, } N = \sum_{i=1}^{n} f_i$$

3. Harmonic Mean (HM)

The harmonic mean of any series is the reciprocal of the arithmetic mean of the reciprocals of the observations.

(i) HM for Ungrouped Data

The harmonic mean of n items $x_1, x_2, \ldots, x_n$ is defined as

$$HM = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}} = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right]^{-1}$$

(ii) HM for Grouped Data

If corresponding frequencies of each observation are $f_1, f_2, \ldots, f_n$, then $HM = \left[\frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i}\right]^{-1}$

4. Median

If the variable of series are arranged in ascending or descending order, then the value of the middle variable is defined as the median.

(i) Median for Ungrouped Data

Let $x_1, x_2, \ldots, x_n$ be the n observations.

If n is odd, then Median $= \frac{n+1}{2}$th observation

If n is even, then

$$\text{Median} = \frac{\left[\left(\frac{n}{2}\right)\text{th} + \left(\frac{n}{2}+1\right)\text{th}\right]\text{observation}}{2}$$

(ii) Median for Grouped Data

If in a continuous distribution, the total frequency be N, then the class whose comulative (comulative) frequency is either equal to $N/2$ or just greater than $N/2$ is called median class.

$$\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - c}{f} \times h,$$

where,

l = Lower limit of median class

f = Frequency of median group

h = Size of median group

c = Cumulative frequency of group before median group

5. Mode

Mode is the observation which has maximum frequency.

(i) Mode for Ungrouped Data

If $x_1, x_2, \ldots, x_n$ are the n observations and corresponding frequencies are $f_1, f_2, \ldots, f_n$, then the corresponding observation of maximum frequency is a modal value.

(ii) Mode for Grouped Data

In a continuous data the interval which has maximum frequency called **modal class**.

Then, $$\text{mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h,$$

where, l = Lower limit of modal class

f_1 = Frequency of modal class

f_0 = Frequency of the class preceding the modal class

f_2 = Frequency of the class succeeding the modal class

h = Class size

Relation between Averages

Let AM, GM and HM are the arithmetic mean, geometric mean and harmonic mean respectively, then the following relations are given below.

- $(AM)(HM) = (GM)^2$
- Mode = 3 Median −2 Mean (Emperical formula)

Measure *of* Dispersion

A measure of dispersion is designed to state the extent to which the individual observations vary from their average. There are following types of dispersion are

1. Range

The difference between the maximum and the minimum observation is called **range**.

i.e., $$\text{Range} = L - S$$

$$\text{Coefficient of range} = \frac{L - S}{L + S}$$

where, L = Maximum observation and S = Minimum observation

2. Quartile Deviation

It is one-half of the difference between the third and the first quartile.

i.e., $$QD = \frac{Q_3 - Q_1}{2} \text{ and coefficient of } QD = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

where, Q_1 = Value of $\frac{n}{4}$th term and Q_3 = Value of $\frac{3n}{4}$th term

3. Mean Deviation (MD)

The mean, (mode or median) of the absolute deviations of the values of the variable from a measure of their average is called **Mean Deviation** (MD).

(i) MD for Ungrouped Data

If $x_1, x_2, \ldots, x_n$ are n observations, then $$MD = \frac{\sum_{i=1}^{n} |x_i - z|}{n}$$

where, z = mean or mode or median

(ii) MD for Grouped Data

If corresponding frequencies of each observation are $f_1, f_2, \dots, f_n$, then

$$\text{MD} = \frac{\sum_{i=1}^{n} f_i |x_i - z|}{N}, \text{ where } \sum_{i=1}^{n} f_i = N$$

$$\text{Coefficient of MD} = \frac{\text{Mean deviation}}{\text{Corresponding mean}} \times 100\%$$

4. Standard Deviation (SD)

The square root of the arithmetic mean of the squares of deviations of the terms from their arithmetic mean is called standard deviation and it is denoted by σ.

(i) SD for Ungrouped Data

If $x_1, x_2, \dots, x_n$ are n observations, then $\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}$

(ii) SD for Grouped Data

If corresponding frequencies of each observation are $f_1, f_2, \dots, f_n$, then

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}} = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i^2\right) - \frac{1}{N}\left(\sum_{i=1}^{n} f_i x_i\right)^2}, \text{ where } \sum_{i=1}^{n} f_i = N$$

5. Variance

The square of SD is called **variance** and it is denoted by σ^2. Coefficient of variation $= \frac{\sigma}{\bar{x}} \times 100\%$

Two different series having n_1 and n_2 observations and whose corresponding means and variance are $\bar{x}_1, \bar{x}_2$ and σ_1, σ_2. Then, **combined variance,**

$$\sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$$

where $d_1 = (\bar{x}_1 - \bar{x}_{12}), d_2 = (\bar{x}_2 - \bar{x}_{12})$ and $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

- *The algebraic sum of the deviation of a set of values from their AM is zero.*
- *Mean deviation from the median is less than that measured from any other mean.*
- *Standard deviation is always less than range.*
- *Standard deviation of n natural numbers is* $\sigma = \left[\frac{1}{12}(n^2 - 1)\right]^{1/2}$.
- *Quartile deviation* $= \frac{2}{3}\sigma$ *and mean deviation* $= \frac{4}{5}\sigma$

Effect of Average and Dispersion on Change of Origin and Scale

	Change of origin	Change of scale
Mean	Dependent	Dependent
Median	Not dependent	Dependent
Mode	Not dependent	Dependent
Standard deviation	Not dependent	Dependent
Variance	Not dependent	Dependent

Practice Zone DAY 34

1. If SD of variate x is σ, then the SD of $\frac{ax+b}{p}, \forall a, b, p \in R$ is
   (a) $\left|\frac{a}{p}\right|\sigma_x$ (b) $\left|\frac{p}{a}\right|\sigma_x$
   (c) $\frac{p}{a}\sigma_x$ (d) None of these

2. The median of 19 observations of a group is 30. If two observations with values 8 and 32 are further included, then the median of the new group of 21 observations will be
   (a) 28 (b) 30
   (c) 32 (d) 34

3. If a variable takes the discrete values $\alpha+4, \alpha-\frac{7}{2}, \alpha-\frac{5}{2}, \alpha-3, \alpha-2, \alpha+\frac{1}{2}, \alpha-\frac{1}{2}, \alpha+5$ (where, $\alpha > 0$), then the median is
   (a) $\alpha-\frac{5}{4}$ (b) $\alpha-\frac{1}{2}$
   (c) $\alpha-2$ (d) $\alpha+\frac{5}{4}$

4. The first of two samples has 100 items with mean 15 and SD = 3. If the whole group has 250 items with mean 15.6 and SD $=\sqrt{13.44}$, then SD of the second group is
   (a) 4 (b) 5
   (c) 6 (d) 3.52

5. If a variable x takes values x_i such that $a \le x_i \le b$, for $i = 1, 2, \ldots, n$, then
   (a) $a^2 \le \text{Var}(x) \le b^2$ (b) $a \le \text{Var}(x) \le b$
   (c) $\frac{a^2}{4} \le \text{Var}(x)$ (d) $(b-a)^2 \ge \text{Var}(x)$

6. The best statistical measure used for comparing two series is
   (a) mean deviation (b) range
   (c) coefficient of variation (d) None of these

7. If the mean age of combined group of boys and girls are 18 yr and the means of the age of boys be 20 and of girls is 16, then percentage of boys in the group is
   (a) 45 (b) 50
   (c) 25 (d) 65

8. If quartile deviation of a sample is 20, then the most likely value of SD is
   (a) 30 (b) 12 (c) 18 (d) 13

9. An aeroplane flies around a squares the sides of which measure 100 miles each. The aeroplane covers at a speed of 100 m/h, the first side at 200 m/h, the second side at 300 m/h, the third side and 400 m/h, the fourth side. The average speed of the aeroplane around the square is
   (a) 190 m/h (b) 195 m/h
   (c) 192 m/h (d) 200 m/h

10. The mean deviation from mean of the observation $a, a+d, a+2d, \ldots, a+2nd$ is
   (a) $\frac{n(n+1)d^2}{3}$ (b) $\frac{n(n+1)}{2}d^2$
   (c) $a+\frac{n(n+1)d^2}{2}$ (d) None of these

11. If the standard deviation of $x_1, x_2, \ldots, x_n$ is 3.5, then the standard deviation of $-2x_1-3, -2x_2-3, \ldots, -2x_n-3$ is
   (a) -7 (b) -4
   (c) 7 (d) 1.75

12. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$. The standard deviation of the observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be
   (a) $\sqrt{10}+20$ (b) $\sqrt{10}+10$
   (c) $\sqrt{10}$ (d) None of these

13. The mean and median of 100 items are 50 and 52, respectively. The value of largest item is 100. It was later found that it is 110 not 100. The true mean and median are
   (a) 50.10, 51.5 (b) 50.10, 52
   (c) 50, 52 (d) None of these

14. If a variable takes the values $0, 1, 2, \ldots, n$ with frequencies proportional to the binomial coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$, then mean of the distribution is
   (a) $\frac{n}{2}$ (b) $\frac{n(n+1)}{2}$
   (c) $\frac{n(n-1)}{2}$ (d) None of these

15. Find the mean deviation from the median of the following data. **[NCERT Exemplar]**

Class interval	0-6	6-12	12-18	18-24	24-30
Frequency	4	5	3	6	2

(a) 7.08 (b) 7 (c) 7.1 (d) 7.05

16. Let $x_1, x_2, \dots, x_n$ be n observations and $w_i = lx_i + k$ for $i = 1, 2, \dots, n$, where l and k are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of w_i's is 55 and standard deviation of w_i's is 15, the values of l and k should be **[NCERT Exemplar]**

(a) $l = 1.25, k = -5$ (b) $l = -1.25, k = 5$
(c) $l = 2.5, k = -5$ (d) $l = 2.5, k = 5$

17. The marks of some students were listed out of 75. The SD of marks was found to be 9. Subsequently the marks were raised to a maximum of 100 and variance of new marks was calculated. The new variance is

(a) 81 (b) 122
(c) 144 (d) None of these

18. Consider any set of 201 observations $x_1, x_2, \dots, x_{200}, x_{201}$. It is given that $x_1 < x_2 < \dots < x_{200} < x_{201}$. Then, the mean deviation of this set of observations about a point k is minimum when k is equal to

(a) $(x_1 + x_2 + \dots + x_{200} + x_{201})/201$
(b) x_1
(c) x_{101}
(d) x_{201}

19. Coefficient of variation of two distributions are 50 and 60 and their arithmetic means are 30 and 25, respectively. Difference of their standard deviation is **[NCERT Exemplar]**

(a) 0 (b) 1 (c) 1.5 (d) 2.5

20. If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then the standard deviation of $x_1, x_2, \dots, x_{18}$ is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$
(c) $\frac{3}{2}$ (d) None of these

21. Mean and standard deviation of 100 observations were found to be 40 and 10, respectively. If at the time of calculation two observations were wrongly taken as 30 and 70 in place of 3 and 27 respectively, find the correct standard deviation. **[NCERT Exemplar]**

(a) 10.20 (b) 10.24
(c) 10.29 (d) 10.27

22. The quartile deviation of daily wages (in ₹) of 7 persons given below 12, 7, 15, 10, 17, 19 and 25 is

(a) 14.5 (b) 5 (c) 9 (d) 4.5

23. The mean of five observations is 4 and their variance is 5.2. If three of them are 1, 2, 6, then other two are

(a) 4, 7 (b) 2, 9
(c) 5, 6 (d) 2, 10

Directions (Q. Nos. 24 to 26) *Let us consider the data of marks of a student whose corresponding frequencies are given.*

x	0-8	8-16	16-24	24-32	32-40	40-48	48-56
f	5	10	13	25	35	19	13

24. The median of the following distribution is

(a) 32.5
(b) 33.6
(c) 36.5
(d) None of these

25. The mode of the following distribution is

(a) 33.07 (b) 36.04
(c) 37.05 (d) None of these

26. The mean of the following distribution is

(a) 33.825 (b) 32.80
(c) 34 (d) None of these

Directions (Q. Nos. 27 to 29) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

27. The algebraic sum of deviation from their mean is defined as $\sum_{i=1}^{n}(x_i - \bar{x})$.

Statement I The algebraic sum of the deviations of 20 observations measured from 30 is 2. The mean value of the observation is 30.

Statement II The sum of deviation from their mean is zero.

28. If n is a natural number, then

Statement I The mean of the squares of first n natural number is $\frac{(n+1)(2n+1)}{6}$.

Statement II $\Sigma n = \frac{n(n+1)}{2}$

29. Standard deviation is not depend on change of origin

Statement I The standard deviation of n items is λ and if each item is decreased by 1, then SD will be $\lambda - n$.

Statement II If each item increased or decreased by some constant, then SD remains unchanged.

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30. If the median and the range of four numbers $\{x, y, 2x+y, x-y\}$, where $0 < y < x < 2y$, are 10 and 28 respectively, then the mean of the numbers is **[JEE Main 2013]**

(a) 18 (b) 10
(c) 5 (d) 14

31. Mean of 5 observations is 7. If four of these observations are 6, 7, 8, 10 and one is missing, then the variance of all the five observations is **[JEE Main 2013]**

(a) 4 (b) 6
(c) 8 (d) 2

32. The mean of a data set consisting of 20 observations is 40. If one observation 53 was wrongly recorded as 33, then the correct mean will be **[JEE Main 2013]**

(a) 41 (b) 49
(c) 40.5 (d) 42.5

33. In a set of $2n$ observations, half of them are equal to a and the remaining half are equal to $-a$. If the standard deviation of all the observations is 2, then the value of $|a|$ is **[JEE Main 2013]**

(a) 2 (b) $\sqrt{2}$
(c) 4 (d) $2\sqrt{2}$

34. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? **[AIEEE 2013]**

(a) Mean (b) Median (c) Mode (d) Variance

35. Let $x_1, x_2, \dots, x_n$ be n observations and let $\bar{x}$ be their arithmetic mean and σ^2 be the variance.

Statement I Variance of $2x_1, 2x_2, \dots, 2x_n$ is $4\sigma^2$.

Statement II Arithmetic mean $2x_1, 2x_2, \dots, 2x_n$ is $4\bar{x}$. **[AIEEE 2012]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

36. If the mean deviations about the median of the numbers $a, 2a, \dots, 5a$ is 50, then $|a|$ is equal to **[AIEEE 2011]**

(a) 3 (b) 4 (c) 5 (d) 2

37. For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is **[AIEEE 2010]**

(a) $\frac{5}{2}$ (b) $\frac{11}{2}$
(c) 6 (d) $\frac{13}{2}$

38. If the mean deviation of numbers $1, 1+d, 1+2d, \dots, 1+100d$ from their mean is 255, then d is equal to **[AIEEE 2009]**

(a) 10.0 (b) 20.0 (c) 10.1 (d) 20.2

39. **Statement I** The variance of first n even natural numbers is $\frac{n^2-1}{4}$.

Statement II The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$. **[AIEEE 2009]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

40. The mean of the numbers a, b, 8, 5, 10 is 6 and the variance is 6.80. Then, which one of the following gives possible values of a and b? **[AIEEE 2008]**

(a) $a=3, b=4$ (b) $a=0, b=7$
(c) $a=5, b=2$ (d) $a=1, b=6$

41. The average marks of boys in a class is 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is **[AIEEE 2007]**

(a) 40% (b) 20% (c) 80% (d) 60%

42. Let a population A has 100 observations 101, 102, .., 200 and another population B has 100 observations 151, 152, ..., 250. If V_A and V_B represent the variances of the two populations respectively, then $\frac{V_A}{V_B}$ is **[AIEEE 2006]**

(a) 9/4 (b) 4/9 (c) 2/3 (d) 1

43. If $x_1, x_2, \dots, x_n$ be n observations such that $\Sigma x_i^2 = 400$ and $\Sigma x_i = 80$. Then, a possible value of n among the following is **[AIEEE 2005]**

(a) 12 (b) 9 (c) 18 (d) 16

44. If in a frequency distribution the mean and median are 21 and 22 respectively, then its mode is approximately **[AIEEE 2005]**

(a) 24.0 (b) 25.5 (c) 20.5 (d) 22.0

45. Consider the following statements.

(i) Mode can be computed from histogram.
(ii) Median is not independent of change of scale.
(iii) Variance is independent of change of origin and scale.

Which of these is /are correct? **[AIEEE 2004]**

(a) Only (i) (b) Only (ii)
(c) (i) and (ii) (d) (i), (ii) and (iii)

46. In a series of $2n$ observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the observations is 2, then $|a|$ is equal to **[AIEEE 2004]**

(a) $\frac{1}{n}$ (b) $\sqrt{2}$

(c) 2 (d) $\frac{\sqrt{2}}{n}$

47. In an experiment with 15 observations on x, the following results were available $\Sigma x^2 = 2830, \Sigma x = 170$. One observation that was 20 was found to be wrong and was replaced by the correct value 30. Then, the corrected variance is **[AIEEE 2003]**

(a) 78.0 (b) 188.66 (c) 177.33 (d) 8.33

Answers

1. (a)	**2.** (b)	**3.** (a)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (d)
11. (c)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (d)	**23.** (a)	**24.** (b)	**25.** (d)	**26.** (d)	**27.** (d)	**28.** (b)	**29.** (d)	**30.** (d)
31. (a)	**32.** (a)	**33.** (a)	**34.** (d)	**35.** (c)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (d)	**40.** (a)
41. (c)	**42.** (d)	**43.** (d)	**44.** (a)	**45.** (c)	**46.** (c)	**47.** (a)			

Hints & Solutions

1. Let $u = \frac{ax+b}{p}$, then $\bar{u} = \frac{a\bar{x}+b}{p}$

$$\therefore \text{SD} = \sqrt{\frac{\Sigma(u-\bar{u})^2}{\Sigma f}} = \sqrt{\frac{a^2}{p^2}\frac{\Sigma(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{a^2}{p^2}\sigma_x^2} = \left|\frac{a}{p}\right|\sigma_x$$

2. Since, there are 19 observations. So, the middle term is 10th. After including 8 and 32, *i.e.*, 8 will come before 30 and 32 will come after 30.
Here, new median will remain 30.

3. Firstly arrange the given data in ascending order.

$$\alpha - \frac{7}{2}, \alpha - 3, \alpha - \frac{5}{2}, \alpha - 2, \alpha - \frac{1}{2}, \alpha + \frac{1}{2}, \alpha + 4, \alpha + 5$$

$\therefore$ Median $= \frac{1}{2}$ [Value of 4th item + Value of 5th item]

$$= \frac{\alpha - 2 + \alpha - \frac{1}{2}}{2} = \frac{2\alpha - \frac{5}{2}}{2} = \alpha - \frac{5}{4}$$

4. We know that, $\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$,

where $d_1 = m_1 - a, d_2 = m_2 - a$, a being the mean of the whole group.

$$\therefore \quad 15.6 = \frac{100 \times 15 + 150 \times m_2}{250} \Rightarrow m_2 = 16$$

$$\text{Thus,} \quad 13.44 = \frac{\begin{bmatrix}(100 \times 9 + 150 \times \sigma^2) \\ + 100 \times (0.6)^2 + 150 \times (0.4)^2\end{bmatrix}}{250}$$

$$\Rightarrow \quad 33.60 = (900 + 150\sigma^2 + 36 + 24)$$

$$\therefore \quad \sigma^2 = 16 \Rightarrow \sigma = 4$$

5. Since, SD < Range $\Rightarrow \sigma \le (b-a)$

$$\sigma^2 \le (b-a)^2$$

$$\Rightarrow (b-a)^2 \ge \text{Var}(x)$$

6. The best statistical measure used for comparing two series is coefficient of variation.

7. Let there be 100 persons and out of 100 there are x boys and $100 - x$ are girls.

According to the question, $\frac{16(100-x)+20(x)}{100} = 18$

$$\Rightarrow \quad 1600 - 16x + 20x = 1800$$

$$\therefore \quad 4x = 200 \Rightarrow x = 50$$

Hence 50% of boys in a group.

8. $\because$ Quartile deviation $= \frac{2}{3} \times$ Standard deviation

$\therefore$ Standard deviation $= \frac{20 \times 3}{2} = 30$

9. Using the harmonic mean formula,

$$\frac{1}{H} = \frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i} \Rightarrow H = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i}}$$

$$\therefore \text{ Average speed} = \frac{400}{100\left(\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400}\right)} = 192 \text{ m/h}$$

10. $\bar{x} = \frac{1}{2n+1}[a + (a+d) + \ldots + (a+2nd)]$

$$= \frac{1}{2n+1}[(2n+1)a + d(1+2+\ldots+2n)]$$

$$= a + d\frac{2n}{2}\cdot\frac{(2n+1)}{2n+1} = a + nd$$

$$\therefore \text{ MD from mean} = \frac{1}{2n+1}\Sigma|x_i - \bar{x}|$$

$$= \frac{1}{2n+1}2|d|(1+2+\ldots+n) = \frac{n(n+1)|d|}{(2n+1)}$$

11. If $d_i = \frac{x_i - A}{h}$, then $\sigma_x = |h|\sigma_d$

Now, $-2x_i - 3 = \dfrac{x_i + \frac{3}{2}}{-\frac{1}{2}}$

Here, $h = -\frac{1}{2}$

$\therefore \quad \sigma_d = \frac{1}{|h|}\sigma_x = 2 \times 3.5 = 7$

12. The new observations are obtained by adding 20 to each previous observation.

Hence, the standard deviation of new observations will be same *i.e.*, $\sqrt{10}$.

13. Given, mean $\bar{x} = \frac{\Sigma x_i}{n} \Rightarrow \Sigma x_i = 100 \times 50 = 5000$

$\therefore$ Corrected mean $= \frac{5000 - 100 + 110}{100} = 50.10$

But median remain same *i.e.*, 52.

14. Here, $N = \sum_{i=1}^{n} f_i = k({}^nC_0 + {}^nC_1 + \dots + {}^nC_n)$

$= k(1+1)^n = k2^n$

where, k is a constant of proportionality.

and $\Sigma f_i x_i = k(1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \dots + n \cdot {}^nC_n)$

$= kn\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \dots + 1\right] = kn2^{n-1}$

$\therefore$ Mean, $\bar{x} = \frac{1}{2^n}(n \cdot 2^{n-1}) = \frac{n}{2}$

15.

Class interval	Mid value x_i	f	cf	$\lvert x - M_d \rvert$	$f\lvert x - M_d\rvert$
0-6	3	4	4	11	44
6-12	9	5	9	5	25
12-18	15	3	12	1	3
18-24	21	6	18	7	42
24-30	27	2	20	13	26
Total		20			140

Now, $\frac{N}{2} = \frac{20}{2} = 10$, which lies in the interval 12-18.

$l = 12, cf = 9, f = 3$

$\therefore \quad M_d = 12 + \frac{10 - 9}{3} \times 6 \quad \left(\because M_d = l + \frac{\frac{N}{2} - cf}{f} \times h\right)$

$= 12 + 2 = 14$

$\therefore$ Mean deviation $= \frac{\Sigma f_i |x_i - M_d|}{N} = \frac{140}{20} = 7$

16. Given, $w_i = lx_i + k$, $M(x_i) = \bar{x} = 48$, $\sigma(x_i) = 12$, $M(w_i) = 55$ and $\sigma(w_i) = 15$

$\because \quad M(w_i) = lM(x_i) + M(k)$

$\Rightarrow \quad 55 = l \times 48 + k$

$\Rightarrow \quad \sigma(w_i) = l\sigma(x_i) + \sigma(k)$

$\Rightarrow \quad 15 = l(12) + 0$

$\Rightarrow \quad l = \frac{15}{12} = 1.25$

From Eq. (i), $\quad 55 = 1.25 \times 48 + k \Rightarrow k = 55 - 60$

$\therefore \quad k = -5$

17. Given, $\sigma = 9$

Let a student obtains x marks out of 75. Then, his marks out of 100 are $\frac{4x}{3}$. Each observation is multiply by $\frac{4}{3}$.

$\therefore \quad$ New SD, $\sigma = \frac{4}{3} \times 9 = 12$

Hence, variance is $\sigma^2 = 144$.

18. Given that, $x_1 < x_2 < x_3 < \dots < x_{201}$

Hence, median of the given observation $= \left(\frac{201+1}{2}\right)$th item $= x_{101}$

Now, deviation will be minimum of taken from the median.

Hence, mean deviation will be minimum, if $k = x_{101}$.

19. Given, coefficient of variation, $C_1 = 50$

and coefficient of variation, $\quad C_2 = 60$

We have, $\quad \bar{x}_1 = 30$ and $\bar{x}_2 = 25$

$\because \quad C = \frac{\sigma}{\bar{x}} \times 100 \Rightarrow 50 = \frac{\sigma_1}{30} \times 100$

$\Rightarrow \quad \sigma_1 = 15$ and $60 = \frac{\sigma_2}{25} \times 100$

$\Rightarrow \quad \sigma_2 = 15$

$\therefore \quad$ Required difference, $\sigma_1 - \sigma_2 = 15 - 15 = 0$

20. Let $\quad d_i = x_i - 8$

$\therefore \quad \sigma_x^2 = \sigma_d^2 = \frac{1}{18}\Sigma d_i^2 - \left(\frac{1}{8}\Sigma d_i\right)^2$

$= \frac{1}{18} \times 45 - \left(\frac{9}{18}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$

$\Rightarrow \quad \sigma_x^2 = \frac{3}{2}$

21. $\therefore \quad$ New mean, $\bar{x} = \frac{100 \times 40 + 3 + 27 - 30 - 70}{100}$

$= \frac{4000 - 70}{100} = \frac{3930}{100} = 39.3$

$\because \quad \Sigma x^2 = N(\sigma^2 + \bar{x}^2)$

$\therefore \quad$ SD $= 100(100 + 600) = 170000$

New $\quad \Sigma x^2 = 170000 - (30)^2 - (70)^2 + (3)^2 + (27)^2$

$= 170000 - 900 - 4900 + 9 + 729$

$= 164938$

$\therefore$ New SD $= \sqrt{\frac{\text{New } \Sigma x^2}{N} - (\text{New } \bar{x})^2} = \sqrt{\frac{164938}{100} - (39.3)^2}$

$= \sqrt{1649.38 - 1544.49} = \sqrt{104.89} = 10.24$

22. Firstly arrange the data in ascending order *i.e.*, 7, 10, 12, 15, 17, 19, 25.

$\therefore \quad Q_1 = \text{Size of } \left(\frac{n+1}{4}\right)\text{th item}$

$= \text{Size of 2nd item} = 10$

$Q_3 = \text{Size of } \left[\frac{3(n+1)}{4}\text{th}\right] \text{item}$

$= \text{Size of 6th item} = 19$

$\therefore \text{Quartile deviation} = \frac{Q_3 - Q_1}{2} = \frac{19-10}{2} = 4.5$

23. Let the two observations be x_1, and x_2.

Then, $\frac{1+2+6+x_1+x_2}{5} = 4$

$\Rightarrow \quad x_1 + x_2 = 11 \quad \ldots(i)$

and $\frac{\left[(1-4)^2 + (2-4)^2 + (6-4)^2 + (x_1-4)^2 + (x_2-4)^2\right]}{5} = 5.2$

$\Rightarrow \quad (x_1-4)^2 + (x_2-4)^2 = 26 - 9 - 4 - 4 = 9 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$x_1 = 4$ and $x_2 = 7$

Solutions (Q. Nos. 24 to 26)

x	f	cf
0-8	5	5
8-16	10	15
16-24	13	28
24-32	25	53
32-40	35	88
40-48	19	107
48-56	13	120

Here, $\frac{N}{2} = \frac{120}{2} = 60$

So, the median class is 32 - 40.

Here, $l = 32, f = 35, c = 53, h = 8$

24. $\therefore \quad \text{Median} = l + \frac{\frac{N}{2} - c}{f} \times h = 32 + \frac{60-53}{35} \times 8$

$= 32 + \frac{7 \times 8}{35} = 32 + 1.6 = 33.6$

25. The maximum frequency is in the interval 32-40, which is a modal class, for this $l = 32, f = 35, f_1 = 25, f_2 = 19, h = 8$

$\therefore \quad \text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} \times h = 32 + \frac{35-25}{70-25-19} \times 8$

$= 32 + \frac{10 \times 8}{26} = 32 + 3.07 = 35.07$

26. $\because \quad 2\,\text{Mean} = 3\,\text{Median} - \text{Mode}$

$= 3(33.6) - 35.07 = 100.8 - 35.07 = 65.73$

$\therefore \quad \text{Mean} = 32.865$

27. Since, $\text{Deviation} = \Sigma(x_i - A) \Rightarrow 2 = \Sigma x_i - 20 \times 30$

$\Rightarrow \quad \Sigma x_i = 602$

$\therefore \quad \text{Mean} = \frac{602}{20} = 30.1$

28. Required mean $= \frac{\Sigma n^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}$

Hence, statement II is also a true statement.

29. Statement II is true and Statement I is false.

30. First we arrange four numbers according to the condition $0 < y < x < 2y$ *i.e.*, $x - y, y, x, 2x + y$

$\text{Median} = \frac{\text{2 nd term} + \text{3 rd term}}{2} = 10$

$\Rightarrow \quad y + x = 20 \quad \ldots(i)$

$\text{Range} = (2x + y) - (x - y) = 28$

$\Rightarrow \quad x + 2y = 28 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get, $x = 12$, $y = 8$

So, for numbers are, 4, 8, 12, 32.

$\therefore \quad \text{Mean} = \frac{4+8+12+32}{4} = \frac{56}{4} = 14$

31. Mean, $7 = \frac{6+7+8+10+x}{5} \Rightarrow x = 4$

$\text{Variance} = \frac{(6-7)^2 + (7-7)^2 + (8-7)^2 + (10-7)^2 + (4-7)^2}{5}$

$= \frac{1^2 + 0 + 1^2 + 3^2 + 3^2}{5} = \frac{20}{5} = 4$

32. Given; $\frac{\Sigma x_{20}}{20} = 40 \Rightarrow \Sigma x_{20} = 20 \times 40 = 800$

$\therefore \quad \Sigma x_{20} = 800 - 33 + 53 = 820 \Rightarrow \frac{\Sigma x_{20}}{20} = \frac{820}{20}$

$\therefore \text{New mean} = 41$

33. $\text{Mean} = \frac{\underbrace{(a + a + \ldots + n \text{ times}) + (-a - a - \ldots - n \text{ times})}}{2n} = 0$

$\therefore \quad \text{SD} = \sqrt{\frac{(x_i - \bar{x})^2}{2n}} = \sqrt{\frac{a^2 + a^2 + \ldots + 2n \text{ times}}{2n}}$

$2 = \sqrt{\frac{(a)^2\, 2n}{2n}} = |a|$

34. If initially all marks were x_i, then $\sigma_1{}^2 = \frac{\Sigma (x_i - \bar{x})^2}{N}$

Now, each is increased by 10.

$\therefore \quad \sigma_2{}^2 = \frac{\Sigma (x_i + 10) - (\bar{x} + 10)]^2}{N} = \sigma_1{}^2$

So, variance will not change whereas mean, median and mode will increase by 10.

35. Statement I $\text{AM} = \frac{2x_1 + 2x_2 + 2x_3 + \ldots + 2x_n}{n}$

$= 2\left(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\right) = 2\bar{x}$

Statement II Hence, it is obvious that Statement II is false.

36. Median of $a, 2a, 3a, 4a, \ldots, 50a$ is $\frac{25a+26a}{2} = (25.5)a$

Mean deviation about median $= \frac{\sum_{i=1}^{50}|x_i - \text{median}|}{n}$

$\Rightarrow \quad 50 = \frac{1}{50}\{2|a|\cdot(0.5+1.5+2.5+\ldots+24.5)\}$

$\Rightarrow \quad 2500 = 2|a|\cdot\frac{25}{2}(2\times 0.5 + 24\times 1) = 2|a|\cdot\frac{25}{2}(25)$

$\therefore \quad |a| = 4$

37. Here $\sigma_x^2 = 4, \sigma_y^2 = 5, \bar{x} = 2$ and $\bar{y} = 4$

Now, $\frac{\Sigma x_i}{5} = 2 \Rightarrow \Sigma x_i = 10; \Sigma y_i = 20$ and $\sigma_x^2 = \left(\frac{1}{5}\Sigma x_i^2\right) - (\bar{x})^2$

$\Rightarrow \Sigma x_i^2 = 5(4+4) = 40$ and $\sigma_y^2 = \left(\frac{1}{5}\Sigma y_i^2\right) - (\bar{y})^2$

and $\Sigma y_i^2 = 5(5+16) \quad \Sigma y_i^2 = 105$

Combined variance, $\sigma_z^2 = \frac{1}{10}(\Sigma x_i^2 + \Sigma y_{y_i}^2) - \left(\frac{\bar{x}+\bar{y}}{2}\right)^2$

$= \frac{1}{10}(40+105) - 9 = \frac{145-90}{10} = \frac{55}{10} = \frac{11}{2}$

38. $(\bar{x}) = \frac{\text{Sum of quantities}}{n} = \frac{\frac{n}{2}(a+l)}{n}$

$= \frac{1}{2}(1+1+100d) = 1+50d$

$\therefore \quad MD = \frac{1}{n}\Sigma|x_i - \bar{x}|$

$\Rightarrow 255 = \frac{1}{101}(50d + 49d + 48d + \ldots + d + 0 + d + \ldots + 50d)$

$= \frac{2d}{101}\left(\frac{50\times 51}{2}\right)$

$\therefore \quad d = \frac{255\times 101}{50\times 51} = 10.1$

39. Statement II It is true.

Statement I Sum of n even natural numbers $= n(n+1)$

Mean $(\bar{x}) = \frac{n(n+1)}{n} = n+1$

Variance $= \left[\frac{1}{n}\Sigma(x_i)^2\right] - (\bar{x})^2 = \frac{1}{n}[2^2 + 4^2 + \ldots + (2n)^2] - (n+1)^2$

$= \frac{1}{n}2^2[1^2 + 2^2 + \ldots + n^2] - (n+1)^2$

$= \frac{4}{n}\frac{n(n+1)(2n+1)}{6} - (n+1)^2$

$= \frac{(n+1)[2(2n+1) - 3(n+1)]}{3} = \frac{(n+1)(n-1)}{3} = \frac{n^2-1}{3}$

Hence, Statement I is false.

40. According to the given condition,

$6.80 = \frac{\left[\begin{array}{c}(6-a)^2 + (6-b)^2 + (6-8)^2 \\ + (6-5)^2 + (6-10)^2\end{array}\right]}{5}$

$\Rightarrow \quad 34 = (6-a)^2 + (6-b)^2 + 4 + 1 + 16$

$\Rightarrow \quad (6-a)^2 + (6-b)^2 = 13 = 9 + 4$

$\Rightarrow \quad (6-a)^2 + (6-b)^2 = 3^2 + 2^2$

$\therefore \quad a = 3, b = 4$

41. Let the number of boys and girls be x and y.

$\therefore \quad 52x + 42y = 50(x+y)$

$\Rightarrow \quad 52x + 42y = 50x + 50y$

$\Rightarrow \quad 2x = 8y$

$\Rightarrow \quad x = 4y$

$\therefore$ Total number of students in the class $= x + y = 4y + y = 5y$

$\therefore$ Required percentage of boys $= \left(\frac{4y}{5y}\times 100\right)\% = 80\%$

42. Since, variance is independent of change of origin. Therefore, variance of observations 101, 102, ..., 200 is same as variance of 151, 152, ..., 250

$\therefore \quad V_A = V_B \Rightarrow \frac{V_A}{V_B} = 1$

43. Given that, $\Sigma x_i^2 = 400$ and $\Sigma x_i = 80$, since $\sigma^2 \geq 0$

$\Rightarrow \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 \geq 0 \Rightarrow \frac{400}{n} - \frac{6400}{n^2} \geq 0$

$\therefore \quad n \geq 16$

44. Given that, mean = 21 and median = 22

Using the relation, Mode = 3 Median − 2 Mean

$\therefore$ Mode $= 3(22) - 2(21) = 66 - 42 = 24$

45. It is true that mode can be computed from histogram and median is not independent of change of scale but variance is independent of change of origin and not of scale.

46. In the $2n$ observations, half of them equal to a and remaining half equal to $-a$. Then, the mean of total $2n$ observations is equal to zero.

$\therefore \quad SD = \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}$

$\Rightarrow \quad 2 = \sqrt{\frac{\Sigma x^2}{2n}} \Rightarrow 4 = \frac{\Sigma x^2}{2n} = \frac{2na^2}{2n}$

$\Rightarrow \quad a^2 = 4$

$\therefore \quad |a| = 2$

47. Given, $n = 15$, $\Sigma x^2 = 2830$ and $\Sigma x = 170$

Since, one observation 20 was replaced by 30, then

$\Sigma x^2 = 2830 - 400 + 900 = 3330$

and $\quad \Sigma x = 170 - 20 + 30 = 180$

$\therefore$ Variance, $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{3330}{15} - \left(\frac{180}{15}\right)^2$

$= \frac{3330 - 2160}{15} = 78.0$

Day 35

Probability

Day 35

Outlines ...

- Sample Space and Event
- Types of Events
- Probability of an Event
- Independent Events
- Baye's Theorem
- Probability Distribution

Sample Space *and* Event

The **sample space** is the set of all possible outcomes of an experiment (tossing a coin, rolling a die, drawing a card from a pack of playing cards).

An event is a subset of S. If a die is rolled, then $S = \{1, 2, 3, 4, 5, 6\}$ is the sample space and getting an odd number $A = \{1, 3, 5\}$ is an event. The set of all events is the power set of S and elements in a power set is 2^n, where n is the number of elements in a set S.

Types *of* Events

The different types of events considered in probability theory are described below

1. **Equally Likely Event**

   The given events are said to be **equally likely**, if none of them is expected to occur in preference to the other.

2. **Mutually Exclusive/Disjoint**

   A set of events is said to be **mutually exclusive**, if occurrence of one of them prevents or denies the occurrence of any of the remaining events.

   If a set of events $E_1, E_2, \ldots, E_n$ for mutually exclusive events, then

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = \phi.$$

3. **Exhaustive Events**

A set of events is said to be **exhaustive**, if the performance of the experiment always results in the occurence of atleast one of them.

4. **Complementary Events**

In a random experiment, let S be the sample space and E be an event. If $E \subseteq S$, the $E^c \subseteq S$ is called the complement of E.

Probability *of* an Event

Probability is a measure or estimation of how likely it is that something will happen or that a statement is true. The higher degree of probability, the more likely the event is to happen or in a longer series of samples the greater the number of times such event is expected to happen.

If the sample space has n points (all possible cases) and an event A has m points (all favourable cases), then the probability of A is $P(A) = \frac{m}{n}$.

(i) Probability of Odds in favour of $A = \frac{P(A)}{P(\bar{A})} = \frac{m}{n-m}$ (ii) Probability of Odds in against $A = \frac{P(\bar{A})}{P(A)} = \frac{n-m}{m}$

Probability of getting a sure event is 1 and probability of getting an impossible event is 0.

- *If a set of events $E_1, E_2, \ldots, E_n$, then for exhaustive events $P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1$.*
- *If an event has more than one sample point, it is called a compound event.*
  e. g., Getting atleast one head in the experiment of tossing two coins simultaneously.

Addition Theorem of Probability

The addition theorem in the probability concept is the process of determination of the probability that either event A or event B occurs or both occur. The notation between two events A and B the addition is denoted as '$\cup$'.

1. **Addition Theorem for Two Events**

If A and B are two events associated with a random experiment,
then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ which is equal to the atleast one of the event is occur.

2. **Addition Theorem for Three Events**

If A, B and C are three events associated with a random experiment, then
$P(A \cup B \cup C) = P(A) + P(B) + (C) - P(A \cap B)$
$-P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

(i) The addition theorem for n events $A_1, A_2, \ldots, A_n$ is

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

(ii) For any two events A and B,
$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$.

Booley's Inequalities

(i) $P(A \cap B) \ge P(A) + P(B) - 1$

(ii) $P(A \cap B \cap C) \ge P(A) + P(B) + P(C) - 2$

(iii) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

- $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$
- $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$
- $P(A) = P(A \cap B) + P(A \cap \bar{B})$
- $P(B) = P(B \cap A) + P(B \cap \bar{A})$
- P (exactly one of E_1, E_2 occurs)
  $= P(E_1 \cap E_2') + P(E_1' \cap E_2)$
  $= P(E_1) - P(E_1 \cap E_2) + P(E_2) - P(E_1 \cap E_2)$
  $= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$
- P (neither E_1 nor E_2)
  $= P(E_1' \cap E_2') = 1 - P(E_1 \cup E_2)$
- $P(E_1' \cup E_2') = 1 - P(E_1 \cap E_2)$

Conditional Probability

If A and B are two events associated with the sample space of a random experiment, then conditional probability of the event A given that B has occurred

i.e., $$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$

$$P(B)\neq 0$$

Properties of Conditional Probability

Let A and B be two events of a sample space S of an experiment, then

(i) $P\left(\frac{S}{A}\right)=P\left(\frac{A}{A}\right)=1$

(ii) $P\left(\frac{A'}{B}\right)=1-P\left(\frac{A}{B}\right)$,

where A' is complement of A.

Multiplication Theorem of Probability

If A and B are two events associated with a random experiment, then

$$P(A\cap B)=P(A)\cdot P\left(\frac{B}{A}\right), P(A)\neq 0$$

or $$P(A\cap B)=P(B)\cdot P\left(\frac{A}{B}\right), P(B)\neq 0$$

Extension of Multiplication Theorem

If $A_1, A_2, \ldots, A_n$ are n events associated with a random experiment, then

$$P(A_1\cap A_2\cap\ldots\cap A_n)$$
$$=P(A_1)P\left(\frac{A_2}{A_1}\right)P\left(\frac{A_3}{A_1\cap A_2}\right)\ldots P\left(\frac{A_n}{A_1\cap A_2\cap\ldots\cap A_{n-1}}\right)$$

Independent Events

Two events are said to be independent, if the occurrence of one does not depend upon the other. If $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1\cap E_2\cap E_3\cap\ldots\cap E_n)=P(E_1)\cdot P(E_2)\ldots P(E_n).$$

If E and F are independent events, then the pairs E and $\bar{F}$, $\bar{E}$ and F, $\bar{E}$ and $\bar{F}$ are also independent.

Properties of Independent Events

If A and B are two independent events, then

(i) A' and B' are also independent events.

(ii) A' and B' are also independent events.

(iii) A' and B' are also independent events.

Important Results

If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) probability that all letters are in right envelopes $=\frac{1}{n!}$

(ii) probability that all letters are not in right envelopes $=1-\frac{1}{n!}$

(iii) probability that no letter is in right envelopes

$$=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+(-1)^n\frac{1}{n!}$$

(iv) probability that exactly r letters are in right envelopes

$$=\frac{1}{r!}\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots+(-1)^{n-r}\frac{1}{(n-r)!}\right)$$

Baye's Theorem

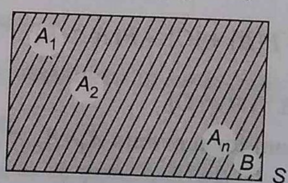
Let the sample space S be the union of n non-empty disjoint subsets (mutually exclusive and exhaustive events).

i.e., $$S=A_1\cup A_2\cup\ldots\cup A_n$$

and $$A_i\cap A_j=\phi, i\neq j$$

For any event B such that

$$B=(A_1\cap B)\cup(A_2\cap B)\cup\ldots\cup(A_n\cap B)$$



Then, $$P(A_i/B)=\frac{P(A_i\cap B)}{P(B)}=\frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{n}P(A_i)P(B/A_i)}$$

Probability Distribution

A random variable is a real valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by the capital letters X,Y,Z,... and so on.
The random variable may be of two types as given below.

1. Discrete Random Variable

A random variable which can take only finite or countably infinite number of values is called a discrete random variable.

2. Continuous Random Variable

A random variable which can take any value between two given limits is called a continuous random variable.

Probability Distribution of a Random Variable

If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

If a random variable X takes values, $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then

X	x_1	x_2	x_3	...	x_n
$P(X)$	p_1	p_2	p_3	...	p_n

is known as the probability distribution of X.

The probability distribution of random variable X is defined only when the various values of the random variable.

e.g., $x_1, x_2, x_3, \ldots, x_n$ together with respective probabilities $p_1, p_2, p_3, \ldots, p_n$ satisfies $\sum_{i=1}^{n} p_i = 1$

Mean

If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then the mean $\bar{X}$ of X is defined as

$$\bar{X} = p_1 x_1 + p_2 x_2 + \ldots + p_i x_i$$

or

$$\bar{X} = \sum_{i=1}^{n} p_1 x_1$$

The mean of a random variable X is also known as its mathematical expectation and it is denoted by E (X).

Variance

If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with the respective probabilities $p_1, p_2, \ldots, p_n$, then variance of X is defined as

$$\text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2.$$

Bernoulli Trial

In a random experiment, if there are any two events, "Success and Failure" and the sum of the probabilities of these two events is one, then any outcome of such experiment is known as a **Bernoulli trial**.

Binomial Distribution

The probability of r successes in n independent Bernoulli Trials is denoted by $P(X = r)$ and is given by

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where, p = Probability of success

and q = Probability of failure

and $p + q = 1$

(i) Mean $= np$

(ii) Variance $= npq$

(iii) Mean is always greater than variance.

Practice Zone DAY 35

1. If 6 objects are distributed at random among 6 persons, the probability that atleast one person does not get any object is

(a) $\frac{313}{324}$ (b) $\frac{315}{322}$ (c) $\frac{317}{324}$ (d) $\frac{319}{324}$

2. A bag contains 14 coins of which 3 are counter-fiet with heads on both sides. The rest are fair coins. One is selected at random from the bag and tossed. The probability of getting a head is

(a) $\frac{10}{21}$ (b) $\frac{10}{19}$ (c) $\frac{11}{21}$ (d) None of these

3. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3 is

(a) $\frac{4}{55}$ (b) $\frac{4}{35}$ (c) $\frac{4}{33}$ (d) $\frac{4}{1155}$

4. Consider two events A and B. If odds against A are as 2 : 1 and those in favour of $A \cup B$ are as 3 : 1, then

(a) $\frac{1}{2} \le P(B) \le \frac{3}{4}$ (b) $\frac{5}{12} \le P(B) \le \frac{3}{4}$ (c) $\frac{1}{4} \le P(B) \le \frac{3}{5}$ (d) None of these

5. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the sum of the numbers obtained is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered $2, 3, 4, \ldots, 12$ is picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is

(a) $\frac{192}{401}$ (b) $\frac{193}{401}$ (c) $\frac{193}{792}$ (d) $\frac{17}{75}$

6. If the letters of the word 'MATHEMATICS' are arranged arbitrarily, the probability that C comes before E, E before H, H before I and I before S, is

(a) $\frac{1}{75}$ (b) $\frac{1}{24}$ (c) $\frac{1}{120}$ (d) $\frac{1}{720}$

7. Nine-digit numbers are formed using $1, 2, 3, \ldots, 9$ without repetition. The probability that the number is divisible by 4 is

(a) $\frac{2}{7}$ (b) $\frac{3}{8}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$

8. A four digit numbers is formed with the digits from 1, 2, 3, 4, 5, without repetition. The probability that it is divisible by 3, is

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

9. A and B stand in ring along with 10 other persons. If the arrangement is at random, the probability that there are exactly 3 person between A and B, is

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{1}{12}$

10. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random one at a time with replacement. The events A, B and C are defined as the first bulb is defective, the second bulb is non-defective, the two bulbs are both defective or non-defective, respectively. Then,

(a) A, B and C are pairwise independent
(b) A, B and C are pairwise not independent
(c) A, B and C are independent
(d) None of the above

11. A person goes to office either by car, scooter, bus or train the probabilities of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$, respectively. The probability that he reaches office late if he takes car, scooter, bus or train is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$, respectively. If he reaches office in time, the probability that he travelled by car is

(a) $\frac{1}{5}$ (b) $\frac{1}{9}$ (c) $\frac{2}{11}$ (d) $\frac{1}{7}$

12. Two dice are tossed. The following two events A and B are

$$A = \{(x, y) : x + y = 11\}, \; B = \{(x, y) : x \ne 5\}$$

where, (x, y) denotes a typical sample point. **[NCERT Exemplar]**

(a) not independent (b) independent (c) mutually exclusive (d) None of these

13. A man throws a fair coin a number of times and gets 2 points for each head he throws and 1 point for each tail he throws. The probability that he gets exactly 6 points is

(a) $\frac{21}{32}$ (b) $\frac{23}{32}$ (c) $\frac{41}{64}$ (d) $\frac{43}{64}$

14. One function is selected from all the functions $F: S \to S$, where $S = \{1, 2, 3, 4, 5, 6\}$. The probability that it is onto function, is

(a) $\frac{5}{324}$ (b) $\frac{7}{324}$ (c) $\frac{5}{162}$ (d) $\frac{5}{81}$

15. In a hurdle race, a runner has probability p of jumping over a specific hurdle. Given that in 5 trials, the runner succeeded 3 times, the conditionals probability that the runner had succeeded in the first trial, is

(a) $\frac{3}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{5}$ (d) None of these

16. Two ball are drawn at random with replacement from a box containing 10 black and 8 red balls. The probability that

(i) both balls are red

(ii) first ball is black and second ball is red

(iii) one of them is black and other is red

are, respectively **[NCERT]**

(a) $\frac{16}{81}, \frac{20}{81}$ and $\frac{40}{81}$ (b) $\frac{40}{81}, \frac{20}{81}$ and $\frac{16}{81}$

(c) $\frac{20}{81}, \frac{16}{81}$ and $\frac{40}{81}$ (d) None of these

17. A group of 10 boys are randomly divided into two teams containing 5 boys each. The probability that the two tallest boys in different team is

(a) $\frac{28}{45}$ (b) $\frac{4}{9}$ (c) $\frac{4}{11}$ (d) $\frac{5}{11}$

18. If 3 numbers are selected from the first 15 natural numbers, then the probability that the numbers are in AP is

(a) $\frac{7}{65}$ (b) $\frac{9}{65}$ (c) $\frac{8}{65}$ (d) $\frac{6}{65}$

19. There are 5 balls numbered 1 to 5 and 5 boxes numbered 1 to 5. The balls are kept in the boxes one in each box. The probability that exactly 2 balls are kept in the corresponding numbered boxes and the remaining 3 balls in the wrong boxes, is

(a) $\frac{1}{5}$ (b) $\frac{1}{6}$ (c) $\frac{1}{10}$ (d) $\frac{1}{12}$

20. A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red, then the probability that the missing card is black, is

(a) $\frac{2}{3}$ (b) $\frac{15}{26}$ (c) $\frac{16}{39}$ (d) $\frac{37}{52}$

21. There are 2 teams with n persons in each. The probability of selecting 2 persons from one team and 1 person from the other team is 6/7, then n is equal to

(a) 3 (b) 4 (c) 5 (d) 6

22. A box contains 8 white and 4 black balls. The balls are drawn randomly one-by-one until those of the same colour left in the box. The probability that they are white, is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

23. Let $x = 33^n$. The index n is given a positive integral value at random. The probability that the value of x will have 3 in the unit's place, is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) None of these

24. If $x_1, x_2, \ldots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \ldots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle numbers, is

(a) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (b) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

(c) $\frac{{}^{19}C_2 \times {}^{31}C_2}{{}^{50}C_5}$ (d) None of these

25. A discrete random variable X has the following probability distribution

X	1	2	3	4	5	6	7
$P(X)$	C	$2C$	$2C$	$3C$	C^2	$2C^2$	$7C^2 + C$

The value of C and the mean of the distribution are **[NCERT Exemplar]**

(a) $\frac{1}{10}$ and 3.66 (b) $\frac{1}{20}$ and 2.66

(c) $\frac{1}{15}$ and 1.33 (d) None of these

26. A draws two cards at random from a pack of 52 cards. After returning them to the pack and shuffling it, B draws two cards at random. The probability that thier draws contain exactly one comman card is

(a) $\frac{25}{546}$ (b) $\frac{50}{663}$ (c) $\frac{25}{663}$ (d) None of these

27. The chances of defective screws in three boxes A, B and C are $\frac{1}{5}, \frac{1}{6}$ and $\frac{1}{7}$, respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, the probability that it come from box A is

(a) $\frac{16}{29}$ (b) $\frac{1}{15}$

(c) $\frac{27}{59}$ (d) $\frac{42}{107}$

28. A fair coin is tossed n times. If X is the number of times heads occur and $P(X=4), P(X=5)$ and $P(X=6)$ are in AP, then n is equal to

(a) 13 (b) 7

(c) 11 (d) None of these

29. The mean and variance of a binomial distribution are 4 and 3 respectively, then the probability of getting exactly six successes in this distribution is

(a) ${}^{16}C_6\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{6}$ (b) ${}^{16}C_6\left(\frac{1}{4}\right)^{6}\left(\frac{3}{4}\right)^{10}$

(c) ${}^{12}C_6\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{6}$ (d) None of these

30. In an examinations, 20 questions of true-false type are asked. Suppose a students tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers true, if it falls tails, he answers false. The probability that he answers atleast 12 questions correctly is **[NCERT]**

(a) $\left(\frac{1}{2}\right)^{20}({}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20})$

(b) $\left(\frac{1}{2}\right)^{10}({}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20})$

(c) $\left(\frac{1}{2}\right)^{20}({}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20})$

(d) None of the above

31. An urn contains 10 coupons marked 1, 2, 3, ..., 10. If two coupons are drawn, then the chance that the difference of the coupons exceeds 3 is equal to

(a) $\frac{8}{15}$ (b) $\frac{7}{15}$

(c) $\frac{1}{10}$ (d) $\frac{3}{21}$

32. Two independent events namely A and B and the probability that both A and B occurs is $\frac{1}{10}$ and the probability that neither of them occurs is $\frac{3}{10}$. Then, the probability of occurrence of event B is

(a) $\frac{4-\sqrt{7}}{3}$ (b) $\frac{4+\sqrt{7}}{3}$

(c) $\frac{4+\sqrt{6}}{2}$ (d) $\frac{4-\sqrt{6}}{10}$

33. A committee of 4 students is selected at random from a group consisting 8 boys and 4 girls.Given that there is atleast one girl on the committee, the probability that there are exactly 2 girls on the committee, is **[NCERT Exemplar]**

(a) $\frac{7}{99}$ (b) $\frac{13}{99}$

(c) $\frac{14}{99}$ (d) None of these

34. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{2}{5}$ (d) $\frac{1}{5}$

35. Two numbers are selected randomly from the set $S=\{1, 2, 3, 4, 5, 6\}$ without replacement one-by-one. The probability that minimum of the two numbers is less than 4, is

(a) $\frac{1}{15}$ (b) $\frac{14}{15}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$

Directions (Q. Nos. 36 and 37) *If a particular experiment be given n (a finite) independent trials.*
If the probability of success in one trial (say) p, we get
The probability of failure, $q=(1-p)$
The probability of r success in n trials $= {}^nC_r p^r q^{n-r}$

36. The probability of man hitting a target in one fire is 1/4. Atleast n times he must fire at the target that the chances of hitting the target atleast once will exceed 2/3, then n is

(a) 2 (b) 4

(c) 6 (d) 8

37. A coin tossed 4 times whose faces are marked with the numbers 5 and 3. The odds in favour of getting a sum less than 15 is

(a) $\frac{5}{16}$ (b) $\frac{5}{11}$ (c) $\frac{16}{5}$ (d) $\frac{11}{5}$

Directions (Q. Nos. 38 and 39) *There are fours boxes A_1, A_2, A_3 and A_4. Box A_i has i cards and on each card a number is printed, the numbers are from 1 to i. A box is selected randomly, the probability of selection of box A_i is $\frac{i}{10}$ and then a card is drawn. Let E_i represents the event that a with number i is drawn. Then,*

38. $P(E_1)$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{1}{10}$

(c) $\frac{2}{5}$ (d) $\frac{1}{4}$

39. $P\left(\frac{A_3}{E_2}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Directions (Q. Nos. 40 to 44) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

40. Consider a natural number x is chosen at random from the first 100 natural numbers.

Statment I The probability that $\frac{(x-10)(x-50)}{(x-30)} > 0$ is 0.69.

Statement II If A is an event, then $0 < P(A) < 1$.

41. Consider the relation $P(A) + P(\bar{A}) = 1$.

Statement I 20 persons are sitting in a row. These persons are selected at random. The probability that two selected persons are not together is 0.7.

Statement II If A is an event, then $P(\text{not } A) = 1 - P(A)$.

42. Statement I If 12 coins are thrown simultaneously, then probability of appearing exactly five head is equal to probability of appearing exactly 7 heads.

Statement II ${}^nC_r = {}^nC_s \Rightarrow$ either $r = s$ or $r + s = n$ and $P(H) = P(T)$ in a single trial.,

43. Consider A and B are two events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$.

Statement I $\frac{1}{6} \le P(A \cap B) \le \frac{1}{2}$.

Statement II $P(A \cup B) \le \max\{P(A), P(B)\}$

and $P(A \cap B) \ge \min\{P(A), P(B)\}$.

44. Consider A and B are two events such that $0 < P(A), P(B) < 1$.

Statement I $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = \frac{3}{2}$.

Statement II $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

and

$P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$.

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45. Let A, B and C try to hit a target simultaneously but independently. Their respective probabilities of hitting the targets are $\frac{3}{4}, \frac{1}{2}$ and $\frac{5}{8}$. The probability that the target is hit by A or B but not by C is **[JEE Main 2013]**

(a) $\frac{21}{64}$ (b) $\frac{7}{8}$
(c) $\frac{7}{32}$ (d) $\frac{9}{64}$

46. Let two independent events, if the probability that exactly one of them occurs is $\frac{26}{49}$ and the probability that none of them occurs is $\frac{15}{49}$, then the probability of more probable of the two events is **[JEE Main 2013]**

(a) $\frac{4}{7}$ (b) $\frac{6}{7}$
(c) $\frac{3}{7}$ (d) $\frac{5}{7}$

47. If two events A and B are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$ and $P(B) = \frac{1-x}{4}$, then the set of possible values of x lies in the interval **[JEE Main 2013]**

(a) $[0, 1]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$
(c) $\left[-\frac{1}{3}, \frac{5}{9}\right]$ (d) $\left[-\frac{7}{9}, \frac{4}{9}\right]$

48. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is **[JEE Main 2013]**

(a) $\frac{17}{3^5}$ (b) $\frac{13}{3^5}$ (c) $\frac{11}{3^5}$ (d) $\frac{10}{3^5}$

49. Three numbers are chosen at random without replacement from $\{1, 2, 3, \ldots, 8\}$. The probability that their minimum is 3, given that their maximum is 6, is **[AIEEE 2012]**

(a) $\frac{3}{8}$ (b) $\frac{1}{5}$ (c) $\frac{1}{4}$ (d) $\frac{2}{5}$

50. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of atleast one failure is greater than or equal to 31/32, then p lies in the interval **[AIEEE 2011]**

(a) $\left(\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$ (c) $\left(\frac{11}{12}, 1\right]$ (d) $\left(\frac{1}{2}, \frac{3}{4}\right]$

51. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours, is **[AIEEE 2010]**

(a) $\frac{1}{3}$ (b) $\frac{2}{7}$ (c) $\frac{1}{21}$ (d) $\frac{2}{23}$

52. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \ldots, 20\}$.

Statement I The probability that the chosen numbers when arranged in some order will form an AP, is $\frac{1}{85}$.

Statement II If the four chosen numbers from an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. **[AIEEE 2010]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

53. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than **[AIEEE 2009]**

(a) $\frac{1}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1}{\log_{10} 4 + \log_{10} 3}$
(c) $\frac{9}{\log_{10} 4 - \log_{10} 3}$ (d) $\frac{4}{\log_{10} 4 - \log_{10} 3}$

54. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals to **[AIEEE 2009]**

(a) $\frac{1}{14}$ (b) $\frac{1}{7}$ (c) $\frac{5}{14}$ (d) $\frac{1}{50}$

55. It is given that, the events A and B are such that $P(A) = \frac{1}{4}$, $P(A \mid B) = \frac{1}{2}$ and $P(B \mid A) = \frac{2}{3}$. Then, $P(B)$ is **[AIEEE 2008]**

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

56. A die is thrown. Let A be the event that the number obtained is greater than 3 and B be the event that the number obtained is less than 5. Then, $P(A \cup B)$ is **[AIEEE 2008]**

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) 0 (d) 1

57. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only, if the first misses the target. The probability that the target is hit by the second plane, is **[AIEEE 2007]**

(a) 0.06 (b) 0.14 (c) 0.32 (d) 0.7

58. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for complement of event A. Then, events A and B are **[AIEEE 2005]**

(a) mutually exclusive and independent
(b) independent but not equally likely
(c) equally likely but not independent
(d) equally likely and mutually exclusive

59. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house, is **[AIEEE 2005]**

(a) $\frac{7}{9}$ (b) $\frac{8}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$

60. A random variable X has the probability distribution

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X \text{ is a prime number}\}$ and $F = \{X < 4\}$, then the probability $P(E \cup F)$ is **[AIEEE 2004]**

(a) 0.87 (b) 0.77 (c) 0.35 (d) 0.50

61. Events A, B and C are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$, $P(B) = \frac{1-x}{4}$ and $P(C) = \frac{1-2x}{2}$. Then, the set of possible values of x are in the interval **[AIEEE 2003]**

(a) $\left[\frac{1}{3}, \frac{1}{2}\right]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$ (c) $\left[\frac{1}{3}, \frac{13}{3}\right]$ (d) $[0, 1]$

Answers

1. (d)	2. (a)	3. (d)	4. (b)	5. (c)	6. (c)	7. (d)	8. (c)	9. (b)	10. (a)
11. (d)	12. (a)	13. (d)	14. (a)	15. (a)	16. (a)	17. (a)	18. (a)	19. (b)	20. (a)
21. (b)	22. (c)	23. (a)	24. (b)	25. (a)	26. (b)	27. (d)	28. (b)	29. (b)	30. (a)
31. (b)	32. (d)	33. (d)	34. (c)	35. (d)	36. (b)	37. (b)	38. (c)	39. (b)	40. (b)
41. (d)	42. (a)	43. (c)	44. (d)	45. (a)	46. (a)	47. (c)	48. (c)	49. (b)	50. (b)
51. (b)	52. (c)	53. (a)	54. (a)	55. (c)	56. (d)	57. (c)	58. (b)	59. (b)	60. (b)
61. (a)									

Hints & Solutions

1. Number of ways of distributing 6 objects to 6 persons $= 6^6$

Number of ways of distributing 1 object to each persons $= 6!$

$\therefore$ Required probability $= 1 - \dfrac{6!}{6^6} = 1 - \dfrac{5!}{6^5} = \dfrac{319}{324}$

2. Suppose, event A is selecting a counterfiet coin and B be the event of getting head.

$\therefore$ Probability of getting a head $= P(A \cap B) + P(\overline{A} \cap B)$

$$= P(A)\,P(B/A) + P(\overline{A})\,P(B/\overline{A})$$

$$= \frac{3}{14} \times 1 + \frac{11}{14} \times \frac{1}{3} = \frac{3}{14} + \frac{11}{42}$$

$$= \frac{9+11}{42} = \frac{20}{42} = \frac{10}{21}$$

3. Here, $n(S) = {}^{100}C_3$

Let E = All three of them are divisible by both 2 and 3.

$\Rightarrow$ Divisible by 6 *i.e.*, $\{6, 12, 18, \ldots, 96\}$

Thus, out of 16 we have to select 3.

$\therefore \quad n(E) = {}^{16}C_3$

$\therefore$ Required probability $= \dfrac{{}^{16}C_3}{{}^{100}C_3} = \dfrac{4}{1155}$

4. Here, $P(A) = \dfrac{1}{3}$ and $P(A \cup B) = \dfrac{3}{4}$

$\therefore \ P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$

$\Rightarrow \quad \dfrac{3}{4} \le \dfrac{1}{3} + P(B) \ \Rightarrow \ P(B) \ge \dfrac{5}{12}$

Also, $B \subseteq A \cup B \Rightarrow P(B) \le P(A \cup B) = \dfrac{3}{4}$

$\therefore \quad \dfrac{5}{12} \le P(B) \le \dfrac{3}{4}$

5. The probability of getting the sum 7 or 8 from two dice is $\dfrac{6}{36} + \dfrac{5}{36} = \dfrac{11}{36}$.

The probability of getting the card with number 7 or 8 is $\dfrac{2}{11}$.

$\therefore$ Required probability $= \dfrac{1}{2} \cdot \dfrac{11}{36} + \dfrac{1}{2} \cdot \dfrac{2}{11} = \dfrac{11}{72} + \dfrac{2}{22} = \dfrac{193}{792}$

6. Total number of arrangement is $\dfrac{11!}{2!2!2!} = \dfrac{11!}{8}$

Number of arrangement in which C, E, H, I and S appear in that order $= ({}^{11}C_5)\dfrac{6!}{2!2!2!} = \dfrac{11!}{8.5!}$

$\therefore$ Required probability $= \dfrac{11!}{8.5!} \div \dfrac{11!}{8} = \dfrac{1}{5!} = \dfrac{1}{120}$

7. Any number is divisible by 4, if last two digits numbers are divisible by 4. Here, 16 number ending with two digits *i.e.*, 12, 16, 24, 28, 32, 36, ..., 96, which are divisible by 4.

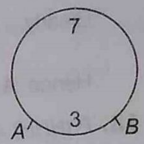
$\therefore$ Required probability $= \dfrac{16 \cdot 7!}{9!} = \dfrac{2}{9}$

8. Four digit numbers $= {}^5P_4 = 120$

It is divisible by 3 in 4! ways $= 24$

$\therefore$ Required probability $= \dfrac{24}{120} = \dfrac{1}{5}$

9. Number of ways of arranging 3 persons between A and $B = 2 \times ({}^{10}P_3) \times 7! = 2 \times 10!$



Total number of arrangement of 12 person is 11!.

$\therefore \quad$ Required probability $= \dfrac{2 \times 10!}{11!} = \dfrac{2}{11}$

10. Here, $P(A) = \dfrac{50}{100} = \dfrac{1}{2}, P(B) = \dfrac{50}{100} = \dfrac{1}{2}$

$$P(C) = P(DD \text{ or } NN) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(A \cap B) = P(DN) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A) \cdot P(B)$$

$$P(A \cap C) = P(DD) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A) \cdot P(C)$$

$$P(B \cap C) = P(NN) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(B) \cdot P(C)$$

So, A, B and C are pairwise independent.

$A \cap B \cap C$ is a null set since, if the first one defective, then both cannot be defective or both non-defective.

$$P(A \cap B \cap C) = P(\phi) = 0 \ne P(A) \cdot P(B) \cdot P(C)$$

Hence, A, B and C are not independent.

11. Let C, S, B and T be the events of the person going by car, scooter, bus and train, respectively.

$$P(C) = \frac{1}{7}, P(S) = \frac{3}{7}, P(B) = \frac{2}{7}, P(T) = \frac{1}{7}$$

Let L be the event that the person reaching the office in time.

Then, $\overline{L}$ be the event that the person reaching the office in late.

$$P(L/C) = \frac{7}{9}, P(L/S) = \frac{8}{9}, P(L/B) = \frac{5}{9}, P(L/T) = \frac{8}{9}$$

$$\therefore \quad P(C/L) = \frac{P(C) \cdot P(L/C)}{\left[\begin{array}{l} P(C) \cdot P(L/C) + P(S) \cdot P(L/S) \\ \quad + P(B) \cdot P(L/B) + P(T) \cdot P(L/T) \end{array}\right]}$$

$$= \frac{\frac{1}{7} \cdot \frac{7}{9}}{\frac{1}{7} \cdot \frac{7}{9} + \frac{3}{7} \cdot \frac{8}{9} + \frac{2}{7} \cdot \frac{5}{9} + \frac{1}{7} \cdot \frac{8}{9}} = \frac{7}{49} = \frac{1}{7}$$

12. Let S be the sample space having $6 \times 6 = 36$ elements.

Given, $\quad A = \{x, y : x + y = 11\}$

$\Rightarrow \quad A = \{(5, 6), (6, 5)\}$

So, $\quad P(A) = \dfrac{n(A)}{n(s)} = \dfrac{2}{36} = \dfrac{1}{18}$

and $\quad B = \{(x, y) : x \ne 5\}$

$\Rightarrow \quad B = S - \{(1, 5), (2, 5), (3, 5), (4, 5), (6, 5)\}$

$n(B) = 36 - 5 = 31$

$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{31}{36}$

and $\quad A \cap B = \{(x, y): x + y = 11, x \neq 5\} = \phi$

$\Rightarrow \quad n(A \cap B) = 0$

$\Rightarrow \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{0}{36} = 0$

Clearly, $P(A) \cdot P(B) = \frac{1}{18} \times \frac{31}{36} \neq P(A \cap B)$

Hence, A and B are not independent events.

13. $P(HHH) + P(HHTT) + P(HTTTT) + P(TTTTTT)$

$$= \frac{1}{2^3} + ({}^4C_2)\frac{1}{2^4} + ({}^5C_1)\frac{1}{2^5} + \frac{1}{2^6} = \frac{43}{64}$$

14. Total number of function is 6^6 and number of onto function = 6!

$\therefore$ Required probability $= \frac{6!}{6^6} = \frac{5}{324}$

15. Let A and B denote the event that the runner succeeds exactly 3 times out of 5 times and event that the runner succeeds on the first trial.

$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$

$\therefore \quad P(B \cap A) = p\ {}^4C_2 p^2 (1-p)^2 = 6p^3(1-p)^2$

$P(A) = {}^5C_3\ p^3(1-p)^2 = 10p^3\ (1-p)^2$

$\therefore \quad P\left(\frac{B}{A}\right) = \frac{6p^3(1-p)^2}{10p^3(1-p)^2} = \frac{3}{5}$

16. Total number of balls = 18, number of red ball = 8

and number of black balls = 10

$\therefore$ Probability of drawing a red ball $= \frac{\text{Number of red balls}}{\text{Total number of balls}} = \frac{8}{18}$

Similarly, probability of drawing a black ball

$= \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{10}{18}$

(i) Probability of getting both red balls = P (both balls are red)

= P (a red ball is drawn at first draw and again a red ball at second draw)

$= \frac{8}{18} \times \frac{8}{18} = \frac{16}{81}$

(ii) P (probability of getting first ball is black and second is red)

$= \frac{10}{18} \times \frac{8}{18} = \frac{20}{81}$

(iii) Probability of getting one black and other red ball

= P (first ball is black and second is red)

+ P (first ball is red and second is black)

$= \frac{10}{18} \times \frac{8}{18} + \frac{8}{18} \times \frac{10}{18} = \frac{40}{81}$

17. Number of ways of forming 2 team is $({}^{10}C_5)\frac{1}{2!}$. The number of ways of forming 2 team without the 2 tallest boys is $({}^8C_4)\frac{1}{2!}$.

$\therefore$ Required probability $= \left[2 \cdot ({}^8C_2)\frac{1}{2!}\right] \div \left[2 \cdot ({}^{10}C_2)\frac{1}{2!}\right]$

$= \frac{28}{45}$

18. Let $a < b < c$ be in AP.

$a + c = 2b$, even number.

Therefore, a and c are both even or both odd.

$\therefore$ Required probability $= \frac{{}^7C_2 + {}^8C_2}{{}^{15}C_3} = \frac{7}{65}$

19. The number of ways of choosing the 2 correct boxes is $({}^5C_2)$.

The number of ways of choosing 3 wrong boxes is $3!\left(-\frac{1}{3!} + \frac{1}{2!}\right)$.

$\therefore$ Required probability $= ({}^5C_2) \cdot 3!\left(-\frac{1}{3!} + \frac{1}{2!}\right) \div 5!$

$= \frac{10 \times 2}{120} = \frac{1}{6}$

20. Let B stand for the event that black card is missing, then $P(B) = P(\bar{B}) = \frac{1}{2}$.

Let E be the event that all the first 13 cards are red.

$\therefore \quad P(E/B) = \frac{26}{51} \cdot \frac{25}{50} \cdots \frac{14}{39}$,

$P(E/\bar{B}) = \frac{25}{51} \cdot \frac{24}{50} \cdots \frac{13}{39}$

$P(B/E) = \frac{P(B) \cdot P(E/B)}{P(B) \cdot P(E/B) + P(\bar{B}) \cdot P(E/\bar{B})}$

$= \frac{26 \cdot 25 \ldots 14}{26 \cdot 25 \ldots 14 + 25 \cdot 24 \ldots 13}$

$= \frac{26}{26 + 13} = \frac{26}{39} = \frac{2}{3}$

21. $\therefore$ Required Probability $= \frac{2({}^nC_2) \cdot ({}^nC_1)}{({}^{2n}C_3)} = \frac{6}{7}$

$\Rightarrow \quad \frac{3n}{2(2n-1)} = \frac{6}{7}$

$\therefore \quad n = 4$

22. After 11 draws a white ball is left in the box.

In the first 11 draws, we must get 7 white and 4 black balls.

$\therefore$ Required probability $\frac{({}^8C_7)({}^4C_4)}{({}^{12}C_{11})} = \frac{8}{12} = \frac{2}{3}$

23. Here, $n(S)$ = Number of ways $(33)^n$ can have a digit in the units's place.

$\Rightarrow \quad n(S) = 4 \quad (\because$ it is 3, 9, 7 or 1)

and $\quad n(E) = 1$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{1}{4}$

24. Here, $n(S) = {}^{50}C_5, n(E) = {}^{30}C_2 \times {}^{19}C_2$

$\therefore \quad P(E) = \frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

25. Since, $\Sigma p_i = 1$, we have

$C + 2C + 2C + 3C + C^2 + 2C^2 + 7C^2 + C = 1$

i.e., $\quad 10C^2 + 9C - 1 = 0$

i.e., $\quad (10C - 1)(C + 1) = 0$

$\Rightarrow \quad C = \frac{1}{10}, C = -1$

Therefore, the permissible value of $C = \frac{1}{10}$.

Mean $= \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{7} x_i p_i$

$= 1 \times \frac{1}{10} + 2 \times \frac{2}{10} + 3 \times \frac{2}{10} + 4 \times \frac{3}{10} + 5\left(\frac{1}{10}\right)^2 + 6 \times 2\left(\frac{1}{10}\right)^2 + 7\left[7\left(\frac{1}{10}\right)^2 + \frac{1}{10}\right]$

$= \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{49}{100} + \frac{7}{10}$

$= 3.66$

26. The probability of both drawing the comman cards x,

$P(X)$ = (Probability of A drawing the card x and any other card y) $\times$ (Probability of B drawing the card x and a card other than y)

$= \frac{{}^{51}C_1}{{}^{52}C_2} \times \frac{{}^{50}C_1}{{}^{52}C_2} \; \forall\, x$, where x has 52 values.

$\therefore$ Required probability $= \Sigma P(X)$

$= 52 \times \frac{51 \times 50 \times 4}{52 \times 51 \times 52 \times 51} = \frac{50}{663}$

27. Here, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$\therefore \quad P\left(\frac{A}{E_1}\right) = \frac{1}{5}, P\left(\frac{A}{E_2}\right) = \frac{1}{6}, P\left(\frac{A}{E_3}\right) = \frac{1}{7}$

$\therefore \quad P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{7}} = \frac{42}{107}$

28. Since, ${}^nC_4 \frac{1}{2^n}, {}^nC_5 \frac{1}{2^n}$ and ${}^nC_6 \frac{1}{2^n}$ are in AP.

Also, ${}^nC_4, {}^nC_5$ and ${}^nC_6$ are in AP.

$\therefore \quad 2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$

Dividing by ${}^nC_5$ both sides, we get

$2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$

$= \frac{5}{n-4} + \frac{n-5}{6} = \frac{n^2 - 9n + 50}{6(n-4)}$

$\Rightarrow \quad n^2 - 21n + 98 = 0 \quad \Rightarrow \quad n = 7, 14$

29. Here, $npq = 3$ and $np = 4$

$\therefore \quad \frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}, p = \frac{1}{4}$ and $n = 16$

$\therefore$ Probability of exactly six success $= {}^{16}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^{10}$

30. Let X denote the number of correct answer given by the student. The repeated tosses of a coin are Bernoulli trials. Since, head on a coin represent the true answer and tail represents the false answer, the correctly answered of the question are Bernoulli trials.

$\therefore \quad p = P$ (a success) $= P$ (coin show up a head) $= \frac{1}{2}$

and $\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

So, X has a binomial distribution with $n = 20$, $p = \frac{1}{2}$

and $\quad q = \frac{1}{2}$

$\therefore \quad P(X = r) = {}^{20}C_r \cdot \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{20-r}$

Hence, P (atleast 12 questions are answered as true)

$= P(X \geq 12) = P(12) + P(13) + P(14) + P(15) + P(16) + P(17) + P(18) + P(19) + P(20)$

$= {}^{20}C_{12}\, p^{12}q^8 + {}^{20}C_{13}\, p^{13}q^7 + {}^{20}C_{14}\, p^{14}q^6 + {}^{20}C_{15}\, p^{15}q^5 + {}^{20}C_{16}\, p^{16}q^4 + {}^{20}C_{17}\, p^{17}q^3 + {}^{20}C_{18}\, p^{18}q^2 + {}^{20}C_{19}\, p^{19}q^1 + {}^{20}C_{20}\, p^{20}$

$= ({}^{20}C_{12} + {}^{20}C_{13} + {}^{20}C_{14} + {}^{20}C_{15} + {}^{20}C_{16} + {}^{20}C_{17} + {}^{20}C_{18} + {}^{20}C_{19} + {}^{20}C_{20}) \cdot \frac{1}{2^{20}}$

$= \left(\frac{1}{2}\right)^{20} ({}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20})$

31. Selecting any two coupons there are several differences possible and their corresponding probability is $\frac{{}^{n-m}C_2}{{}^nC_2}$, where m can have any value less than n.

$\therefore \quad \frac{{}^{10-3}C_2}{{}^{10}C_2} = \frac{7!}{2!\,5!} \times \frac{2! \times 8!}{10!} = \frac{7}{15}$

32. $P(A \cap B)$ is equal to $\frac{1}{10}$ and $P(\overline{A \cup B}) = \frac{3}{10}$.

Then, $\quad P(A \cup B) = 1 - P(\overline{A \cup B}) = \frac{7}{10}$

$\because \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \quad P(A) + P(B) = P(A \cup B) + P(A \cap B)$

$\Rightarrow \quad P(A)+P(B)=\frac{7}{10}+\frac{1}{10}=\frac{4}{5}$ ...(i)

$\because \quad P(A)\cdot P(B)=\frac{1}{10}$

$\therefore \quad P(A)=\frac{1}{10P(B)}$

From Eq. (i),

$$P(B)+\frac{1}{10\{P(B)\}}=\frac{4}{5} \Rightarrow 10\{P(B)\}^2+1=8P(B)$$

Let $P(B)=t$, then $10t^2+1=8t \Rightarrow 10t^2-8t+1=0$

$$\therefore \quad t=\frac{8\pm\sqrt{64-4\times10\times1}}{2\times10}=\frac{8\pm2\sqrt{6}}{20}$$

So, $P(B)=\frac{4-\sqrt{6}}{10}$ is possible.

33. Let A denote the event that atleast one girl will be chosen and B the event the exactly 2 girls will be chosen. We require $P(B/A)$

Since, A denotes the event that atleast one girl will be chosen. and A denotes that no girl is chosen *i.e.*, 4 boys are chosen.

Then, $P(A')=\frac{^8C_4}{^{12}C_4}=\frac{70}{495}=\frac{14}{99}$ and $P(A)=1-\frac{14}{99}=\frac{85}{99}$

Now, $P(A\cap B)=P$ (2 boys and 2 girls)

$$=\frac{^8C_2\cdot{}^2C_2}{^{12}C_4}=\frac{6\times28}{495}=\frac{56}{165}$$

Thus, $P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{56}{165}\times\frac{99}{85}=\frac{168}{425}$

34. Let E = Event when each American man is seated adjacent to his wife and A = Event when Indian man is seated adjacent of his wife.

Now, $n(A\cap E)=(4!)\times(2!)^5$

Event when each American man is seated adjacent to his wife. Again, $n(E)=(5!)\times(2!)^4$

$$\therefore \quad P\left(\frac{A}{E}\right)=\frac{n(A\cap E)}{n(E)}=\frac{(4!)\times(2!)^5}{(5!)\times(2!)^4}=\frac{2}{5}$$

35. Here, two numbers are selected from {1, 2, 3, 4 ,5 ,6}.

$\Rightarrow n(S)=6\times5$ (as one-by-one without replacement)

Favourable events $\Rightarrow$ The minimum of the two numbers is less than 4.

$\therefore n(E)=6\times4$ (as the minimum of the two is less than 4)

We can select one from (1, 2, 3, 4) and other from (1, 2, 3, 4, 5, 6).

$\therefore$ Required probability $=\frac{n(E)}{n(S)}=\frac{24}{30}=\frac{4}{5}$

36. Here, $p=\frac{1}{4}, q=1-\frac{1}{4}=\frac{3}{4}$

Since, 1 − (probability of not hitting the target) > 2/3

$$\Rightarrow \quad 1-{}^nC_n\left(\frac{3}{4}\right)^n>\frac{2}{3} \text{ or } \left(\frac{1}{3}\right)>\left(\frac{3}{4}\right)^n$$

Hence, the minimum value of n is 4.

37. Here, $p=1/2, q=1/2$

For getting total sum less than 15 in 4 trails, we will throw coins to get

(i) $3+3+3+3<15$ (ii) $5+3+3+3<15$

$\therefore$ Probability of this event,

$$P(E)={}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1+{}^4C_4\left(\frac{1}{2}\right)^4=\frac{4}{16}+\frac{1}{16}=\frac{5}{16}$$

$$\therefore \text{ Odds (in favour)}=\frac{P(E)}{P(\bar{E})}=\frac{\frac{5}{16}}{1-\frac{5}{16}}=\frac{5}{11}$$

38. $P(E_1)=\frac{1}{10}\times1+\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}=\frac{2}{5}$

39. $P\left(\frac{A_3}{E_2}\right)=\frac{\frac{3}{10}\times\frac{1}{3}}{\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}}=\frac{1}{3}$

40. Since, $\frac{(x-10)(x-50)}{(x-30)}>0 \Rightarrow 10<x<30$ or $x>50$

$\therefore \quad x=11,\ldots,29$ or $x=51,52,\ldots,100$

$n(x)=69$

$\therefore$ Required probability $=\frac{69}{100}=0.69$

41. Since, the number of ways of selecting two persons out of 20 is $^{20}C_2=190$.

So, the number of ways in which two selected persons together is 19.

$\therefore$ Required probability $=1-\frac{19}{190}=0.9$

42. Here, $p=q=\frac{1}{2}$

$\therefore$ Probability of appearing exactly five heads

$$={}^{12}C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7$$

$$={}^{12}C_{12-5}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5$$

$$={}^{12}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5$$

Probability of appearing exactly seven heads.

43. We have, $P(A\cup B)\ge\max\{P(A),P(B)\}=\frac{2}{3}$

$\Rightarrow \quad P(A\cup B)\ge\frac{2}{3}$

$\Rightarrow P(A\cap B)=P(A)+P(B)-P(A\cup B)\ge P(A)+P(B)-1$

$$=\frac{1}{2}+\frac{2}{3}-1=\frac{1}{6}$$

$\Rightarrow \quad P(A\cap B)\ge\frac{1}{6}$ ...(i) and $P(A\cap B)\le \min\{P(A),P(B)\}=\frac{1}{2}$

$$P(A\cap B)\le \frac{1}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{6}\le P(A\cap B)\le \frac{1}{2}$$

44. I. $P\left(\frac{A}{\bar B}\right)+P\left(\frac{\bar A}{\bar B}\right)=\frac{P(A\cap \bar B)}{P(\bar B)}+\frac{P(\bar A\cap \bar B)}{P(\bar B)}$

$$=\frac{P(A\cap \bar B)+P(\bar A\cap \bar B)}{P(\bar B)}=\frac{P(\bar B)}{P(\bar B)}=1$$

II. $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$ (by definition)

$$P(\bar B)=\{(A\cup \bar A)\cap \bar B\}=P\{(A\cap \bar B)\cup(\bar A\cap \bar B)\}$$
$$=P(A\cap \bar B)+P(\bar A\cap \bar B)$$

45. $P(A\cup B\cap \bar C)=P(A\cup B)\times P(\bar C)$

$$=[P(A)+P(B)-P(A\cap B)]\times P(\bar C)$$
$$=\left(\frac{3}{4}+\frac{1}{2}-\frac{3}{8}\right)\times \frac{3}{8}$$
$$\left[\because P(A\cap B)=P(A)\cdot P(B)=\frac{3}{4}\cdot\frac{1}{2}=\frac{3}{8}\right]$$
$$=\frac{7}{8}\cdot\frac{3}{8}=\frac{21}{64}$$

46. $P(A\cap \bar B)+P(\bar A\cap B)=\frac{26}{49}$

$\Rightarrow$ $P(A)\,P(\bar B)+P(\bar A)P(B)=\frac{26}{49}$ ...(i)

Also, $P(\bar A\cap \bar B)=\frac{15}{49}$

$\therefore$ $P(\bar A)\,P(\bar B)=\frac{15}{49}$

47. $\because$ $0\le \frac{3x+1}{3}\le 1,\ 0\le \frac{1-x}{4}\le 1$ and $0\le \frac{3x+1}{3}+\frac{1-x}{4}\le 1$

$\Rightarrow$ $0\le 3x+1\le 3,\ 0\le 1-x\le 4,\ 0\le \frac{12x+4+3-3x}{12}\le 1$

$\Rightarrow$ $-1\le 3x\le 2,\ 0\le 1-x\le 4,\ 0\le 9x+7\le 12$

$\Rightarrow$ $-\frac{1}{3}\le x\le \frac{2}{3},\ -1\le -x\le 3,\ -7\le 9x\le 5$

$\Rightarrow$ $-\frac{1}{3}\le x\le \frac{2}{3},\ -3\le x\le 1,\ -\frac{7}{9}\le x\le \frac{5}{9}$

$\therefore$ $-\frac{1}{3}\le x\le \frac{5}{9}$

48. $\because$ Probability of guessing a correct answer, $p=\frac{1}{3}$

and probability of guessing a wrong answer, $q=\frac{2}{3}$

$\therefore$ The probability of guessing a 4 or more correct answer

$$={}^5C_4\left(\frac{1}{3}\right)^4\cdot\frac{2}{3}+{}^5C_5\left(\frac{1}{3}\right)^5=5\cdot\frac{2}{3^5}+\frac{1}{3^5}=\frac{11}{3^5}$$

49. Let A denotes the event that the minimum of the three selected numbers is 3 and B denotes the event that the maximum of the three selected numbers is 6.

If the maximum number is 6, then

(i) 6 should be one of the three selected numbers (1 way), and

(ii) The remaining two numbers should be less than 6 (*i.e.*, any 2 from 1 to 5) (${}^5C_2$ ways).

$\therefore$ $P(B)=\frac{1\times {}^5C_2}{{}^8C_3}=\frac{10}{{}^8C_3}$

Similarly, $P(A\cap B)=$ The probability that the minimum number is 3 and the maximum number is 6. If the minimum number is 3 and the maximum number 6 is, then

(i) 6 should be one of the 3 selected numbers (1 way).

(ii) 3 should be one of the three selected numbers (1 ways), and

(iii) The remaining 1 number should lie between 3 and 6 (*i.e.*, any one of 4 and 5) (2 ways).

$\therefore$ $P(A\cap B)=\frac{1\times 1\times {}^2C_1}{{}^8C_3}$

$\therefore$ $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{2}{10}=\frac{1}{5}$

50. We have, $n=5$ and $r\ge 1$

$\therefore$ $P(X=r)={}^nC_r p^{n-r}q^r$,

$\therefore$ $P(X\ge 1)=1-P(X=0)=1-{}^5C_0.p^5.q^0\ge \frac{31}{32}$

$\Rightarrow$ $p^5\le 1-\frac{31}{32}=\frac{1}{32}$

$\therefore$ $p\le \frac{1}{2}$ and $p\ge 0 \Rightarrow p\in\left[0,\frac{1}{2}\right]$

51. Total number of cases $={}^9C_3=84$

Number of favourable cases $={}^3C_1\cdot{}^4C_1\cdot{}^2C_1=24$

$\therefore$ $p=\frac{24}{84}=\frac{2}{7}$

52. $n(S)={}^{20}C_4$

Statement I

When common difference is 1; total number of cases = 17
When common difference is 2; total number of cases = 14
When common difference is 3; total number of cases = 11
When common difference is 4; total number of cases = 8
When common difference is 5; total number of cases = 5
When common difference is 6; total number of cases = 2
Hence, required probability

$$=\frac{17+14+11+8+5+2}{{}^{20}C_4}$$
$$=\frac{57}{4845}=\frac{1}{85}$$

53. According to the given condition, $1-\left(\frac{3}{4}\right)^n \ge \frac{9}{10}$

$\Rightarrow \quad \left(\frac{3}{4}\right)^n \le 1-\frac{9}{10}=\frac{1}{10} \Rightarrow \left(\frac{4}{3}\right)^n \ge 10$

$\Rightarrow n(\log_{10} 4-\log_{10} 3) \ge \log_{10} 10 = 1$

$\therefore \quad n \ge \frac{1}{\log_{10} 4-\log_{10} 3}$

54. $S=\{00, 01, 02, \ldots, 49\}$

Let A be the event that sum of the digits on the selected ticket is 8, then $A=\{08, 17, 26, 35, 44\}$

Let B be the event that the product of the digits is zero.

$B=\{00, 01, 02, 03, \ldots, 09, 10, 20, 30, 40\}$

$\therefore \quad A \cap B=\{08\}$

$\therefore$ Required probability $=P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{50}}{\frac{14}{50}}=\frac{1}{14}$

55. Given that, $\quad P(A)=\frac{1}{4}, P\left(\frac{A}{B}\right)=\frac{1}{2}$

and $\quad P\left(\frac{B}{A}\right)=\frac{2}{3}$

We know that, $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$ ...(i)

and $\quad P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}$ ...(ii)

$\therefore \quad P(B)=\frac{P\left(\frac{B}{A}\right)\cdot P(A)}{P\left(\frac{A}{B}\right)}=\frac{\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)}=\frac{1}{3}$

56. Since, $\quad A=\{4, 5, 6\}$ and $B=\{1, 2, 3, 4\}$

$\therefore \quad A\cap B=\{4\}$

$\therefore \quad P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{3}{6}+\frac{4}{6}-\frac{1}{6}=1$

57. Let the events be $\quad A =$ Ist aeroplane hit the target

$B =$ IInd aeroplane hit the target

and their corresponding probabilities are

$P(A)=0.3$ and $P(B)=02$

$\Rightarrow \quad P(\bar{A})=0.7$ and $P(\bar{B})=0.8$

$\therefore$ Required probability $=P(\bar{A})P(B)+P(\bar{A})P(\bar{B})P(\bar{A})P(B)+\ldots$

$=(0.7)(0.2)+(0.7)(0.8)(0.7)(0.2)$

$+(0.7)(0.8)(0.7)(0.8)(0.7)(0.2)+\ldots$

$=0.14\,[1+(0.56)+(0.56)^2+\ldots]$

$=014\left(\frac{1}{1-0.56}\right)=\frac{0.14}{0.44}=0.32$

58. Since, $\quad 1-P(A\cup B)=\frac{1}{6} \Rightarrow P(\bar{A})-P(B)+\frac{1}{4}=\frac{1}{6}$

$\Rightarrow \quad P(B)=\frac{1}{3}$ and $P(A)=\frac{3}{4}$

Now, $\quad P(A\cap B)=\frac{1}{4}=P(A)\,P(B)$

Hence, the events A and B are independent events but not equally likely.

59. All the three persons have three options to apply a house.

$\therefore$ Total number of cases $=3^3$

Now, favourable cases = 3 (an either all has applied for house 1 or 2 or 3)

$\therefore$ Required probability $=\frac{3}{3^3}=\frac{1}{9}$

60. Given, $\quad E=\{X$ is a prime number$\}=\{2, 3, 5, 7\}$

$\therefore \quad P(E)=P(X=2)+P(X=3)+P(X=5)+P(X=7)$

$\Rightarrow \quad P(E)=0.23+0.12+0.20+0.07=0.62$

and $\quad F=\{X<4\}=\{1, 2, 3\}$

$\Rightarrow \quad P(F)=P(X=1)+P(X=2)+P(X=3)$

$\Rightarrow \quad P(F)=0.15+0.23+0.12=0.5$

and $E\cap F=\{X$ is prime number as well as $<4\}=\{2, 3\}$

$P(E\cap F)=P(X=2)+P(X=3)$

$=0.23+0.12=0.35$

$\therefore$ Required probability, $P(E\cup F)=P(E)+P(F)-P(E\cap F)$

$=0.62+0.5-0.35=0.77$

61. Since, $0\le P(A)\le 1, 0\le P(B)\le 1, 0\le P(C)\le 1$

and $0\le P(A)+P(B)+P(C)\le 1$

$\therefore \quad 0\le \frac{3x+1}{3}\le 1 \Rightarrow -\frac{1}{3}\le x\le \frac{2}{3}$ ...(i)

$0\le \frac{1-x}{4}\le 1$

$\Rightarrow \quad -3\le x\le 1$ ...(ii)

$0\le \frac{1-2x}{2}\le 1$

$\Rightarrow \quad -\frac{1}{2}\le x\le \frac{1}{2}$ ...(iii)

and $\quad 0\le \frac{3x+1}{3}+\frac{1-x}{4}+\frac{1-2x}{2}\le 1$

$\Rightarrow \quad 0\le 13-3x\le 12$

$\Rightarrow \quad \frac{1}{3}\le x\le \frac{13}{3}$ ...(iv)

From Eqs. (i), (ii), (iii) and (iv), $\frac{1}{3}\le x\le \frac{1}{2}$

Day 36

Mathematical Reasoning

Day 36 Outlines ...

- Concept of Reasoning
- Sentence
- Statement (Proposition)
- Truth Value and Truth Table
- Logical Connectives or Sentencial Connectives
- Elementry Operations of Logic

Concept of Reasoning

Logic is the subject which deals with the principles of reasoning. Logic is sometimes defined as the Science of proof. Mathematics and other Science subjects deals with the reasoning and the arguments and every student of Mathematics and other Science should know the principles of logic.

Sentence

A sentence is a relatively independent grammatical unit. It can stand alone or it can be combined with other sentences to form a text, a story etc.

There are following types of statements

1. Declarative/Assertive Sentence

A declarative sentence 'declares' or states a fact, arrangement or opinion. Declarative sentence can be either positive or negative.

e.g., Gopal Pandey is an intelligent student.

2. Imperative Sentence

The imperative commands (sometimes requests). The imperative takes no subject as 'you' is the implied subject. The imperative form ends with a period (.) or an exclamation point (!).

e.g., Please give me a cold drink.

3. Exclamatory Sentence

In this type of sentence, there is a sudden burst of feeling. The feelings may be full of joys, sorrows etc.

e.g., Hurrah ! we win hockey match.

4. Interrogative Sentence

A sentence that asks a question is an interrogative sentence.

The interrogative form ends with a question mark (?).

e.g., Do you think that you win the prize today?

5. Optative Sentence

A sentence that expresses a wish is an optative sentence.

e.g., Wish you best of luck!

Statement (Proposition)

*We convey our daily views in the form of sentence which is a collection of words. This collection of words is called **statements**, if it has some sense. Therefore, "A declarative sentence, whose truth or falsity can be decided is called a statement of logical sentence but the sentence should not be imperative, interrogative and exclamatory."*

Statements are denoted by $p, q, r, \ldots$ etc.

e.g., "Delhi is the capital of India" is a statement, while "Do your work", is not a statement.

The working nature of statement in logic is same as nature of switch in circuit.

i.e., Switch ⟨ ON (1) / OFF (0) and Statement ⟨ True (T) / False (F)

There are following types of statements

1. Simple Statement

A statement, if cannot be broken into two or more sentences, is a simple statement. The truth value of the simple statement does not explicitly depend on any other statement.

Generally, small letters $p, q, r, \ldots$ denote simple statements.

2. Compound Statement

A statement formed by two or more simple statement by the words such as "and", "or", "not", "if then", "if and only if", then the resulting statement is called a compound statement.

This is also called logical connectives.

Two circuits can be connected either by series or parallel. i.e., two statements can be connected either by '$\wedge$' or '$\vee$'.

3. Substatements

Simple statements which when combined form a compound statement are called substatements, also called components.

- *A true statement is known as a valid statement.*
- *A false statement is known as an invalid statement.*
- *Imperative, exclamatory, interrogative, optative sentences are not statements.*
- *Mathematical identities are considered to be statements because they can either be true or false but not both.*

4. Open Statement

A sentence which contains one or more variables such that when certain values are given to the variable it becomes a statement, is called an open statement.

e.g., "He is a great man" is an open sentence because in this sentence "He" can be replaced by any person.

Truth Value and Truth Table

A statement can be either 'true' or 'false' which are called true values of a statement and it is represented by the symbols 'T' and 'F', respectively.

A table that shows the relationship between the truth value of compound statement, $S(p, q, r, \ldots)$ and the truth values of its substatements $p, q, r, \ldots$ etc., is called the truth table of statement S.

- For a single statement p, number of rows $= 2^1 = 2$

p
T
F

- For two statements p and q, number of rows $= 2^2 = 4$

p	q
T	T
T	F
F	T
F	F

- For three statements p, q and r.

  Number of rows $= 2^3 = 8$

p	q	r
T	F	F
T	F	T
T	T	F
T	T	T
F	F	F
F	F	T
F	T	F
F	T	T

If a compound statement has simply n substatements, then there are 2^n rows representing logical possibilities.

Logical Connectives or Sentencial Connectives

Two or more statements are combined to form a compound statement by using symbols. These symbols are called logical connectives.

Logical connectives are given below

Words	Symbols
and	$\wedge$
or	$\vee$
implies that (if ..., then)	$\Rightarrow$
If and only if (implies and is implied by)	$\Leftrightarrow$

Elementary Operations of Logic

Formation of compound sentences from simple sentence using logical connectives are termed as elementary operation of logic. There are five such operations discussed below.

1. Negation (Inversion) of Statement

A statement which is formed by changing the truth value of a given statement by using word like 'no' or 'not' is called negation of a given statement. It is representalley the symbol '$\sim$'.

e.g., Let p : The number 2 is greater than 7.

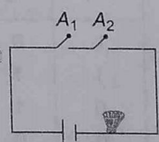
Then, $\sim p$: The number 2 is not greater than 7.

If p is statement, then negation of p is denoted by '$\sim p$'. The truth table for NOT is given by

p	$\sim p$
T	F
F	T

2. Conjunction

A compound sentence formed by two simple sentences p and q using connective "and" is called the conjunction of p and q and is represented by $p \wedge q$.



e.g., Let p : Ramesh is a student.

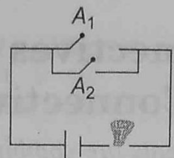
and q : Ramesh belongs to Allahabad. Then,

e.g., $p \wedge q \equiv$ Ramesh is a student and he belongs to Allahabad.

The truth table for operation 'and' is given by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Above conjunction can be explained through circuit.



Here, denote 1 ≈ T, 0 ≈ F

A_1	A_2	Result (Bulb)
1	0	0
1	1	1
0	1	0
0	0	0

⟶ $p \vee q$

p	q	$p \wedge q$
T	F	F
T	T	T
F	T	F
F	F	F

- *The statement $p \wedge q$ is true, if both p and q are true.*
- *The statement $p \wedge q$ is false, if atleast one of p and q or both are false.*

3. Disjunction (Alternation)

A compound sentence formed by two simple sentences p and q using connective "or" is called the disjunction of p and q and is represented by $p \vee q$.

e.g., Let p : Bus left early and q : My watch is going slow.
Then, $p \vee q$ = Bus left early or my watch is going slow.

The truth table for 'operation' OR is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Above disjunction can be explained through circuit.

Here, denote 1 ≈ T, 0 ≈ F

A_1	A_2	Result (Bulb)
1	0	1
1	1	1
0	1	1
0	0	0

⟶

p	q	$p \vee q$
T	F	T
T	T	T
F	T	T
F	F	F

- *The statement $p \vee q$ is true, if atleast one of p and q or both are true.*
- *The statement $p \wedge q$ is false, if both p and q are false.*

4. Implication (Conditional)

A compound sentence formed by two simple sentences p and q using connective "if ... then ..." is called the implication of p and q and represented by $p \Rightarrow q$ which is read as "p implies q". Here, p is called antecedent or hypothesis and q is called consequent or conclusion. *e.g.*, Let p : Train reaches in time.

and q = I can attend the meeting. Then, $p \Rightarrow q \equiv$ If train reaches in time, then I can attend the meeting.

The truth table for if ... then is given by

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It is clear from the truth table that Column III is equal to Column V. *i.e.*, statement $p \Rightarrow q$ is equivalent to $\sim p \vee q$.

5. Biconditional Statement

Two simple sentences connected by the phrase "if and only if," form a biconditional statement. It is represented by the symbol '$\Leftrightarrow$'.

e.g., Let p : ΔABC is an isosceles triangle

and q : Two sides of a triangle are equal.

Then, $p \Leftrightarrow q$: ΔABC is an isosceles triangle if and only if two sides of a triangle are equal.

The truth table for if and only if is given by

p	q	$p \Leftrightarrow q$	$\sim p$	$\sim q$	$\sim p \vee q$	$p \vee \sim q$	$(\sim p \vee q) \wedge (p \vee \sim q)$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T

- *It is clear form the truth table that Column III is equal to column VIII. i.e., statement $p \Leftrightarrow q$ is equivalent to $(\sim p \sim \vee q) \wedge (p \vee \sim q)$.*
- *The statement $p \Leftrightarrow q$ has true, if either both are true or both are false.*
- *The statement $p \Leftrightarrow q$ has false, if exactly one of them is false.*

Table for Basic Logical Connections

p	q	$\sim p$	$p \wedge q$	$q \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Converse, Inverse and Contrapositive of an Implication

If p and q are two statements, then here is a quick definition of their reverse, converse, inverse and contrapositive.
Reverse, Statement : if then, Converse : if then ,
Inverse : if not then not, Contrapositive : if not , then not

- The reverse of $p \Rightarrow q$ is $q \Leftarrow p$
- The converse of $p \Rightarrow q$ is $q \Rightarrow p$
- The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
- The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

By definition, the reverse of an implication means the same as the original implication itself. Each implication implies its contropositive, even intaitionstically. In classical logic an implication is logically equivalent to its contra positive and moreover, its inverse is logically equivalent to its converse. *e.g.*, If $2+2=4$, then Jawahar Lal Nehru is the first Prime Minister of India.

Let $p \equiv 2+2=4$

and $q \equiv$ Jawahar Lal Nehru is the first Prime Minister of India. Bicondintial statement $p \Rightarrow q$

(i) The converse of $p \Rightarrow q$ is $q \Rightarrow p$
i.e., if Jawahar Lal Nehru is the first Prime Minister of India, then $2+2=4$

(ii) The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
i.e., if Jawahar Lal Nehru is not the first Prime Minister of India, then $2+2=4$

(iii) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$ *i.e.*, if $2+2 \neq 4$, then Jawahar Lal Nehru is not the first Prime Minister of India.

- $\sim(p \Rightarrow q) \equiv \sim(\sim p \vee q) = \{p \vee (\sim q)\}$
  $\therefore \sim(p \Leftrightarrow q) \equiv \{p \wedge (\sim q)\}$
- $\sim(p \Rightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$
- $p \Rightarrow q = \sim p \vee q$
- $(p \Leftrightarrow q) \Leftrightarrow r = p \Leftrightarrow (q \Leftrightarrow r)$
- $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow q)$

Tautology

A compound statement is called a tautology, if it has truth value T whatever may be the truth value of its compounds. *e.g.*, Statement $(p \Rightarrow q) \wedge p \Rightarrow q$ is a tautology.

The truth table is prepared as follows.

p	q	$p \Rightarrow q$	$p \Rightarrow q \wedge p$	$(p \Rightarrow q) \wedge p \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Contradiction (Fallacy)

A compound statement is called contradiction, if its truth value is F whatever may be the truth value of its components. *e.g.*, Statement $\sim p \wedge p$ is a contradiction.

The truth table is prepared as follows

p	$\sim p$	$\sim p \wedge p$
T	F	F
F	T	F

A statement which is neither a tautology nor a contradiction is a contigency.

Algebra of Statement

Some of the important laws considered under the category of algebra of statement are given as.

1. **Idempotent Laws**
   For any statement p, we have
   (a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$
2. **Commutative Laws**
   For any two statements p and q, we have
   (a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$
3. **Associative Laws**
   For any three statements p, q and r, we have
   (a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
4. **Distributive Laws**
   (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. **Involution Laws**
   For any statement p, we have $\sim(\sim p) \equiv p$
6. **De-morgan's Laws**
   (a) $\sim(p \wedge q) = \sim p \vee \sim q$ (b) $\sim(p \vee q) = \sim p \wedge \sim q$
7. **Complement Laws**
   For any statement p, we have
   (a) $p \vee \sim p \equiv T$ (b) $p \wedge \sim p \equiv F$
   (c) $\sim T \equiv F$ (d) $\sim F \equiv T$
8. **Identity Laws**
   For any statement p, we have
   (a) $p \wedge T \equiv p$ (b) $p \vee F \equiv p$
   (c) $p \vee T \equiv T$ (d) $p \wedge F \equiv F$
   where, T and F are the true and false statement.
9. **Duality**
   Two compound statements S_1 and S_2 are said to be duals of each other, if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other.
   Symbolically, it can be written as , if $S(p,q) = p \wedge q$, then its dual is $S^*(p,q) = p \vee q$.

Practice Zone DAY 36

1. If p and q are simple propositions, then $p \Leftrightarrow \sim q$ is true when

(a) p is true and q is true (b) both p and q are false
(c) p is false and q is true (d) None of the above

2. The contrapositive and converse of the statement "I go to beach whenever it is a sunny day" is **[NCERT]**

(a) (i) If it is not a sunny day, then I do not go to beach.
(ii) If it is a sunny day, then I go to beach
(b) (i) If it is a sunny day, then I do not go to beach
(ii) If it is a sunny day, then I go to beach
(c) (i) If it is not a sunny day, then I go to beach.
(ii) If it is not a sunny day, then I go to beach
(d) None of the above

3. Which of the following is true?

(a) $p \Rightarrow q \equiv \sim p \Rightarrow \sim q$
(b) $\sim(p \Rightarrow \sim q) \equiv \sim p \wedge q$
(c) $\sim(\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$
(d) $\sim(p \Rightarrow q) \equiv [\sim(p \Rightarrow q)] \wedge \sim(q \Rightarrow p)$

4. $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to

(a) $\sim p$ (b) p
(c) q (d) $\sim q$

5. The inverse of the proposition $(p \wedge \sim q) \Rightarrow r$ is

(a) $\sim r \Rightarrow \sim p \vee q$ (b) $\sim p \vee q \Rightarrow \sim r$
(c) $r \Rightarrow p \wedge \sim q$ (d) None of these

6. An equivalent expression for $(p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s)$ which contains neither the biconditional nor the conditional is

(a) $(\sim p \vee q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s))$
(b) $(\sim p \wedge q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s))$
(c) $(\sim p \vee q \wedge r) \wedge ((\sim r \vee s) \vee (r \vee \sim s))$
(d) None of the above

7. Which of the following is the converse of the statement; 'If $x > 4$, then $x + 2 > 5$'?

(a) If $x + 2 \le 5$, then $x < 4$
(b) If x not greater than 4, then $x + 2$ is not greater than 5
(c) If $x + 2 > 5$, then $x > 4$
(d) If $x + 2$ is not greater than 5, then x is not greater than 4

8. If p : a natural number n is odd and q : natural number n is not divisible by 2, then the biconditional statement $p \Leftrightarrow q$ is **[NCERT]**

(a) A natural number n is odd if and only if it is divisible by 2
(b) A natural number n is odd if and only if it is not divisible by 2
(c) If a natural number n is odd, then it is not divisible by 2
(d) None of the above

9. The logically equivalent proposition of $p \Leftrightarrow q$ is

(a) $(p \wedge q) \vee (p \wedge q)$ (b) $(p \Rightarrow q) \wedge (q \Rightarrow p)$
(c) $(p \wedge q) \vee (q \Rightarrow p)$ (d) $(p \wedge q) \Rightarrow (p \vee q)$

10. If both p and q are false, then

(a) $p \wedge q$ is true (b) $p \vee q$ is true
(c) $p \Rightarrow q$ is true (d) $p \Leftrightarrow q$ is false

11. If $p \Rightarrow (\sim p \vee q)$ is false, the truth value of p and q are respectively.

(a) F, T (b) F, F
(c) T, F (d) T, T

12. The contrapositive of the inverse of $p \Rightarrow \sim q$ is

(a) $\sim q \Rightarrow p$ (b) $p \Rightarrow q$
(c) $\sim q \Rightarrow \sim p$ (d) $\sim p \Rightarrow \sim q$

13. If p : 4 is an even prime number, q : 6 is divisor of 12 and r : the HCF of 4 and 6 is 2, then which one of the following is true?

(a) $(p \wedge q)$ (b) $(p \vee q) \wedge \sim r$
(c) $\sim(q \wedge r) \vee p$ (d) $\sim p \vee (q \wedge r)$

14. If p = He is intelligent and q = He is strong.

Then, symbolic form of statement :

'It is wrong that he is intelligent or strong,' is

(a) $\sim p \vee \sim p$ (b) $\sim(p \wedge q)$
(c) $\sim(p \vee q)$ (d) $p \vee \sim q$

15. A compound sentence formed by two simple statements p and q using connective 'and' is called

(a) conjunction (b) disjunction
(c) implication (d) None of these

16. If $p = \Delta ABC$ is an equilateral and q = each angle is 60°. Then, symbolic form of statement

"ΔABC is an equilateral, iff each of its angle is of 60°," is

(a) $p \vee q$ (b) $p \wedge q$
(c) $p \Rightarrow q$ (d) $p \Leftrightarrow q$

17. The equivalent form of the conditional statement $p \Rightarrow q$ is

(a) q is necessary for p (b) if q, then p
(c) q is sufficient for p (d) None of these

18. Let truth value of p be F and q be T. Then, truth value of $\sim(\sim p \vee q)$ is

(a) T (b) F
(c) either T or F (d) neither T nor F

19. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q and r are respectively

(a) F, T, T (b) T, T, F (c) T, F, F (d) F, F, F

20. Logically equivalent to $\sim(\sim p \Rightarrow q)$ is

(a) $p \wedge q$ (b) $p \wedge \sim q$ (c) $\sim p \wedge q$ (d) $\sim p \wedge \sim q$

21. $\sim S(p, q)$ is equivalent to

(a) $S^*(\sim p, \sim q)$ (b) $S^*(p, \sim q)$
(c) $S^*(\sim p, q)$ (d) None of these

22. The dual of the statement $(p \vee q) \vee r$ is

(a) $(p \wedge q) \vee r$ (b) $(p \wedge q) \wedge r$
(c) $(p \vee q) \wedge r$ (d) None of these

23. Negation of '4 + 3 = 7 and 10 < 12' is

(a) $4 + 3 \neq 7$ and < 10 (b) $4 + 3 = 7$ and $10 \nless 12$
(c) $4 + 3 \neq 7$ or $10 \nless 12$ (d) None of these

24. Negation of 'London is in England and Berut is in Lebnan' is

(a) London is in Lebnan and Berut is in England
(b) London is not in England or Berut is not in Lebnan
(c) Berut is in England or London is in Lebnan
(d) None of the above

25. Let p and q stand for the statements '$2 \times 4 = 8$' and '4 divides 7', respectively. Then, what are the truth values for following biconditional statements?

(i) $p \leftrightarrow q$ (ii) $\sim p \leftrightarrow q$
(iii) $\sim q \leftrightarrow p$ (iv) $\sim p \leftrightarrow \sim q$

(a) TTTT (b) FTTT (c) FTFF (d) FTTF

26. Find the contrapositive of "If two triangles are identical, then these are similar".

(a) If two triangles are not similar, then these are not identical
(b) If two triangles are not identical, then these are not similar
(c) If two triangles are not identical, then these are similar
(d) If two triangles are not similar, then these are identical

27. $\sim(p \Rightarrow q) \Leftrightarrow \sim p \vee \sim q$ is

(a) a tautology
(b) a contradiction
(c) Neither a tautology nor a contradiction
(d) Cannot come to any conclusion

28. $(p \wedge \sim q) \wedge (\sim p \vee q)$ is

(a) a contradiction (b) a tautology
(c) Either (a) or (b) (d) Neither (a) nor (b)

29. The proposition $p \Rightarrow \sim(p \wedge \sim q)$ is

(a) contradiction (b) a tautology
(c) Either (a) or (b) (d) Neither (a) nor (b)

30. Select the tautology from the compound statements.

(i) $p \vee \sim p$ (ii) $p \rightarrow p \vee q$
(iii) $(p \wedge q)$ (iv) $p \wedge \sim p$

(a) (i) and (ii) (b) (i) and (iii)
(c) (ii) and (iii) (d) None of these

31. Write down the contradiction or fallacy for above question.

(a) (i) and (iii) (b) (i) and (ii)
(c) (iv) (d) None of these

32. Which of the following is logically equivalent to $\sim(\sim p \Rightarrow q)$?

(a) $p \wedge q$ (b) $p \wedge \sim q$ (c) $\sim p \wedge q$ (d) $\sim p \wedge \sim q$

33. $\sim(p \Leftrightarrow q)$ is

(a) $\sim p \wedge \sim q$ (b) $\sim p \vee \sim q$
(c) $(p \wedge \sim q) \vee (\sim p \wedge q)$ (d) None of these

34. Which of the following is true for any two statements p and q?

(a) $\sim[p \vee (\sim q)] \equiv (\sim p) \wedge q$
(b) $(p \vee q) \vee (\sim q)$ is a tautology
(c) $(p \wedge q) \wedge (\sim q)$ is a contradiction
(d) $\sim[p \wedge (\sim p)]$ is a tautology

35. If $(p \wedge \sim r) \Rightarrow (\sim p \vee q)$ is false, then the truth values of p, q and r respectively

(a) T, F and F (b) F, F and T
(c) F, T and T (d) T, F and T

36. $\sim p \wedge q$ is logically equivalent to

(a) $p \Rightarrow q$ (b) $q \Rightarrow p$
(c) $\sim(p \Rightarrow q)$ (d) $\sim(q \Rightarrow p)$

37. $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is

(a) a tautology
(b) a contradiction
(c) tautology and contradiction
(d) Neither a tautology nor a contradiction

38. If p and q are two statements, then $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a

(a) contradiction (b) tautology
(c) Neither (a) nor (b) (d) None of these

39. Which of the following is not correct?

(a) $\sim(p \wedge q) = (\sim p) \vee (\sim q)$
(b) Truth value of $p \wedge q$ = truth value of $q \wedge p$
(c) $\sim(\sim p) = p$
(d) $p \Leftrightarrow q \equiv (p \Rightarrow q) \vee (q \Rightarrow p)$

Directions (Q. Nos. 40 and 41) *Let us define a truth table*

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

40. The statement $\sim(\sim p \vee \sim q)$ is a

(a) tautology (b) contradiction
(c) contingency (d) None of these

41. If $\sim(p \wedge q)$ is false, then corresponding values of p and q are, respectively

(a) T, T (b) T, F
(c) F, T (d) None of these

Directions (Q. Nos. 42 and 43) *Let us define statements. p : Ramesh is tall, q : Ramesh is handsome. Solve the following problems.*

42. The symbolic form of the statement "Being tall is sufficient condition to be handsome" is

(a) $p \Rightarrow q$ (b) $q \Rightarrow p$
(c) $p \Leftrightarrow q$ (d) None of these

43. The symbolic form of the statement "Ramesh is tall and handsome" is

(a) $(p \vee \sim p) \vee q$
(b) $p \wedge q$
(c) $p \vee (p \wedge \sim q)$
(d) $\sim(\sim p \wedge q)$

Directions (Q. Nos. 44 to 52) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

44. If p = '11 is an integer', q = 'apple is sour', then

Statement I The symbolic form of 'if apple is not sour, then 11 is an odd integer' is $\sim q \Rightarrow p$.
Statement II It is a biconditional statement.

45. Statement I 'Ram is rich or happy' is a statement.
Statement II It is disjunction of the pair of statements.

46. Consider the statement $p \vee (\sim q) \wedge (\sim p)$, then

Statement I Its dual is $p \wedge (\sim q) \vee (\sim p)$
Statement II In a dual statement we interchange their operators $\wedge$ and $\vee$.

47. Consider the statement $[p \wedge (p \to q)] \to q$

Statement I It is tautology.
Statement II If all truth values of a statement is true, then the statement is a tautology.

48. Let p : ice is cold and q : blood is green be two statements, then

Statement I $p \vee q$ ice is cold or blood is green.
Statement II $p \wedge q$ ice is not cold or blood is green.

49. Let p and q be two statements.

Statement I $p \wedge \sim p$ is a contradiction.
Statement II $p \to q = \sim p \vee q$

50. If $p \to q$ be any conditional statement.

Statement I The converse of $p \to q$ is the statement $q \to p$.
Statement II The inverse of $p \to q$ is the statement $\sim q \to \sim p$.

51. If p, q and r be any three statements.

Statement I The statement $p \to (q \to r)$ is a tautology.
Statement II $(p \wedge q) \to r$ and $p \to (q \to r)$ are identical.

52. Consider the statements $(p \vee q) \wedge \sim p$ and $\sim p \wedge q$

Statement I Both are logically equivalent.
Statement II The end columns of the truth table of both statements are identical.

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53. The statement $p \to (q \to p)$ is equivalent to **[JEE Main 2013]**

(a) $p \to q$
(b) $p \to (p \vee q)$
(c) $p \to (p \to q)$
(d) $p \to (p \wedge q)$

54. For integers m and n, both greater than 1, consider the following three statements **[JEE Main 2013]**

P : m divides n

Q : m divides n^2

R : m is prime, then

(a) $Q \wedge R \to P$ (b) $P \wedge Q \to R$
(c) $Q \to R$ (d) $Q \to P$

55. Let p and q be any two logical statements and $r : p \to (\sim p \vee q)$. If r has a truth value F, then the truth values of p and q are respectively

(a) F, F (b) T, T
(c) T, F (d) F, T

56. Statement I The statement $A \to (B \to A)$ is equivalent to $A \to (A \vee B)$.

Statement II The statement $\sim\{(A \wedge B) \to (\sim A \vee B)\}$ is tautology. **[JEE Main 2013]**

(a) Statement I is true, Statement II is false.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(d) Statement I is false, Statement II is true.

57. Consider

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.

Statement II $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology. **[JEE Main 2013]**

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

58. The negation of the statement "If I become a teacher, then I will open a school", is **[AIEEE 2012]**

(a) I will become a teacher and I will not open a school
(b) Either I will not become a teacher or I will not open a school
(c) Neither I will become a teacher nor I will open a school
(d) I will not become a teacher or I will open a school

59. The only statement among the followings that is a tautology is **[AIEEE 2011]**

(a) $B \to [A \wedge (A \to B)]$ (b) $A \wedge (A \vee B)$
(c) $A \vee (A \wedge B)$ (d) $[A \wedge (A \to B)] \to B$

60. Consider the following statements

P : Suman is brilliant.
Q : Suman is rich.
R : Suman is honest.

The negative of the statement 'Suman is brilliant and dishonest if and only if Suman is rich.' can be expressed as **[AIEEE 2011]**

(a) $\sim Q \leftrightarrow (P \wedge \sim R)$ (b) $\sim Q \leftrightarrow P \wedge R$
(c) $\sim (P \wedge \sim R) \leftrightarrow Q$ (d) $\sim P \wedge (Q \leftrightarrow \sim R)$

61. Let S be a non-empty subset of R. Consider the following statement

P : There is a rational number $x \in S$ such that $x > 0$.

Which of the following statements is the negation of the statement P? **[AIEEE 2010]**

(a) There is a rational number $x \in S$ such that $x \le 0$
(b) There is no rational number $x \in S$ such that $x \le 0$
(c) Every rational number $x \in S$ satisfies $x \le 0$
(d) $x \in S$ and $x \le 0 \Rightarrow x$ is not rational

Directions (Q. Nos. 62 and 63) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

62. Statement I $\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Statement II $\sim (p \leftrightarrow \sim q)$ is a tautology. **[AIEEE 2009]**

63. Let p be the statement 'x is an irrational number', q be the statement 'y is a transcendental number' and r be the statement 'x is a rational number iff y is a transcendental number'. **[AIEEE 2008]**

Statement I r is equivalent to either q or p.

Statement II r is equivalent to $\sim (p \leftrightarrow \sim p)$.

64. The statement $p \to (q \to p)$ is equivalent to **[AIEEE 2008]**

(a) $p \to (p \leftrightarrow q)$ (b) $p \to (p \to q)$
(c) $p \to (p \vee q)$ (d) $p \to (p \wedge q)$

Answers

1. (c)	**2.** (a)	**3.** (c)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (c)
11. (c)	**12.** (a)	**13.** (d)	**14.** (c)	**15.** (a)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (c)	**20.** (c)
21. (a)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (d)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (d)	**30.** (a)
31. (c)	**32.** (d)	**33.** (c)	**34.** (a)	**35.** (a)	**36.** (d)	**37.** (b)	**38.** (b)	**39.** (d)	**40.** (c)
41. (a)	**42.** (b)	**43.** (b)	**44.** (c)	**45.** (b)	**46.** (a)	**47.** (a)	**48.** (c)	**49.** (b)	**50.** (c)
51. (d)	**52.** (b)	**53.** (b)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (b)	**58.** (a)	**59.** (d)	**60.** (a)
61. (c)	**62.** (c)	**63.** (*)	**64.** (c)						

Hints & Solutions

1. $p \Leftrightarrow \sim q$ is true iff $p, \sim q$ are both true or both false. ($\because q$ true $\Rightarrow \sim q$ false, so $p \sim q$ are both false)

2. **Contrapositive statement** If it is not a sunny day, then I do not go to beach.

Converse statement If it is a sunny day, then I go to beach.

3. $\sim(p \Rightarrow q) \equiv p \wedge \sim q$

$\therefore \quad \sim(\sim p \Rightarrow \sim q) \equiv \sim p \wedge \sim(\sim q) \equiv \sim p \wedge q$

Thus, $\quad \sim(\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$

4. $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$\equiv \sim p \wedge (\sim q \vee q) \equiv \sim p \wedge t \equiv \sim p$

5. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

$\therefore$ Inverse of $(p \wedge \sim q) \Rightarrow r$ is

$\sim(p \wedge \sim q) \Rightarrow \sim r$

i.e., $\quad (\sim p \vee q) \Rightarrow \sim r$

6. $(p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s)$

$\equiv (p \Rightarrow q \wedge r) \vee [(\sim r \vee s) \wedge (r \vee \sim s)]$

$\equiv (\sim p \vee q \wedge r) \vee [(\sim r \vee s) \wedge (r \vee \sim s)]$

$[\because p \Rightarrow q \wedge r \equiv \sim p \vee (q \wedge r)]$

7. The converse of the given statement is 'if $x + 2 > 5$, then $x > 4$'.

8. Given, p : A natural number n is odd and q : natural number n is not divisible by 2. The biconditional statement $p \Leftrightarrow q$ *i.e.*, "A natural number n is odd if and only if it is not divisible by 2".

9. $(p \Rightarrow q) \wedge (q \Rightarrow p)$ means $p \Leftrightarrow q$

Hence, option (b) is correct.

10. If both p and q are false, then $p \Rightarrow q$ is true.

11. $p \Rightarrow (\sim p \vee q)$ is false means p is true and $\sim p \vee q$ is false.

$\Rightarrow p$ is true and both $\sim p$ and q are false.

Hence, p is true and q is false.

12. The inverse of $p \Rightarrow \sim q$ is $\sim p \Rightarrow q$

The contrapositive of $\sim p \Rightarrow q$ is $\sim q \Rightarrow p$.

($\because$ contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$)

13. Given that, p : 4 is an even prime number.

q : 6 is a divisor of 12 and r : the HCF of 4 and 6 is 2.

So, the truth value of the statements.

p, q and r are F, T and T, respectively.

Hence, $\sim p \vee (q \wedge r)$ is true.

14. The symbolic meaning of the given statement is $\sim(p \vee q)$.

15. A compound sentence formed by two simple statements p and q using connective 'and' is called conjunction.

16. The symbolic form of given statement is $p \Leftrightarrow q$.

17. The conditional statement $p \Rightarrow q$ reflects that whenever it is known that p is true, it will have to follow that q is also true. So, q is necessary for p.

18. The truth value of $\sim(\sim p \vee q)$ is F.

19. Proposition $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q and r are respectively T, F, F.

20. $\because \sim(p \Rightarrow q) = p \vee \sim q$

$\therefore \quad \sim(\sim p \Rightarrow q) = \sim p \wedge \sim(\sim q) = \sim p \wedge q$

21. $\because \quad \sim S\ (p, q) = \sim(p \wedge q) = (\sim p) \vee (\sim q) = S^*(\sim p, \sim q)$

22. According to definition let $s(p, q) = p \wedge q$ be a compound statement. Then, $s^*(p, q) = p \vee q$,

Therefore, the dual of $(p \vee q) \vee r$ is $(p \wedge q) \wedge r$.

23. Let $\quad p : 4 + 3 = 7$

$q : 10 < 12$

The proposition is : $p \wedge q$.

Its negation is $\sim(p \wedge q) = \sim p \vee \sim q$

We have, $4 + 3 \neq 7$ or $10 \nless 12$

24. Let p : London is in England.

q : Berut is in Lebnan.

We have, $p \wedge q$

Its negation is $\sim(p \wedge q) = \sim p \vee \sim q$

i.e., London is not in England or Berut is not in Lebnan.

25. Since, p is true and q is false $\Rightarrow np$ is false and $\sim q$ is true.

$p \leftrightarrow q$ is F (since, p is true, q is false)

$\sim p \leftrightarrow q$ is T (since, $\sim p$ is false, q is false)

$\sim q \leftrightarrow p$ is T (since, $\sim q$ is true, p is true)

$\sim p \leftrightarrow \sim q$ is F (since, $\sim p$ is false, $\sim q$ is true)

26. Consider the following statements

p : Two triangles are identical.

q : Two triangles are similar.

Clearly, the given statement in symbolic form is $p \Rightarrow q$.

So, its contrapositive is given by $\sim q \Rightarrow \sim p$.

Now, $\sim p$: Two triangles are not identical

$\sim q$: Two triangles are not similar

So, $\sim q \Rightarrow \sim p$: If two triangles are not similar, then these are not identical.

27.

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \Rightarrow q) \Leftrightarrow \sim p \vee \sim q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	F
F	F	T	F	T	T	T	F

Last column shows that result is neither a tartology nor a contradiction.

28.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Clearly, $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction *i.e.*, option (a) is correct.

29.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \Rightarrow \sim(p \wedge \sim q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Result is neither tautology nor contradiction.

30. It is true for all possible truth values

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

31.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

So, statement $(p \wedge \sim p)$ is contradiction.

32. It is clear from table that $\sim(\sim p \Rightarrow q)$ is equivalent to $\sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$	$\sim p \Rightarrow q$	$\sim(\sim p \Rightarrow q)$
T	T	F	F	T	F	F	F	T	F
T	F	F	T	F	T	F	F	T	F
F	T	T	F	F	F	T	F	T	F
F	F	T	T	F	F	F	T	F	T

33. It is clear from the table that $\sim(p \Leftrightarrow q)$ is equivalent to $(p \wedge \sim q) \vee (\sim p \wedge q)$.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim p \vee \sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$p \Leftrightarrow q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$	$\sim(p \Leftrightarrow q)$
T	T	F	F	F	F	F	F	T	F	F
T	F	F	T	F	T	T	F	F	T	T
F	T	T	F	F	T	F	T	F	T	T
F	F	T	T	T	T	F	F	T	F	F

34.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	$\sim p \wedge q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

$\therefore \sim[p \vee (\sim q)] \equiv (\sim p) \wedge q$.

35. Truth table

p	q	r	$\sim p$	$\sim r$	$p \wedge \sim r$	$\sim p \vee q$	$(p \wedge \sim r) \Rightarrow (\sim p \vee q)$
T	T	T	F	F	F	T	T
T	T	F	F	T	T	T	T
T	F	T	F	F	F	F	T
T	F	F	F	T	T	F	F
F	T	T	T	F	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	F	F	T	T
F	F	F	T	T	F	T	T

Since, $(p \wedge \sim r) \Rightarrow (\sim p \vee q)$ is F. Then, $p = T, q = F, r = F$

So, P, Q and R are true statements.

$\therefore Q \wedge R = T \wedge T \rightarrow T = P$

36.

p	q	$\sim p$	$\sim p \wedge q$	$p \Rightarrow q$	$q \Rightarrow p$	$\sim(p \Rightarrow q)$	$\sim(q \Rightarrow p)$
T	T	F	F	T	T	F	F
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	F	T	F	T	T	F	F

It is clear from the above table that columns 4 and 8 are equal.
Hence, $\sim p \wedge q$ is equivalent to $\sim(q \Rightarrow p)$

37.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

As we see from the last column of truth table all entries are F. Hence, given proposition is a contradiction.

38.

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Hence, given proposition is a tautology.

39.

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$q \wedge p$	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \vee (q \Rightarrow p)$
T	T	F	F	T	F	F	T	T	T	T	T
T	F	F	T	F	T	T	F	F	F	T	T
F	T	T	F	F	T	T	F	F	T	F	T
F	F	T	T	F	T	T	F	T	T	T	T

It is clear from the table that false statement is $p \Leftrightarrow q \equiv (p \Rightarrow q) \vee (q \Rightarrow p)$
Hence, it is clear from the table that $p \Leftrightarrow q$ and $(p \Rightarrow q) \vee (q \Rightarrow p)$ is not logically equilibrium.

40. The values of $\sim(\sim p \vee \sim q)$ all are neither true nor false, so it is a contingency.

41. If $\sim(p \wedge q)$ is false, then corresponding values of p and q are respectively T and T.

42. The symbolic form of the given statement is $q \Rightarrow p$.

43. The symbolic form of the given statement is $(p \wedge q)$.

44. Statement I is a true statement but Statement II is not a true statement.

45. Statement I is a true statement. Statement II is also a true statement but it is not a proper explanation for Statement I.

46. Both statements are true and Statement II is the correct explanation for Statement I.

47. Truth table for $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

All values of columns are true.
Therefore, $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

48.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p : Ice is cold q : Blood is green.
$p \vee q$: Ice is cold or blood is green.
$p \wedge q$: Ice is cold and blood is green.

49. Truth table for $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

So, statement $(p \wedge \sim p)$ is a contradiction.
Truth table for statement

$$p \to q$$

and $$\sim p \vee q$$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence, statement $(p \to q)$ is logically equivalent to $\sim p \vee q$.

50. If $p \to q$ be any conditional statement, then
(i) the converse of $p \to q$ is the statement $q \to p$.
(ii) the inverse of $p \to q$ is the statement $\sim p \to \sim q$.

51.

p	q	r	$p \to r$	$p \wedge q$	$q \to r$	$p \to (q \to r)$	$(p \wedge q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	T	F	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

From table columns of $p \to (q \to r)$ is not a tautology and column of $p \to (q \to r)$ and $(p \wedge q) \to r$ are identical. Hence, Statement I is false, Statement II is true.

52.

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$\sim p \wedge q$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	F	F

So, the statement $(p \vee q) \wedge \sim p$ is logically equivalent to $\sim p \wedge q$.

53.

p	q	$q \to p$	$p \to (q \to p)$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T

So, statement $p \to (q \to p)$ is logically equivalent to $p \to (p \vee q)$.

54. $P : \frac{n}{m}; Q : \frac{n^2}{m}; R : m$ is prime.

Let
$$m = 5, p = 10, n^2 = 100$$
$$m = 3, n = 12, n^2 = 144$$
$$m = 7, n = 14, n^2 = 196$$

So, P, Q and R are true statements.
$\therefore Q \wedge R = T \wedge T \to T = P$

55.

p	q	$\sim p$	$\sim p \vee q$	$p \to (p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

So, from the above table, if $p \to (\sim p \vee q)$ has truth value F, then the truth values of P and q are T are F respectively.

56.

A	B	$A \vee B$	$B \to A$	$A \wedge B$	$\sim A$	$\sim A \vee B$	$A \to (A \vee B)$
T	T	T	T	T	F	T	T
T	F	T	T	F	F	F	T
F	T	T	F	F	T	T	T
F	F	F	T	F	T	T	T

$A \to (B \to A)$	$(A \wedge B) \to (\sim A \vee B)$	$\sim[(A \wedge B) \to (\sim A \vee B)]$
T	T	F
T	T	F
T	T	F
T	T	F

So, $A \to (B \to A)$ is equivalent to $A \to (A \vee B)$.

But $\sim[A \wedge B) \to (\sim A \vee B)]$ is not a tautology *i.e.*, it is contradiction.

57. **Statement II** $(p \to q) \leftrightarrow (\sim q \to \sim p)$

$\equiv (p \to q) \leftrightarrow (p \to q)$

which is always true, so Statement II is true.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$

$\equiv p \wedge \sim q \wedge \sim p \wedge q$

$\equiv p \wedge \sim p \wedge \sim q \wedge q$

$\equiv f \wedge f \equiv f$

Hence, it is a fallacy statement.

So, Statement I is true.

Alternate Solution

Statement II $(p \to q) \leftrightarrow (\sim q \to \sim p)$

$\sim q \to \sim p$ is contrapositive of $p \to q$

Hence, $(p \to q) \leftrightarrow (p \to q)$ will be a tautology.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, it is a fallacy.

58. Let us assume that p : 'I become a teacher' and q : I will open a school. Then, we can easily ascertain that

Negation of $(p \to q)$ is $\sim(p \to q) = p \wedge \sim q$, which means that 'I will become a teacher and I will not open a school.'

59.

A	B	$A \vee B$	$A \wedge B$	$A \vee (A \vee B)$	$A \vee (A \wedge B)$	$A \to B$	$A \wedge (A \to B)$	$A \wedge (A \to B) \to B$	$B \to (A \wedge (A \to B))$
T	T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	F	T	T
F	T	T	F	T	F	T	F	T	F
F	F	F	F	F	F	T	F	T	T

Since, the truth value of all the elements in the column $A \wedge (A \to B) \to B$

So, $A \wedge (A \to B) \to B$ is tautology.

60. Suman is brilliant and dishonest, if and only if Suman is rich, is expressed as,

$$Q \leftrightarrow (P \wedge \sim R)$$

So, negation of it will be $\sim (Q \leftrightarrow (P \wedge \sim R))$.

61. P : There is rational number $x \in S$ such that $x > 0$.

$\sim P$: Every rational number $x \in S$ satisfies $x \le 0$.

62.

p	q	$p \leftrightarrow q$	$\sim q$	$(p \leftrightarrow \sim q)$	$\sim(p \leftrightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	T	T	F	T

63. Here, $p : x$ is an irrational number

$q : y$ is a transcendental number

$r : x$ is a rational number iff y is a transcendental number

$\Rightarrow \quad r : \sim p \leftrightarrow q$

$S_1 : r \equiv q \vee p$ and $\quad S_2 : r \equiv \sim(p \leftrightarrow \sim q)$

p	q	$\sim p$	$\sim q$	$r : \sim p \leftrightarrow q$	$q \vee p$	$(p \leftrightarrow \sim q)$	$\sim(p \leftrightarrow \sim q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	T	T	F
F	T	T	F	T	T	T	F
F	F	T	T	F	F	F	T

It is clear from the table that r is not equivalent to either of the statements.

Hence, none of the given options is correct.

64.

q	p	$q \to p$	$p \to (q \to p)$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	T	F	T

So, statement $p \to (q \to p)$ is equivalent to $p \to (p \vee q)$.

Unit Test 6

(Statistics Probability & Mathematical Reasoning)

DAY 37

1. One mapping is selected at random from all the mappings of the set $A = \{1, 2, 3, \dots, n\}$ into itself. The probability that the mapping selected is one to one is given by

(a) $\frac{1}{n^n}$ (b) $\frac{1}{n!}$

(c) $\frac{(n-1)!}{n^{n-1}}$ (d) None of these

2. A natural number is selected at random from the set $X = \{x : 1 \le x \le 100\}$. The probability that the number satisfies the inequation $x^2 - 13x \le 30$ is

(a) $\frac{9}{50}$ (b) $\frac{3}{20}$

(c) $\frac{2}{11}$ (d) None of these

3. A fair coin is tossed 100 times. The probability of getting tails an odd number of times is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$

(c) $\frac{3}{8}$ (d) None of these

4. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. The probability that it is actually a six is

(a) $\frac{3}{8}$ (b) $\frac{1}{5}$

(c) $\frac{3}{4}$ (d) None of these

5. A letter is taken at random from the letters of the word 'STATISTICS' and another letter is taken at random from the letters of the word 'ASSISTANT'. The probability that they are the same letters is

(a) $\frac{1}{45}$ (b) $\frac{13}{90}$

(c) $\frac{19}{90}$ (d) $\frac{5}{18}$

6. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)P(B)$, then

(a) $P\left(\frac{A}{B}\right) = 0$ (b) $P\left(\frac{B}{A}\right) = 0$

(c) $P(A' \cap B') = P(A')P(B')$ (d) $P(A/B) + P(B/A) = 1$

7. An ellipse of eccentricity $\frac{2\sqrt{2}}{3}$ is inscribed in a circle and a point within the circle is chosen at random. The probability that this point lies outside the ellipse is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{1}{9}$ (d) $\frac{2}{9}$

8. The probability that when 12 balls are distributed among three boxes, the first box will contain three balls, is

(a) $\frac{2^9}{3^{12}}$ (b) $\frac{^{12}C_3 \cdot 2^9}{3^{12}}$

(c) $\frac{^{12}C_3 \cdot 2^{12}}{3^{12}}$ (d) None of these

9. Four-digit numbers are formed using each of the digit $1, 2, \dots, 8$ only once. One number from them is picked up at random. The probability that the selected number contains unity is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{8}$ (d) None of these

10. While shuffling a pack of playing cards, four are accidently dropped. The probability that the cards are dropped one from each suit is

(a) $\frac{1}{256}$ (b) $\frac{2197}{20825}$

(c) $\frac{3}{20825}$ (d) None of these

11. A boy is throwing stones at a target. The probability of hitting the target at any trial is $\frac{1}{2}$. The probability of hitting the target 5th time at the 10th throw is

(a) $\frac{5}{2^{10}}$ (b) $\frac{63}{2^9}$

(c) $\frac{^{10}C_5}{2^{10}}$ (d) $\frac{^{10}C_4}{2^{10}}$

12. An automobile driver travels from plane to a hill station 120 km distant at an average speed of 30 km/h. Then, he makes the return trip at an average speed of 25 km/h. He covers another 120 km distance on plane at an average speed of 50 km/h. His average speed over the entire distance of 360 km will be

(a) $\frac{30+25+50}{3}$ km/h (b) $\frac{25+35+15}{3}$ km/h

(c) $\frac{3}{\frac{1}{30}+\frac{1}{25}+\frac{1}{50}}$ km/h (d) None of these

13. The mean of 10 numbers is 12.5, the mean of the first six is 15 and the last five is 10. The sixth number is

(a) 12 (b) 15

(c) 18 (d) None of these

14. The geometric mean of numbers $7, 7^2, 7^3, \ldots, 7^n$ is

(a) $7^{7/n}$ (b) $7^{n/7}$

(c) $7^{(n-1)/2}$ (d) $7^{(n+1)/2}$

15. For a frequency distribution consisting of 18 observations, the mean and the standard deviation were found to be 7 and 4, respectively. But on comparison with the original data, it was found that a figure 12 was miscopied as 21 in calculations. The correct mean and standard deviation are

(a) 6.7, 2.7 (b) 6.5, 2.5

(c) 6.34, 2.34 (d) None of these

16. If runs of two players A and B in 10 cricket matches are such that player A has mean 50 and variance 36 and player B has mean 60 and variance 81 of runs, then the player more consistent in runs is

(a) A (b) B

(c) Both are equally consistent (d) None of the above

17. The upper quartile for the following distribution

Size of items	1	2	3	4	5	6	7
Frequency	2	4	5	8	7	3	2

is given by the size of

(a) $\left(\frac{31+1}{4}\right)$th item (b) $\left[2\left(\frac{31+1}{4}\right)\right]$th item

(c) $\left[3\left(\frac{31+1}{4}\right)\right]$th item (d) $\left[4\left(\frac{31+1}{4}\right)\right]$th item

18. The marks obtained by 60 students in a certain test are given below

Marks	Number of students
10–20	2
20–30	3
30–40	4
40–50	5
50–60	6
60–70	12
70–80	14
80–90	10
90–100	4

Find the median of the above data.

(a) 68.33 (b) 70

(c) 71.11 (d) None of these

19. If the mean of a binomial distribution is 25, then its standard deviation lies in the interval given below

(a) $[0, 5)$ (b) $(0, 5]$ (c) $[0, 25)$ (d) $(0, 25)$

20. Five-digit numbers are formed using the digits 0, 2, 4, 6, 8 without repeating the digits. If a number so formed is chosen at random, the probability that it is divisible by 20 is

(a) 1/2 (b) 1/3 (c) 1/4 (d) 2/5

21. Let S be the universal set and $n(X) = k$. The probability of selecting two subsets A and B of the set X such that $B = \overline{A}$, is

(a) $\frac{1}{2}$ (b) $\frac{1}{2^{k-1}}$ (c) $\frac{1}{2^k}$ (d) $\frac{1}{3^k}$

22. Let p: She is intelligent and q : She is studious. The symbolic form of "it is not true that she is not intelligent or she is not studious" is

(a) $p \wedge \sim q$ (b) $\sim p \wedge q$

(c) $p \wedge q$ (d) None of these

23. If p : Ajay works hard, q : Ajay gets good marks, then proposition $\sim p \Rightarrow \sim q$ is equivalent to

(a) Ajay does not work hard and yet he gets good marks

(b) Ajay work hard if and only if he gets good marks

(c) if Ajay does not work hard, then he does not get good marks

(d) None of the above

24. $\sim[(p \vee q) \wedge \sim (p \wedge q)]$ is equivalent to

(a) $p \Leftrightarrow q$ (b) $\sim p \wedge q$

(c) $\sim(p \Leftrightarrow q)$ (d) None of these

25. Assuming $(p \vee q)$ is true and $(p \wedge q)$ is false, state which of the following proposition have true values?

(a) $\sim p \wedge q$ (b) $\sim p \vee \sim q$

(c) $p \Leftrightarrow q$ (d) None of these

26. The dual statement of the compound statement "Rohan and Mohan cannot speak French".

(a) Rohan or Mohan cannot speak French
(b) Rohan speak French and Mohan speak French
(c) Rohan and Mohan both speak French
(d) None of the above

27. The proposition of $(p \vee r) \wedge (q \vee r)$ is equivalent to

(a) $(p \wedge q) \vee r$ (b) $(p \vee q) \wedge r$ (c) $p \wedge (q \vee r)$ (d) $p \vee (q \wedge r)$

28. The dual of $[p \vee (\sim q)] \wedge (\sim p)$ is

(a) $(p \vee q) \wedge p$ (b) $(\sim p \vee q) \vee (\sim p)$
(c) $(p \wedge \sim p) \vee \sim q$ (d) None of these

29. If the arithmetic mean of two unequal positive real numbers a and b (where, $a > b$) be twice as their geometric mean, then $a : b$ is equal to

(a) $(2+\sqrt{3}) : (\sqrt{2}+3)$ (b) $(2-\sqrt{3}) : (2+\sqrt{3})$
(c) $(2+\sqrt{3}) : (2-\sqrt{3})$ (d) None of these

30. The variates x and u are related by $hu = x - a$. then correct relation between σ_x and σ_u is

(a) $\sigma_x = h\sigma_u$ (b) $\sigma_u = h\sigma_x$
(c) $\sigma_x = a + h\sigma_u$ (d) $\sigma_u = a + h\sigma_x$

31. Given that, $x \in [0, 1]$ and $y \in [0, 1]$. If A be the event of (x, y) satisfying $y^2 \le x$ and B be the event of (x, y) satisfying $x^2 \le y$. Then,

(a) $P(A \cap B) = \frac{1}{3}$ (b) A, B are exhaustive
(c) A, B are mutually exclusive (d) A, B are independent

32. There are two independent events A and B. The probability that both A and B occurs is $\frac{1}{8}$ and the probability that neither of them occurs is $\frac{1}{4}$. Then, the probability of the two events are, respectively

(a) $\frac{7 \pm \sqrt{17}}{16}, \frac{2}{7 \pm \sqrt{17}}$ (b) $\frac{5 \pm \sqrt{14}}{16}, \frac{3}{5 \pm \sqrt{14}}$
(c) $\left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{3}\right)$ (d) None of these

Directions (Q. Nos. 33 and 34) *Let p : lines l and m are perpendicular to each other.*

q : R is a point on line m.

and r : R is a point in the line m which is perpendicular to l.

33. The proposition of this statement is

(a) $r \equiv p \wedge q$ (b) $r \equiv p \vee q$
(c) $r \equiv (p \wedge q) \wedge \sim q$ (d) None of these

34. The negation of above statement is

(a) $\sim r \equiv p \vee \sim q$ (b) $\sim r \equiv \sim p \vee q$
(c) $\sim r \equiv \sim p \vee \sim q$ (d) None of these

Directions (Q. Nos. 35 to 37) *A box contains n coins. Let $P(E_i)$ be the probability that exactly i out of n coins are biased. If $P(E_i)$ is directly proportional to $i(i+1), 1 \le i \le n$.*

35. Proportionality constant k is equal to

(a) $\frac{3}{n(n^2+1)}$ (b) $\frac{1}{(n^2+1)(n+2)}$
(c) $\frac{3}{n(n+1)(n+2)}$ (d) None of these

36. If P be the probability that a coin selected at random is biased, then P is equal to

(a) $\frac{(3n-1)(n+2)}{4n(n+2)}$ (b) $\frac{(3n+1)(n+2)}{4n(n+2)}$
(c) $\frac{(3n+1)}{5n(n+2)}$ (d) None of these

37. If a coin selected at random is found to be biased, then the probability that it is the only biased coin in the box is

(a) $\frac{1}{n(n+1)(n+2)(3n+1)}$ (b) $\frac{24}{n(n+1)(n+2)(2n+1)}$
(c) $\frac{24}{n(n+1)(n+2)(3n+1)}$ (d) None of these

Directions (Q. Nos. 38 to 41) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

38. Suppose two groups of scores A and B are such that $A = (x, x+2, x+4)$ and $B = (x-2, x+2, x+6)$

Statement I Group B has more variability than group A.

Statement II The value of mean for group B is more than that of group A.

39. Let p : He is poor. q : He is happy.

Statement I The symbolic form of the statement "It is not true that if he is poor, then he is happy" is $\sim(p \Rightarrow q)$.

Statement II The negation of the above statement is $(p \Rightarrow \sim q)$.

40. Let A and B be two independent events.

Statement I If $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is $\frac{2}{7}$.

Statement II $P(\bar{E}) = 1 - P(E)$, where E is any event.

41. Statement I The statements $(p \vee q) \wedge \sim p$ and $\sim p \wedge q$ are logically equivalent.

Statement II The end columns of the truth table of both statements are identical.

Answer *with* Solutions

1. (*c*) Total number of cases $= n^n$

$\therefore$ The number of favourable cases

$$= n(n-1)\ldots 2\cdot 1$$
$$= n!$$

$\therefore$ Required probability $= \dfrac{n!}{n^n} = \dfrac{(n-1)!}{n^{n-1}}$

2. (*b*) Total number of ways = 100

Given, $X^2 - 13X \le 30$

$$\Rightarrow \left(X - \frac{13}{2}\right)^2 \le \frac{289}{4}$$

$$\Rightarrow -\frac{17}{2} \le X - \frac{13}{2} \le \frac{17}{2}$$

$$\Rightarrow -2 \le X \le 15$$

$$\because X \in N$$

$$\therefore X = \{1, 2, 3, \ldots, 15\}$$

$\therefore$ Required probability $= \dfrac{15}{100} = \dfrac{3}{20}$

3. (*a*) $P(X = \text{odd number})$

$$= P(X=1) + P(X=3) + \ldots + P(X=99)$$

$$= {}^{100}C_1\left(\frac{1}{2}\right)^{100} + {}^{100}C_3\left(\frac{1}{2}\right)^{100} + \ldots + {}^{100}C_{99}\left(\frac{1}{2}\right)^{100}$$

$$= ({}^{100}C_1 + {}^{100}C_3 + \ldots + {}^{100}C_{99})\left(\frac{1}{2}\right)^{100} = 2^{99}\cdot\left(\frac{1}{2}\right)^{100} = \frac{1}{2}$$

4. (*a*) Let E = The event that six occurs

and A = The event that the man reports that it is a six

$$\therefore P\left(\frac{E}{A}\right) = \frac{P(E)P(A/E)}{P(E)P(A/E) + P(E')P(A/E')}$$

$$= \frac{(1/6)(3/4)}{(1/6)(3/4) + (5/6)(1/4)} = \frac{3}{8}$$

5. (*c*) Letters of the word STATISTICS are A, C, I, I, S, S, S, T, T, T. Letters of the word ASSISTANT are A, A, I, N, S, S, S, T, T. Common letters are A, I, S and T.

Probability of choosing A is $\dfrac{1}{10} \times \dfrac{2}{9} = \dfrac{2}{90}$.

Probability of choosing I is $\dfrac{2}{10} \times \dfrac{1}{9} = \dfrac{2}{90}$

Probability of choosing S is $\dfrac{3}{10} \times \dfrac{3}{9} = \dfrac{9}{90}$.

Probability of choosing T is $\dfrac{3}{10} \times \dfrac{2}{9} = \dfrac{6}{90}$.

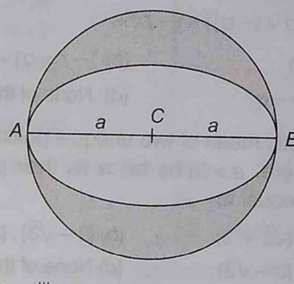
$\therefore$ Probability of required event $= \dfrac{2}{90} + \dfrac{2}{90} + \dfrac{9}{90} + \dfrac{6}{90} = \dfrac{19}{90}$

6. (*c*) Given, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$$\therefore P(A \cap B) = P(A)P(B)$$

$$\Rightarrow P(A' \cap B') = P(A')\,P(B')$$

7. (*b*) Let equation of ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (where, $a > b$)



$\therefore$ Area of an ellipse $= \pi ab$

$$= \pi a \times a\sqrt{1-e^2} = \pi a^2\sqrt{1-\frac{8}{9}} = \frac{\pi a^2}{3}$$

$\therefore$ Area of circle $= \pi a^2$

$\therefore$ Required probability $= \dfrac{\pi a^2 - \dfrac{\pi a^2}{3}}{\pi a^2} = \dfrac{2}{3}$

8. (*b*) Since, each ball can be put into any one of the three boxes, so that total number of ways in which 12 balls can be put into three boxes is 3^{12}.

The three balls can be chosen in ${}^{12}C_3$ ways and remaining 9 balls can be put in the remaining 2 boxes in 2^9 ways.

$\therefore$ Required probability $= \dfrac{{}^{12}C_3 \cdot 2^9}{3^{12}}$

9. (*a*) The total number of four-digit numbers formed with the digits $1, 2, \ldots, 8$ is ${}^8C_4 \times 4!$.

Now, the total number of four digits numbers formed with the digits 1, 2, ..., 8 and containing unity as one of the digits is ${}^7C_3 \times 4!$.

$\therefore$ Required probability $= \dfrac{{}^7C_3 \times 4!}{{}^8C_4 \times 4!} = \dfrac{1}{2}$

10. (*b*) Required probability $= \dfrac{13^4}{{}^{52}C_4} = \dfrac{2197}{20825}$

11. (*b*) Here, $p = \dfrac{1}{2}$ and $q = \dfrac{1}{2}$

$\therefore$ Required probability $= {}^9C_4\left(\dfrac{1}{2}\right)^4\left(\dfrac{1}{2}\right)^5 \times \dfrac{1}{2}$

$$= {}^9C_4\left(\frac{1}{2}\right)^{10} = \frac{9 \times 8 \times 7 \times 6}{(1 \times 2 \times 3 \times 4) \times 2^{10}} = \frac{63}{2^9}$$

12. (c) Average speed $= \dfrac{120+120+120}{\dfrac{120}{30}+\dfrac{120}{25}+\dfrac{120}{50}}$

$= \dfrac{3}{\dfrac{1}{30}+\dfrac{1}{25}+\dfrac{1}{50}}$ km/h

13. (b) Let the mean of the last four digit be A_2.

Then, $12.5 = \dfrac{6\times 15 + 4\times A_2}{6+4}$

$\Rightarrow \quad 125 = 90 + 4A_2 \Rightarrow A_2 = \dfrac{35}{4}$

Let the sixth number be x, then

$10 = \dfrac{1\times x + 4\times \dfrac{35}{4}}{1+4}$

$\Rightarrow \quad 50 = x + 35 \Rightarrow x = 15$

14. (d) $\text{GM} = (7\cdot 7^2 \ldots 7^n)^{1/n} = \left(7^{\frac{n(n+1)}{2}}\right)^{1/n} = 7^{\frac{n+1}{2}}$

15. (b) Given, $\dfrac{\Sigma x_i}{18} = 7 \Rightarrow \Sigma x_i = 126$

Also, $\Sigma x_i = 126 - 21 + 12 = 117$

$\therefore$ True mean $= \dfrac{\Sigma x_i}{18} = \dfrac{117}{18} = 6.5$

Since, $\dfrac{\Sigma x_i^2}{18} - (\text{Mean})^2 = 16$

$\Rightarrow \quad \dfrac{\Sigma x_i^2}{18} = 4^2 + (7)^2$

$\Rightarrow \quad \Sigma x_i^2 = 1170$

$\Rightarrow \quad \Sigma x_i^2 = 1170 - 21^2 + 12^2 = 873$

$\therefore$ True variance $= \dfrac{\Sigma x_i^2}{18} - (\text{Mean})^2$

$= \dfrac{873}{18} - (6.5) = 48.5 - 42.25 = 6.25$

$\therefore$ True standard deviation $= \sqrt{\text{True variance}} = \sqrt{6.25} = 2.5$

16. (a) Given that, mean and variance of players A and B is

$\bar{x}_1 = 50, \sigma_1^2 = 36$ or $\sigma_1 = 6$

$\bar{x}_2 = 60, \sigma_2^2 = 81$ or $\sigma_2 = 9$

From the following formula, we can check consistency

coefficient of variation $= \dfrac{\text{SD}}{\text{mean}} \times 100$

Therefore, coefficient of variation of player,

$A = \dfrac{\sigma_1}{\bar{x}_1}\times 100 = \dfrac{6}{50}\times 100 = 12\%$

and coefficient of variation of player,

$B = \dfrac{\sigma_2}{\bar{x}_2}\times 100 = \dfrac{9}{60}\times 100 = 15\%$

Since, coefficient of variation of player A is less, hence it is more consistent in comparision to player B.

17. (c) Upper quartile = Size of $\left[3\left(\dfrac{n+1}{4}\right)\text{th}\right]$ item

= Size of $\left[3\left(\dfrac{31+1}{4}\right)\right]$th item

18. (a)

Class	x_i	f_i	cf
10-20	15	2	2
20-30	25	3	5
30-40	35	4	9
40-50	45	5	14
50-60	55	6	20
60-70	65	12	32
70-80	75	14	46
80-90	85	10	56
90-100	95	4	60

Here, $N = 60 \Rightarrow \dfrac{N}{2} = 30$ which lies in the interval 60-70.

$\therefore \quad \text{Median} = l + \dfrac{\dfrac{N}{2} - c}{f}\times h$

$= 60 + \dfrac{30-20}{12}\times 10$

$= 60 + \dfrac{100}{12} = 68.33$

19. (a) Since, $0 \le \sqrt{npq} < \sqrt{np}$

$\Rightarrow \quad 0 \le \text{SD} < 5 \qquad (\because p \ne 0)$

$\Rightarrow \quad \text{SD} \in [0, 5)$

20. (c) The total number of numbers formed is $5! - 4! = 96$.

If a number is divisible by 20, then the last digit is 0. The number of numbers with zero at the end is $4! = 24$.

$\therefore$ Required probability $= \dfrac{24}{96} = \dfrac{1}{4}$

21. (b) Total number of subsets of X is 2^k.

$\therefore \quad n(S) = {}^{2^k}C_2$

Let $n(E)$ = The number of selections of two non-intersecting subsets whose union is X.

$= \dfrac{1}{2}({}^kC_0 + {}^kC_1 + {}^kC_2 + \ldots) = \dfrac{1}{2}\times 2^k$

$\therefore$ Required probability $= \dfrac{\dfrac{1}{2}\times 2^k}{{}^{2^k}C_2} = \dfrac{2^{k-1}}{2^k\left(\dfrac{2^k-1}{2}\right)} = \dfrac{1}{2^k-1}$

22. (c) Here, $\sim p$: She is not intelligent.

$\sim q$: She is not studious.

So, the required symbolic form is $\sim(\sim p \vee \sim q)$ or $p \wedge q$.

23. (c) Given, proposition is equivalent to "If Ajay does not work hard, then he does not get good marks".

24. (a) It is clear from the table that column IIIrd and IVth are identical.

p	q	$p \Leftrightarrow q$	$\sim[(p \vee q) \wedge \sim(p \wedge q)]$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

25. (b)

p	q	$p \vee q$	$p \wedge q$	$\sim p \wedge q$	$\sim p \vee \sim q$	$p \Leftrightarrow q$
T	F	T	F	F	T	F
F	T	T	F	T	T	F

26. (a) Rohan or Mohan cannot speak French.

27. (a) Using Distributive law,

$$(p \vee r) \wedge (q \vee r) \equiv (p \wedge q) \vee r$$

28. (d) The dual of $[p \vee (\sim q) \wedge (\sim p)]$ is $[p \wedge (\sim q)] \vee (\sim p)$.

29. (c) According to the question,

We have given that,

Arithmetic mean = 2 × Geometric mean

$$\Rightarrow \quad \frac{a+b}{2} = 2\sqrt{ab}$$

$$\Rightarrow \quad a + b = 4\sqrt{ab} \quad \ldots \text{(i)}$$

We know that, $(a-b)^2 = (a+b)^2 - 4ab$

On substituting the values of $(a+b)$ from Eq. (i), we get

$$(a-b)^2 = (4\sqrt{ab})^2 - 4ab = 12ab$$

$$\Rightarrow \quad a - b = 2\sqrt{3}\sqrt{ab} \quad \ldots \text{(ii)}$$

Now, $a + b = 4\sqrt{ab}$

and $a - b = 2\sqrt{3}\sqrt{ab}$

On adding above equations, we get

$$2a = \sqrt{ab}(4 + 2\sqrt{3})$$

$$\Rightarrow \quad 2a = 2\sqrt{ab}(2 + \sqrt{3})$$

$$\Rightarrow \quad a = \sqrt{ab}(2 + \sqrt{3})$$

On putting the value of a in Eq. (i), we get

$$\sqrt{ab}(2 + \sqrt{3}) + b = 4\sqrt{ab}$$

$$\Rightarrow \quad b = 4\sqrt{ab} - \sqrt{ab}(2 + \sqrt{3})$$

$$\Rightarrow \quad b = \sqrt{ab}(4 - 2 - \sqrt{3})$$

$$= \sqrt{ab}(2 - \sqrt{3})$$

$\therefore$ Required ratio, $a : b = \dfrac{\sqrt{ab}(2+\sqrt{3})}{\sqrt{ab}(2-\sqrt{3})} = (2+\sqrt{3}) : (2-\sqrt{3})$

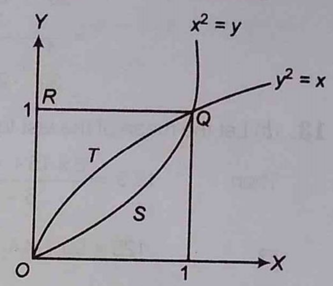
30. (a) Here, $u = \dfrac{x}{h} - \dfrac{a}{h}$

We know that standard deviation is not dependent on change of origin.

$$\therefore \quad \sigma_u = \frac{\sigma_x}{h} \Rightarrow \sigma_x = h\sigma_u$$

31. (a) A = The event of (x, y) belonging to the area $OTQPO$

and B = The event of (x, y) belonging to the area $OSQRO$



$$P(A) = \frac{\text{ar}(OTQPO)}{\text{ar}(OPQRO)} = \frac{\int_0^1 \sqrt{x}\,dx}{1 \times 1} = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$$

$$P(B) = \frac{\text{ar}(OSQRO)}{\text{ar}(OPQRO)} = \frac{\int_0^1 \sqrt{y}\,dy}{1 \times 1} = \frac{2}{3}$$

$$P(A \cap B) = \frac{\text{ar}(OTQS)}{\text{ar}(OPQRO)} = \frac{\int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx}{1 \times 1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$P(A) + P(B) = \frac{2}{3} + \frac{2}{3} \neq 1$$

So, A and B are not exhaustive.

Then, $P(A) \cdot P(B) = \dfrac{2}{3} \cdot \dfrac{2}{3} = \dfrac{4}{9} \neq P(A \cap B)$

So, A and B are not independent.

Then, $P(A \cup B) = 1,\ P(A) + P(B) = \dfrac{2}{3} + \dfrac{2}{3} \neq P(A \cup B)$

So, A and B are not mutually exclusive.

32. (a) Given, $P(A \cap B) = \dfrac{1}{8}$

and $P(\bar{A} \cap \bar{B}) = \dfrac{1}{4}$

$\therefore \quad P(A) \cdot P(B) = \dfrac{1}{8}$

and $P(\overline{A \cup B}) = \dfrac{1}{4}$

$\Rightarrow \quad P(A) \cdot P(B) = \dfrac{1}{8}$

and $P(A \cup B) = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$[\because P(A \cup B) = P(A) + P(B) - P(A \cap B)]$

$$\frac{3}{4} = P(A) + P(B) - \frac{1}{8}$$

$$\Rightarrow \quad P(A) + P(B) = \frac{7}{8}$$

Let $P(A) = x$, then $P(B) = \dfrac{1}{8x}$

$$\therefore \quad x + \frac{1}{8x} = \frac{7}{8} \Rightarrow \frac{8x^2 + 1}{8x} = \frac{7}{8}$$

$$\Rightarrow \quad 8x^2 + 1 = 7x \quad \text{or} \quad 8x^2 - 7x + 1 = 0$$

$$\therefore \quad x = \frac{7 \pm \sqrt{49 - 32}}{16}$$

$$P(A) = \frac{7 \pm \sqrt{17}}{16}$$

$$\Rightarrow \quad P(B) = \frac{2}{7 \pm \sqrt{17}}$$

$$\frac{3}{4} = P(A) + P(B) - \frac{1}{8}$$

$$\Rightarrow \quad P(A) + P(B) = \frac{7}{8}$$

Let $P(A) = x$, then $P(B) = \frac{1}{8x}$

$$\therefore \quad x + \frac{1}{8x} = \frac{7}{8}$$

$$\Rightarrow \quad \frac{8x^2 + 1}{8x} = \frac{7}{8}$$

$$\Rightarrow \quad 8x^2 + 1 = 7x$$

or $\quad 8x^2 - 7x + 1 = 0$

$$\therefore \quad x = \frac{7 \pm \sqrt{49 - 32}}{16}$$

$$P(A) = \frac{7 \pm \sqrt{17}}{16}$$

$$\Rightarrow \quad P(B) = \frac{2}{7 \pm \sqrt{17}}$$

33. (*a*) It is clear that proposition of given statement is

$$r \equiv (p \wedge q)$$

34. (*c*) $\quad \sim r \equiv \sim (p \wedge q)$

or $\quad \sim r \equiv \sim p \vee \sim q$

35. (*c*) Since, $P(E_i) = ki(i + 1)$

Also, $\quad P(E_1) + \ldots + P(E_n) = 1$

$$\therefore \quad k \sum_{i=1}^{n} i(i + 1) = 1$$

$$\Rightarrow \quad k\left[\frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2}\right] = 1$$

$$\Rightarrow \quad \frac{n(n + 1)}{2} k\left(\frac{2n + 1}{3} + 1\right) = 1$$

$$\Rightarrow \quad k = \frac{3}{n(n + 1)(n + 2)}$$

36. (*b*) $P(E) = \sum_{i=1}^{n} P(E_i) \times P\left(\frac{E}{E_i}\right)$

$$= \sum_{i=1}^{n} ki(i + 1)\frac{i}{n}$$

$$= \frac{k}{n} \sum_{i=1}^{n} (i^3 + i^2)$$

$$= \frac{k}{n}\left[\left\{\frac{n(n + 1)}{2}\right\}^2 + \frac{n(n + 1)(2n + 1)}{6}\right]$$

$$= \frac{k}{n} \cdot \frac{n(n + 1)}{2} \frac{(3n^2 + 7n + 2)}{6}$$

$$= \frac{(3n + 1)(n + 2)}{4n(n + 2)}$$

37. (*c*) $P\left(\frac{E_1}{E}\right) = \frac{P(E_1) \cdot P\left(\frac{E}{E_1}\right)}{P(E)}$

$$= \frac{k \times 2 \times \frac{1}{n}}{\frac{(3n + 1)(n + 2)}{4n(n + 2)}}$$

$$= \frac{24}{n(n + 1)(n + 2)(3n + 1)}$$

38. (*c*) Since, $A = (x, x + 2, x + 4)$

and $\quad B = (x - 2, x + 2, x + 6)$

$\therefore \quad$ Mean of $A = \frac{x + x + 2 + x + 4}{3} = x + 2$

and mean of $B = \frac{x - 2 + x + 2 + x + 6}{3} = x + 2$

Hence, group B has more variability than group A.

Since, from the given data difference in scores of group A is 2 but difference in scores of group B is 4.

39. (*c*) Hence, Statement I is true and Statement II is false.

40. (*a*) $P(A \cup \bar{B}) = 1 - \overline{(A \cup \bar{B})}$

$\Rightarrow 0.8 \quad = 1 - P(\bar{A} \cap B)$

$\Rightarrow 0.8 \quad = 1 - P(\bar{A})\, P(B)$

$\Rightarrow \quad 0.8 = 1 - 0.7 \times P(B)$

$\therefore \quad P(B) = \frac{2}{7}$

41. (*b*)

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim q$	$\sim p \wedge q$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	F	F

It is clear from the table both statements are true but Statement II is not correct explanation for Statement I.

Day 38

Mock TEST 1

(Based on Complete Syllabus)

Instructions

1. The test consists of 30 questions.
2. Candidates will be awarded marks for correct response of each question. 1/4 (one-fourth) marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
3. There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response.

1. Let $f(x)$ satisfies the requirements of Lagrange's mean value theorem in [0, 2]. If $f(0)=0$ and $|f'(x)| \le \frac{1}{2}$ for all x in [0, 2], then

(a) $f(x) \le 2$ (b) $|f(x)| \le 1$ (c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in [0, 2]

2. The value of $f(0)$, so that $f(x) = \dfrac{(4^x - 1)^3}{\sin\left(\frac{x}{4}\right)\log\left(1+\frac{x^2}{3}\right)}$ is continuous everywhere, is equal to

(a) $3(\log 4)^3$ (b) $4(\log 4)^3$ (c) $12(\log 4)^3$ (d) $15(\log 4)^3$

3. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is

(a) 0 (b) 1 (c) 2 (d) 4

4. The statement $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to

(a) p (b) $\sim p$ (c) q (d) $\sim q$

5. If $A = \{\theta : 2\cos^2\theta + \sin\theta \le 2\}$ and $B = \left\{\theta : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$, then $A \cap B$ is equal to

(a) $\left\{\theta : \pi \le \theta \le \frac{3\pi}{2}\right\}$ (b) $\left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{2}\right\}$

(c) $\left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{6} \text{ or } \pi \le \theta \le \frac{3\pi}{2}\right\}$

(d) None of the above

6. The coefficient of x^{50} in $(1+x)^{41}(1-x+x^2)^{40}$ is

(a) 0 (b) 1 (c) 2 (d) 3

7. The value of $\log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}}$ is

(a) $3\log_2 7$ (b) $3\log_7 2$ (c) $1 - 3\log_7 2$ (d) $1 - 3\log_2 7$

8. The integral $\int_{-1/2}^{1/2}\left[[x] + \log\left(\frac{1+x}{1-x}\right)\right]dx$ is equal to

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2\log\frac{1}{2}$

9. A square $OABC$ is formed by line pairs $xy = 0$ and $xy + 1 = x + y$, where O is the origin. A circle with centre C_1 inside the square is drawn to touch the line pair $xy = 0$ and another circle with centre C_2 and radius twice that of C_1, is drawn to touch the circle C_1 and the other line pair. The radius of the circle with centre C_1 is

(a) $\frac{\sqrt{2}}{\sqrt{3}(\sqrt{2}+1)}$ (b) $\frac{2\sqrt{2}}{\sqrt{2}(\sqrt{2}+1)}$

(c) $\frac{\sqrt{2}}{3(\sqrt{2}+1)}$ (d) $\frac{\sqrt{2}+1}{3\sqrt{2}}$

10. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x-1)^3$ (b) $(x+1)^3$
(c) $(x+1)^2$ (d) $(x-1)^2$

11. A natural number x is chosen at random from the first one hundred natural numbers. The probability that $\frac{(x-20)(x-40)}{(x-30)}<0$

(a) $\frac{9}{50}$ (b) $\frac{3}{50}$ (c) $\frac{7}{25}$ (d) None of these

12. If $A_r=\begin{bmatrix} r & r-1 \\ r-1 & r\end{bmatrix}$, where r is a natural number, then $|A_1|+|A_2|+\ldots+|A_{2010}|$ must be equal to

(a) 2010 (b) $(2010)^2$ (c) 2011 (d) $(2010)^3$

13. If z is non-real and $i=\sqrt{-1}$, then $\sin^{-1}\left(\frac{1}{i}(z-1)\right)$ be the angle of a triangle, if

(a) Re $(z)=1$, Im $(z)=2$ (b) Re$(z)=1$, $-1\le$ Im$(z)\le 1$
(c) Re (z) + Im $(z)=0$ (d) None of these

14. If $\lim_{x\to 0}\frac{\sin 2x-a\sin x}{x^3}$ is finite, then the value of a and the limit are respectively

(a) $-1, 1$ (b) $2, -1$ (c) $2, 1$ (d) $-2, -1$

15. The ends A and B of a rod of length $\sqrt{5}$ are sliding along the curve $y=2x^2$. Let x_A and x_B be the x-coordinate of the ends. At the moment when A is at (0, 0) and B is at (1, 2) the derivative $\frac{dx_B}{dx_A}$ has the value equal to

(a) 1/3 (b) 1/5 (c) 1/8 (d) 1/9

16. Suppose the function $g_n(x)=x^{2n+1}+a_nx+b_n$; $(n\in N)$ satisfies the equation

$$\int_{-1}^{1}(px+q)g_n(x)\,dx=0$$

for all linear functions $(px+q)$, then

(a) $a_n=b_n=0$ (b) $b_n=0, a_n=-\frac{3}{2n+3}$
(c) $a_n=0, b_n=-\frac{3}{2n+3}$ (d) $a_n=\frac{3}{2n+3}, b_n=-\frac{3}{2n+3}$

17. Given, $f(x)=\begin{vmatrix} 0 & x^2-\sin x & \cos x-2 \\ \sin x-x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0\end{vmatrix}$, then $\int f(x)dx$ is equal to

(a) $\frac{x^3}{3}-x^2\sin x+\sin 2x+C$ (b) $\frac{x^3}{3}-x^2\sin x-\cos 2x+C$
(c) $\frac{x^3}{3}-x^2\cos x-\cos 2x+C$ (d) None of these

18. The two vertices of a triangle are $(4,-3)$ and $(-2,5)$. If the orthocentre of the triangle is at (1, 2), then the third vertex is

(a) $(-33,-26)$ (b) (33, 26)
(c) (26, 33) (d) None of these

19. Normals AO, AA_1, AA_2 are drawn to parabola $y^2=8x$ from the point $A(h,0)$. If ΔOA_1A_2 is equilateral, then possible values of h is

(a) 24 (b) 25
(c) 28 (d) 27

20. The unit vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular and the unit vector $\mathbf{c}$ is inclined at an angle θ to both $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{c}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma(\mathbf{a}\times\mathbf{b})$, then which is not true ?

(a) $\gamma^2=1-2\alpha^2$ (b) $\alpha=2\beta$
(c) $\gamma^2=-\cos 2\theta$ (d) $\beta^2=\frac{1+\cos 2\theta}{2}$

21. If $0°\le\theta\le 180°$ and $81^{\sin^2\theta}+81^{\cos^2\theta}=30$, then θ is

(a) 30° (b) 45° (c) 120° (d) 150°

22. The sum of n terms of the following series $1+(1+x)+(1+x+x^2)+\ldots$ will be

(a) $\frac{1-x^n}{1-x}$ (b) $\frac{x(1-x^n)}{1-x}$
(c) $\frac{n(1-x)-x(1-x^n)}{(1-x)^2}$ (d) None of these

23. Let function $f: R\to R$ be defined by $f(x)=2x+\sin x$ for $x\in R$, then f is

(a) one-one and onto (b) one-one but not onto
(c) onto but not one-one (d) Neither one-one nor onto

24. For the arithmetic progression $a, a+d, a+2d, a+3d, \ldots, a+2nd$, the mean deviation from mean is

(a) $\frac{n(n+1)d}{2n-1}$ (b) $\frac{n(n+1)d}{2n+1}$
(c) $\frac{n(n-1)d}{2n+1}$ (d) None of these

25. A dictionary is printed consisting of 7 lettered words only that can be made with a letter of the word CRICKET. If the words are printed in alphabetical order, as in an ordinary dictionary, then the number of words before the word CRICKET is

(a) 480 (b) 481
(c) 530 (d) 531

Directions (Q. Nos. 26 to 28) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below .*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

26. Consider $g(x) = f(x) - 1$ and $f(x) + f(1-x) = 2, \forall\, x \in R$,

Statement I $g(x)$ is symmetrical about the point $\left(\frac{1}{2}, 0\right)$.

Statement II If $g(a-x) = -g(a+x), \forall\, x \in R$, then $g(x)$ is symmetrical about the point $(a, 0)$.

27. The equation of two straight lines are $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{-3}$ and $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z+3}{2}$

Statement I The given lines are coplanar.

Statement II The equation

$2x_1 - y_1 = 1,\ x_1 + 3y_1 = 4,\ 3x_1 + 2y_1 = 5$ are consistent.

28. Statement I For $a = -\frac{1}{\sqrt{3}}$, the volume of the parallelopiped formed by vectors $\mathbf{i} + a\mathbf{j}$, $a\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{j} + a\mathbf{k}$ is maximum.

Statement II The volume of the parallelopiped having three coterminous edges a, b and $c = |[a\, b\, c]|$.

29. The solution of the differential equation $x^3 \frac{dy}{dx} + 4x^2 \tan y = e^x \sec y$ satisfying $y(1) = 0$, is

(a) $\tan y = (x-2)e^x \log 3$
(b) $\sin y = e^x(x-1)x^{-4}$
(c) $\tan y = (x-1)e^x x^{-3}$
(d) $\sin y = e^x(x-1)x^{-3}$

30. At a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, tangent PQ is drawn. If the point Q be at a distance $\frac{1}{p}$ from the point P, where p is distance of the tangent from the origin, then the locus of the point Q is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{1}{a^2b^2}$
(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{1}{a^2b^2}$
(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2b^2}$
(d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2b^2}$

Answer *with* Solutions

1. (*b*) Since, $f(x)$ satisfy the Lagrange's mean value theorem.

$\therefore \quad f'(c) = \frac{f(x) - f(0)}{x - 0}$

where, $0 < c < x < 2$ *i.e.*, $0 < c < 2$

$\Rightarrow \quad f(x) = xf'(c)$

$\Rightarrow \quad |f(x)| = |xf'(c)| = |x||f'(c)| \le 2 \cdot \frac{1}{2} = 1$

$\Rightarrow \quad |f(x)| \le 1$

2. (*c*) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(4^x - 1)^3}{\sin\left(\frac{x}{4}\right) \log\left(1 + \frac{x^2}{3}\right)}$

$$= \lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)^3 \frac{4 \cdot \frac{x}{4}}{\sin\left(\frac{x}{4}\right)} \cdot \frac{1}{\frac{1}{x^2} \log\left(1 + \frac{x^2}{3}\right)}$$

$$= \lim_{x \to 0} \left(\frac{4^x - 1}{x}\right)^3 \cdot 4 \cdot \left(\frac{\frac{x}{4}}{\sin\frac{x}{4}}\right) \cdot \frac{1}{\frac{1}{3} \log\left(1 + \frac{x^2}{3}\right)^{3/x^2}}$$

$$= (\log 4)^3 \cdot 4 \cdot 3 \cdot \frac{1}{1} = 12(\log 4)^3$$

3. (*a*) Since, $\cos(\alpha - \beta) = 1$

$\Rightarrow \quad \alpha - \beta = n\pi$

But $\quad -2\pi < \alpha - \beta < 2\pi \quad \{\because \alpha, \beta \in (-\pi, \pi)\}$

$\therefore \quad \alpha - \beta = 0$

Thus, $\quad \cos(\alpha + \beta) = \frac{1}{e}$

$\Rightarrow \cos 2\alpha = \frac{1}{e} < 1$, which is true for four values of α, as $-2\pi < 2\alpha < 2\pi$.

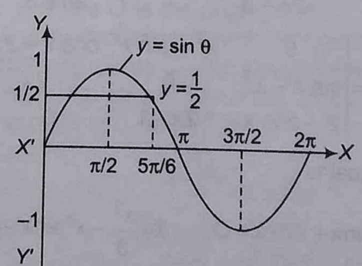
4. (*b*) $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$\equiv \sim p \wedge (\sim q \vee q) \equiv \sim p$

5. (*c*) Now, $2\cos^2\theta + \sin\theta \le 2$ and $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$\Rightarrow 2 - 2\sin^2\theta + \sin\theta \le 2 \Rightarrow \sin\theta\,(2\sin\theta - 1) \ge 0$

***Case* I** $\sin\theta \ge 0$ and $2\sin\theta - 1 \ge 0$



$\therefore \quad \sin\theta \ge 0$ and $\sin\theta \ge \frac{1}{2}$

$\Rightarrow \quad \sin\theta \ge \frac{1}{2}$

$\Rightarrow \quad \frac{\pi}{2} \le \theta \le \frac{5\pi}{6} \quad \ldots$(i)

Case II $\sin\theta \le 0$ and $2\sin\theta - 1 \le 0$

$\therefore$ $\sin\theta \le 0$ and $\sin\theta \le \frac{1}{2}$

$\Rightarrow$ $\sin\theta \le 0$

$\Rightarrow$ $\pi \le \theta \le \frac{3\pi}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$A \cap B = \left\{\theta : \frac{\pi}{2} \le \theta \le \frac{5\pi}{6} \text{ or } \pi \le \theta \le \frac{3\pi}{2}\right\}$$

6. (*a*) Coefficient of x^{50} in $(1+x)^{41}(1-x+x^2)^{40}$

$= $ Coefficient of x^{50} in $(1+x)(1+x^3)^{40}$

$= $ Coefficient of x^{50} in

$(1+x)(1+{}^{40}C_1x^3+\ldots+{}^{40}C_{16}(x^3)^{16}+{}^{40}C_{17}(x^3)^{17}+\ldots)$

$= 0$

7. (*c*) $\log_7 \log_7 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \log_7\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)$

$= \log_7\left(\frac{7}{8}\right) = 1 - \log_7 2^3 = 1 - 3\log_7 2$

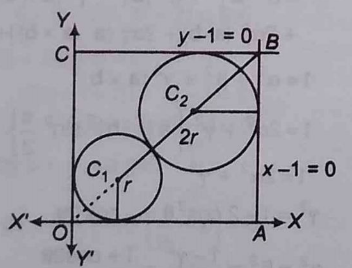
8. (*a*) Let $I = \int_{-1/2}^{1/2}\left[[x] + \log\left(\frac{1+x}{1-x}\right)\right]dx$

$= \int_{-1/2}^{0}[x]\,dx + \int_{0}^{1/2}[x]\,dx + 0$

$\left[\because \log\left(\frac{1+x}{1-x}\right) \text{is an odd function}\right]$

$= \int_{-1/2}^{0} -1\,dx + \int_{0}^{1/2} 0\,dx = -[x]_{-1/2}^{0} + 0 = -\frac{1}{2}$

9. (*c*) Diagonal of the square $= \sqrt{2}$



Also, $d = r\sqrt{2} + 3r + 2\sqrt{2}r$

$\Rightarrow$ $\sqrt{2} = 3\sqrt{2}r + 3r$

$\Rightarrow$ $r = \frac{\sqrt{2}}{3(\sqrt{2}+1)}$

10. (*a*) Given that, $f''(x) = 6(x-1)$

$f'(x) = 3(x-1)^2 + C_1$ ...(i)

But at point (2, 1) the line $y = 3x - 5$ is tangent to the graph $y = f(x)$.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x=2)} = 3$ or $f'(2) = 3$

From Eq. (i), $f'(2) = 3(2-1)^2 + C_1$

$\Rightarrow$ $3 = 3 + C_1$

$\Rightarrow$ $C_1 = 0$

$\therefore$ $f'(x) = 3(x-1)^2$

Given, $f(2) = 1$

$\Rightarrow$ $f(x) = (x-1)^3 + C_2$

$\Rightarrow$ $f(2) = 1 + C_2$

$\Rightarrow$ $1 = 1 + C_2 \Rightarrow C_2 = 0$

Hence, $f(x) = (x-1)^3$

11. (*c*) Since, $\frac{(x-20)(x-40)}{(x-30)} < 0$

$\Rightarrow$ $x \in (-\infty, 20) \cup (30, 40)$

$\therefore$ $E = (1, 2, 3, \ldots, 19, 31, 32, \ldots, 39)$

$\therefore$ $n(E) = 28$

$\therefore$ Required probability $= \frac{28}{100} = \frac{7}{25}$

12. (*b*) $|A_r| = r^2 - (r-1)^2$

$\therefore$ $|A_1| + |A_2| + \ldots + |A_{2010}| = \sum_{r=1}^{2010}\{r^2 - (r-1)^2\}$

$= (2010)^2 - (0)^2 = (2010)^2$

13. (*b*) By the properties of inverse trigonometric function $\frac{z-1}{i} = \text{real}$

$\Rightarrow$ $\frac{x-1+iy}{i} = \text{real} \Rightarrow \frac{x-1}{i} + y = \text{real}$

$\Rightarrow$ $x - 1 = 0 \Rightarrow x = 1$

$\therefore$ $\sin^{-1}\left(\frac{z-1}{i}\right) = \sin^{-1}(y)$

So, $-1 \le y \le 1$

$\therefore$ $\text{Re}(z) = x = 1, -1 \le \text{Im}(z) \le 1$

14. (*b*) $\lim_{x\to 0}\frac{\sin 2x - a\sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{2\cos 2x - a\cos x}{3x^2}$

Since, the limit is finite number, so the numerator should be finite, $a = 2$

$= \lim_{x\to 0}\frac{2\cos 2x - 2\cos x}{3x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{-4\sin 2x + 2\sin x}{6x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 0}\frac{-8\cos 2x + 2\cos x}{6} = -1$

15. (*d*) We have, $y = 2x^2$

$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$

$\Rightarrow$ $(x_B - x_A)^2 + 4(x_B^2 - x_A^2)^2 = 5$

On differentiating w.r.t. x_A and denoting, $\frac{dx_B}{dx_A} = D$

$2(x_B - x_A)(D-1) + 8(x_B^2 - x_A^2)(2x_BD - 2x_A) = 0$

On putting $x_A = 0$; $x_B = 1$, then

$2(1-0)(D-1)+8(1-0)(2D-0)=0$

$\Rightarrow \quad 2D-2+16D=0$

$\Rightarrow \quad D=\frac{1}{9}$

16. (*b*) We have, $\int_{-1}^{1}(px+q)(x^{2n+1}+a_nx+b_n)dx=0$

Equating the odd component to be zero and integrating, we get

$$\frac{2p}{2n+3}+\frac{2a_np}{3}+2b_nq=0 \text{ for all } p, q$$

Hence, $\quad b_n=0$ and $a_n=-\frac{3}{2n+3}$

17. (*d*) $f(x)=\begin{vmatrix} 0 & x^2-\sin x & \cos x-2 \\ -(x^2-\sin x) & 0 & 1-2x \\ -(\cos x-2) & -(1-2x) & 0 \end{vmatrix}$

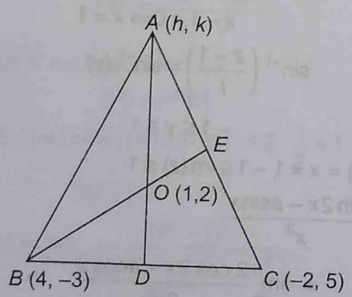
Hence, it is a skew-symmetric matrix of odd order, then

$f(x)=0$

$\therefore \quad \int f(x)\,dx = 0+$ constant

18. (*b*) Let the third vertex be (h, k).

Now, the slope of AO or AD is $\frac{k-2}{h-1}$.



Slope of BC is $\frac{5+3}{-2-4}=-\frac{4}{3}$

Slope of BE is $\frac{-3-2}{4-1}=-\frac{5}{3}$ and slope of AC is $\frac{k-5}{h+2}$.

Since, $AD \perp BC \Rightarrow \frac{k-2}{h-1}\times\left(-\frac{4}{3}\right)=-1$

$\Rightarrow \quad 3h=4k-5 \quad \ldots(i)$

Again, since $BE \perp AC$

$\Rightarrow \quad -\frac{5}{3}\times\frac{k-5}{h+2}=-1$

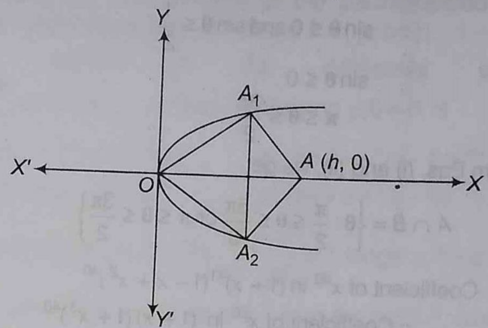
$\Rightarrow \quad 3h=5k-31 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$h=33$

and $\quad k=26$

19. (*c*) Let $A_1=(2t_1^2, 4t_1)$, $A_2=(2t_1^2, -4t_1)$



Clearly, $\angle A_1OA=\frac{\pi}{6} \Rightarrow \frac{2}{t_1}=\frac{1}{\sqrt{3}} \Rightarrow t_1=2\sqrt{3}$

Equation of normal at $A(h, 0)$ is

$y=-t_1x+4t_1+2t_1^3$

$\Rightarrow \quad h=4+2t_1^2=4+2\cdot 12=28$

20. (*d*) Here, $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=1, \mathbf{a}\cdot\mathbf{b}=0$ and

$\cos\theta=\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}$

Now, $\quad \mathbf{c}=\alpha\,\mathbf{a}+\beta\,\mathbf{b}+\gamma\,(\mathbf{a}\times\mathbf{b}) \quad \ldots(i)$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{c}=\alpha(\mathbf{a}\cdot\mathbf{a})+\beta\,(\mathbf{a}\cdot\mathbf{b})+\gamma\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\}$

$\Rightarrow \quad \cos\theta=\alpha|\mathbf{a}|^2 \Rightarrow \cos\theta=\alpha$

Similarly, by taking dot product on both sides of Eq. (i) by $\mathbf{b}$, we get

$\beta=\cos\theta$

$\therefore \quad \alpha=\beta$

$\therefore$ From Eq. (i), $|\mathbf{c}|^2=|\alpha\,\mathbf{a}+\beta\mathbf{b}+\gamma(\mathbf{a}\times\mathbf{b})|^2$

$=\alpha^2|\mathbf{a}|^2+\beta^2|\mathbf{b}|^2+\gamma^2|\mathbf{a}\times\mathbf{b}|^2$

$+2\alpha\beta\,(\mathbf{a}\cdot\mathbf{b})+2\alpha\gamma\,\{\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})\}+2\beta\gamma\,\{\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})\}$

$\Rightarrow \quad 1=\alpha^2+\beta^2+\gamma^2|\mathbf{a}\times\mathbf{b}|^2$

$\Rightarrow \quad 1=2\alpha^2+\gamma^2\left\{|\mathbf{a}|^2\cdot|\mathbf{b}|^2\sin^2\frac{\pi}{2}\right\}$

$\Rightarrow \quad 1=2\alpha^2+\gamma^2$

$\Rightarrow \quad \gamma^2=1-2\cos^2\theta=-\cos 2\theta$

$\Rightarrow \quad \alpha^2=\beta^2=\frac{1-\gamma^2}{2}=\frac{1+\cos 2\theta}{2}$

21. (*a*) Let $81^{\sin^2\theta}=t$

Given, $\quad 81^{\sin^2\theta}+81^{\cos^2\theta}=30$

$\therefore \quad t+\frac{81}{t}=30$

$\Rightarrow \quad t^2-30t+81=0$

$\Rightarrow \quad (t-27)(t-3)=0$

$\Rightarrow \quad t=27, 3$

$\Rightarrow \quad 81^{\sin^2\theta}=3^{4\sin^2\theta}=3^3, 3^1$

$\Rightarrow \quad 4\sin^2\theta=3, 4\sin^2\theta=1$

$\Rightarrow \quad \sin\theta=\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2} \quad \Rightarrow \quad \theta=\frac{\pi}{6}, \frac{\pi}{3}$

22. (c) $\because 1+(1+x)+(1+x+x^2)+\ldots$
$+(1+x+x^2+x^3+\ldots+x^{n-1})+\ldots$

$$=\frac{1}{(1-x)}\{(1-x)+(1-x^2)+(1-x^3)$$
$$+(1-x^4)+\ldots\}+\text{upto } n \text{ terms}$$
$$=\frac{1}{(1-x)}[n-\{(x+x^2+x^3+\ldots+\text{upto } n \text{ terms})\}]$$
$$=\frac{1}{(1-x)}\left[n-\frac{x(1-x^n)}{1-x}\right]=\frac{n(1-x)-x(1-x^n)}{(1-x)^2}$$

23. (a) $\because f(x)=2x+\sin x$

$\therefore\ f'(x)=2+\cos x>0$ for all x

Since, $f(x)$ is strictly increasing. So, f is one-one.

Here, $\lim_{x\to-\infty} f(x)=-\infty$ and $\lim_{x\to\infty} f(x)=\infty$

Hence, f is onto.

24. (b) $\because$ Mean $\bar{x}=\frac{1}{(2n+1)}[a+(a+d)+(a+2d)+\ldots+(a+2nd)]$

$$=\frac{1}{(2n+1)}\left[\frac{2n+1}{2}(a+a+2nd)\right]=a+nd$$

$\therefore$ Mean deviation from mean

$$=\frac{1}{(2n+1)}\sum_{r=0}^{2n}|(a+rd)-(a+nd)|$$
$$=\frac{1}{(2n+1)}\sum_{r=0}^{2n}|(r-n)d|$$
$$=\frac{1}{(2n+1)}\times 2d\,(1+2+\ldots+n)=\frac{n(n+1)}{2n+1}d$$

25. (c) The number of words before the word CRICKET is

$$4\times 5!+2\times 4!+2!=530$$

26. (a) Since, $f(x)+f(1-x)=2$

$\Rightarrow\quad f(x)-1+f(1-x)-1=0$

$\Rightarrow\quad g(x)+g(1-x)=0$

Replacing x by $x+\frac{1}{2}$, we get

$$g\left(x+\frac{1}{2}\right)+g\left(\frac{1}{2}-x\right)=0$$

Hence, it is symmetrical about $\left(\frac{1}{2},0\right)$.

27. (a) Any point on the first line is $(2x_1+1, x_1-3, -3x_1+2)$. Any point on the second line is $(y_1+2, -3y_1+1, 2y_1-3)$. If two lines are coplanar, then $2x_1-y_1=1, x_1+3y_1=4, 3x_1+2y_1=5$ are consistent.

28. (d) Let
$$V=\begin{vmatrix}1 & a & 0\\ a & 1 & 1\\ 0 & 1 & a\end{vmatrix}=a-1-a^3$$

$$\therefore\quad \frac{dV}{da}=1-3a^2=0\Rightarrow a=\pm\frac{1}{\sqrt{3}}$$

Now, $\frac{d^2V}{da^2}=-6a$

$$\Rightarrow\left(\frac{d^2V}{da^2}\right)_{\left(a=\frac{1}{\sqrt{3}}\right)}=-\frac{6}{\sqrt{3}}$$

Hence, it is maximum at $a=\frac{1}{\sqrt{3}}$.

29. (b) We have, $x^3\frac{dy}{dx}+4x^2\tan y=e^x\sec y$

$$\Rightarrow\quad x^2\cos y\frac{dy}{dx}+4x^2\sin y=e^x$$
$$\Rightarrow\quad x^4\cos y\frac{dy}{dx}+4x^2\sin y\,dx=xe^x\,dx$$
$$\Rightarrow\quad d\,(x^4\sin y)=xe^x\,dx$$

On intergrating, we get

$$x^4\sin y=(x-1)e^x+C \quad \ldots\text{(i)}$$

It is given that $y=0$, where $x=1$

On putting $x=1$ and $y=0$ in Eq. (i), we get

$$c=0$$

On putting $C=0$ in Eq. (i), we get

$$x^4\sin y=(x-1)e^x$$
$$\sin y=e^x(x-1)x^{-4}$$

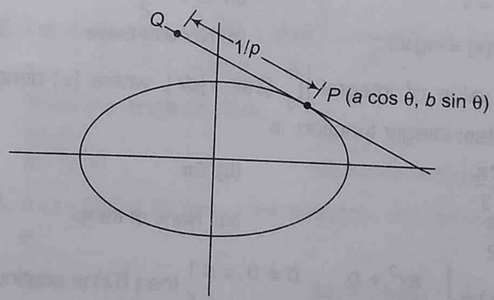
30. (a) Equation of the tangent at P is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}$$

The distance of the tangent from the origin is

$$p=\left|\frac{ab}{\sqrt{b^2\cos\theta+a^2\sin^2\theta}}\right|$$

$$\Rightarrow\quad \frac{1}{p}=\frac{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}{ab}$$



Now, the coordinates of the point Q are given as follows

$$\frac{\frac{x-a\cos\theta}{-a\sin\theta}}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\frac{y-b\sin\theta}{b\cos\theta}}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$
$$=\frac{1}{p}=\frac{\sqrt{b^2\cos^2+a^2\sin^2\theta}}{ab}$$

$$\Rightarrow\quad x=\cos\theta-\frac{a\sin\theta}{ab}\text{ and } y=b\sin\theta+\frac{b\cos\theta}{ab}$$

$$\Rightarrow\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1+\frac{1}{a^2b^2}\text{ is the required locus.}$$

Day 39

Mock TEST 2

(Based on Complete Syllabus)

Instructions

1. The test consists of 30 questions.
2. Candidates will be awarded marks for correct response of each question. 1/4 (one-fourth) marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
3. There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response.

1. If $\int x \log\left(1+\frac{1}{x}\right)dx = f(x)\log(x+1) + g(x)x^2 + xL + C$, then

(a) $L = 1$ (b) $f(x) = \frac{x^2}{2}$
(c) $g(x) = \log x$ (d) None of these

2. The value of integral $\left|\int_0^{\pi} [2\sin x]dx\right|$, where $[x]$ denotes greatest integer function, is

(a) $\frac{2\pi}{3}$ (b) 2π
(c) $\frac{\pi}{2}$ (d) None of these

3. If $f(x) = \begin{cases} ax^2 + b, & b \neq 0, x \leq 1 \\ bx^2 + ax + c & x > 1 \end{cases}$, then $f(x)$ is continuous and differentiable at $x = 1$, if

(a) $c = 0, a = 2b$ (b) $a = b, c \in R$
(c) $a = b, c = 0$ (d) $a = b, c \neq 0$

4. If $\int f(x)\sin x \cos x \, dx = \frac{1}{2(b^2 - a^2)}\log[f(x)] + C$, then $f(x)$ is equal to

(a) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x}$ (b) $\frac{1}{a^2\sin^2 x - b^2\cos^2 x}$
(c) $\frac{1}{a^2\cos^2 x - b^2\sin^2 x}$ (d) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$

5. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is

(a) reflexive but not symmetric
(b) reflexive but not transitive
(c) neither symmetric nor transitive
(d) symmetric and transitive

6. The solution of differential equation $2y\sin x\left(\frac{dy}{dx}\right) = 2\sin x\cos x - y^2\cos x$ at $x = \frac{\pi}{2}$, $y = 1$ is

(a) $y^2 = \sin x$ (b) $y = \sin^2 x$
(c) $y^2 = \cos x + 1$ (d) None of these

7. An experiment succeeds twice as often as it fails. Then, the probability that in the next 4 trials there will be atleast 2 successes, is

(a) $\frac{1}{9}$ (b) $\frac{8}{9}$ (c) $\frac{5}{9}$ (d) $\frac{2}{9}$

8. The value of $\sin\{3\sin^{-1}(0.8)\}$ is

(a) $\sin(2)$ (b) $\sin(1.88)$
(c) $-\sin(0.88)$ (d) None of these

9. For $n \in N, 10^{n-2} \geq 81n$, if

(a) $n > 5$ (b) $n \geq 5$ (c) $n < 5$ (d) $n > 8$

10. The two consecutive terms in the expansion of $(3+2x)^{74}$ whose coefficients are equal, are

(a) 11, 12 (b) 7, 8
(c) 30, 31 (d) None of these

11. A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2-12x-4y+4=0$. The equation of the parabola is

(a) $y^2-8x-6y+25=0$ (b) $y^2-6x+8y-25=0$
(c) $x^2-6x-8y+25=0$ (d) $x^2+6x-8y-25=0$

12. If p, p' denote the lengths of the perpendiculars from the focus and the centre of an ellipse with semi- major axis of length a respectively on a tangent to the ellipse and r denotes the focal distance of the point, then

(a) $ap'=rp+1$ (b) $rp=ap'$
(c) $ap=rp'+\frac{1}{\sqrt{3}}$ (d) $ap=rp'$

13. A variable plane at a constant distances P, from origin cuts axes at A, B and C, then the locus of centroid of tetrahedron $OABC$ is

(a) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{P^2}$ (b) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{2P^2}$
(c) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{3}{P^2}$ (d) None of these

14. The equation of the locus of the pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, of any tangent line to the auxiliary circle is the curve $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\lambda^2$, where

(a) $\lambda^2=a^2$ (b) $\lambda^2=\frac{1}{a^2}$ (c) $\lambda^2=b^2$ (d) $\lambda^2=\frac{1}{b^2}$

15. The solution set for $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2x^5}\le 0$

(a) $\left(-\infty,-\frac{3}{2}\right)\cup\left(0,\frac{4}{3}\right)\cup(4,\infty)$
(b) $\left(-\frac{3}{2},0\right)\cup\left(\frac{4}{3},4\right)$
(c) $(-\infty,0)\cup(2,\infty)$
(d) None of the above

16. A parallelogram is constructed on the vectors $\mathbf{a}=3\alpha-\beta$, $\mathbf{b}=\alpha+3\beta$, if $|\alpha|=|\beta|=2$ and angle between α and β is $\frac{\pi}{3}$, then length of a diagonal of the parallelogram is

(a) $4\sqrt{5}$ (b) $4\sqrt{3}$
(c) $4\sqrt{17}$ (d) None of these

17. $f(x)=x-\cot^{-1}x-\log(x+\sqrt{1+x^2})$ is increasing in

(a) $(-\infty,\infty)$ (b) $(-\infty,2)$
(c) $(2,5)$ (d) $(-\infty,10)$

18. The contrapositive of the statement, 'if x is a prime number and x divides ab, then x divides a or x divides b'. Can be symbolically represented using logical connectives, on appropriately defined statements p, q, r and s as appropriately defined statements p, q, r and s as

(a) $(\sim r\vee\sim s)\to(\sim p\wedge\sim q)$
(b) $(r\wedge s)\to(\sim p\wedge\sim q)$
(c) $(\sim r\wedge\sim s)\to(\sim p\vee\sim q)$
(d) $(r\vee s)\to(\sim p\vee\sim q)$

19. If [.] denotes the greatest integer less than or equal to the real number under consideration and $-1\le x<0$; $0\le y<1$ and $1\le z<2$, then the value of the determinant $\begin{vmatrix}[x]+1 & [y] & [z]\\ [x] & [y]+1 & [z]\\ [x] & [y] & [z]+1\end{vmatrix}$ is

(a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

20. Area of a triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2), where a, x_1 and x_2 are in GP with common ratio r and b, y_1 and y_2 are in GP with common ratio s, is given by

(a) $\frac{1}{2}ab(r-1)(s-1)(s-r)$ (b) $\frac{1}{2}ab(r+1)(s+1)(s-r)$
(c) $ab(r-1)(s-1)(s-r)$ (d) None of these

21. The equation of perpendicular bisectors of sides AB and AC of a ΔABC are $x-y+5=0$ and $x+2y=0$, respectively. If the coordinates of vertex A are $(1,-2)$, then equation of BC is

(a) $14x+23y-40=0$ (b) $14x-23y+40=0$
(c) $23x+14y-40=0$ (d) $23x-23y+40=0$

22. Let $A=\begin{bmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{bmatrix}$, where $0\le\theta\le 2\pi$.

Then, the range of $|A|$ is

(a) 0 (b) {2, 4}
(c) [2, 4] (d) None of these

23. If $|z-25i|\le 15$, then $|\max \text{amp}(z)-\min \text{amp}(z)|$ is equal to

(a) $\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\pi-2\cos^{-1}\left(\frac{3}{5}\right)$
(c) $\frac{\pi}{2}+\cos^{-1}\left(\frac{3}{5}\right)$ (d) $\sin^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{3}{5}\right)$

24. If $ABCD$ is a cyclic quadrilateral such that $12\tan A-5=0$ and $5\cos B+3=0$, then the quadratic equation whose roots are $\cos C$ and $\tan D$ is

(a) $39x^2+88x+48=0$ (b) $39x^2-16x-48=0$
(c) $39x^2-88x+48=0$ (d) None of the above

25. A mirror and a source of light are situated at the origin O and a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the DR's of the normal to the plane of mirror are 1, −1, 1, then DC's for the reflected ray are

(a) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$
(b) $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$
(c) $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$
(d) $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

26. n locks and 'n' corresponding keys are available. But the actual combination is not know. The maximum numbers of trials that are needed to assign the key to their corresponding locks are

(a) ${}^nC_2$
(b) ${}^nC_3$
(c) $n!$
(d) ${}^{n+1}C_2$

27. The value of the expression $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1-\sin x}$ is/are

(a) $\sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$
(b) $\sqrt{2}\cos\left(\frac{\pi}{4} + x\right)$
(c) $\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)$
(d) None of these

Directions (Q. Nos. 28 to 30) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

28. Consider matrix $[A]_{3\times 3}$ is defind as $a_{ij} = \frac{i-j}{i+2j}$

Statement I The matrix A cannot be expressed as a sum of symmetric and skew-symmetric matrix.

Statement II Matrix A is neither symmetric nor skew-symmetric matrix.

29. Suppose, $f(x) = x\sin x - \frac{1}{2}\sin^2 x, x \in \left(0, \frac{\pi}{2}\right)$

Statement I Range of $f(x)$ is $\left(0, \frac{\pi-1}{2}\right)$.

Statement II Range of $f(x)$ is not determined.

30. Statement I If both roots of the equation $4x^2 - 2x + a = 0$, $a \in R$ lies in the interval (−1, 1), then $-2 < a \le \frac{1}{4}$.

Statement II If $f(x) = 4x^2 - 2x + a$, then $D \ge 0$, $f(-1) > 0$ and $f(1) > 0 \Rightarrow -2 < a \le \frac{1}{4}$.

Answer with Solutions

1. (*d*) Let $I = \int x\log\left(1 + \frac{1}{x}\right)dx$

$$= \log\left(1+\frac{1}{x}\right)\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{\left(1+\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)dx$$

$$= \frac{x^2}{2}\log\left(1+\frac{1}{x}\right) + \frac{1}{2}\int\frac{x}{(1+x)}dx$$

$$= \frac{x^2}{2}\log\left(1+\frac{1}{x}\right) + \frac{1}{2}\left(\int dx - \int\frac{1}{1+x}dx\right)$$

$$= \frac{x^2}{2}\log\left(\frac{x+1}{x}\right) + \frac{1}{2}x - \log(1+x) + C$$

$$= \log(1+x)\left(\frac{x^2}{2} - 1\right) - \frac{x^2}{2}\log x + \frac{x}{2} + C$$

$\therefore \quad f(x) = \left(\frac{x^2}{2} - 1\right), g(x) = -\frac{\log x}{2}$ and $L = \frac{1}{2}$

2. (*a*) Let $I = \int_0^{\pi}[2\sin x]dx = \int_0^{\pi/6}[2\sin x]dx + \int_{\pi/6}^{\pi/2}[2\sin x]dx + \int_{\pi/2}^{5\pi/6}[2\sin x]dx + \int_{5\pi/6}^{\pi}[2\sin x]dx$

$$= 0 + \int_{\pi/6}^{\pi/2}1\,dx + \int_{\pi/2}^{5\pi/6}1\,dx + 0$$

$$= [x]_{\pi/6}^{\pi/2} + [x]_{\pi/2}^{5\pi/6}$$

$$= \frac{\pi}{2} - \frac{\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{3}$$

3. (*a*) Given that, $f(x) = \begin{cases} ax^2 + b, & b \ne 0, x \le 1 \\ bx^2 + ax + c, & x > 1 \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} 2ax, & b \ne 0, x < 1 \\ 2bx + a, & x > 1 \end{cases}$

Since, $f(x)$ is continuous at $x = 1$.

$\therefore \quad \lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \quad a + b = b + a + c \Rightarrow c = 0$

Also, $f(x)$ is differentiable at $x = 1$.

$\therefore \quad$ (LHD at $x = 1$) = (RHD at $x = 1$)

$\Rightarrow \quad 2a = 2b(1) + a \Rightarrow a = 2b$

4. (*a*) Given that, $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2 - a^2)}\log[f(x)] + C$

On differentiating both sides, we get

$$f(x)\sin x\cos x = \frac{d}{dx}\left\{\frac{\log[f(x)]}{2(b^2-a^2)}+C\right\}$$

$$\Rightarrow \quad f(x)\sin x\cos x = \frac{1}{2(b^2-a^2)}\cdot\frac{1}{f(x)}f'(x)$$

$$\Rightarrow \quad 2(b^2-a^2)\sin x\cos x = \frac{f'(x)}{[f(x)]^2}$$

On integrating both sides, we get

$$-b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$$

$$\therefore \quad f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$$

5. (*a*) Since, (1, 1), (2, 2), (3, 3) $\in R$, therefore R is reflexive.
Again, (1, 2) $\in R$, but (2, 1) $\notin R$, therefore R is not symmetric.
Now, (1, 2), (2, 3) $\in R$
$\Rightarrow$ (1, 3) $\in R$, so R is transitive.

6. (*a*) Given, equation can be rewritten as

$$2y\frac{dy}{dx} + y^2\cot x = 2\cos x$$

Put $\quad y^2 = v \Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \quad \frac{dv}{dx} + v\cot x = 2\cos x$$

$$\therefore \quad \text{IF} = e^{\int\frac{\cos x}{\sin x}dx} = e^{\log\sin x} = \sin x$$

$\therefore$ Solution is $v\sin x = \int 2\cos x\sin x\,dx + C$

$$\Rightarrow \quad y^2\sin x = \sin^2 x + C$$

when $x = \frac{\pi}{2}$, $y = 1$

$$1 = 1 + C \Rightarrow C = 0$$

$$\therefore \quad y^2 = \sin x$$

7. (*b*) Given, $p = 2q$

$\because \quad p + q = 1 \quad \Rightarrow \quad p = \frac{2}{3}$ and $q = \frac{1}{3}$

$\therefore$ Required probability

$$= {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + {}^4C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 + {}^4C_4\left(\frac{2}{3}\right)^4$$

$$= 6\times\frac{4}{81} + \frac{4\times 8}{81} + \frac{1\times 16}{81} = \frac{72}{81} = \frac{8}{9}$$

8. (*b*) $\sin\{3\sin^{-1}(0.8)\} = \sin\{3(0.8) - (0.8)^3\}$

$$= \sin(2.4 - 0.512) = \sin(1.88)$$

9. (*b*) Let $P(n): 10^{n-2} \geq 81n$

For $n = 4, 10^2 \ngeq 81\times 4$

For $n = 5, 10^3 \geq 81\times 5$

Hence, by mathematical induction for $n \geq 5$ the proposition is true.

10. (*c*) General term of $(3+2x)^{74}$ is

$$T_{r+1} = {}^{74}C_r(3)^{74-r}2^r x^r$$

Let two consecutive terms are T_{r+1}th and T_{r+2}th terms.
According to the question,
Coefficient of T_{r+1} = Coefficient of T_{r+2}

$$\Rightarrow \quad {}^{74}C_r 3^{74-r}2^r = {}^{74}C_{r+1}3^{74-(r+1)}2^{r+1}$$

$$\Rightarrow \quad \frac{{}^{74}C_{r+1}}{{}^{74}C_r} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{74-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow \quad 148 - 2r = 3r + 3$$

$$\therefore \quad r = 29$$

Hence, two consecutive terms are 30 and 31.

11. (*c*) Given, equation can be rewritten as

$$(y-2)^2 = 12x$$

Here, vertex and focus are (0, 2) and (3, 2).
$\therefore$ Vertex of the required parabola is (3, 2) and focus is (3, 4).
The axis of symmetry is $x = 3$ and latusrectum $= 4\cdot 2 = 8$
Hence, required equation is

$$(x-3)^2 = 8(y-2)$$

$$\Rightarrow \quad x^2 - 6x - 8y + 25 = 0$$

12. (*d*) Tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \ldots\text{(i)}$$

$\therefore$ p = perpendicular distance from focus $(ae, 0)$ to the line (i)

$$= \frac{\left|\frac{ae}{a}\cos\theta + 0 - 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} = \frac{1-e\cos\theta}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} \quad \ldots\text{(ii)}$$

Also, p' = perpendicular distance from centre (0, 0) to the line (i)

$$= \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} \quad \ldots\text{(iii)}$$

Again, $\quad r = SP = a(1 - e\cos\theta)$

$$\therefore \quad ap = \frac{a - ae\cos\theta}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} = rp'$$

13. (*a*) Let the centroid of tetrahedron $OABC$ is (α, β, γ) such that

$$\alpha = \frac{0+a+0+0}{4}, \beta = \frac{0+0+b+0}{4}, \gamma = \frac{0+0+0+c}{4}$$

$$\Rightarrow \quad a = 4\alpha, b = 4\beta, c = 4\gamma$$

So, distance of plane from origin is P,

$$P=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{P^2}$$

On putting the values of a, b and c, then

$$\frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}=\frac{1}{P^2}$$

Hence, the locus of centroid is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{P^2}$$

14. (*b*) The equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Let (h, k) be the pole, then equation of the polar of (h, k) with respect to the given ellipse is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

Since, this is tangent to the circle.

$$\frac{|0+0-1|}{\sqrt{\left(\frac{h}{a^2}\right)^2+\left(\frac{k}{b^2}\right)^2}}=\pm a$$

$$\Rightarrow \frac{h^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$$

Hence, locus of (h, k) is

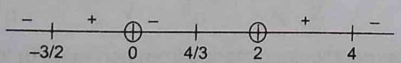
$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}$$

15. (*d*) Given inequality can be rewritten as

$$\frac{(2x+3)(3x-4)^3(x-4)}{(x-2)^2x^5}\ge 0$$

$\Rightarrow (2x+3)(3x-4)^3(x-4)(x-2)^2x^5 \ge 0, x \ne 0, 2$

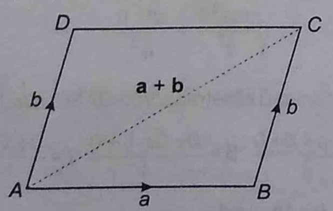
$$x=-\frac{3}{2}, \frac{4}{3}, 4, 2, 0$$



In a sign scheme when we multiply the sign in each interval, we get the positive interval.

$$\left(-\infty, -\frac{3}{2}\right] \cup \left(0, \frac{4}{3}\right] \cup [4, \infty)$$

16. (*b*) $\because \mathbf{AC} = \mathbf{a} + \mathbf{b}$



$\Rightarrow |\mathbf{AC}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}$

$$\Rightarrow |\mathbf{AC}|^2 = \{|3\alpha-\beta|^2 + |\alpha+3\beta|^2 + 2(3\alpha-\beta)(\alpha+3\beta)\}$$
$$= 9\alpha^2 + \beta^2 - 6\alpha\beta + \alpha^2 + 9\beta^2 + 6\alpha\cdot\beta$$
$$+ 6\alpha^2 - 6\beta^2 + 16\alpha\cdot\beta$$
$$\Rightarrow |\mathbf{AC}|^2 = 16\alpha^2 + 4\beta^2 + 16\alpha.\beta$$
$$\Rightarrow |\mathbf{AC}|^2 = 64 + 16 + 16|\alpha||\beta|\cos\frac{\pi}{3}$$
$$\Rightarrow |\mathbf{AC}|^2 = 64 + 16 + 16\times 2\times 2\times\frac{1}{2}$$
$$\Rightarrow |\mathbf{AC}| = 4\sqrt{7}$$

Similarly, $|\mathbf{BD}| = |\mathbf{a} - \mathbf{b}| = 4\sqrt{3}$

17. (*a*) Given that,

$$f(x)=x-\cot^{-1}x-\log(x+\sqrt{1+x^2})$$

On differentiating w.r.t. x, we get

$$f'(x)=1+\frac{1}{1+x^2}-\frac{1}{(x+\sqrt{1+x^2})}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$
$$=1+\frac{1}{1+x^2}-\frac{1}{\sqrt{1+x^2}}$$
$$\Rightarrow \frac{1+x^2+1-\sqrt{1+x^2}}{1+x^2}>0, \forall x\in R$$
$$\Rightarrow \frac{2+x^2-\sqrt{1+x^2}}{1+x^2}>0, \forall x\in R$$

So, $f(x)$ is an increasing function in $(-\infty, \infty)$.

18. (*c*) Let $p = x$ is a prime number

$q = x$ divides ab

$r = x$ divides a

and $s = x$ divides b

The given statement becomes in logical form

$$p \wedge q \to r \vee s$$

Its contrapositive is

$$\sim(r \vee s) \to \sim(p \wedge q)$$
$$\Rightarrow (\sim r \wedge \sim s) \to (\sim p \vee \sim q)$$

19. (*c*) Since, $-1 \le x < 0$, then $[x] = -1$

Also, $0 \le y < 1, [y] = 0$

and $1 \le z < 2, [z] = 1$

Therefore, given determinant becomes

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

20. (*a*) Since, a, x_1 and x_2 are in GP with common ratio r.

$\therefore \quad x_1 = ar, x_2 = ar^2$

Also, b, y_1 and y_2 are in GP with common ratio s.

$\therefore \quad y_1 = bs, y_2 = bs^2$

The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab\begin{vmatrix} 1 & 1 & 1 \\ r & s & 1 \\ r^2 & s^2 & 1 \end{vmatrix}$$

$$= \frac{ab}{2}\begin{vmatrix} 1 & 0 & 0 \\ r & s-r & 1-r \\ r^2 & s^2-r^2 & 1-r^2 \end{vmatrix}$$

$$= \frac{ab}{2}\{(s-r)(1-r^2)-(1-r)(s^2-r^2)\}$$

$$= \frac{ab}{2}(s-r)(1-r)\{1+r-(s+r)\}$$

$$= \frac{ab}{2}(s-r)(1-r)(1-s)$$

$$= \frac{ab}{2}(s-r)(r-1)(s-1)$$

21. (*a*) Let $B(x_1, y_1)$ and $C(x_2, y_2)$ be two vertices and $P\left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$ lies on perpendicular bisector

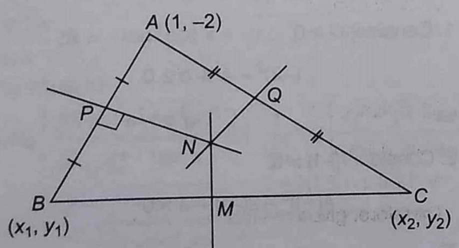
$$x - y + 5 = 0.$$

$$\therefore \quad \frac{x_1+1}{2} - \frac{y_1-2}{2} = -5$$

$$\Rightarrow \quad x_1 - y_1 = -13 \quad \ldots\text{(i)}$$

Also, PN is perpendicular to AB.

$$\therefore \quad \frac{y_1+2}{x_1-1} \times 1 = -1$$



$$\Rightarrow \quad y_1 + 2 = -x_1 + 1$$

$$\Rightarrow \quad x_1 + y_1 = -1 \quad \ldots\text{(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x_1 = 7, y_1 = 6$$

So, the coordinates of B are (−7, 6).

Similarly, the coordinates of C are $\left(\frac{11}{5}, \frac{2}{5}\right)$.

Hence, equation of BC is $y - 6 = \dfrac{\frac{2}{5}-6}{\frac{11}{5}+7}(x+7)$

$$\Rightarrow \quad y - 6 = -\frac{14}{23}(x+7)$$

$$\Rightarrow \quad 14x + 23y - 40 = 0$$

22. (*c*) Since, $|A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

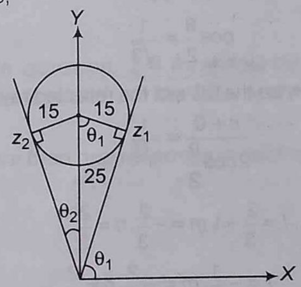
$$\Rightarrow \quad |A| = 2(1+\sin^2\theta)$$

Now, $\quad 0 \le \sin^2\theta \le 1$, for all $\theta \in [0, 2\pi]$.

$$\Rightarrow \quad 2 \le 2 + 2\sin^2\theta \le 4, \text{ for all } \theta \in [0, 2\pi)$$

So, the range of $|A|$ is [2, 4].

23. (*b*) We have,



$$\max \text{amp}(z) = \text{amp}(z_2) \text{ and } \min \text{amp}(z) = \text{amp}(z_1)$$

Now, $\quad \text{amp}(z_1) = \theta_1 = \cos^{-1}\left(\frac{15}{25}\right) = \cos^{-1}\left(\frac{3}{5}\right)$

$$\text{amp}(z_2) = \frac{\pi}{2} + \theta_2 = \frac{\pi}{2} + \sin^{-1}\left(\frac{15}{25}\right) = \frac{\pi}{2} + \sin^{-1}\left(\frac{3}{5}\right)$$

$$\therefore \quad |\max \text{amp}(z) - \min \text{amp}(z)|$$

$$= \left|\frac{\pi}{2} + \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right|$$

$$= \left|\frac{\pi}{2} + \frac{\pi}{2} - \cos^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right| = \pi - 2\cos^{-1}\left(\frac{3}{5}\right)$$

24. (*b*) $\because \tan A = \frac{5}{12} = -\tan C, \cos B = -\frac{3}{5} = -\cos D$

In cyclic quadrilateral,

$$A + C = \pi, B + D = \pi$$

$$\therefore \quad \tan C = -\frac{5}{12} \Rightarrow \cos C = -\frac{12}{13} = \alpha$$

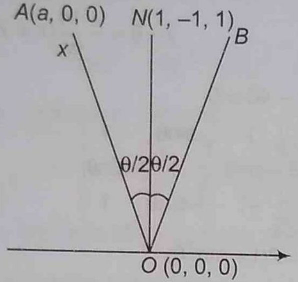
$$\cos D = \frac{3}{5} \Rightarrow \tan D = \frac{4}{3} = \beta$$

$\therefore$ Required equation is $x^2 - (\alpha + \beta)x + \alpha\beta x = 0$

$$x^2 - \left(-\frac{12}{13} + \frac{4}{3}\right)x + \left(-\frac{12}{13} \times \frac{4}{3}\right) = 0$$

$$\Rightarrow \quad 39x^2 - 16x - 48 = 0$$

25. (*d*) Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$. Let AO be the incident ray and OB be the reflected ray, ON is the normal to the mirror at O.



$\therefore \quad \angle AON = \angle NOB = \dfrac{\theta}{2}$ (say)

DR's of OA are $(a, 0, 0)$ and so its DC's are $(1, 0, 0)$.

DC's of ON are $\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

$$\therefore \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

Let l, m and n, be the DC's of the reflected ray OB.

Then, $$\frac{n+0}{2\cos\frac{\theta}{2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad l = \frac{2}{3} - 1, m = -\frac{2}{3}, n = \frac{2}{3}$$

$$\Rightarrow \quad l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$$

Hence, DC's of the reflected ray are $-\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}$.

26. (*a*) Firstly key will be tried for at the most $(n-1)$ locks. Second key will be tried at the most $(n-2)$ locks and so on. Thus, the maximum number of trials needed.

$$= (n-1) + (n-2) + \dots\dots + 1$$
$$= \frac{n(n-1)}{2}$$
$$= {}^nC_2$$

27. (*a*) Let $A = \dfrac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1+\cos x)(1-\sin x)}$

$$= \frac{(\sin^3 x + \cos^3 x) + (\cos x + \sin x)(\cos^2 x + \sin^2 x)(\cos x - \sin x)}{(1+\cos x)(1-\sin x)}$$

$$= \frac{(\sin x + \cos x)\{(1 - \sin x \cos x) + (\cos x - \sin x)\}}{1 + \cos x - \sin x \cos x}$$

$$\Rightarrow \quad A = \sin x + \cos x \quad \dots(i)$$

$$= \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{4} + x\right)$$

Again from Eq. (i), we get

$$A = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4} - x\right)$$

28. (*d*) We have,

$$a_{ij} = \frac{i-j}{i+2j}$$

$$\therefore \quad A = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{2}{7} \\ \frac{1}{4} & 0 & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{7} & 0 \end{bmatrix}$$

which is neither symmetric nor skew-symmetric but every matrix can be expressed as a sum of symmetric and skew-symmetric matrix.

29. (*c*) Let $f(x) = x\sin x - \dfrac{1}{2}\sin^2 x$

$$f'(x) = x\cos x + \sin x - \sin x \cos x$$
$$= \sin x(1 - \cos x) + x\cos x$$

For $x \in \left(0, \dfrac{\pi}{2}\right)$, $\sin x > 0, (1 - \cos x) > 0, \cos x > 0$

$$\Rightarrow \quad f'(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$\Rightarrow f(x)$ is strictly increasing in $\left(0, \dfrac{\pi}{2}\right)$.

Now, $\quad \lim_{x \to 0} f(x) = 0$

and $\quad \lim_{x \to \frac{\pi}{2}} f(x) = \dfrac{\pi - 1}{2}$

$$\therefore \quad \text{Range of } f(x) = \left(0, \frac{\pi - 1}{2}\right)$$

30. (*a*) Let $f(x) = 4x^2 - 2x + a$

Since, both roots of $f(x) = 0$ lie in the interval $(-1, 1)$, we can take $D \geq 0, f(-1) > 0$ and $f(1) > 0$

1. Consider $D \geq 0$,

$$(-2)^2 - 4 \cdot 4 \cdot a \geq 0$$
$$\Rightarrow \quad a \leq 1/4 \quad \dots(i)$$

2. Consider $f(-1) > 0$,

$$4(-1)^2 - 2(-1) + a > 0$$
$$\Rightarrow \quad a > -6 \quad \dots(ii)$$

3. Consider $f(1) > 0$,

$$4(1)^2 - 2(1) + a > 0$$
$$\Rightarrow \quad a > -2 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$-2 < a \leq 1/4$$

Day 40

Mock TEST 3

(Based on Complete Syllabus)

Instructions

1. *The test consists of 30 questions.*
2. *Candidates will be awarded marks for correct response of each question. 1/4 (one-fourth) marks will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.*
3. *There is only one correct response for each question. Filling up more than one response in each question will be treated as wrong response.*

1. Two circles in complex plane are

$C_1 : |z - i| = 2$

$C_2 : |z - 1 - 2i| = 4$. Then,

(a) C_1 and C_2 touch each other
(b) C_1 and C_2 intersect at two distinct points
(c) C_1 lies within C_2
(d) C_2 lies within C_1

2. If $\int \frac{\sqrt{\cos 2x}}{\sin x}\, dx = -\log|\cot x + \sqrt{\cot^2 x - 1}| + A + C$, then A is equal to

(a) $\frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}}\right|$ (b) $\frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2} + \sqrt{1 - \sec^2 x}}{\sqrt{2} - \sqrt{1 - \sec^2 x}}\right|$

(c) $\frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2} + \sqrt{1 - \sin^2 x}}{\sqrt{2} - \sqrt{1 - \sin^2 x}}\right|$ (d) None of the above

3. If m_1 and m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$, then the area of the triangle formed by the lines $y = m_1 x$, $y = -m_2 x$ and $y = 1$ is

(a) $\frac{1}{2}\left(\frac{\sqrt{3} + 2}{\sqrt{3} - 1}\right)$ (b) $\frac{1}{2}\left(\frac{\sqrt{3} + 2}{\sqrt{3} + 1}\right)$

(c) $\frac{1}{2}\left(\frac{-\sqrt{3} + 2}{\sqrt{3} - 1}\right)$ (d) None of these

4. The area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$, is

(a) $\left(\frac{4}{\log 2} + \frac{8}{3}\right)$ sq units (b) $\left(\frac{4}{\log 2} - \frac{8}{3}\right)$ sq units

(c) $\left(\frac{8}{\log 3} - \frac{4}{3}\right)$ sq units (d) $\left(\frac{3}{\log 2} - \frac{4}{3}\right)$ sq units

5. A circle is drawn to pass through the extremities of the latusrectum of the parabola $y^2 = 8x$. It is given that, this circle also touches the directrix of the parabola. The radius of this circle is equal to

(a) 4 (b) $\sqrt{21}$
(c) 3 (d) $\sqrt{26}$

6. The equation of the ellipse whose axes are coincident with the coordinate axes and which touches the straight lines $3x - 2y - 20 = 0$ and $x + 6y - 20 = 0$, is

(a) $\frac{x^2}{5} + \frac{y^2}{8} = 1$

(b) $\frac{x^2}{40} + \frac{y^2}{10} = 10$

(c) $\frac{x^2}{40} + \frac{y^2}{10} = 1$

(d) $\frac{x^2}{10} + \frac{y^2}{40} = 1$

Directions (Q. Nos. 7 to 9) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

7. The product of odd and even function is an add function.

Statement I The function $f(x) = x^2 e^{-x^2} \sin|x|$ is an even function.

Statement II Product of two odd functions is an even function.

8. Statement I If $\mathbf{r} \cdot \mathbf{a} = 0, \mathbf{r} \cdot \mathbf{b} = 0, \mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar vectors.

Statement II If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

9. If three points are collinear, then there is no circle passing through these points.

Statement I Number of circles passing through (1, 2), (4, 8) and (0, 0) is one.

Statement II Every triangle has one circumcircle.

10. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, (where $pq \neq 0$) are bisected by the x-axis, then

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

11. If A and B are different matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$, then

(a) det $(A^2 + B^2)$ must be zero
(b) det $(A - B)$ must be zero
(c) det $(A^2 + B^2)$ as well as det $(A - B)$ must be zero
(d) Atleast one of det $(A^2 + B^2)$ or det $(A - B)$ must be zero

12. A fair coin is tossed 100 times. The probability of getting tails 1, 3, ..., 49 times is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) $\frac{1}{8}$ (d) $\frac{1}{16}$

13. If $f'(x) = \sin(\log x)$ and $y = f\left(\frac{2x+3}{3-2x}\right)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) 6 sin log (5) (b) 5 sin log (6)
(c) 12 sin log (5) (d) 5 sin log (12)

14. The distance between the line

$$\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

and the plane $\mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ is

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$
(c) $\frac{10}{3\sqrt{3}}$ (d) $\frac{10}{9}$

15. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is –1, is

(a) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = 1$
(b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$
(c) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$
(d) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

16. Which of the following statements is negation of the statement "All differentiable functions are continuous"?

(a) It is not true that all differentiable functions are continuous
(b) Not all differentiable functions are continuous
(c) There exist a differentiable function which is not continuous
(d) All of the above

17. If x, y be reals satisfying $\sin x + \cos y = 1$ and $\sin y + \cos x = -1$. Then, which of the following option must be correct?

(a) $\sin(x + y) = 0$ (b) $\cos(x + y) = 0$
(c) $\cos(x - y) = 0$ (d) $\cos 2x = \cos y$

18. The value of $\tan\left\{\cos^{-1}\left(\frac{-2}{7}\right) - \frac{\pi}{2}\right\}$ is

(a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

19. A boat is being rowed away from a cliff of 150 m height. At the top of the cliff the angle of depression of boat changes from 60° to 45° in 2 min. Then, the speed of the boat (in m/h) is

(a) $\frac{4500}{\sqrt{3}}$ (b) $\frac{4500}{\sqrt{3}}(\sqrt{3} - 1)$
(c) $\frac{4300}{\sqrt{3}}$ (d) $\frac{4500}{\sqrt{3}}(\sqrt{3} + 1)$

20. The equation of a line of intersection of planes $4x + 4y - 5z = 12$ and $8x + 12y - 13z = 32$ can be written as

(a) $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{4}$ (b) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$
(c) $\frac{x}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ (d) $\frac{x}{2} = \frac{y}{3} = \frac{z-2}{4}$

21. If $f(x)$ is a function satisfying $f(x+y)=f(x)f(y)$ for all $x, y \in N$ such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$, then the value of n is

(a) 4 (b) 5
(c) 6 (d) None of these

22. If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then

(a) $abc > 1$ (b) $abc > -8$
(c) $abc < -8$ (d) $abc > -2$

23. A bag contains a white and b black balls. Two players A and B alternately draw a ball from the bag, replacing the ball each time after the draw. A begins the game.

If the probability of A winning (that is drawing a white ball) is twice the probability of B winning (that is drawing a black ball), then the ratio $a:b$ is equal to

(a) 1 : 2 (b) 2 : 1
(c) 1 : 1 (d) None of these

24. Number of solutions of the equation $\sin x = [x]$, where [.] denotes the largest integer function, is

(a) 0 (b) 1
(c) 2 (d) None of these

25. If $(5+2\sqrt{6})^n = I + f;\ n, I \in N$ and $0 \le f < 1$, then I equals to

(a) $\frac{1}{f} - f$ (b) $\frac{1}{1+f} - f$
(c) $\frac{1}{1+f} + f$ (d) $\frac{1}{1-f} - f$

26. The value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is

(a) 0 (b) $\frac{2}{105}$ (c) $\frac{22}{7} - \pi$ (d) $\frac{71}{15} - \frac{3\pi}{2}$

27. The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$ is

(a) $18\sqrt{3}$ (b) $10\sqrt{3}$
(c) $11\sqrt{3}$ (d) $12\sqrt{3}$

28. The median of a set of 11 distinct observations is 20.5. If each of the last 5 observations of the set is increased by 4, then the median of the new set

(a) is increased by 2
(b) is decreased by 2
(c) is two times the original a median
(d) remains the same as that of the original set

29. Solution of the differential equation $\frac{\sqrt{x}dx + \sqrt{y}dy}{\sqrt{x}dx - \sqrt{y}dy} = \sqrt{\frac{y^3}{x^3}}$ is given by

(a) $\frac{3}{2}\log\left(\frac{y}{x}\right) + \log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right| + \tan^{-1}\left(\frac{y}{x}\right)^{3/2} + c = 0$

(b) $\frac{2}{3}\log\left(\frac{y}{x}\right) + \log\left|\frac{x^{3/2}+y^{3/2}}{x^{3/2}}\right| + \tan^{-1}\left(\frac{y}{x}\right) + c = 0$

(c) $\frac{2}{3}\log\left(\frac{y}{x}\right) + \log\left(\frac{x+y}{x}\right) + \tan^{-1}\left(\frac{y^{3/2}}{x^{3/2}}\right) + c = 0$

(d) None of the above

30. If the function $f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$, $x \ne 0$ is continuous at each point of its domain, them the value of $f(0)$ is

(a) 2 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{1}{3}$

Answer *with* Solutions

1. (*c*) Given equations can be rewritten as

$$x^2 + (y-1)^2 = 2^2$$

and $$(x-1)^2 + (y-2)^2 = 4^2$$

Here, centres are $C_1\ (0, 1)$ and $C_2\ (1, 2)$ and radii are $r_1 = 2$, $r_2 = 4$.

Now, $$C_1C_2 = \sqrt{(1-0)^2 + (2-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\because$ $$|r_1 - r_2| = |2-4| = 2$$

$$C_1C_2 < |r_1 - r_2|$$

2. (*a*) Let $$I = \int \sqrt{\frac{\cos 2x}{\sin^2 x}}dx = \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}dx = \int \sqrt{\cot^2 x - 1}\,dx$$

On putting $\cot x = \sec\theta$

and $-\text{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$, we get

$$I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta\tan\theta}{-\text{cosec}^2 x}d\theta = -\int \frac{\sec\theta\tan^2\theta}{1+\sec^2\theta}d\theta$$

$$= -\int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= -\int \frac{(1+\cos^2 \theta) - 2\cos^2 \theta}{\cos \theta\,(1+\cos^2 \theta)} d\theta$$

$$= -\int \sec \theta\, d\theta + 2\int \frac{d(\sin \theta)}{1+\cos^2 \theta}$$

$$= -\log|\sec \theta + \sqrt{\sec^2 \theta - 1}|$$

$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1-\cos^2 \theta}}{\sqrt{2} - \sqrt{1-\cos^2 \theta}} \right| + C$$

$$= -\log|\cot x + \sqrt{\cot^2 x - 1}|$$

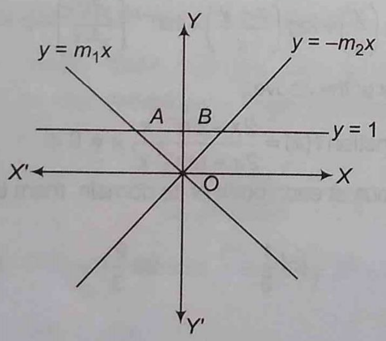
$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1-\tan^2 x}}{\sqrt{2} - \sqrt{1-\tan^2 x}} \right| + C$$

But $I = -\log|\cot x + \sqrt{\cot^2 x - 1}| + A + C$ (given)

$$\therefore \quad A = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1-\tan^2 x}}{\sqrt{2} - \sqrt{1-\tan^2 x}} \right|$$

3. (*a*) Since, m_1 and m_2 are the roots of the equation

$$x^2 + (\sqrt{3}+2)x + (\sqrt{3}-1) = 0.$$



$\therefore \quad m_1 + m_2 = -(2+\sqrt{3})$

and $\quad m_1 m_2 = \sqrt{3} - 1$

$\Rightarrow \quad m_1 < 0, m_2 < 0$

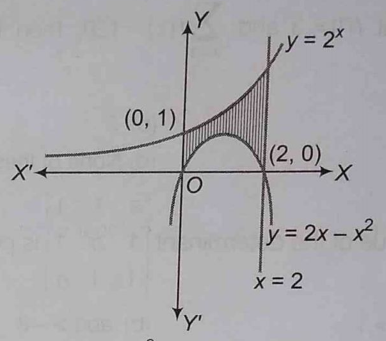
So, the point of intersections are $A\left(\frac{1}{m_1}, 1\right)$ and $B\left(-\frac{1}{m_2}, 1\right)$ and $O(0, 0)$.

$$\therefore \text{Area of } \Delta OAB = -\frac{1}{2} \left| \begin{vmatrix} \frac{1}{m_1} & 1 & 1 \\ \frac{-1}{m_2} & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \frac{1}{m_1} + \frac{1}{m_2} \right|$$

$$= \frac{1}{2} \left| \frac{m_1 + m_2}{m_1 m_2} \right| = \frac{1}{2} \left(\frac{2+\sqrt{3}}{\sqrt{3}-1} \right)$$

4. (*d*) $\therefore$ Required area $= \int_0^2 (y_1 - y_2)\, dx$



$$= \int_0^2 (2^x - 2x + x^2)\, dx$$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3} \right]_0^2$$

$$= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2}$$

$$= \left(\frac{3}{\log 2} - \frac{4}{3} \right) \text{ sq units}$$

5. (*a*) Extremities of the latusrectum of the parabola are (2, 4) and (2, − 4).

Since, any circle drawn with any focal chord at its diameter touches the directrix, thus equation of required circle is

$$(x+2)(x-2)^2 + (y-4)(y+4) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 12 = 0$$

$$\therefore \quad \text{Radius} = \sqrt{(2)^2 + 12} = 4$$

6. (*c*) The general equation of the tangent to the ellipse is

$$y = mx \pm \sqrt{a^2 m^2 + b^2} \quad \ldots \text{(i)}$$

Since, $y = \frac{3}{2}x - 10$ is a tangent to the ellipse, therefore its comparing with Eq. (i), we get

$$m = \frac{3}{2} \quad \text{and} \quad a^2 m^2 + b^2 = 100$$

$$\Rightarrow \quad 9a^2 + 4b^2 = 400 \quad \ldots \text{(ii)}$$

Similarly, line $\quad y = -\frac{1}{6}x + \frac{10}{3}$

is a tangent to the ellipse, therefore its comparing with Eq. (i), we get

$$m = -\frac{1}{6} \quad \text{and} \quad a^2 m^2 + b^2 = \frac{100}{9}$$

$$\Rightarrow \quad a^2 + 36b^2 = 400 \quad \ldots \text{(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$a^2 = 40 \quad \text{and} \quad b^2 = 10$$

Hence, required equation of the ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1.$$

7. (*b*) I. Given, $f(x) = x^2e^{-x^2}\sin|x|$

Here, $x^2e^{-x^2}$ and $\sin|x|$ are even function.

II. The product of two odd functions is always an even function.

8. (*b*) We have, $\mathbf{r}\cdot\mathbf{a} = 0 \Rightarrow \mathbf{r}\perp\mathbf{a}$

$\mathbf{r}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{r}\perp\mathbf{b}$

and $\mathbf{r}\cdot\mathbf{c} = 0 \Rightarrow \mathbf{r}\perp\mathbf{c}$

Since, vectors **a**, **b** and **c** are coplanar vectors.

Also, $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$

$\therefore \quad \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = \mathbf{0}$

$\Rightarrow \quad \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) = 0$

$\Rightarrow \quad 0+[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+0 = 0$

$\Rightarrow \quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

Hence, **a**, **b** and **c** are coplanar vectors.

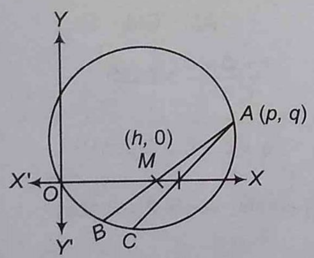
9. (*d*) I. Now, $\begin{vmatrix} 1 & 2 & 1 \\ 4 & 8 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(8-8) = 0$

So, it is collinear, hence no circle is passing through these points.

II. It is a true statement.

10. (*d*) Let AB be a chord of the circle through $A\,(p,q)$ and $M\,(h, 0)$ be the mid-point of AB.

Therefore, the coordinates of B are $(-p+2h, -q)$.



Since, B lies on the circle $x^2+y^2 = px+qy$, then

$$(-p+2h)^2+(-q)^2 = p(-p+2h)+q(-q)$$

$\Rightarrow \quad 2p^2+2q^2-6ph+4h^2 = 0$

$\Rightarrow \quad 2h^2-3ph+(p^2+q^2) = 0 \quad \ldots(\text{i})$

As, there are two distinct chords AB and AC from $A(p,q)$ which are bisected on X-axis there must be two distinct values of h satisfying Eq. (i), then

$D = (b^2-4ac) > 0$, we have

$(-3p)^2-4(2)(p^2+q^2) > 0 \Rightarrow p^2 > 8q^2$

11. (*d*) Since, $A^3 = B^3$ and $A^2B = B^2A$

$\therefore \quad A^3-A^2B = B^3-B^2A$

$\Rightarrow \quad (A^2+B^2)(A-B) = O$

$\Rightarrow \quad \det(A^2+B^2)\det(A-B) = 0$

12. (*b*) Here, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$\therefore P(X=1)+P(X=3)+\ldots+P(X=49)$

$$= {}^{100}C_1\left(\frac{1}{2}\right)^{100}+{}^{100}C_3\left(\frac{1}{2}\right)^{100}+\ldots+{}^{100}C_{49}\left(\frac{1}{2}\right)^{100}$$

$$= \left(\frac{1}{2}\right)^{100}({}^{100}C_1+{}^{100}C_3+\ldots+{}^{100}C_{49})$$

$\left[\begin{array}{l}\because ({}^{100}C_1+{}^{100}C_3+\ldots+{}^{100}C_{99}) = 2^{99} \\ \text{but } {}^{100}C_{99} = {}^{100}C_1 \\ {}^{100}C_{97} = {}^{100}C_3, \ldots, {}^{100}C_{51} = {}^{100}C_{49} \\ \therefore 2({}^{100}C_1+{}^{100}C_3+\ldots+{}^{100}C_{49}) = 2^{99}\end{array}\right]$

$$= \left(\frac{1}{2}\right)^{100}\times 2^{98} = \frac{1}{4}$$

13. (*c*) Given that, $y = f\left(\frac{2x+3}{3-2x}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = f'\left(\frac{2x+3}{3-2x}\right)\frac{d}{dx}\left(\frac{2x+3}{3-2x}\right)$$

$$= \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]\left[\frac{(3-2x)(2)-(2x+3)(-2)}{(3-2x)^2}\right]$$

$$= \frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x=1)} = \frac{12}{(3-2)^2}\sin\log(5)$$

$$= 12\sin\log(5)$$

14. (*c*) Line is parallel to plane as

$$(\hat{\mathbf{i}}-\hat{\mathbf{j}}+4\hat{\mathbf{k}})\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}) = 0$$

General point on the line is $(\lambda+2, -\lambda-2, 4\lambda+3)$. For $\lambda = 0$ point on this line is $(2, -2, 3)$ and distance from

$$\mathbf{r}\cdot(\hat{\mathbf{i}}+5\hat{\mathbf{j}}+\hat{\mathbf{k}}) = 5$$

or $x+5y+z = 5$, is

$$d = \frac{|2+5(-2)+3-5|}{\sqrt{1+25+1}}$$

$$\Rightarrow \quad d = \frac{|-10|}{3\sqrt{3}} = \frac{10}{3\sqrt{3}}$$

15. (*d*) Let a and b be intercepts on the coordinate axes.

Given that, $a+b = -1$

$\Rightarrow \quad b = -a-1 = -(a+1)$

Equation of line is $\frac{x}{a}+\frac{y}{b} = 1$

$\Rightarrow \quad \frac{x}{a}-\frac{y}{a+1} = 1 \quad \ldots(\text{i})$

Since, this line passes through the point (4, 3).

$\therefore \quad \frac{4}{a} - \frac{3}{a+1} = 1$

$\Rightarrow \quad a + 4 = a^2 + a$

$\therefore \quad a = \pm 2$

$\therefore$ Equation of line from Eq. (i) is

$$\frac{x}{2} - \frac{y}{3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1.$$

16. (*d*) All statements are correct.

17. (*a*) On squaring given equations, we get

$\sin^2 x + \cos^2 y + 2\sin x \cos y = 1$ ...(i)

and $\cos^2 x + \sin^2 y + 2\sin y \cos x = 1$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2 + 2\sin(x+y) = 2$

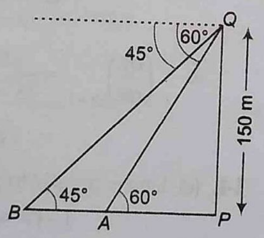
$\therefore \quad \sin(x+y) = 0$

18. (*a*) $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\} = \tan\left\{\pi - \cos^{-1}\left(\frac{2}{7}\right) - \frac{\pi}{2}\right\}$

$= \tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$

$= \tan\left\{\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right)\right\} = \frac{2}{3\sqrt{5}}$

19. (*b*) Let $PQ = 150$ m



In ΔAPQ, $\tan 60° = \frac{PQ}{AP}$

$\Rightarrow \quad AP = \frac{150}{\sqrt{3}}$ ...(i)

and in ΔBPQ, $\tan 45° = \frac{PQ}{AB + AP}$

$\Rightarrow \quad AB + \frac{150}{\sqrt{3}} = 150$

$\Rightarrow \quad AB = \frac{150}{\sqrt{3}}(\sqrt{3} - 1)$

$\therefore$ Speed of boat $= \frac{AB}{2} = \frac{1}{2} \times \frac{150}{\sqrt{3}}(\sqrt{3} - 1) \times 60$

$= \frac{4500}{\sqrt{3}}(\sqrt{3} - 1)$ m/h

20. (*b*) Given equation of planes are

$4x + 4y - 5z = 12$ ...(i)

and $8x + 12y - 13z = 32$ ...(ii)

Let DR's of required line be (l, m, n).

From Eqs. (i) and (ii), we get

$4l + 4m - 5n = 0$

and $8l + 12m - 13n = 0$

$\Rightarrow \quad \frac{l}{8} = \frac{m}{12} = \frac{n}{16}$

$\Rightarrow \quad \frac{l}{2} = \frac{m}{3} = \frac{n}{4}$

Now, we take intersection point with $z = 0$ is given by

$4x + 4y = 12$ ...(iii)

and $8x + 12y = 32$ ...(iv)

On solving Eqs. (i) and (ii), we get the point (1, 2, 0).

$\therefore$ Required line is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$.

21. (*a*) $f(x) = f(1 + 1 + 1 + \ldots + x \text{ times})$

$= f(1)f(1)f(1)\ldots\ldots x \text{ times} = [f(1)]^x = 3^x$

$\therefore \quad \sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} 3^x = 3^1 + 3^2 + \ldots + 3^n = \frac{3^1 - 3^n \cdot 3}{1-3}$

$= \frac{3^{n+1} - 3}{2} \quad \left(\because \text{sum} = \frac{a - lr}{1-r}\right)$

$\therefore \quad \frac{3^{n+1} - 3}{2} = 120$

$\Rightarrow \quad 3^{n+1} = 243 = 3^5$

$\Rightarrow \quad n + 1 = 5$

$\therefore \quad n = 4$

22. (*b*) Let $\Delta' = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = abc + 2 - a - b - c > 0$

or $\quad abc + 2 > a + b + c$ ...(i)

$\because \quad$ AM > GM

$\Rightarrow \quad \frac{a+b+c}{3} > (abc)^{\frac{1}{3}}$

$\Rightarrow \quad a + b + c > 3(abc)^{\frac{1}{3}}$ ...(ii)

From Eqs. (i) and (ii), $abc + 2 > 3(abc)^{\frac{1}{3}}$

Let $\quad (abc)^{\frac{1}{3}} = x,$

Then $\quad x^3 + 2 > 3x$

$\Rightarrow \quad (x-1)^2(x+2) > 0$

$\therefore \quad x + 2 > 0 \Rightarrow x > -2$

$\Rightarrow \quad x^3 > -8$

$\Rightarrow \quad abc > -8$

23. (*c*) Here, $P(W) = \frac{a}{a+b}$ and $P(B) = \frac{b}{a+b}$

$\therefore$ Probability of *A* winning

$= P(W) + P(\overline{W})P(\overline{B})P(W) + \ldots$

$= \frac{P(W)}{1 - P(\overline{W})P(\overline{B})} = \frac{\frac{a}{a+b}}{1 - \frac{b}{a+b} \cdot \frac{a}{a+b}}$

$$= \frac{a(a+b)}{a^2+b^2+ab} = P_1 \quad \text{(say)}$$

and probability of B winning

$$= 1 - P_1 = 1 - \frac{a^2+ab}{a^2+b^2+ab}$$

$$= \frac{b^2}{a^2+b^2+ab} = P_2 \quad \text{(say)}$$

Given, $P_1 = 2P_2$

$\Rightarrow \quad \dfrac{a^2+ab}{a^2+b^2+ab} = \dfrac{2b^2}{a^2+b^2+ab}$

$\Rightarrow \quad a^2 + ab - 2b^2 = 0$

$\Rightarrow \quad (a-b)(a+2b) = 0$

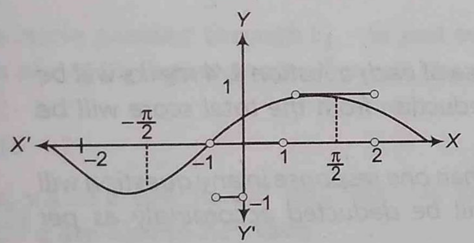
$\Rightarrow \quad a - b = 0 \quad (\because a + 2b \neq 0)$

$\Rightarrow \quad a = b$

$\therefore \quad a : b = 1 : 1$

24. (*c*) $y = \sin x = [x]$

Graphs of $y = \sin x$ and $y = [x]$ are as shown.



Hence, two solutions are $x = 0$ and $x = \frac{\pi}{2}$.

25. (*d*) $I + f + f' = (5+2\sqrt{6})^n + (5-2\sqrt{6})^n = 2k$ (even integer)

$\therefore \quad f + f' = 1$

Now, $(I+f)f' = (5+2\sqrt{6})^n (5-2\sqrt{6})^n = (1)^n = 1$

$\Rightarrow \quad (I+f)(1-f) = 1 \Rightarrow I = \dfrac{1}{1-f} - f$

26. (*c*) Let $I = \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \int_0^1 x^4 \left(x^2 - 4x + 5 - \frac{4}{1+x^2}\right) dx$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4)\, dx - 4\int_0^1 \frac{x^4}{(1+x^2)} dx$$

$$= \int_0^1 (x^6 - 4x^5 + 5x^4) - 4\int_0^1 \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx$$

$$= \left(\frac{x^7}{7} - \frac{4x^6}{6} + x^5\right)_0^1 - 4\left(\frac{x^3}{3} - x + \tan^{-1} x\right)_0^1$$

$$= \left(\frac{1}{7} - \frac{4}{6} + 1\right) - 4\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) = \frac{22}{7} - \pi$$

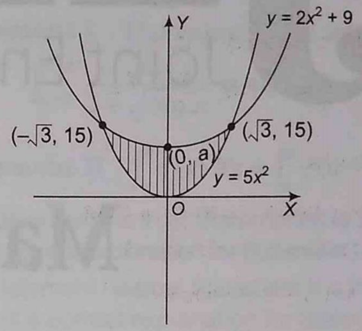
27. (*d*) Given parabolas are $5x^2 - y = 0$ and $2x^2 - y + 9 = 0$

Now eliminating y from above equations, we get

$$5x^2 - (2x^2 + 9) = 0$$

$\Rightarrow \quad 3x^2 = 9 \Rightarrow x = \pm\sqrt{3}$

Given parabolas interset at $(\sqrt{3}, 15)$ and $(-\sqrt{3}, 15)$.



The two parabolas are $x^2 = \frac{1}{5} y,\ x^2 = \frac{1}{2}(y-9)$

$\therefore \quad \text{Area} = 2\displaystyle\int_0^{\sqrt{3}} (y_1 - y_2)\, dx$

$$= 2\int_0^{\sqrt{3}} [(2x^2+9) - 5x^2)]\, dx = 2\int_0^{\sqrt{3}} (9 - 3x^2)\, dx$$

$$= 2[9x - x^3]_0^{\sqrt{3}}$$

$\therefore \quad \text{Area} = 12\sqrt{3}$

28. (*d*) Since, $n = 11$, then median term $= \left(\dfrac{11+1}{2}\right)$th term $=$ 6th term

As, last five observations are increased by 4.

Hence, the median of the 6th observations will remain same.

29. (*d*) We have, $\dfrac{\sqrt{x}\, dx + \sqrt{y}\, dy}{\sqrt{x}\, dx - \sqrt{y}\, dy} = \sqrt{\dfrac{y^3}{x^3}}$

$\Rightarrow \quad \dfrac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \dfrac{y^{3/2}}{x^{3/2}}$

$\Rightarrow \quad \dfrac{du + dv}{du - dv} = \dfrac{v}{u}$, where $u = x^{3/2}$ and $v = y^{3/2}$

$\Rightarrow \quad u\, du + u\, dv = v\, du - v\, dv$

$\Rightarrow \quad u\, du + v\, dv = v\, du - u\, dv$

$\Rightarrow \quad \dfrac{u\, du + v\, dv}{u^2 + v^2} = \dfrac{v\, du - u\, dv}{u^2 + v^2}$

$\Rightarrow \quad \dfrac{d(u^2 + v^2)}{u^2 + v^2} = -2d \tan^{-1}\left(\dfrac{v}{u}\right) + c$

On integrating, we get $\log(u^2 + v^2) = -2\tan^{-1}\left(\dfrac{v}{u}\right) + c$

$\Rightarrow \quad \dfrac{1}{2}\log(x^3 + y^3) + \tan^{-1}\left(\dfrac{y}{x}\right)^{3/2} = \dfrac{c}{2}$

30. (*b*) Since, $f(x)$ is continuous

$\therefore \quad \displaystyle\lim_{x \to 0} \left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}\right) = f(0)$

$\therefore \quad f(0) = \displaystyle\lim_{x \to 0} \frac{\left(2 - \frac{1}{\sqrt{1-x^2}}\right)}{2 + \frac{1}{1+x^2}} = \frac{2 - \frac{1}{\sqrt{1}}}{2 + \frac{1}{1}} = \frac{1}{3}$