Day 5

Matrices

Day 5 Outlines ...

- Matrix
- Algebra of Matrices
- Transpose of a Matrix
- Types of Matrices
- Adjoint of a Matrix
- Inverse of a Matrix

Matrix

A **matrix** is an arrangement of numbers in rows and columns. A matrix of order $m \times n$ is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Its element in the ith row and jth column is a_{ij}. If $m = n$, then matrix is a **square matrix**. In a square matrix A, the diagonal elements are $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$.

Some Definitions

1. If all elements of a matrix are zero, then it is called a **null zero matrix.**
2. A matrix which has only one row and any number of columns is called a **row matrix** and if it has only one column and any number of rows is called a **column matrix.**
3. If in a matrix, the number of rows is less/greater than the number of columns, then it is called **horizontal/vertical matrix**.
4. If all elements except the principal diagonal in a square matrix are zero, it is called a **diagonal matrix**. *The number of zeros in a diagonal matrix is given by $n^2 - n$, where n is the order of matrix.*
5. In a square matrix, if non-diagonal elements are zero and diagonal elements are all unity, then it is called an **unit** (identity) **matrix.**
6. In a square matrix, if non-diagonal elements are zero and diagonal elements are all same real number, then it is called a **scalar matrix**.

7. In a square matrix, if $a_{ij} = 0, \forall\, i > j$, then it is called an upper triangular matrix and if $a_{ij} = 0, \forall i < j$, then it is called a lower triangular matrix.

8. **Equality of Matrices** Two matrices A and B are said to be equal, if they are of same order and all the corresponding elements are equal *i.e.*, $A = B$.

Algebra of Matrices

1. **Addition of Matrices** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of same order, then

$$A + B = [a_{ij} + b_{ij}],\ \forall\ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Let A, B and C are three matrices of same order, then the properties of addition of matrices are

(i) Commutative $A + B = B + A$
(ii) Associative $(A + B) + C = A + (B + C)$
(iii) Additive Identity $A + O = A = O + A$
(iv) Additive Inverse $A + (-A) = O = (-A) + A$
(v) Cancellation Laws
(a) $A + B = A + C \Rightarrow B = C$ (left cancellation law)
(b) $B + A = C + A \Rightarrow B = C$ (right cancellation law)

2. **Subtraction of Matrices** Let A and B are two matrices of same order $m \times n$, then $A - B = [a_{ij} - b_{ij}]_{m \times n}$.

3. **Multiplication of a Matrix by a Scalar** Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k.

Let A and B be two matrices of same order, then the properties of multiplication of a matrix by scalar are

(i) $k(A + B) = kA + kB$
(ii) $(k_1 + k_2)A = k_1A + k_2A$

4. **Multiplication of Matrices** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that number of columns of A is equal to the number of rows of B, then the product matrix is $C = [c_{ij}]$ of order $m \times p$,

where, $$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

Let A, B and C are three matrices of order $m \times n, n \times p$ and $n \times k$, then the properties of multiplication of matrices are

(i) Non-commutative $AB \neq BA$
(ii) Associative $(AB)C = A(BC)$
(iii) Multiplicative Identity $IA = A = AI$
(iv) Multiplicative Distributive $A(B + C) = AB + AC$

Transpose of a Matrix

Let A be $m \times n$ matrix. Then, $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^C or A^T.

Important Results

- If A and B be two matrices of order $m \times n$, then $(A \pm B)' = A' \pm B'$.
- If k be a scalar, then $(kA)' = kA'$
- $(A')' = A$
- $(AB)' = B'A'$
- $(A^n)' = (A')^n$

Types of Matrices

There are several ways of classifying matrices depanding on symmetry, sparsity etc.

The list of types of matrices and the situation in which they may arise is given below

1. **Idempotent Matrix** A square matrix A is called an idempotent matrix, if it satisfies the relation $A^2 = A$.

If A and B are idempotent matrices, then A + B is an idempotent iff AB = BA.

2. **Nilpotent Matrix** A square matrix A is called nilpotent matrix, if it satisfies the relation $A^k = O$ and $A^{k+1} \neq O$.

3. **Involutory Matrix** A square matrix A is called an involutory matrix, if it satisfies the relation $A^2 = I$.

4. **Symmetric Matrix** A square matrix A is called symmetric matrix, if it satisfies the relation $A' = A$.

- *If A and B are symmetric matrices of the same order, then*
  (i) *AB is symmetric if and only if AB = BA.*
  (ii) *$A \pm B$, $AB + BA$ are also symmetric matrices.*
- *If A is symmetric matrix, then A^{-1} will also be symmetric matrix.*

5. **Skew-symmetric Matrix** A square matrix A is called skew-symmetric matrix, if it satisfies the relation $A' = -A$. Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrix.

i.e., $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

- *If A and B are two skew-symmetric matrices, then*
  (i) *$A \pm B$, $AB - BA$ are skew-symmetric matrices*
  (ii) *$AB + BA$ is a symmetric matrix.*
- *Determinant of skew-symmetric matrix of odd order is zero.*

6. **Orthogonal Matrix** A square matrix A is called an orthogonal matrix, if it satisfies the relation given as $AA' = I$.

If A and B are orthogonal matrices, then AB is also an orthogonal matrix. Every orthogonal matrix is invertible.

Trace of a Matrix

The sum of the diagonal elements of a square matrix A is called the trace of A and is denoted by $\text{tr}(A)$.

(i) $\text{tr}(\lambda A) = \lambda\, \text{tr}(A)$

(ii) $\text{tr}(A) = \text{tr}(A')$

(iii) $\text{tr}(AB) = \text{tr}(BA)$

Adjoint of a Matrix

Let $A = [a_{ij}]_{m \times n}$ be a square matrix of order n and C_{ij} be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A is defined as the transpose of the cofactor matrix and it is denoted by adj (A).

Properties

Let A be a matrix of order n, then

(i) $(\text{adj } A)A = A\,(\text{adj } A) = |A| \cdot I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$, if $|A| \neq 0$

(iii) $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$

(iv) If $|A| = 0$, then $(\text{adj } A)\, A = A\,(\text{adj } A) = O$

(v) $\text{adj}(A^T) = (\text{adj } A)^T$

(vi) $|\text{adj}\,(\text{adj } A)| = |A|^{(n-1)^2}$

(vii) Adjoint of a diagonal matrix is a diagonal matrix.

(viii) $\text{adj}\,(\text{adj } A) = |A|^{n-2} A$

Equivalent Matrices

Two matrices A and B are said to be equivalent, if one is obtained from the other by one or more elementary operations and we write $A \sim B$.

Following elementary operations are given below

1. Interchanging any two rows (or columns). This transformation is indicated by
$$R_i \leftrightarrow R_j \ (C_i \leftrightarrow C_j)$$
2. Multiplication of the elements of any row (column) by a non-zero scalar quantity and indicated as
$$R_i \leftrightarrow kR_i \ (C_i \leftrightarrow kC_i)$$
3. Addition of constant multiple of the elements of any row (column) to the corresponding element of any other row, indicated as
$$R_i \rightarrow R_i + kR_j \ (C_i \rightarrow C_i + kC_j)$$

This elementary transformation is used to determine the inverse of a matrix A. *i.e.*, $AB = BA = I \Rightarrow B = A^{-1}$

In this transformation, we use either row (column) transformation.

Inverse of a Matrix

Let A be a non-singular (where, $|A| \neq 0$) square matrix. Then, a square matrix B such that $AB = BA = I$ is called inverse of A. *i.e.*, $A^{-1} = \dfrac{1}{|A|}\,\text{adj}\,(A)$.

Properties

Let A and B are square matrices of same order, then

(i) A square matrix is invertible if and only if it is non-singular.

(ii) $(A')^{-1} = (A^{-1})'$ (iii) $(AB)^{-1} = B^{-1}A^{-1}$ (iv) $|A^{-1}| = |A|^{-1}$

Solution of a System of Linear Equations

Let system of linear equations in three variables are
$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2 \text{ and } a_3x + b_3y + c_3z = d_3$$
It can be written in the matrix form
$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B \quad \Rightarrow \quad X = A^{-1}B$$

Test of Consistency

Non-homogeneous Equations $(B \neq 0)$

- If $|A| \neq 0$, then the system of equations is **consistent** and has a unique solution given by $X = A^{-1}B$.
- If $|A| = 0$ and $(\text{adj } A) \cdot B \neq 0$, then the system of equations is inconsistent and has no solution.
- If $|A| = 0$ and $(\text{adj } A) \cdot B = 0$, then the system of equations is consistent and has an infinite number of solutions.

Homogeneous Equations $(B = 0)$

- If $|A| \neq 0$, then system of equations have only trivial solution and it has one solution.
- If $|A| = 0$, then system of equations has non-trivial solution and it has infinite number of solutions.
- If number of equations is less than to number of unknowns, then it has non-trivial solution.

A homogeneous system of equations is never inconsistent.

Practice Zone DAY 5

1. If $A=\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, then which of the following is correct? **[NCERT Exemplar]**

(a) $(A+B)\cdot(A-B)=A^2+B^2$
(b) $(A+B)\cdot(A-B)=A^2-B^2$
(c) $(A+B)\cdot(A-B)=I$
(d) None of the above

2. If A and B are square matrices such that $A^2=A, B^2=B$ and A, B commute, then

(a) $(AB)^2=I$ (b) $(AB)^2=AB$
(c) $(AB)^2=O$ (d) None of these

3. If $A=\begin{bmatrix}1 & 2 & -1\\-1 & 1 & 2\\2 & -1 & 1\end{bmatrix}$, then $\det\{\text{adj}(A)\}$ is

(a) $(14)^2$ (b) $(13)^2$
(c) $(14)^3$ (d) $(13)^3$

4. If $A=\begin{bmatrix}\alpha & 2\\2 & \alpha\end{bmatrix}$ and $\det A^3=125$, then α is equal to

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 5

5. If $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\\tan\frac{\alpha}{2} & 0\end{bmatrix}$ and I is the identity matrix of order 2, then $(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{bmatrix}$ is equal to **[NCERT Exemplar]**

(a) A
(b) I
(c) $I+A$
(d) None of the above

6. If $A=\begin{bmatrix}2 & 1\\-4 & -2\end{bmatrix}$, then $I+2A+3A^2+\ldots\infty$ is equal to

(a) $\begin{bmatrix}4 & 1\\-4 & 0\end{bmatrix}$ (b) $\begin{bmatrix}3 & 1\\-4 & -1\end{bmatrix}$
(c) $\begin{bmatrix}5 & 2\\-8 & -3\end{bmatrix}$ (d) $\begin{bmatrix}5 & 2\\-3 & -8\end{bmatrix}$

7. If $3A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\x & 2 & y\end{bmatrix}$ and $A^TA=AA^T=I$, then xy is equal to

(a) -1 (b) 1
(c) 2 (d) -2

8. If $A=\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}$ and $A^{-1}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\-5 & 0 & \alpha\\1 & -2 & 1\end{bmatrix}$, then α is equal to

(a) -2 (b) 5
(c) 2 (d) -1

9. Choose the correct answer.

(a) every scalar matrix is an identity matrix
(b) every identity matrix is a scalar matrix
(c) every diagonal matrix is an identity matrix
(d) a square matrix whose each element is 1, is an identity matrix

10. If the equations $a(y+z)=x, b(z+x)=y, c(x+y)=z$ have non-trivial solution, then $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$ is equal to

(a) 1 (b) 2
(c) -1 (d) -2

11. If $A=\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}$, then the value of A^3-6A^2+7A is equal to **[NCERT Exemplar]**

(a) I (b) 0
(c) $-2I$ (d) $2I$

12. If $A=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$, then A^n is equal to

(a) $2^{n-1}A-(n-1)I$ (b) $nA-(n-1)I$
(c) $2^{n-1}A+(n-1)I$ (d) $nA+(n-1)I$

13. If A is skew-symmetric and $B=(I-A)^{-1}(I+A)$, then B is

(a) singular (b) symmetric
(c) skew-symmetric (d) orthogonal

14. If $A = \begin{bmatrix} -1 & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then $I + A + A^2 + \ldots \infty$ is equal to

(a) $\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$ (b) $\frac{2}{7}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

(c) $\frac{2}{7}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ (d) undefined

15. The value of λ such that the system $x - 2y + z = -4$, $2x - y + 2z = 2$, $x + y + \lambda z = 4$ has no solution, is

(a) 0 (b) 1
(c) $\neq 1$ (d) 3

16. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then k is equal to

(a) 0, 1 (b) 1, −1
(c) −1, 2 (d) 2, −2

17. If $x = cy + bz$, $y = az + cx$ and $z = bx + ay$, where x, y and z are not all zero, then $a^2 + b^2 + c^2$ is equal to

(a) $1 + 2abc$ (b) $1 - 2abc$
(c) $1 + abc$ (d) $abc - 1$

18. If $A = \begin{bmatrix} \frac{-1 + i\sqrt{3}}{2i} & \frac{-1 - i\sqrt{3}}{2i} \\ \frac{1 + i\sqrt{3}}{2i} & \frac{1 - i\sqrt{3}}{2i} \end{bmatrix}$, $i = \sqrt{-1}$ and $f(x) = x^2 + 2$, then $f(A)$ is equal to

(a) $\left(\frac{5 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\left(\frac{3 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $(2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

19. If $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is zero matrix, then θ and ϕ differ by

(a) even multiple of $\frac{\pi}{2}$ (b) odd multiple of $\frac{\pi}{2}$
(c) even multiple of π (d) odd multiple of π

20. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(a) idempotent (b) nilpotent
(c) involutary (d) orthogonal

21. If $A = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$, then A^{-1} is equal to

(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

22. Let a, b and c be positive real numbers. The following system of equations in x, y and z

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has

(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

Directions (Q. Nos. 23 and 24) *Let A and B are two matrices of same order 3 × 3, where* $A = \begin{bmatrix} 1 & 3 & \lambda + 2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 4 \\ 3 & 2 & 5 \\ 2 & 1 & 4 \end{bmatrix}$.

23. If A is singular matrix, then tr $(A + B)$ is equal to

(a) 6 (b) 12
(c) 24 (d) 17

24. If matrix $2A + 3B$ is singular, then the value of 2λ is

(a) 11 (b) 13
(c) 15 (d) 17

Directions (Q. Nos. 25 to 27) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

25. Statement I If A is skew-symmetric of order 3, then its determinant should be zero.

Statement II If A is square matrix, then

$$\det(A) = \det(A') = \det(-A')$$

26. Let A be a square matrix of order 3 satisfying $AA' = I$, then

Statement I $A^{-1}A = I$

Statement II $(AB)' = B'A'$

27. Statement I If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj (adj A) = A.

Statement II $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, A be n rowed non-singular matrix.

28. If $A(\pi/6) = \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$B(\pi/4) = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $[A(\pi/6)\cdot B(\pi/4)]^{-1}$ is equal to

(a) $\begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 1 & 1 & 0 \\ 3 & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (d) None of these

29. The system of equations $x + y + 3z = 1$, $2x + y + 2z = 3$, $3x + 2y + 5z = 3$ have

(a) unique solution (b) infinite solution
(c) inconsistent (d) None of these

30. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ and $A^n = I$, then the value of n is

(a) 2 (b) 4
(c) 6 (d) 3

31. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^TQ^{2005}P$ is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2 + \sqrt{3} & 1 \\ -1 & 2 - \sqrt{3} \end{bmatrix}$ (d) $\frac{1}{4}\begin{bmatrix} 2005 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2005 \end{bmatrix}$

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32. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, is equal to **[JEE Main 2013]**

(a) $4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $4\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$

(c) $32\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (d) $32\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

33. If the system of linear equations **[JEE Main 2013]**

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then

(a) $a = 8$, b can be any real number
(b) $b = 15$, a can be any real number
(c) $a \in R - \{8\}$ and $b \in R - \{15\}$
(d) $a = 8$, $b = 15$

34. If p, q, r are 3 real numbers satisfying the matrix equation,

$[p\, q\, r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ 0\ 1]$, then $2p + q - r$ is equal to **[JEE Main 2013]**

(a) -3 (b) -1 (c) 4 (d) 2

35. Let A, other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A **[JEE Main 2013]**

Statement I tr $(A) = 0$

Statement II det $(A) = -1$

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

36. Consider the system of equations $x + ay = 0$, $y + az = 0$ and $z + ax = 0$. Then, the set of all real values of 'a' for which the system has a unique solution is **[JEE Main 2013]**

(a) $R - \{1\}$ (b) $R - \{-1\}$
(c) $\{1, -1\}$ (d) $\{1, 0, -1\}$

37. The number of values of k, for which the system of equations

$$(k + 1)x + 8y = 4k$$
and
$$kx + (k + 3)y = 3k - 1$$

has no solution, is **[JEE Main 2013]**

(a) infinite (b) 1
(c) 2 (d) 3

38. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

(a) 4 (b) 11
(c) 5 (d) 0

39. Let P and Q be 3×3 matrices $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then the determinant of $(P^2 + Q^2)$ is **[AIEEE 2012]**

(a) -2 (b) 1
(c) 0 (d) -1

40. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and u_1, u_2 are column matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $u_1 + u_2$ is equal to **[AIEEE 2012]**

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

41. Let A and B be two symmetric matrices of order 3.

Statement I $A(BA)$ and $(AB)A$ are symmetric matrices.
Statement II AB is symmetric matrix, if matrix multiplication of A with B is commutative. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

42. The number of values of k for which the linear equations $4x + ky + 2z = 0$, $kx + 4y + z = 0$ and $2x + 2y + z = 0$ posses a non-zero solution is **[AIEEE 2011]**

(a) 2 (b) 1
(c) zero (d) 3

43. Consider the system of linear equations

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 1$$

The system has **[AIEEE 2010]**

(a) infinite number of solutions
(b) exactly 3 solutions
(c) a unique solution
(d) no solution

44. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is **[AIEEE 2010]**

(a) less than 4 (b) 5
(c) 6 (d) atleast 7

Directions *(Q. Nos. 45 to 47) Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

45. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix.

Define $\text{tr}(A) =$ sum of diagonal elements of A and $|A| =$ determinant of matrix A.

Statement I $\text{tr}(A) = 0$
Statement II $|A| = 1$ **[AIEEE 2010]**

46. Let A be 2×2 matrix.

Statement I $\text{adj}(\text{adj } A) = A$
Statement II $|\text{adj } A| = A$ **[AIEEE 2009]**

47. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by $\text{tr}(A)$, the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement I If $A \neq I$ and $A \neq -I$, then $\det(A) = -1$
Statement II If $A \neq I$ and $A \neq -I$, then $\text{tr}(A) \neq 0$ **[AIEEE 2008]**

48. If A be a square matrix all of whose entries are integers. Then, which one of the following is true? **[AIEEE 2008]**

(a) If $\det(A) = \pm 1$, then A^{-1} need not exist
(b) If $\det(A) = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers
(c) If $\det(A) \neq \pm 1$, then A^{-1} exists and all its entries are non-integers
(d) If $\det(A) = \pm 1$, then A^{-1} exists and all its entries are integers

49. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $\det(A^2) = 25$, then $|\alpha|$ is equal to **[AIEEE 2007]**

(a) 1 (b) $\frac{1}{5}$
(c) 5 (d) 5^2

50. If A and B are 3×3 matrices such that $A^2 - B^2 = (A - B)(A + B)$, then **[AIEEE 2006]**

(a) either A or B is zero matrix
(b) either A or B is unit matrix
(c) $A = B$
(d) $AB = BA$

51. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $a, b, \in N$, then **[AIEEE 2006]**

(a) there exists exactly one B such that $AB = BA$
(b) there exists infinitely many B's such that $AB = BA$
(c) there cannot exist any B such that $AB = BA$
(d) there exist more than but finite number of B 's such that $AB = BA$

52. The system of equations

$$\alpha x + y + z = \alpha - 1,\ x + \alpha y + z = \alpha - 1,\ x + y + \alpha z = \alpha - 1$$

[AIEEE 2005]

has no solution, if α is

(a) -2 or 1 (b) -2
(c) 1 (d) -1

Answers

1. (d)	**2.** (b)	**3.** (a)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (b)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (c)	**15.** (c)	**16.** (b)	**17.** (b)	**18.** (d)	**19.** (b)	**20.** (b)
21. (a)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (b)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (a)
31. (a)	**32.** (a)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (d)
41. (a)	**42.** (a)	**43.** (d)	**44.** (d)	**45.** (c)	**46.** (b)	**47.** (c)	**48.** (d)	**49.** (b)	**50.** (d)
51. (d)	**52.** (b)								

Hints & Solutions

1. Now, $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

$$A - B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1 & 0+1 \\ 0+1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and $B^2 = B \cdot B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\therefore \quad A^2 - B^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

and $(A + B)(A - B) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

Hence, $(A + B)(A - B) \neq A^2 - B^2$

2. Now, $(AB)^2 = (AB) \cdot (AB) = A(BA)B = A(AB)B$ $\quad (\because AB = BA)$

$$= (A \cdot A)(B \cdot B)$$
$$= A^2 \cdot B^2 = AB$$

3. $\because \quad |\text{adj}(A)| = |A|^{n-1}$

$\because \quad |A| = 14$

$\therefore \quad |\text{adj}(A)| = (|A|)^2 = (14)^2$

4. $125 = \det(A^3) = (\det A)^3 = (\alpha^2 - 4)^3$

$\Rightarrow \quad \alpha^2 - 4 = 5$

$\Rightarrow \quad \alpha^2 = 9$

$\Rightarrow \quad \alpha = \pm 3$

5. Here, $A = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$, where $t = \tan\left(\frac{\alpha}{2}\right)$

Now, $\cos\alpha = \dfrac{1 - \tan^2\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \dfrac{1 - t^2}{1 + t^2}$

and $\sin\alpha = \dfrac{2\tan\left(\frac{\alpha}{2}\right)}{1 + \tan^2\left(\frac{\alpha}{2}\right)} = \dfrac{2t}{1 + t^2}$

Now, we have $(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-t^2+2t^2}{1+t^2} & \frac{-2t+t(1-t^2)}{1+t^2} \\ \frac{-t(1-t^2)+2t}{1+t^2} & \frac{2t^2+1-t^2}{1+t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+t^2}{1+t^2} & \frac{-t(1+t^2)}{1+t^2} \\ \frac{t(1+t^2)}{1+t^2} & \frac{1+t^2}{1+t^2} \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

Now, $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$

$$(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = I + A$$

6. $A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore \; I + 2A + 3A^2 + \ldots = I + 2A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$$

7. A is orthogonal. Each row is orthogonal to the other rows.

$R_1 \cdot R_3 = 0 \Rightarrow x + 4 + 2y = 0$

and $R_2 \cdot R_3 = 0 \Rightarrow 2x + 2 - 2y = 0$

On solving, we get $x = -2$, $y = -1$

$\therefore \quad xy = 2$

8. Here, $AA^{-1} = I$

If R_1 of A is multiplied by C_3 of A^{-1}, we get

$2 - \alpha + 3 = 0 \Rightarrow \alpha = 5$

9. Every identity matrix is a scalar matrix.

10. Here, $\begin{vmatrix} -1 & a & a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} = 0$

Applying, $C_2 \to C_2 - C_1; C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} -1 & a+1 & a+1 \\ b & -(b+1) & 0 \\ c & 0 & -(1+c) \end{vmatrix} = 0$$

Applying $R_1 \to \dfrac{R_1}{a+1}, R_2 \to \dfrac{R_2}{b+1}, R_3 \to \dfrac{R_3}{c+1}$,

$$\begin{vmatrix} -\dfrac{1}{a+1} & 1 & 1 \\ \dfrac{b}{b+1} & -1 & 0 \\ \dfrac{c}{c+1} & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad -\dfrac{1}{a+1} + \dfrac{b}{b+1} + \dfrac{c}{c+1} = 0$

$\therefore \quad -\dfrac{1}{a+1} + 1 - \dfrac{1}{b+1} + 1 - \dfrac{1}{c+1} = 0$

$\Rightarrow \quad \dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1} = 2$

11. $A^2 = A \times A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 7A + 2I =$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 - 6A^2 + 7A = -2I$

12. $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

....

....

$A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$= nA - (n-1)I$

13. Now, $BB^T = (I - A)^{-1}(I + A)(I + A)^T[(I - A)^{-1}]^T$

$= (I - A)^{-1}(I + A)(I - A)(I + A)^{-1}$

$= (I - A)^{-1}(I - A)(I + A)(I + A)^{-1} = I \cdot I = I$

Hence, B is an orthogonal matrix.

14. $\therefore I + A + A^2 + \ldots \infty = (I - A)^{-1}$

$$= \begin{bmatrix} 2 & -\dfrac{3}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}^{-1} = \frac{4}{7}\begin{bmatrix} \dfrac{1}{2} & \dfrac{3}{2} \\ -\dfrac{1}{2} & 2 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

15. For no solution, $\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & \lambda \end{vmatrix} \neq 0 \Rightarrow \lambda \neq 1$

16. For non-zero solution, $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} 1 & -k & -1 \\ k-1 & k-1 & 0 \\ 0 & k+1 & 0 \end{vmatrix} = 0 \quad \begin{pmatrix} R_2 \to R_2 - R_1, \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$\Rightarrow \quad k^2 - 1 = 0 \Rightarrow k = 1, -1$

17. Given, $-x + cy + bz = 0, cx - y + az = 0$

and $bx + ay - z = 0$

$\Rightarrow \quad \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 = 1 - 2abc$

18. Thus, $A = \begin{bmatrix} \frac{\omega}{i} & \frac{\omega^2}{i} \\ -\frac{\omega^2}{i} & \frac{-\omega}{i} \end{bmatrix} = \frac{\omega}{i}\begin{bmatrix} 1 & \omega \\ -\omega & -1 \end{bmatrix}$

$\therefore \quad A^2 = -\omega^2 \begin{bmatrix} 1-\omega^2 & 0 \\ 0 & 1-\omega^2 \end{bmatrix}$

$= \begin{bmatrix} -\omega^2+\omega^4 & 0 \\ 0 & -\omega^2+\omega^4 \end{bmatrix}$

$= \begin{bmatrix} -\omega^2+\omega & 0 \\ 0 & -\omega^2+\omega \end{bmatrix}$

$\because \quad f(x) = x^2 + 2$

(given)

$\therefore \quad f(A) = A^2 + 2I$

$= \begin{bmatrix} -\omega^2+\omega & 0 \\ 0 & -\omega^2+\omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= (-\omega^2+\omega+2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= (3+2\omega)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (2+i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad \left[\text{if } \theta-\phi = (2n+1)\frac{\pi}{2}, n \in I\right]$

20. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

$\therefore \; A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, A is nilpotent matrix of index 2.

21. Since, $|A| = 1$

Given, $A = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$

$\therefore \quad A^{-1} = \frac{1}{1}\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(-x)$

22. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$

Then, $X+Y-Z=1, X-Y+Z=1, -X+Y+Z=1$

$\therefore$ Coefficient matrix is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Here, $|A| \neq 0$

Hence, it has unique solution.

23. Since, A is singular, therefore $\begin{vmatrix} 1 & 3 & \lambda+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{vmatrix} = 0$

$\Rightarrow 1(40-40) - 3(20-24) + (\lambda+2)(10-12) = 0$

$\Rightarrow \quad \lambda = 4$

Now, $A + B = \begin{bmatrix} 4 & 5 & 10 \\ 5 & 6 & 13 \\ 5 & 6 & 14 \end{bmatrix}$

$\therefore \quad \text{tr}(A+B) = 4+6+14 = 24$

24. Now, $2A+3B = \begin{bmatrix} 11 & 12 & 2\lambda+16 \\ 13 & 14 & 31 \\ 12 & 13 & 32 \end{bmatrix}$

Since, $|2A+3B| = 0$

$\Rightarrow \quad 11(448-403) - 12(416-372) + (2\lambda+16)(169-168) = 0$

$\Rightarrow \quad 2\lambda = 17$

25. Statement I is true.

But $\det(A') = \det(-A')$ is not true.

26. Statement I Given, $AA' = I \Rightarrow |A|^2 = 1$

Hence, A is an invertiable matrix.

$\Rightarrow \quad A' = A^{-1} \Rightarrow A'A = A^{-1}A = I$

27. Since, $\text{adj}(\text{adj}\, A) = |A|^{n-2} A$

Here, $n = 3$

$\therefore \quad \text{adj}(\text{adj}\, A) = |A| A$

Now, $|A| = 3(-3+4) + 3(2) + 4(-2) = 1$

$\therefore \quad \text{adj}(\text{adj}\, A) = A$

28. $\because \quad (AB)^{-1} = B^{-1} \cdot A^{-1}$

$\therefore \quad [A(\pi/6) \cdot B(\pi/4)]^{-1} = [B(\pi/4)]^{-1}[A(\pi/6)]^{-1}$

It is known that, $[A(\pi/6)]^{-1} = A(-\pi/6)$

$\therefore \quad [A(\pi/6) \cdot B(\pi/4)]^{-1} = B(-\pi/4) \cdot A(-\pi/6)]$

$= \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & 0 & \sin\left(-\frac{\pi}{4}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\left(-\frac{\pi}{6}\right) & \sin\left(-\frac{\pi}{6}\right) & 0 \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right) & \left[\frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)\right] & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. The coefficient matrix for the linear equations is

$$A=\begin{bmatrix}1&1&3\\2&1&2\\3&2&5\end{bmatrix}$$

where, $|A|=0$ and $(\text{adj } A)\,B\neq O$

Hence, the given equation is inconsistent.

30. Since, square of involuntary matrix is equal to identity matrix.

i.e., $$A\cdot A=\begin{bmatrix}1&0&0\\0&1&0\\3&4&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&4&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=A^n$$

On comparing, $n=2$

31. P is orthogonal matrix as $P^TP=I$

$$Q^{2005}=(PAP^T)(PAP^T)\dots(PAP^T)=PA^{2005}P^T$$
$$\therefore P^TQ^{2005}P=P^T\cdot PA^{2005}P^T\cdot P=A^{2005}$$

Now, $$A^2=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$$
$$A^3=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&3\\0&1\end{bmatrix}$$
$$\Rightarrow A^{2005}=\begin{bmatrix}1&2005\\0&1\end{bmatrix}$$

32. $\because A=\begin{bmatrix}1&2\\4&-3\end{bmatrix}, A^2=A\cdot A=\begin{bmatrix}1&2\\4&-3\end{bmatrix}\cdot\begin{bmatrix}1&2\\4&-3\end{bmatrix}=\begin{bmatrix}9&-4\\-8&+7\end{bmatrix}$

$$\therefore A^2+4A-5I=\begin{bmatrix}9&-4\\-8&17\end{bmatrix}+\begin{bmatrix}4&8\\16&-12\end{bmatrix}+\begin{bmatrix}-5&0\\0&-5\end{bmatrix}$$
$$=\begin{bmatrix}8&4\\8&0\end{bmatrix}=4\begin{bmatrix}2&1\\2&0\end{bmatrix}$$

33. For consistent in $|A|=0$ and $(\text{adj } A)\,B=0$

$$\begin{vmatrix}1&2&3\\1&3&5\\2&5&a\end{vmatrix}=0$$

$\Rightarrow 1(3a-25)-2(a-10)+3(5-6)=0 \Rightarrow a=8$

On solving, $(\text{adj } A)B=0$, we get $b=15$

34. $[3p+3q+2r,\ 4p+2q+0,\ p+3q+2r]=[3\ 0\ 1]$

$\Rightarrow 3p+3q+2r=3,\ 4p+2q=0,\ p+3q+2r=1$

$\Rightarrow p=1, q=-2, r=3$

$\therefore 2p+q-r=2-2-3=-3$

35. Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $A^2=I$ (given)

$$\therefore \begin{bmatrix}a&b\\c&d\end{bmatrix}^2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$\Rightarrow$ $b(a+d)=0, c(a+d)=0$

and $a^2+bc=1, bc+d^2=0$

$\Rightarrow$ $a=1, d=-1, b=c=0$

If $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, then $A^2=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$

$A\neq I, A\neq -I$

$\det(A)=-1$ and $\text{tr}(A)=1-1=0$

36. $$\begin{vmatrix}1&a&0\\0&1&a\\a&0&1\end{vmatrix}\neq 0$$

$\therefore 1(1-0)-a(0-a^2)\neq 0$

$\Rightarrow$ $1+a^3\neq 0 \quad\Rightarrow a\neq -1$

Hence, for unique solution $a\in R-\{-1\}$.

37. Given equations can be written in matrix form as $AX=B$

where, $A=\begin{bmatrix}k+1&8\\k&k+3\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}$

and $B=\begin{bmatrix}4k\\3k-1\end{bmatrix}$

For no solution, $|A|=0$ and $(\text{adj } A)\,B\neq 0$

Now, $|A|=\begin{vmatrix}k+1&8\\k&k+3\end{vmatrix}=0$

$\Rightarrow (k+1)(k+3)-8k=0$

$\Rightarrow k^2+4k+3-8k=0$

$\Rightarrow k^2-4k+3=0$

$\Rightarrow (k-1)(k-3)=0$

$\therefore k=1, k=3$

Now, $\text{adj } A=\begin{bmatrix}k+3&-8\\-k&k+1\end{bmatrix}$

$$\therefore (\text{adj } A)\,B=\begin{bmatrix}k+3&-8\\-k&k+1\end{bmatrix}\begin{bmatrix}4k\\3k-1\end{bmatrix}$$
$$=\begin{bmatrix}(k+3)(4k)-8(3k-1)\\-4k^2+(k+1)(3k-1)\end{bmatrix}$$
$$=\begin{bmatrix}4k^2-12k+8\\-k^2+2k-1\end{bmatrix}$$

Put $k=1$, then

$(\text{adj } A)\,B=\begin{bmatrix}4-12+8\\-1+2-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ not true

Put $k=3$, then

$(\text{adj } A)B=\begin{bmatrix}36-36+8\\-9+6-1\end{bmatrix}=\begin{bmatrix}8\\-4\end{bmatrix}\neq 0$ true.

Hence, the required value of k is 3.

Alternate Solution

Condition for the system of equations has no solution,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$$
$$\therefore \frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4k}{3k-1}$$

Take $\frac{k+1}{k} = \frac{8}{k+3}$

$\Rightarrow k^2 + 4k + 3 = 8k$

$\Rightarrow k^2 - 4k + 3 = 0$

$\Rightarrow (k-1)(k-3) = 0$

$\therefore k = 1, 2$

If $k = 1$, then $\frac{8}{1+3} \neq \frac{4 \cdot 1}{2}$, (false)

and $k = 3$, then $\frac{8}{6} \neq \frac{4 \cdot 3}{9-1}$, (true)

$\therefore k = 3$

Hence, only one value of k exist.

38. Given, $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

$\therefore |P| = 1(12-12) - \alpha(4-6) + 3(4-6) = 2\alpha - 6$

$\because P = \text{adj}(A)$ (given)

$\therefore |P| = |\text{adj}\, A| = |A|^2 = 16$

$\therefore 2\alpha - 6 = 16 \Rightarrow 2\alpha = 22$

$\Rightarrow \alpha = 11$

39. On subtracting the given equation we get

$P^3 - P^2Q = Q^3 - Q^2P$

$\Rightarrow P^2(P-Q) = Q^2(Q-P)$

$\Rightarrow (P-Q)(P^2+Q^2) = O$

$\therefore |P^2 + Q^2| = 0$

40. Since, both Au_1 and Au_2 are given, hence adding them, we get

$$Au_1 + Au_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+1 \\ 0+0 \end{bmatrix}$$

$$\Rightarrow A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since, A is non-singular matrix, *i.e.*, $|A| \neq 0$, hence, multiplying both sides by A^{-1}, we get

$$A^{-1}A(u_1 + u_2) = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1(10-0) - 0 + 0 = 1$

$\therefore$ Adjoint of $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ $(\because |A| = 1)$

From Eq. (i),

$$u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & +0+0 \\ -2 & +1+0 \\ 1 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

41. Given, $A^T = A$ and $B^T = B$

Statement I $[A(BA)]^T = (BA)^T \cdot A^T$

$= (A^T B^T)A^T$

$= (AB)A = A(BA)$

So, $A(BA)$ is symmetric matrix.

Similarly, $(AB)A$ is symmetric matrix.

Hence, Statement I is true.

Also, Statement II is true, as if A and B is symmetric.

Hence, (AB) is symmetric iff $(AB = BA)$ *i.e.*, AB is commutative.

42. Since, equation has non-zero solution.

$\therefore \Delta = 0$

$\therefore \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow 4(4-2) - k(k-2) + 2(2k-8) = 0$

$\Rightarrow 8 - k^2 + 2k + 4k - 16 = 0$

$\Rightarrow k^2 - 6k + 8 = 0$

$\Rightarrow (k-2)(k-4) = 0 \Rightarrow k = 2, 4$

So, number of values of k is 2.

43. The given system of linear equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -8 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} \quad \begin{pmatrix} R_2 \to -R_2 \\ R_3 \to R_3 - R_2 \end{pmatrix}$$

Clearly, the given system of equations has no solution.

Aliter Subtracting the addition of first two equations from third equation, we get $0 = -5$, which is an absurd result. Hence, the given system of equation has no solution.

44. Consider $\begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}$. By placing 1 in anyone of the 6 * position and 0 else where, we get 6 non-singular matrices.

Similarly, $\begin{pmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{pmatrix}$ gives atleast one non-singular matrix.

45. A satisfies $A^2 - \text{tr}(A)\cdot A + (\det A)I = 0$

On comparing with $A^2 - I = 0$, we get

$$\text{tr}(A) = 0, |A| = -1$$

Aliter

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a,b,c,d \neq 0$

Now, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$\Rightarrow a^2 + bc = 1, bc + d^2 = 1$

and $ab + bd = ac + cd = 0$

Also, $c \neq 0$ and $b \neq 0 \Rightarrow a + d = 0$

$\therefore \text{tr}(A) = a + d = 0$

and $|A| = ad - bc = -a^2 - bc = -1$

46. Statement I $\text{adj}(\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$

Statement II $|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

47. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(\because A^2 = I)$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow b(a+d) = 0,\ c(a+d) = 0$

and $a^2 + bc = 1, bc + d^2 = 0$

$\Rightarrow a^2 = 1, d = -1, b = c = 0$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A \neq I,\ A \neq -I$

$\det(A) = -1$

Hence, Statement I is true.

Statement II $\text{tr}(A) = 1 - 1 = 0$

Hence, Statement II is false.

48. As $\det(A) = \pm 1$, A^{-1} exists

and $A^{-1} = \dfrac{1}{\det(A)}(\text{adj } A) = \pm(\text{adj } A)$

All entries in adj (A) are integers.

So, A^{-1} has integer entries.

49. $\det(A^2) = (\det A)^2 = \begin{vmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{vmatrix}^2 = 25$

$\Rightarrow (25\alpha)^2 = 25$

$\Rightarrow \alpha^2 = \dfrac{1}{25}$

$\therefore |\alpha| = \dfrac{1}{5}$

50. $A^2 - B^2 = (A-B)(A+B) = A^2 + AB - BA - B^2$

$\therefore AB = BA$

51. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

$\Rightarrow AB = BA$

$\Rightarrow a = b = 1, 2, 3, \ldots$

52. Let $\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \alpha^3 - 3\alpha + 2 = 0$

$\Rightarrow (\alpha - 1)(\alpha^2 + \alpha - 2) = 0$

$\Rightarrow (\alpha - 1)^2(\alpha + 2) = 0 \Rightarrow \alpha = 1 \text{ or } -2$

$\therefore \alpha = 1$

$x + y + z = 0$ gives infinite number of solutions.

Hence, $\alpha = -2$ has no solution.

Day 6

Determinants

Day 6 Outlines ...

- Concept of Determinant
- Properties of Determinants
- Cyclic Determinant
- Minors and Cofactors
- System of Linear Equations in Three and Two Variables

Concept of Determinant

Every square matrix A is associated with a number, called its determinant and it is denoted by det (A) or $|A|$. Only square matrices have determinants. The marices which are square do not have determinants.

Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

e.g., (i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(ii) If $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$

$$= a\begin{vmatrix} q & r \\ v & w \end{vmatrix} - b\begin{vmatrix} p & r \\ u & w \end{vmatrix} + c\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

$$= a(qw - vr) - b(pw - ur) + c(pv - uq)$$

Properties of Determinants

1. If each element of a row (column) is a zero, then $\Delta = 0$.
2. If two rows (columns) are proportional, then $\Delta = 0$.
3. If any two rows (columns) are interchanged odd times, then Δ becomes $-\Delta$.
4. If the rows and columns are interchanged, then Δ is unchanged *i.e.*, $|A^T| = |A|$.
5. If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant

   i.e., $\begin{vmatrix} ka & kb & kc \\ p & q & r \\ u & v & w \end{vmatrix} = k \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$

6. $\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$

7. If a scalar multiple of any row (column) is added to another row (column), then Δ is unchanged

   i.e., $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p+ka & q+kb & r+kc \\ u & v & w \end{vmatrix}$

8. Let $\Delta(x)$ be a 3rd order determinant having polynomials as its elements.
   (i) If $\Delta(a)$ has two rows (columns) proportional, then $(x-a)$ is a factor of $\Delta(x)$.
   (ii) If $\Delta(a)$ has three rows (columns) proportional, then $(x-a)^2$ is a factor of $\Delta(x)$.

9. Product of two determinants

   i.e., $|AB| = |A||B| = |BA| = |AB^T| = |A^TB| = |A^TB^T|$

10. If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ a & b & c \end{vmatrix}$, then

    (i) $\sum_{x=1}^{n} \Delta(x) = \begin{vmatrix} \sum_{x=1}^{n} f_1(x) & \sum_{x=1}^{n} f_2(x) & \sum_{x=1}^{n} f_3(x) \\ \sum_{x=1}^{n} g_1(x) & \sum_{x=1}^{n} g_2(x) & \sum_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$

    (ii) $\prod_{x=1}^{n} \Delta(x) = \begin{vmatrix} \prod_{x=1}^{n} f_1(x) & \prod_{x=1}^{n} f_2(x) & \prod_{x=1}^{n} f_3(x) \\ \prod_{x=1}^{n} g_1(x) & \prod_{x=1}^{n} g_2(x) & \prod_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$

11. $\det(kA) = k^n \det(A)$, if A is of order $n \times n$.
12. $\det(A^n) = (\det A)^n$, if $n \in I^+$
13. $|A^T| = |A|$, where A^T is a transpose of a matrix.

Symmetric Determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

Skew-symmetric Determinant

$$\begin{vmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{vmatrix} = 0$$

In a skew-symmetric matrix of odd order, the value of determinant is zero.

Cyclic Determinant

The determinant of the circulant matrix is described below.

A circulant matrix is one in which each row vector is rotated one element to the right relative to the preceding row vector.

1. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

2. $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

3. $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
   $= -(a^3+b^3+c^3-3abc)$

5. $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$
   $= abc(a-b)(b-c)(c-a)$

6. $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$

7. $\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$

8. $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix} = 0$

$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$

Minors and Cofactors

The **minor** M_{ij} of the element a_{ij} is the determinant obtained by deleting the ith row and jth column of Δ.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minors are

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ etc.}$$

The **cofactor** C_{ij} of the element a_{ij} is $(-1)^{i+j} M_{ij}$.

Here, cofactors of Δ are

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ etc.}$$

$$\therefore \sum_{i=1}^{3} a_{ij} C_{ik} = \begin{cases} 0, & \text{if } j \neq k \\ \Delta, & \text{if } j = k \end{cases} \quad \text{or} \quad \sum_{j=1}^{3} a_{ij} C_{kj} = \begin{cases} 0, & \text{if } i \neq k \\ \Delta, & \text{if } i = k \end{cases}$$

> *The sum of products of the element of any row with their corresponding cofactors is equal to the value of determinant.*
> i.e., $\Delta = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$
> $= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$
> $= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$

Area of Triangle by using Determinant

Let three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

$$\therefore \text{ Area of } \Delta ABC = \frac{1}{2} \left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\|$$

$$= \frac{1}{2} |[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$$

If three points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

System of Linear Equations in Three and Two Variables

Determinant can be used for solving the system of linear equations in two and three variables and checking the consistency of the system of linear equations.

Let the system of non-homogeneous equations in three variables be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Cramer's Rule

In this method to solve such a linear equations by using Cramer's rule method.

Firstly, determine

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Then, use $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$

Similarly, in this method we can determine the system of linear equation in **two variables**.

Test of Consistency

- If $D \neq 0$, then it is consistent with unique solution.
- If $D = D_1 = D_2 = D_3 = 0$, then it is consistent with infinitely many solutions and it is also called as **non-trivial solution.**
- If $D = 0$ and atleast one of D_1, D_2 and D_3 is non-zero, then it is inconsistent (no solution).

Practice Zone

DAY 6

1. The value of third order determinant is 11, then the value of the square of the determinant formed by the cofactors will be

(a) 11 (b) 121 (c) 1331 (d) 14641

2. If $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then k is equal to

(a) 0 (b) 1 (c) 2 (d) 3

3. $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if

(a) x, y and z are in AP (b) x, y and z are in GP
(c) x, y and z are in HP (d) xy, yz and zx are in AP

4. If a, b and c are positive and not all equal, then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) > 0 (b) ≥ 0
(c) < 0 (d) ≤ 0

5. If the coordinates of the vertices of an equilateral triangle with sides of length a are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is equal to **[NCERT Exemplar]**

(a) $\frac{a^4}{4}$ (b) $\frac{3a^2}{4}$
(c) $\frac{5a^4}{4}$ (d) $\frac{3a^4}{4}$

6. If $A = \begin{bmatrix} 23 & 1+i & -i \\ 1-i & -31 & 4-5i \\ i & 4+5i & 17 \end{bmatrix}$, then det (A) is

(a) complex number with positive real part
(b) complex number with negative imaginary part
(c) pure imaginary
(d) real

7. If $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to **[NCERT Exemplar]**

(a) 0 (b) -1
(c) 2 (d) 3

8. $\begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{(n+1)}P_{(n+1)} & {}^{(n+2)}P_{(n+2)} \\ {}^nC_n & {}^{(n+1)}C_{(n+1)} & {}^{(n+2)}C_{(n+2)} \end{vmatrix}$ is equal to

(a) $n(n!)$
(b) $(n+1)(n+1)!$
(c) $(n+2)(n+2)!$
(d) $(n^2+n+1)n!$

9. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, then the constant term is

(a) 0 (b) 1
(c) -1 (d) 2

10. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = k(a+b+c)^3$, then k is equal to

(a) 0 (b) 1
(c) 2 (d) 3

11. The value of $\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ is

(a) $\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}^2$ (b) $\begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}^2$

(c) $\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\beta & 0 & \cos\beta \\ 0 & \cos\gamma & \sin\gamma \end{vmatrix}^2$ (d) None of these

12. If a, b and c are cube roots of unity, then

$$\begin{vmatrix} e^a & e^{2a} & e^{3a}-1 \\ e^b & e^{2b} & e^{3b}-1 \\ e^c & e^{2c} & e^{3c}-1 \end{vmatrix}$$ is equal to

(a) 0
(b) e
(c) e^2
(d) e^3

13. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then abc is equal to

(a) 2
(b) −1
(c) 1
(d) 0

14. The number of distinct real roots of the equation $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is

(a) 0
(b) 1
(c) 2
(d) 3

15. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value −1. Then,

(a) C is empty
(b) B and C have the same number of elements
(c) $A = B \cup C$
(d) B has twice as many elements as C

16. If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$ is equal to

[NCERT Exemplar]

(a) 0
(b) −1
(c) 1
(d) None of these

17. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(50)$ is equal to

(a) 0
(b) 50
(c) 1
(d) −50

18. If ω is an imaginary cube root of unity, then the value of

$$\begin{vmatrix} a & b\omega^2 & a\omega \\ b\omega & c & b\omega^2 \\ c\omega^2 & a\omega & c \end{vmatrix}$$ is

(a) $a^3 + b^3 + c^2 - 3abc$
(b) $a^2b - b^2c$
(c) 0
(d) $a^2 + b^2 + c^2$

19. If x, y and z are positive, then

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$ is equal to

(a) 0
(b) 1
(c) −1
(d) None of these

20. If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$, then θ is equal to

(a) $\frac{7\pi}{24}, \frac{11\pi}{24}$
(b) $\frac{5\pi}{24}, \frac{7\pi}{24}$
(c) $\frac{11\pi}{24}, \frac{\pi}{24}$
(d) $\frac{\pi}{24}, \frac{7\pi}{24}$

21. If $px^4 + qx^3 + rx^2 + sx + t$

$$= \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix}$$, where p, q, r, s and t are constants, then t is equal to

(a) 0
(b) 1
(c) 2
(d) −1

22. If α, β and γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 0
(b) −2
(c) 2
(d) 4

Directions (Q. Nos. 23 to 25) *Consider the determinant*

$$\Delta = \begin{vmatrix} p & q & r \\ x & y & z \\ l & m & n \end{vmatrix}$$

M_{ij} denotes the minor of an element in ith row and jth column and C_{ij} denotes the cofactor of an element in ith row and jth column.

23. The value of $p \cdot C_{21} + q \cdot C_{22} + r \cdot C_{23}$ is

(a) 0
(b) $-\Delta$
(c) Δ
(d) Δ^2

24. The value of $x \cdot C_{21} + y \cdot C_{22} + z \cdot C_{23}$ is

(a) 0
(b) $-\Delta$
(c) Δ
(d) Δ^2

25. The value of $q \cdot M_{12} - y \cdot M_{22} + m \cdot M_{32}$ is

(a) 0
(b) $-\Delta$
(c) Δ
(d) Δ^2

Directions (Q. Nos. 26 and 27) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

26. Consider the determinant of a skew-symmetric matrix of even order may or may not be zero.

Statement I The determinants of a matrix $A = [a_{ij}]_{5 \times 5}$, where $a_{ij} + a_{ji} = 0$ for i and j are non zero.

Statement II The determinant of a skew-symmetric matrix of odd order is zero.

27. Consider set of rational equations
$x + ky + 3z = 0, 3x + ky - 2z = 0$ and $2x + 3y - 4z = 0$.

Statement I The system of equations posses a non-trivial solution over the set of above rationals equations, then the value of k is $3\frac{1}{2}$.

Statement II For non-trivial solution, $\Delta = 0$.

28. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ {}^3C_1 & {}^4C_1 & {}^5C_1 \\ {}^3C_2 & {}^4C_2 & {}^5C_2 \end{vmatrix}$ is equal to

(a) 1 (b) 2 (c) -1 (d) None of these

29. Determine the value of x, if $\begin{vmatrix} 4+x & 4 & x \\ 4-x & 4 & x \\ 4-x & 4 & -x \end{vmatrix} = 0$

(a) $x = 1/2$ (b) $x = 0$ (c) $x = -1$ (d) None of these

30. If ω is a cube root of unity, then a root of the following $\begin{vmatrix} x-\omega-\omega^2 & \omega & \omega^2 \\ \omega & x-\omega-1 & 1 \\ \omega^2 & 1 & x-1-\omega^2 \end{vmatrix} = 0$ is

(a) $x = 0$ (b) $x = -1$
(c) $x = \omega$ (d) None of these

31. If $\Delta_r = \begin{vmatrix} 2r & x & N(N+1) \\ 6r^2-1 & y & N^2(2N+3) \\ 4r^3-2Nr & z & N^3(N+1) \end{vmatrix}$, where $N \in$ natural number. Then, $\sum_{r=1}^{N} \Delta_r$ is equal to

(a) N (b) N^2
(c) Zero (d) None of these

32. If a, b and c are pth, qth and rth terms of an HP, then $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) term containing a, b, c, p, q and r
(b) a constant
(c) zero
(d) None of the above

33. The points $A(a, b+c)$, $B(b, c+a)$ and $C(c, a+b)$ are

(a) vertices of an isosceles triangle
(b) vertices of an equilateral triangle
(c) collinear
(d) None of the above

34. If the system of equations $x + ay = 0, az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(a) 0 (b) -1
(c) 1 (d) No real values

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35. If a, b and c are sides of a scalene triangle, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is [JEE Main 2013]

(a) non-negative (b) negative (c) positive (d) non-positive

36. Statement I The system of linear equations

$$x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos a)y + (\sin a)z = 0$$
$$x - (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution for only one value of α lying in the interval $\left(0, \frac{\pi}{2}\right)$.

Statement II The equation in α

$$\begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

has only one solution lying in the interval $\left(0, \frac{\pi}{2}\right)$. [JEE Main 2013]

(a) Statement I is true, Statement II is true, Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

37. **Statement I** Determinant of a skew-symmetric matrix of order 3 is zero.

Statement II For any matrix A, $\det(A^T) = \det(A)$ and $\det(-A) = -\det(A)$.

where, $\det(B)$ denotes the determinant of matrix B. Then, **[JEE Main 2013]**

(a) Statement I is true and Statement II is false
(b) Both statements are true
(c) Both statements are false
(d) Statement I is false and Statement II is true

38. Let a, b and c be such that $(b+c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0.$$

Then, the value of 'n' is **[AIEEE 2009]**

(a) zero
(b) any even integer
(c) any odd integer
(d) any integer

39. Let a, b and c be any real numbers. If there are real numbers x, y and z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE 2008]**

(a) 1 (b) 2 (c) -1 (d) 0

40. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $xy \neq 0$, then D is divisible by **[AIEEE 2007]**

(a) Both x and y
(b) x but not y
(c) y but not x
(d) Neither x nor y

41. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree **[AIEEE 2005]**

(a) 0 (b) 1
(c) 2 (d) 3

42. If $a_1, a_2, a_3, \ldots$ are in GP, then

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$ is equal to **[AIEEE 2004]**

(a) 0 (b) 1
(c) 2 (d) 4

Answers

1. (d)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (d)	**6.** (d)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (b)
11. (b)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (a)
21. (a)	**22.** (a)	**23.** (a)	**24.** (c)	**25.** (b)	**26.** (a)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (a)
31. (c)	**32.** (c)	**33.** (c)	**34.** (b)	**35.** (b)	**36.** (d)	**37.** (a)	**38.** (c)	**39.** (a)	**40.** (a)
41. (c)	**42.** (a)								

Hints & Solutions

1. Required determinant $|\text{adj } A|^2 = (11^2)^2 = 14641$

2. Let $\Delta = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} + \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

3. Applying $C_1 \to C_1 - pC_2 - C_3$,

$$\begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2 + 2yp + z) & xp + y & yp + z \end{vmatrix} = 0$$

$\Rightarrow \quad 0 = (xp^2 + 2yp + z)(y^2 - xz)$

$\therefore \quad y^2 = xz$

4. Expanding by the first row, we get

$$a(bc - a^2) - b(b^2 - ca) + c(ab - c^2) = 3abc - a^3 - b^3 - c^3$$
$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= -\frac{(a+b+c)}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

which is negative.

5. If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of a triangle, then

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \ldots(i)$$

Also, we know that, if a be the length of an equilateral triangle, then

$$\text{Area} = \frac{\sqrt{3}}{4}a^2 \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{3}}{4}a^2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}a^2 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

On squaring both sides, we get

$$\frac{3}{4}a^4 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

6. $\bar{A} = \begin{bmatrix} 23 & 1-i & i \\ 1+i & -31 & 4+5i \\ -i & 4-5i & 17 \end{bmatrix}$

$$(\bar{A})^T = A$$

$$\Rightarrow \quad \det(\bar{A})^T = \det(A)$$

$$\Rightarrow \quad \det(\bar{A}) = \det(A)$$

$$\Rightarrow \quad \overline{\det(A)} = \det(A)$$

So, $\det(A)$ is real.

7. Given that, $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$

$$f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t - \cos t & 0 & 2t-1 \\ \sin t - \cos t & 0 & t-1 \end{vmatrix}$$

Expanding along C_2,

$$f(t) = -t\{(2\sin t - \cos t)(t-1) - (2t-1)(\sin t - \cos t)\}$$

$$= -t\{2t\sin t - t\cos t - 2\sin t + \cos t - 2t\sin t + 2t\cos t + \sin t - \cos t\}$$

$$= -t(t\cos t - \sin t)$$

$$\Rightarrow \quad f(t) = t\sin t - t^2\cos t$$

$$\Rightarrow \quad \frac{f(t)}{t^2} = \frac{\sin t}{t} - \cos t$$

$$\Rightarrow \quad \lim_{t \to 0}\frac{f(t)}{t^2} = \lim_{t \to 0}\frac{f(t)}{t^2}$$

$$= \lim_{t \to 0}\left\{\frac{\sin t}{t} - \cos t\right\} \quad \left(\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right)$$

$$= \lim_{t \to 0}\frac{\sin t}{t} - \lim_{t \to 0}\cos t$$

$$= 1 - \cos(0)$$

$$= 1 - 1 = 0$$

8. Let $\Delta = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} n & 1 & 1 \\ n! & n \cdot n! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{pmatrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \end{pmatrix}$$

$$= (n+1)(n+1)! - n \cdot n!$$

$$= [(n+1)^2 - n]\, n! = (n^2 + n + 1) \cdot n!$$

9. Put $x = 0$, then we get $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1$

10. Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3$$

$$\therefore k = 1$$

11. $\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$

12. $\Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$

$$= e^a \cdot e^b \cdot e^c \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$$= 0 \quad (\because a+b+c = 0 \Rightarrow e^{a+b+c} = 1)$$

13. Now, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 + abc = 0 \quad \text{(since, determinant} \neq 0)$$

$$\therefore \quad abc = -1$$

14. Applying $C_1 \to C_1 + C_2 + C_3$ and taking common from C_1,

$$\Rightarrow (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$$\Rightarrow (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & (\sin x - \cos x) & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

$$\Rightarrow \quad (\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$$

$$\Rightarrow \quad \tan x = 1$$

$$\therefore \quad x = \frac{\pi}{4}$$

15. If we interchange any two rows of a determinant in the set B, its value becomes -1 and hence it is in C. Likewise, for every determinant in C, there is corresponding determinant in B.

So, B and C have the same number of elements.

16. Given, A, B and C are the angles of a triangles.

$$\because \quad A + B + C = \pi$$

$$\Rightarrow \quad A + B = \pi - C$$

Now, let $\cos(A+B) = \cos(\pi - C) = -\cos C$

$$\Rightarrow \quad \cos A \cos B - \sin A \sin B = -\cos C$$

$$\Rightarrow \quad \cos A \cos B + \cos C = \sin A \sin B \quad \ldots\text{(i)}$$

Similarly, $\cos A \cos C + \cos B = \sin A \sin C \quad \ldots$(ii)

and $\sin(B + C) = \sin(\pi - A) = \sin A \quad \ldots$(iii)

$$\text{Now, let } \Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = -1(1 - \cos^2 A) + \cos C(\cos C + \cos A \cos B) + \cos B(\cos B + \cos A \cos C)$$

$$= -\sin^2 A + \cos C(\sin A \sin B) + \cos B(\sin A \sin C)$$

[using Eqs. (i) and (ii) and $\cos^2 A + \sin^2 A = 1$]

$$= -\sin^2 A + \sin A(\sin B \cos C + \cos B \sin C)$$

$$= -\sin^2 A + \sin A \sin(B + C)$$

[$\because \sin(x + y) = \sin x \cos y + \cos x \sin y$]

$$= -\sin^2 A + \sin^2 A = 0$$

[using Eq. (iii)]

17. Taking common factors x from C_2, $(x+1)$ from C_3 and $(x-1)$ from R_3, we get

$$f(x) = x(x^2 - 1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

$$= x(x^2 - 1)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -(x+1) & 1 \\ 3x & -2(x+1) & 2 \end{vmatrix} \quad \begin{pmatrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \end{pmatrix}$$

$$= 0$$

$$\therefore \quad f(50) = 0$$

18.

$$\Delta = \begin{vmatrix} a(1+\omega) & b\omega^2 & a\omega \\ b(\omega + \omega^2) & c & b\omega^2 \\ c(\omega^2 + 1) & a\omega & c \end{vmatrix} \quad (C_1 \to C_1 + C_3)$$

$$= \begin{vmatrix} -a\omega^2 & b\omega^2 & a\omega \\ -b & c & b\omega^2 \\ -c\omega & a\omega & c \end{vmatrix} = \omega^2 \cdot \omega \begin{vmatrix} -a & b & a\omega^2 \\ -b & c & b\omega^2 \\ -c & a & c\omega^2 \end{vmatrix}$$

$$= -\omega^3 \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} = 0$$

19.

$$\begin{vmatrix} \dfrac{\log x}{\log x} & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & \dfrac{\log y}{\log y} & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & \dfrac{\log z}{\log z} \end{vmatrix}$$

By taking common factors from the columns and the rows, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

20. Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta + 1 + 4\sin 4\theta = 0 \quad \text{(expanding along } R_3\text{)}$$

$$\Rightarrow \quad \sin 4\theta = -\frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

21. Put $x = 0$ in the given equation, we get

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 12 = 0$$

22. Since, $\alpha + \beta + \gamma = 0$

Applying $C_1 \to C_1 + C_2 + C_3$ in the given determinant,

$$\Delta = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

23. Since, p, q and r the entries of first row and C_{21}, C_{22} and C_{23} are cofactors of second row.

$\therefore \quad p \cdot C_{21} + q \cdot C_{22} + r \cdot C_{23} = 0$

24. Since, x, y and z are the entries of second row and C_{21}, C_{22} and C_{23} are cofactors of second row.

$\therefore \quad x \cdot C_{21} + y \cdot C_{22} + z \cdot C_{23} = \Delta$

25. $q \cdot M_{12} - y \cdot M_{22} + m \cdot M_{32} = -q \cdot C_{12} - y \cdot C_{22} - m \cdot C_{32}$

$= -(q \cdot C_{12} + y \cdot C_{22} + m \cdot C_{32})$

$= -\Delta$

26. $\because \; a_{ij} + a_{ji} = 0 \Rightarrow A = -A^T$

$\Rightarrow \quad |A| = -|A^T| = -|A|$

$\Rightarrow \quad 2|A| = 0$

$\therefore \quad |A| = 0$

27. Let $\Delta = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$, then

$$\Delta = \begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix}$$

$\Rightarrow \; 20k + 33 - 22k = 0$

$\therefore \quad k = \dfrac{33}{2}$

28. Applying $C_3 \to C_3 - C_2, C_2 \to C_2 - C_1$, and using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$$\begin{vmatrix} 1 & 0 & 0 \\ {}^3C_1 & {}^3C_0 & {}^4C_0 \\ {}^3C_2 & {}^3C_1 & {}^4C_1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 3 & 3 & 4 \end{vmatrix} = 1$$

29. Applying row operation $R_3 \to R_3 - R_2$

$$\begin{vmatrix} 4+x & 4 & x \\ 4-x & 4 & x \\ 0 & 0 & -2x \end{vmatrix} = 0$$

$\Rightarrow \quad -16x^2 = 0$

$\therefore \quad x = 0$

30. Applying $C_1 \to C_1 + C_2 + C_3$,

$$\begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix} = 0 \qquad (\because 1 + \omega + \omega^2 = 0)$$

$$\Rightarrow \quad x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix} = 0$$

$\therefore \quad x = 0$

31. Now, $\sum_{r=1}^{N} 2r = N(N+1)$

$$\sum_{r=1}^{N} (6r^2 - 1) = 6N\left[\frac{(N+1)(2N+1)}{6}\right] - N = 2N^3 + 3N^2$$

$$\sum_{r=1}^{N} (4r^3 - 2Nr) = \frac{4N^2(N+1)^2}{4} - \frac{2N \cdot N(N+1)}{2} = N^3(N+1)$$

$$\therefore \quad \sum_{r=1}^{N} \Delta_r = \begin{vmatrix} N(N+1) & x & N(N+1) \\ 2N^3 + 3N^2 & y & N^2(2N+3) \\ N^3(N+1) & z & N^3(N+1) \end{vmatrix} = 0$$

32. Let A be the first term and D be the common difference of the corresponding HP, then

$\dfrac{1}{a} = A + (p-1)D, \dfrac{1}{b} = A + (q-1)D$ and $\dfrac{1}{c} = A + (r-1)D$

Now, $\quad \Delta = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

$$= abc \begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \quad [\because R_1 \to R_1 - D(R_2) - (A-D)R_3]$$

$= 0$

33. Now, area of $(\Delta ABC) = \dfrac{1}{2}\left\|\begin{matrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{matrix}\right\|$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$

$(C_2 \to C_2 + C_1)$

$= 0$

34. For infinite solution, $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 1 + a^3 = 0 \Rightarrow a = -1$

35. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= -(a^3 + b^3 + c^3 - 3abc)$

$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\dfrac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

< 0; where $a \neq b \neq c$

36. **Statement I** The coefficient matrix $A = \begin{bmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{bmatrix}$

For non-trivial solution, put $|A| = 0$

$\Rightarrow \quad \cos 2\alpha = 0 = \cos\frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$

Statement II $\begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$

$\Rightarrow \quad 2\cos\alpha(2\cos^2\alpha - 1) = 0$

$\Rightarrow \quad \cos\alpha = 0, \pm\frac{1}{\sqrt{2}}$

$\therefore \quad \alpha = \frac{\pi}{2}, \pm\frac{\pi}{4}$

37. Determinant of skew-symmetric matrix of odd order is zero and of even order is perfect square.

$\det(A^T) = \det(A)$ and $\det(-A) = (-1)^n \det(A)$

Hence, Statement II is false.

38. $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$

$(\because |A'| = |A|)$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$

$= [1 + (-1)^{n+2}] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$

This is equal to zero only, if $n+2$ is an odd *i.e.*, n is an odd integer.

39. Given, equations are $x - cy - bz = 0$,
$cx - y + az = 0$
and $bx + ay - z = 0$.

For non-zero solution, $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$

$\Rightarrow \quad 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow \quad 1 - a^2 - c^2 - abc - abc - b^2 = 0$

$\therefore \quad a^2 + b^2 + c^2 + 2abc = 1$

40. Applying $R_3 \to R_3 - R_1, R_2 \to R_2 - R_1$, we get

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$

which is divisible by both x and y.

41. Applying $C_1 \to C_1 + C_2 + C_3$ and put $a^2 + b^2 + c^2 = -2$

$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$,

$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (x-1)^2$

42. $a_n = a_1 r^{n-1}$, where r is the common ratio.

$\log a_n = \log a_1 + (n-1)\log r$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ in given determinant,

$\Rightarrow \quad \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ 3\log r & 3\log r & 3\log r \\ 6\log r & 6\log r & 6\log r \end{vmatrix} = 0$

Since, R_2 and R_3 are proportional.

Day 7

Binomial Theorem and Mathematical Induction

Day 7

Outlines ...

- Binomial Theorem
- Binomial Theorem for Positive Index
- Applications of Binomial Theorem
- Binomial Theorem for Rational Index
- Principle of Mathematical Induction

Binomial Theorem

In Binomial theorem describes the algebraic expansion of powers of a Binomial. According to this theorem, it is possible to expand the power $(x+y)^n$ into a sum involving terms of the form $ax^b y^c$, where the exponents b and c are non-negative integers with $b+c=n$ and the coefficient a of each term is a specific positive integer depending on n and b. When an exponent is zero, then the corresponding power is usually omitted from the term. The coefficient a in the term $ax^b y^c$ is known as the Binomial coefficient $\binom{n}{b}$.

Binomial Theorem for Positive Index

An algebraic expression consisting of two terms with + ve or –ve sign between them, then it is called binomial expression.

If n is any positive integer, then $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + \ldots + {}^nC_na^n$ $= \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$, where x and a are **real** (complex) **numbers**.

(i) In the expansion $(x+a)^n$ it contains $(n+1)$ terms.

(ii) In the expansion $(x+a)^n$, the sum of the powers of x and a in each term is equal to n.

(iii) The coefficient of terms equidistant from the beginning and the end are equal.

(iv) The values of the binomial coefficients steadily increase to maximum and then steadily decrease.

(v) $(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + \ldots + (-1)^n \, {}^nC_na^n$

(vi) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

(vii) If n is a positive integer, then the number of terms in $(x+y+z)^n$ is $\dfrac{(n+1)(n+2)}{2}$.

> ▸▸ The number of terms in the expansion of
> $$(x+a)^n + (x-a)^n = \begin{cases} \dfrac{n+2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is even} \end{cases}$$
> ▸▸ The number of terms in the expansion of
> $$(x+a)^n - (x-a)^n = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

General Term and Middle Term

1. Let $(r+1)$th term be the **general term** in the expansion of $(x+a)^n$. $T_{r+1} = {}^nC_rx^{n-r}a^r$

   If expansion is $(x-a)^n$, then the **general term** is
   $$(-1)^n \cdot {}^nC_r\, x^{n-r}a^r.$$

2. The **middle term** in the expansion of $(a+x)^n$.

   Case I If n is even, then $\left(\dfrac{n}{2}+1\right)$th term is middle term.

   Case II If n is odd, then $\dfrac{(n+1)}{2}$th term and $\dfrac{(n+3)}{2}$th terms are middle terms.

3. In the binomial expansion of $(x+a)^n$, the r th term from the end is $(n+1-r+1)$ *i.e.*, $(n-r+2)$th term

> ▸▸ Coefficient of x^m in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ is the coefficient of T_{r+1}, where $r = \dfrac{np-m}{p+q}$
>
> ▸▸ The term independent of x in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ is T_{r+1}, where $r = \dfrac{np}{p+q}$
>
> ▸▸ If the coefficients of rth, $(r+1)$ th, $(r+2)$ th term of $(1+x)^n$ are in AP, then $n^2-(4r+1)n+4r^2=2$

Greatest Term

If T_r and T_{r+1} be the rth and $(r+1)$th terms in the expansion of $(1+x)^n$, then $\dfrac{T_{r+1}}{T_r} = \dfrac{{}^nC_rx^r}{{}^nC_{r-1}x^{r-1}} = \dfrac{n-r+1}{r}\,x$

Let numerically, T_{r+1} be the greatest term in the above expansion.

Then, $T_{r+1} \ge T_r$ or $\dfrac{T_{r+1}}{T_r} \ge 1$

$$\therefore \quad \frac{n-r+1}{r}|x| \ge 1 \text{ or } r \le \frac{(n+1)}{(1+|x|)}|x| \quad \ldots\text{(i)}$$

Now, substituting values of n and x in Eq. (i), we get $r \le m+f$ or $r \le m$, where m is a positive integer and f is a fraction such that $0<f<1$.

When n is even T_{m+1} is the greatest term, when n is odd T_m and T_{m+1} are the greatest terms and both are equal. The coefficients of the middle terms in $(a+x)^n$ are called **greatest coefficients.**

Properties of Binomial Coefficients

In the Binomial expansion of $(1+x)^n$,

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots {}^nC_nx^n,$$

where, ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are the coefficients of various powers of x are called **binomial coefficient** and it is also written as $C_0, C_1, \ldots, C_n$ or $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}$

- ${}^nC_r = {}^nC_s \Rightarrow r = s$ or $r+s=n \Rightarrow {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^{n+1}C_{r+1} = \dfrac{n+1}{r+1}\,{}^nC_r$
- $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
- $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
- $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
- $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n \cdot C_n = 0$
- $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2}$
- $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r} = \dfrac{(2n)!}{(n-r)!(n+r)!}$

  $C_1 - 2C_2 + 3C_3 - \ldots = 0$
- $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)\cdot C_n = (n+2)2^{n-1}$
- $C_0 + \dfrac{C_1}{2}x + \dfrac{C_2}{3}x^2 + \dfrac{C_3}{4}x^3 + \ldots + \dfrac{C_n}{n+1}\cdot x^n = \dfrac{(1+x)^{n+1}-1}{(n+1)\,n}$
- $C_0 - C_2 + C_4 - C_6 + \ldots = \sqrt{2}^n \cos\dfrac{n\pi}{4}$
- $C_1 - C_3 + C_5 - C_7 + \ldots = \sqrt{2}^n \sin\dfrac{n\pi}{4}$

Applications of Binomial Theorem

1. R-f Factor Relation

Here, we are going to discuss problems involving $(\sqrt{A}+B)^n = I + f$, where I and n are positive integers $0 \le f \le 1, |A - B^2| = k$ and $|\sqrt{A} - B| < 1$.

Approach for these type of problems can be learnt from following examples.

2. Divisibility Problem

In the expansion,

$$(1+\alpha)^n = 1 + {}^nC_1\alpha + {}^nC_2\alpha^2 + \ldots + {}^nC_n\alpha^n.$$

We can conclude that,

$(1+\alpha)^n - 1 = {}^nC_1\alpha + {}^nC_2\alpha^2 + \ldots + {}^nC_n\alpha^n$ is divisible by α *i.e.*, it is a multiple of α

3. Differentiability Problem

Sometimes to generalise the result we use the differentiation.

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$$

On differentiating w.r.t., we get

$$n(1+x)^{n-1} = 0 + {}^nC_1 + 2\cdot x\cdot {}^nC_2 + \ldots + n\cdot {}^nC_nx^{n-1}$$

Put $x = 1$, we get

$$n(1+1)^{n-1} = {}^nC_1 + 2\cdot {}^nC_2 + \ldots + n\cdot {}^nC_n$$

$$\Rightarrow \quad n2^{n-1} = {}^nC_1 + 2\cdot {}^nC_2 + \ldots + n\cdot {}^nC_n$$

Binomial Theorem for Rational Index

If n is any rational number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$$

(i) For $|x| < 1$,

(a) $(1-x)^{-3} = 1 + \binom{3}{1}x + \binom{4}{2}x^2 + \ldots\infty$

(b) $(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{n}x^2 + \ldots\infty$

(ii) (a) The coefficient of x^{n-1} in the expansion of $(x-1)(x-2)\ldots(x-n) = -\dfrac{n(n+1)}{2}$

(b) The coefficient of x^{n-1} in the expansion of $(x+1)(x+2)\ldots(x+n) = \dfrac{n(n+1)}{2}$

General Term

Let $(r+1)$th term be the general term in the expansion of $(1+x)^n$.

Then, $$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

(i) If in the above expansion, n is any positive integer, then the series in RHS is finite otherwise infinite.

(ii) $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{1\cdot 2}x^2 + \dfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \ldots$

If n is a positive integer, then

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

- *If n is a positive integer, then $(1+x)^n$ contains $(n+1)$ terms i.e., a finite number of terms. When n is general exponent, then the expansion of $(1+x)^n$ contains infinitely many terms.*
- *When n is a positive integer, the expansion of $(1+x)^n$ is valid for all values of x. If n is general exponent, the expansion of $(1+x)^n$ is valid for the values of x satisfying the condition $|x| < 1$.*

Multinomial Theorem

For any $n \in N$,

(i) $$(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1!\,r_2!} x_1^{r_1} x_2^{r_2}$$

(ii) $$(x_1 + x_2 + \ldots + x_n)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!\,r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2}\ldots x_k^{r_k}$$

The general term in the above expansion is

$$\frac{n!}{r_1!\,r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2}\ldots x_k^{r_k}.$$

Principle of Mathematical Induction

In an algebra, there are certain results that are formulated in terms of n, where n is a positive integer. Such results can be proved by specific techniques, which is known as the principle of mathematical induction.

First Principle of Mathematical Induction

It consists of the following three steps

Step I Actual verification of the proposition for the starting value i.

Step II Assuming the proposition to be true for k, $k \geq i$ and proving that it is true for the value $(k+1)$ which is next higher integer.

Step III To combine the above two steps. Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true *i.e.*, $p(n)$ is true for $n = 1$.

(ii) $p(m+1)$ is true, whenever $p(m)$ is true *i.e.*, $p(m)$ is true $\Rightarrow p(m+1)$ is true. Then, $p(n)$ is true for all natural numbers n.

Second Principle of Mathematical Induction

The second principle of mathematical induction consists of following steps

Step I Actual verification of the proposition for the starting value i and $(i+1)$.

Step II Assuming the proposition to be true for $k-1$ and k and then proving that it is true for the value $k+1$; $k \geq i+1$.

Step II Combining the above two steps. Let $p(n)$ be a statement involving the natural number n such that

(i) $p(1)$ is true *i.e.*, $p(n)$ is true for $n = 1$ and

(ii) $p(m+1)$ is true, whenever $p(m)$ is true for all n, where $i \leq n \leq m$. Then, $p(n)$ is true for all natural numbers.

For $a \neq b$, the expression $a^n - b^n$ is divisible by

(a) $a+b$, if n is even.

(b) $a-b$, if n is odd or even.

- *Product of r consecutive integers is divisible by r!*
- *The sum of first n natural numbers* $= \Sigma n = \dfrac{n(n+1)}{2}$
- *The sum of squares of first n natural numbers* $= \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- *The sum of cubes of first n natural numbers* $= \sum n^3 = \dfrac{n^2(n+1)^2}{4}$

Practice Zone DAY 7

1. If the term from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, then the value of k is **[NCERT Exemplar]**

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

2. The greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

(a) $\binom{20}{7}\frac{1}{27}$ (b) $\binom{20}{6}\frac{1}{81}$
(c) $\frac{1}{9}\binom{20}{9}$ (d) None of these

3. If $\binom{n}{r+1}=56, \binom{n}{r}=28, \binom{n}{r-1}=8$, then $n+r$ is equal to

(a) 8 (b) 10
(c) 12 (d) 9

4. If $(1+x-2x^2)^6=1+a_1x+a_2x^2+\ldots+a_{12}x^{12}$, then the expression $a_2+a_4+\ldots+a_{12}$ is equal to

(a) 32 (b) 63
(c) 64 (d) 31

5. The least positive integer n such that

$$\binom{n-1}{3}+\binom{n-1}{4}>\binom{n}{3} \text{ is}$$

(a) 6 (b) 7 (c) 8 (d) 9

6. The term independent of x in the expansion of $\left(3x-\frac{2}{x^2}\right)^{15}$ is **[NCERT Exemplar]**

(a) $-3003(3^{10})(2^5)$ (b) $-3003(3^{10})(2^4)$
(c) $3003(3^{10})(2^5)$ (d) None of these

7. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, then the value of p is **[NCERT Exemplar]**

(a) ± 3 (b) ± 1
(c) ± 2 (d) None of these

8. The constant term in the expansion of $\left(1+x+\frac{2}{x}\right)^6$ is

(a) 479 (b) 517 (c) 569 (d) 581

9. If the $(r+1)$th term in the expansion of $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ has the same power of a and b, then the value of r is

(a) 9 (b) 10
(c) 8 (d) 6

10. If in the expansion of $\left(2^{1/3}+\frac{1}{3^{1/3}}\right)^n$, the ratio of 7th term from the beginning to the 7th term from the end is 1 : 6, then n is equal to

(a) 7 (b) 8
(c) 9 (d) 10

11. If $C_r=\binom{10}{r}$ then $\sum_{r=1}^{10} C_{r-1}C_r$ is equal to

(a) $\binom{20}{9}$ (b) $\binom{20}{10}$
(c) $\binom{20}{13}$ (d) $\binom{20}{8}$

12. If $n \in N$, then $121^n-25^n+1900^n-(-4)^n$ is divisible by

(a) 1904
(b) 2000
(c) 2002
(d) 2006

13. If $x=(2+\sqrt{3})^n, n \in N$ and $f=x-[x]$, then $\frac{f^2}{1-f}$ is

(a) an irrational number
(b) a non-integer rational number
(c) an odd number
(d) an even number

14. $\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{m}\binom{m}{p}$ is equal to

(a) 3^n (b) 2^n
(c) 3^n+2^n (d) 3^n-2^n

15. If $(1+x)^n=\sum_{r=0}^{n} C_rx^r$, then $\sum_{s=0}^{r}(-1)^s(r-s+1)C_s$ is equal to

(a) $\binom{n-1}{r}$ (b) $(-1)^r\binom{n-1}{r}$
(c) $\binom{n-2}{r}$ (d) $(-1)^r\binom{n-2}{r}$

16. The value of $\binom{40}{31}+\sum_{r=0}^{10}\binom{n+r}{30}$ is

(a) $2\binom{45}{15}$ (b) $2\binom{50}{20}$ (c) $\binom{50}{20}$ (d) $\binom{51}{20}$

17. $C_1-\frac{C_2}{2}+\frac{C_3}{3}-\ldots+(-1)^{n-1}\frac{C_n}{n}$ is equal to

(a) $1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{(-1)^{n-1}}{n}$ (b) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$

(c) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}$ (d) None of these

18. The coefficient of x in the expansion of $(1-3x+7x^2)(1-x)^{16}$ is **[NCERT Exemplar]**

(a) 19 (b) −19 (c) 18 (d) −18

19. $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots \text{ to } m \text{ terms}\right]$ is equal to

(a) $\frac{1-2^{-2mn}}{1-2^{-n}}$ (b) $\frac{1+2^{-mn}}{1-2^{-n}}$ (c) $\frac{1-2^{-mn}}{2^n-1}$ (d) $\frac{1-2^{-mn}}{1+2^n}$

20. If $a_n=\sqrt{7+\sqrt{7+\sqrt{7\ldots}}}$ having n radical signs, then by method of mathematical induction, which of the following is true?

(a) $a_n>7, \forall n\ge 1$ (b) $a_n>3, \forall n\ge 1$

(c) $a_n<4, \forall n\ge 1$ (d) $a_n<3, \forall n\ge 1$

21. If $a_n=2^{2^n}+1$, then for $n>1$, last digit of a_n is

(a) 3 (b) 5 (c) 7 (d) 8

22. For each $n\in N, 2^{3n}-1$ is divisible by

(a) 8 (b) 16 (c) 32 (d) None of these

23. If $P(n)=2+4+6+\ldots+2n, n\in N$, then $P(k)=k(k+1)+2$ $\Rightarrow P(k+1)=(k+1)(k+2)+2$ for all $k\in N$. So, we can conclude that $P(n)=n(n+1)+2$ for

(a) all $n\in N$ (b) $n>1$

(c) $n>2$ (d) Nothing can be said

24. For all $n\in N$ $\cos\alpha\cos2\alpha\cos4\alpha\ldots\cos(2^{n-1}\alpha)$ is equal to

(a) $\frac{\sin(2^n\alpha)}{2\sin\alpha}$ (b) $\frac{\sin(2^n\alpha)}{2^n\sin\alpha}$

(c) $\frac{\cos(2^n\alpha)}{2^n\cos\alpha}$ (d) None of these

25. The statement

$P(n): 1\times1!+2\times2!+3\times3!+\ldots+n\times n!=(n+1)!-1$ is

(a) true for all $n>1$ (b) not true for any

(c) true for all $n\in N$ (d) None of these

26. The minimum value of $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}, \forall n\in N$ is

(a) $2(\sqrt{n}-1)$ (b) $(\sqrt{n+1}-1)$

(c) $(\sqrt{n+1}+3)$ (d) None of these

Directions (Q. Nos. 27 and 28) *Let us consider the series* $S_n=2\cdot7^n+3\cdot5^n-5$.

27. S_n is divisible by the multiple of

(a) 5 (b) 7

(c) 24 (d) None of these

28. If S_n is divisible for every n, then S_n is

(a) >0 (b) >1

(c) >5 (d) None of these

Directions (Q. Nos. 29 to 32) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

29. Consider a binomial theorem

$(a+b)^n={}^nC_0\,a^n+{}^nC_1\,a^{n-1}\,b+\ldots+{}^nC_n\,b^n$

Statement I If n is even, then

$${}^{2n}C_1+{}^{2n}C_3+\ldots+{}^{2n}C_{n-1}=2^{2n-1}.$$

Statement II ${}^{2n}C_1+{}^{2n}C_3+\ldots+{}^{2n}C_{2n-1}=2^{2n-1}$.

30. Statement I The coefficient of x^n in the binomial expansion of $(1-x)^{-2}$ is $(n+1)$.

Statement II The coefficient of x^r in $(1-x)^{-n}$ when $n\in N$ is ${}^{n+r-1}C_r$.

31. Statement I $\frac{(n+2)!}{(n-1)!}$ is divisible by 6.

Statement II Product of 3 consecutive integers is divisible by 3!.

32. Consider a mathematical induction such that it is true for $P(1)$ and $P(k)$.

Statement I For all $n\in N$, $3\cdot5^{2n+1}+2^{3n+1}$ is divisible by 17.

Statement II For all $n\in N$, $10^{2n-1}+1$ is divisible by 11.

33. The coefficient of x^3 in the expression of $(1+x+x^2+x^3)^{10}$ is

(a) 201 (b) 220

(c) 211 (d) None of these

34. The value of $({}^{12}C_0)^2-({}^{12}C_1)^2+({}^{12}C_2)^2-\ldots+({}^{12}C_{12})^2$ is

(a) ${}^{12}C_5$ (b) ${}^{12}C_7$

(c) ${}^{12}C_6$ (d) None of these

35. If the coefficient of x^5 in $\left[ax^2 + \frac{1}{bx}\right]^{10}$ is a times and equal to the coefficient of x^{-5} in $\left[ax - \frac{1}{b^2x^2}\right]^{10}$, then the value of ab is

(a) $(b)^{-3}$
(b) $-(b)^6$
(c) $(b)^{-1}$
(d) None of the above

36. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$ is equal to

(a) $\binom{30}{11}$ (b) $\binom{60}{10}$ (c) $\binom{30}{10}$ (d) $\binom{65}{55}$

37. If ${}^{n-1}C_r = (k^2 - 3) \cdot {}^nC_{r+1}$, then k belongs to

(a) $(-\infty, -2]$ (b) $[2, \infty)$
(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

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38. The ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^{15}$ is **[JEE Main 2013]**

(a) 7 : 16 (b) 7 : 64
(c) 1 : 4 (d) 1 : 32

39. If the 7th term in the binomial expansion of $\left(\frac{3}{\sqrt[3]{84}} + \sqrt{3} \ln x\right)^9$, $x > 0$ is equal to 729, then x can be **[JEE Main 2013]**

(a) e^2 (b) e (c) $\frac{e}{2}$ (d) $2e$

40. The sum of the rational terms in the binomial expansion of $(2^{1/2} + 3^{1/5})^{10}$ is **[JEE Main 2013]**

(a) 25 (b) 32 (c) 9 (d) 41

41. If for positive integers $r > 1, n > 2$, the coefficients of the $(3r)$ th and $(r + 2)$ th powers of x in the expansion of $(1 + x)^{2n}$ are equal, then n is equal to **[JEE Main 2013]**

(a) $2r + 1$ (b) $2r - 1$ (c) $3r$ (d) $r + 1$

42. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is **[JEE Main 2013]**

(a) 4 (b) 120 (c) 210 (d) 310

43. If n is a positive integer, then $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$ is **[AIEEE 2012]**

(a) an irrational number
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

44. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is **[AIEEE 2011]**

(a) -132 (b) -144
(c) 132 (d) 144

45. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is **[AIEEE 2009]**

(a) 0 (b) 2 (c) 7 (d) 8

Directions (Q. Nos. 46 and 47) *Each of these questions contains two statements : Statement I (Assertion) and statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answers. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

46. Statement I For every natural number, $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

Statement II For every natural number, $n \geq 2$.

$$\sqrt{n(n+1)} < n + 1$$ **[AIEEE 2008]**

47. Statement I $\sum_{r=0}^{n} (r + 1) \cdot {}^nC_r = (n + 2)^{n-1}$

Statement II $\sum_{r=0}^{n} (r + 1)\, {}^nC_r \cdot x^r = (1 + x)^n + nx(1 + x)^{n-1}$ **[AIEEE 2008]**

48. In the expansion of $(a - b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ is equal to **[AIEEE 2007]**

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$
(c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$

49. If the expansion, in powers of x of the function $\frac{1}{(1 - ax)(1 - bx)}$ is $a_0 + a_1x + a_2x^2 + \ldots$, then a_n is **[AIEEE 2006]**

(a) $\frac{a^n - b^n}{b - a}$ (b) $\frac{a^{n+1} - b^{n+1}}{b - a}$
(c) $\frac{b^{n+1} - a^{n+1}}{b - a}$ (d) $\frac{b^n - a^n}{b - a}$

50. If the coefficients of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ are equal, then **[AIEEE 2005]**

(a) $a+b=1$ (b) $a-b=1$ (c) $ab=-1$ (d) $ab=1$

51. If $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then which one of the following holds for all $n\ge 1$, by the principle of mathematical induction? **[AIEEE 2005]**

(a) $A^n=2^{2n-1}A+(n-1)I$ (b) $A^n=nA+(n-1)I$

(c) $A^n=2^{n-1}A-(n-1)I$ (d) $A^n=nA-(n-1)I$

52. If $S(k)=1+3+5+\ldots+(2k-1)=3+k^2$. Then, which of the following is true? **[AIEEE 2004]**

(a) $S(1)$ is correct

(b) $S(k)\Rightarrow S(k+1)$

(c) $S(k)\nRightarrow S(k+1)$

(d) Principle of mathematical induction can be used to prove the formula

53. The coefficients of x^n in the expansion of $(1+x)(1-x)^n$ is **[AIEEE 2004]**

(a) $n-1$

(b) $(-1)^n(1-n)$

(c) $(-1)^{n-1}(n-1)^2$

(d) $(-1)^{n-1}x$

54. The coefficients of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same, if α is equal to **[AIEEE 2004]**

(a) $-\frac{5}{3}$ (b) $\frac{3}{5}$

(c) $-\frac{3}{10}$ (d) $\frac{10}{3}$

55. The number of integral terms in the expansion of $(\sqrt{3}+5^{1/8})^{256}$ is **[AIEEE 2003]**

(a) 32 (b) 33

(c) 34 (d) 35

Answers

1. (c)	**2.** (a)	**3.** (b)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (c)
11. (a)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (d)	**16.** (d)	**17.** (b)	**18.** (b)	**19.** (c)	**20.** (c)
21. (c)	**22.** (d)	**23.** (d)	**24.** (b)	**25.** (c)	**26.** (a)	**27.** (c)	**28.** (a)	**29.** (d)	**30.** (a)
31. (a)	**32.** (b)	**33.** (b)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (d)	**38.** (d)	**39.** (b)	**40.** (d)
41. (a)	**42.** (c)	**43.** (a)	**44.** (b)	**45.** (b)	**46.** (b)	**47.** (a)	**48.** (b)	**49.** (c)	**50.** (d)
51. (d)	**52.** (b)	**53.** (b)	**54.** (c)	**55.** (b)					

Hints & Solutions

1. General term of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$

$$T_{r+1}={}^{10}C_r(\sqrt{x})^{10-r}\left(-\frac{k}{x^2}\right)^r$$

$$={}^{10}C_r x^{\frac{10-r}{2}-2r}(-k)^r={}^{10}C_r(-k)^r x^{\frac{10-5r}{2}}$$

The term free from x.

Put $\frac{10-5r}{2}=0\Rightarrow r=2$

$\therefore {}^{10}C_2(-k)^2=405\Rightarrow \frac{10\times 9}{2}\times k^2=405\Rightarrow k=\pm 3$

2. Greatest term in the expansion of $(1+x)^n$ is T_{r+1}

where, $r=\left[\frac{(n+1)x}{1+x}\right]$

Here, $n=20$, $x=\frac{1}{\sqrt{3}}$

$\therefore \quad r=\left[\frac{21}{\sqrt{3}+1}\right]=[10.5(\sqrt{3}-1)]=(7.69)\approx 7$

Hence, greatest term is $\sqrt{3}\binom{20}{7}\left(\frac{1}{\sqrt{3}}\right)^7=\binom{20}{7}\frac{1}{27}$.

3. Now, $\binom{n}{r+1}\div\binom{n}{r}=\frac{56}{28}=2$

$\Rightarrow \quad \frac{n-r}{r+1}=2$

$\Rightarrow \quad n=3r+2$ ...(i)

and $\binom{n}{r}\div\binom{n}{r-1}=\frac{28}{8}=\frac{7}{2}$

$\Rightarrow \quad \frac{n-r+1}{r}=\frac{7}{2}$

$\Rightarrow \quad 2n=9r-2$ ...(ii)

On solving Eqs. (i) and (ii), we get

$n=8,\ r=2$

$\therefore \quad n+r=10$

4. On putting $x=1$ and $x=-1$ respectively, we get

$0=1+a_1+a_2+\ldots+a_{12}$

and $64=1-a_1+a_2-\ldots+a_{12}$

On adding, we get

$64=2(1+a_2+a_4+\ldots+a_{12})$

$\therefore \quad a_2+a_4+\ldots+a_{12}=31$

5. Given, $\binom{n-1}{3}+\binom{n-1}{4}>\binom{n}{3}$

$\Rightarrow \binom{n}{4}>\binom{n}{3}$

$\Rightarrow \dfrac{\binom{n}{4}}{\binom{n}{3}}>1 \Rightarrow \dfrac{n-4+1}{4}>1 \Rightarrow n>7$

6. The general term of $\left(3x-\dfrac{2}{x^2}\right)^{15}$ is

$T_{r+1}={}^{15}C_r\,(3x)^{15-r}\left(\dfrac{-2}{x^2}\right)^r={}^{15}C_r(3)^{15-r}(-2)^r x^{15-3r}$

For independent term of x,

put $15-3r=0 \Rightarrow r=5$

Coefficient $={}^{15}C_5(3)^{15-5}(-2)^5=-3003(3^{10})(2^5)$

7. Given expression is $\left(\dfrac{p}{2}+2\right)^8$

Here, $n=8$ (even)

$\Rightarrow$ Middle term $=\left(\dfrac{8}{2}+1\right)$th $=5$th

$T_5={}^8C_4(p/2)^{8-4}(2^4)$

$\Rightarrow \dfrac{8\times7\times6\times5}{4\times3\times2\times1}\times\dfrac{p^4}{2^4}\times2^4=1120$

$\Rightarrow p^4=16$

$\Rightarrow p=\pm2$

8. $\left(1+x+\dfrac{2}{x}\right)^6=1+\binom{6}{1}\left(x+\dfrac{2}{x}\right)+\binom{6}{2}\left(x+\dfrac{2}{x}\right)^2$

$+\ldots+\binom{6}{6}\left(x+\dfrac{2}{x}\right)^6$

Constant term is $1+\binom{6}{2}\binom{2}{1}2^1+\binom{6}{4}\binom{4}{2}2^2+\binom{6}{6}\binom{6}{3}2^3$

$=1+60+360+160$

$=581$

9. $\therefore$ General term is $T_{r+1}={}^{21}C_r\left(\sqrt[3]{\dfrac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\dfrac{b}{\sqrt[3]{a}}}\right)^r$

$={}^{21}C_r\,a^{7-\frac{r}{2}}\,b^{\frac{2r}{3}-\frac{7}{2}}$

$\because$ Power of a = Power of b (given)

$\Rightarrow 7-\dfrac{r}{2}=\dfrac{2}{3}r-\dfrac{7}{2}$

$\therefore r=9$

10. Here, $\binom{n}{6}\left(2^{\frac{1}{3}}\right)^{n-6}\left(\dfrac{1}{3^{1/3}}\right)^6:\binom{n}{n-6}(2^{1/3})^6\left(\dfrac{1}{3^{1/3}}\right)^{n-6}=1:6$

$\Rightarrow 6\cdot2^{\frac{n}{3}-4}\cdot3^{\frac{n}{3}-4}=1$

$\Rightarrow 6^{\frac{n}{3}-3}=1=6^0$

$\therefore n=9$

11. Since, $(1+x)^{10}=C_0+C_1x+C_2x^2+\ldots+C_{10}x^{10}$

and $\left(1+\dfrac{1}{x}\right)^{10}=C_0+\dfrac{C_1}{x}+\dfrac{C_2}{x^2}+\ldots+\dfrac{C_{10}}{x^{10}}$

So, $C_0C_1+C_1C_2+C_2C_3+\ldots+C_9C_{10}$ is the coefficient of x in the product $(1+x)^{10}\left(1+\dfrac{1}{x}\right)^{10}=\dfrac{(1+x)^{20}}{x^{10}}$

or the coefficient of x^{11} in $(1+x)^{20}$ is

$\binom{20}{11}=\binom{20}{9}$

12. *Case* I $121^n-25^n=(96+25)^n-25^n$ is divisible by 96.

$1900^n-(-4)^n=(1904-4)^n-(-4)^n$ is divisible by 1904.

Both are divisible by 16.

Case II $121^n-(-4)^n=(125-4)^n-(-4)^n$ is divisible by 125 and $1900^n-25^n=(1875+25)^n-25^n$ is divisible by 1875.

Hence, both are divisible by 125.

$\therefore$ Given number is divisible by $16\times125=2000$

13. $(2+\sqrt{3})^n=[x]+f$

Let $(2-\sqrt{3})^n=f',\ 0<f'<1$

$\therefore [x]+f+f'=(2+\sqrt{3})^n+(2-\sqrt{3})^n$

$=2\left[\binom{n}{0}2^n+\binom{n}{2}2^{n-2}3+\ldots\right]$, an even integer

$\therefore [x]$ is an odd integer, $f'=1-f$

But $([x]+f)f'=(2+\sqrt{3})^n(2-\sqrt{3})^n=1$

$\therefore [x]=\dfrac{1}{f'}-f=\dfrac{1}{1-f}-f$

$=\dfrac{1-f+f^2}{1-f}$

So, $\dfrac{f^2}{1-f}=[x]-1$, an even number.

14. Since, $\binom{n}{m}\binom{m}{p}=\dfrac{n!}{(n-m)!\,p!\,(m-p)!}=\binom{n}{p}\binom{n-p}{m-p}$

$\therefore$ Given series can be rewritten as,

$\sum_{p=1}^{n}\sum_{m=p}^{n}\binom{n}{p}\binom{n-p}{m-p}$

$=\sum_{p=1}^{n}\binom{n}{p}\sum_{m=p}^{n}\binom{n-p}{m-p}$

$=\sum_{p=1}^{n}\binom{n}{p}\sum_{t=0}^{n-p}\binom{n-p}{t}$

$=\sum_{p=1}^{n}\binom{n}{p}2^{n-p}$ (put $m-p=t$)

$=2^n\sum_{p=1}^{n}\binom{n}{p}\cdot\dfrac{1}{2^p}$

$=2^n\left[\left(1+\dfrac{1}{2}\right)^n-1\right]=3^n-2^n$

51. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$, $a, b \in N$, then

52. The system of equations

15. From infinite GP, we have

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

On differentiating, we get

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$$

But $\quad (1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots$

On multiplying the above two and considering the coefficient of x^r, we get

$$(r+1)C_0 - rC_1 + (r-1)C_2 - (r-2)C_3 + \dots + (-1)^r C_r$$

$$= \text{Coefficient of } x^r \text{ in } (1-x)^{n-2} = (-1)^r \binom{n-2}{r}$$

16. Given series is

$$\binom{40}{31} + \sum_{r=0}^{10} \binom{40+r}{30} = \binom{40}{31} + \text{Coefficient of } x^{30} \text{ in}$$

$$[(1+x)^{40} + (1+x)^{41} + \dots + (1+x)^{50}]$$

$$= \binom{40}{31} + \text{Coefficient of } x^{30} \text{ in } (1+x)^{40}\frac{[(1+x)^{11}-1]}{x} (\because \text{GP sum})$$

$$= \binom{40}{31} + \text{Coefficient of } x^{31} \text{ in } [(1+x)^{51} - (1+x)^{40}]$$

$$= \binom{40}{31} + \binom{51}{31} - \binom{40}{31} = \binom{51}{31} = \binom{51}{20}$$

17. Since, $(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots$

$$\Rightarrow 1 - (1-x)^n = C_1x - C_2x^2 + C_3x^3 - \dots$$

$$\Rightarrow \frac{1-(1-x)^n}{x} = C_1 - C_2x + C_3x^2 - \dots$$

$$\Rightarrow \int_0^1 (C_1 - C_2x + C_3x^2 - \dots)\,dx = \int_0^1 \frac{1-(1-x)^n}{1-(1-x)}\,dx$$

$$\Rightarrow \frac{C_1}{1} - \frac{C_2}{2} + \frac{C_3}{3} - \dots = \int_0^1 \frac{1-x^n}{1-x}\,dx$$

$$\left[\because \int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx\right]$$

$$= \int_0^1 (1 + x + x^2 + \dots + x^{n-1})\,dx$$

$$= \left[x + \frac{x^2}{2} + \dots + \frac{x^n}{n}\right]_0^1 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

18. $(1-3x+7x^2)(1-x)^{16}$

$$= (1-3x+7x^2)({}^{16}C_0 - {}^{16}C_1x + {}^{16}C_2x^2 + \dots)$$

After multiplying the term containing x is $-{}^{16}C_1x - 3\,{}^{16}C_0x$.

Coefficient of $x = -16 - 3 = -19$

19. Given, series is $\sum (-1)^r \binom{n}{r}\frac{1}{2^r} + \sum (-1)^r \binom{n}{r}\left(\frac{3}{4}\right)^r + \dots$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \dots$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \dots \text{ to } m \text{ terms}$$

$$= \frac{1}{2^n}\frac{\left(1-\frac{1}{2^{mn}}\right)}{1-\frac{1}{2^n}} = \frac{1-2^{-mn}}{2^n-1} \quad \text{(GP series)}$$

20. We have, $a_1 = \sqrt{7} < 4$

Assume $a_k < 4$ for some natural number k.

$$\therefore \quad a_{k+1} = \sqrt{7+a_k} < \sqrt{7+4} < 4$$

$$\Rightarrow \quad a_n < 4, \forall n \ge 1$$

21. For $n = 2$, $a_2 = 2^{2^2} + 1 = 16 + 1 = 17$

Assume that, $a_k = 2^{2^k} + 1 = 10m + 7$, where $k > 1$ and m is some positive integer.

Now, $\quad a_{k+1} = 2^{2^{k+1}} + 1 = \left(2^{2^k}\right)^2 + 1$

$$= (10m+6)^2 + 1$$

$$= 10(10m^2 + 12m + 3) + 7$$

Hence, last digit of a_n is 7, $\forall n > 1$.

22. Now, $2^{3n} - 1 = (2^3)^n - 1$

$$= (1+7)^n - 1$$

$$= 1 + {}^nC_1(7) + {}^nC_2(7)^2 + \dots + {}^nC_n(7)^n - 1$$

$$= 7[{}^nC_1 + {}^nC_2 7 + \dots + {}^nC_n 7^{n-1}]$$

Hence, 7 divides $2^{3n} - 1$ for all $n \in N$.

23. Here, $P(1) = 2$ and from the given equation,

$$P(k) = k(k+1) + 2$$

$$\Rightarrow \quad P(1) = 4$$

So, $P(1)$ is not true.

Similarly, for $P(2)$ is not true.

Hence, mathematical induction is not applicable.

24. Put $n = 1$ in option (b), we get $\dfrac{\sin 2\alpha}{2\sin\alpha} = \cos\alpha$

Again, put $n = 2$, we get

$$\frac{\sin 4\alpha}{4\sin\alpha} = \frac{4\sin\alpha\cos\alpha\cos 2\alpha}{4\sin\alpha} = \cos\alpha\cos 2\alpha$$

25. Given, $P(n): 1\times 1! + 2\times 2! + 3\times 3! + \dots + n\times n!$

$$= (n+1)! - 1$$

Put $n = 1, 2, 3, \dots, n$ is RHS, which is equal to LHS, respectively.

26. Let $P(n) = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dots + \dfrac{1}{\sqrt{n}}, \ \forall n \in N$

For $n = 1$,

$$P(1) = 1 > 2(\sqrt{2} - 1)$$

Again for $n = 2$,

$$P(2) = \frac{1+\sqrt{2}}{\sqrt{2}} > 2(\sqrt{3} - 1)$$

Similarly for k,

$$P(k) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{k}}$$

$$= \frac{\sqrt{2\times 3\times 4\times\dots\times k} + \sqrt{1\times 3\times 4\times\dots\times k} + \dots + \sqrt{1\times 2\times 3\times\dots\times(k-1)}}{\sqrt{2\times 3\times 4\times 5\times\dots\times k}} > 2(\sqrt{k} - 1)$$

Hence, it is true for all natural number N.

27. Let $P(n): 2\cdot 7^n + 3\cdot 5^n - 5$

Then, $P(1): 2\cdot 7 + 3\cdot 5 - 5 = 24$

Let $P(m)$ be true

$\therefore \quad P(m): 2\cdot 7^m + 3\cdot 5^m - 5 = 24k,\ k \in N$ ...(i)

Now, $P(m+1) - P(m)$:

$2(7^{m+1} - 7^m) + 3(5^{m+1} - 5^m)$

$= 2\cdot 7^m(7-1) + 3\cdot 5^m(5-1) = 12(7^m + 5^m)$

Since, 7^m and 5^m are odd integer and for all $m \in N$, their sum must be even say $7^m + 5^m = 2\lambda, \lambda \in N$

$\therefore \quad P(m+1) - P(m): 24\lambda$

$\Rightarrow \quad P(m+1): 24\lambda + P(m)$

$\Rightarrow \quad P(m+1): 24(\lambda + k)$ [from Eq. (i)]

Here, $P(m+1)$ is divisible by 24.

28. Hence, S_n is divisible for every n greater than zero.

29. Since, n is even, put $n = 2$

LHS $= {}^4C_1 = 4$ and RHS $= 2^3 = 8$

Hence, Statement I is false, but Statement II is true.

30. Since, coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$

$\therefore$ Coefficient of x^n in $(1-x)^{-2} = {}^{2+n-1}C_n = {}^{n+1}C_n = (n+1)$

31. $\dfrac{(n+2)!}{(n-1)!} = \dfrac{(n+2)(n+1)n(n-1)!}{(n-1)!}$

$= (n+2)(n+1)n$

It is a product of 3 consecutive integrals and it is divisible by 3!.

32. Let $P_1(n): 3\cdot 5^{2n+1} + 2^{3n+1}$

$P_1(1): 3\cdot 5^3 + 2^4 = 3(125) + 16 = 391$

$P_1(2): 3\cdot 5^5 + 2^7 = 3\cdot 3125 + 128 = 9503$

So, it is divisible by 17.

Let $P_2(n): 10^{2n-1} + 1$

$P_2(1): 10 + 1 = 11$

$P_2(2): 10^3 + 1 = 1001$

So, it is divisible by 11.

33. $(1 + x + x^2 + x^3)^{10} = (1+x)^{10}(1+x^2)^{10}$

$= ({}^{10}C_0 + {}^{10}C_1 x + {}^{10}C_2 x^2 + \dots)$

$\times ({}^{10}C_0 + {}^{10}C_1 x^2 + {}^{10}C_2 x^4 + \dots)$

The coefficient of x^3 in given expression is

${}^{10}C_1 \cdot {}^{10}C_1 + {}^{10}C_3 \cdot {}^{10}C_0 = 10\times 10 + 120 \times 1 = 220$

34. $\because (x+1)^{12} = {}^{12}C_0 x^{12} + {}^{12}C_1 x^{11} + {}^{12}C_2 x^{10} + \dots + {}^{12}C_{12}$ ...(i)

and $(1-x)^{12} = {}^{12}C_0 - {}^{12}C_1 x + {}^{12}C_2 x^2 - \dots + {}^{12}C_{12} x^{12}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$(x^2-1)^{12} = ({}^{12}C_0 x^{12} + {}^{12}C_1 x^{11} + {}^{12}C_2 x^{10} + \dots + {}^{12}C_{12})$

$\times ({}^{12}C_0 - {}^{12}C_1 x + {}^{12}C_2 x^2 - \dots + {}^{12}C_{12} x^{12})$

On comparing the coefficient of x^{12}, we get

$${}^{12}C_6 = ({}^{12}C_0)^2 - ({}^{12}C_1)^2 + \dots - ({}^{12}C_{11})^2 + ({}^{12}C_{12})^2$$

35. The general term is

$$T_{r+1} = {}^{10}C_r \cdot (ax^2)^{10-r} \cdot \left(\frac{1}{bx}\right)^r$$

$$= {}^{10}C_r \cdot (a)^{10-r}\left(\frac{1}{b}\right)^r (x)^{20-3r}$$

Since, x^5 occurs in T_{r+1}.

$\therefore \quad 20 - 3r = 5 \Rightarrow 3r = 15$

$\Rightarrow \quad r = 5$

So, the coefficient of x^5 is ${}^{10}C_5 (a)^5 (b)^{-5}$.

Again, let x^{-5} occurs in T_{r+1} of $\left[ax - \dfrac{1}{b^2x^2}\right]^{10}$ is

$${}^{10}C_r (ax)^{10-r}\left(-\frac{1}{b^2x^2}\right)^r = {}^{10}C_r (a)^{10-r}\left(-\frac{1}{b^2}\right)^r (x)^{10-3r}$$

$10 - 3r = -5 \Rightarrow 15 = 3r \Rightarrow r = 5$

So, the coefficient of x^{-5} is $-{}^{10}C_5 \dfrac{a^5}{b^{10}}$.

According to the given condition,

$${}^{10}C_5 \frac{a^5}{b^5} = -a\,{}^{10}C_5 \frac{a^5}{b^{10}}$$

$\Rightarrow \quad -b^5 = a \Rightarrow -b^6 = ab$

36. Since, $(1-x)^{30} = \binom{30}{0} - \binom{30}{1}x + \binom{30}{2}x^2 - \dots + \binom{30}{30}x^{30}$

and $(1+x)^{30} = \binom{30}{0}x^{30} + \binom{30}{1}x^{29} + \binom{30}{2}x^{28}$

$+ \dots + \binom{30}{10}x^{20} + \dots + \binom{30}{30}$

Multiply above two and consider the coefficients of x^{20} on both sides, we get the given series is the coefficient of x^{20} in $(1-x)^{30}(x+1)^{30} = (1-x^2)^{30}$ which is $\binom{30}{20}$ or $\binom{30}{10}$.

37. Since, ${}^{n-1}C_r = (k^2 - 3)\dfrac{n}{r+1} \cdot {}^{n-1}C_r$

$\Rightarrow \quad k^2 - 3 = \dfrac{r+1}{n}$

$\Rightarrow \quad 0 < k^2 - 3 \le 1 \quad \left[\because n \ge r \Rightarrow \dfrac{r+1}{n} \le 1 \text{ and } n, r > 0\right]$

$\Rightarrow \quad 3 < k^2 \le 4$

$\Rightarrow \quad k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

38. $T_{r+1} = {}^{15}C_r (x^2)^{15-r} \cdot \left(\dfrac{2}{x}\right)^r = {}^{15}C_r\, x^{30-2r} \cdot 2^r \cdot x^{-r}$

$= {}^{15}C_r \cdot x^{30-3r} \cdot 2^r$ ...(i)

For coefficient of x^{15}, put $30 - 3r = 15 \Rightarrow 3r = 15 \Rightarrow r = 5$

$\therefore$ Coefficient of $x^{15} = {}^{15}C_5 \cdot 2^5$

For coefficient of independent of x i.e., x^0, put $30 - 3r = 0$

$\Rightarrow r = 10$

$\therefore$ Coefficient of $x^0 = {}^{15}C_{10} \cdot 2^{10}$

51. If $A = \begin{bmatrix} 1 & 2 \\ \cdot & \cdot \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ \cdot & \cdot \end{bmatrix}$, $a, b, \in N$, then

52. The system of equations

By condition $\Rightarrow \dfrac{\text{Coefficient of } x^{15}}{\text{Coefficient of } x^0} = \dfrac{{}^{15}C_6 \cdot 2^5}{{}^{15}C_{10} \cdot 2^{10}}$

$$= \frac{{}^{15}C_{10} \cdot 2^5}{{}^{15}C_{10} \cdot 2^{10}} = 1:32$$

39. $T_7 = {}^9C_6 \left(\dfrac{3}{3\sqrt{84}}\right)^3 (\sqrt{3} \ln x)^6 = 729$

$\Rightarrow \dfrac{84 \times 3^3}{84} \times 3^3 \times (\ln x)^6 = 729$

$= (\ln x)^6 = 1 \Rightarrow x = e$

40. General term, $T_{r+1} = {}^{10}C_r (2^{1/2})^{10-r} (3^{1/5})^r$

$$= {}^{10}C_r\, 2^{\left(5 - \frac{r}{2}\right)} \cdot 3^{r/5}$$

Clearly, T_{r+1} will be independent of radical sign if $\pi/2$ and $r/5$ are integers, where $0 \le r \le 10$

$\therefore \quad r = 0, 10.$

$T_{0+1} = T_1 = {}^{10}C_0 2^5 \cdot 3^0 = 32,$

$T_{10+1} = T_{11} = {}^{10}C_{10} \cdot 2^0 \cdot 3^2 = 9$

$\therefore$ Required Sum $= 32 + 9 = 41$

42. $\left[\dfrac{x+1}{x^{2/3} - x^{1/3} + 1} - \dfrac{(x-1)}{x - x^{1/2}}\right]^{10}$

$$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)}\right]^{10}$$

$$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}}\right]^{10}$$

$$= (x^{1/3} - x^{-1/2})^{10}$$

So, the general term is

$$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r = {}^{10}C_r (-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$$

For independent of x, put

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0$$

$\Rightarrow \quad 20 = 5r \Rightarrow r = 4$

$\therefore \quad T_5 = {}^{10}C_4 = \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$

43. $(\sqrt{3} + 1)^{2n} = {}^{2n}C_0 (\sqrt{3})^{2n} + {}^{2n}C_1 (\sqrt{3})^{2n-1}$

$+ {}^{2n}C_2 (\sqrt{3})^{2n-2} + \ldots + {}^{2n}C_{2n} (\sqrt{3})^{2n-2n}$

and $(\sqrt{3} - 1)^{2n} = {}^{2n}C_0 (\sqrt{3})^{2n} (-1)^0 + {}^{2n}C_1 (\sqrt{3})^{2n-1} (-1)^1$

$+ {}^{2n}C_2 (\sqrt{3})^{2n-2} (-1)^2 + \ldots + {}^{2n}C_{2n} (\sqrt{3})^{2n-2n} (-1)^{2n}$

$\therefore (\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n} = 2\,[{}^{2n}C_1 (\sqrt{3})^{2n-1}$

$+ {}^{2n}C_3 (\sqrt{3})^{2n-3} + {}^{2n}C_5 (\sqrt{3})^{2n-5}$

$+ \ldots + {}^{2n}C_{2n-1} (\sqrt{3})^{2n-(2n-1)}]$

which is most certainly an irrational number because of odd powers of $\sqrt{3}$ in each of the terms.

44. Here, $(1 - x - x^2 + x^3)^6 = \{(1 - x) - x^2 (1 - x)\}^6$

$= \{(1 - x)(1 - x^2)\}^6 = (1 - x)^6 \cdot (1 - x^2)^6$

$= \left\{\sum_{r=0}^{6} (-1)^r\, {}^6C_r \cdot x^r\right\} \left\{\sum_{s=0}^{6} (-1)^s\, {}^6C_s \cdot x^{2s}\right\}$

$= \sum_{r=0}^{6} \sum_{s=0}^{6} (-1)^{r+s} \cdot {}^6C_r \cdot {}^6C_s \cdot x^{r+2s}$

For coefficient of x^7, $+ r + 2s = 7$

i.e., $(s = 1, r = 5)$ or $(s = 2, r = 3)$ or $(s = 3, r = 1)$

$\therefore$ Coefficient of x^7 is

$\{(-1)^{5+1} \cdot {}^6C_5 \cdot {}^6C_1\} + \{(-1)^{3+2} \cdot {}^6C_3 \cdot {}^6C_2\}$

$+ \{(-1)^{1+3} \cdot {}^6C_1 \cdot {}^6C_3\}$

$= (36) - (20)(15) + 6(20)$

$= 36 - 300 + 120 = -144$

45. $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (63 - 1)^{2n+1}$

$= (1 + 63)^n + (1 - 63)^{2n+1}$

$= [1 + {}^nC_1\, 63 + {}^nC_2\, (63)^2 + \ldots + (63)^n]$

$+ [1 - {}^{(2n+1)}C_1\, 63 + {}^{(2n+1)}C_2\, (63)^2$

$- \ldots + (-1)(63)^{(2n+1)}]$

$= 2 + 63\,[{}^nC_1 + {}^nC_2\, (63) + \ldots + (63)^{n-1}$

$- {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 (63) - \ldots - (63)^{(2n)}]$

So, the remainder is 2.

46. Let $\quad P(n) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{n}}$

Now, $P(2) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} > \sqrt{2}$

Let us assume that $P(k) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{k}} > \sqrt{k}$ is true.

$\therefore P(k + 1) = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{\sqrt{k}} + \dfrac{1}{\sqrt{k+1}} > \sqrt{k+1}$ has to be true

LHS $> \sqrt{k} + \dfrac{1}{\sqrt{k+1}} = \dfrac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}}$

Since, $\quad \sqrt{k(k+1)} > k \; (\forall\, k \le 0)$

$\therefore \quad \dfrac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} > \dfrac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$

Hence, Statement I is correct.

Let $\quad P(n) = \sqrt{n(n+1)} < n + 1$

Now, $\quad P(2) = \sqrt{2 \times 3} < 3$

If $P(k) = \sqrt{k(k+1)} < (k + 1)$ is true

Now, $P(k + 1) = \sqrt{(k+1)(k+2)} < k + 2$ has to be true.

Since, $\quad (k + 1) < k + 2$

$\therefore \quad \sqrt{(k+1)(k+2)} < (k + 2)$

Hence, Statement II is not a correct explanation of Statement I.

47. Since, $\sum_{r=0}^{n} {}^nC_r \cdot x^r = (1+x)^n$

On multiplying by x, we get

$$\sum_{r=0}^{n} {}^nC_r \cdot x^{r+1} = x(1+x)^n$$

On differentiating w.r.t. x, we get

$$\sum_{r=0}^{n} (r+1) \cdot {}^nC_r \cdot x^r = (1+x)^n + nx(1+x)^{n-1}$$

Hence, Statement II is true.

If $x = 1$, then

$$\sum_{r=0}^{n} (r+1) \cdot {}^nC_r = 2^n + n(2)^{n-1} = (n+2)2^{n-1}$$

So, Statement I is true, Statement II is true; Statement II is a correct explanation of Statement I.

48. Since, $T_5 = -T_6$

$$\Rightarrow \binom{n}{4} a^{n-4}(-b)^4 = -\binom{n}{5} a^{n-5}(-b)^5$$

$$\therefore \quad \frac{a}{b} = \binom{n}{5} \Big/ \binom{n}{4} = \frac{n-5+1}{5} = \frac{n-4}{5}$$

49. Since, a_n = Coefficient of x^n in $(1-ax)^{-1}(1-bx)^{-1}$

$$= a^0 b^n + ab^{n-1} + \ldots + a^n b^0$$

$$= a^0 b^n \left(1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \ldots\right)$$

$$= a^0 b^n \left(\frac{(a/b)^{n+1} - 1}{(a/b) - 1}\right)$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

50. T_{r+1} in $\left(ax^2 + \frac{1}{bx}\right)^{11} = \binom{11}{r}\left(\frac{1}{bx}\right)^r (ax^2)^{11-r}$

$$= \binom{11}{r} a^{11-r} \cdot b^{-r} \cdot x^{22-3r}$$

$$\therefore \quad 22 - 3r = 7 \Rightarrow r = 5$$

So, the coefficient of x^7 is $\binom{11}{5} a^6 b^{-5}$. ...(i)

Also, T_{r+1} in $\left(ax - \frac{1}{bx^2}\right)^{11} = \binom{11}{r} a^{11-r}(-b)^{-r} x^{11-3r}$

$$11 - 3r = -7$$

$$\Rightarrow \quad r = 6$$

So, the coefficient is $\binom{11}{6} a^5 b^{-6}$. ...(ii)

From Eqs. (i) and (ii),

$$a = b^{-1}$$

$$\Rightarrow \quad ab = 1$$

51. Here, $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Similarly, $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ can be verified by induction.

Now, taking options

(b) $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} + \begin{bmatrix} n-1 & 0 \\ n & n-1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \neq \begin{bmatrix} 2n-1 & 0 \\ n & 2n-1 \end{bmatrix}$$

(d) $nA - (n-1)I = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$$

52. $S(k) = 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$

Put $k = 1$ in both sides, we get

LHS = 1 and RHS = 3 + 1 = 4

$\Rightarrow$ LHS $\neq$ RHS

Put $(k+1)$ in both sides on the place of k.

LHS $= 1 + 3 + 5 + \ldots + (2k-1) + (2k+1)$

RHS $= 3 + (k+1)^2 = 3 + k^2 + 2k + 1$

Let LHS = RHS

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = 3 + k^2 + 2k + 1$$

$$\Rightarrow \quad 1 + 3 + 5 + \ldots + (2k-1) = 3 + k^2$$

If $S(k)$ is true, then $S(k+1)$ is also true.

Hence, $S(k) \Rightarrow S(k+1)$

53. The coefficient of x^n in

$$(1+x)\left[1 - \binom{n}{1}x + \ldots + (-1)^{n-1} nx^{n-1} + (-1)^n x^n\right]$$

is $(-1)^n + n(-1)^{n-1} = (-1)^n(1-n)$

54. Since, $\binom{4}{2}\alpha^2 = \binom{6}{3}(-\alpha)^3$

$$\Rightarrow \quad 6\alpha^2 = -20\alpha^3$$

$$\Rightarrow \quad \alpha = -\frac{3}{10}$$

55. $T_{r+1} = \binom{256}{r} 3^{\frac{256-r}{2}} \cdot 5^{r/8}$

Gives integer for $r = 0, 8, 16, \ldots, 256$

The number of integral terms $= \frac{256}{8} + 1 = 32 + 1 = 33$

Day 8

Permutation and Combination

Day 8

Outlines ...

- Fundamental Principle of Counting
- Factorial Notation
- Permutations
- Combinations
- Applications of Permutations and Combinations

Fundamental Principle of Counting

The fundamental principle of counting is a way to figure out the total number of ways different events can occur. *If a certain work A can be done in m ways and another work B in n ways, then*

(i) the number of ways of doing the work A or B is $m + n$. (addition principle)

(ii) the number of ways of doing both the works is mn. (multiplication principle)

Factorial Notation

The product of first n natural numbers is denoted by $n!$ and read as 'factorial n'.

Thus $\quad n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$

e.g., $\quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

and $\quad 4! = 4 \times 3! = 4 \times 3 \times 2!$

$\quad = 4 \times 3 \times 2 \times 1 = 24$

Properties of Factorial Notation

1. $0! = 1! = 1$
2. Factorials of negative integers and fractions are not defined.
3. $n! = n(n-1)! = n(n-1)(n-2)!$
4. $\dfrac{n!}{r!} = n(n-1)(n-2)\ldots(r+1)$

Permutations

Permutations means **arrangement** of things. In other words, the number of permutations of n different things taken r at a time is ${}^nP_r$.

Meaning of ${}^nP_r$

${}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \dfrac{n!}{(n-r)!}, 0 \le r \le n$ and

${}^nP_0 = 1, {}^nP_1 = n, {}^nP_n = n!$

e.g., The permutation of a, b and c taken 2 at a time is ${}^3P_2 = 3 \cdot 2 = 6$ i.e., ab, ac, bc, ba, ca and cb.

(i) ${}^nP_r + r \cdot {}^nP_{r-1} = {}^{n+1}P_r$

(ii) ${}^nP_r = n\,{}^{n-1}P_{r-1}$

(iii) ${}^nP_r = (n-r+1)\,{}^nP_{r-1}$

Important Results on Permutations

1. Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement is $r \cdot {}^{n-1}P_{r-1}$.
2. Number of permutations of n different things taken r at a time, when a particular thing is never taken in each arrangement is ${}^{n-1}P_r$.
3. The number of permutations of n different things taken r at a time, allowing repetitions is n^r. e.g., The number of 3-digit number formed with the digits 1, 2, 3, 4 and 5 with repetitions of digits is $5^3 = 125$.
4. The permutations of n things of which p are identical of one sort, q are identical of a second sort, r are identical of a third sort is $\dfrac{n!}{p!\,q!\,r!}$, where $p + q + r = n$

5. Arrangements

(i) The number of ways in which m different things and n different things $(m + 1 \ge n)$ can be arranged in a row so that no two things of second kind come together is $m!\,{}^{(m+1)}P_n$.

(ii) The number of ways in which n different things and $(n-1)$ different things can be arranged in a row, so that no two things of same type come together is $n!\,(n-1)!$.

6. Dearrangement

Suppose n letters are to be kept in n directed envelopes, the number of ways in which they can be placed, if none of the letters goes into its own envelope is

$$n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots (-1)^n \frac{1}{n!}\right].$$

7. Sum of Digits

(i) Sum of numbers formed by taking all the given n digits (excluding 0) is (sum of all the n digits) $\times (n-1)! \times (111\dots n$ times).

(ii) Sum of the numbers formed by taking all the given n digits (including 0) is (sum of all the n digits) $\times [(n-1)! \times (111\dots n$ times$) - (n-2)\ \{111\dots(n-1)$ times$\}]$.

(iii) Sum of all r digit numbers formed by taking the given n digits (without zero) is (sum of all the n digits) $\times {}^{(n-1)}P_{r-1} \times (111\dots r$ times).

(iv) Sum of all the r-digit numbers formed by taking the given n digits (including 0) is (sum of all the n digits) $\times [{}^{(n-1)}P_{r-1} \times (111\dots r$ times$)$ $- {}^{(n-2)}P_{r-2} \times \{111\dots(r-1)$ times$\}]$

- *If there are m items of one kind, n items of another kind and so on. Then, the number of ways of choosing r items out of these items = coefficient of x^r in*

$$(1 + x + x^2 + \dots + x^m)(1 + x + x^2 + \dots + x^n)\dots$$

- *If there are m items of one kind, n items of another kind and so on. Then, the number of ways of choosing r items out of these items such that atleast one item of each kind is included in every selection = coefficient of x^r in*

$$(x + x^2 + \dots + x^m)(x + x^2 + \dots + x^n)\dots$$

Circular Permutations

The number of circular permutations of n different things is $\dfrac{n!}{n} = (n-1)$. These includes both clockwise and anti-clockwise permutations. The number of circular permutations of n different things taken r at a time is $\dfrac{1}{2r}\,{}^nP_r$ or $\dfrac{1}{r}\,{}^nP_r$ according as the arrangement is clockwise or anti-clockwise is considered or not.

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Important Results on Circular Permutations

1. The number of ways in which m different things and n different things (where, $m \geq n$) can be arranged in a circle so that no two things of second kind come together is $(m-1)!\,{}^mP_n$.
2. The number of ways in which m different things and n different things can be arranged in a circle so that all the second type of things come together is $m!\,n!$.
3. The number of ways in which m different things and n different things (where, $m \geq n$) can be arranged in the form of garland so that no two things of second kind come together is $(m-1)!\,{}^mP_n/2$.
4. The number of ways in which m different things and n different things can be arranged in the form of garland so that all the second type of things come together is $m!\,n!\,2$.

Combinations

Combination means **selection** of things or the number of combinations of n different things taken r at a time is ${}^nC_r$ or $\binom{n}{r}$.

Meaning of ${}^nC_r$

$$^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$$

e.g., The combinations of a, b, c and d taken 3 at a time are $\binom{4}{3} = 4$, namely abc, abd, acd and bcd.

(i) ${}^nC_r = {}^nC_{n-r}$ (ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$ (iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Important Results on Combinations

1. The number of combinations of n different things, taken r at a time, where p particular things occur is ${}^{n-p}C_{r-p}$.
2. The number of combinations of n different things, taken r at a time, where p particular things never occur is ${}^{n-p}C_4$
3. The number of combinations of n identical things taking $r\,(r \leq n)$ at a time is 1.
4. The number of combinations of n different things taken r at a time allowing repetitions is ${}^{n+r-1}C_r$ or $\binom{n+r-1}{r}$.
   e.g., The combination of a, b and c taken 2 at a time allowing repetitions are 6, namely aa, bb, cc, ab, ac and bc.
5. The number of ways of dividing x identical things among r persons such that each one gets atleast one is $\binom{n-1}{r-1}$, which is also the number of positive integer solutions of the equation $x_1 + x_2 + x_3 + \ldots + x_r = n$.

- *The number of ways of selecting r items from a group of n items in which p are identical, is* ${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_0$, *if* $r \leq p$ *and* ${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots + {}^{n-p}C_{r-p}$ *if* $r > p$
- *The number of ways of answering one or more of n questions is* ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1$.
- *The number of ways of answering one or more n questions when each question has an alternative* $= 2^n$

Applications of Permutations *and* Combinations

The functional and the geometrical applications of permutaions and combinations are given below.

Functional Applications

1. The number of all permutations (arrangements) of n different objects taken r at a time,
   (i) when a particular object is to be always included in each arrangement is ${}^{n-1}C_{r-1} \times r!$
   (ii) when a particular object is never taken in each arrangement is ${}^{n-1}C_r \times r!$

2. If the sets A has m elements and B has n elements, then
   (i) the number of functions from A to B is n^m.
   (ii) the number of one-one functions from A to B is ${}^nP_m$, $m \le n$.
   (iii) the number of onto functions from A to B is
   $$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \dots, m \le n.$$
   (iv) the number of increasing (decreasing) functions from A to B is $\binom{n}{m}$, $m \le n$.
   (v) the number of non-decreasing (non-increasing) functions from A to B is $\binom{m+n-1}{m}$, $m \le n$.
   (vi) the number of bijections from A to B is $n!$, if $m = n$.
   (vii) the number of bijections from A to A such that $f(x) \ne x, \forall x \in A$, is $m!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^m}{m!}\right]$.

Geometrical Applications

1. Number of triangle formed from n points, when no three points are collinear is ${}^nC_3$.
2. Out of n non-concurrent and non-parallel straight line, the points of intersection are ${}^nC_2$.
3. Number of parallelogram in two system of parallel lines (when Ist system contains m parallel lines and IInd set contains n parallel lines) $= {}^nC_2 \times {}^mC_2$.
4. The number of diagonals in a polygon of n sides is ${}^nC_2 - n$.
5. The number of total triangles formed by joining the n points on a plane of which m are collinear is ${}^nC_3 - {}^mC_3$.
6. The number of total different straight lines formed by joining the n points on a plane of which m are collinear is ${}^nC_2 - {}^mC_2 + 1$.
7. The number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

Prime Factors

Any integer > 1 can be expressed as product of primes

$$7 = 7^1, 12 = 2^2 \cdot 3^1, 360 = 2^3 \cdot 3^2 \cdot 5^1$$

Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \dots p_r^{\alpha_r}$, where $p_i, i = 1, 2, \dots, r$ are distinct primes and $\alpha_i, i = 1, 2, \dots, r$ are positive integers.

(i) Number of divisor of n is $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\dots(\alpha_r + 1)$

(ii) Sum of divisor of n is $\frac{(p_1^{\alpha_1+1} - 1)}{(p_1 - 1)} \frac{(p_2^{\alpha_2+1} - 1)}{(p_2 - 1)} \dots \frac{(p_r^{\alpha_r+1} - 1)}{(p_r - 1)}$

(iii) If p is a prime and p^r divides !. Then, $r = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \dots$

Division of Objects into Groups

The division of objects into groups are taken place as when the objects are different and identical as given below.

Objects are Different

1. The number of ways of dividing n different objects into 3 groups of p, q and r $(p + q + r = n)$ is
   (i) $\frac{n!}{p!q!r!}$; p, q and r are unequal.
   (ii) $\frac{n!}{p!2!(q!)^2}$; $q = r$
   (iii) $\frac{n!}{3!(p!)^3}$; $p = q = r$
2. The number of ways of dividing n different objects into r groups is
   $$\frac{1}{r!}\left[r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \binom{r}{3}(r-3)^n + \dots\right]$$
3. The number of ways of dividing n different objects into r groups taking into account the order of the groups and also the order of objects in each group is
   $${}^{(n+r-1)}P_n = r(r+1)(r+2)\dots(r+n-1).$$

Objects are Identical

1. The number of ways of dividing n identical objects among r persons such that each gets 1, 2, 3, ... or k objects is the coefficient of x^{n-r} in the expansion of $(1 + x + x^2 + \dots + x^{k-1})^r$.
2. The number of ways of dividing n identical objects among r persons such that each one may get atmost n objects is $\binom{n+r-1}{r-1}$. In other words, the total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.
3. The total number of ways of dividing n identical objects among r persons, each one of whom, receives atleast one item is ${}^{n-1}C_{r-1}$. In other words, the number of ways in which n identical things can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.
4. The total number of selections of some or all out of $p + q + r$ items, where p are alike of one kind, q are alike of second kind and rest are alike of third kind is $[(p+1)\cdot(q+1)(r+1) - 1]$.

Practice Zone DAY 8

1. ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$ is equal to **[NCERT Exemplar]**
(a) ${}^{n+1}C_r$ (b) ${}^{n+1}C_{r+1}$
(c) ${}^{n+2}C_r$ (d) ${}^{n+2}C_{r+1}$

2. A sports team of 11 students is to be constituted, choosing atleast 5 from Class XI and atleast 5 from Class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted? **[NCERT Exemplar]**
(a) ${}^{20}C_5 \times {}^{20}C_6$ (b) $2({}^{20}C_5 \times {}^{20}C_6)$
(c) $2({}^{20}C_5)^2$ (d) None of these

3. The vertices of a regular polygon of 12 sides are joined to form triangles. The number of triangles which do not have their sides as the sides of the polygon is
(a) 96 (b) 108
(c) 112 (d) 220

4. Out of 8 sailors on a boat, 3 can work only one particular side and 2 only the other side. Then, number of ways in which the sailors can be arranged on the boat is
(a) 2718 (b) 1728
(c) 7218 (d) None of these

5. There are 4 balls of different colours and 4 boxes of same colours as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour is
(a) 8 (b) 9
(c) 7 (d) 10

6. The number of words which can be formed with two different consonants and one vowel out of 7 different consonants and 3 different vowels, the vowels should be between the two consonants is
(a) 92 (b) 172
(c) 126 (d) 136

7. The number of zeroes at the end of 2007! is
(a) 499 (b) 500
(c) 501 (d) 502

8. If 4 dice are rolled, then the number of ways of getting the sum 10 is
(a) 56 (b) 64 (c) 72 (d) 80

9. The number of positive integer solution (x, y, z) of the equation $xyz = 24$ is
(a) 18 (b) 20
(c) 24 (d) 30

10. A guard of 12 men is formed from a group of n soldiers in all possible ways. If the number of times two particular soldiers A and B are together on guard is thrice the number of times three particular soldiers C, D, E are together on guard, then n is equal to
(a) 18 (b) 24 (c) 32 (d) 36

11. In a cricket match between two teams X and Y, the team X requires 10 runs to win in the last 3 balls. If the possible runs that can be made from a ball be 0, 1, 2, 3, 4, 5 and 6 the number of sequence of runs made by the batsman is
(a) 12 (b) 18
(c) 21 (d) 36

12. A person is permitted to select atleast one and atmost n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, then n is equal to
(a) 4 (b) 8 (c) 16 (d) 32

13. The number of ways in which a mixed double game can be arranged from amongst 9 married couples, if no husband and wife play in the same game is
(a) 756 (b) 1512
(c) 3024 (d) None of these

14. The sum of all the 4-digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 without repitition of the digits is
(a) 399960 (b) 288860
(c) 301250 (d) 420210

15. The total number of natural numbers of 6 digits that can be made with digits 1, 2, 3 and 4, if all digits are to appear in the same number atleast once is
(a) 1560 (b) 840 (c) 1080 (d) 480

16. The number of ways of arranging the letters of the word 'NALGONDA' such that letters of the word 'GOD' occur in that order (G before O and O before D) is
(a) 1250 (b) 1440
(c) 1560 (d) 1680

17. Out of 5 apples, 10 mangoes and 15 oranges, the number of ways of distributing 15 fruits each to two persons is
(a) 56 (b) 64
(c) 66 (d) 72

18. The number of divisors of the form $4n+1, n \geq 0$ of the number $10^{10}\, 11^{11}\, 13^{13}$ is
(a) 750 (b) 840
(c) 924 (d) 1024

19. The number of positive integers formed with atmost 10 digits using 0, 1 and 2 is
(a) 59048 (b) 57512
(c) 56011 (d) 7431

20. Two players P_1 and P_2 play a series of $2n$ games. Each game can result in either a win or loss for P_1. Total number of ways in which P_1 can win the series of these games is equal to
(a) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$
(c) $\frac{1}{2}(2^{n} - {}^{2n}C_n)$ (d) None of these

21. Total number of function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ that are onto and $f(i) \neq i$ is equal to
(a) 9 (b) 44
(c) 16 (d) None of these

22. The number of ways in which an examiner can assign 30 marks to 8 questions giving not less than 2 marks to any question is
(a) 108120 (b) 124320
(c) 116280 (d) 144240

23. The number of n-digit numbers which contain the digits 2 and 7, but not the digits 0, 1, 8, 9 is
(a) $6^n - 2 \cdot 5^n + 4^n$ (b) $6^n - 5^n + 4^n$
(c) $6^n - 5^n - 4^n$ (d) None of these

24. A student is allowed to select atmost n books from a collection of $(2n+1)$ books. If the number of ways in which he can select atleast one book is 63. Then, n is equal to
(a) 3 (b) 4
(c) 6 (d) 5

25. The number of ways in which any four letters can be selected out of the letters of the word PCBDCC is
(a) 8 (b) 5
(c) 7 (d) 6

26. Given that n is odd, the number of ways in which three numbers in AP can be selected from $1, 2, 3, 4, \ldots, n$ is
(a) $\frac{(n-1)^2}{2}$ (b) $\frac{(n+1)^2}{4}$
(c) $\frac{(n+1)^2}{2}$ (d) $\frac{(n-1)^2}{4}$

27. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. Find the number of different ways of doing questions. **[NCERT Exemplar]**
(a) 779 (b) 781 (c) 780 (d) 782

28. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 18 (c) 12 (d) 9

Directions (Q. Nos. 29 and 30) *Consider the letters of the word 'MATHEMATICS'. There are eleven letters some of them are identical. Letters are classified as repeating and non-repeating letters. Set of repeating letters = $\{M, A, T\}$. Set of non-repeating letters = $\{H, E, I, C, S\}$*

29. Possible number of words taking all letters at a time such that atleast one repeating letter is at odd position in each word is
(a) $\frac{9!}{2!2!2!}$ (b) $\frac{11!}{2!2!2!}$ (c) $\frac{11!}{2!2!}$ (d) $\frac{9!}{2!2!}$

30. Possible number of words, taking all letters at a time such that in each word both M's are together and both T's are together but both A's are not together, is
(a) $7! \cdot {}^8C_2$ (b) $\frac{11!}{2!2!2!} - \frac{10!}{2!2!}$
(c) $\frac{6!\,4!}{2!2!}$ (d) $\frac{9!}{2!2!2!}$

31. If the letters of the word 'SACHIN' are arranged in all possible ways and these words are written in dictionary order, then the word 'SACHIN' appears at serial number
(a) 600 (b) 601 (c) 602 (d) 603

32. The number of ways of distributing 8 identical balls in 3 distinct boxes so that no box is empty, is
(a) 5 (b) $\binom{8}{3}$ (c) 3^8 (d) 21

Directions (Q. Nos. 33 to 36) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

33. Consider a rectangle chess board, having 8 horizontal and 8 vertical lines.

Statement I Number of rectangle on a chess board is ${}^8C_2 \times {}^8C_2$.

Statement II To form a rectangle, we have to select any two of the horizontal line and any two of the vertical line.

34. Consider ten-digit numbers, with the help at 2, 3, 4, 5

Statement I The sum of the digits in the ten places by using the above numbers all at a time is 84.

Statement II The sum of the digits in the units place of all numbers formed with the help of $a_1, a_2, \ldots, a_n$ taken all at a time is $(n-1)!\,(a_1 + a_2 + \ldots + a_n)$ (repetition of digits is not allowed).

35. Consider a group of some men and women in a team.

Statement I From a group of 8 men and 4 women a team of 5 members, including atleast one woman can be formed in 736 ways.

Statement II Number of ways of selecting atleast one woman from m men and n women is ${}^{m+n}C_n - {}^{m}C_n$.

36. Statement I A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 with non-repetition. The total number formed are 216.

Statement II If sum of digits of any number is divisible by 3, then the number must be divisible by 3.

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37. A committee of 4 persons is to be formed from 2 ladies, 2 old men and 4 young men such that in includes atleast 1 lady, atleast 1 old man and at most 2 young men. Then the total number of ways in which this committee can be formed is **[JEE Main 2013]**

(a) 40 (b) 41 (c) 16 (d) 32

38. The number of ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question, is **[JEE Main 2013]**

(a) ${}^{30}C_7$ (b) ${}^{21}C_8$ (c) ${}^{21}C_7$ (d) ${}^{30}C_8$

39. On the sides AB, BC, BC, CA of a ΔABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are **[JEE Main 2013]**

(a) 210 (b) 205 (c) 215 (d) 220

40. 5-digit numbers are to be formed using 2, 3, 5, 7, 9 without repeating the digits. If p be the number of such numbers that exceed 20000 and q be the number of those that lie between 30000 and 90000, then $p:q$ is **[JEE Main 2013]**

(a) 6 : 5 (b) 3 : 2 (c) 4 : 3 (d) 5 : 3

41. Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22} \right\}$

Then, the number of non-singular matrices in the set S is **[JEE Main 2013]**

(a) 27 (b) 24 (c) 10 (d) 20

42. Let A and B two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is **[JEE Main 2013]**

(a) 256 (b) 220 (c) 219 (d) 211

43. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is **[JEE Main 2013]**

(a) 7 (b) 5 (c) 10 (d) 8

44. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is **[AIEEE 2012]**

(a) 880 (b) 629 (c) 630 (d) 879

45. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can formed such that $Y \subseteq X, Z \subseteq X$ and $Y \cap Z$ is empty, is **[AIEEE 2012]**

(a) 5^2 (b) 3^5 (c) 2^5 (d) 5^3

46. Statement I The number of ways distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.

Statement II The number of ways of choosing any 3 places. From 9 different places is ${}^9C_3$. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

47. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done, is **[AIEEE 2010]**

(a) 3 (b) 36 (c) 66 (d) 108

Directions (Q. Nos. 48 and 49) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.* **[AIEEE 2008]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

48. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$

Statement I $S_3 = 55 \times 2^9$

Statement II $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$

49. In a shop there are five types of ice-creams available. A child buys six ice-creams.

Statement I The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.

Statement II The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

50. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent? **[AIEEE 2008]**

(a) $7 \cdot {}^6C_4 \cdot {}^8C_4$ (b) $8 \cdot {}^6C_4 \cdot {}^7C_4$
(c) $6 \cdot 7 \cdot {}^8C_4$ (d) $6 \cdot 8 \cdot {}^7C_4$

51. The set $S = \{1, 2, 3, \ldots, 12\}$ is to be partitioned into three sets A, B and C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is **[AIEEE 2007]**

(a) $12!/3!(4!)^3$ (b) $12!/3!(3!)^4$ (c) $12!/(4!)^3$ (d) $12!/(3!)^4$

52. At an election, a voter may vote for any number of candidates not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for atleast one candidate, then the number of ways in which he can vote, is **[AIEEE 2006]**

(a) 6210 (b) 385 (c) 1110 (d) 5070

Answers

1. (c)	**2.** (b)	**3.** (c)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (c)	**10.** (d)
11. (a)	**12.** (c)	**13.** (a)	**14.** (a)	**15.** (d)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (b)	**22.** (c)	**23.** (a)	**24.** (a)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (b)	**29.** (b)	**30.** (a)
31. (b)	**32.** (d)	**33.** (d)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (c)	**38.** (b)	**39.** (d)	**40.** (b)
41. (a)	**42.** (d)	**43.** (c)	**44.** (d)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (d)	**50.** (b)
51. (a)	**52.** (b)								

Hints & Solutions

1. Total number of triangles is $\binom{12}{3} = 220$, the number of triangles with two sides common with polygon is 12. The number of triangles with one side common with polygon $= 8 \cdot 12 = 96$

So, the required number of ways is $220 - (12 + 96) = 112$.

2. Using the number of dearrangements

i.e., $n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right\}$

Here, $n = 4$

So, the required number of ways $= 4!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$

$= 12 - 4 + 1 = 9$

3. Selection of 2 consonants from 7 consonants $= {}^7C_2 = 21$

Selection of one vowel from 3 different vowels $= {}^3C_1 = 3$

Now, we have 2 consonants and a vowel and the vowel should be between consonants

i.e., consonant-vowel-consonant

So, two consonants can change their positions in 2! ways.

Hence, by the product rule, number of words $= 21 \times 3 \times 2 = 126$

4. $\left[\frac{2007}{5}\right] = 401, \left[\frac{401}{5}\right] = 80, \left[\frac{80}{5}\right] = 16, \left[\frac{16}{5}\right] = 3$

$\therefore \quad 2007! = (5^{3+16+80+401}) \cdot 2^\alpha \cdot 3^\beta \ldots = 5^{500} \cdot 2^\alpha \cdot 3^\beta \ldots$

2007! ends with 500 zeroes.

5. ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2} = {}^nC_r + {}^nC_{r-1} + {}^nC_{r-1} + {}^nC_{r-2}$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$.

6. Let the particular side on which 3 particular sailors can work be named A and on the other side by B on which 2 particular sailors can work. Thus, we are left with 3 sailors only selection of one sailor for side $A = {}^3C_1 = 3$ and then we are left with 2 sailors for the other side. Now, on each side, 4 sailors can be arranged in 4! ways.

$\therefore$ Total number of arrangements $= 3 \times 24 \times 24 = 1728$

7. Coefficient of x^{10} in $(x + x^2 + \ldots + x^6)^4$

$=$ Coefficient of x^6 in $(1 + x + \ldots + x^5)^4$

$= (1 - x^6)^4 (1 - x)^{-4} = (1 - 4x^6 + \ldots)\left[1 + \binom{4}{1}x + \ldots\right]$

Hence, coefficient of x^6 is $\binom{9}{6} - 4 = 80$

8. $24 = 1 \cdot 1 \cdot 24 = 1 \cdot 2 \cdot 12 = 1 \cdot 3 \cdot 8 = 1 \cdot 4 \cdot 6 = 2 \cdot 2 \cdot 6 = 2 \cdot 3 \cdot 4$

Each of (1, 2, 12), (1, 3, 8), (1, 4, 6) and (2, 3, 4) can be permuted in 3! ways.

Each of (1, 1, 24) and (2, 2, 6) can be permuted in 3 ways.

So, required number of ways is $4 \times 6 + 2 \times 3 = 30$.

9. Number of times A and B are together on guard is $\binom{n-2}{10}$.

Number of times C, D and E are together on guard is $\binom{n-3}{9}$.

According to the question, $\binom{n-2}{10} = 3\binom{n-3}{9}$

$\Rightarrow \quad n - 2 = 30 \Rightarrow n = 32$

10. Required number is the coefficient of x^{10} in

$$(1 + x + x^2 + \ldots + x^6)^3$$

$$= (1 - x^7)^3 (1 - x)^{-3} = (1 - 3x^7 + \ldots)\left[1 + \binom{3}{1}x + \binom{4}{2}x^2 + \ldots\right]$$

Hence, coefficient of x^{10} is $\binom{12}{10} - 3\binom{5}{3} = 36$.

11. We have, ${}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = 255$ ...(i)

But ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{2n+1}$

$= (1 + 1)^{2n+1}$

$\therefore {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n)$

$+ {}^{2n+1}C_{2n+1} = 2^{2n+1}$

$\Rightarrow \quad 1 + 2(255) + 1 = 2^{2n+1}$

[from Eq. (i)]

$\Rightarrow \quad 1 + 255 = 2^{2n}$

$\Rightarrow \quad 2^8 = 2^{2n} \Rightarrow n = 4$

12. Two men can be chosen in ${}^9C_2$ ways. Since, no husband and wife are to play in the same game, so we have to select two women from the remaining 7 women. This can be done in ${}^7C_2$ ways. If M_1, M_2 and W_1, W_2 are chosen, then a team can be constituted in 4 ways *i.e.*, M_1W_2, M_1W_1, M_2W_1 and M_2W_2. Hence, the number of ways of arranging the game

$$= {}^9C_2 \times {}^7C_2 \times 4 = 36 \times 21 \times 4 = 3024$$

13. Required sum = (Sum of all the n digits $\times\, {}^{n-1}P_{r-1} \times (111 \ldots r$ times)

$$= (1 + 2 + 3 + 4 + 5)\,{}^4P_3 \times (1111) = 15 \times 24 \times 1111 = 399960$$

14. There are two cases arises

Case I Any one of the digits 1, 2, 3, 4 appears thrice and the remaining digits only once *i.e.*, of the type 1, 2, 3, 4, 4, 4 etc.

$\therefore$ Number of ways of selection of digits which appears thrice $= {}^4C_1$

$\therefore$ Number of numbers of this type $= \dfrac{6!}{3!} \times {}^4C_1 = 480$

Case II Any of the digits 1, 2, 3, 4 appears twice and the remaining two only once,

i.e., of the type 1, 2, 3, 3, 4, 4 etc.

$\therefore$ Number of ways of selection of 2 digits which appears twice $= {}^4C_2$

$\therefore$ Number of numbers of this type $= \dfrac{6!}{2!\,2!} \times {}^4C_2 = 1080$

$\therefore$ Required number of numbers $= 480 + 1080 = 1560$

15. G,O,D can be selected in $\binom{8}{3} = 56$ ways

A,A,N,N,L can be arranged in $\dfrac{5!}{2!\,2!} = 30$ ways

$\therefore$ Required number of ways $= 56 \times 30 = 1680$

16. Required number is the coefficient of x^{15} in the product

$$(1 + x + x^2 + \ldots + x^5)(1 + x + x^2 + \ldots + x^{10})(1 + x + x^2 + \ldots + x^{15})$$

$$= (1 - x^6)(1 - x^{11})(1 - x^{16})(1 - x)^{-3}$$

$$= (1 - x^6 - x^{11} + \ldots)\left(1 + \binom{3}{1}x + \binom{4}{2}x^2 + \ldots\right)$$

which is $\binom{17}{15} - \binom{11}{9} - \binom{6}{4} = \binom{17}{2} - \binom{11}{2} - \binom{6}{2}$

$= 136 - 55 - 15 = 66$

17. $2^{10}5^{10}11^{11}13^{13}$ has a divisor of the form $2^\alpha \cdot 5^\beta \cdot 11^\gamma \cdot 13^\delta$, where

$\alpha = 0, 1, 2, \ldots, 10;\ \beta = 0, 1, 2, \ldots, 10;\ \gamma = 1, 2, \ldots, 11;\ \delta = 1, 2, \ldots, 13$

It is of the form $4n + 1$, if

$\alpha = 0;\ \beta = 0, 1, 2, \ldots, 10;\ \gamma = 0, 2, 4, \ldots, 10;\ \delta = 0, 1, 2, \ldots, 13.$

$\therefore$ Number of divisors $= 11 \times 6 \times 14 = 924$

18. Each of the 10 digits can assume any of the 3 values. The number, so formed includes all zeroes.

$\therefore$ Required number of positive integer $= 3^{10} - 1 = 59048$

19. P_1 must win atleast $(n + 1)$ games.

Let P_1 win $n + r$ games ($r = 1, 2, \ldots, n$)

$\therefore$ Number of ways $= \sum_{r=1}^{n} {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + \ldots + {}^{2n}C_{2n}$

$$= \frac{2^{2n}}{2} - {}^{2n}C_n = \frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$$

20. There are two cases arises

Case I Choose six from class XI and five from class XII

$\therefore$ Number of ways $= {}^{20}C_6 \times {}^{20}C_5$

Case II Choose five from class XI and six from class XII

$\therefore$ Number of ways $= {}^{20}C_5 \times {}^{20}C_6$

$\therefore$ Required number of ways $= 2({}^{20}C_5 \times {}^{20}C_6)$

21. Total number of required functions

= Number of dearrangement of 5 objects.

$$= 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

22. $x_1 + x_2 + \ldots + x_8 = 30$

$X_i = x_i - 1 \Rightarrow X_1 + X_2 + \ldots + X_8 = 22$

Number of ways $= {}^{22-1}C_{8-1} = {}^{21}C_7 = 116280$

23. Total numbers without any restriction containing the digits 2, 3, 4, 5, 6 and 7, $n(S) = 6^n$

Total number of numbers which contain 3, 4, 5, 6 and 7, $n(A) = 5^n$

Total number of numbers which contain 2, 3, 4, 5 and 6, $n(B) = 5^n$

Total number of numbers which contain 3, 4, 5 and 6,

$$n(A \cap B) = 4^n$$

Total numbers which does not contain digit 2 and 7

$$= 5^n + 5^n - 4^n$$

$\therefore$ Total numbers which contain 2 and 7 $= 6^n - 5^n - 5^n + 4^n$

$= 6^n - 2 \cdot 5^n + 4^n$

24. He can select either 1,2, ... , or n books.

The number of ways is

$$\binom{2n+1}{1}+\binom{2n+1}{2}+\ldots+\binom{2n+1}{n}$$
$$=\frac{1}{2}\left[\binom{2n+1}{1}+\binom{2n+1}{2}+\ldots+\binom{2n+1}{n}+\binom{2n+1}{n+1}+\ldots+\binom{2n+1}{2n}\right]$$
$$=\frac{1}{2}\left[2^{2n+1}-\binom{2n+1}{0}-\binom{2n+1}{2n+1}\right]$$
$$\Rightarrow \quad 2^{2n}-1=63 \quad \text{(given)}$$
$$\Rightarrow \quad 2^{2n}=64=2^6 \Rightarrow n=3$$

25. As, there are three letters alike out of six letters and the three others are different.

Selection can be done in the following manner

(i) All different ${}^4C_1=1$

(ii) 2 alike, 2 different ${}^1C_1 \cdot {}^3C_2=3$

(iii) 3 alike and 1 different ${}^1C_1 \cdot {}^3C_1=3$

Therefore, total number of ways $=1+3+3=7$

26. Let $n=2m+1$. If a, b and c are in AP, then $2b=a+c$

i.e., sum of two numbers is even.

Hence, $m+1$ is odd and m is even number.

Since, sum of two numbers is even, then both numbers are even or odd.

$\therefore$ Required number of ways

$$={}^mC_2+{}^{m+1}C_2=\frac{m(m-1)}{2}+\frac{m(m+1)}{2}$$
$$=m^2=\left(\frac{n-1}{2}\right)^2=\frac{(n-1)^2}{4}$$

27. Total number of ways

= (Attempt 3 from group I and 4 from group II)

+ (Attempt 4 from group I and 3 from group II)

+ (Attempt 5 from group I and 2 from group II)

+ (Attempt 2 from group I and 5 from group II)

$$={}^6C_3\times{}^6C_4+{}^6C_4\times{}^6C_3+{}^6C_5\times{}^6C_2+{}^6C_2\times{}^6C_5$$
$$=2\,({}^6C_3\times{}^6C_4)+2\,({}^6C_5\times{}^6C_2)=2(20\times15)+2\,(6\times15)$$
$$=600+180=780$$

28. Total number of parallelogram formed $={}^4C_2\times{}^3C_2=6\times3=18$

29. Since, there are 5 even places and 4 pairs of repeated letters, therefore atleast one of these must be at an odd place.

$\therefore$ Number of ways $=\dfrac{11!}{2!2!2!}$

30. Make a group of both M's and another of T's. Then, except A's we have S letters remaining so M's, T's and the letters except A's can be arranged in 7! ways.

$\therefore$ Total number of arrangements $=7!\times{}^8C_2$

31. The letters of given word are A, C, H, I, N, S.

If first letter is A, C, H, I, N, then are $5\cdot5!=600$ words

Then, next word is SACHIN.

So, the required serial number is 601.

32. Required number of ways is equal to the number of positive integer solutions of the equation

$x+y+z=8$ which $\binom{8-1}{3-1}=\binom{7}{2}=21$

33. In a chess board 9 horizontal and 9 vertical lines.

Number of rectangles of any size are ${}^9C_2\times{}^9C_2$.

34. Sum of the digits in the tens place

$$=\text{Sum of the digits in the unit's place}$$
$$=(4-1)!\,(2+3+4+5)=6\cdot14=84$$

35. Required number of ways $={}^4C_1\times{}^8C_4+{}^4C_2\times{}^8C_3+{}^4C_3\times{}^8C_2+{}^4C_4\times{}^8C_1$

$$=280+336+112+8=736$$

Hence, Statement I is true. Statement II is false

36. **Statement I** Number form by using 1, 2, 3, 4, 5 = 5! = 120

Number formed by using 0, 1, 2, 4, $5=4\cdot4\cdot3\cdot2\cdot1=96$

$\therefore$ Required number of ways $=120+96=216$

37. Ladies = 2, old men = 2, young men = 4

Case I 1 lady, 1 old man, 2 young man

$$={}^2C_2\cdot{}^2C_1\cdot{}^4C_2=2\cdot2\cdot6=24$$

Case II 2 ladies, 1 old man, 1 young man

$$={}^2C_2\cdot{}^2C_1\cdot{}^4C_1=1\cdot2\cdot4=8$$

Case III 1 lady, 2 old man, 1 young man

$$={}^2C_1\cdot{}^2C_2\cdot{}^4C_1=2\cdot1\cdot4=8$$

Case IV 2 lady, 2 old man, 0 young man

$$={}^2C_2\cdot{}^2C_1\cdot{}^4C_0=1\cdot1\cdot1=1$$

$\therefore$ Required number of ways $=24+8+8+1=41$

38. Let $x_1, x_2, \ldots, x_8$ denote the question.

$\therefore \quad x_1+x_2+\ldots+x_8=30$

Also, $\quad x_1, x_2, \ldots, x_8\geq2$

Let $\quad u_1=x_1-2, u_2=x_2-2 \ldots u_8=x_8-2$

$\therefore \quad (u_1+2+u_2+2+\ldots+u_8+2)=30$

$\Rightarrow \quad u_1+u_2+\ldots+u_8=14$

$\therefore$ Total number of solutions $={}^{14+8-1}C_{8-1}={}^{21}C_7$

39. Required number of triangles that can be constructed using these chosen points as vertices $={}^{12}C_3-{}^3C_3-{}^4C_3-{}^5C_3$

(here, we subtract those triangle in which points are collinear)

$$=220-1-4-10=220-15=205$$

40. p = The number of such numbers that exceeds 20000 = 5! = 120

q = The number of those that lie between 30000 and 90000, then,

$$=5!-4!-4!=120-24-24=72$$

$$\therefore \quad \frac{p}{q}=\frac{120}{72}=\frac{5}{3}$$

41. A matrix whose determinant is non-zero is called a non-singular matrix. There are total 27 combinations, in those of them 20 such combination, in which determinant is non-zero.

42. Given, $n(A) = 2$, $n(B) = 4$, hence $n(A \times B) = 8$

The number of subsets of $A \times B$ having 3 or more elements

$$= {}^8C_3 + {}^8C_4 + \ldots + {}^8C_8$$
$$= ({}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$$
$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 = 256 - 1 - 8 - 28 = 219$$

$(\because 2^n = {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n)$

43. $T_n = {}^nC_3$, hence $T_{n+1} = {}^{n+1}C_3$

So, $T_{n+1} - T_n \Rightarrow {}^{n+1}C_3 - {}^nC_3 = 10$ (given)

$$\Rightarrow \frac{(n+1)\,n\,(n-1)}{3!} - \frac{n\,(n-1)\,(n-2)}{3!} = 10$$
$$\Rightarrow \frac{n\,(n-1)}{3!}(n+1-n+2) = 10$$
$$\Rightarrow \frac{n\,(n-1)}{3!} \times 3 = 10$$
$$\Rightarrow n^2 - n - 20 = 0 \Rightarrow n = 5$$

44. The number of ways to choose zero or more white balls

$= (10 + 1)$ $\quad$ ($\because$ all white balls are mutually identical)

Number of ways to choose zero or more green balls

$= (9 + 1)$ $\quad$ ($\because$ all green balls are mutually identical)

Number of ways to choose zero or more black balls

$= (7 + 1)$ $\quad$ ($\because$ all black balls are mutually identical)

Hence, number of ways to choose zero or more balls of any colour $= (10 + 1)(9 + 1)(7 + 1)$

Also, number of ways to choose a total of zero balls = 1

Hence, the number of ways to choose atleast one ball (irrespective of any colour)

$$= (10 + 1)(9 + 1)(7 + 1) - 1$$
$$= 880 - 1 = 879$$

45. The number of different ordered pairs (Y, Z) such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z = \phi$. Since, $Y \subseteq X$, $Z \subseteq X$, hence we can only use the elements of X to construct sets Y and Z.

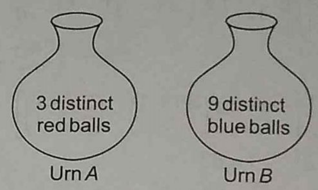
$n(Y)$	Number of ways to make Y	Number of ways to make Z such that $Y \cap Z = \phi$
0	${}^5C_0$	2^5
1	${}^5C_1$	2^4
2	${}^5C_2$	2^3
3	${}^5C_3$	2^2
4	${}^5C_4$	2^1
5	${}^5C_5$	2^0

Hence, total number of ways to construct sets Y and Z such that $Y \cap Z = \phi$

$$= {}^5C_0 \times 2^5 + {}^5C_1 \times 2^{5-1} + \ldots + {}^5C_5 \times 2^{5-5}$$
$$= (2 + 1)^5 = 3^5$$

46. Let the number of ways of distributing n identical objects, among r persons such that each person gets atleast one object is same as the number of ways of selecting $(r - 1)$ places out of $(n - 1)$ different places, *i.e.*, ${}^{n-1}C_{r-1}$.

47.



The number of ways in which two balls from urn A and two balls from urn B can be selected $= {}^3C_2 \times {}^9C_2 = 3 \times 36 = 108$

48. $\because$ $\quad S_1 = \sum_{j=1}^{10} j(j-1)\frac{10!}{j(j-1)(j-2)!(10-j)!}$

$$= 90\sum_{j=2}^{10} \frac{8!}{(j-2)!\{8-(j-2)\}!} = 90 \cdot 2^8$$

and $\quad S_2 = \sum_{j=1}^{10} j\frac{10!}{j(j-1)!\{9-(j-1)\}!}$

$$= 10\sum_{j=1}^{10} \frac{9!}{(j-1)!\{9-(j-1)\}!} = 10 \cdot 2^9$$

Also, $\quad S_3 = \sum_{j=1}^{10} [j(j-1) + j]\frac{10!}{j!(10-j)!}$

$$= \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j + \sum_{j=1}^{10} j\,{}^{10}C_j$$
$$= 90 \cdot 2^8 + 10 \cdot 2^9 = 90 \cdot 2^8 + 20 \cdot 2^8 = 110 \cdot 2^8 = 55 \cdot 2^9$$

49. Since, the number of ways that child can buy the six ice-creams is equal to the number of different ways of arranging $6A$'s and $4B$' s in a row.

$\therefore$ Number of ways to arrange $6A$'s and $6B$' s in row

$$= \frac{10!}{6!4!} = {}^{10}C_4$$

and number of integral solution of the equation

$x_1 + x_2 + x_3 + x_4 + x_5 = 6$ is ${}^{6+5-1}C_{5-1} = {}^{10}C_4 \neq {}^{10}C_5$

Hence, Statement I is false and Statement II is true.

50. Given word is MISSISSIPPI.

Here, I = 4 times, S = 4 times, P = 2 times, M = 1 time

_M_I_I_I_I_P_P_

$\therefore$ Required number of words $= {}^8C_4 \times \frac{7!}{4!2!}$

$$= {}^8C_4 \times \frac{7 \times 6!}{4!2!} = 7 \cdot {}^8C_4 \cdot {}^6C_4$$

51. Required number of ways $= {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$$= \frac{12!}{8! \times 4!} \times \frac{8!}{4! \times 4!} \times 1 = \frac{12!}{(4!)^3}$$

52. Total number of ways $= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

$$= 10 + 45 + 120 + 210 = 385$$

Unit Test 1 (Algebra)

DAY 9

1. Which of the following is the empty set?

(a) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(d) $\{x : x$ is a real number and $x^2 = x + 2\}$

2. From 50 students taking examinations in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. Atmost 19 passed Mathematics and Physics, atmost 29 passed Mathematics and Chemistry and atmost 20 passed Physics and Chemistry. The largest possible number that could have passed all three examinations is

(a) 11 (b) 12
(c) 13 (d) 14

3. Let A be the non-empty set of children in a family. The relation 'x is a brother of y' in A is

(a) reflexive
(b) symmetric
(c) transitive
(d) an equivalence relation

4. The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$
(c) $\text{Re}(z) > 3$ (d) None of these

5. If X be the set of all complex numbers z such that $|z| = 1$ and define relation R on X by $z_1 R z_2$ is $|\arg z_1 - \arg z_2| = \frac{2\pi}{3}$, then R is

(a) reflexive (b) symmetric
(c) transitive (d) anti-symmetric

6. If $1, \omega$ and ω^2 be the three cube roots of unity, then $(1+\omega)(1+\omega^2)(1+\omega^4)\ldots 2n$ factors is equal to

(a) 1 (b) -1
(c) 0 (d) None of these

7. The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(a) $-1, \omega$ (b) $-1, \omega^2$
(c) ω, ω^2 (d) None of these

8. Let z_1, z_2 and z_3 be three points on $|z| = 1$. If θ_1, θ_2 and θ_3 be the arguments of z_1, z_2 and z_3 respectively, then $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)$

(a) $\geq \frac{3}{2}$ (b) $\geq -\frac{3}{2}$
(c) $\leq \frac{-3}{2}$ (d) None of these

9. If the roots of the equation

$$(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$$

are real, then a^2, bd and c^2 are in

(a) AP (b) GP
(c) HP (d) None of these

10. If $a < 0$, then the positive root of the equation $x^2 - 2a|x - a| - 3a^2 = 0$ is

(a) $a(-1-\sqrt{6})$ (b) $a(1-\sqrt{2})$ (c) $a(1-\sqrt{6})$ (d) $a(1+\sqrt{2})$

11. If α and β are the roots of the equation $ax^2 - 2bx + c = 0$, then $\alpha^3\beta^3 + \alpha^2\beta^3 + \alpha^3\beta^2$ is equal to

(a) $\frac{c^2}{a^3}(c - 2b)$ (b) $\frac{c^2}{a^3}(c + 2b)$
(c) $\frac{bc^2}{a^3}$ (d) None of these

12. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$ (b) $2 \leq a \leq 3$ (c) $3 \leq a \leq 4$ (d) $a > 4$

13. The integer k for which the inequality $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ is valid for any x, is

(a) 2 (b) 3
(c) 4 (d) None of these

14. If $b > a$, then the equation $(x-a)(x-b)-1=0$, has

(a) both the roots in $[a, b]$
(b) both the roots in $(-\infty, a]$
(c) both the roots in (b, ∞)
(d) one root in $(-\infty, a)$ and other in (b, ∞)

15. Let a, b and $c \in R$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation of $a^2x^2 + 2bx + 2c = 0$ has a root γ, that always satisfies

(a) $\gamma = \alpha$ (b) $\gamma = \beta$
(c) $\gamma = (\alpha + \beta)/2$ (d) $\alpha < \gamma < \beta$

16. The value of x satisfying $\log_2(3x-2) = \log_{1/2} x$ is

(a) $-\frac{1}{3}$ (b) 2
(c) $\frac{1}{2}$ (d) None of these

17. If $f(x, n) = \sum_{r=1}^{n} \log_x \left(\frac{r}{x}\right)$, then the value of x satisfying the equation $f(x, 11) = f(x, 12)$ is

(a) 10 (b) 11
(c) 12 (d) None of these

18. The set of all values of x satisfying $x^{\log_x (1-x)^2} = 9$ is

(a) a finite set containing atleast three elements
(b) a subset of R containing Z (set of all integers)
(c) a finite set containing atleast two elements
(d) a finite set

19. If $\log_{0.5}(x-1) < \log_{0.25}(x-1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(3, \infty)$ (c) $(-\infty, 0)$ (d) $(0, 3)$

20. The maximum sum of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + 18 + \ldots$ is

(a) 310 (b) 290
(c) 320 (d) None of these

21. The number of common terms to the two sequences $17, 21, 25, \ldots, 417$ and $16, 21, 26, \ldots, 466$ is

(a) 21 (b) 19 (c) 20 (d) 91

22. Between two numbers whose sum is $2\frac{1}{6}$ an even number of arithmetic means are inserted. If the sum of these means exceeds their number by unity, then the number of means are

(a) 12 (b) 10
(c) 8 (d) None of these

23. If the sum of the first three terms of a GP is 21 and the sum of the next three terms is 168, then the first term and the common ratio is

(a) 3, 4 (b) 2, 4
(c) 3, 2 (d) None of these

24. The three numbers a, b and c between 2 and 18 are such that their sum is 25, the numbers $2, a$ and b are consecutive terms of an AP and the numbers b, c and 18 are consecutive terms of a GP. The three numbers are

(a) 3, 8, 14 (b) 2, 9, 14
(c) 5, 8, 12 (d) None of these

25. If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$ and $b \neq a + c$, then a, b and c are in

(a) GP (b) HP
(c) AP (d) None of these

26. Sum of n terms of series $12 + 16 + 24 + 40 + \ldots$ will be

(a) $2(2^n - 1) + 8n$ (b) $2(2^n - 1) + 6n$
(c) $3(2^n - 1) + 8n$ (d) $4(2^n - 1) + 8n$

27. Let R be a relation defined by $R = \{(x, x^3) : x$ is a prime number $< 10\}$, then which of the following is true?

(a) $R = \{(1, 1), (2, 8), (3, 27), (4, 64), (5, 125), (6, 216), (7, 343), (8, 512), (9, 729)\}$
(b) $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
(c) $R = \{(2, 8), (4, 64), (6, 216), (8, 512)\}$
(d) None of the above

28. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Then, the number of days in which the work was completed is

(a) 29 days (b) 24 days
(c) 25 days (d) 26 days

29. The sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$, is

(a) $\frac{n^2+n}{2(n^2+n+1)}$ (b) $\frac{n^2-n}{2(n^2+n+1)}$
(c) $\frac{n^2+n}{2(n^2-n+1)}$ (d) None of these

30. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then which of the following is not true?

(a) $\lim_{n \to \infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ (b) $\lim_{n \to \infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$
(c) $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}, \forall n \neq N$ (d) None of these

31. If C is a skew-symmetric matrix of order n and X is $n \times 1$ column matrix, then $X' C X$ is a

(a) scalar matrix (b) unit matrix
(c) null matrix (d) None of these

32. Which of the following is correct?

(a) Skew-symmetric matrix of an even order is always singular.
(b) Skew-symmetric matrix of an odd order is non-singular
(c) Skew-symmetric matrix of an odd order is singular
(d) None of the above

33. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$, then the value of

$$\Delta = \begin{vmatrix} b_2c_3 - b_3c_2 & a_3c_2 - a_2c_3 & a_2b_3 - a_3b_2 \\ b_3c_1 - b_1c_3 & a_1c_3 - a_3c_1 & a_3b_1 - a_1b_3 \\ b_1c_2 - b_2c_1 & a_2c_1 - a_1c_2 & a_1b_2 - a_2b_1 \end{vmatrix} \text{ is}$$

(a) 5 (b) 25 (c) 125 (d) 0

34. If A and B are square matrices such that $B = -A^{-1}BA$, then

(a) $AB + BA = O$
(b) $(A+B)^2 = A^2 - B^2$
(c) $(A+B)^2 = A^2 + 2AB + B^2$
(d) $(A+B)^2 = A + B$

35. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if

(a) x, y, z are in AP (b) x, y, z are in GP
(c) x, y, z are in HP (d) xy, yz, zx are in AP

36. If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim_{x \to 0} \frac{f'(x)}{x}$ is

(a) 1 (b) −1
(c) 2 (d) −2

37. The value of k, for which the system of equations

$x + ky + 3z = 0$, $3x + ky - 2z = 0$ and $2x + 3y - 4z = 0$

possess a non-trivial solution over the set of rationals, is

(a) $-\frac{33}{2}$ (b) $\frac{33}{2}$
(c) 11 (d) None of these

38. If α, β and γ are the roots of the equation $x^2(x+a) = e(x+1)$.

Then, the value of the determinant $\begin{vmatrix} 1+\alpha & 1 & 1 \\ 1 & 1+\beta & 1 \\ 1 & 1 & 1+\gamma \end{vmatrix}$ is

(a) −1 (b) 1
(c) 0 (d) $1 + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

39. If $x = -5$ is a root of $\begin{vmatrix} 2x+1 & 4 & 8 \\ 2 & 2x & 2 \\ 7 & 6 & 2x \end{vmatrix} = 0$, then the other two roots are

(a) 3, 3.5 (b) 1, 3.5
(c) 3, 6 (d) 2, 6

40. The 8th term of $\left(3x + \frac{2}{3x^2}\right)^{12}$ when expanded in ascending power of x, is

(a) $\frac{228096}{x^3}$ (b) $\frac{228096}{x^9}$
(c) $\frac{328179}{x^9}$ (d) None of these

41. The greatest term in the expansion of $(3-5x)^{11}$ when $x = \frac{1}{5}$, is

(a) 55×3^9 (b) 46×3^9
(c) 55×3^6 (d) None of these

42. For all natural number $n > 1$, $2^{4n} - 15n - 1$ is divisible by

(a) 225 (b) 125
(c) 325 (d) None of these

43. If x is so small that its two and higher power can be neglected and if $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1 + kx$, then k is equal to

(a) −2 (b) 1 (c) 10 (d) 11

44. If n is an integer greater than 1, then $a - {}^nC_1(a-1) + {}^nC_2(a-2) + \ldots + (-1)^n(a-n)$ is equal to

(a) a (b) 0 (c) a^2 (d) 2^n

45. The greatest integer less than or equal to $(\sqrt{2}+1)^6$ is

(a) 196 (b) 197
(c) 198 (d) 199

46. In an examination a candidate has to pass in each of the papers to be successful. If the total number of ways to fail is 63, how many papers are there in the examination?

(a) 6 (b) 8 (c) 10 (d) 12

47. The number of seven letter words that can be formed by using the letters of the word 'SUCCESS' so that the two C are together but no two S are together, is

(a) 24 (b) 36
(c) 54 (d) None of these

48. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q, so that $P \cap Q$ contains exactly two elements is

(a) $9 \cdot {}^nC_2$ (b) $3^n - {}^nC_2$
(c) $2 \cdot {}^nC_n$ (d) None of these

49. The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if atleast 1 green and 1 blue ball is to be included is

(a) 3700 (b) 3720
(c) 4340 (d) None of these

50. If the sets A and B are defined as

$$A=\{(x,y): y=\frac{1}{x}, 0\neq x\in R\}$$

and $B=\{(x,y): y=-x, x\in R\}$, then

(a) $A\cap B=A$ (b) $A\cap B=B$
(c) $A\cap B=\phi$ (d) None of these

51. There are 16 points in a plane no three of which are in a straight line except 8 which are all in a straight line. The number of triangles can be formed by joining them equals to

(a) 1120 (b) 560
(c) 552 (d) 504

52. The value of the natural numbers n such that inequality $2^n > 2n+1$ is valid, is

(a) for $n\geq 3$ (b) for $n<3$
(c) for mn (d) for any n

Directions (Q. Nos. 53 and 54) *Let* $w=\cos\left(\frac{2\pi}{7}\right)+i\sin\left(\frac{2\pi}{7}\right)$ *and* $\alpha=w+w^2+w^4$ *and* $\beta=w^3+w^5+w^6$.

53. $\alpha+\beta$ is equal to

(a) 0 (b) -1
(c) -2 (d) 1

54. α and β are the roots of the equation

(a) $x^2+x+1=0$ (b) $x^2+x+2=0$
(c) $x^2+3x+5=0$ (d) None of these

Directions (Q. Nos. 55 and 56)
Let $F(x)=f(x)+g(x)$, $G(x)=f(x)-g(x)$ *and* $H(x)=\frac{f(x)}{g(x)}$, *where* $f(x)=1-2\sin^2 x$ *and* $g(x)=\cos 2x, \forall f: R\to[-1,1]$ *and* $g: R\to[-1,1]$.

55. Domain and range of $H(x)$ are respectively

(a) R and $\{1\}$
(b) R and $\{0,1\}$
(c) $R\sim\{(2n+1)\frac{\pi}{4}\}$ and $\{1\}, n\in I$
(d) $R\sim\left\{(2n+1)\frac{\pi}{2}\right\}$ and $\{0,1\}, n\in I$

56. If $F: R\to[-2,2]$, then

(a) $F(x)$ is one-one function
(b) $F(x)$ is many-one function
(c) $F(x)$ is onto function
(d) $F(x)$ is into function

Directions (Q. Nos. 57 to 61) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

57. Statement I The number of natural numbers which divide 10^{2009} but not 10^{2008} is 4019.

Statement II If p is a prime, then number of divisors of p^n is $p^{n+1}-1$.

58. Suppose $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ and $B=\begin{bmatrix}2&0\\0&2\end{bmatrix}$. Let X be a 2×2 matrices such that $X'AX=B$.

Statement I X is non-singular and $\det(X)=\pm 2$.

Statement II X is a singular matrix.

59. The general term in the expansion of $(a+x)^n$ is ${}^nC_r\, a^{n-r}x^r$.

Statement I The third term in the expansion of $\left(2x+\frac{1}{x^2}\right)^m$ does not contain x. The value of x for which that term equal to the second term in the expansion of $(1+x^3)^{30}$ is 2.

Statement II $(a+x)^n=\sum_{r=0}^{n}{}^nC_r\, a^{n-r}x^r$.

60. Sets A and B have four and eight elements, respectively.

Statement I The minimum number of elements in $A\cup B$ is 8.

Statement II $A\cap B=5$

61. Let $a\neq 0, p\neq 0$ and

$$\Delta=\begin{vmatrix}a&b&c\\0&p&q\\p&q&0\end{vmatrix}$$

Statement I If the equations $ax^2+bx+c=0$ and $px+q=0$ have a common root, then $\Delta=0$.

Statement II If $\Delta=0$, then the equations $ax^2+bx+c=0$ and $px+q=0$ have a common root.

Answer *with* Solutions

1. (*b*) Since, $x^2+1=0 \Rightarrow x=\pm i$

As x is real, so it is a empty set.

2. (*d*) Given, $n(M\cup P\cup C)=50, n(M)=37, n(P)=24, n(C)=43$

$n(M\cap P)\le 19, n(M\cap C)\le 29, n(P\cap C)\le 20$

$\therefore \quad n(M\cup P\cup C)=n(M)+n(P)+n(C)-n(M\cap P)-n(M\cap C)-n(P\cap C)+n(M\cap P\cap C)$

$\Rightarrow \quad 50=37+24+43-n(M\cap P)-n(M\cap C)-n(P\cap C)+n(M\cap P\cap C)$

$\Rightarrow n(M\cap P\cap C)=n(M\cap P)+n(M\cap C)+n(P\cap C)-54$

$\therefore n(M\cap P\cap C)\le 19+29+20-54=14$

3. (*c*)

(i) **Reflexive** Let $x\in A$, if x is a girl, then we cannot say that x is a brother of x. It is not reflexive.

(ii) Let $(x, y)\in R$, then x is a brother of y. But y may or may not be a boy.
Hence, we cannot say that $(y, x)\in R$.
$\therefore R$ is not symmetric.

(iii) Let $(x, y)\in R$ and $(y, z)\in R$
$\therefore x$ is a brother of y and y is a brother of z.
It implies x is a brother of $z \Rightarrow (x, z)\in R$
Hence, R is transitive.

4. (*c*) $|(x-4)+iy|^2<|(x-2)+iy|^2$ (let $z=x+iy$)

$\Rightarrow \quad (x-4)^2-y^2<(x-2)^2-y^2$

$\Rightarrow \quad -4x<-12$

$\Rightarrow \quad x>3$

$\therefore \quad \text{Re}(z)>3$

5. (*b*) $\because |z|=1 \Rightarrow z=\cos\theta+i\sin\theta$

$\therefore \quad \arg(z)=\theta$

Then, $\arg(z_1)=\theta_1$ and $\arg(z_2)=\theta_2$

$\therefore \quad z_1Rz_2 \Rightarrow |\arg z_1-\arg z_2|=\dfrac{2\pi}{3}$

$\Rightarrow \quad z_1Rz_2 \Rightarrow |\theta_1-\theta_2|=\dfrac{2\pi}{3}$

$\Rightarrow \quad |\theta_2-\theta_1|=\dfrac{2\pi}{3}$

$\Rightarrow \quad z_2Rz_1$ but $z_1\ne z_2$

($\because$ when $z_1=z_2$, then $0=\dfrac{2\pi}{3}$ is not possible)

Hence, it is symmetric.

6. (*a*) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ to $2n$ factors

$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots$ to $2n$ factors

$=[(1+\omega)(1+\omega)\ldots$ to n factors$]$
$[(1+\omega^2)(1+\omega^2)\ldots$ to n factors$]$

$=(1+\omega)^n(1+\omega^2)^n=(1+\omega+\omega^2+\omega^3)^n$

$=(0+\omega^3)^n=\omega^{3n}=1$

7. (*c*) $\quad z^3+2z^2+2z+1=0$

$\Rightarrow \quad (z+1)(z^2+z+1)=0$

$\Rightarrow \quad z=-1, \omega, \omega^2$

But $z=-1$ does not satisfy the second equation.
Hence, common roots are ω and ω^2.

8. (*b*) We have, $|z_1|=|z_2|=|z_3|=1$

Now, $\quad |z_1+z_2+z_3|\ge 0$

$\Rightarrow |z_1|^2+|z_2|^2+|z_3|^2+2\text{Re}(z_1\bar{z}_2+z_2\bar{z}_3+z_3\bar{z}_1)\ge 0$

$\Rightarrow \quad 3+2[\cos(\theta_1-\theta_2)+\cos(\theta_2-\theta_3)+\cos(\theta_3-\theta_1)]\ge 0$

$\Rightarrow \cos(\theta_1-\theta_2)+\cos(\theta_2-\theta_3)+\cos(\theta_3-\theta_1)\ge -\dfrac{3}{2}$

9. (*b*) Here, $D\ge 0$

$\therefore \quad 4(bc+ad)^2-4(a^2+b^2)(c^2+d^2)\ge 0$

$\Rightarrow b^2c^2+a^2d^2+2abcd-a^2c^2-a^2d^2-b^2c^2-b^2d^2\ge 0$

$\Rightarrow \quad (ac-bd)^2\le 0$

$\Rightarrow \quad ac-bd=0$

($\because$ square of any expression cannot be negative)

$\therefore \quad b^2d^2=a^2c^2$

Hence, a^2, bd and c^2 are in GP.

10. (*b*) If $x\ge a$, then $x^2-2a(x-a)-3a^2=0$

$\Rightarrow x^2-2ax-a^2=0$

$\therefore \quad x=\dfrac{2a\pm\sqrt{4a^2+4a^2}}{2}=a(1\pm\sqrt{2})$

Since, $x\ge a$

$\therefore x=a(1+\sqrt{2})$, it is impossible because $a<0$

$\therefore x=a(1-\sqrt{2})$

If $x<a$, then $x^2+2a(x-a)-3a^2=0$

$\Rightarrow \quad x^2+2ax-5a^2=0$

$\therefore \quad x=(-1\pm\sqrt{6})a$

(impossible $x<a$ and $a<0$)

11. (*b*) Here, $\alpha+\beta=\dfrac{2b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

Now, $(\alpha\beta)^3+\alpha^2\beta^2(\beta+\alpha)=\left(\dfrac{c}{a}\right)^3+\dfrac{c^2}{a^2}\left(\dfrac{2b}{a}\right)=\dfrac{c^2(c+2b)}{a^3}$

12. (*a*) According to the question,

$D\ge 0$ and $f(3)>0$

$\Rightarrow \quad 4a^2-4(a^2+a-3)\ge 0$

and $\quad 3^2-2a(3)+a^2+a-3>0$

$\Rightarrow \quad -a+3\ge 0$ and $a^2-5a+6>0$

$\Rightarrow \quad a\le 3$ and $a<2$ or $a>3$

$\therefore \quad a<2$

13. (*b*) Let $f(x) = x^2 - 2(4k-1)x + 15k^2 - 2k - 7$, then $f(x) > 0$

$\therefore \quad D < 0$

$\Rightarrow \quad 4(4k-1)^2 - 4(15k^2 - 2k - 7) < 0$

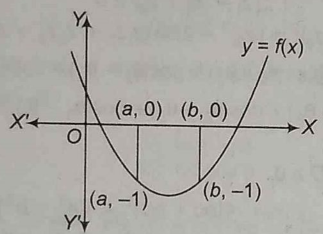
$\Rightarrow \quad k^2 - 6k + 8 < 0$

$\Rightarrow \quad 2 < k < 4$

Hence, required integer value of k is 3.

14. (*d*) Let $f(x) = (x-a)(x-b) - 1$

We observe that the coefficient of x^2 in $f(x)$ is positive and $f(a) = f(b) = -1$. Thus, the graph of $f(x)$ is as shown in figure given below



It is evident from the graph that one of the roots of $f(x) = 0$ lies in $(-\infty, a)$ and the other root lies in (b, ∞).

15. (*d*) Let $f(x) = a^2x^2 + 2bx + 2c$

$\therefore \quad f(\alpha) = a^2\alpha^2 + b\alpha + c = 0$

and $\quad a^2\beta^2 - b\beta - c = 0$

Now, $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = b\alpha + c = -a^2\alpha^2$

$f(\beta) = a^2\beta^2 + 2b\beta + 2c = 3(b\beta + c) = 3a^2\beta^2$

But $0 < \alpha < \beta \Rightarrow \alpha, \beta$ are real number.

$\therefore \quad f(\alpha) < 0, f(\beta) > 0$

Hence, $\alpha < \gamma < \beta$

16. (*d*) Given, $\log_2(3x-2) = \log_{1/2} x = -\log_2 x = \log_2 x^{-1}$

$\Rightarrow \quad 3x - 2 = x^{-1}$

$\Rightarrow \quad 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (3x+1)(x-1) = 0$

$\Rightarrow \quad x = 1 \text{ or } x = -\frac{1}{3}$

$\therefore \quad x = 1$

($\because$ negative of x cannot satisfy the given equation)

17. (*c*) $f(x, n) = \sum_{r=1}^{n} (\log_x r - \log_x x)$

$= \sum_{r=1}^{n} (\log_x r - 1) = \log_x(1 \cdot 2 \ldots n) - n = \log_x n! - n$

Given, $\quad f(x, 11) = f(x, 12)$

$\Rightarrow \quad \log_x(11!) - 11 = \log_x(12!) - 12$

$\Rightarrow \quad \log_x\left(\frac{12!}{11!}\right) = 1$

$\Rightarrow \quad \log_x(12) = 1$

$\therefore \quad x = 12$

18. (*d*) Taking logarithm with base 3, we get

$\log_x(1-x)^2 \log_3 x = 2$

$\Rightarrow \quad \frac{\log_3(1-x)^2}{\log_3 x} \log_3 x = 2$

$\Rightarrow \quad \log_3(1-x)^2 = 2$

$\Rightarrow \quad (1-x)^2 = 9$

$\Rightarrow \quad x = 4, -2$

$\therefore \quad x = 4 \qquad (\because x > 0)$

19. (*a*) Given, $\log_{0.5}(x-1) < \log_{0.25}(x-1)$

$\Rightarrow \quad \log_{0.5}(x-1) < \log_{(0.5)^2}(x-1)$

$= \frac{1}{2}\log_{0.5}(x-1)$

$\Rightarrow \quad \log_{0.5}(x-1) < 0$

$\Rightarrow \quad x - 1 > 1$

$\therefore \quad x > 2$

20. (*a*) Here, $a = 20, d = -\frac{2}{3}$

As the common difference is negative, the terms will become negative after some stage. So, the sum is maximum, if only positive terms are added.

Now, $T_n = 20 + (n-1)\left(-\frac{2}{3}\right) \geq 0 \Rightarrow 60 - 2(n-1) \geq 0$

$\Rightarrow \quad 62 \geq 2n$

$\Rightarrow \quad 31 \geq n$

$\therefore$ The first 31 terms are non-negative.

$\therefore$ Maximum sum,

$S_{31} = \frac{31}{2}\left[2 \times 20 + (31-1)\left(-\frac{2}{3}\right)\right]$

$= \frac{31}{2}(40 - 20) = 310$

21. (*c*) First series has common difference 4 and second series has common difference 5.

Hence, the series with common terms has common difference is equal to the LCM of 4 and 5 *i.e.*, 20.

$\therefore$ The first common term is 21.

$\therefore$ The series will be 21, 41, 61, ..., 411 which has 20 terms.

22. (*a*) Let $2n$ arithmetic means be $A_1, A_2, \ldots, A_{2n}$ between a and b. Then,

$A_1 + A_2 + \ldots + A_{2n} = \frac{a+b}{2} \times 2n$

$= \frac{13/6}{2} \times 2n = \frac{13n}{6}$

Given, $\quad A_1 + A_2 + \ldots + A_{2n} = 2n + 1$

$\Rightarrow \quad 2n + 1 = \frac{13n}{6}$

$\Rightarrow \quad 12n + 6 = 13n$

$\therefore \quad n = 6$

Hence, the number of means $= 2 \times 6 = 12$

23. (c) Given, $a_1 + a_2 + a_3 = 21$

$\Rightarrow a(1 + r + r^2) = 21$

and $a_4 + a_5 + a_6 = 168$

$\Rightarrow ar^3(1 + r + r^2) = 168$

$\therefore r^3 = 8 \Rightarrow r = 2$

and $a(1 + 2 + 4) = 21$

$\therefore a = 3$

24. (c) Given, $a + b + c = 25$ ...(i)

$\because$ 2, a, b are in AP $\therefore 2a = 2 + b$ ...(ii)

$\because$ $b, c, 18$ are in GP $\therefore c^2 = 18b$ ...(iii)

From Eqs. (i) and (ii), $3b = 48 - 2c$

From Eq. (iii), $c^2 = 6(48 - 2c) = 288 - 12c$

$\Rightarrow c^2 + 12c - 288 = 0$

$\Rightarrow (c + 24)(c - 12) = 0$

$\Rightarrow c = 12$ as $c \neq -24$

$\therefore b = 8$ and $a = 5$

25. (b) Given, $\frac{1}{a} + \frac{1}{c-b} = \frac{1}{b-a} - \frac{1}{c}$

$\Rightarrow \frac{a+c-b}{a(c-b)} = \frac{c-b+a}{c(b-a)}$

$\Rightarrow a(c-b) = c(b-a)$

$\Rightarrow 2ac = ab + bc$

$\therefore \frac{2ac}{a+c} = b$

26. (d) Let $S_n = 12 + 16 + 24 + \ldots + T_n$

Again, $S_n = +12 + 16 + \ldots + T_n$

$0 = (12 + 4 + 8 + 16 + \ldots$ upto n terms$) - T_n$

$\therefore T_n = 12 + \frac{4(2^{n-1} - 1)}{2-1} = 2^{n+1} + 8$

On putting $n = 1, 2, 3, \ldots$, we get

$T_1 = 2^2 + 8, T_2 = 2^3 + 8, T_3 = 2^4 + 8 \ldots$

$\therefore S_n = T_1 + T_2 + \ldots + T_n$

$= (2^2 + 2^3 + \ldots$ upto n terms$)$ $+ (8 + 8 + \ldots$ upto n terms$)$

$= \frac{2^2(2^n - 1)}{2-1} + 8n = 4(2^n - 1) + 8n$

27. (b) Given, x is a prime < 10

$\therefore x = \{2, 3, 5, 7\}$

Now, from $R = \{(x, x^3) : x = 2, 3, 5, 7\}$

$= \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

28. (c) Here, $a = 150$ and $d = -4$

$S_n = \frac{n}{2}[2 \times 150 + (n-1)(-4)]$

$= n(152 - 2n)$

Had the workers not dropped, then the work would have finished is $(n - 8)$ days with 150 workers working on each day.

$\therefore n(152 - 2n) = 150(n - 8)$

$\Rightarrow n^2 - n - 600 = 0$

$\Rightarrow (n - 25)(n + 24) = 0$

$\therefore n = 25$ ($\because n$ cannot be negative)

29. (a) $T_r = \frac{r}{1 + r^2 + r^4}, r = 1, 2, 3, \ldots, n$

$= \frac{r}{(r^2 + r + 1)(r^2 - r + 1)} = \frac{1}{2}\left[\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right]$

$\therefore \sum_{r=1}^{n} T_r = \frac{1}{2}\left[\sum_{r=1}^{n} \frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right]$

$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1}\right)\right]$

$= \frac{1}{2}\left[1 - \frac{1}{n^2 + n + 1}\right] = \frac{n^2 + n}{2(n^2 + n + 1)}$

30. (b) $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow (A^{-1})^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Similarly, $(A^{-1})^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

Now, $\lim_{n \to \infty} \frac{1}{n} A^{-n} = \lim_{n \to \infty} \begin{bmatrix} 1/n & 0 \\ -1 & 1/n \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

and $\lim_{n \to \infty} \frac{1}{n^2} A^{-n} = \lim_{n \to \infty} \begin{bmatrix} 1/n^2 & 0 \\ -\frac{1}{n} & 1/n^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

31. (c) Here, X is $n \times 1$, C is $n \times n$ and X' is $1 \times n$ order matrix. Therefore, $X'CX$ is 1×1 order matrix. Let $X'CX = K$

Then, $(X'CX)' = X'C'X''$

$= X'(-C)X = -K$

$\Rightarrow 2K = O$

$\therefore K = O$

32. (c) Since, the determinant of a skew-symmetric matrix of an odd order is zero. Therefore, the matrix is singular.

33. (b) We know that, if A is a square matrix of order n and B is the matrix of cofactors of elements of A. Then,

$|B| = |A|^{n-1}$

$\therefore \Delta = |A|^{3-1} = 5^{3-1} = 25$

34. (a) Given, $B = -A^{-1}BA$

$\Rightarrow AB = -A(A^{-1}BA)$

$\Rightarrow AB = -I(BA)$

$\therefore AB + BA = O$

35. (*b*) Applying $R_3 \to R_3 - pR_1 - R_2$

$$\Rightarrow \begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ -(xp^2+2yp+z) & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow -(xp^2+2yp+z)(xz-y^2)=0$$

Hence, x, y and z are in GP.

36. (*d*) Applying $R_1 \to R_1 - R_3$

$$f(x)=\begin{vmatrix} \cos x-\tan x & 0 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

$$=(\cos x-\tan x)(x^2-2x^2)$$

$$=-x^2(\cos x-\tan x)$$

$$\therefore \quad f'(x)=-2x(\cos x-\tan x)-x^2(-\sin x-\sec^2 x)$$

$$\therefore \quad \lim_{x\to 0}\frac{f'(x)}{x}=\lim_{x\to 0}[-2(\cos x-\tan x)+\lim_{x\to 0}x(\sin x+\sec^2 x)]$$

$$=-2\times 1=-2$$

37. (*b*) For non-trivial solution, we must have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix}=0$$

Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$,

$$\begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix}=0$$

$$\Rightarrow 20k+11(3-2k)=0 \Rightarrow k=\frac{33}{2}$$

38. (*c*) Given, equation is $x^3+ax^2-ex-e=0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$,

$$\Delta=\begin{vmatrix} (1+\alpha) & 1 & 1 \\ -\alpha & \beta & 0 \\ 0 & -\beta & \gamma \end{vmatrix}$$

$$=(1+\alpha)(\beta\gamma-0)+\alpha(\gamma)+\alpha\beta$$

$$=\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\beta\gamma=-e+e=0$$

39. (*b*) Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 2x+10 & 2x+10 & 2x+10 \\ 2 & 2x & 2 \\ 7 & 6 & 2x \end{vmatrix}=0$$

Taking $2x+10$ common from R_1 and applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$2(x+5)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2x-2 & 0 \\ 7 & -1 & 2x-7 \end{vmatrix}=0$$

$$\Rightarrow 2(x+5)(2x-2)(2x-7)=0$$

$$\therefore \quad x=-5, 1, 3.5$$

40. (*a*) For ascending power of x, we take the expression

$$\left(\frac{2}{3x^2}+3x\right)^{12}$$

$$\therefore T_8 \text{ in } \left(\frac{2}{3x^2}+3x\right)^{12}={}^{12}C_7\left(\frac{2}{3x^2}\right)^{12-7}(3x)^7$$

$$=\frac{12!}{7!5!}\left(\frac{2}{3x^2}\right)^5(3x)^7$$

$$=\frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2}\times\frac{2^5\times 3^2}{x^3}$$

$$=\frac{228096}{x^3}$$

41. (*a*) $(3-5x)^{11}=3^{11}\left(1-\frac{5x}{3}\right)^{11}=3^{11}\left(1-\frac{1}{3}\right)^{11}$ $\quad \left(\because x=\frac{1}{5}\right)$

$$\therefore \text{ Greatest term}=\frac{|x|(n+1)}{(|x|+1)}$$

$$=\frac{\left|-\frac{1}{3}\right|(11+1)}{\left|-\frac{1}{3}\right|+1}=3$$

Now, $\quad T_3=3^{11}\cdot{}^{11}C_2\left(-\frac{1}{3}\right)^2$

$$=3^{11}\left(\frac{11\cdot 10}{1\cdot 2}\times\frac{1}{9}\right)=55\times 3^9$$

42. (*a*) Now, $\quad 2^{4n}=(1+15)^n$

$$=1+{}^nC_1\cdot 15+{}^nC_2\cdot 15^2+{}^nC_3\cdot 15^3+\ldots$$

$$\therefore 2^{4n}-1-15n=15^2[{}^nC_2+{}^nC_3\cdot 15+\ldots]=225k$$

Hence, it is divisible by 225.

43. (*d*) $(1-2x)^{-1/2}(1-4x)^{-5/2}$

$=(1+x)(1+10x)$ (neglecting higher power)

$=1+11x$ (neglecting higher power)

$=1+kx$

$\therefore \quad k=11$

44. (*b*) LHS $=a[C_0-C_1+C_2-C_3+\ldots+(-1)^n C_n]$

$$+[C_1-2C_2+3C_3-\ldots+(-1)^{n-1}\cdot nC_n]$$

$$=a\cdot 0+0=0$$

45. (*b*) Let $(\sqrt{2}+1)^6=I+F$, where I is an integer and $0\le F<1$

Let $\quad f=(\sqrt{2}-1)^6$

Now, $\quad \sqrt{2}-1=\frac{1}{\sqrt{2}+1} \Rightarrow 0<\sqrt{2}-1<1$

Also, $\quad I+F+f=(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$

$$=2[{}^6C_0\cdot 2^3+{}^6C_2\cdot 2^2+{}^6C_4\cdot 2+{}^6C_6]$$

$$=2(8+60+30+1)=198$$

Hence, $\quad F+f=198-I$ is an integer.

But $\quad 0<F+f<2$

$\therefore \quad F+f=1$ and thus $I=197$

46. (*a*) Let the number of papers be n.

$\therefore$ Total number of ways to fail or pass

$$= {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n$$

$\therefore$ Total number of ways to fail $= 2^n - 1$

(since, there is only one way to pass)

According to the question,

$$2^n - 1 = 63 \Rightarrow 2^n = 2^6 \Rightarrow n = 6$$

47. (*a*) Considering CC as single letter, U,CC,E can be arranged in 3! ways

Here, × U× CC× E×

Hence, the required number of ways $= 3! \cdot {}^4C_3 = 24$

48. (*d*) Let $A = \{a_1, a_2, \ldots, a_n\}, 1 \le i \le n$

(i) $a_i \in P, a_i \in Q$

(ii) $a_i \notin P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$

(iv) $a_i \in P, a_i \notin Q$

So, $P \cap Q$ contains exactly two elements, taking 2 elements in Eq. (i) and $(n-2)$ elements in Eqs. (ii), (iii) and (iv).

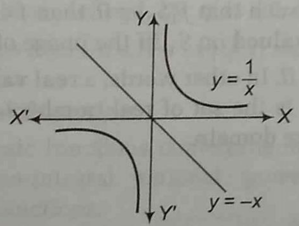
$\therefore$ Required number of ways $= {}^nC_2 \times 3^{n-2}$

49. (*b*) Atleast one green ball can be selected out of 5 green balls in $2^5 - 1$ *i.e.*, 31 ways. Similarly, atleast one blue ball can be selected from 4 blue balls in $2^4 - 1 = 15$ ways and atleast one red or not red can be selected in $2^3 = 8$ ways.

Hence, the required number of ways $= 31 \times 15 \times 8 = 3720$

50. (*c*) It is clear from the graph that two curves do not intersect anywhere.

$\therefore A \cap B = \phi$



51. (*d*) $\therefore$ Required number of ways $= {}^{16}C_3 - {}^8C_3 = 560 - 56 = 504$

52. (*a*) Check through options, the condition $2^n > 2n + 1$ is valid for $n \ge 3$.

53. (*b*) Here, $\alpha + \beta = \sum_{k=1}^{6} w^k = \frac{w(1 - w^6)}{1 - w} = -1 \quad (\because w^7 = 1)$

54. (*b*) Now, $\alpha\beta = (w + w^2 + w^4)(w^3 + w^5 + w^6)$

$$= 3 + (\alpha + \beta) = 2$$

$\therefore$ Required equation is

$$x^2 - (-1)x + 2 = 0$$

or

$$x^2 + x + 2 = 0$$

55. (*c*) $\because H(x) = \frac{f(x)}{g(x)} = \frac{1 - 2\sin^2 x}{\cos 2x} = \frac{\cos 2x}{\cos 2x} = 1$

But $\cos 2x \ne 0$

$\Rightarrow \quad 2x \ne n\pi + \frac{\pi}{2}, n \in I$

$\therefore \quad x \in R \sim \left\{(2n+1)\frac{\pi}{4}, n \in I\right\}$

and range $= \{1\}$

56. (*c*) $\because F(x) = f(x) + g(x) = 1 - 2\sin^2 x + \cos 2x$

$= 2\cos 2x \quad (\because -1 \le \cos 2x \le 1 \Rightarrow -2 \le 2\cos 2x \le 2)$

$\therefore$ Range of $F(x)$ = codomain of $F(x)$

Hence, $F(x)$ is onto function.

57. (*c*) The number of divisor of $10^m = 2^m 5^m$ is $(m+1)^2$.

$\therefore$ Number of divisors which divide 10^{2009} but not 10^{2008} is

$$(2010)^2 - (2009)^2 = 4019$$

58. (*c*) If $X = O$, then $X'AX = O \Rightarrow B = O$, a contradiction.

Let $\det(X) = a$, then $\det(X') = a$

$\therefore \quad \det(X'AX) = \det(B)$

$\Rightarrow a(-1)a = -4 \quad [\because \det(X'AX) = \det(X')\det(A)\det(X)]$

$\therefore \quad a = \pm 2$

As $\det(X) \ne 0$, X cannot be a singular matrix.

59. (*b*) $T_3 = {}^mC_2(2x)^{m-2}\left(\frac{1}{x^2}\right)^2 = {}^mC_2(2)^{m-2} \cdot x^{m-6}$

For independent of x, put $m - 6 = 0 \Rightarrow m = 6$

$\therefore \quad T_3 = {}^6C_2(2)^{6-2} = 15 \times 16 = 240$

According to the question, ${}^{30}C_1 x^3 = 240$

$\Rightarrow \quad x^3 = 8 \Rightarrow x = 2$

60. (*c*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 4 + 8 - n(A \cap B)$

$= 12 - n(A \cap B)$

Since, maximum number of element is $n(A \cap B) = 4$.

$\therefore$ Minimum number of element is

$$n(A \cup B) = 12 - 4 = 8$$

61. (*b*) If λ is a common root of $ax^2 + bx + c = 0$ and $px + q = 0$, then

$a\lambda^2 + b\lambda + c = 0, p\lambda + q = 0$ and $p\lambda^2 + q\lambda = 0$

Eliminating λ, we obtained $\Delta = 0$.

For Statement II, expanding Δ along C_1, we obtain

$$aq^2 + p(bq - cp) = 0$$

or

$$a\left(-\frac{q}{p}\right)^2 + b\left(-\frac{q}{p}\right) + c = 0$$

Thus, $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root.

Day 10

Real Function

Day 10 Outlines ...

- Real Valued Function
- Basic Functions
- Composition of Function
- Inverse Function

Real Valued Function

Let $f : S \to T$ be a function and $S_1 \subseteq S$, such that $f(S_1) \subseteq R$, then f is said to be real valued on S_1. *i.e.*, f is defined as **real valued** on S_1 iff the image of S_1 under f lies entirely within the set of real numbers R. In other words, a **real valued function** is a function $f : S \to R$ whose codomain is the set of real numbers R. *i.e.*, f is real valued iff it is real valued over its entire domain.

Domain

The domain of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Method for Finding Domain

If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then

(i) The domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$, while domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

(ii) Domain of $[\sqrt{f(x)}] = D_1 \cap \{x : f(x) > 0\}$

Range

Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

Method for Finding Range

First of all find the domain of $y = f(x)$.

(i) If domain $\in$ finite number of points. $\Rightarrow$ Range $\in$ set of corresponding $f(x)$ values.

(ii) If domain $\in R$ or $R - \{\text{some finite points}\}$, then express x in terms of y. From this find y for x to be defined.

(iii) If domain $\in$ a finite interval, find the least and greatest value or range using monotonically.

Basic Functions

Different basic functions have been given below

1. Algebraic Function

Those functions which are made from square root, power of independent variable and four fundamental operations (addition, subtraction, multiplication and division) are called algebraic function.

Any polynomial function $f : R \to R$ is onto, if degree of f is odd and into, if degree of f is even.

Some algebraic functions are given below

(i) Polynomial Function

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$

where, $a_0, a_1, a_2, \dots, a_n$ are real numbers and defined by $a_0 \neq 0$, $n \in N$ is known as polynomial function and n is a degree of polynomial function. Its domain $\in R$.

(ii) Rational Function

$f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and defined by $q(x) \neq 0$ is called rational function. Its domain $\in R - \{x \mid q(x) = 0\}$.

(iii) Irrational Function

The algebraic functions containing one or more terms having non-integral rational power x are called irrational functions.

e.g., $y = f(x) = 2\sqrt{x} - \sqrt[3]{x} + 6$

2. Reciprocal Function

If x is non-negative real numbers, then $\dfrac{1}{x}$ is the reciprocal function of x. Its domain $\in R - \{0\}$ and range $\in R - \{0\}$.

3. Constant Function

If only single element in the range of the function f, then f is known as constant function. Its domain $\in R$ and range $\in \{k\}$, where k is a constant. It is periodic with no fundamental period.

4. Identity Function

Let $f(x) = x, \forall \ x \in R$, known as identity function. Its domain $\in R$ and range $\in R$. It is not a periodic function.

5. Equal and Identical Function

Two functions f and g are said to be equal, if

(i) the domain of f = the domain of g

(ii) the range of f = the range of g

(iii) $f(x) = g(x), \forall \ x \in$ domain

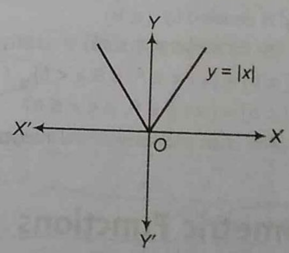
6. Power Function

A function $f : R \to R$ defined as $f(x) = x^{2m}$, where n is finite number, known as power function. Its domain $\in R$ and range $\in R^+ \cup \{0\}$. It is not a periodic function.

7. Absolute Valued Function (Modulus Function)

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Its Domain $\in R$ and Range $\in [0, \infty]$. It is not a periodic function.



Properties of Modulus Function

(i) $|x| \leq a \Rightarrow -a \leq x \leq a \ (a \geq 0)$

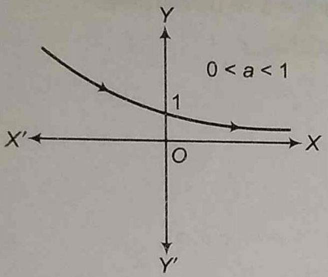
(ii) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a \ (a \geq 0)$

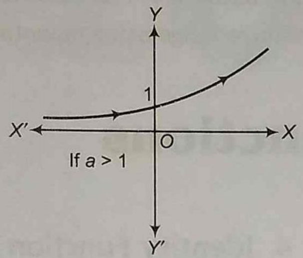
(iii) $|x \pm y| \leq |x| + |y|$

(iv) $|x \pm y| \geq ||x| - |y||$

8. Exponential Function

The function $f(x) = a^x$, $a > 0, a \neq 1, a \in$ constant, is said to be an exponential function. Its Domain $\in R$ and Range $\in (0, \infty)$.

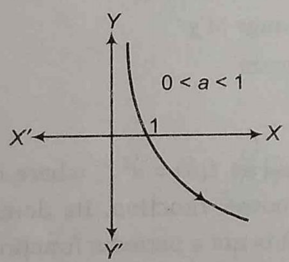


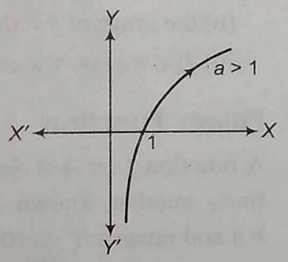


It is a one-one into function.

9. Logarithmic Function

The function $f(x) = \log_a x$, $(x, a > 0)$ and $a \neq 1$ is a logarithmic function. Its Domain $\in (0, \infty)$ and Range $\in R$. It is a one-one into function



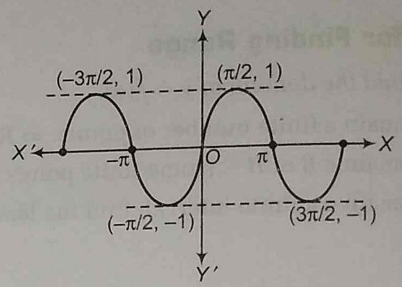


> **Intervals of a Function**
>
> - The set of real numbers x, such that $a \leq x \leq b$ is called a closed interval and denoted by $[a, b]$ i.e., $\{x : x \in R, a \leq x \leq b\}$.
> - The set of real number x, such that $a < x < b$ is called open interval and is denoted by (a, b) i.e., $\{x : x \in R, a < x < b\}$
> - Intervals $[a, b) = \{x : x \in R, a \leq x < b\}$ and $(a, b] = \{x : x \in R, a < x \leq b\}$ are called semi-open and semi-closed intervals.

10. Trigonometric Functions

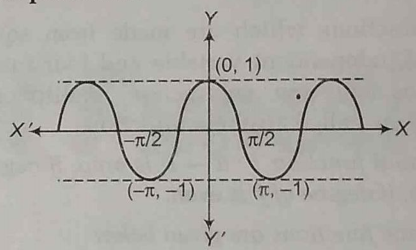
(i) Sine Function

Function $f(x) = \sin x$, the domain of sine function is R and the range is $[-1, 1]$. It is a periodic function with **period** 2π.



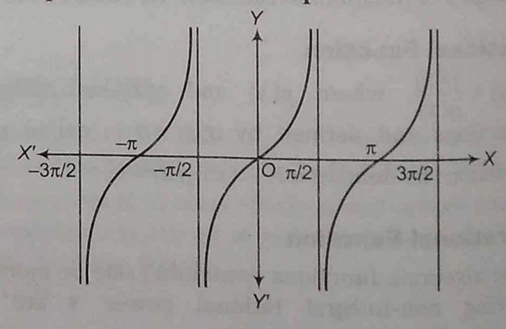
(ii) Cosine Function

Function $f(x) = \cos x$, the domain of cosine function is R and the range is $[-1, 1]$. It is a periodic function with **period** 2π.



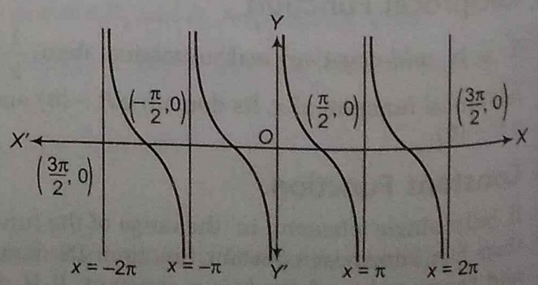
(iii) Tangent Function

Function $f(x) = \tan x$, the domain of tangent function is $R - \left\{\frac{(2n+1)\pi}{2}, n \in I\right\}$ and range is R. It is a periodic function with **period** π.



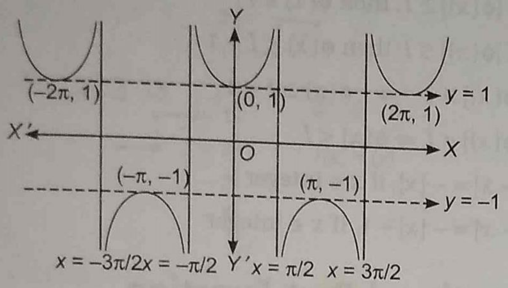
(iv) Cotangent Function

Function $f(x) = \cot x$, the domain $\in R - \{n\pi \mid n \in I\}$ and range $\in R$. It is a periodic function with **period** π.



(v) Secant Function

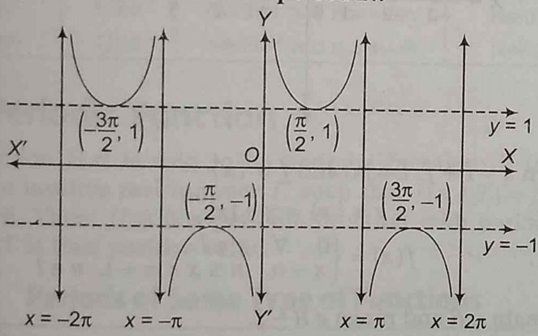
Function $f(x) = \sec x$ the domain $\in R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$.



Range $\in R - (-1, 1)$. It is a periodic function with **period 2π**.

(vi) Cosecant Function

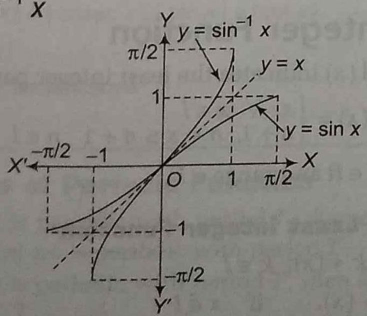
Function $f(x) = \operatorname{cosec} x$, the domain $\in R - \{n\pi : n \in I\}$ and range $\in R - (-1, 1)$. It is a periodic function with **period 2π**



- *A function is said to be an explicit function, if it is expressed in the form $y = f(x)$.*
- *A function is said to be an implicit function, if it is expressed in the form $f(x, y) = C$, where C is constant. e.g., $\sin(x + y) - \cos(x + y) = 2$*

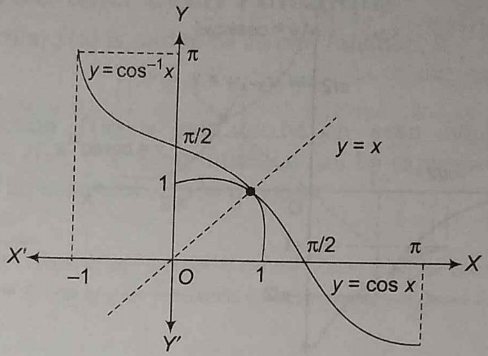
11. Inverse Trigonometric Function

(i) $y = \sin^{-1} x$



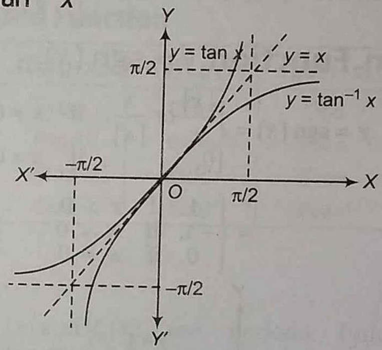
Here, domain $\in [-1, 1]$ and range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(ii) $y = \cos^{-1} x$



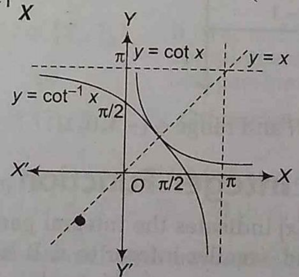
Here, domain $\in [-1, 1]$ and range $\in [0, \pi]$

(iii) $y = \tan^{-1} x$



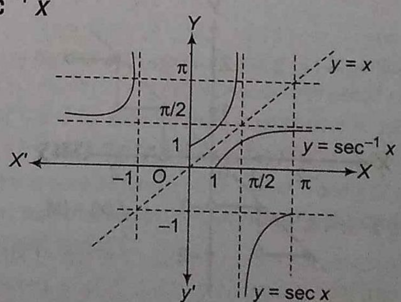
Here, domain $\in R$ and range $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $y = \cot^{-1} x$



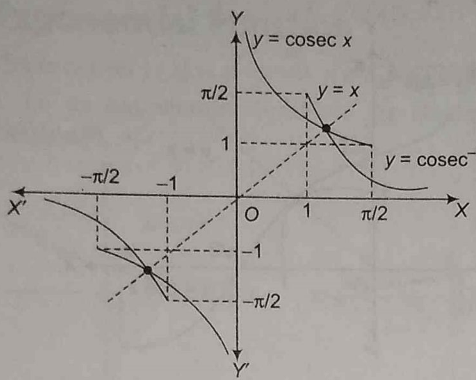
Here, domain $\in R$ and range $\in (0, \pi)$

(v) $y = \sec^{-1} x$



Here, domain $\in R - (-1, 1)$ and range $\in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(vi) $y = \text{cosec}^{-1} x$

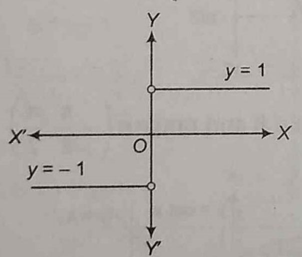


Here, domain $\in R - (-1, 1)$ and range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

12. Signum Function $y = \text{sgn}(x)$

$$y = \text{sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

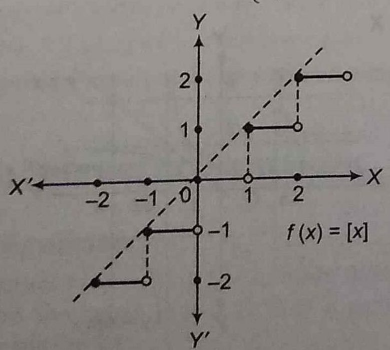
$$= \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$



Its domain $\in R$ and range $\in \{-1, 0, 1\}$.

13. Greatest Integer Function

The symbol $[x]$ indicates the integral part of x which is nearest and smaller integer to x. It is also known as floor of x. $f(x) = [x] = \begin{cases} x \ \forall \ x \in I \\ n, \ n \le x < n+1, \ n \in I \end{cases}$



Its domain $\in R$ and range $\in I$.

Properties of Greatest Integer Function

(i) $[x + I] = [x] + I$, where I is an integer.

(ii) $[x + y] \ge [x] + [y]$

(iii) If $[\phi(x)] \ge I$, then $\phi(x) \ge I$

(iv) If $[\phi(x)] \le I$, then $\phi(x) < I + 1$

(v) $[\phi(x)] > I \Rightarrow \phi(x) \ge I + 1$

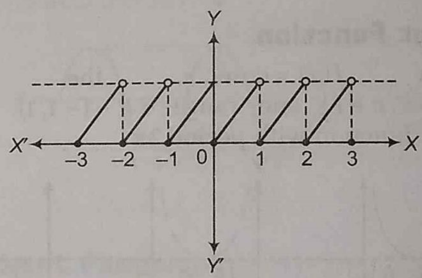
(vi) $[\phi(x)] < I \Rightarrow \phi(x) < I$

(vii) $[-x] = -[x]$, if $x \in$ integer

(viii) $[-x] = -[x] - 1$, if $x \notin$ integer

14. Fractional Part Function

The symbol $\{x\}$. t indicates fractional part of x.



In $x = I + f$, $I = [x]$ and $f = \{x\}$

$\therefore \quad y = \{x\} = x - [x]$

$\therefore \quad f(x) = \begin{cases} 0, & \forall \ x \in I \\ x - n, & n \le x < n+1, \ n \in I \end{cases}$

Its domain $\in R$ and range $\in R - I$.

Properties of Fractional Part Function

(i) $\{x\} = \begin{cases} x, & \text{if } x \in [0, 1) \\ 0, & \text{if } x \in I \end{cases}$

(ii) $\{-x\} = 1 - \{x\}$, if $x \notin I$

(iii) $\{x\} + \{-x\} = \begin{cases} 0, & \text{if } x \in I \\ 1, & \text{if } x \notin I \end{cases}$

15. Least Integer Function

The symbol (x) indicates the least integer part of x,

$$f(x) = (x) = \begin{cases} x, \ \forall \ x \in I \\ n+1, \ n < x \le n+1 \quad n \in I \end{cases}$$

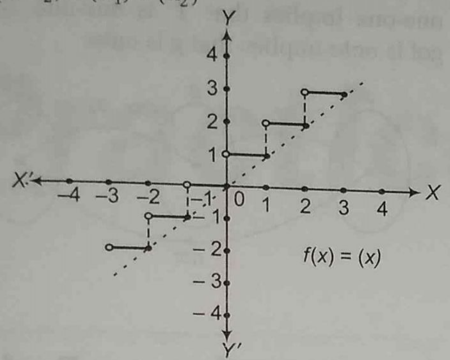
Its domain $\in R$ and range $\in I$.

Properties of Least Integer Function

(i) $(x + k) = k + (x)$, $k \in I$

(ii) $(-x) = \begin{cases} -(x), & \text{if } x \in I \\ -(x) + 1, & \text{if } x \notin I \end{cases}$

(iii) $(x_1 + x_2) \le (x_1) + (x_2)$



16. Odd and Even Functions

A function $f(x)$ is said to be an odd function, if

$$f(-x) = -f(x), \forall x.$$

A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$, $\forall x$. Every function can be expressed as the sum of an even and an odd function

- *Graph of odd function is symmetrical in opposite quadrants.*
- *Graph of even function is always symmetrical about Y-axis.*

Different Conditions for Even and Odd Function

f(x)	g(x)	f(x) + g(x)	f(x) − g(x)	f(x) g(x)	f(x) / g(x)	(gof) (x)	(fog) (x)
Odd	Odd	Odd	Odd	Even	Even	Odd	Odd
Even	Even	Even	Even	Even	Even	Even	Even
Odd	Even	Neither odd nor even	Neither odd nor even	Odd	Odd	Even	Even
Even	Odd	Neither odd nor even	Neither odd nor even	Odd	Odd	Even	Even

17. Periodic Function

A function $f(x)$ is said to be periodic function, if there exists a positive real number T, such that $f(x+T) = f(x)$, $\forall\ x \in R$. Then, $f(x)$ is a periodic function with period T, where T is least positive value.

Periods of Some Type of Functions

S. No.	Function	Periods
1.	$\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$, $(\sin x)^{2n+1}$, $(\cos x)^{2n+1}$, $(\sec x)^{2n+1}$, $(\operatorname{cosec} x)^{2n+1}$	2π
2.	$\tan x$, $\cot x$, $\tan^n x$, $\cot^n x$, $(\sin x)^{2n}$, $(\cos x)^{2n}$, $(\sec x)^{2n}$, $(\operatorname{cosec} x)^{2n}$, $\lvert \sin x \rvert$, $\lvert \cos x \rvert$, $\lvert \tan x \rvert$, $\lvert \cot x \rvert$, $\lvert \sec x \rvert$, $\lvert \operatorname{cosec} x \rvert$	π
3.	$\lvert \sin x + \cos x \rvert$, $\sin^4 x + \cos^4 x$, $\lvert \sec x \rvert + \lvert \operatorname{cosec} x \rvert$, $\lvert \tan x \rvert + \lvert \cot x \rvert$	$\frac{\pi}{2}$
4.	$x - [x]$	1
5.	Algebraic functions	Period does not exist

Properties of Periodic Function

(i) If $f(x)$ is periodic with period T, then $cf(x)$, $f(x+c)$ and $f(x) \pm c$ is periodic with period T.

(ii) If $f(x)$ is periodic with period T, then $kf(cx+d)$ has period $\frac{T}{|c|}$.

(iii) If $f_1(x)$ and $f_2(x)$ are periodic functions with periods T_1 and T_2 respectively, then we have $h(x) = f_1(x) + f_2(x)$ has period as

$$= \begin{cases} \frac{1}{2}\text{LCM of } \{T_1, T_2\}, & \text{if } f_1(x) \text{ and } f_2(x) \text{ are complementary pairwise comparable even functions.} \\ \quad \text{or} & \\ \text{LCM of } \{T_1, T_2\}, & \text{Otherwise} \end{cases}$$

(a) $\text{LCM of } \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{\text{LCM of } (a, c, e)}{\text{HCF of } (b, d, f)}$

(b) LCM of rational with rational is possible. LCM of irrational with irrational is possible. But LCM of rational and irrational is not possible.

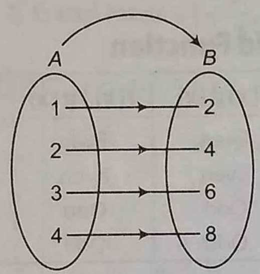
- *If $f(x)$ is periodic with period T_1 and $g(x)$ is periodic with period T_2, then $f(x) + g(x)$ is periodic with period equal to LCM of T_1 and T_2, provided there is no positive K, such that $f(k+x) = g(x)$ and $g(k+x) = f(x)$.*
- *If $f(x)$ is a periodic function with period T and $g(x)$ is any function, such that domain of C domain of g, then gof is also periodic with period T.*

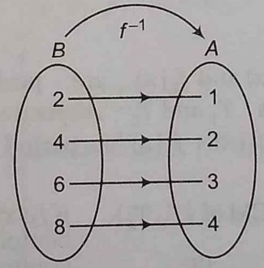
Testing the Periodicity of a Function

1. Put $f(T+x) = f(x)$ and solve this equation to find the positive values of T independent of x.
2. If no positive value of T independent of x is obtained, then $f(x)$ is a non-periodic function.
3. If positive values of T independent of x are obtained, then $f(x)$ is a periodic function and the least positive value of T is the period of the function $f(x)$.

18. Inverse Function

Let $f : A \to B$ be a one-one and onto function, then there exists a unique function.





$g : B \to A$, such that $f(x) = y \Leftrightarrow g(y) = x, \forall\, x \in A$ and $y \in B$. Then, g is said to be inverse of f.

Thus, $g \equiv f^{-1} : B \to A$

$= \{(f(x), x) \mid (x, f(x)) \in f\}$

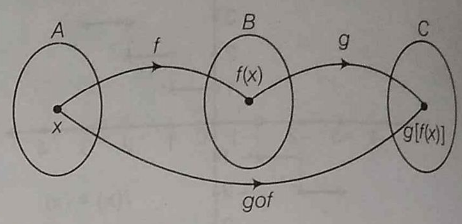
Inverse of an even function is not defined and an even function cannot be strictly monotonic.

Composition of Function

Let $f : A \to B$ and $g : B \to C$ be any two functions, then the composition of f and g denoted by the function *gof*, defined by *gof* $: A \to C : gof(x) = g[f(x)], \forall\, x \in A$.

Also, *gof* is defined only when range $(f) \in$ domain (g). Clearly, domain (gof) = domain (f)

It can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.



Operations on Real Functions

Let $f : X \to R$ and $g : X \longrightarrow R$ be two real functions, then

- **Sum** The sum of the functions f and g is defined as
  $f + g : X \longrightarrow R$ such that
  $(f+g)(x) = f(x) + g(x)$.
- **Product** The product of the functions f and g is defined as
  $fg : X \longrightarrow R$,
  such that $(fg)(x) = f(x)\, g(x)$

Clearly, $f+g$ and fg are defined only, if f and g have the same domain. In case, the domain of f and g are different. Then, Domain of $f+g$ or fg = Domain of $f \cap$ Domain of g.

- **Multiplication by a Number**
  Let $f : X \longrightarrow R$ be a function and let c be a real number.
  Then, we define
  $cf : X \longrightarrow R$, such that
  $(cf)(x) = cf(x), \forall\, x \in X$.
- **Composition** (Function of Function)
  Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions.
  We define $gof : A \longrightarrow C$
  such that $gof(c) = g\{f(x)\}, \forall\, x \in A$
  Alternate *There exists $y \in B$, such that if $f(x) = y$ and $g(y) = z$, then $gof(x) = z$.*

Practice Zone DAY 10

1. Let $f: R \to R$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and –3, respectively are **[NCERT Exemplar]**
   (a) $\phi, \{4, -4\}$
   (b) $\{3, -3\}, \phi$
   (c) $\{4, -4\}, \phi$
   (d) $\{4, -4\}, \{2, -2\}$

2. If $e^x = y + \sqrt{1+y^2}$, then y is equal to
   (a) $\frac{e^x + e^{-x}}{2}$
   (b) $\frac{e^x - e^{-x}}{2}$
   (c) $e^x + e^{-x}$
   (d) $e^x - e^{-x}$

3. The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30, \forall$ real $x \neq 1$. The value of $f(7)$ is
   (a) 8
   (b) 4
   (c) – 8
   (d) 11

4. If $f: R \to R$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$, then $(fofo)(x)$ is **[NCERT Exemplar]**
   (a) $\frac{x}{\sqrt{1+x^2}}$
   (b) $\frac{x}{\sqrt{1+3x^2}}$
   (c) x,
   (d) 1

5. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$ is
   (a) $\frac{r}{\{-1, -2\}}$
   (b) $(-2, \infty)$
   (c) $\frac{R}{\{-1, -2, -3\}}$
   (d) $\frac{(-3, \infty)}{\{-1, -2\}}$

6. If $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then, $f(\theta)$
   (a) ≥ 0 only, when $\theta \geq 0$
   (b) $\leq 0, \forall$ real θ
   (c) $\geq 0, \forall$ real θ
   (d) ≤ 0 only, when $\theta \leq 0$

7. If $f: R \to R$ satisfies $f(x+y) = f(x) + f(y), \ \forall \ x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is
   (a) $\frac{7n}{2}$
   (b) $\frac{7(n+1)}{2}$
   (c) $7n(n+1)$
   (d) $\frac{7n(n+1)}{2}$

8. The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is given by
   (a) $\log_e\left(\frac{x-2}{x-1}\right)^{1/2}$
   (b) $\log_e\left(\frac{x-1}{3-x}\right)^{1/2}$
   (c) $\log_e\left(\frac{x}{2-x}\right)^{1/2}$
   (d) $\log_e\left(\frac{x-1}{x+1}\right)^{-2}$

9. The function $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is
   (a) an odd function
   (b) an even function
   (c) a periodic function
   (d) None of these

10. The period of the function $f(x) = \sin^3 x + \cos^3 x$ is
    (a) 2π
    (b) π
    (c) $\frac{2\pi}{3}$
    (d) None of these

11. Domain of $f(x) = \sqrt{\frac{x-1}{x-2\{x\}}}$, where $\{\cdot\}$ denotes the fractional part of x, is
    (a) $(-\infty, 0) \cup (0, 2]$
    (b) $[1, 2)$
    (c) $(-\infty, \infty) \sim [0, 2)$
    (d) $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

12. If $[x^2] + x - a = 0$ has a solution, where $a \in N$ and $a \leq 20$, then total number of different values of 'a' can be
    (a) 2
    (b) 3
    (c) 4
    (d) 6

13. Range of $f(x) = [|\sin x| + |\cos x|]$, where $[\cdot]$ denotes the greatest integer function, is
    (a) $\{0\}$
    (b) $\{0, 1\}$
    (c) $\{1\}$
    (d) None of these

14. Let $f: (2, 3) \to (0, 1)$ be defined by $f(x) = x - [x]$, then $f^{-1}(x)$ is equal to
    (a) $x - 2$
    (b) $x + 1$
    (c) $x - 1$
    (d) $x + 2$

15. Total number of solutions of $[x]^2 = x + 2\{x\}$, where [·] and {·} denotes the greatest integer function and fractional part respectively, is equal to

(a) 2 (b) 4
(c) 6 (d) None of these

16. If $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$ for $x \in R$, then $f(2002)$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

17. For a real number x, $[x]$ denotes the integral part of x. The value of $\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{99}{100}\right]$ is

(a) 49 (b) 50 (c) 48 (d) 51

Directions (Q. Nos. 18 and 19) *Let $f(x) = x^2 - 2x - 3$, $g(x) = f(|x|)$, $h(x) = |g(x)|$ are three functions.*

18. The interval in which $f(x) > 0$ is

(a) $R - [-1, 3]$ (b) $[-1, 3]$
(c) $(-\infty, \infty)$ (d) None of these

19. Number of solutions of $g(x) = 0$ is/are

(a) 2 (b) 3 (c) 4 (d) 0

Directions (Q. Nos. 20 and 21) *Let $f : X \to Y$ be a bijection. We define $g : Y \to X$ such that $f(x) = y \Leftrightarrow g(y) = x$, $x \in X, y \in Y$, then g is called the inverse of f and is denoted by f^{-1}.*

20. If the function $f : (1, \infty) \to (1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is equal to

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1 + \sqrt{4\log_2 x + 1})$
(c) $\frac{1}{2}(1 - \sqrt{4\log_2 x + 1})$ (d) not defined

21. If $f : R \to (-\infty, 1)$ such that $f(x) = 1 - 2^{-x}$, then $f^{-1}(x)$ is

(a) $1 + \log_2(-x)$ (b) $1 - \log_2(-x)$
(c) $\log_2(1 - x)$ (d) $-\log_2(1 - x)$

Directions (Q. Nos. 22 to 25) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

22. Statement I $f(x) = |x - 2| + |x - 3| + |x - 5|$ is an odd function for all values of x lie between 3 and 5.

Statement II For odd function $f(-x) = -f(x)$

23. Statement I The period of $f(x) = 2\cos\frac{1}{3}(x - \pi) + 4\sin\frac{1}{3}(x - \pi)$ is 3π.

Statement II If T is the period of $f(x)$, then the period of $f(ax + b)$ is $\frac{T}{|a|}$.

24. If domain of $f(x)$ and $g(x)$ are D_1 and D_2 respectively, then domain of $f(x) + g(x)$ is $D_1 \cap D_2$, then

Statement I The domain of the function $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$ is $[-1, 1]$.

Statement II $\sin^{-1} x$ and $\cos^{-1} x$ is defined in $|x| \le 1$ and $\tan^{-1} x$ is defined for all x.

25. If the range of $f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain, then

Statement I The range of $\log\left(\frac{1}{1 + x^2}\right)$ is $(-\infty, \infty)$.

Statement II When $0 < x \le 1$, $\log x \in (-\infty, 0]$.

26. If $f(x) = \dfrac{x^4 + x^2 + 1}{x^2 - x + 1}$, then find the value of $f(\omega^n)$, where 'ω' is the non-real root of the equation $\omega^3 = 1$ and n is a multiple of 3

(a) 0 (b) 3
(c) 9 (d) Does not exist

27. If $f^2(x) \cdot f\left(\frac{1 - x}{1 + x}\right) = x^3$, [where, $x \ne -1, 1$ and $f(x) \ne 0$], then find $|[f(-2)]|$ (where [.] is the greatest integer function)

(a) $\frac{1}{x}$ (b) $1 - x$ (c) 1 (d) 2

28. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g\{f(x)\}$ is invertible in the domain

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

29. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued x, is

(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

30. Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is reflection of the graph of $f(x)$ w.r.t. the line $y = x$, then $g(x)$ is equal to

(a) $-\sqrt{x} - 1, x \ge 0$ (b) $\frac{1}{(x + 1)^2}, x > -1$
(c) $\sqrt{x + 1}, x \ge -1$ (d) $\sqrt{x} - 1, x \ge 0$

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31. If $f(1) = -1$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$, then the possible value of $f(6)$ lies in the interval **[JEE Main 2013]**

(a) $[15, 19)$ (b) $(-\infty, 12)$
(c) $[12, 15)$ (d) $[19, \infty)$

32. If $A = \sin^2 x + \cos^4 x$, then for all real x **[AIEEE 2011]**

(a) $\frac{13}{16} \leq A \leq 1$ (b) $1 \leq A \leq 2$
(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

33. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is **[AIEEE 2011]**

(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty) - (0)$ (d) $(-\infty, \infty)$

34. Let $f(x) = (x + 1)^2 - 1, x \geq -1$ **[AIEEE 2009]**

Statement I The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Statement II f is a bijection.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

35. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$.

Show that f is invertible and its inverse is **[AIEEE 2008]**

(a) $g(y) = (y - 3)/4$
(b) $g(y) = (3y + 4)/3$
(c) $g(y) = 4 + \frac{y + 3}{4}$
(d) $g(y) = \frac{y + 3}{4}$

36. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$ is defined, is **[AIEEE 2007]**

(a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$

37. The domain of definition of the function $f(x) = \sqrt{\log_{10}\left(\frac{5x - x^2}{4}\right)}$ is **[AIEEE 2002]**

(a) $[1, 4]$ (b) $[1, 0]$ (c) $[0, 5]$ (d) $[5, 0]$

Answers

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (d)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (b)	**10.** (a)
11. (d)	**12.** (c)	**13.** (c)	**14.** (d)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (a)	**20.** (b)
21. (d)	**22.** (b)	**23.** (d)	**24.** (a)	**25.** (d)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (a)	**30.** (d)
31. (d)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (a)	**36.** (d)	**37.** (a)			

Hints & Solutions

1. Let $y = x^2 + 1$

$\Rightarrow \quad x = \pm\sqrt{y - 1}$

$\therefore \quad f^{-1}(x) = \pm\sqrt{x - 1}$

$\therefore f^{-1}(17) = \pm\sqrt{17 - 1} = \pm 14$

and $f^{-1}(-3) = \pm\sqrt{-3 - 1}$

$= \pm\sqrt{-4} \in R$

$= \phi$

2. Given, $\quad e^x = y + \sqrt{1 + y^2}$

$\Rightarrow \quad e^x - y = \sqrt{1 + y^2}$

$\Rightarrow e^{2x} + y^2 - 2ye^x = 1 + y^2$

$\Rightarrow \quad e^{2x} - 1 = 2ye^x$

$\Rightarrow \quad 2y = \frac{e^{2x} - 1}{e^x} = e^x - e^{-x}$

$\therefore \quad y = \frac{e^x - e^{-x}}{2}$

3. $3f(x) + 2f\left(\frac{x + 59}{x - 1}\right) = 10x + 30$

For $x = 7$, $3f(7) + 2f(11) = 100$

For $x = 11$, $3f(11) + 2f(7) = 140$

$\therefore \quad \frac{f(7)}{20} = \frac{f(11)}{-220} = \frac{1}{9 - 4}$

$\Rightarrow \quad f(7) = 4$

4. Given that, $f(x) = \frac{x}{\sqrt{1+x}}$

$(fof)x = \{f(f(x))\}$

$$= f\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}} = \frac{x}{\sqrt{1+2x^2}}$$

Now, $(fofof)x = \{f(fof)x\}x$

$$= f\left(\frac{x}{\sqrt{1+2x^2}}\right) = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}$$

$$= \frac{x}{\sqrt{1+3x^2}}$$

5. For numerator, $x + 3 > 0$

$\Rightarrow \quad x > -3 \quad \ldots(i)$

and for denominator $(x+1)(x+2) \neq 0$

$\Rightarrow \quad x \neq -1, -2 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get $\frac{(-3, \infty)}{\{-1, -2\}}$

6. $f(\theta) = (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta$

$= \sin^2\theta\,(4 - 4\sin^2\theta)$

$= 4\sin^2\theta\cos^2\theta = (\sin 2\theta)^2 \geq 0$

7. $\sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \ldots + f(n)$

$= f(1) + 2f(1) + 3f(1) + \ldots + nf(1)[\because f(x+y) = f(x) + f(y)]$

$= (1 + 2 + 3 + \ldots + n)\, f(1)$

$= \frac{n(n+1)}{2} \cdot 7$

$= \frac{7n(n+1)}{2}$

8. Given, $y = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

$\Rightarrow \quad y = \frac{e^{2x} - 1}{e^{2x} + 1} + 2$

$\Rightarrow \quad e^{2x} = \frac{1-y}{y-3} = \frac{y-1}{3-y}$

$\Rightarrow \quad x = \frac{1}{2}\log_e\left(\frac{y-1}{3-y}\right)$

$\Rightarrow \quad f^{-1}(y) = \log_e\left(\frac{y-1}{3-y}\right)^{1/2}$

$\therefore \quad f^{-1}(x) = \log_e\left(\frac{x-1}{3-x}\right)^{1/2}$

9. Given, $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = \frac{x + xe^x}{2(e^x - 1)} + 1$

Now, $f(-x) = \frac{-x - xe^{-x}}{2(e^{-x} - 1)} + 1 = \frac{x + xe^x}{2(e^x - 1)} + 1$

$\therefore \quad f(-x) = f(x)$

10. $f(x) = \left[\frac{3\sin x - \sin 3x}{4} + \frac{3\cos x + \cos 3x}{4}\right]$

$\therefore$ Period of $f(x)$ = LCM of $\{\sin x, \cos x, \sin 3x, \cos 3x\}$

$= \frac{\text{LCM of } \{2\pi, 2\pi\}}{\text{HCF of } \{1, 3\}} = 2\pi$

11. We must have $\frac{x-1}{x-2\{x\}} \geq 0$, there are two cases arise

Case I $x \geq 1$ and $x > 2\{x\}$

$\Rightarrow \quad x \geq 2$

Som the Common part is $x \in [2, \infty)$.

Case II $x \leq 1$ and $x < 2\{x\}$

$\Rightarrow \quad x < 1$ and $x \neq 0$.

Common part is $x \in (-\infty, 0) \cup (0, 1)$.

Finally, $x = 1$ is also a point of the domain.

12. $[x^2] + x - a = 0$

Since, x has to be an integer.

$\Rightarrow \quad a = x^2 + x = x(x+1)$

Thus, a can be 2, 6, 12, 20.

13. $y = |\sin x| + |\cos x|$

$\Rightarrow \quad y^2 = 1 + |\sin 2x|$

$\Rightarrow \quad 1 \leq y^2 \leq 2$

$\Rightarrow \quad y \in [1, \sqrt{2}]$

$\therefore \quad f(x)\,[y] = 1, \forall x \in R$

14. $f : (2, 3) \to (0, 1)$ and $f(x) = x - [x]$

$\therefore \quad f(x) = y = x - 2$

$\Rightarrow \quad x = y + 2$

$\Rightarrow \quad f^{-1}(x) = x + 2$

15. $[x]^2 = x + 2\{x\}$

$\Rightarrow \quad [x]^2 = [x] + 3\{x\}$

$\Rightarrow \quad \{x\} = \frac{[x]^2 - [x]}{3} \Rightarrow 0 \leq \frac{[x]^2 - [x]}{3} < 1$

$\Rightarrow \quad [x] \in \left(\frac{1-\sqrt{13}}{2}, 0\right] \cup \left[1, \frac{1+\sqrt{13}}{2}\right)$

$\Rightarrow \quad [x] = -1, 0, 1, 2 \Rightarrow \{x\} = \frac{2}{3}, 0, 0, \frac{2}{3}$

$\therefore \quad x = -\frac{1}{3}, 0, 1, \frac{8}{3}$

16. Given, $f(x) = \frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$

$\Rightarrow \quad f(x) = \frac{\cos^2 x + \sin^2 x(1 - \cos^2 x)}{\sin^2 x + \cos^2 x(1 - \sin^2 x)}$

$\Rightarrow \quad f(x) = \frac{\sin^2 x + \cos^2 x - \sin^2 x \cdot \cos^2 x}{\sin^2 x + \cos^2 x - \sin^2 x \cdot \cos^2 x}$

$\Rightarrow \quad f(x) = 1$

$\therefore \quad f(2002) = 1$

17. $\because [x]$ denotes the integral part of x.

Hence, after term $\left[\frac{1}{2}+\frac{50}{100}\right]$ each term will be one. Hence, the sum of given series will be 50.

18. Since, $f(x)=x^2-2x-3=(x+1)(x-3)0$

for $f(x)>0$, then $x\in(-\infty,-1)\cup(3,\infty)$

$\therefore \quad f(x)\in R-[-1,3]$

19. Since, $g(x)=f(|x|)$

$\Rightarrow x^2-2|x|-3=0$

$\Rightarrow (|x|+1)(|x|-3)=0$

$\Rightarrow |x|=3, |x|\neq -1$

$\therefore x=\pm 3$

20. Since, $y=2^{x(x-1)}$

$\Rightarrow x(x-1)-\log_2 y=0$

$\Rightarrow x^2-x-\log_2 y=0$

$\therefore x=\dfrac{1\pm\sqrt{1+4\log_2 y}}{2}$

$\Rightarrow x=\dfrac{1+\sqrt{1+4\log_2 y}}{2} \quad (\because x>1)$

$\therefore f^{-1}(x)=\dfrac{1+\sqrt{1+4\log_2 x}}{2}$

21. Let $y=1-2^{-x} \Rightarrow 2^{-x}=1-y$

$\Rightarrow -x=\log_2(1-y)$

$\Rightarrow x=-\log_2(1-y)$

$\therefore f^{-1}(x)=-\log_2(1-x)$

22. Here, $f(x)=\begin{cases}-3x+10, \forall x\le 2\\ -x+6, \forall 2<x\le 3\\ x, \forall 3<x\le 5\\ 3x-10, \quad \forall x>5\end{cases}$

$\therefore f(x)=x, \forall 3<x<5$

$\Rightarrow f(-x)=-x=-f(x)$

23. Period of $2\cos\frac{1}{3}(x-\pi)$ and

$4\sin\frac{1}{3}(x-\pi)$ are $\frac{2\pi}{1/3}, \frac{2\pi}{1/3}$ or $6\pi, 6\pi$

$\therefore$ Period of their sum $=6\pi$

24. Since, $\sin^{-1}x$ is defined in $[-1,1]$,

$\cos^{-1}x$ is defined in $[-1,1]$

and $\tan^{-1}x$ is defined in R.

Hence, $f(x)$ is defined in $[-1,1]$.

25. Range of $\frac{1}{1+x^2}$ is $(0, 1)$ and domain R

$\therefore \log\left(\frac{1}{1+x^2}\right)\in(-\infty,0]$

26. Now, $f(x)=\dfrac{x^4+x^2+1}{x^2-x+1}=\dfrac{(x^2+x+1)(x^2-x+1)}{(x^2-x+1)}$

$\Rightarrow f(x)=x^2+x+1$

Now, $f(\omega^n)=\omega^{2n}+\omega^n+1=3$

$(\because \omega^n=1$, when n is a multiple of 3)

27. $f^2(x)\cdot f\left(\frac{1-x}{1+x}\right)=x^3 \quad \ldots(i)$

Replacing x by $\frac{1-x}{1+x}$, we get

$f^2\left(\frac{1-x}{1+x}\right)f(x)=\left(\frac{1-x}{1+x}\right)^3 \quad \ldots(ii)$

From Eqs. (i) and (ii),

$f^3(x)=x^6\left(\frac{1+x}{1-x}\right)^3 \Rightarrow f(x)=x^2\left(\frac{1+x}{1-x}\right)$

$f(-2)=\frac{4}{3}\Rightarrow [f(-2)]=-2$

$\therefore |[f(-2)]|=2$

28. $g\{f(x)\}=(\sin x+\cos x)^2-1$ is invertible.

$\Rightarrow g\{f(x)\}=\sin 2x$

We know that, $\sin x$ is bijective only when $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Thus, $g\{f(x)\}$ is bijective, if $-\frac{\pi}{2}\le 2x\le\frac{\pi}{2}$.

$\therefore -\frac{\pi}{4}\le x\le\frac{\pi}{4}$

29. For $f(x)$ to be defined, $\sin^{-1}(2x)+\frac{\pi}{6}\ge 0$

$\Rightarrow -\frac{\pi}{6}\le\sin^{-1}(2x)\le\sin\frac{\pi}{2}$

$\Rightarrow \sin\left(-\frac{\pi}{6}\right)\le 2x\le\sin\left(\frac{\pi}{2}\right)$

$\Rightarrow -\frac{1}{4}\le x\le\frac{1}{2}$

$\Rightarrow x\in\left[-\frac{1}{4},\frac{1}{2}\right]$

30. Let $y=(x+1)^2$ for $x\ge -1$

$\Rightarrow \pm\sqrt{y}=x+1$

$\Rightarrow \sqrt{y}=x+1$

$\Rightarrow y\ge 0, x+1\ge 0$

$\Rightarrow x=\sqrt{y}-1$

$\Rightarrow f^{-1}(y)=\sqrt{y}-1$

$\Rightarrow f^{-1}(x)=\sqrt{x}-1, x\ge 0$

31. $\because f'(x)\ge 4.2$

$\Rightarrow f(x)\ge 4.2x$

$\Rightarrow f(6)\ge 4.2\times 6$

$\Rightarrow f(6)\ge 25\in[19,\infty)$

32. $A = \sin^2 x + \cos^4 x$

$\Rightarrow$ $A = 1 - \cos^2 x + \cos^4 x$

$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$

$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$ ...(i)

where, $0 \le \left(\cos^2 x - \frac{1}{2}\right)^2 \le \frac{1}{4}$ ...(ii)

$\therefore$ $\frac{3}{4} \le A \le 1$

33. $y = \frac{1}{\sqrt{|x| - x}}$

For domain, $|x| - x > 0$

$\Rightarrow$ $|x| > x$

i.e., only possible, if $x < 0$.

$\therefore$ $x \in (-\infty, 0)$

34. Given, $f(x) = (x+1)^2 - 1, x \ge -1$

$\Rightarrow$ $f'(x) = 2(x+1) \ge 0$ for $x \ge -1$

$\Rightarrow f(x)$ is one-one.

Since, codomain of the given function is not given, hence it can be considered as R the set of reals and consequently R is not onto.

Hence, f is not bijective. Statement II is false.

Also, $f(x) = (x+1)^2 - 1 \ge -1$ for $x \ge -1$

$\Rightarrow$ $R_f = [-1, \infty)$

Clearly, $f(x) = f^{-1}(x)$ at $x = 0$ and $x = -1$

Hence, Statement I is true.

35. Since, $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$

$\therefore$ $Y = \{7, 8, \dots \infty\}$

Let $y = 4x + 3$

$\Rightarrow$ $x = \frac{y-3}{4}$

Inverse of $f(x)$ is

$g(y) = \frac{y-3}{4}$

36. $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$

Since, 4^{-x^2} is defined for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$\cos^{-1}\left(\frac{x}{2} - 1\right)$ is defined, if $-1 \le \frac{x}{2} - 1 \le 1$

$\Rightarrow$ $0 \le x \le 4$

and $\log(\cos x)$ is defined, if $\cos x > 0$

$\Rightarrow$ $x \in \left[0, \frac{\pi}{2}\right)$

37. For domain of $f(x)$,

$\log_{10}\left(\frac{5x - x^2}{4}\right) \ge 0$

$\Rightarrow$ $\frac{5x - x^2}{4} \ge 1$

$\Rightarrow$ $x^2 - 5x + 4 \le 0$

$\Rightarrow$ $(x-1)(x-4) \le 0$

$\therefore$ $x \in [1, 4]$

Day 11

Limits, Continuity and Differentiability

Day 11

Outlines ...

- Limits
- Fundamental Theorems on Limits
- Methods of Evaluating Limits
- Algebraic Limit
- Continuity
- Differentiability

Limits

Let $y = f(x)$ be a function of x. If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to a, then the unique number, so obtained called the **limit** of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

Left Hand and Right Hand Limits

Consider the values of the functions at the points which are very near to a on the left. If these values tend to definite unique number as x tends to a, then the unique number, so obtained is called the **left hand limit** of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

Similarly, **right hand limit** can be expressed as

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Existence of Limit

$\lim_{x \to a} f(x)$ exists when

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.*, LHL and RHL both exist.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.*, LHL = RHL

Fundamental Theorems on Limits

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ (where, l and m are real numbers), then

(i) $\lim_{x\to a} \{f(x) + g(x)\} = l + m$ (sum rule)

(ii) $\lim_{x\to a} \{f(x) - g(x)\} = l - m$ (difference rule)

(iii) $\lim_{x\to a} \{f(x) \cdot g(x)\} = l \cdot m$ (product rule)

(iv) $\lim_{x\to a} k \cdot f(x) = k \cdot l$ (constant multiple rule)

(v) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (quotient rule)

(vi) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

(vii) $\lim_{x\to a} \log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$

(viii) If $f(x) \le g(x), \forall x$, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(ix) $\lim_{x\to a} [f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(x) $\lim_{x\to a} f\{g(x)\} = f\{\lim_{x\to a} g(x)\} = f(m)$ provided f is continuous at $\lim_{x\to a} g(x) = m$.

Important Results on Limit

1. Trigonometric Limits

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(ii) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(iii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$

(vi) $\lim_{x\to 0} \cos x = 1$

(vii) $\lim_{x\to a} \frac{\sin(x-a)}{x-a} = 1$

(viii) $\lim_{x\to a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a, |a| \le 1$

(x) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a, |a| \le 1$

(xi) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

(xii) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x\to\infty} \frac{\sin 1/x}{1/x} = 1$

- $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} = \frac{m^2}{n^2}$; $\lim_{x\to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \frac{a^2 - b^2}{c^2 - d^2}$
- $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} = \frac{n^2 - m^2}{2}$
- $\lim_{x\to 0} \frac{\sin^p mx}{\sin^p nx} = \left(\frac{m}{n}\right)^p$; $\lim_{x\to 0} \frac{\tan^p mx}{\tan^p nx} = \left(\frac{m}{n}\right)^p$
- $\lim_{x\to 0} (\cos x + a \sin bx)^{\frac{1}{x}} = e^{ab}$
- $\lim_{x\to\infty} \cos\frac{x}{2} \cos\frac{x}{4} \cos\frac{x}{8} \dots \cos\frac{x}{2^n} = \frac{\sin x}{x}$
- $\lim_{x\to\infty} \sin x$ oscillates between -1 to 1.

2. Logarithmic Limits

We use the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty$, where $-1 < x \le 1$ and expansion is true only, if base is e.

(i) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(ii) $\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$

(iii) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e;\ a > 0, \neq 1$

3. Exponential Limits

We use the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$.

(i) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

(iii) $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda;\ \lambda \neq 0$

- $\lim_{x\to a} \frac{x^a - a^x}{x^x - a^a} = \frac{1 - \log a}{1 + \log a}$
- $\lim_{x\to 0} \frac{(1+x)^m - 1}{(1+x)^n - 1} = \frac{m}{n}$
- $\lim_{x\to 0} (1+ax)^{b/x} = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$
- $\lim_{x\to\infty} \left(\frac{x \pm a}{x \pm b}\right)^{x+c} = e^{(a \mp b)}$
- $\lim_{x\to\infty} \frac{x^n}{e^x} = 0, \forall n$
- $\lim_{m\to\infty} \left(\cos\frac{x}{m}\right)^m = 1$

4. Based on the Form 1^∞

To evaluate the exponential form 1^∞, we use following results

(i) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\{1+f(x)\}^{1/g(x)} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

or

(ii) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a}\{f(x)\}^{g(x)} = e^{\lim_{x\to a}\{f(x)-1\}g(x)}$$

Important Results

- $\lim_{x\to 0}(1+x)^{1/x} = e$
- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^\lambda$
- $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^\lambda$
- $\lim_{x\to\infty} a^x = \begin{cases} 0, & 0\le a<1 \\ 1, & a=1 \\ \infty, & a>1 \\ \text{Does not exist}, & a<0 \end{cases}$

Methods of Evaluating Limits

1. Determinate Forms (Limits by Direct Substitution)

To find $\lim_{x\to a} f(x)$, we substitute $x=a$ in the function. If the value comes out to be a definite value, it is the limit. That is $\lim_{x\to a} f(x) = f(a)$ provided it exists.

2. Indeterminate Form

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 0^0, \infty^0, \infty-\infty$ and 1^∞, where zero does not stand for exactly zero but a quantity approaching towards zero.

L' Hospital's Rule

If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

(ii) both are continuous at $x=a$.

(iii) both are differentiable at $x=a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x=a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g(a)\neq 0$.

Above rule is also applicable, if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embeded in L'Hospital's rule.

We can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$.

Algebraic Limit

Let $f(x)$ be an algebraic function and a be a real number. Then, $\lim_{x\to a} f(x)$ is known as an algebraic limit.

$$\lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1}+\dots}{b_0x^n + b_1x^{n-1}+\dots} = \begin{cases} \frac{a_0}{b_0}, & \text{when } m=n \\ 0, & \text{when } m<n \\ \infty, & \text{when } m>n \text{ and } \frac{a_0}{b_0}>0 \\ -\infty, & \text{when } m>n \text{ and } \frac{a_0}{b_0}<0 \end{cases}$$

Newton-Leibnitz's Formula

Let us consider the definite integral

$$I(x) = \int_{\phi(x)}^{\psi(x)} f(t)\,dt$$

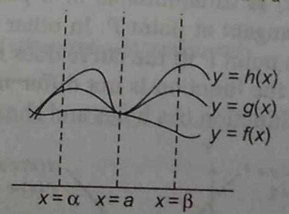
Newton-Leibnitz's formula states that,

$$\frac{d}{dx}\{I(x)\} = f\{\psi(x)\}\cdot\left\{\frac{d}{dx}\psi(x)\right\} - f\{\phi(x)\}\left\{\frac{d}{dx}\phi(x)\right\}$$

Sandwich Theorem

If $f(x)\le g(x)\le h(x)$, $\forall x\in(\alpha,\beta)-\{a\}$

and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$

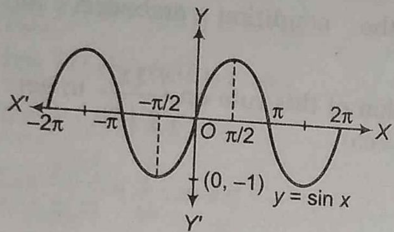


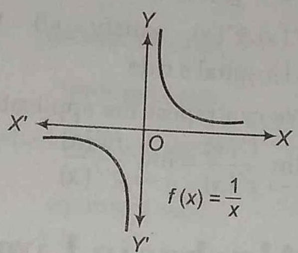
Then, $\lim_{x\to a} g(x) = l$, where $a\in(\alpha,\beta)$

Continuity

If the graph of a function has no break or gap, then it is **continuous** *Fig.(a). A function which is not continuous is called a* **discontinuous** *Fig. (b) function.*

e.g., Graphs of functions sin x, x and e^x etc., are continuous while tan x and sec x etc., are discontinuous.





Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if and only if it satisfies the given condition

(i) $f(a)$ exists ('a' lies in the domain of f)

(ii) $\lim_{x\to a} f(x)$ exist *i.e.*, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

or RHL = LHL

(iii) $\lim_{x\to a} f(x) = f(a)$.

Continuity in an Open Interval

A function $f(x)$ is said to be continuous in an open interval (a, b), if it is continuous at each and every point of (a, b).

Continuity in a Closed Interval

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if (i) it is continuous in (a, b).

(ii) value of the function at b is equal to left hand limit at b *i.e.*, $f(b) = \lim_{x\to b^-} f(x)$.

(iii) value of the function at a is equal to right hand limit at a *i.e.*, $f(a) = \lim_{x\to a^+} f(x)$.

Results on Continuous Functions

Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$, then

(i) $f(x) \pm g(x)$ is continuous at $x = a$.

(ii) $f(x) \cdot g(x)$ is continuous at $x = a$.

(iii) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

(iv) function $f(x)$ is said to everywhere continuous, if it is a continuous on the entire real line *i.e.*, $(-\infty, \infty)$.

(v) integral function of a continuous function is a continuous function.

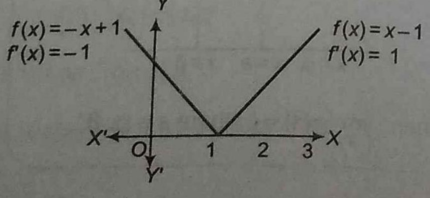
(vi) the composition of function f and g, $(fog)(x)$ is continuous at $x = a$.

(vii) $f(x)$ is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there is atleast one value of x for which $f(x)$ vanishes *i.e.*, $f(a) > 0, f(b) < 0 \Rightarrow \exists\, c \in (a, b)$ such that $f(c) = 0$.

- *If $\lim_{x\to a} f(x)$ does not exist, then we cannot remove this discontinuity. So, this become a non–removable discontinuity or essential discontinuity.*
- *If $f(x)$ is continuous at $x = c$ and $g(x)$ is discontinuous at $x = a$, then*
  (i) $f + g$ and $f - g$ are discontinuous.
  (ii) $f \cdot g$ may be continuous.
- *If f and g are discontinuous at $x = c$, then $f + g$, $f - g$ and $f \cdot g$ may still be continuous.*

Differentiability

The function $f(x)$ is differentiable at a point P iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point *i.e.*, the function is not differentiable at those points on which function has holes and sharp edges.



Let us consider the function $f(x) = |x - 1|$. It is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Differentiability of a Function at a Point

A function F is said to be differentiable at $x = c$, if **left hand** and **right hand** derivatives at c exist and are equal.

Right Hand and Left Hand Derivative

Right hand derivative of $f(x)$ at $x = a$ denoted by $f'(a+0)$ or $f'(a^+)$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Left hand derivative of $f(x)$ at $x = a$ denoted by $f'(a-0)$ or $f'(a^-)$ is $\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$.

A function is said to be **differentiable** (finitely) at $x = a$, if $f'(a+0) = f'(a-0) =$ finite. The common limit is called the **derivative** of $f(x)$ at $x = a$ denoted by $f'(a)$ *i.e.*, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Results on Differentiability

1. Every polynomial, constant and exponential function is differentiable at each $x \in R$.
2. The logarithmic, trigonometric and inverse trigonometric function are differentiable in their domain.
3. The sum, difference, product and quotient of two differentiable functions is differentiable.
4. If f is derivable in the open interval (a, b) and also at the end points a and b, then f is said to be derivable in the closed interval $[a, b]$.
5. A function f is said to be a differentiable function, if it is differentiable at every point of its domain.
6. If a function is differentiable at a point, it is necessarily continuous at that point but the converse is not necessarily true.
7. Absolute functions are always continuous throughout, but not differentiable at their critical points.

> *If a function is differentiable at a point, then it is continuous also at that point.*
>
> *If $f(x)$ and $g(x)$ both are not differentiable at $x = a$, then the product of function $f(x) \cdot g(x)$ can be still differentiable at $x = a$*
>
> *If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the sum function $f(x) + g(x)$ also not differentiable at $x = a$*
>
> *If $f(x)$ and $g(x)$ both are not differentiable at $x = a$, then the sum function may be a differentiable function.*

Continuity and Differentiability of Some Standard Functions

Type of Functions	Curve	Continuity and Differentiability
Identity function	$f(x) = x$	Continuous and differentiable in their domain
Exponential function	$f(x) = a^x, a > 0, a \neq 1$	
Logarithmic function	$f(x) = \log_a x;\ x, a > 0$ and $a \neq 1$	
Root function	$f(x) = \sqrt{x}$	Continuous and differentiable in $(0, \infty)$
Greatest integer function	$f(x) = [x]$	Other than integral value it is continuous and differentiable
Least integer function	$f(x) = (x)$	
Fractional part function	$f(x) = \{x\} = x - [x]$	
Signum function	$f(x) = \frac{\lvert x \rvert}{x} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$	Continuous and differentiable everywhere except at $x = 0$
Sine function	$y = \sin x$	Continuous and differentiable in their domain
Cosine function	$y = \cos x$	
Tangent function	$y = \tan x$	
Cosecant function	$y = \operatorname{cosec} x$	
Secant function	$y = \sec x$	
Cotangent function	$y = \cot x$	
Arc sine function	$y = \sin^{-1} x$	Continuous and differentiable in their domain
Arc cosine function	$y = \cos^{-1} x$	
Arc tangent function	$y = \tan^{-1} x$	
Arc cosecant function	$y = \operatorname{cosec}^{-1} x$	
Arc secant function	$y = \sec^{-1} x$	
Arc cotangent function	$y = \cot^{-1} x$	

Practice Zone DAY 11

1. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$

(c) log2 (d) e^4

2. If $f(x)=3x^{10}-7x^8+5x^6-21x^3+3x^2-7$, then the value of $\lim_{h\to 0}\frac{f(1-h)-f(1)}{h^3+3h}$ is

(a) $\frac{53}{3}$ (b) $\frac{22}{3}$

(c) 13 (d) $\frac{22}{13}$

3. If $f'(2)=6$ and $f'(1)=4$, then $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$ is equal to

(a) 3 (b) $-3/2$

(c) $3/2$ (d) Does not exist

4. If $f(x)=\begin{vmatrix}\sin x & \cos x & \tan x\\ x^3 & x^2 & x\\ 2x & 1 & 1\end{vmatrix}$, then $\lim_{x\to 0}\frac{f(x)}{x^2}$ is equal to

(a) 3 (b) -1

(c) 0 (d) 1

5. If $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$, where $0<a<\frac{1}{2}$, is equal to

(a) $\frac{3}{2(1+a^2)}$ (b) $\frac{3}{2(1+x^2)}$

(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

6. If $f(x)=\begin{cases}e^x, & x\le 0\\ |1-x|, & x>0\end{cases}$, then

(a) $f(x)$ is differentiable at $x=0$

(b) $f(x)$ is continuous at $x=0, 1$

(c) $f(x)$ is differentiable at $x=1$

(d) None of the above

7. The function defined by $f(x)=\begin{cases}|x-3|, & x\ge 1\\ \frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4}, & x<1\end{cases}$ is

(a) continuous at $x=1$ (b) continuous at $x=3$

(c) differentiable at $x=1$ (d) All of these

8. $\lim_{n\to\infty}\sin[\pi\sqrt{n^2+1}]$ is equal to

(a) ∞ (b) 0

(c) Does not exist (d) None of these

9. $f(x)=\begin{cases}|x|\cos\left(\frac{1}{x}\right), & x\ne 0\\ 0, & x=0\end{cases}$ is **[NCERT Exemplar]**

(a) discontinuous at $x=0$ (b) continuous at $x=0$

(c) Does not exist (d) None of these

10. If f is strictly increasing function, then $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

11. For what value of k, the function $f(x)=\begin{cases}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \frac{2x+1}{x-2}, & \text{if } 0\le x\le 1\end{cases}$ is continuous at $x=0$? **[NCERT]**

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{3}{2}$ (d) $-\frac{1}{2}$

12. If $x>0$ and g is a bounded function, then $\lim_{n\to\infty}\frac{f(x)\cdot e^{nx}+g(x)}{e^{nx}+1}$ is equal to

(a) 0 (b) $f(x)$

(c) $g(x)$ (d) None of these

13. The value of $\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$ is

(a) $e^{-1/3}$ (b) $e^{-2/3}$

(c) e^{-1} (d) e^{-2}

14. $\lim_{x\to 0} |x|^{[\cos x]}$, where [ ] is the greatest integer function, is

(a) 1 (b) 0
(c) Does not exist (d) None of these

15. If $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0\end{cases}$, then

(a) $f(0+0)=1$ (b) $f(0-0)=1$
(c) $f(x)$ is continuous at $x=0$ (d) None of these

16. If $\lim_{x\to 0}\frac{\log(a+x)-\log a}{x}+k\lim_{x\to 0}\frac{\log x-1}{x-e}=1$, then

(a) $k=e\left(1-\frac{1}{a}\right)$ (b) $k=e(1+a)$
(c) $k=e(2-a)$ (d) Equality is not possible

17. If $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$ and $g(x)=\sin x+\cos x$, then the points of discontinuity of $f\{g(x)\}$ in $(0,2\pi)$ is

(a) $\left\{\frac{\pi}{2},\frac{3\pi}{4}\right\}$ (b) $\left\{\frac{3\pi}{4},\frac{7\pi}{4}\right\}$ (c) $\left\{\frac{2\pi}{3},\frac{5\pi}{3}\right\}$ (d) $\left\{\frac{5\pi}{4},\frac{7\pi}{3}\right\}$

18. If $f:R\to R$ be a differentiable function having $f(2)=6$ and $f'(2)=\frac{1}{48}$. Then, $\lim_{x\to 2}\frac{\int_6^{f(x)}4t^3dt}{x-2}$ is equal to

(a) 12 (b) 18 (c) 24 (d) 36

19. The value of $\lim_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)$ is

(a) 1 (b) $\frac{\sin x}{x}$
(c) $\frac{x}{\sin x}$ (d) None of these

20. The domain of the derivative of the function $f(x)=\begin{cases}\tan^{-1}x, & |x|\le 1\\ \frac{1}{2}(|x|-1), & |x|>1\end{cases}$ is

(a) $R-\{-1\}$ (b) R
(c) $R-\{-3\}$ (d) None of these

21. $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$ is [NCERT]

(a) continous at $x=-3$ and discontinuous at $x=3$
(b) continuous at $x=-3, 3$
(c) discontinuous at $x=-3, 3$
(d) continuous at $x=3$ and discontinuous at $x=-3$

22. The value of $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$ (where $a, b, c>0$) is

(a) $(abc)^3$ (b) abc
(c) $(abc)^{1/3}$ (d) None of these

23. The value of the constant α and β such that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-\alpha x-\beta\right)=0$ are respectively

(a) (1, 1) (b) (−1, 1)
(c) (1, −1) (d) (0, 1)

24. If $f(x)=[\sin x]+[\cos x]$, $x\in[0,2\pi]$, where [ ] denotes the greatest integer function. Then, the total number of points, where $f(x)$ is non-differentiable, is

(a) 2 (b) 3
(c) 5 (d) 4

Directions (Q. Nos. 25 and 26) *If $f(x+y)=f(x)+f(y)$ for all $x,y\in R$ and $f(1)=1$ and $g(0)=\lim_{x\to 0}\frac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2 f(\sin x)}$, then*

25. the value of $f(x)$ is

(a) x (b) x^2
(c) $3x$ (d) None of these

26. the value of $g(x)$ is

(a) $\log_e 2$ (b) $\frac{1}{2}\log_e 2$
(c) $2\log_e 2$ (d) $\log_e\left(\frac{1}{2}\right)$

Directions (Q. Nos. 27 and 28)

Let $f(x)=\begin{cases}x+a, & x<0\\ |x-1|, & x\ge 0\end{cases}$

and $g(x)=\begin{cases}x+1, & x<0\\ (x-1)^2+b, & x\ge 0\end{cases}$

where, a and b are non-negative real numbers.

27. The value of a, if $(gof)x$ is continuous for all real x, is

(a) −1 (b) 0
(c) 1 (d) 2

28. The value of b, if $gof(x)$ is continuous for all real x, is

(a) −1 (b) 0
(c) 1 (d) 2

Directions (Q. Nos. 29 to 32) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

29. **Statement I** The function $f(x) = (3x-1)|4x^2 - 12x + 5| \cos \pi x$ is differentiable at $x = \frac{1}{2}$ and $\frac{5}{2}$.

Statement II $\cos(2n+1)\frac{\pi}{2} = 0, \forall n \in I$.

30. **Statement I** $\lim_{x \to 0} \frac{\sqrt{1-\cos 2x}}{x}$ does not exist.

Statement II $|\sin x| = \begin{cases} \sin x, & 0 < x < \pi/2 \\ -\sin x, & -\pi/2 < x < 0 \end{cases}$

31. Consider the function $f(x) = \frac{1}{\{x\}}$, where { } denotes the fractional part function.

Statement I $f(x)$ is discontinuous for integral values of x.

Statement II For integral values of $x, f(x)$ is not defined.

32. **Statement I** $f(x) = |\log x|$ is differentiable at $x = 1$.

Statement II Both $\log x$ and $-\log x$ are differentiable at $x = 1$.

33. The limit of the following is

$$\lim_{x \to 3} \frac{\sqrt{1-\cos(x^2 - 10x + 21)}}{(x-3)}$$

(a) $-(2)^{3/2}$ (b) $(2)^{1/2}$ (c) $(2)^{-3/2}$ (d) 3

34. $\lim_{x \to \frac{\pi}{4}} \frac{\int_0^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ is equal to

(a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$ (c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4f(2)$

35. If f be twice differentiable function satisfying $f(1) = 1, f(2) = 4, f(3) = 9$, then

(a) $f''(x) = 2, \forall x \in (R)$
(b) $f'(x) = 5, f(x)$ for some $x \in (1, 3)$
(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$
(d) None of the above

36. If $\lim_{x \to 0} \frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$
(c) n (d) $n + \frac{1}{n}$

37. If $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then, $\lim_{x \to 0} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ is equal to

(a) 1 (b) $e^{1/2}$
(c) e^2 (d) e^3

38. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k, k$ an integer, is

(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

39. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then, g is

(a) onto, if f is onto
(b) one-one, if f is one-one
(c) continuous, if f is continuous
(d) differentiable, if f is differentiable

AIEEE & JEE Main Archive

40. The value of $\lim_{x \to 0} \frac{1}{x}\left[\tan^{-1}\left(\frac{x+1}{2x+1}\right) - \frac{\pi}{4}\right]$ is **[JEE Main 2013]**

(a) 1 b) $-\frac{1}{2}$ (c) 2 (d) 0

41. Let $f(x) = -1 + |x-2|$ and $g(x) = 1 - |x|$, then the set of all points where fog is discontinuous is **[JEE Main 2013]**

(a) {0, 2} (b) {0, 1, 2}
(c) {0} (d) an empty set

42. Let f be a composite function of x defined by $f(u) = \frac{1}{u^2 + u - 2}, u(x) = \frac{1}{x-1}$. Then, the number of points x where f is discontinuous is **[JEE Main 2013]**

(a) 4 (b) 3 (c) 2 (d) 1

43. Consider the function

$f(x) = [x] + |1-x|, -1 \le x \le 3$ where, $[x]$ is the greatest integer function.

Statement I f is not continuous at $x = 0, 1, 2$ and 3.

Statement II $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \\ 1+x, & 1 \le x < 2 \\ 2+x, & 2 \le x \le 3 \end{cases}$ **[JEE Main 2013]**

(a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

44. $\lim_{x\to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to **[JEE Main 2013]**

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 1 (d) 2

45. Consider the function $f(x) = |x-2| + |x-5|, x \in R$.

Statement I $f'(4) = 0$

Statement II f is continuous in $[2, 5]$ and differentiable in $(2, 5)$ and $f(2) = f(5)$. **[AIEEE 2012]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

46. If $f: R \to R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is **[AIEEE 2012]**

(a) continuous for every real x

(b) discontinuous only at $x = 0$

(c) discontinuous only at non-zero integral values of x

(d) continuous only at $x = 0$

47. $\lim_{x\to 2}\left(\frac{\sqrt{1-\{\cos 2(x-2)\}}}{x-2}\right)$ is equal to **[AIEEE 2011]**

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) Does not exist

48. The values of p and q for which the function

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$$

is continuous for all x in R, are **[AIEEE 2011]**

(a) $p = \frac{5}{2}, q = \frac{1}{2}$ (b) $p = -\frac{3}{2}, q = \frac{1}{2}$

(c) $p = \frac{1}{2}, q = \frac{3}{2}$ (d) $p = \frac{1}{2}, q = -\frac{3}{2}$

49. If $f: (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f\{2f(x)+2\}]^2$. Then, $g'(0)$ is equal to **[AIEEE 2010]**

(a) 4 (b) -4

(c) 0 (d) -2

50. If $f: R \to R$ be a positive increasing function with $\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$. Then, $\lim_{x\to\infty} \frac{f(2x)}{f(x)}$ is equal to **[AIEEE 2010]**

(a) 1 (b) 2/3

(c) 3/2 (d) 3

51. Let $f: R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$.

Statement I $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement II $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in R$. **[AIEEE 2010]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

52. Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1}, & \text{if } x \ne 1 \\ 0, & \text{if } x = 1 \end{cases}$

Then, which one of the following is true? **[AIEEE 2008]**

(a) f is differentiable at $x = 1$ but not at $x = 0$

(b) f is neither differentiable at $x = 0$ nor at $x = 1$

(c) f is differentiable at $x = 0$ and $x = 1$

(d) f is differentiable at $x = 0$ but not at $x = 1$

53. If $f: R \to R$ be function defined by $f(x) = \min\{x+1, |x|+1\}$. Then, which of the following is true? **[AIEEE 2007]**

(a) $f(x) \ge 1$ for all $x \in R$

(b) $f(x)$ is not differentiable at $x = 1$

(c) $f(x)$ is differentiable everywhere

(d) $f(x)$ is not differentiable

54. The set of points, where $f(x) = \frac{x}{1+|x|}$ is differentiable, is **[AIEEE 2006]**

(a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$

(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

55. $\lim_{n\to\infty}\left(\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \ldots + \frac{n}{n^2}\sec^2 1\right)$ is equal to **[AIEEE 2005]**

(a) $\frac{1}{2}\tan 1$ (b) $\tan 1$

(c) $\frac{1}{2}\operatorname{cosec} 1$ (d) $\frac{1}{2}\sec 1$

56. Let $f(x) = \frac{1-\tan x}{4x-\pi}, x \ne \frac{\pi}{4}, x \in \left[0, \frac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is equal to **[AIEEE 2004]**

(a) 1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) -1

57. If $\lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b are **[AIEEE 2004]**

(a) $a \in R, b \in R$ (b) $a = 1, b \in R$

(c) $a \in R, b = 2$ (d) $a = 1, b = 2$

58. If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0 \\ 0, & x=0\end{cases}$, then $f(x)$ is [AIEEE 2003]

(a) continuous as well as differentiable for all x
(b) continuous for all x but not differentiable at $x=0$
(c) neither differentiable nor continuous at $x=0$
(d) discontinuous everywhere

59. $\lim_{x\to\infty}\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$ is equal to [AIEEE 2002]

(a) $\frac{1}{p+1}$ (b) $\frac{1}{1-p}$
(c) $\frac{1}{p}-\frac{1}{p-1}$ (d) $\frac{1}{p+2}$

Answers

1. (b)	2. (a)	3. (a)	4. (d)	5. (d)	6. (b)	7. (b)	8. (b)	9. (b)	10. (c)
11. (d)	12. (b)	13. (b)	14. (a)	15. (c)	16. (a)	17. (b)	18. (b)	19. (b)	20. (a)
21. (a)	22. (d)	23. (c)	24. (c)	25. (a)	26. (b)	27. (c)	28. (b)	29. (a)	30. (a)
31. (a)	32. (d)	33. (a)	34. (a)	35. (c)	36. (d)	37. (c)	38. (a)	39. (c)	40. (b)
41. (d)	42. (d)	43. (b)	44. (d)	45. (b)	46. (c)	47. (d)	48. (b)	49. (b)	50. (a)
51. (a)	52. (d)	53. (c)	54. (b)	55. (a)	56. (c)	57. (b)	58. (b)	59. (a)	

Hints & Solutions

1. $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

$=\lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1}$

$=\frac{1}{2}$

2. $\lim_{h\to 0}\frac{f(1-h)-f(1)}{h^3+3h}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}\cdot\frac{-1}{h^2+3}$

$=f'(1)\cdot\left(\frac{-1}{3}\right)=\frac{53}{3}$

3. $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

$=\lim_{h\to 0}\frac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}=\frac{f'(2)\times 2}{f'(1)\times 1}=\frac{6\times 2}{4\times 1}=3$

4. $\because$ $f(x)=x(x-1)\sin x-(x^3-2x^2)\cos x-x^3\tan x$

$=x^2\sin x-x^3\cos x-x^3\tan x+2x^2\cos x-x\sin x$

$\therefore \lim_{x\to 0}\frac{f(x)}{x^2}=\lim_{x\to 0}\left(\sin x-x\cos x-x\tan x+2\cos x-\frac{\sin x}{x}\right)$

$=2-1=1$

5. $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

On putting $x=\tan\theta$ in both equations, we get

$f(\theta)=\cot^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$\Rightarrow$ $f(\theta)=\cot^{-1}(\tan 3\theta)$

$\Rightarrow$ $f(\theta)=\cot^{-1}\left\{\cot\left(\frac{\pi}{2}-3\theta\right)\right\}=\frac{\pi}{2}-3\theta$

$\therefore$ $f'(\theta)=-3$ ...(i)

and $g(\theta)=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\cos^{-1}(\cos 2\theta)=2\theta$

$\therefore$ $g'(\theta)=2$

Now, $\lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right)=\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\times\frac{1}{\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)}$

$=f'(x)\cdot\frac{1}{g'(x)}=-3\times\frac{1}{2}=-\frac{3}{2}$

6. $f(x)=\begin{cases} e^x, & x\le 0 \\ 1-x, & 0<x\le 1 \\ x-1, & x>1\end{cases}$

$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{1-h-1}{h}=-1$

$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$

So, it is not differentiable at $x=0$.

Similarly, it is not differentiable at $x=1$ but it is continuous at $x=0$ and 1.

7. Since, $|x-3| = x-3$, if $x \ge 3$ and $|x-3| = -x+3$, if $x<3$

$\therefore$ The given function can be defined as

$$f(x) = \begin{cases} \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}, & x<1 \\ 3-x, & 1 \le x < 3 \\ x-3, & x \ge 3 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} \frac{x}{2} - \frac{3}{2}, & x<1 \\ -1, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

At $x = 1$,

LHD $= f'(1) = -1$ and RHD $= f'(1) = -1$

8. $\lim_{n\to\infty} \sin\left\{n\pi\left(1+\frac{1}{n^2}\right)^{1/2}\right\}$

$$= \lim_{n\to\infty} \sin\left\{n\pi\left(1+\frac{1}{2n^2} - \frac{1}{8n^4} + \ldots\infty\right)\right\}$$

$$= \lim_{n\to\infty} \sin\left\{n\pi + \frac{\pi}{2n} - \frac{\pi}{8n^3} + \ldots\infty\right\}$$

$$= \lim_{n\to\infty} (-1)^n \sin\pi\left(\frac{1}{2n} - \frac{1}{8n^3} + \ldots\infty\right)$$

$$= 0$$

9. We want to check the continuity at $x = 0$.

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0^-} f(0-h) = \lim_{h\to 0^-} |-h|\cos\left(\frac{1}{-h}\right)$$

$$= \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0 \times \cos\frac{1}{0} = 0 \quad (\because -1 \le \cos x \le 1, \forall x \in R)$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0^-} |0+h| \cos\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} |h|\cos\left(\frac{1}{h}\right) = \lim_{h\to 0} h\cos\frac{1}{h} = 0 \times \cos\frac{1}{0} = 0$$

and $f(0) = 0$

Thus, LHL = RHL = $f(0) = 0$

Hence, the function is continuous at $x = 0$.

10. $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{2xf'(x^2) - f'(x)}{f'(x)} \quad \text{(using L' Hospital's rule)}$$

$$= -1 + \lim_{x\to 0} \frac{2xf'(x^2)}{f'(x)} = -1$$

($\because f'(0) \ne 0$ as f is strictly increasing function)

11. Given that, the function is continuous.

At $x = 0$,

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0^-} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{1-kh} - \sqrt{1+kh})}{-h} \times \frac{(\sqrt{1-kh} + \sqrt{1+kh})}{(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$= \lim_{h\to 0} \frac{(1-kh) - (1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})} = \frac{2k}{2} = k$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{2h+1}{h-2} = \frac{-1}{2}$$

Since, $f(x)$ is continuous, so $k = -\frac{1}{2}$.

12. Given, $x > 0$ and g is a bounded function. Then,

$$\lim_{n\to\infty} \frac{f(x)\cdot e^{nx} + g(x)}{e^{nx} + 1} = \lim_{n\to\infty} \left[\frac{f(x)}{1 + \left(\frac{1}{e^{nx}}\right)} + \frac{g(x)}{e^{nx}+1}\right]$$

$$= \frac{f(x)}{1+0} + \frac{\text{Finite}}{\infty} = f(x)$$

13. $\lim_{x\to\infty} \left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}} = \lim_{x\to\infty} \left(\frac{3x+2-6}{3x+2}\right)^{\frac{x+1}{3}}$

$$= \lim_{x\to\infty} \left(1 - \frac{6}{3x+2}\right)^{\frac{x+1}{3}}$$

$$= \lim_{x\to\infty} \left[\left(1 - \frac{6}{3x+2}\right)^{\frac{3x+2}{-6}}\right]^{\frac{-6}{3x+2}\cdot\frac{x+1}{3}}$$

$$= \lim_{x\to\infty} e^{\frac{-2(x+1)}{3x+2}} = e^{-2/3}$$

$$\left[\because \lim_{x\to\infty} \frac{-2(x+1)}{3x+2} = \frac{-2}{3}\right]$$

14. $\text{RHL} = \lim_{x\to 0^+} = \lim_{x\to 0^+} |x|^0 = 1$

$\text{LHL} = \lim_{x\to 0^-} |-x|^0 = \lim_{x\to 0^-} 1 = 1$

$\because$ RHL = LHL

$\therefore \lim_{x\to 0} |x|^{[\cos x]} = 1$

15. $\lim_{x\to 0^+} f(x) = x^2 \sin\frac{1}{x}, -1 \le \sin\frac{1}{x} \le 1, x \to 0$

$\therefore \lim_{x\to 0^+} f(x) = 0, \lim_{x\to 0^-} f(x) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

16. Let $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

Therefore, given function $= f'(a) + kf'(e) = 1$

$$\Rightarrow \quad \frac{1}{a} + \frac{k}{e} = 1$$

$$\therefore \quad k = e\left(\frac{a-1}{a}\right)$$

17. $f\{g(x)\} = \begin{cases} 1, & 0 < x < 3\pi/4 \quad \text{or} \quad 7\pi/4 < x < 2 \\ 0, & x = 3\pi/4, 7\pi/4 \\ -1, & 3\pi/4 < x < 7\pi/4 \end{cases}$

Clearly, $[f\{g(x)\}]$ is not continuous at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

18. $\lim_{x\to 2} \frac{\int_6^{f(x)} 4t^3 dt}{x-2} = \lim_{x\to 2} \frac{4\{f(x)\}^3 \times f'(x)}{1}$ $\left(\frac{0}{0}\text{ form}\right)$

$= 4\{f(2)\}^3 \times f'(2) = 18$

19. We know that,

$\cos A \cdot \cos 2A \cdot \cos 4A \ldots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \sin A}$

Take $A = \frac{x}{2^n}$, then

$$\cos\left(\frac{x}{2^n}\right)\cdot\cos\left(\frac{x}{2^{n-1}}\right)\ldots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$\therefore \lim_{n\to\infty} \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\ldots\cos\left(\frac{x}{2^{n-1}}\right)\cdot\cos\left(\frac{x}{2^n}\right)$$

$$= \lim_{n\to\infty} \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)} = \lim_{n\to\infty} \frac{\sin x}{x}\cdot\frac{x/2^n}{\sin(x/2^n)} = \frac{\sin x}{x}$$

20. $f(x) = \begin{cases} \frac{1}{2}(-x-1), & x < -1 \\ \tan^{-1} x, & -1 \le x \le 1 \\ \frac{1}{2}(x-1), & x > 1 \end{cases}$

$f'(x) = \begin{cases} -\frac{1}{2}, & x < -1 \\ \frac{1}{1+x^2}, & -1 < x < 1 \\ \frac{1}{2}, & x > 1 \end{cases}$

$\therefore \quad f'(-1-0) = -\frac{1}{2}$

$f'(-1+0) = \frac{1}{1+(-1+0)^2} = \frac{1}{2}$

$f'(1-0) = \frac{1}{1+(1-0)^2} = \frac{1}{2}$

and $\quad f'(1+0) = \frac{1}{2}$

Hence, $f'(-1)$ does not exist.

$\therefore$ Domain of $f'(x) = R - \{-1\}$

21. Polynomial function is always continuous. So, we have to check the continuity only at $x = -3$ and 3.

Here, $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$

For $x < -3$, $f(x) = |x| + 3$; $-3 < x < 3$, $f(x) = -2x$ and $x > 3$, $f(x) = 6x+2$ is a polynomial function, so it is continuous in a given interval. So, we have to check the continuity only at $x = -3$ and 3.

At $x = -3$, LHL $= \lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (|x| + 3)$

$\therefore \quad \lim_{h\to 0} [|-3-h| + 3] = \lim_{h\to 0} (6+h) = 6$

RHL $= \lim_{h\to -3^+} f(x) = \lim_{h\to -3^+} (-2x)$

$\therefore \quad \lim_{h\to 0} -2(-3+h) = \lim_{h\to 0} (6-2h) = 6$

Also, $f(-3) = |-3| + 3 = 6$ $\quad (\because f(x) = |x| + 3)$

$\therefore$ LHL = RHL = $f(-3)$

Thus, $f(x)$ is continuous at $x = -3$.

At $x = 3$, LHL $= \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-2x)$

$\therefore \quad \lim_{h\to 0} -2(3-h) = \lim_{h\to 0} (-6+2h) = -6$

RHL $= \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (6x+2)$

$\therefore \quad \lim_{h\to 0} [6(3+h)+2] = \lim_{h\to 0} (18+6h+2)$

$= \lim_{h\to 0} (20+6h) = 20$

$\therefore \quad$ LHL $\ne$ RHL

Thus, $f(x)$ is discontinuous at $x = 3$.

22. Let $\quad y = \lim_{x\to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{2}{x} \log\left(\frac{a^x + b^x + c^x}{3}\right)$

$= 2\lim_{x\to 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x}$

Apply L'Hospital's rule,

$$= 2\lim_{x\to 0} \frac{\frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}}{1}$$

$\log y = \log(abc)^{2/3}$

$\Rightarrow \quad y = (abc)^{2/3}$

23. $\lim_{x\to\infty} \left(\frac{x^2+1}{x+1} - \alpha x - \beta\right) = 0$

$\Rightarrow \quad \lim_{x\to\infty} \frac{x^2(1-\alpha) - x(\alpha+\beta) + 1 - \beta}{x+1} = 0$

$\therefore \quad 1 - \alpha = 0, \alpha + \beta = 0 \Rightarrow \alpha = 1, \beta = -1$

24. $[\sin x]$ is non-differentiable at $x = \frac{\pi}{2}, \pi, 2\pi$ and $[\cos x]$ is non-differentiable at $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ and 2π.

Thus, $f(x)$ is definitely non-differentiable at $x = \pi, \frac{3\pi}{2}, 0$.

Also, $f\left(\frac{\pi}{2}\right) = 1, f\left(\frac{\pi}{2} - 0\right) = 0, f(2\pi) = 1, f(2\pi - 0) = -1$

Thus, $f(x)$ is also non-differentiable at $x = \frac{\pi}{2}$ and 2π.

25. Since, $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

$\Rightarrow \quad f(x) = xf(1)$ for all $x \in R$ $\quad \{\because x = 1 + 1 + \ldots + 1 \text{ (x times)}\}$

$\Rightarrow \quad f(x) = x$ for all $x \in R$ $\quad [\because f(1) = 1, \text{ given}]$

26. Now, $g(0) = \lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)} = \lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \cdot \sin x}$

$= \lim_{x\to 0} \frac{(2^{\tan x - \sin x} - 1)\cdot 2^{\sin x}}{x^2 \sin x}$

$= \lim_{x\to 0} \frac{(2^{\tan x - \sin x} - 1)}{(\tan x - \sin x)} \cdot \frac{\tan x - \sin x}{x^2 \sin x} \cdot 2^{\sin x}$

$= (\log_e 2)\left(\frac{1}{2} \cdot 2^0\right) \qquad \left(\because \lim_{x\to 0} \frac{\tan x - \sin x}{x^2 \sin x} = \frac{1}{2}\right)$

$= \frac{1}{2} \log_e 2$

27. Since, $(gof)x$ is continuous.

So, $g(x)$ and $f(x)$ are also continuous.

Then, $\quad f(0) = \lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h)$

$\Rightarrow \quad 1 = \lim_{h\to 0} (0 - h + a) = a$

$\therefore \quad a = 1$

28. $g(0) = \lim_{x\to 0-} g(x) = \lim_{h\to 0} g(0-h) = \lim_{h\to 0} (0 - h + 1)$

$\Rightarrow \quad 1 + b = 1$

$\therefore \quad b = 0$

29. Statement I is correct as though $|4x^2 - 12x + 5|$ is non-differentiable at $x = \frac{1}{2}$ and $\frac{5}{2}$ but $\cos \pi x = 0$ those points.

So, $f'\left(\frac{1}{2}\right)$ and $f'\left(\frac{5}{2}\right)$ exists.

30. Since, $\lim_{x\to 0} \frac{\sqrt{1 - \cos 2x}}{x} = \sqrt{2} \lim_{x\to 0} \frac{|\sin x|}{x}$

$= \sqrt{2} \begin{cases} \lim_{x\to 0} \frac{\sin x}{x}, & 0 < x < \pi/2 \\ -\lim_{x\to 0} \frac{\sin x}{x}, & -\pi/2 < x < 0 \end{cases}$

$= \sqrt{2} \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & -\pi/2 < x < 0 \end{cases}$

Hence, LHL $\neq$ RHL

31. If $x \in I$, then $\{x\} = 0$

Hence, $f(x)$ is discontinuous for integral values of x.

32. $f(x) = \begin{cases} -\log x, x < 1 \\ \log x, x \geq 1 \end{cases}$

$f'(x) = \begin{cases} -1/x, & x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$

$\therefore \quad f'(1^-) = -1$ and $f'(1^+) = 1$

Hence, $f(x)$ is not differentiable.

33. $\lim_{x\to 3} \frac{\sqrt{1 - \cos(x^2 - 10x + 21)}}{(x-3)}$

$= \lim_{x\to 3} \frac{\sqrt{2} \sin \frac{(x-3)(x-7)}{2}}{(x-3)}$

$= \lim_{x\to 3} \frac{\sqrt{2} \sin \frac{(x-3)(x-7)}{2}}{\frac{(x-3)\cdot(x-7)}{2}} \cdot \frac{(x-7)}{2}$

$= \lim_{x\to 3} (x-7) \cdot \lim_{x\to 3} \frac{\sin \frac{(x-3)(x-7)}{2}}{\frac{(x-3)(x-7)}{2}} \times \frac{1}{\sqrt{2}}$

$= -(2)^{3/2}$

34. $\lim_{x\to \pi/4} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}} = \frac{\lim_{x\to \frac{\pi}{4}} f(\sec^2 x) 2 \sec x \cdot \sec x \tan x}{2x}$

$= \frac{2f(2)}{\pi/4} = \frac{8}{\pi} f(2)$

35. Let $g(x) = f(x) - x^2$

So, $g(x)$ has atleast real roots which are $x = 1, 2$ and 3.

Since, $g'(x)$ has atleast two real roots in $x \in (1, 3)$.

So, $g''(x)$ has atleast one real root in $x \in (1, 3)$.

$\Rightarrow f''(x) - 2 \cdot 1 = 0$

Hence, $f''(x) = 2$ for atleast one $x \in (1, 3)$.

36. Since, $\lim_{x\to 0} \left[(a-n)n - \frac{\tan x}{x}\right] \cdot \frac{\sin nx}{x} = 0$

$\Rightarrow \quad [(a-n)n - 1]\, n = 0$

$\Rightarrow \quad (a-n)n = 1$

$\therefore \quad a = n + \frac{1}{n}$

37. Let $y = \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$

$\Rightarrow \quad \log y = \frac{1}{x}[\log f(1+x) - \log f(1)]$

$\Rightarrow \quad \lim_{x\to 0} \log y = \lim_{x\to 0} \left[\frac{1}{f(1+x)} f'(1+x)\right]$

$\Rightarrow \quad \lim_{x\to 0} \log y = \frac{f'(1)}{f(1)} = \frac{6}{3}$

$\therefore \quad \lim_{x\to 0} y = e^2$

38. If x is just less than k, then $[x] = k - 1$

$\therefore \quad f(x) = (k-1)\sin \pi x$

LHD of $\quad f(x) = \lim_{x\to k} \frac{(k-1)\sin \pi x - k \sin \pi k}{x - k}$

$= \lim_{x\to k} \frac{(k-1)\sin \pi x}{x - k}$, where $x = k - h$

$= \lim_{h\to 0} \frac{(k-1)\sin \pi (k-h)}{-h}$

$= (k-1)(-1)^k \pi$

39. Let $h(x) = |x|$. Then, $g(x) = |f(x)| = h\{f(x)\}$

Since, composition of two continuous functions is continuous and g is continuous, if f is continuous.

40. $\lim_{x\to 0}\frac{1}{x}\left\{\tan^{-1}\left(\frac{x+1}{2x+1}\right)-\frac{\pi}{4}\right\}$

$$= \lim_{x\to 0}\frac{1}{x}\left\{\tan^{-1}\left(\frac{x+1}{2x+1}\right)-\tan^{-1}(1)\right\}$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\tan^{-1}\left\{\frac{\frac{x+1}{2x+1}-1}{1+\frac{x+1}{2x+1}}\right\}$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\tan^{-1}\left(\frac{x+1-2x-1}{2x+1+x+1}\right)$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot\tan^{-1}\left(\frac{-x}{4x+2}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

(using L'hospital rule)

$$= \lim_{x\to 0}\frac{1}{1+\frac{x^2}{(4x+2)^2}}\times\left[-\left(\frac{4x+2-4x}{(4x+2)^2}\right)\right]$$

$$= \lim_{x\to 0}-\frac{(+2)}{x^2+(4x+2)^2}$$

$$= -\frac{(+2)}{0+(0+2)^2}=\frac{-2}{4}=\frac{-1}{2}$$

41. $g(x)=\begin{cases}1+x, & x<0\\ 1-x, & x\ge 0\end{cases}$

$$\therefore\quad f\{g(x)\}=\begin{cases}1+|x-1|, & x<0\\ 1+|-x-1|, & x\ge 0\end{cases}$$

$$=\begin{cases}1+1-x, & x<0\\ 1+x+1, & x\ge 0\end{cases}=\begin{cases}2-x, & x<0\\ 2+x, & x\ge 0\end{cases}$$

It is a polynomial function, so it is continuous in everywhere except at $x=0$.

Now, LHL $= \lim_{x\to 0} 2-x=2$

RHL $= \lim_{x\to 0} 2+x=2$

Also, $f(0)=2+0=2$

Hence, it is continuous everywhere.

42. The function $u=f(x)=\frac{1}{x-1}$ is discontinuous at the point $x=1$.

The function $y=g(u)=\frac{1}{u^2+u-2}=\frac{1}{(u+2)(v-1)}$

is discontinuous at $u=-2$ and $u=1$.

When $\quad u=-2\Rightarrow\frac{1}{x-1}=-2\Rightarrow x=\frac{1}{2}$,

When $\quad u=1\Rightarrow\frac{1}{x-1}=1\Rightarrow x=2$

Hence, the composite function $y=gf(x)$ is discontinuous at three points, $x=\frac{1}{2}$, $x=1$ and $x=2$.

43. I. We know $[x]$ is discontinuous for every integer values of x.

$\therefore\quad f(x)$ is discontinuous for $x=0, 1, 2$ and 3.

II. $$f(x)=\begin{cases}-1+1-x, & -\infty\le x<0\\ 0+1-x, & 0\le x<1\\ 1+x-1, & 1\le x<2\\ 2+x-1, & 2\le x<3\\ 3+2, & x=3\end{cases}$$

$$=\begin{cases}-x, & -\infty\le x<0\\ 1-x, & 0\le x<1\\ x, & 1\le x<2\\ x+1, & 2\le x<3\\ 5, & x=3\end{cases}$$

44. Let $I=\lim_{x\to 0}\frac{(1-\cos 2x)}{x^2}\cdot\frac{(3+\cos x)}{1}\cdot\frac{x}{\tan 4x}$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\cdot\frac{3+\cos x}{1}\cdot\frac{x}{\tan 4x}$$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}(3+\cos x)\cdot\lim_{x\to 0}\frac{4x}{4\tan 4x}$$

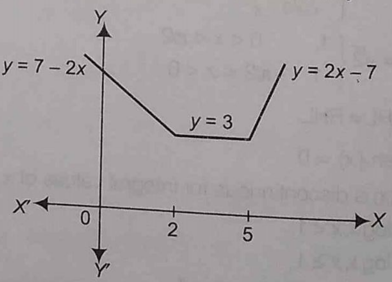
$$=2\cdot 4\cdot\frac{1}{4}=2$$

45. $\therefore\ f(x)=|x-2|+|x-5|$

$$=\begin{cases}(2-x)+(5-x), & x<2\\ (x-2)+(5-x), & 2\le x\le 5\\ (x-2)+(x-5), & x>5\end{cases}$$

$$=\begin{cases}7-2x, & x<2\\ 3, & 2\le x\le 5\\ 2x-7, & x>5\end{cases}$$

Now, we can draw the graph of f very easily.



Statement I $f'(4)=0$

It is obviously clear that, f is constant around $x=4$, hence, $f'(4)=0$.

Hence, Statement I is correct.

Statement II It can be clearly seen that

(i) f is continuous, $\forall\, x\in[2, 5]$

(ii) f is differentiable, $\forall\, x\in(2, 5)$

(iii) $f(2)=f(5)=3$

Hence, Statement II is also correct but obviously not a correct explanation of Statement I.

46. Now, $\cos x$ is continuous, $\forall x \in R \Rightarrow \cos\pi\left(x-\frac{1}{2}\right)$ is also continuous, $\forall x \in R$.

Hence, the continuity of f depends upon the continuity of $[x]$, then $[x]$ is discontinuous, $\forall x \in I$.

So, we should check the continuity of f at $x = n$, $\forall n \in I$

LHL at $x = n$ is given by

$$f(n^-) = \lim_{x\to n^-} f(x)$$

$$= \lim_{x\to n^-}[x]\cos\pi\left(x-\frac{1}{2}\right) = (n-1)\cos\frac{(2n-1)\pi}{2} = 0$$

RHL at $x = n$ is given by

$$f(n^+) = \lim_{x\to n^+} f(x) = \lim_{x\to n^+}[x]\cos\pi\left(x-\frac{1}{2}\right)$$

$$= (n)\cos\frac{(2n-1)\pi}{2} = 0$$

Also, value of the function at $x = n$ is

$$f(n) = [n]\cos\pi\left(n-\frac{1}{2}\right) = (n)\cos\frac{(2n-1)\pi}{2} = 0$$

$\therefore f(n^+) = f(n^-) = f(n)$

Hence, f is continuous at $x = n$, $\forall n \in I$.

47. $$\lim_{x\to 2}\frac{\sqrt{1-\cos 2(x-2)}}{(x-2)} = \lim_{x\to 2}\frac{\sqrt{2\cdot\sin^2(x-2)}}{(x-2)}$$

$$= \frac{\sqrt{2}\,|\sin(x-2)|}{(x-2)} = \lim_{x\to 2}\frac{\sqrt{2}\,|\sin(x-2)|}{(x-2)}$$

RHL at $x = 2$,

$$\lim_{h\to 0}\frac{\sqrt{2}\,|\sin(2+h-2)|}{(2+h)-2} = \lim_{h\to 0}\frac{\sqrt{2}\,|\sin h|}{h} = \lim_{h\to 0}\frac{\sqrt{2}\sin h}{h} = \sqrt{2}$$

LHL at $x = 2$,

$$\lim_{h\to 0}\frac{\sqrt{2}\,|\sin(2-h-2)|}{(2-h)-2} = \lim_{h\to 0}\frac{\sqrt{2}\,|\sin(-h)|}{-h}$$

$$= \lim_{h\to 0}\frac{\sqrt{2}\sin h}{-h} = -\sqrt{2}$$

So, the limit does not exist.

48. Here, $f(x) = \begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q, & x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{\frac{3}{2}}}, & x>0\end{cases}$

Since, $f(x)$ is continuous for $x \in R$.

So, the function is continuous at $x = 0$.

RHL at $x = 0$,

$$\lim_{h\to 0}\frac{\sqrt{h+h^2}-\sqrt{h}}{h^{3/2}} = \lim_{h\to 0}\frac{\sqrt{h}\{\sqrt{h+1}-1\}}{h\sqrt{h}}$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{h+1}-1}{h}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}\right)$$

$$= \lim_{h\to 0}\frac{(h+1)-1}{h\{\sqrt{h+1}+1\}} = \lim_{h\to 0}\frac{1}{\sqrt{h+1}+1} = \frac{1}{2} \quad \ldots(i)$$

LHL at $x = 0$,

$$\lim_{h\to 0}\frac{\sin(p+1)(-h)+\sin(-h)}{-h} = \lim_{h\to 0}\frac{\sin(p+1)h}{h}+\frac{\sin h}{h}$$

$$= (p+1)+1 = (p+2) \quad \ldots(ii)$$

and $\quad f(0) = q \quad \ldots(iii)$

From Eqs. (i), (ii) and (iii), we get

$$\frac{1}{2} = q = p+2$$

$\therefore \quad p = -\frac{3}{2}, q = \frac{1}{2}$

49. We have, $f:(-1,1)\longrightarrow R$

$f(0) = -1 \quad f'(0) = 1$

$g(x) = [f\{2f(x)+2\}]^2$

$\Rightarrow \quad g'(x) = 2[f\{2f(x)+2\}]\times f'\{2f(x)+2\}\times 2f'(x)$

$\therefore \quad g'(0) = 2[f\{2f(0)+2\}]\times f'\{2f(0)+2\}\times 2f'(0)$

$= 2[f(0)]\times f'(0)\times 2f'(0) = 2\times(-1)\times 1\times 2\times 1 = -4$

50. Since, $f(x)$ is a positive increasing function.

$\therefore \quad 0 < f(x) < f(2x) < f(3x)$

$\Rightarrow \quad 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$

$\Rightarrow \quad \lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f(2x)}{f(x)} \le \lim_{x\to\infty}\frac{f(3x)}{f(x)}$

By Sandwich theorem, $\lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$

51. $\because \quad f(x) = \dfrac{1}{e^x+\dfrac{2}{e^x}}$

Using $\quad$ AM $\ge$ GM,

$$\frac{e^x+\frac{2}{e^x}}{2} \ge \left(e^x\cdot\frac{2}{e^x}\right)^{1/2}, \quad \text{as } e^x > 0$$

$\Rightarrow \quad e^x+\frac{2}{e^x} \ge 2\sqrt{2} \Rightarrow 0 < \dfrac{1}{e^x+\frac{2}{e^x}} \le \dfrac{1}{2\sqrt{2}}$

$\therefore \quad 0 < f(x) \le \frac{1}{2\sqrt{2}}, \quad \forall x \in R$

Hence, Statement II is true and Statement I as for some c.

$\therefore \quad f(c) = \frac{1}{3}$

Alternate Solution

$$f(x) = \frac{1}{e^x+2e^{-x}} = \frac{e^x}{e^{2x}+2}$$

$\therefore \quad f'(x) = \dfrac{(e^{2x}+2)e^x-2e^{2x}\cdot e^x}{(e^{2x}+2)^2}$

$\Rightarrow \quad f'(x) = 0 \Rightarrow e^{2x}+2 = 2e^{2x}$

$\Rightarrow \quad e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

Maximum value of $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

$$0 < f(x) \le \frac{1}{2\sqrt{2}}, \ \forall\, x \in R$$

Since, $\quad 0 < \frac{1}{3} < \frac{1}{2\sqrt{2}}$ for some $c \in R$

$\therefore \quad f(c) = \frac{1}{3}$

52. Now, $\quad f'(1^-) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h \to 0} \frac{(1-h-1)\cdot \sin\left(\frac{1}{1-h-1}\right) - 0}{-h}$$

$$= \lim_{h \to 0} \sin\left(-\frac{1}{h}\right) = -\lim_{h \to 0} \sin\frac{1}{h}$$

and $\quad f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{(1+h-1)\sin\left(\frac{1}{1+h-1}\right) - 0}{h} = \lim_{h \to 0} \sin\frac{1}{h}$$

$\therefore \quad f'(1^-) \ne f'(1^+)$

Hence, f is not differentiable at $x = 1$.

Again, now $f'(0^-) = \lim_{h \to 0} \frac{(0+h+1)\cdot \sin\left(\frac{1}{0+h+1}\right) - \sin 1}{-h}$

$$= \lim_{h \to 0} \frac{\left[-\left\{(h+1)\cos\left(\frac{1}{h+1}\right) \times \left(\frac{1}{(h+1)^2}\right)\right\} + \sin\left(\frac{1}{h+1}\right)\right]}{-1}$$

(using L' Hospital's rule)

$$= \cos 1 - \sin 1$$

and $f'(0^+) = \lim_{h \to 0} \frac{(0+h-1)\cdot \sin\left(\frac{1}{0+h-1}\right) - \sin 1}{h}$

$$= \lim_{h \to 0} \frac{(h-1)\cos\left(\frac{1}{h-1}\right)\left(\frac{-1}{(h-1)^2}\right) + \sin\left(\frac{1}{h-1}\right)}{1}$$

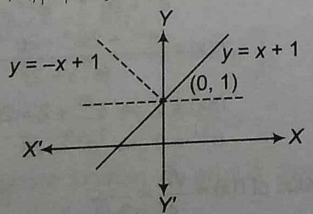
(using L' Hospital's rule)

$$= \cos 1 - \sin 1$$

$\Rightarrow \quad f'(0^-) = f'(0^+)$

Hence, f is differentiable at $x = 0$.

53. $f(x) = \min\{x+1, |x|+1\} = x+1, \forall x \in R$



Hence, $f(x)$ is differentiable everywhere.

54. Since, $\quad f(x) = \frac{x}{1+|x|} = \frac{g(x)}{h(x)}$ (say)

It is clear that $g(x)$ and $h(x)$ are differentiable on $(-\infty, \infty)$ and $(-\infty, 0) \cup (0, \infty)$.

Now, $\quad \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x}{1+|x|} - 0}{x} = 1$

Hence, $f(x)$ is differentiable on $(-\infty, \infty)$.

55. $\lim_{n \to \infty} \frac{1}{n}\left[\frac{1}{n}\sec^2\left(\frac{1}{n}\right)^2 + \frac{2}{n}\sec^2\left(\frac{2}{n}\right)^2 + \ldots + \frac{n}{n}\sec^2\left(\frac{n}{n}\right)^2\right]$

$$= \lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)\sec^2\left(\frac{r}{n}\right)^2 = \int_0^1 x \sec^2(x^2)\, dx$$

Put $\quad x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow \quad x\,dx = \frac{1}{2}dt$

$\therefore \quad A = \frac{1}{2}\int_0^1 \sec^2 t\, dt = \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}\tan 1$

56. $\lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{4x - \pi} \quad \left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{4} = -\frac{2}{4} = -\frac{1}{2}$$

For $f(x)$ to be continuous at $x = \frac{\pi}{4}, f\left(\frac{\pi}{4}\right) = -\frac{1}{2}$.

57. $\lim_{x \to \infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = \lim_{x \to \infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x\left(\frac{a/x + b/x^2}{a/x + b/x^2}\right)}$

$$= e^{\lim_{x \to \infty} 2x\left(\frac{a}{x} + \frac{b}{x^2}\right)} = \lim_{x \to \infty} e^{2\left(a + \frac{b}{x}\right)}$$

$\Rightarrow \quad e^2 = e^{2a}$ (given)

$\therefore \quad a = 1, b \in R$

58. RHL $= \lim_{h \to 0} (0+h)e^{-2/h} = \lim_{h \to 0} \frac{h}{e^{2/h}} = 0$

LHL $= \lim_{h \to 0} (0-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} = 0$

Hence, $f(x)$ is continuous at $x = 0$.

Now, $\quad Rf'(x) = \lim_{h \to 0} \frac{(0+h)e^{-\left(\frac{1}{h} + \frac{1}{h}\right)} - 0}{h}$

$$= \lim_{h \to 0} e^{-2/h} = \infty$$

and $\quad Lf'(x) = \lim_{h \to 0} \frac{(0-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} - 0}{-h} = \lim_{h \to 0} e^{-0} = 1$

$\therefore \quad Lf'(x) \ne Rf'(x)$

Hence, $f(x)$ not differentiable at $x = 0$.

59. $\lim_{n \to \infty} \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}} = \lim_{n \to \infty} \frac{\left(1 + \frac{1}{n}\right)^{p-1}}{p+1} = \frac{1}{p+1}$

Day 12

Differentiation

Day 12 Outlines ...

- Derivative (Differential Coefficient
- Some Standard Differentiations
- Derivative of Functions
- Second Order Derivative
- Differential Coefficient using Trigonometrical Substitution

Derivative (Differential Coefficient)

The rate of change of a quantity y with respect another quantity x is called the **derivative** or **differential coefficient** of y with respect to x. The process of finding derivative of a function called **differentiation**.

Some Standard Differentiations

1. Differentiation of Algebraic Functions

(i) $\frac{d}{dx} x^n = nx^{n-1}3$

(ii) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

(iii) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$

2. Differentiation of Trigonometric Functions

(i) $\frac{d}{dx}(\sin x) = \cos x$

(ii) $\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(v) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vi) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

3. Differentiation of Logarithmic and Exponential Functions

(i) $\frac{d}{dx}(\log x) = \frac{1}{x}$, for $x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \log a$, for $a > 0$

(iv) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$, for $x > 0, a > 0, a \neq 1$

4. Differentiation of Inverse Trigonometric Functions

Sometimes the given function can be deducted with the help of inverse trigonometrical substitution and then to find the differential coefficient is very easy.

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(iii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(iv) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(v) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for $x \in R$

(vi) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for $x \in R$

(vii) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

(viii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1) = \cos^{-1}(1 - 2x^2)$

(ix) $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right) \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right) \\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \end{cases}$

(x) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(xi) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(xii) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

(xiii) $\cos^{-1} x + \sin^{-1} x = \pi/2$

(xiv) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(xv) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$

(xvi) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$

(xvii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{(1-x^2)(1-y^2)}]$

(xviii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$

5. Differentiation using Substitution

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
(iii)	$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$	$x^2 = a^2\cos 2\theta$
(iv)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$
(v)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(vi)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(ix)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

Geometrically Meaning of Derivative at a Point

Geometrically derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $\{c, f(c)\}$.

Slope of tangent at $P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left\{\frac{df(x)}{dx}\right\}_{x=c}$ or $f'(c)$.

Derivative of Functions

1. Sum or Difference Rule

$$\frac{d}{dx}\{f(x) \pm g(x)\} = f'(x) \pm g'(x)$$

2. Product Rule

$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

3. **Quotient Rule**

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\{g(x)\}^2}, g(x) \neq 0$$

4. **Logarithmic Differentiation Rule**

$$y = \{f(x)\}^{g(x)}$$

$$\Rightarrow \frac{dy}{dx} = \{f(x)\}^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + \log f(x)\cdot g'(x)\right]$$

5. **Parametric Differentiation Rule**

If $x = \phi(t)$ and $y = \phi(t)$, where t is parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

6. **Chain/Composite Rule**

If $y = f\{g(x)\}$, then

$$\frac{dy}{dx} = f'\{g(x)\}\, g'(x)$$

Second Order Derivative

Suppose $y = f(x)$ and it is differentiate twicly, then it is said to be **second order derivative.**

e.g., $$y = x^3 \Rightarrow \frac{d^2y}{dx^2} = 6x$$

Differentiation of a Determinant

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, then $\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$

Differentiation of Integrable Functions

If $g_1(x)$ and $g_2(x)$ are defined in $[a, b]$, differentiable at $x \in [a, b]$ and $f(t)$ is continuous for $g_1(a) \leq f(t) \leq g_2(b)$, then

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)\,dt = f[g_2(x)]\frac{d}{dx}[g_2(x)] - f[g_1(x)]\frac{d}{dx}[g_1(x)]$$

Derivative of Special Types of Functions

- If $y = f(x)^{\{f(x)\}^{\dots\infty}}$, then
$$\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$
- If $y = \sqrt{\frac{1+g(x)}{1-g(x)}}$, then
$$\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2}\sqrt{\frac{1-g(x)}{1+g(x)}}$$
- If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots\infty}}}$,
then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$
- If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then
$$\frac{dy}{dx} = \frac{-f'(x)}{g'(y)[1+\log f(x)]^2}$$
- $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then
$$\frac{dy}{dx} = \frac{g(y)}{f(x)}\cdot\frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$$

Practice Zone

DAY 12

1. If $x^2 + y^2 = t - \frac{1}{t}, x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $x^3 y \frac{dy}{dx}$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

2. If $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{2t}{t^2+1}$ (b) $\frac{2t}{t^2-1}$
(c) $\frac{2t}{1-t^2}$ (d) None of these

3. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then $\frac{dy}{dx}$ is equal to

(a) $\sqrt{\frac{1-x^2}{1-y^2}}$ (b) $\sqrt{\frac{1-y^2}{1-x^2}}$ (c) $\sqrt{\frac{x^2-1}{1-y^2}}$ (d) $\sqrt{\frac{y^2-1}{1-x^2}}$

4. If $y = \frac{a^{\cos^{-1} x}}{1+a^{\cos^{-1} x}}$ and $z = a^{\cos^{-1} x}$, then $\frac{dy}{dz}$ is equal to

(a) $-\frac{1}{1+a^{\cos^{-1} x}}$ (b) $\frac{1}{1+a^{\cos^{-1} x}}$
(c) $\frac{1}{(1+a^{\cos^{-1} x})^2}$ (d) None of these

5. If $y = \log x \cdot e^{(\tan x + x^2)}$, then $\frac{dy}{dx}$ is equal to

(a) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + x)\log x\right]$
(b) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x - x)\log x\right]$
(c) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$
(d) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x - 2x)\log x\right]$

6. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{\sqrt{1-x^3}}$ (b) $-\frac{x}{\sqrt{1-x^2}}$
(c) $\frac{2x}{\sqrt{1-x^4}}$ (d) None of these

7. If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to **[NCERT Exemplar]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$
(c) $\frac{1}{2}$ (d) $2\sqrt{2}$

8. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) 1 (b) -1
(c) 0 (d) None of these

9. If $f(x) = |x-1|$ and $g(x) = f[f\{f(x)\}]$, then for $x > 2$, $g'(x)$ is equal to

(a) -1, if $2 \le x < 3$ (b) 1, if $2 \le x < 3$
(c) 1, if $x > 2$ (d) None of these

10. If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) 1 (d) $\frac{\sqrt{3}}{2}$

11. The derivative of $y = (1-x)(2-x)\ldots(n-x)$ at $x = 1$ is

(a) 0 (b) $(-1)(n-1)!$
(c) $n! - 1$ (d) $(-1)^{n-1}(n-1)!$

12. If f and g be differentiable function satisfying $g'(a) = 2, g(a) = b$ and $fog = I$ (identity function). Then, $f'(b)$ is equal to

(a) $\frac{1}{2}$ (b) 2
(c) $\frac{2}{3}$ (d) None of these

13. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then, the value of $[f(3)]^2 - [\phi(3)]^2$ is

(a) 0 (b) 9
(c) 41 (d) None of these

14. If $3f(x)-2f(1/x)=x$, then $f'(2)$ is equal to

(a) $\frac{2}{7}$ (b) $\frac{1}{2}$

(c) 2 (d) $\frac{7}{2}$

15. If $f(x)=x^n$, then the value of $f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots+\frac{(-1)^n f^n(1)}{n!}$ is

(a) 2^n (b) 0

(c) 2^{n-1} (d) None of these

16. If $f(x)=(\cos x+i\sin x)\cdot(\cos 2x+i\sin 2x)(\cos 3x+i\sin 3x)\ldots(\cos nx+i\sin nx)$ and $f(1)=1$, then $f''(1)$ is equal to

(a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$

(c) $-\left[\frac{n(n+1)}{2}\right]^2$ (d) None of these

17. If $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$, where P is a constant. Then, $\frac{d^3}{dx^3}\{f(x)\}$ at $x=0$ is equal to

(a) P (b) $P+P^2$

(c) $P+P^3$ (d) independent of P

18. If $y=\sin^{-1}(x\sqrt{1-x}+\sqrt{x}\sqrt{1-x^2})$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{-2x}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ (b) $\frac{-1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$

(c) $\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ (d) None of these

19. If $f(x)=\sqrt{1+\cos^2(x^2)}$, then $f'\left[\left(\frac{\sqrt{\pi}}{2}\right)\right]$ is equal to

(a) $\sqrt{\pi}/6$ (b) $-\sqrt{\pi/6}$ (c) $1/\sqrt{6}$ (d) $\pi/\sqrt{6}$

20. If $y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{x}+1}$

(c) 1 (d) None of these

21. The solution set of $f'(x)>g'(x)$, where $f(x)=\frac{1}{2}(5)^{2x+1}$ and $g(x)=5^x+4x\log_e 5$ is

(a) $(1,\infty)$ (b) $(0,1)$ (c) $(\infty,0)$ (d) $(0,\infty)$

22. If $y=f(x)$ is an odd differentiable function defined on $(-\infty,\infty)$ such that $f'(3)=-2$, then $f'(-3)$ is equal to

(a) 4 (b) 2

(c) −2 (d) 0

23. If $x=a\cos t\sqrt{\cos 2t}$ and $y=a\sin t\sqrt{\cos 2t}$ (where, $a>0$), then $\left|\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right|$ at $\frac{\pi}{6}$ is given by

(a) $\frac{a}{3}$ (b) $a\sqrt{2}$

(c) $\frac{\sqrt{2}}{3a}$ (d) $\frac{\sqrt{2}a}{3}$

24. If $\sqrt{x^2+y^2}=ae^{\tan^{-1}\left(\frac{y}{x}\right)}$, $a>0$ assuming $y>0$, then $y''(0)$ is equal to

(a) $\frac{2}{a}e^{-\pi/2}$ (b) $-\frac{2}{a}e^{\pi/2}$

(c) $-\frac{2}{a}e^{-\pi/2}$ (d) None of these

Directions (Q. Nos. 25 to 27) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

25. Statement I If $u=f(\tan x), v=g(\sec x)$ and $f'(1)=2$, $g'(\sqrt{2})=4$, then $\left(\frac{du}{dv}\right)_{x=\pi/4}=\frac{1}{\sqrt{2}}$.

Statement II If $u=f(x), v=g(x)$, then the derivative of f with respect to g is $\frac{du}{dv}=\frac{du/dx}{dv/dx}$.

26. Statement I For $x<0, \frac{d}{dx}(\ln|x|)=-\frac{1}{x}$.

Statement II For $x<0, |x|=-x$.

27. Consider, if $u=f(x)$, $v=g(x)$, then the derivative of f with respect to g is $\frac{du}{dv}=\frac{du/dx}{dv/dx}$.

Statement I Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is 1 for $0<x<1$.

Statement II $\sin^{-1}\left(\frac{2x}{1+x^2}\right)\neq\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $-1\le x\le 1$.

28. Derivative of $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x is **[NCERT Exemplar]**

(a) $(x\cot x + \log\sin x) + \dfrac{1}{2\sqrt{x - x^2}}$

(b) $(x\cot x + \log\sin x) + \dfrac{1}{\sqrt{x - x^2}}$

(c) $(\sin x)(x\cot x + \log x) + \dfrac{1}{\sqrt{x - x^2}}$

(d) $(\sin x)^x(x\cot x + \log\sin x) + \dfrac{1}{2\sqrt{x - x^2}}$

29. If $f(x) = |x|^3$, then the value of $f''(x)$ is

(a) $f''(x) = \begin{cases} 6x, x \ge 0 \\ 6x, x < 0 \end{cases}$

(b) $f''(x) = \begin{cases} -6x, x \ge 0 \\ 6x, x < 0 \end{cases}$

(c) $f''(x)$ does not exist

(d) None of these

30. If $y = (1 - x)(1 + x^2)(1 + x^4)\dots(1 + x^{2n})$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(a) -1 (b) $\dfrac{1}{(1+x)^2}$ (c) $\dfrac{x}{(1+x^2)}$ (d) $\dfrac{x}{(1-x)^2}$

31. If $y = |\sin x|^{|x|}$, then the value of $\dfrac{dy}{dx}$ at $x = -\dfrac{\pi}{6}$ is

(a) $\dfrac{2^{\frac{-\pi}{6}}}{6}[6\log 2 - \sqrt{3}\pi]$ (b) $\dfrac{2^{\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$

(c) $\dfrac{2^{-\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$ (d) None of these

32. Let $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$.

If $F(x) = \left[f\left(\dfrac{x}{2}\right)\right]^2 + \left[g\left(\dfrac{x}{2}\right)\right]^2$ and $F(5) = 5$, then $F(10)$ is equal to

(a) 0 (b) 5 (c) 10 (d) 25

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33. If $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$, where $x \ne 0, -2$, then $\dfrac{d}{dx}[f^{-1}(x)]$ (wherever it is defined) is equal to **[JEE Main 2013]**

(a) $\dfrac{-1}{(1-x)^2}$ (b) $\dfrac{3}{(1-x)^2}$ (c) $\dfrac{1}{(1-x)^2}$ (d) $\dfrac{-3}{(1-x)^2}$

34. For $a > 0, t \in \left(0, \dfrac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$. Then, $1 + \left(\dfrac{dy}{dx}\right)^2$ equals to **[JEE Main 2013]**

(a) $\dfrac{x^2}{y^2}$ (b) $\dfrac{y^2}{x^2}$ (c) $\dfrac{x^2 + y^2}{y^2}$ (d) $\dfrac{x^2 + y^2}{x^2}$

35. If $f(x) = \sin(\sin x)$ and $f''(x) + \tan x f'(x) + g(x) = 0$, then $g(x)$ is equal to **[JEE Main 2013]**

(a) $\cos^2 x\cos(\sin x)$ (b) $\sin^2 x\cos(\cos x)$

(c) $\sin^2 x\sin(\cos x)$ (d) $\cos^2 x\sin(\sin x)$

36. If $y = \sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at $x = 1$ is equal to **[JEE 2013]**

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) 1 (d) $\sqrt{2}$

37. $\dfrac{d^2x}{dy^2}$ is equal to **[AIEEE 2011]**

(a) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (b) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

(c) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (d) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

38. If y be an implicit function of x defined by $x^{2x} - 2x^x\cot y - 1 = 0$. Then, $y'(1)$ is equal to **[AIEEE 2009]**

(a) -1 (b) 1

(c) $\log 2$ (d) $-\log 2$

39. If $f(x) = x|x|$ and $g(x) = \sin x$

Statement I gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement II gof is twice differentiable at $x = 0$. **[AIEEE 2009]**

(a) Statement I is true, Statement II is true; Statement II is true correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

40. If $x^m y^n = (x + y)^{m+n}$, then $\dfrac{dy}{dx}$ is equal to **[AIEEE 2006]**

(a) $\dfrac{x+y}{xy}$ (b) xy

(c) $\dfrac{x}{y}$ (d) $\dfrac{y}{x}$

41. If $y = (x + \sqrt{1 + x^2})^n$, then $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is equal to **[AIEEE 2002]**

(a) n^2y (b) $-n^2y$

(c) $-y$ (d) $2x^2y$

Answers

1. (a)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (a)	**10.** (a)
11. (b)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (c)	**19.** (b)	**20.** (a)
21. (d)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (a)	**26.** (a)	**27.** (c)	**28.** (d)	**29.** (a)	**30.** (a)
31. (a)	**32.** (b)	**33.** (d)	**34.** (d)	**35.** (d)	**36.** (a)	**37.** (c)	**38.** (a)	**39.** (c)	**40.** (d)
41. (a)									

Hints & Solutions

1. $x^4 + y^4 = \left(t - \frac{1}{t}\right)^2 + 2 = (x^2 + y^2)^2 + 2$

$\Rightarrow \quad x^2y^2 = -1 \Rightarrow y^2 = -\frac{1}{x^2}$

On differentiating, we get $2y\frac{dy}{dx} = \frac{2}{x^3} \Rightarrow x^3y\frac{dy}{dx} = 1$

2. We have, $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

Put $t = \tan\theta$

$\therefore \quad x = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

and $\quad y = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{2\cos 2\theta} = -\tan 2\theta$

$= \frac{-2\tan\theta}{1-\tan^2\theta} = \frac{-2t}{1-t^2} = \frac{2t}{t^2-1}$

3. On putting $x = \sin\theta$ and $y = \sin\phi$, we get

Given equation beocmes

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow \quad 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = a\left\{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right\}$

$\Rightarrow \quad \frac{\theta-\phi}{2} = \cot^{-1}a$

$\Rightarrow \quad \theta - \phi = 2\cot^{-1}a$

$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

$\Rightarrow \quad \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

4. $y = \frac{a^{\cos^{-1}x}}{1+a^{\cos^{-1}x}}, z = a^{\cos^{-1}x} \Rightarrow y = \frac{z}{1+z}$

$\Rightarrow \quad \frac{dy}{dz} = \frac{(1+z)1 - z(1)}{(1+z)^2} = \frac{1}{(1+z)^2} = \frac{1}{(1+a^{\cos^{-1}x})^2}$

5. Given, $y = \log x \cdot e^{(\tan x + x^2)}$

$\therefore \quad \frac{dy}{dx} = e^{(\tan x + x^2)} \cdot \frac{1}{x} + \log x \cdot e^{(\tan x + x^2)}(\sec^2 x + 2x)$

$= e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$

6. Given, $y = \tan^{-1}\left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$. Put $x^2 = \cos 2\theta$

$\therefore \quad y = \tan^{-1}\left[\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right] = \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right] = \frac{\pi}{4} - \theta = \frac{\pi}{4} - \cos^{-1}x^2$

$\therefore \quad \frac{dy}{dx} = 0 + \frac{2x}{\sqrt{1-x^4}} = \frac{2x}{\sqrt{1-x^4}}$

7. When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$, so that $|\cos x| = -\cos x$,

i.e., $\quad f(x) = -\cos x, f'(x) = \sin x$

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

8. When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore \quad \cos x - \sin x > 0$

Also, when $\frac{\pi}{4} < x < \pi, \cos x < \sin x$

When $\frac{\pi}{4} < x < \pi, \cos x - \sin x < 0$

$\therefore \quad |\cos x - \sin x| = -(\cos x - \sin x)$, when $\frac{\pi}{4} < x < \pi$

$\Rightarrow f'(x) = \sin x + \cos x \Rightarrow f'\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$

9. We have, $f(x) = |x - 1|$ $\quad (\because x > 2)$

$f[f(x)] = f(x-1) = |(x-1) - 1| = |x-2|$

$g(x) = f[f\{f(x)\}] = f(x-2)$

$= |(x-2) - 1| = |x-3| = \begin{cases} x-3, & \text{if } x \ge 3 \\ -x+3, & \text{if } 2 \le x < 3 \end{cases}$

$\therefore g'(x) = \begin{cases} 1, & \text{if } x \ge 3 \\ -1, & \text{if } 2 \le x < 3 \end{cases}$

10. $f(x) = \frac{2\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x}{2\sin x}$

$= \frac{\sin 2x \cos 2x \cos 4x \cdot \cos 8x \cdot \cos 16x}{2\sin x} = \frac{\sin 32x}{2^5 \sin x}$

$\therefore \quad f'(x) = \frac{1}{32} \cdot \frac{32\cos 32x \cdot \sin x - \cos x \cdot \sin 32x}{\sin^2 x}$

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{32 \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times 0}{32\left[\frac{1}{\sqrt{2}}\right]^2} = \sqrt{2}$

11. $\frac{dy}{dx} = -[(2-x)(3-x)\ldots(n-x) + (1-x)(3-x)\ldots(n-x) + \ldots(1-x)(2-x)\ldots(n-1-x)]$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = -[(n-1)! + 0 + \ldots + 0]$

$= (-1)(n-1)!$

12. Since, $fog = I \Rightarrow fog(x) = x$ for all x

$\Rightarrow f'(g(x))g'(x) = 1$ for all x

$\Rightarrow f'(g(a)) = \frac{1}{g'(a)} = \frac{1}{2}$

$\Rightarrow f'(b) = \frac{1}{2}$ $[\because g(a) = b]$

13. $\frac{d}{dx}\{[f(x)]^2 - [\phi(x)]^2\} = 2[f(x)\cdot f'(x) - \phi(x)\cdot\phi'(x)]$

$= 2[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)]$

$= 0$ $[\because f'(x) = \phi(x)$ and $\phi'(x) = f(x)]$

$\Rightarrow [f(x)]^2 - [\phi(x)]^2 =$ Constant

$\therefore [f(3)]^2 - [\phi(3)]^2 = [f(3)]^2 - [f'(3)]^2$

$= 25 - 16 = 9$

14. $3f(x) - 2f(1/x) = x$ ...(i)

Let $1/x = y$, then

$3f(1/y) - 2f(y) = 1/y$

$\Rightarrow -2f(y) + 3f(1/y) = 1/y$

$\Rightarrow -2f(x) + 3f(1/x) = 1/x$ ...(ii)

On multiplying Eq. (i) by 3 and Eq. (ii) by 2 and adding, we get

$5f(x) = 3x + \frac{2}{x}$

$\Rightarrow f(x) = \frac{1}{5}\left(3x + \frac{2}{x}\right)$

$\Rightarrow f'(x) = \frac{1}{5}\left(3 - \frac{2}{x^2}\right)$

$\Rightarrow f'(2) = \frac{1}{5}\left(3 - \frac{2}{4}\right) = \frac{1}{2}$

15. We have, $f(x) = x^n \Rightarrow f(1) = 1 = {}^nC_0$

$\frac{f'(1)}{1!} = \frac{n}{1!} = {}^nC_1$

$\frac{f''(1)}{2!} = \frac{n(n-1)}{2!} = {}^nC_2$

$\frac{f'''(1)}{3!} = \frac{n(n-1)(n-2)}{3!} = {}^nC_3$

$\vdots \qquad \vdots$

$\frac{f^n(1)}{n!} = \frac{n!}{n!} = {}^nC_n$

$\therefore f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$

$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\,{}^nC_n$

$= (1-1)^n = 0$

16. $f(x) = (\cos x + i\sin x)(\cos 2x + i\sin 2x)(\cos 3x + i\sin 3x) \ldots (\cos nx + i\sin nx)$

$= \cos(x + 2x + 3x + \ldots + nx) + i\sin(x + 2x + 3x + \ldots + nx)$

$= \cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x$

$\Rightarrow f'(x) = \left[\frac{n(n+1)}{2}\right]\left[-\sin\frac{n(n+1)}{2}x + i\cos\frac{n(n+1)}{2}x\right]$

$f''(x) = -\left[\frac{n(n+1)}{2}\right]^2\left[\cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x\right]$

$= -\left[\frac{n(n+1)}{2}\right]^2\cdot f(x)$

$\therefore f''(1) = -\left[\frac{n(n+1)}{2}\right]^2 f(1) = -\left[\frac{n(n+1)}{2}\right]^2$

17. $f'''(x) = \begin{vmatrix} \frac{d^3}{dx^3}(x^3) & \frac{d^3}{dx^3}(\sin x) & \frac{d^3}{dx^3}(\cos x) \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$

$= \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$

$\therefore f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix} = 0,$

which is independent of P.

18. On putting $x = \sin A$ and $\sqrt{x} = \sin B$

$y = \sin^{-1}\left(\sin A\sqrt{1 - \sin^2 B} + \sin B\sqrt{1 - \sin^2 A}\right)$

$= \sin^{-1}(\sin A\cos A + \sin B\cos A)$

$= \sin^{-1}[\sin(A + B)]$

$= A + B = \sin^{-1} x + \sin^{-1}\sqrt{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2\sqrt{x - x^2}}$

19. $f(x) = \sqrt{1 + \cos^2(x^2)}$

$f'(x) = \frac{1}{2\sqrt{1 + \cos^2(x^2)}}(2\cos x^2)\cdot(-\sin x^2)\cdot(2x)$

$\Rightarrow f'(x) = \frac{-x\sin 2x^2}{\sqrt{1 + \cos^2(x^2)}}$

At $x = \frac{\sqrt{\pi}}{2}$, $f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{-\frac{\sqrt{\pi}}{2}\cdot\sin\frac{2\pi}{4}}{\sqrt{1 + \cos^2\frac{\pi}{4}}} = \frac{-\frac{\sqrt{\pi}}{2}\cdot 1}{\sqrt{\frac{3}{2}}}$

$\therefore f'\left(\frac{\sqrt{\pi}}{2}\right) = -\sqrt{\frac{\pi}{6}}$

20. $\because$ $y = \sec^{-1}\left[\frac{\sqrt{x}+1}{\sqrt{x}-1}\right] + \sin^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right]$

$= \cos^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right] + \sin^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right] = \frac{\pi}{2}$

$\Rightarrow \frac{dy}{dx} = 0 \quad \left(\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right)$

21. Since, $f'(x) > g'(x)$

$\Rightarrow \left(\frac{1}{2}\right)5^{2x+1}\log_e 5 \times 2 > 5^x \log_e 5 + 4\log_e 5$

$\Rightarrow 5^{2x}\cdot 5 > 5^x + 4$

$\Rightarrow 5\cdot 5^{2x} - 5^x - 4 > 0$

$\Rightarrow (5^x - 1)(5\cdot 5^x + 4) > 0$

$\therefore 5^x > 1$

$\Rightarrow x > 0$

22. Since, $f(x)$ is odd.

$\therefore f(-x) = -f(x)$

$\Rightarrow f'(-x)(-1) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$

$\therefore f'(-3) = f'(3) = -2$

23. We have,

$\frac{dx}{dt} = a\left[-\sin t\sqrt{\cos 2t} - \frac{\cos t\cdot\sin 2t}{\sqrt{\cos 2t}}\right] = \frac{-a\sin 3t}{\sqrt{\cos 2t}}$

and $\frac{dy}{dt} = a\left[\cos t\sqrt{\cos 2t} - \frac{\sin t\cdot\sin 2t}{\sqrt{\cos 2t}}\right] = \frac{a\cos 3t}{\sqrt{\cos 2t}}$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\cot 3t$

$\Rightarrow \frac{d^2y}{dx^2} = 3\operatorname{cosec}^2 3t\cdot\frac{dt}{dx}$

$= \frac{-3\operatorname{cosec}^2 3t\cdot\sqrt{\cos 2t}}{a\sin 3t}$

$= -\left(\frac{3}{a}\right)\operatorname{cosec}^3 3t\cdot\sqrt{\cos 2t}$

$\therefore \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \bigg/ \frac{d^2y}{dx^2}$

$= (1+\cot^2 3t)^{3/2} \bigg/ \left(\frac{-3}{a}\right)\operatorname{cosec}^3 3t\sqrt{\cos 2t}$

$\Rightarrow \left|\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \bigg/ \frac{d^2y}{dx^2}\right|$ at $t = \frac{\pi}{6}$ is $\frac{a}{3\sqrt{\cos\frac{\pi}{3}}} = \frac{\sqrt{2}a}{3}$

24. When $x = 0$, $y > 0$

$\Rightarrow y = ae^{\pi/2}$

Taking log on both sides of the given equation, we get

$\frac{1}{2}\log(x^2+y^2) = \log a + \tan^{-1}\left(\frac{y}{x}\right)$

On differentiating both the sides w.r.t. x, we get

$\frac{1}{2}\times\frac{2x+2yy'}{x^2+y^2} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{xy'-y}{x^2}$

$\Rightarrow x + yy' = xy' - y \quad \ldots$(i)

On differentiating again w.r.t. x, we get

$1 + (y')^2 + yy'' = xy'' + y' - y'$

$\Rightarrow 1 + (y')^2 = (x-y)y''$

$\Rightarrow y'' = \frac{1+(y')^2}{x-y}$

When $x = 0$, we get from Eq. (i), $y' = -1$

$\Rightarrow y''(0) = \frac{2}{-ae^{\pi/2}} = \frac{-2}{a}e^{-\pi/2}$

25. Given, $u = f(\tan x) \Rightarrow \frac{du}{dx} = f'(\tan x)\sec^2 x$

and $v = g(\sec x)$

$\Rightarrow \frac{dv}{dx} = g'(\sec x)\sec x\tan x$

$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{f'(\tan x)}{g'(\sec x)}\cdot\frac{1}{\sin x}$

$\therefore \left(\frac{du}{dv}\right)_{x=\pi/4} = \frac{f'(1)}{g'(\sqrt{2})}\cdot\sqrt{2} = \frac{2}{4}\cdot\sqrt{2} = \frac{1}{\sqrt{2}}$

26. For $x < 0, \frac{d}{dx}(\ln|x|) = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)}(-1) = \frac{1}{x}$

27. For $0 < x < 1, \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\therefore \frac{du}{dv} = 1 \quad (\because u = v)$

28. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

and $u = (\sin x)^x$, $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Now, $u = (\sin x)^x$

$\log u = x\log(\sin x)$

$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}(x)$

$= \frac{x}{\sin x}\cos x + \log(\sin x)$

$\frac{du}{dx} = u[x\cot x + \log\sin x] = (\sin x)^x[x\cot x + \log\sin x]$

Again, $v = \sin^{-1}\sqrt{x}$

$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(x^{1/2}) = \frac{1}{\sqrt{1-x}}\frac{1}{2}x^{-1/2}$ (using chain rule)

$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} \Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}}$

$\therefore \frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$

29. Here, $f(x) = |x|^3$

When $x \geq 0$, $f(x) = |x|^3 = x^3$

On differentiating both sides w.r.t.x, we get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^3) \Rightarrow f'(x) = 3x^2$$

Again, differentiating both sides w.r.t.x, we get

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(3x^2) \Rightarrow f''(x) = 6x$$

When $x < 0$, then

$$f(x) = |x|^3 = -x^3$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(-x^3)$$

$$\Rightarrow \quad f'(x) = -3x^2$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(-3x^2) = -6x$$

$$\frac{d}{dx} f''(x) = -6x$$

Hence, $f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$

30. Given, $\quad y = (1-x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$

or $\quad y = \dfrac{(1-x^2)(1+x^2)\ldots(1+x^{2n})}{(1+x)}$

$$= \frac{1-(x)^{4n}}{(1+x)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+x)\cdot(0-4n\cdot x^{4n-1})-(1-x^{4n})\cdot 1}{(1+x)^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = -1$$

31. Given, $y = |\sin x|^{|x|}$

In the neighbourhood of $-\dfrac{\pi}{6}$, $|x|$ and $|\sin x|$ both are negative *i.e.*,

$y = (-\sin x)^{(-x)}$

Taking log on both sides, we get

$$\log y = (-x)\cdot\log(-\sin x)$$

On differentiating both sides, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (-x)\left(\frac{1}{-\sin x}\right)\cdot(-\cos x) + \log(-\sin x)\cdot(-1)$$

$$= -x\cot x - \log(-\sin x)$$

$$= -[x\cot x + \log(-\sin x)]$$

$$\Rightarrow \quad \frac{dy}{dx} = -y\,[x\cot x + \log(-\sin x)]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=-\frac{\pi}{6}} = \frac{(2)^{\frac{-\pi}{6}}}{6}[6\log 2 - \sqrt{3}\pi]$$

32. Given, $\dfrac{d}{dx}\{f'(x)\} = -f(x)$

$$\Rightarrow \quad g'(x) = -f(x) \quad [\because g(x) = f'(x), \text{ given}]$$

Also, given $F(x) = \left\{f\left(\dfrac{x}{2}\right)\right\}^2 + \left\{g\left(\dfrac{x}{2}\right)\right\}^2$

$$\Rightarrow \quad F'(x) = 2\left\{f\left(\frac{x}{2}\right)\right\} f'\left(\frac{x}{2}\right)\cdot\frac{1}{2} + 2\left\{g\left(\frac{x}{2}\right)\right\}\cdot g'\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$= 0$$

Hence, $f(x)$ is constant. Therefore, $F(10) = 5$

33. Let $\quad y = \dfrac{x^2 - x}{x^2 + 2x}$

$$\Rightarrow \quad x = \frac{2y+1}{-y+1};\ x \neq 0$$

$$\Rightarrow \quad f^{-1}(x) = \frac{2x+1}{-x+1}$$

$$\therefore \quad \frac{d}{dx}\{f^{-1}(x) = \frac{(-x+1)\cdot 2 - (2x+1)(-1)}{(-x+1)^2}$$

$$= \frac{3}{(-x+1)^2}$$

34. $\because \quad \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{a^{\sin^{-1}t}}}\left(a^{\sin^{-1}t} \times \dfrac{1}{\sqrt{1-t^2}}\right)$

and $\quad \dfrac{dy}{dt} = \dfrac{1}{2\sqrt{a^{\cos^{-1}t}}}\left(a^{\cos^{-1}t} \times \dfrac{-1}{\sqrt{1-t^2}}\right)$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{a^{\sin^{-1}t}}}{\sqrt{a^{\cos^{-1}t}}}\left(\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} \times 1\right) = -\frac{a^{\sqrt{\cos^{-1}t}}}{a^{\sqrt{\sin^{-1}t}}}$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}}$$

$$= 1 + \frac{y^2}{x^2}$$

$$= \frac{x^2 + y^2}{x^2}$$

35. $\quad f(x) = \sin(\sin x)$

$$\Rightarrow \quad f'(x) = \cos\cdot\cos(\sin x)$$

$$\Rightarrow \quad f''(x) = -\sin x\cdot\cos(\sin x) - \cos^2 x\cdot\sin(\sin x)$$

Now, $g(x) = -[f''(x) + f'(x)\cdot\tan x]$

$$= \sin x\cdot\cos(\sin x) + \cos^2 x\cdot\sin(\sin x) - \tan x\cdot\cos x\cdot\cos(\sin x)$$

$$= \sin x\cdot\cos(\sin x) + \cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$$

$$= \cos^2 x\cdot\sin(\sin x)$$

36. Given, $\quad y = \sec(\tan^{-1} x)$

Let $\quad \tan^{-1} x = \theta$

$$\Rightarrow \quad x = \tan\theta$$

$$\therefore \quad y = \sec\theta = \sqrt{1+x^2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1+x^2}}\cdot 2x$$

At $x=1$,

$$\frac{dy}{dx}=\frac{1}{\sqrt{2}}$$

37. Since, $\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}$

$$\Rightarrow \frac{d^2x}{dy^2}=-\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\cdot\frac{dx}{dy}$$

$$=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

38. $x^{2x}-2x^x\cot y-1=0$ ...(i)

Now, $x=1$, $1-2\cot y-1=0$

$$\Rightarrow \cot y=0$$

$$\Rightarrow y=\frac{\pi}{2}$$

On differentiating Eq. (i) w.r.t. x, we get

$$2x^{2x}(1+\log x)-2\left[x^x(-\text{cosec}^2 y)\frac{dy}{dx}+\cot y\, x^x(1+\log x)\right]=0$$

At $\left(1,\frac{\pi}{2}\right)$,

$$2(1+\log 1)-2\left\{1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}+0\right\}=0$$

$$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}=0$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}=-1$$

39. $f(x)=x|x|$ and $g(x)=\sin x$

$$gof(x)=\sin(x|x|)=\begin{cases}-\sin x^2, & x<0\\ \sin x^2, & x>0\end{cases}$$

$$\therefore (gof)'(x)=\begin{cases}-2x\cos x^2, & x<0\\ 2x\cos x^2, & x>0\end{cases}$$

Clearly, $L\,(gof)'(0)=0=R\,(gof)'(0)$

Hence, gof is differentiable at $x=0$ and also its derivative is continuous at $x=0$.

Now, $(gof)''(x)=\begin{cases}-2\cos x^2+4x^2\sin x^2, & x<0\\ 2\cos x^2-4x^2\sin x^2, & x\geq 0\end{cases}$

$\therefore$ $L\,(gof)''(0)=-2$ and $R\,(gof)''(0)=2$

$\therefore$ $L\,(gof)''(0)\neq R\,(gof)''(0)$

Hence, $gof\,(x)$ is not twice differentiable at $x=0$.

40. Taking log on the given equation, we get

$$m\log x+n\log y=(m+n)\log(x+y)$$

$$\Rightarrow \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{(m+n)}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{my+ny-nx-ny}{y(x+y)}\right\}=\frac{mx+my-mx-nx}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

41. $\frac{d}{dx}(y)=n\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$

$$\Rightarrow \left(\sqrt{1+x^2}\right)\frac{dy}{dx}=n\left(x+\sqrt{1+x^2}\right)^n$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(\sqrt{1+x^2}\right)+\frac{dy}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right)=n^2\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{\frac{d^2y}{dx^2}(1+x^2)+\frac{dy}{dx}\cdot x}{\sqrt{1+x^2}}=\frac{n^2\left(x+\sqrt{1+x^2}\right)^n}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2\left(x+\sqrt{1+x^2}\right)^n$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

Day 13

Applications of Derivatives

Day 13 Outlines ...

- Derivatives as the Rate of Change
- Tangent and Normal of a Curve
- Angle of Intersection of Two Curves
- Increasing and Decreasing Function
- Rolle's Theorem
- Lagrange's Mean Value Theorem

Derivatives as the Rate of Change

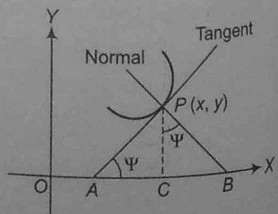
If a variable quantity y is some function of time t *i.e.*, $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change $= \frac{\Delta y}{\Delta t}$, when limit $\Delta t \to 0$ is applied the rate of change becomes instantaneous and we get the rate of change with respect to at the instant x, *i.e.*, $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$

- *Derivative of an even function is always an odd function and derivative of an odd function is always an even function.*
- $\frac{dy}{dx}$ *is nothing but the rate of change of y relative to x.*

Tangent and Normal of a Curve

1. If a **tangent** is drawn to the curve $y = f(x)$ at a point $P(x_1, y_1)$ and this tangent makes an angle ψ with positive x-direction, then



(i) The slope of the tangent is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan\psi$$

(ii) Equation of tangent is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

(iii) Length of tangent $PA = y \operatorname{cosec} \psi = \left| y \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} \right|$

(iv) Length of subtangent $AC = y \cot \psi = \left| \frac{y}{dy/dx} \right|$

2. The normal to a curve at a point $P(x_1, y_1)$ is a line perpendicular to tangent at P and passing through P, then

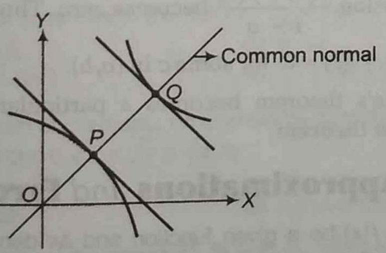
(i) The slope of the normal is $-\frac{1}{\frac{dy}{dx}}$

(ii) Equation of normal is $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$

(iii) Length of normal $PB = y \sec \psi = \left| y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right|$

(iv) Length of subnormal $BC = y \tan \psi = \left| y \left(\frac{dy}{dx}\right) \right|$

» *The shortest distance between two curves lie along the normal i. e., at the points where, tangents are parallel to each other. In a figure PQ is a shortest distance.*



» *If at any point on the curve, the subtangent and subnormal are equal, then length of tangent is $\sqrt{2}$ time the ordinate.*

» *If tangent and normal is parallel to X-axis.*

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

and $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$, *respectively.*

» *If tangent and normal is parallel to Y-axis.*

$\Rightarrow \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$

and $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$, *respectively.*

Angle of Intersection of Two Curves

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection. Thus, the angle between the tangents of the two curves $y = f_1(x)$ and $y = f_2(x)$ is given by

$$\tan \phi = \left| \frac{\left(\frac{dy}{dx}\right)_{\mathrm{I}(x_1, y_1)} - \left(\frac{dy}{dx}\right)_{\mathrm{II}(x_1, y_1)}}{1 + \left(\frac{dy}{dx}\right)_{\mathrm{I}(x_1, y_1)} \left(\frac{dy}{dx}\right)_{\mathrm{II}(x_1, y_1)}} \right| \quad \text{or} \quad \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Orthogonal Curves

If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally. The curves are called orthogonal curves.

If $\phi = \frac{\pi}{2}; m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{\mathrm{I}} \left(\frac{dy}{dx}\right)_{\mathrm{II}} = -1$

Increasing *and* Decreasing Function

A function f is said to be an increasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \forall x_1, x_2 \in]a, b[$.

A function f is said to be a decreasing function in $]a, b[$, if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \forall x_1, x_2 \in]a, b[$.

$f(x)$ is known as **non-decreasing**, if $f'(x) \ge 0$ and **non-increasing**, if $f'(x) \le 0$. e.g., $f(x) = \tan x, - x \ \forall x \in R$ is always increasing function in their domain.

Strictly Increasing and Strictly Decreasing Function

A function $f(x)$ is known as **strictly increasing** function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

A function $f(x)$ is known as **strictly decreasing** function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Hence, $f(x)$ is known as strictly increasing, if $f'(x) > 0$ and strictly decreasing, if $f'(x) < 0$.

» *A function $f(x)$ is said to be increasing (decreasing) at point x_0, if there is an interval $(x_0 - h, x_0 + h)$ containing x_0, such that $f(x)$ is increasing (decreasing) on $(x_0 - h, x_0 + h)$.*

» *A function $f(x)$ is said to be increasing on $[a, b]$, if it is increasing (decreasing) on (a, b) and it is also increasing at $x = a$ and $x = b$.*

Monotonic Function

A function f is said to be monotonic in an interval, if is either increasing or decreasing in that interval.

We summarize the result as shown below

$f'(a_1)$	$f''(a_1)$	$f'''(a_1)$	Behaviour of f at a_1
+			Increasing
−			Decreasing
0	+		Minimum
0	−		Maximum
0	0	±	Inflection
0	0	0	Find further derivative

Blank space indicates that the function may have any value at a_1.

Results on Monotonic Function

1. If $f(x)$ is a strictly increasing function on an interval $[a,b]$, then f^{-1} exists and it is also a strictly increasing function.
2. If $f(x)$ is strictly increasing function on an interval $[a,b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.
3. If $f(x)$ is continuous on $[a,b]$ such that $f'(c) \geq 0$ $[f'(c) > 0]$ for each $c \in (a,b)$, then $f(x)$ is monotonically increasing on $[a,b]$.
4. If $f(x)$ is continuous on $[a,b]$ such that $f'(c) \leq 0$ $[f'(c) < 0]$ for each $c \in [a,b]$, then $f(x)$ is monotonically decreasing function on $[a,b]$.
5. If $f(0) = 0$ and $f'(x) \geq 0$, $\forall\, x \in R$, then $f(x) \leq 0, \forall\, x \in (-\infty, 0)$ and $f(x) \geq 0, \forall\, x \in (0, \infty)$.
6. If $f(0) = 0$ and $f'(x) \leq 0, \forall\, x \in R$, then $f(x) \geq 0$, $\forall\, x \in (-\infty, 0)$ and $f(x) \leq 0, \forall\, x \in (0, \infty)$.

Monotonic function have atmost one root.

Rolle's Theorem

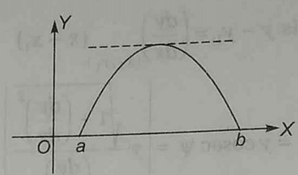
Let f be a real-valued function defined in the closed interval $[a,b]$, such that

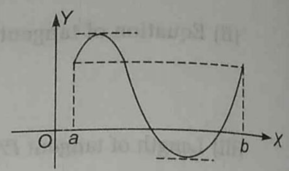
(i) $f(x)$ is continuous in the closed interval $[a,b]$.

(ii) $f(x)$ is differentiable in the open interval (a,b).

(iii) $f(a) = f(b)$, then there is some point c in the open interval (a,b), such that $f'(c) = 0$.

Geometrically Under the assumptions of Rolle's theorem, the graph of $f(x)$ starts at point $(a, 0)$ and ends at point $(b, 0)$ as shown in figures





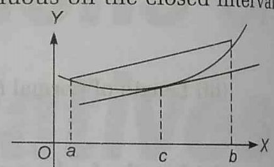
The conclusion is that there is atleast one point c between a and b, such that the tangent to the graph at $\{c, f(c)\}$ is parallel to the X-axis.

Algebraic Interpretation of Rolle's Theorem

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem

Let f be a real function, continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b). Then, there is atleast one point c in the open interval (a,b), such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Geometrically Any chord of the curve $y = f(x)$, there is a point on the graph, where the tangent is parallel to this chord.

Remarks In the particular case, where $f(a) = f(b)$.

The expression $\dfrac{f(b) - f(a)}{b - a}$ becomes zero. Thus, when $f(a) = f(b), f'(c) = 0$ for some c in (a,b).

Thus, Rolle's theorem becomes a particular case of the **mean value theorem**.

Approximations and Errors

- Let $y = f(x)$ be a given function and Δx denotes a small increment in x, corresponding which y increases by Δy. Then, for small increments, we assume that $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$ (symbol ≈ stands for "approximately equal to")

  $\therefore \Delta y = \dfrac{dy}{dx} \Delta x$. Thus, $y + \Delta y = f(x + \Delta x) = f(x) + \left(\dfrac{dy}{dx}\right) \Delta x$

- Let Δx be the error in the measurement of independent variable x and Δy is corresponding error in the measurement of dependent variable y. Then, $\Delta y = \left(\dfrac{dy}{dx}\right) \Delta x$

  where, Δy = Absolute error in measurement of y,

  $\dfrac{\Delta y}{y}$ = Relative error in measurement of y

  $\dfrac{\Delta y}{y} \times 100$ = Percentage error in measurement of y

Practice Zone

DAY 13

1. An angle θ, $0<\theta<\frac{\pi}{2}$, which increases twice as fast as its sine, is **[NCERT Exemplar]**

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

2. If the curve $y=a^x$ and $y=b^x$ intersect at angle α, then $\tan\alpha$ is equal to

(a) $\frac{a-b}{1+ab}$ (b) $\frac{\log a-\log b}{1+\log a\log b}$

(c) $\frac{a+b}{1-ab}$ (d) $\frac{\log a+\log b}{1-\log a\log b}$

3. If the normal to the curve $y^2=5x-1$ at the point $(1,-2)$ is of the form $ax-5y+b=0$, then a and b are

(a) $4,-14$ (b) $4,14$

(c) $-4,14$ (d) $4,2$

4. Coordinates of a point of the curve $y=x\log x$ at which the normal is parallel to the line $2x-2y=3$ are

(a) $(0,0)$ (b) (e,e)

(c) $(e^2,2e^2)$ (d) $(e^{-2},-2e^{-2})$

5. The sum of intercepts on coordinate axes made by tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, is

(a) a (b) $2a$

(c) $2\sqrt{a}$ (d) None of these

6. Line joining the points (0, 3) and (5, −2) is a tangent to the curve $y=\frac{ax}{1+x}$, then

(a) $a=1\pm\sqrt{3}$ (b) $a=\phi$

(c) $a=-1\pm\sqrt{3}$ (d) $a=-2\pm2\sqrt{3}$

7. The function $f(x)=(x-3)^2$ satisfies all the conditions of mean value theorem in $[3,4]$. A point on $y=(x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is

(a) $\left(\frac{7}{2},\frac{1}{2}\right)$ (b) $\left(\frac{7}{2},\frac{1}{4}\right)$

(c) $(1,4)$ (d) $(4,1)$

8. In the mean value theorem, $f(b)-f(a)=(b-a)f'(c)$, if $a=4$, $b=9$ and $f(x)=\sqrt{x}$, then the value of c is

(a) 8.00 (b) 5.25 (c) 4.00 (d) 6.25

9. The function $f(x)=\cos x-2px$ is monotonically decreasing for

(a) $p<\frac{1}{2}$ (b) $p>\frac{1}{2}$ (c) $p<2$ (d) $p>2$

10. The abscissa of the points of the curve $y=x^3$ in the interval $[-2,2]$, where the slope of the tangents can be obtained by mean value theorem for the interval $[-2,2]$, are

(a) $\pm\frac{2}{\sqrt{3}}$ (b) $+\sqrt{3}$ (c) $\pm\frac{\sqrt{3}}{2}$ (d) 0

11. The equation of tangents to the curve

$$y=\cos(x+y),\ -2\pi\le x\le 2\pi$$

that are parallel to the line $x+2y=0$ is/are **[NCERT]**

(a) $3x+4y+3\pi=0,\ 2x+4y+\pi=0$

(b) $2x+4y+3\pi=0,\ 2x+4y-\pi=0$

(c) $x+2y+2\pi=0,\ x+y+\pi=0$

(d) None of the above

12. The values of a for which the function $(a+2)x^3-3ax^2+9ax-1=0$ decreases monotonically throughout for all real x, are

(a) $a<-2$ (b) $a>-2$

(c) $-3<a<0$ (d) $-\infty<a\le-3$

13. If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$, where $0<x\le1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing functions

(b) both $f(x)$ and $g(x)$ are decreasing functions

(c) $f(x)$ is an increasing function

(d) $g(x)$ is an increasing function

14. If $f(x)=x^3+bx^2+cx+d$ and $0<b^2<c$, then in $(-\infty,\infty)$

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is a strictly decreasing function

(d) $f(x)$ is unbounded

15. $f(x)=\int_0^x|\log_2[\log_3\{\log_4(\cos t+a)\}]|\,dt$. If $f(x)$ is increasing for all real values of x, then

(a) $a\in(-1,1)$ (b) $a\in(1,5)$

(c) $a\in(1,\infty)$ (d) $a\in(5,\infty)$

16. The parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect orthogonally at point $P(x_1, y_1)$, where $x_1, y_1 \neq 0$, then
(a) $b = a^2$
(b) $b = a^3$
(c) $b^3 = a^2$
(d) None of these

17. $y = f(x)$ is a parabola, having its axis parallel to Y-axis. If the line $y = x$ touches this parabola at $x = 1$, then
(a) $f''(1) + f'(0) = 1$
(b) $f''(0) - f'(1) = 1$
(c) $f''(1) - f'(0) = 1$
(d) $f''(0) + f'(1) = 1$

18. If $f'(x) > 0$ and $f''(x) > 0, \forall x \in R$, then for any two real numbers x_1 and $x_2, (x_1 \neq x_2)$
(a) $f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$
(b) $f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$
(c) $f'\left(\frac{x_1 + x_2}{2}\right) > \frac{f'(x_1) + f'(x_2)}{2}$
(d) $f'\left(\frac{x_1 + x_2}{2}\right) < \frac{f'(x_1) + f'(x_2)}{2}$

19. If $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in
(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(b) $\left(0, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$
(d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

20. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/s. How fast is the area decreasing when the two equal sides are equal to the base? **[NCERT]**
(a) $3b$ cm²/s
(b) $\sqrt{2}b$ cm²/s
(c) $\frac{b}{\sqrt{2}}$ cm²/s
(d) $\sqrt{3}b$ cm²/s

21. The function $\frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ is decreasing, if
(a) $ad - bc > 0$
(b) $ad - bc < 0$
(c) $ab - cd > 0$
(d) $0 \leq x \leq -2$

22. A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m? **[NCERT Exemplar]**
(a) 8 m/s
(b) 12 m/s
(c) 16 m/s
(d) 19 m/s

23. The function $f(x) = x(x + 3)e^{-(1/2)x}$ satisfies all the conditions of Rolle's theorem in $[-3, 0]$. The value of c is
(a) 0
(b) -1
(c) -2
(d) -3

24. The tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at
(a) $(6, 7)$
(b) $(-6, 7)$
(c) $(6, -7)$
(d) $(-6, -7)$

25. If $f(x)$ be a monotonic polynomial of $2m - 1$ degree, where $m \in N$, then the equation $[f(x) + f(3x) + f(5x) + \ldots + f(2m - 1)x] = 2m - 1$ has
(a) atleast one real root
(b) $2m$ roots
(c) exactly one real root
(d) $(2m + 1)$ roots

26. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(a) $\frac{\pi}{3}$
(b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$
(d) π

27. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
(a) increasing on $\left[-\frac{1}{2}, 1\right]$
(b) decreasing on R
(c) increasing on R
(d) decreasing on $\left[-\frac{1}{2}, 1\right]$

28. If $f(x) = \int e^x (x - 1)(x - 2)\, dx$, then f decreases in the interval
(a) $(-\infty, -2)$
(b) $(-2, -1)$
(c) $(1, 2)$
(d) $(2, \infty)$

29. If $f(x)$ satisfy all the conditions of mean value theorem in $[0, 2]$. If $f(0) = 0$ and $|f'(x)| \leq \frac{1}{2}$ for all x in $[0, 2]$, then
(a) $f(x) < 2$
(b) $|f(x)| \leq 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in $[0, 2]$

30. If $a + b = 4$, $a < 2$ and $g(x)$ be a monotonically increasing function of x. Then, $f(x) = \int_0^a g(x)dx + \int_0^b g(x)dx$
(a) increases with increase in $(b - a)$
(b) decreases with increase in $(b - a)$
(c) increases with decrease in $(b - a)$
(d) None of the above

31. The function which is neither decreasing nor increasing in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is
(a) $\operatorname{cosec} x$
(b) $\tan x$
(c) x^2
(d) $|x - 1|$

Directions (Q. Nos. 32 to 34) *To find the point of contact $P \equiv (x_1, y_1)$ of a tangent to the graph of $y = f(x)$ passing through origin O, we equate the slope of tangent to $y = f(x)$ at P to the slope of OP. Hence, we solve the equation $f'(x_1) = \frac{f(x_1)}{x_1}$ to get x_1 and y_1.*

32. The equation $|\log mx| = px$, where m is a positive constant has a single root for
(a) $0 < p < \frac{m}{e}$
(b) $p < \frac{e}{m}$
(c) $0 < p < \frac{e}{m}$
(d) $p > \frac{m}{e}$

33. The equation $|\log mx| = px$, where m is a positive constant have exactly two roots for

(a) $p = \frac{m}{e}$ (b) $p = \frac{e}{m}$ (c) $0 < p \le \frac{e}{m}$ (d) $0 < p \le \frac{m}{e}$

34. The equation $|\log mx| = px$, where m is a positive constant have exactly three roots for

(a) $p < \frac{m}{e}$ (b) $0 < p < \frac{m}{e}$

(c) $0 < p < \frac{e}{m}$ (d) $p < \frac{e}{m}$

Directions (Q. Nos. 35 and 36) *Consider the function* $f : (-\infty, \infty) \to (-\infty, \infty)$ *defined by*

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1};\ 0 < a < 2.$$

35. Which of the following is true?

(a) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(b) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(c) $f'(1) f'(-1) = (2 - a)^2$

(d) $f'(1) f'(-1) = -(2 + a)^2$

36. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}\, dt$. Which of the following is true?

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(c) $g'(x)$ change sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ does not change sign on $(-\infty, \infty)$

Directions (Q. Nos. 37 to 42) *Each of these questions contains two statements : Statement I (Assertion) and Statement II(Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

37. Statement I If Rolle's theorem be applied in $f(x)$, then Lagrange Mean Value Theorem (LMVT) is also applied in $f(x)$.

Statement II Both Rolle's theorem and LMVT cannot be applied in $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

38. Consider the ordinate of a point describing the circle $x^2 + y^2 = 25$ decreases at the rate of 1.5 cm/s.

Statement I Then the rate of change of the abscissa of the point when ordinate equals 4 cm is 2 cm/s.

Statement II $x\,dx + y\,dx = 0$.

39. Statement I The points on the curve $y^2 = x + \sin x$ at which the tangent is parallel to x-axis lies on a straight line.

Statement II Tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$ or $\frac{dx}{dy} = \infty$.

40. Statement I The ratio of length of tangent to length of normal is inversely proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Statement II Length of normal and tangent to a curve $y = f(x)$ is $|y\sqrt{1+m^2}|$ and $\left|\frac{y\sqrt{1+m^2}}{m}\right|$, where $m = \frac{dy}{dx}$.

41. If $g(x)$ is a differentiable function $g(1) \ne 0, g(-1) \ne 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2-1}{g(x)}$ in $[-1,1]$, then

Statement I $g(x)$ has atleast one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$, Rolle's theorem is applicable for $x \in (a, b)$.

42. Statement I Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

43. Equation of normal to the curve $y = \cos^2 x$ at point $\left(\frac{\pi}{4}, 0\right)$ is

(a) $x - y = \frac{\pi}{4}$ (b) $x + y = \frac{\pi}{2}$

(c) $x + y = \frac{\pi}{4}$ (d) None of these

44. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$

(a) on the left of $x = c$

(b) on the right of $x = c$

(c) at no point

(d) at all points

AIEEE & JEE Main Archive

45. **Statement I** The equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between I and II.

Statement II The function $f(x) = x \log x$ is an increasing function in $[1, 2]$ and $g(x) = 2 - x$ is a decreasing function in $[1, 2]$ and the graphs represented by these functions intersect at a point in $[1, 2]$. [JEE Main 2013]

(a) Statement I is true; Statement II is false.
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is false; Statement II is true.

46. If the surface area of a sphere of radius r is increasing uniformly at the rate $8\text{ cm}^2/\text{s}$, then the rate of change of its volume is [JEE Main 2013]

(a) constant (b) proportional to $\sqrt{r}$
(c) proportional to r^2 (d) proportional to r

47. **Statement I** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.

Statement II The function x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b). [JEE Main 2013]

(a) Statement I is true; Statement II is false.
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I.
(d) Statement I is false; Statement II is true.

48. If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then a value of α is [JEE Main 2013]

(a) 2 (b) $\frac{4}{3}$
(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

49. A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in the surface area (in cm^2/min) of the balloon when its diameter is 14 cm, is [JEE Main 2013]

(a) 10 (b) $\sqrt{10}$
(c) 100 (d) $10\sqrt{10}$

50. If an equation of a tangent to the curve, $y = \cos(x + y)$, where $-1 \le x \le 1 + \pi$, is $x + 2y = k$, then k is equal to [JEE Main 2013]

(a) 1 (b) 2
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

51. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employees 25 more workers, then the new level of production of items is [JEE Main 2013]

(a) 2500 (b) 3000
(c) 3500 (d) 4500

52. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to [JEE Main 2013]

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

53. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$ [JEE Main 2013]

(a) lies between 1 and 2
(b) lies between 2 and 3
(c) lies between -1 and 0
(d) Does not exist

54. A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began is [AIEEE 2012]

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$

55. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X-axis, is [AIEEE 2010]

(a) $y = 0$ (b) $y = 1$ (c) $y = 2$ (d) $y = 3$

56. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? [AIEEE 2008]

(a) 5 (b) 7 (c) 1 (d) 3

57. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is [AIEEE 2007]

(a) $2\log_3 e$ (b) $\frac{1}{2}\log_e 3$
(c) $\log_3 e$ (d) $\log_e 3$

58. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in [AIEEE 2007]

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

59. A spherical iron ball 10 cm in radius is coated with a layer ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice 15 cm, then the rate at which the thickness of ice decreases, is **[AIEEE 2005]**

(a) $\frac{5}{6\pi}$ cm/min (b) $\frac{1}{54\pi}$ cm/min
(c) $\frac{1}{18\pi}$ cm/min (d) $\frac{1}{36\pi}$ cm/min

60. Let $f(a) = g(a) = k$ and their nth derivatives $f^n(a), g^n(a)$ exist and are not equal for some n.
Further, if $\lim_{x \to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4$,
then the value of k is equal to **[AIEEE 2003]**

(a) 4 (b) 2
(c) 1 (d) 0

Answers

1. (d)	2. (b)	3. (a)	4. (d)	5. (a)	6. (b)	7. (b)	8. (d)	9. (b)	10. (a)
11. (b)	12. (d)	13. (c)	14. (a)	15. (d)	16. (d)	17. (a)	18. (b)	19. (b)	20. (d)
21. (b)	22. (a)	23. (c)	24. (d)	25. (a)	26. (a)	27. (a)	28. (c)	29. (b)	30. (a)
31. (a)	32. (d)	33. (a)	34. (b)	35. (a)	36. (d)	37. (b)	38. (a)	39. (d)	40. (a)
41. (c)	42. (d)	43. (a)	44. (a)	45. (a)	46. (d)	47. (c)	48. (b)	49. (a)	50. (d)
51. (c)	52. (a)	53. (d)	54. (c)	55. (d)	56. (c)	57. (a)	58. (b)	59. (c)	60. (a)

Hints & Solutions

1. Given, $2\frac{d}{dt}(\sin\theta) = \frac{d\theta}{dt} \Rightarrow 2 \times \cos\theta \frac{d\theta}{dt} = \frac{d\theta}{dt}$

$\Rightarrow \quad 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

2. Clearly, the point of intersection of curves is (0, 1).

Now, slope of tangent of first curve,

$$m_1 = \frac{dy}{dx} = a^x \log a$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$$

Slope of tangent of second curve,

$$m_2 = \frac{dy}{dx} = b^x \log b$$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$$

$$\therefore \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}$$

3. We have, $y^2 = 5x - 1$ ...(i)

At (1, – 2); $\frac{dy}{dx} = \left(\frac{5}{2y}\right)_{(1,-2)} = \frac{-5}{4}$

∴ Equation of normal at the point (1, – 2) is

$$[y - (-2)]\left(\frac{-5}{4}\right) + x - 1 = 0$$

$$\therefore 4x - 5y - 14 = 0 \quad \text{...(ii)}$$

As the normal is of the form

$$ax - 5y + b = 0$$

On comparing this with Eq. (ii), we get

$a = 4$ and $b = -14$

4. $y = x\log x \Rightarrow \frac{dy}{dx} = 1 + \log x$

The slope of the normal $= -\frac{1}{(dy/dx)} = \frac{-1}{1 + \log x}$

The slope of the line $2x - 2y = 3$ is 1.

$$\therefore \frac{-1}{1 + \log x} = 1$$

$$\Rightarrow \log x = -2$$

$$\Rightarrow x = e^{-2}$$

$$\therefore y = -2e^{-2}$$

So, the coordinate of the point is $(e^{-2}, -2e^{-2})$.

5. $\sqrt{x} + \sqrt{y} = \sqrt{a}$; $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Hence, tangent at (x, y) is

$$Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$$

$$\Rightarrow X(\sqrt{y} + Y\sqrt{x}) = \sqrt{xy}(\sqrt{x} + \sqrt{y})\sqrt{axy}$$

$$\Rightarrow \frac{X}{\sqrt{a}\sqrt{x}} + \frac{y}{\sqrt{a}\sqrt{y}} = 1$$

Clearly, its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of intercepts $= \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a} \cdot \sqrt{a} = a$

6. Equation of line joining the points (0, 3) and (5, –2) is $y = 3 - x$. If this line is tangent to $y = \frac{ax}{(x+1)}$, then $(3 - x)(x + 1) = ax$ should have equal roots. Thus, $(a - 2)^2 + 12 = 0 \Rightarrow$ no value of $a \Rightarrow a \in \phi$.

7. Let the point be (x_1, y_1). Therefore, $y_1 = (x_1 - 3)^2$ ...(i)

Now, slope at the tangent at (x_1, y_1) is $2(x_1 - 3)$ but it is equal to 1.

Therefore, $2(x_1 - 3) = 1$

$\Rightarrow \quad x_1 = \frac{7}{2}$

$\therefore \quad y_1 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

8. $f(x) = \sqrt{x}$

$\therefore f(a) = \sqrt{4} = 2$

$f(b) = \sqrt{9} = 3; \quad f'(x) = \frac{1}{2\sqrt{x}}$

Also $\quad f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3-2}{9-4} = \frac{1}{5}$

$\therefore \quad \frac{1}{2\sqrt{c}} = \frac{1}{5} \Rightarrow c = \frac{25}{4} = 6.25$

9. $f(x)$ will be monotonically decreasing, if $f(x) < 0$.

$\Rightarrow \quad f'(x) = -\sin x - 2p < 0$

$\Rightarrow \quad \frac{1}{2}\sin x + p > 0$

$\Rightarrow \quad p > \frac{1}{2} \quad (\because -1 \le \sin x \le 1)$

10. Given that, equation of curve $y = x^3 = f(x)$

So, $\quad f(2) = 8$

and $\quad f(-2) = -8$

Now, $\quad f'(x) = 3x^2$

$\Rightarrow \quad f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$

$\Rightarrow \quad \frac{8 - (-8)}{4} = 3x^2$

$\therefore \quad x = \pm \frac{2}{\sqrt{3}}$

11. On differentiating $y = \cos(x + y)$ w.r.t x, we have

$$\frac{dy}{dx} = \frac{-\sin(x+y)}{1 + \sin(x+y)}$$

Slope of tangent at $(x, y) = \frac{-\sin(x+y)}{1+\sin(x+y)}$

Since, the tangents to the given curve are parallel to the line $x + 2y = 0$, whose slope is $\frac{-1}{2}$.

$\therefore \quad \frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$

$\Rightarrow \quad \sin(x+y) = 1$

$\Rightarrow \quad x + y = n\pi + (-1)^n \frac{\pi}{2}, n \in Z$

Then, $\quad y = \cos(x+y) = \cos\left\{n\pi + (-1)^n \frac{\pi}{2}\right\}, n \in Z$

$= 0$, for all $n \in Z$

Also, since $-2\pi \le x \le 2\pi$, we get $x = \frac{-3\pi}{2}$ and $x = \frac{\pi}{2}$.

Thus, tangents to the given curve are parallel to the line $x + 2y = 0$ only at points $\left(\frac{-3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$.

$\therefore$ The required equation of tangents are

$y - 0 = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right) \Rightarrow 2x + 4y + 3\pi = 0$

and $y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right) \Rightarrow 2x + 4y - \pi = 0$

12. Let $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow \quad 3(a+2)x^2 - 6ax + 9a \le 0$ for all $x \in R$

$\Rightarrow \quad (a+2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow \quad a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow \quad a < -2, -8a^2 - 24a \le 0$

$\Rightarrow \quad a < -2$ and $a(a+3) \ge 0$

$\Rightarrow \quad a < -2, a \le -3$ or $a \ge 0$

$\Rightarrow \quad a \le -3$

$\therefore \quad -\infty < a \le -3$

13. Now, $\quad f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

$= \frac{\cos x(\tan x - x)}{\sin^2 x}$

$\therefore \quad f'(x) > 0$ for $0 < x \le 1$

So, $f(x)$ is an increasing function.

Now, $\quad g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$

$= \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$

Again, $\frac{d}{dx}(\sin 2x - 2x) = 2\cos 2x - 2$

$= 2(\cos 2x - 1) < 0$

So, $\sin 2x - 2x$ is decreasing.

$\Rightarrow \quad \sin 2x - 2x < 0$

$\therefore \quad g'(x) < 0$

So, $g(x)$ is decreasing.

14. We have, $f(x) = x^3 + bx^2 + cx + d$

$\Rightarrow \quad f'(x) = 3x^2 + 2bx + c$

Let D_1 be the discriminant of $f'(x) = 3x^2 + 2bx + c$.

Then, $\quad D_1 = 4b^2 - 12c = 4(b^2 - c) - 8c < 0$

$(\because b^2 < c$ and $c > 0)$

$\therefore \quad f'(x) > 0$ for all $x \in (-\infty, \infty)$

Hence, $f(x)$ is strictly increasing function on $(-\infty, \infty)$.

15. $f(x) = |\log_2 [\log_3 \{\log_4 (\cos x + a)\}]|$

Clearly, $f(x)$ is increasing for all values of x, if

$\log_2 [\log_3 \{\log_4 (\cos x + a)\}]$ is defined for all values of x.

$\Rightarrow \log_3 [\log_4 (\cos x + a)] > 0, \forall x \in R$

$\Rightarrow \log_4 (\cos x + a) > 1, \forall x \in R$

$\Rightarrow \quad \cos x + a > 4, \forall x \in R$

$\therefore \quad a > 5$

16. On solving, $y^2 = 4ax$ and $x^2 = 4by$, we get

$$x = 0 \text{ or } x^3 = 64ab^2$$

Slope of the curves at the common points are $\frac{2a}{y}$ and $\frac{x}{2b}$, respectively. If these parabola intersect orthogonally, then $\frac{2a}{y} \cdot \frac{x}{2b} = -1$

$\Rightarrow \quad ax + by = 0$

$\Rightarrow \quad ax + \frac{x^2}{4} = 0$

$\Rightarrow \quad x = -4a$

$\Rightarrow \quad -x^3 = 64a^3 \qquad (\because x \neq 0)$

$\Rightarrow \quad 64ab^2 + 64a^3 = 0$

$\Rightarrow \quad a^2 + b^2 = 0$

which is not possible.

17. Let $y = f(x) = ax^2 + bx + c$, we have

$$f(1) = 1 \Rightarrow a + b + c = 1.$$

Also, $ax^2 + bx + c = x$ should have $x - 1$ as it's repeated root.

$\Rightarrow \quad ax^2 + (b-1)x + c = a(x-1)^2$

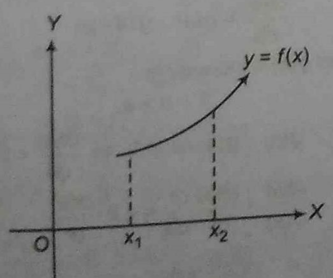
$\Rightarrow \quad 1 - b = 2a, a = c$

We have, $\quad f'(x) = 2ax + b, f''(x) = 2a$

$\Rightarrow \quad f''(1) = 2a, f'(0) = b$

$\therefore \quad f''(1) + f'(0) = 1$

18. Let $A \equiv (x_1, f(x_1))$ and $B \equiv (x_2, f(x_2))$ be any two points on the graph of $y = f(x)$. Chord AB will be lie completely above the graph of $y = f(x)$.



Hence, $\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right)$

19. $\quad g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x$

$\Rightarrow g''(x) = -f'(\sin x) \cdot \sin x + \cos^2 x f''(\sin x)$

$\qquad + f''(\cos x) \cdot \sin^2 x - f'(\cos x) \cdot \cos x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$. Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow g'(x) > 0, \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0, \forall x \in \left(0, \frac{\pi}{4}\right)$

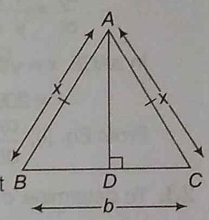
Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$.

20. Let ΔABC be isosceles triangle, where BC is the base of fixed length b.

Let the length of two equal sides of ΔABC be x.

Draw $AD \perp BC$ in figure.

Now, in ΔADC, by applying the Pythagoras theorem, we have



$$AD = \sqrt{x^2 - \left(\frac{b}{2}\right)^2} = \sqrt{x^2 - \frac{b^2}{4}}$$

$\therefore$ Area of triangle $(A) = \frac{1}{2} \times$ base $\times$ height

$$\Rightarrow \quad A = \frac{b}{2}\sqrt{x^2 - \frac{b^2}{4}}$$

The rate of change of the area A w.r.t. time t is given by

$$\frac{dA}{dt} = \frac{1}{2} b \times \frac{1}{2} \frac{2x}{\sqrt{x^2 - \frac{b^2}{4}}} \times \frac{dx}{dt} = \frac{xb}{\sqrt{4x^2 - b^2}} \frac{dx}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm/s.

$\therefore \quad \frac{dx}{dt} = -3$ cm/s $\qquad$ (negative sign use for decreasing)

$\therefore \quad \frac{dA}{dt} = \frac{-3xb}{\sqrt{4x^2 - b^2}}$ cm^2/s

When $x = b$, we have $\frac{dA}{dt} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$ cm^2/s

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}b$ cm^2/s.

Note If the rate of change is increasing, we take positive sign and if the rate of change is decreasing, we take negative sign.

21. Let $y = \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$

The function will be decreasing when $\frac{dy}{dx} < 0$.

$$\frac{\left[\begin{array}{l}(c \sin x + d \cos x)(a \cos x - b \sin x) \\ \quad - (a \sin x + b \cos x)(c \cos x - d \sin x)\end{array}\right]}{(c \sin x + d \cos x)^2} < 0$$

$\Rightarrow ac \sin x \cos x - bc \sin^2 x + ad \cos^2 x$

$\qquad - bd \sin x \cos x - ac \sin x \cos x + ad \sin^2 x$

$\qquad\qquad - bc \cos^2 x + bd \sin x \cos x < 0$

$\Rightarrow \quad ad(\sin^2 x + \cos^2 x) - bc(\sin^2 x + \cos^2 x) < 0$

$\qquad (ad - bc) < 0$

22. Let AB be the position of boy who is flying the kite and C be the position of the kite at any time t.

Let $BD = x$ and $AC = y$, then $AE = x$

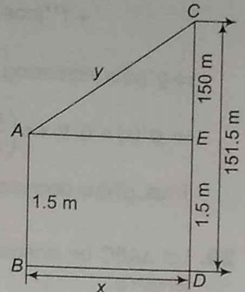
Given, $AB = 1.5$ m, $CD = 151.5$ m

$\therefore CE = 150$ m

Given, $\frac{dx}{dt} = 10$ m/s

Here, we have to find $\frac{dy}{dt}$,

when $y = 250$ m



Now, from ΔCAE, $y^2 = x^2 + 150^2$

On differentiating, we get

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{x}{y}\cdot\frac{dx}{dt} = \frac{x}{y}\cdot 10 \quad \ldots(i)$$

In ΔACE, $x = \sqrt{250^2 - 150^2}$ $(\because y = 250)$

$= 200$ m

From Eq. (i), $\frac{dy}{dt} = \frac{200}{250} \times 10 = 8$ m/s

23. To determine 'c' in Rolle's theorem, $f'(c) = 0$.

Here, $f'(x) = (x^2 + 3x)e^{-(1/2)x}\left(-\frac{1}{2}\right) + (2x + 3)e^{-(1/2)x}$

$$= e^{-\left(\frac{1}{2}\right)x}\left\{-\frac{1}{2}(x^2 + 3x) + 2x + 3\right\}$$

$$= -\frac{1}{2}e^{-(x/2)}(x^2 - x - 6)$$

$\because$ $f'(c) = 0$

$\Rightarrow$ $c^2 - c - 6 = 0$

$\Rightarrow$ $c = 3, -2$

$\therefore$ $c = 3 \notin [-3, 0]$

24. The tangent to the parabola $x^2 = y - 6$ at $(1, 7)$ is

$$x(1) = \frac{1}{2}(y + 7) - 6$$

$\Rightarrow$ $y = 2x + 5$

which is also a tangent to the given circle.

i.e., $x^2 + (2x + 5)^2 + 16x + 12(2x + 5) + c = 0$

$\Rightarrow$ $(5x^2 + 60x + 85 + c = 0)$ must have equal roots.

Let the roots be $\alpha = \beta$.

$\therefore$ $\alpha + \beta = -\frac{60}{5} \Rightarrow \alpha = -6$

$\therefore$ $x = -6$ and $y = 2x + 5 = -7$

25. Given that, $f(x)$ is monotonic.

$\Rightarrow f'(x) = 0$ or $f'(x) > 0, \forall x \in R$

$\Rightarrow f'(px) < 0$ or $f'(px) > 0, \forall x \in R$

So, $f'(px)$ is also monotonic.

Hence, $f(x) + f(3x) + \ldots + f[(2m - 1)x]$ is a monotonic.

Polynomial of odd degree $(2m - 1)$, so it will attain all real values only once.

26. Let $f(x) = \sin 3x$

Thus, $\sin x$ is increasing in $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ is of length π.

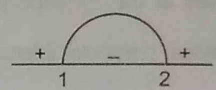
Hence, the length of largest interval in which $f(x) = \sin 3x$ is increasing, is $\frac{\pi}{3}$.

27. $f'(x) = e^{x(1-x)} + xe^{x(1-x)}(1 - 2x)$

$$= e^{x(1-x)}(1 + x - 2x^2) = -e^{x(1-x)}(x - 1)(2x + 1)$$

Hence, $f(x)$ is increasing in $\left[-\frac{1}{2}, 1\right]$.

28. $f'(x) = e^x(x - 1)(x - 2)$



$\therefore$ $f'(x) < 0$, for $1 < x < 2$

Hence, $f(x)$ is decreasing for $x \in (1, 2)$.

29. Since, $\frac{f(2) - f(0)}{2 - 0} = f'(x)$

$\Rightarrow$ $\frac{f(2) - 0}{2} = f'(x) \Rightarrow \frac{df(x)}{dx} = \frac{f(2)}{2}$

$\Rightarrow$ $f(x) = \frac{f(2)}{2}x + C$

$\because$ $f(0) = 0 \Rightarrow C = 0$

$\therefore$ $f(x) = \frac{f(2)}{2}x$

Also, $|f'(x)| \le \frac{1}{2} \Rightarrow \left|\frac{f(2)}{2}\right| \le \frac{1}{2}$

From Eq. (i), $|f(x)| = \left|\frac{f(2)}{2}x\right| = \left|\frac{f(2)}{2}\right||x| \le \frac{1}{2}|x|$ [from Eq. (ii)]

In interval $[0, 2]$, for maximum x

$$|f(x)| \le \frac{1}{2}\cdot 2 \Rightarrow |f(x)| \le 1 \quad (\because x = 2)$$

30. Here, $a + b = 4 \Rightarrow b = 4 - a$ and $b - a = 4 - 2a = t$ (say)

Now, $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx = \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx = I(a)$

$\Rightarrow$ $\frac{dI(a)}{da} = g(a) - g(4 - a)$

As $a < 2$ and $g(x)$ is increasing.

$\Rightarrow$ $4 - a > a$

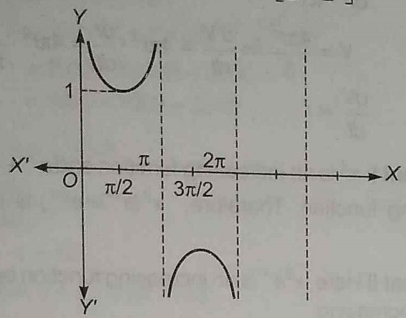
$\Rightarrow$ $g(a) - g(4 - a) < 0 \Rightarrow \frac{dI(a)}{da} < 0$

Now, $\frac{dI(a)}{d(a)} = \frac{dI(a)}{dt}\,\frac{dt}{da} = -2\cdot\frac{dI(a)}{dt}$

$\Rightarrow$ $\frac{dI(a)}{dt} > 0$

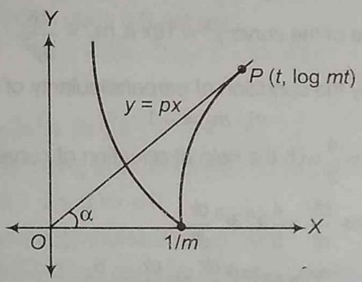
Thus, $I(a)$ is an increasing function of t. Hence, the given expression increasing with $(b - a)$.

31. The graph of cosec x is opposite in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



Solutions (Q. Nos. 32 to 34)

Slope of tangent at P = Slope of OP



$$\Rightarrow \quad \frac{1}{t} = \frac{\log mt}{t} \Rightarrow t = \frac{e}{m} \Rightarrow p = \left(\frac{e}{m}, 1\right)$$

$$\therefore \quad \tan\alpha = p = \frac{m}{e}$$

35. $\because f(x) = \frac{(x^2 + ax + 1) - 2ax}{x^2 + ax + 1} = 1 - \frac{2ax}{x^2 + ax + 1}$

$$\therefore f'(x) = -\left[\frac{(x^2 + ax + 1)\cdot 2a - 2ax(2x + a)}{(x^2 + ax + 1)^2}\right] = -\left[\frac{-2ax^2 + 2a}{(x^2 + ax + 1)^2}\right]$$

$$= 2a\left[\frac{(x^2 - 1)}{(x^2 + ax + 1)^2}\right] \quad \dots \text{(i)}$$

$$\text{and } f''(x) = 2a\left[\frac{(x^2 + ax + 1)^2(2x) - 2(x^2 - 1)(x^2 + ax + 1)(2x + a)}{(x^2 + ax + 1)^4}\right]$$

$$= 2a\left[\frac{2x(x^2 + ax + 1) - 2(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}\right]$$

Now, $f''(1) = \frac{4a(a + 2)}{(a + 2)^3} = \frac{4a}{(a + 2)^2}$

and $f''(-1) = \frac{4a(a - 2)}{(2 - a)^3} = -\frac{4a}{(a - 2)^2}$

$\therefore \quad (2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 4a - 4a = 0$

36. $\because g'(x) = \frac{f'(e^x)}{1 + (e^x)^2} \cdot e^x = 2a\left[\frac{e^{2x} - 1}{(e^{2x} + ae^x + 1)^2}\right]\left(\frac{e^x}{1 + e^{2x}}\right)$

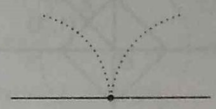
$g'(x) = 0$, if $e^{2x} - 1 = 0 \Rightarrow x = 0$

If $x < 0, e^{2x} < 1 \Rightarrow g'(x) < 0$ and if $x > 0, e^{2x} > 1 \Rightarrow g'(x) > 0$

37. For Rolle's theorem and LMVT, $f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b).

Hence, Statement I is true.

Since, $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is non-differentiable at $x = 0$.



Hence, Statement II is also true.

38. Given, $x^2 + y^2 = 25$

$$\Rightarrow \quad 2x\,dx + 2y\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad \frac{dy/dt}{dx/dt} = -\frac{x}{y}$$

$$\Rightarrow \quad -\frac{1.5}{dx/dt} = -\frac{3}{4}$$

$$\therefore \quad \frac{dx}{dt} = \frac{1.5 \times 4}{3}$$

$$= 2 \text{ cm/s}$$

39. Given, $y^2 = x + \sin x \quad \dots \text{(i)}$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 1 + \cos x$$

Here, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \cos x = -1$$

$$\Rightarrow \quad \sin x = 0$$

From Eq. (i), $y^2 = x$

40. Let the slope of the tangent be denoted by $\tan\psi$.

Length of tangent = $y \operatorname{cosec} \psi$

Length of normal = $y \sec \psi$

$$\therefore \quad \frac{\text{Length of tangent}}{\text{Length of normal}} = \cot\psi \propto \frac{1}{y}$$

Hence, Statement I is true.

Length of normal = $y \sec\psi = |y\sqrt{1 + m^2}|$

Length of tangent = $y \operatorname{cosec} \psi = \left|\frac{y\sqrt{1 + m^2}}{m}\right|$

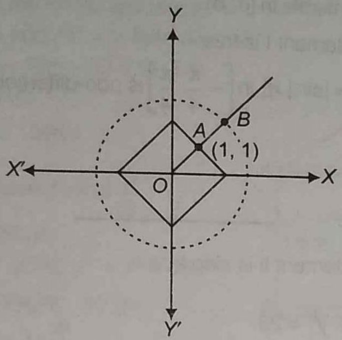
Hence, Statement II is true and explains Statement I.

41. Statement I As $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies $f(x)$ is either discontinuous or $f'(x)$ does not exist atleast one point in $(-1, 1)$.

$g(x) = 0$ at atleast one value of x in $(-1, 1)$.

Hence, Statement II is false.

42. Common normal is $y = x$.



On solving $x + y = 2 \Rightarrow (1, 1)$

and $x^2 + y^2 = 16 \Rightarrow (2\sqrt{2}, 2\sqrt{2})$

The distance between AB is $(4 - \sqrt{2})$ but as curve are not smooth, check at slope points. The coordinates in Ist quadrant are (2, 0) and (4, 0) and here distance = 2.

Hence, $4 - \sqrt{2}$ is not shortest.

43. Given, $y = \cos^2 x$

$\therefore \quad \frac{dy}{dx} = -2\cos x \cdot \sin x = -\sin 2x$

$\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is equal to -1.

So, the slope of normal is equal to 1.

$\therefore$ Equation of normal is $(y - 0) = \left(x - \frac{\pi}{4}\right) \Rightarrow x - y = \frac{\pi}{4}$

44. The equation of the tangent to the curve $y = e^x$ at (c, e^c) is

$$y - e^c = e^c\ (x - c) \qquad \ldots\text{(i)}$$

Equation of the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ is

$$y - e^{c-1} = e^c \cdot \frac{(e - e^{-1})}{2} \cdot [x - (c-1)]$$

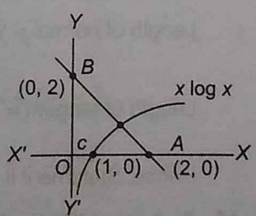
$\Rightarrow [x - (c-1)][2 - (e - e^{-1})] = 2e^{-1}$ [from Eq. (i)]

$$\Rightarrow \quad x - c = \frac{e + e^{-1} - 2}{2 - (e - e^{-1})} < 0$$

$$x < c$$

45. Curve of $g(x) = 2 - x$

and $f(x) = x \log x$



From graph, we observe that $f(x)$ and $g(x)$ are increasing and decreasing in the internal [1, 2] respectively atleast one point in [1, 2] will be exist, where the both curve intersect each other.

46. Surface area, $S = 4\pi r^2$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt} \Rightarrow 8\pi r \cdot \frac{dr}{dt} = 8 \qquad \left(\because \frac{ds}{dt} = 8\ \text{cm}^2/\text{s}\right)$$

$$\frac{1}{dt} = \frac{1}{\pi r}$$

Now, $V = \frac{4\pi r^3}{3} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{1}{\pi r} = 4r$

$\therefore \quad \frac{dV}{dt} \propto r$

47. Statement I x^2 is an increasing function and $e^x + e^{-x}$ is also an increasing function. Therefore, $x^2(e^x + e^{-x})$ is an increasing function.

Statement II Here, x^2e^x is an increasing function but x^2e^{-x} is not always increasing.

48. Slope of the curve $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ is $m_1 = \frac{-4x}{\alpha y}$.

Now, slope of the curve $y^3 = 16x$ is $m_2 = \frac{16}{3y^2}$.

Now, apply the condition of perpendicularity of two curves, *i.e.*, $m_1 \cdot m_2 = -1$

and get $\alpha = \frac{4}{3}$ with the help of equation of curves.

49. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi^3 3r^2 \frac{dr}{dt}$

$\Rightarrow \quad 35 = 4\pi(7)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{28\pi}$

Surface area of balloon, $S = 4\pi r^2$

$\therefore \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \times 7 \times \frac{5}{28\pi} = 10\ \text{cm}^2/\text{min}$

50. $\because \quad y = \cos(x + y)$

$\therefore \quad \frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \quad \frac{dy}{dx}[1 + \sin(x + y)] = -\sin(x + y)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2}$

$$\left(\because x + 2y = k \Rightarrow m = \frac{dy}{dx} = \frac{-1}{2}\right)$$

$\Rightarrow \quad \sin(x + y) = 1$

$\Rightarrow \quad x + y = \frac{\pi}{2}$

$\therefore \quad y = \cos\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \quad x = \frac{\pi}{2}$

$\because \quad x + 2y = k$

$\therefore \quad \frac{\pi}{2} + 2(0) = k \Rightarrow k = \frac{\pi}{2}$

51. Given, $\frac{dP}{dx} = (100 - 12\sqrt{x}) \Rightarrow dP = (100 - 12\sqrt{x})dx$

On integrating both sides, we get

$$\int dP = \int (100 - 12\sqrt{x})dx$$

$$P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000 \Rightarrow C = 2000$

Now, when $x = 25$, then

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$
$$= 2500 - 8 \times 125 + 2000$$
$$= 4500 - 1000 = 3500$$

52. Given, $y = \int_0^x |t|\,dt$

$\therefore \quad \frac{dy}{dx} = |x| = 2$

$\Rightarrow \quad x = \pm 2$

$\therefore$ Points, $y = \int_0^{\pm 2} |t|\,dt = \pm 2$

$\therefore$ Equation of tangent is

$$y - 2 = 2(x-2) y + 2 = 2(x+2)$$

For x-intercept put $y = 0$, we get

$$0 - 2 = 2(x-2) 0 + 2 = 2(x+2)$$

$\therefore \quad x = \pm 1$

53. Let $f(x) = 2x^3 + 3x + k$

On differentiating w.r.t. x, we get

$f'(x) = 6x^2 + 3 > 0, \forall x \in R$

So, $f(x)$ is strictly increasing function.

Hence $f(x) = 0$ has only one real root, so two roots are not possible.

54. Since, the balloon is spherical in shape, hence the volume of the balloon is $V = \frac{4}{3}\pi r^3$.

On differentiating both the sides w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \times \frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} \quad \ldots \text{(i)}$$

Now, to find $\frac{dr}{dt}$ at the rate $t = 49$ min, we require $\frac{dV}{dt}$ the radius (r) at that stage

$$\frac{dV}{dt} = -72\,\pi \text{m}^3/\text{min}$$

Also, amount of volume lost in 49 min $= 72\,\pi \times 49\,\text{m}^3$

$\therefore$ Final volume at the end of 49 min

$$= (4500\,\pi - 3528\pi)\,\text{m}^3$$
$$= 972\,\pi\text{m}^3$$

If r is the radius at the end of 49 min, then

$$\frac{4}{3}\pi r^3 = 972\,\pi$$

$\Rightarrow \quad r^3 = 729$

$\Rightarrow \quad r = 9$

Radius of the balloon at the end of 49 min $= 9$ m

From Eq. (i),

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

$$\Rightarrow \quad \left(\frac{dr}{dt}\right)_{t=49} = \frac{\left(\frac{dV}{dt}\right)_{t=49}}{4\pi\,(r^2)_{t=49}}$$

$$\left(\frac{dr}{dt}\right)_{t=49} = \frac{72\,\pi}{4\pi(9^2)} = \frac{2}{9} \text{ m/min}$$

55. We have, $y = x + \frac{4}{x^2}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

Since, the tangent is parallel to X-axis, therefore

$$\frac{dy}{dx} = 0 \Rightarrow x^3 = 8$$

$\therefore \quad x = 2$ and $y = 3$

56. Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$$f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall x \in R$$

Since, $f(x)$ is increasing.

So, $f(x) = 0$ has only one solution.

57. Using mean value theorem,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \quad \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{3 - 1}$$

$$\therefore \quad c = \frac{2}{\log_e 3} = 2\log_3 e$$

58. $f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x) = \frac{\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$

For $f(x)$ to be increasing,

$$\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0 \quad \Rightarrow \quad -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$$

$$\therefore \quad \frac{-3\pi}{4} < x < \frac{\pi}{4}$$

59. Since, $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50$

$$\Rightarrow \quad 3r^2 \cdot \frac{dr}{dt} = \frac{50 \times 3}{4\pi} \Rightarrow \frac{dr}{dt} = \frac{50}{4\pi r^2}$$

$$\therefore \quad \left(\frac{dr}{dt}\right)_{r=15} = \frac{50}{4\pi \times 225} = \frac{1}{18\pi} \text{ cm/min}$$

60. Using L' Hospitals rule, we get

$$\lim_{x \to a} \frac{f(a)\,g'(x) - g(a)\,f'(x)}{g'(x) - f'(x)} = 4$$

$$\Rightarrow \quad \lim_{x \to a} \frac{kg'(x) - kf'(x)}{g'(x) - f'(x)} = 4$$

$\therefore \quad k = 4$

Day 14

Maxima and Minima

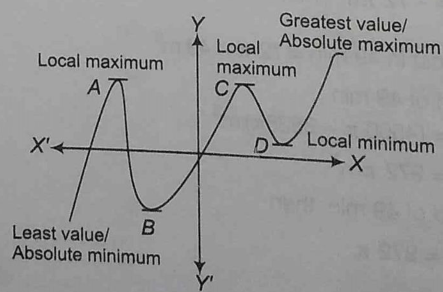
Day 14 Outlines ...

- Maxima and Minima of a Function
- Method to Find Local Maxima or Local Minima
- Concept of Global Maximum/Minimum

Maxima and Minima of a Function

A function $f(x)$ is said to attains a maximum at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$, $x \neq a$ *i.e.*, $f(x) - f(a) < 0, \forall\ x \in (a-\delta, a+\delta), x \neq a$

In such a case $f(a)$ is said to be the maximum value of $f(x)$ at $x = a$.



A function $f(x)$ is said to attain a minimum at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ such that $f(x) > f(a), \forall x \in (a-\delta, a+\delta), x \neq a$.

In such a case $f(a)$ is said to be the minimum value of $f(x)$ at $x = a$. The points at which a function attains either the maximum or the minimum values are known as the **extreme points** or **turning points** and both minimum and maximum values of $f(x)$ are called extreme values. The turning points A and C are called **local maximum** and points B and D are called **local minimum.**

Critical Point

A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a **critical point** of f. Note that, if f is continuous at point c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.

e.g., If $f(x) = x^3$, than $f'(x) = 3x^2$ and so $f'(0) = 0$ but O is neither a point of local maxima nor a point of local minima.

Let f be a function defined on an open interval I. Suppose $c \in I$ be any point. If f has a local maxima or a local minina at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c. The converse of above theorem need not be true, that is a point at which the derivative vanishes need not be a point of local maxima or local minima.

Method to Find Local Maxima or Local Minima

There are two test under which local maxima and local minima can be find out. These two tests are first derivate and second derivate test. *Both the tests are given below.*

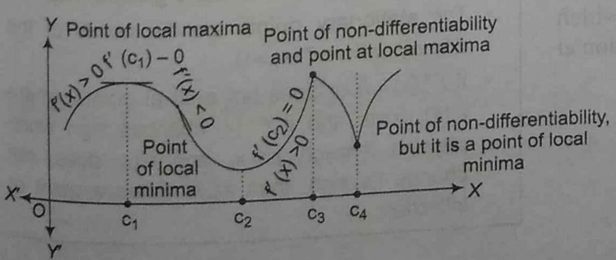
First Derivative Test

Let f be a function defined on an open interval I and f be continuous at a critical point c in I. Then,

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, *i.e.*, if $f'(x) > 0$ at every point sufficiently close to and to the left of c and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of **local maxima**.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through point c, *i.e.*, if $f'(x) < 0$ at every point sufficiently close to and to the left of c and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of **local minima**.

(iii) If $f'(x)$ does not changes sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called **point of inflection**.



If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(t)$ is a local minimum value of f.

Second or Higher Order Derivative Test

1. Find $f'(x)$ and equate it to zero. Solve $f'(x) = 0$ let its roots are $x = a_1, a_2, \ldots$
2. Find $f''(x)$ and at $x = a_1$,
   (i) if $f''(a_1)$ is positive, then $f(x)$ is minimum at $x = a_1$.
   (ii) if $f''(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$.
3. (i) If at $x = a_1, f''(a_1) = 0$, then find $f'''(x)$. If $f'''(a_1) \neq 0$, then $f(x)$ is neither maximum nor minimum at $x = a$.
   (ii) If $f'''(a_1) = 0$, then find $f^{iv}(x)$.
   (iii) If $f^{iv}(x)$ is positive (minimum value) and $f^{iv}(x)$ is negative (maximum value).
4. If at $x = a_1, f^{iv}(a_1) = 0$, then find $f^{v}(x)$ and proceed similarly.

Point of Inflection

- Consider function $f(x) = x^3$. At $x = 0$, $f'(x) = 0$. Also, $f''(x) = 0$ at $x = 0$. Such point is called point of inflection, where 2nd derivative is zero. Consider another function $f(x) = \sin x$, $f''(x) = -\sin x$. Now, $f''(x) = 0$ when $x = n\pi$, then this points are called point of inflection.
- At point of inflection
  (i) It is not necessary that 1st derivative is zero.
  (ii) 2nd derivative must be zero or 2nd derivative changes sign in the neighbourhood of point of inflection.

n th Derivative Test

Let f be a differentiable function on an interval I and a be an interior point of I such that

(i) $f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and
(ii) $f^n(a)$ exists and is non-zero.

- *If n is even and $f^n(a) < 0 \Rightarrow x = a$ is a point of local maximum.*
- *If n is even and $f^n(a) > 0 \Rightarrow x = a$ is a point of local minimum.*
- *If n is odd $\Rightarrow x = a$ is a point of local maximum nor a point of local minimum.*

Test for Local Maximum/Minimum at = , if is not Differentiable at =

Case I When $f(x)$ is continuous at $x = a$ and $f'(a - h)$ and $f'(a + h)$ exist and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$, if $f'(a - h)$ and $f'(a + h)$ are of opposite signs.

If $f'(a-h)>0$ and $f'(a+h)<0$, then $x=a$ will be a point of local maximum.

If $f'(a-h)<0$ and $f'(a+h)>0$, then $x=a$ will be a point of local minimum.

Case II When $f(x)$ is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the information about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case III If $f(x)$ is not continuous at $x=a$ and $f'(a-h)$ and /or $f'(a+h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x=a$.

- *If a function is strictly increasing in $[a, b]$, then* $\begin{cases} f(a) \text{ is local minimum} \\ f(b) \text{ is local maximum} \end{cases}$
- *If a function is strictly decreasing in $[a, b]$, then* $\begin{cases} f(a) \text{ is local maximum} \\ f(b) \text{ is local minimum} \end{cases}$
- *$y=f(x)$ is maximum or minimum according as $z=\dfrac{1}{f(x)}$ is minimum or maximum.*
- *The function $f(x)=\dfrac{ax+b}{cx+d}$ had no local maximum or minimum regardless of values of a, b, c and d.*
- *The function $f(\theta)=\sin^m\theta\cdot\cos^n\theta$ attains maximum values at $\theta=\tan^{-1}\left(\sqrt{\dfrac{m}{n}}\right)$.*
- *If AB is diameter of circle and C be any point on the circumference, then area of the ΔABC will be maximum, if triangle is isosceles.*

Concept of Global Maximum/Minimum

Let $y=f(x)$ be a given function with domain D and $[a,b]\subseteq D$, then global maximum/minimum of $f(x)$ in $[a,b]$ is basically the greatest/least value of $f(x)$ in $[a,b]$.

Global maxima/minima in $[a,b]$ would always occur at critical points of $f(x)$ with in $[a,b]$ or at end points of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a,b]$, find out all critical points of $f(x)$ in $[a,b]$ [*i.e.*, all points at which $f'(x)=0$] and let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at these points.

Then, $M_1 \to$ Global maxima or greatest value.

and $M_2 \to$ Global minima or least value.

where $M_1=\max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2=\min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Then, M_1 is the greatest value or global maxima in $[a,b]$ and M_2 is the least value or global minima in $[a,b]$.

Greatest and Least Values or Local Maximum/ Minimum of a Function in the Closed Interval

By maximum/minimum of local maximum/ or local minimum value of a function $f(x)$ at a point $c\in[a,b]$, we mean the greatest or least value in the intermediate neighbourhood of $x=c$.

It does not mean the greatest or absolute maximum (or the least or absolute minimum) of $f(x)$ in the interval $[a,b]$. A function may have a number of local maxima or local minima in a given interval and even a local minimum may be greater than a relative maximum.

Thus, a local maximum value may not be the greatest (absolute maximum) and local minimum value may not be the least (absolute minimum). Value of the function in any given interval, however, if a function $f(x)$ is continuous on a closed interval $[a,b]$, then it attains the absolute maximum (absolute minimum) at critical points or at the end points of $[a,b]$.

To find the absolute maximum/minimum value of the function, we choose the largest and smallest amongst the numbers

$$f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$$

where, $x=c_1, c_2, \ldots, c_n$ are critical points.

Results on Maxima and Minima

- Maxima and minima occur alternatively *i.e.*, between two maxima there is one minimum and *vice-versa*.
- If $f(x)\to\infty$ as $x\to a$ or b and $f'(x)=0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.
- If $f(x)\to-\infty$ as $x\to a$ or b, then $f(c)$ is necessarily the maximum and greatest value.
- The **stationary points** are the points of the domain, where $f'(x)=0$.
- If $f''(x)=0$ or does not exist at points, where $f'(x)$ exists and if $f''(x)$ changes sign when passing through $x=x_0$ and $f'(x)$ does not change its sign, then x_0 is called a point of inflection.

Practice Zone

DAY 14

1. x^x has a stationary point at
(a) $x = e$ (b) $x = \frac{1}{e}$
(c) $x = 1$ (d) $x = \sqrt{e}$

2. If the sum of two numbers is 3, then the maximum value of the product of the first and the square of second is **[NCERT Exemplar]**
(a) 4 (b) 1 (c) 3 (d) 0

3. The minimum intercepts made by the axes on the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is
(a) 25 (b) 7
(c) 1 (d) None of these

4. The curved surface of the cone inscribed in a given sphere is maximum, if
(a) $h = \frac{4R}{3}$ (b) $h = \frac{R}{3}$
(c) $h = \frac{2R}{3}$ (d) None of these

5. The minimum value of $9x + 4y$, where $xy = 16$ is
(a) 48 (b) 28 (c) 38 (d) 18

6. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a is equal to
(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$

7. If $y = a \log x + bx^2 + x$ has its extremum value at $x = 1$ and $x = 2$, then (a, b) is equal to
(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 2\right)$
(c) $\left(2, \frac{-1}{2}\right)$ (d) $\left(\frac{-2}{3}, \frac{-1}{6}\right)$

8. $f(x) = \begin{cases} 4x - x^3 + \log(a^2 - 3a + 3), & 0 \le x < 3 \\ x - 18, & x \ge 3 \end{cases}$
Complete the set of values of 'a' such that $f(x)$ has a local maxima at $x = 3$, is
(a) $[-1, 2]$ (b) $(-\infty, 1) \cup (2, \infty)$
(c) $[1, 2]$ (d) $(-\infty, -1) \cup (2, \infty)$

9. The function $f(x) = \frac{ax + b}{(x-1)(x-4)}$ has a local maxima at $(2, -1)$, then
(a) $b = 1, a = 0$ (b) $a = 1, b = 0$
(c) $b = -1, a = 0$ (d) $a = -1, b = 0$

10. If $f(x) = \frac{x}{1 + x \tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then
(a) $f(x)$ has exactly one point of minima
(b) $f(x)$ has exactly one point of maxima
(c) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
(d) $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

11. A straight line is drawn through the point $P(3, 4)$ meeting the positive direction of coordinate axes at the points A and B. If O is the origin, then minimum area of ΔOAB is equal to
(a) 12 sq units
(b) 6 sq units
(c) 24 sq units
(d) 48 sq units

12. If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segment BC, CA and AB, respectively. Then, maximum area of such parallelogram is
(a) $\frac{1}{2}$(area of ΔABC) (b) $\frac{1}{4}$(area of ΔABC)
(c) $\frac{1}{6}$(area of ΔABC) (d) $\frac{1}{8}$(area of ΔABC)

13. If $y = f(x)$ be a parametrically defined expression such that $x = 3t^2 - 18t + 7$ and $y = 2t^3 - 15t^2 + 24t + 10, \forall x \in [0, 6]$. Then, the minimum and maximum values of $y = f(x)$ are
(a) 36, 3 (b) 46, 6
(c) 40, −6 (d) 46, −6

14. The value of a, so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value is
(a) 2 (b) 1
(c) 3 (d) 0

15. If $a^2x^4 + b^2y^4 = c^6$, then the maximum value of xy is

(a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^3}{ab}$

(c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c^3}{2ab}$

16. The perimeter of a sector is P. The area of the sector is maximum when its radius is

(a) $\sqrt{P}$ (b) $\frac{1}{\sqrt{P}}$

(c) $\frac{P}{2}$ (d) $\frac{P}{4}$

17. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle, then the length of the hypotenuse is **[NCERT Exemplar]**

(a) $\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^{\frac{3}{2}}$ (b) $\left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)^{\frac{3}{2}}$

(c) $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$ (d) $d\left(a^{\frac{3}{2}} + b^{\frac{3}{2}}\right)^{\frac{2}{3}}$

18. If $P = (1, 1)$, $Q = (3, 2)$ and R is a point on x-axis, then the value of $PR + RQ$ will be minimum at

(a) $\left(\frac{5}{3}, 0\right)$ (b) $\left(\frac{1}{3}, 0\right)$

(c) $(3, 0)$ (d) $(1, 0)$

19. The volume of the largest cone that can be inscribed in a sphere of radius R is **[NCERT Exemplar]**

(a) $\frac{3}{8}$ of the volume of the sphere

(b) $\frac{8}{27}$ of the volume of the sphere

(c) $\frac{2}{7}$ of the volume of the sphere

(d) None of the above

20. The minimum value of $4e^{2x} + 9e^{-2x}$ is

(a) 11 (b) 12

(c) 10 (d) 14

21. If $f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$, then $f(x)$ has

(a) more than one minimum (b) exactly one minimum

(c) atleast one maximum (d) None of these

22. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) $\sqrt{ab}$ (b) $\frac{a}{b}$

(c) $2ab$ (d) ab

23. If $f(x) = x^2 - 4|x|$ and
$$g(x) = \begin{cases} \min\{f(t): -6 \le t \le x\}, & x \in [-6, 0] \\ \max\{f(t): 0 < t \le x\}, & x \in (0, 6] \end{cases}$$, then $g(x)$ has

(a) exactly one point of local minima

(b) exactly one point of local maxima

(c) no point to local maxima but exactly one point of local minima

(d) neither a point of local maxima nor minima

24. The function $f(x) = \frac{x^2 - 2}{x^2 - 4}$ has

(a) no point of local minima

(b) no point of local maxima

(c) exactly one point of local minima

(d) exactly one point of local maxima

25. If $f(x) = \int_0^x (t^2 - 1)\cos t \, dt$, $x \in (0, 2\pi)$. Then, $f(x)$ attains local maximum value at

(a) $x = \frac{\pi}{2}$ (b) $x = 1$

(c) $x = \frac{3\pi}{2}$ (d) None of these

Directions (Q. Nos. 26 to 29) *The absolute maximum and minimum values of function can be found by their monotonic and asymptotic behaviour provided they exist. We may agree that finite limiting values may be regarded as absolute maximum or minimum. For instance the absolute, maximum value of $\frac{1}{1+x^2}$ is unity. It is attained at $x = 0$ while absolute minimum value of the same function is zero which is a limiting value of ($x \to \infty$ or $x \to -\infty$).*

26. The function $f(x) = x^4 - 4x + 1$ will have

(a) absolute maximum value

(b) absolute minimum value

(c) both absolute maximum and minimum values

(d) None of the above

27. The absolute minimum value of the function $\frac{x-2}{\sqrt{x^2+1}}$ is

(a) -1 (b) $\frac{1}{2}$

(c) $-\sqrt{5}$ (d) None of these

28. The absolute minimum and maximum values of the function $\frac{x^2 - x + 1}{x^2 + x + 1}$ are

(a) 1 and 3 (b) $\frac{1}{2}$ and 3

(c) $\frac{1}{3}$ and 3 (d) None of these

29. The function $f(x) = \frac{4}{x-1} - \frac{9}{x+1}$ will

(a) have absolute maximum value $-\frac{1}{2}$

(b) have absolute minimum value $-\frac{25}{2}$

(c) not lie between $\frac{-25}{2}$ and $\frac{-1}{2}$

(d) always be negative

Directions (Q. Nos. 30 and 31) *A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass. Suppose that, y is the length and x is the breadth of the rectangular portion and P is the perimeter.*

30. The ratio of the sides $y : x$ of the rectangle, so that the window transmits the maximum light is

(a) $3:2$
(b) $6:6+\pi$
(c) $6+\pi:6$
(d) $1:2$

31. If μ is the amount of light per square metre for the coloured glass and L is the total light transmitted, then $\left(\frac{dL}{dy}\right)_{y=1}$ is equal to

(a) $\frac{\mu}{2}\left(3P - 12 - \frac{5\pi}{2}\right)$
(b) $\mu\left(2P - 6 - \frac{3\pi}{2}\right)$
(c) $\mu\left(P - 2 - \frac{\pi}{2}\right)$
(d) $2\mu\left(3P - \frac{\pi}{2}\right)$

Directions (Q. Nos. 32 to 36) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

32. If $f'(x) = (x-1)^3(x-2)^8$, then

Statement I $f(x)$ has neither maximum nor minimum at $x = 2$.

Statement II $f'(x)$ changes sign from negative to positive at $x = 2$.

33. Consider the cubic expression $y = x^3 + ax^2 + bx + c$

Statement I The graph of y has extremum, if $a^2 < 3b$.

Statement II y is either increasing or decreasing, $\forall x \in R$.

34. Consider the cubic expression $y = x^3 + ax^2 + bx + c$

Statement I $a^2 < 3b$, then function y have no critical points.

Statement II Either y is increasing function or decreasing function for all $x \in R$.

35. Consider the function $f(x) = -x^2 + 4x + 1 + \sin^{-1}\left(\frac{x}{2}\right)$

Statement I Minimum value of $f(x)$ in interval $[-1, 1]$ is $\left(-4, -\frac{\pi}{6}\right)$.

Statement II Minimum value of $f(x)$ in interval $[-1, 1]$ is $\min\{f(-1), f(1)\}$.

36. Statement I The minimum distance of the fixed point $(0, y_0)$, where $0 \le y_0 \le \frac{1}{2}$, from the curve $y = x^2$ is y_0.

Statement II Maxima and minima of a function is always a root of the equation $f'(x) = 0$.

37. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(a) 0
(b) 1
(c) 2
(d) 3

38. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that minimum $f(x) >$ maximum $g(x)$, then the relation between b and c is

(a) $0 < c < b\sqrt{2}$
(b) $|c| < |b|\sqrt{2}$
(c) $|c| > |b|\sqrt{2}$
(d) No real values of b and c

39. The minimum radius vector of the curve $\frac{4}{x^2} + \frac{9}{y^2} = 1$ is of length

(a) 1
(b) 5
(c) 7
(d) None of these

40. If $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \frac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$, then the points, where $f(x)$ takes maximum and minimum values, are

(a) 1, 4
(b) 0, 4
(c) 2, 4
(d) None of these

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41. The maximum area of a right angled triangle with hypotenuse h is **[JEE Main 2013]**

(a) $\frac{h^3}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$

(c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$

42. The cost of running a bus from A to B, is ₹ $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be ₹ 75 while at 40 km/h, it is ₹ 65. Then, the most economical speed (in km/h) of the bus is **[JEE Main 2013]**

(a) 45 (b) 50

(c) 60 (d) 40

43. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax,\ x \neq 0$ has extreme values at $x = -1$ and $x = 2$. **[AIEEE 2012]**

Statement I f has local maximum at $x = -1$ and at $x = 2$.

Statement II $a = 1/2$ and $b = -1/4$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

44. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a ΔOPQ, where O is the origin, if the area of the $\Delta\ OPQ$ is least, then the slope of the line PQ is **[AIEEE 2012]**

(a) $-1/4$ (b) -4

(c) -2 (d) $-1/2$

45. Let f be a function defined by

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement I $x = 0$ is point of minima of f.

Statement II $f'(0) = 0$. **[AIEEE 2011]**

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

(c) Statement I is true; Statement II is false.

(d) Statement I is false; Statement II is true.

46. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\, dt$. Then, f has **[AIEEE 2011]**

(a) local minimum at π and 2π

(b) local minimum at π and local maximum at 2π

(c) local maximum at π and local minimum at 2π

(d) local maximum at π and 2π

47. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is **[AIEEE 2009]**

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

48. Let $f: R \longrightarrow R$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k, is **[AIEEE 2009]**

(a) 1 (b) 0 (c) $-\frac{1}{2}$ (d) -1

49. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$ **[AIEEE 2009]**

(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(d) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

50. Suppose the cubic $x^3 - px + q$ has three distinct real roots, where $p > 0$ and $q > 0$. Then, which one of the following holds? **[AIEEE 2008]**

(a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

51. If x is real, then the maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is **[AIEEE 2006]**

(a) 41 (b) 1

(c) $\frac{17}{7}$ (d) $\frac{1}{4}$

Answers

1. (b)	**2.** (a)	**3.** (b)	**4.** (a)	**5.** (a)	**6.** (c)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (b)
11. (c)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (c)	**16.** (d)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (b)	**22.** (c)	**23.** (d)	**24.** (d)	**25.** (a)	**26.** (b)	**27.** (c)	**28.** (c)	**29.** (c)	**30.** (b)
31. (a)	**32.** (c)	**33.** (a)	**34.** (a)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (c)	**39.** (b)	**40.** (b)
41. (d)	**42.** (c)	**43.** (b)	**44.** (c)	**45.** (b)	**46.** (c)	**47.** (a)	**48.** (d)	**49.** (b)	**50.** (b)
51. (a)									

Hints & Solutions

1. Let $y = x^x \Rightarrow \log y = x \log x, (x > 0)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = x^x (1 + \log x)$$

Put $\frac{dy}{dx} = 0$, then

$$\log x = -1$$

$\Rightarrow$ $x = e^{-1} = \frac{1}{e}$

Stationary point is $x = \frac{1}{e}$.

2. Let two numbers are x and $(3 - x)$.

Product $P = x(3 - x)^2$

$$\frac{dP}{dx} = -2x(3 - x) + (3 - x)^2$$

$$\frac{dP}{dx} = (3 - x)(3 - 3x)$$

and $\frac{d^2P}{dx^2} = 6x - 12$

For maxima or minima, put $\frac{dP}{dx} = 0$

$\Rightarrow$ $(3 - x)(3 - 3x) = 0$

$\Rightarrow$ $x = 3, 1$

At $x = 3$,

$\frac{d^2P}{dx^2} = 18 - 12 = 6 > 0$ (minima)

At $x = 1$, $\frac{d^2P}{dx^2} = -6 < 0$

So, P is maximum at $x = 1$.

$\therefore$ Maximum value of $P = 1(3 - 1)^2 = 4$

3. Any tangent to the ellipse is $\frac{x}{4}\cos t + \frac{y}{3}\sin t = 1$, where the point of contact is $(4\cos t, 3\sin t)$

or $\frac{x}{4\sec t} + \frac{y}{3\operatorname{cosec} t} = 1$,

It means the axes Q $(4\sec t, 0)$ and $R(0, 3\operatorname{cosec} t)$.

$\therefore$ The distance of the line segment QR is

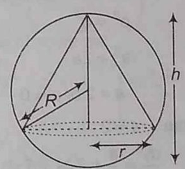
$$QR^2 = D = 16\sec^2 t + 9\operatorname{cosec}^2 t$$

So, the minimum value of D is $(4 + 3)^2$ or $QR = 7$.

4. Let S be the curved surface area of a cone.

$\therefore$ $S = \pi rl = \pi(\sqrt{2Rh - h^2})(\sqrt{h^2 + r^2})$

$= (\pi\sqrt{2Rh - h^2})(\sqrt{2Rh})$



Let $S^2 = P$

$\therefore$ $P = \pi^2\, 2R(2Rh^2 - h^3)$

Since, S is maximum, if P is maximum, then

$$\frac{dP}{dh} = 2\pi^2 R(4Rh - 3h^2) = 0$$

$\therefore$ $h = 0, \frac{4R}{3}$

Again, differentiating $\frac{dP}{dh}$, we get

$$\frac{d^2P}{dh^2} = 2\,\pi^2 R(4R - 6h)$$

$$\frac{d^2P}{dh^2} < 0 \text{ at } h = \frac{4R}{3}$$

5. Let $S = 9x + 4y$

Since, $xy = 16$ is given.

$\therefore$ $y = \frac{16}{x}$

or $S = 9x + \frac{64}{x}$

On differentiating both sides, we get

$$\frac{dS}{dx} = 9 - \frac{64}{x^2} \quad \text{...(i)}$$

$\because$ $\frac{dS}{dx} = 0 \Rightarrow \frac{64}{x^2} = 9$

$\Rightarrow$ $x = \pm\frac{8}{3}$

Again, differentiating Eq. (i) w.r.t. x, we get

$$\frac{d^2S}{dx^2} = +\frac{128}{x^3}$$

Hence, it is minimum at $x = \frac{8}{3}$ and minimum value of S is

$$S_{min} = 9\left(\frac{8}{3}\right) + 4(6) = 48$$

6. $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$

$f''(x) = 12x - 18a$

For maximum and minimum,

$6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$\Rightarrow \quad x = a$ or $x = 2a$,

At $x = a$ maximum and at $x = 2a$ minimum.

$\because \quad p^2 = q$

$\therefore \quad a^2 = 2a$

$\Rightarrow \quad a = 2$ or $a = 0$

But $a > 0$, therefore $\quad a = 2$

7. $\because \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow \quad a = -2b - 1$

and $\quad \left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0$

$\Rightarrow \quad \frac{-2b-1}{2} + 4b + 1 = 0$

$\Rightarrow \quad -b + 4b + \frac{1}{2} = 0$

$\Rightarrow \quad 3b = \frac{-1}{2}$

$\Rightarrow \quad b = \frac{-1}{6}$ and $a = \frac{1}{3} - 1 = \frac{-2}{3}$

8. Clearly, $f(x)$ in increasing just before $x = 3$ and decreasing after $x = 3$. For $x = 3$ to be the point of local maxima.

$f(3) \geq f(3 - 0)$

$\Rightarrow -15 \geq 12 - 27 + \log(a^2 - 3a + 3)$

$\Rightarrow \quad 0 < a^2 - 3a + 3 \leq 1$

$\Rightarrow \quad 1 \leq a \leq 2$

9. Clearly, $f(2) = -1$

$-1 = \frac{2a+b}{(2-1)(2-4)}$

$\Rightarrow \quad 2a + b = 2$

Now, $\quad f'(x) = \frac{4a + 5b - 2bx - ax^2}{(x-1)^2(x-4)^2}, f'(2) = 0$

$\Rightarrow \quad b = 0 \quad \Rightarrow \quad a = 1$

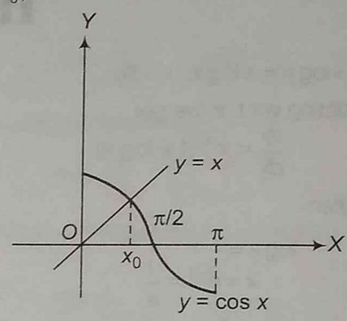
$\Rightarrow \quad f'(x) = -\frac{(x-2)(x+2)}{(x-1)^2(x-4)^2}$

Clearly, for $x > 2$, $f'(x) < 0$ and for $x < 2$, $f'(x) > 0$. Thus, $x = 2$ is indeed the point of local maxima for $y = f(x)$.

10. $f'(x) = \frac{1 - x^2 \sec^2 x}{(1 + x\tan x)^2}$

$= \frac{\sec^2 x(\cos x + x)(\cos x - x)}{(1 + x\tan x)^2}$

Clearly, $f'(x_0) = 0$



and $f'(x) > 0, \forall x \in (0, x_0)$, $f'(x) < 0, \forall x \in \left(x_0, \frac{\pi}{2}\right)$

Thus, $x = x_0$ is the only point of local maxima for $y = f(x)$.

11. Let the equation of drawn line be $\frac{x}{a} + \frac{y}{b} = 1$, where $a > 3, b > 4$ as the line passes through (3, 4) and meets the positive direction of coordinate axes.

We have, $\frac{3}{a} + \frac{4}{b} = 1 \Rightarrow b = \frac{4a}{(a-3)}$

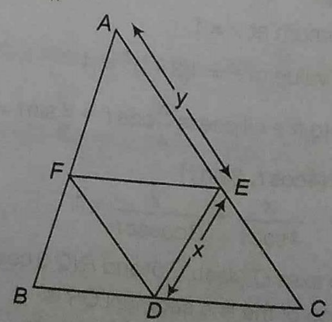
Now, area of $\Delta AOB, \Delta = \frac{1}{2}ab = \frac{2a^2}{(a-3)}$

$\frac{d\Delta}{da} = \frac{2a(a-6)}{(a-3)^2}$

Clearly, $a = 6$ is the point of minima for Δ.

Thus, $\quad \Delta_{min} = \frac{2 \times 36}{3} = 24$ sq units

12. $AF \parallel DE$ and $AE \parallel FD$



Now, in ΔABC and ΔEDC,

$\angle DEC = \angle BAC$, $\angle ACB$ is common.

$\Rightarrow \quad \Delta ABC \equiv \Delta EDC$

Now, $\quad \dfrac{b-y}{b} = \dfrac{x}{c} \Rightarrow x = \dfrac{c}{b}(b-y)$

Now, $\quad S$ = Area of parallelogram

$AFDE = 2$ (area of ΔAEF)

$$\Rightarrow \quad S = 2\left(\frac{1}{2} \times y \sin A\right) = \frac{c}{b}(b-y)\sin A$$

$$\frac{dS}{dy} = \left(\frac{c}{b}\sin A\right)(b-2y)$$

Sign scheme of $\dfrac{dx}{dy}$,

$+ \quad b/2 \quad -$

Hence, S is maximum when $y = \dfrac{b}{2}$.

$$\therefore \quad S_{max} = \frac{c}{b}\left(\frac{b}{2}\right) \times \frac{b}{2}\sin A$$

$$= \frac{1}{2}\left(\frac{1}{2}bc\sin A\right) = \frac{1}{2}(\text{area of } \Delta ABC)$$

13. We have, $\quad \dfrac{dy}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4)$

and $\quad \dfrac{dx}{dt} = 6t - 18 = 6(t-3)$

Thus, $\quad \dfrac{dy}{dx} = \dfrac{(t-1)(t-4)}{(t-3)}$

which indicates that $t = 1, 3$ and 4 are the critical points of $y = f(x)$.

Now, $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx} = \dfrac{t^2 - 6t + 11}{(t-3)^2} \times \dfrac{1}{6(t-3)}$

At $(t = 1)$, $\dfrac{d^2y}{dx^2} < 0 \Rightarrow t = 1$ is a point of local maxima.

At $(t = 4)$, $\dfrac{d^2y}{dx^2} > 0 \Rightarrow t = 4$ is a point of local minima.

At $(t = 3)$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are not defined and change its sign.

$\dfrac{d^2y}{dx^2}$ is unknown in the vicinity of $t = 3$, thus $t = 3$ is a point of neither maxima nor minima.

Finally, maximum and minimum values of expression $y = f(x)$ are 46 and -6, respectively.

14. Let α and β be the roots of the equation

$$x^2 - (a-2)x - a + 1 = 0.$$

Then, $\quad \alpha + \beta = a - 2, \alpha\beta = -a + 1$

Let $\quad z = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= (a-2)^2 + 2(a-1) = a^2 - 2a + 2$$

$$\Rightarrow \quad \frac{dz}{da} = 2a - 2$$

Put $\dfrac{dz}{da} = 0$, then

$$\Rightarrow \quad a = 1$$

$$\therefore \quad \frac{d^2z}{da^2} = 2 > 0$$

So, z has minima at $a = 1$.

So, $\alpha^2 + \beta^2$ has least value for $a = 1$. This is because we have only one stationary value at which we have minima. Hence, $a = 1$.

15. Given, $a^2x^4 + b^2y^4 = c^6$

$$\Rightarrow \quad y = \left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$$

Hence, $\quad f(x) = xy = x\left(\dfrac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$$\Rightarrow \quad f(x) = \left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{4}\left[\frac{c^6x^4 - a^2x^8}{b^2}\right]^{-3/4}\left[\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right]$$

Put $\quad f'(x) = 0$, then

$$\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$$

$$\Rightarrow \quad x^4 = \frac{c^6}{2a^2}$$

$$\therefore \quad x = \pm\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$$

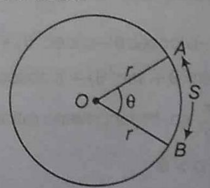
At $x = \dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum.

So, $\quad f\left(\dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\dfrac{c^{12}}{2a^2b^2} - \dfrac{c^{12}}{4a^2b^2}\right)^{1/4}$

$$= \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4} = \frac{c^3}{\sqrt{2ab}}$$

16. Perimeter of sector $= P$

Let AOB be the sector with radius r. If angle of the sector be θ radians, then area of sector



$$A = \frac{1}{2}r^2\theta \qquad \text{...(i)}$$

Length of arc $\quad (S) = r\theta$ or $\theta = \dfrac{S}{r}$

Therefore, perimeter of the sector,

$$P = r + S + r = 2r + S \qquad \text{...(ii)}$$

On substituting $\theta = \frac{S}{r}$ in Eq. (i), we get

$$A = \left(\frac{1}{2}r^2\right)\left(\frac{S}{r}\right) = \frac{1}{2}rS$$

$$\Rightarrow \quad S = \frac{2A}{r}$$

Now, substituting the value of S in Eq. (ii), we get

$$P = 2r + \left(\frac{2A}{r}\right)$$

$$\Rightarrow \quad 2A = Pr - 2r^2$$

On differentiating w.r.t. r, we get $2\frac{dA}{dr} = P - 4r$

For the maximum value of area,

$$\frac{dA}{dr} = 0$$

$$\Rightarrow \quad P - 4r = 0 \Rightarrow r = \frac{P}{4}$$

17. Let P be a point on the hypotenuse AC of right angled ΔABC.

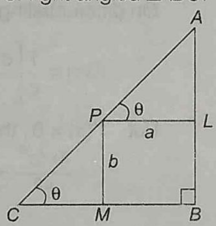
Such that at $PL \perp AB = a$

and $PM \perp BC = b$

Let $\angle APL = \angle ACB = \theta$ (say)

$AP = a\sec\theta$

$PC = b\,\text{cosec}\,\theta$



Let l be the length of the hypotenuse.

Then, $l = AP + PC$

$$\Rightarrow \quad l = a\sec\theta + b\,\text{cosec}\,\theta,\ 0 < \theta < \frac{\pi}{2}$$

On differentiating w. r. t. θ, we get

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta$$

For maxima or minima, put $\frac{dl}{d\theta} = 0$

$$\Rightarrow a\sec\theta\tan\theta = b\,\text{cosec}\,\theta\cot\theta$$

$$\Rightarrow \quad \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta} \Rightarrow \frac{\sin^3\theta}{\cos^3\theta} = \frac{b}{a}$$

$$\Rightarrow \quad \tan^3\theta = \frac{b}{a} \quad \Rightarrow \quad \tan\theta = \left(\frac{b}{a}\right)^{1/3}$$

Now, $\frac{d^2l}{d\theta^2} = a(\sec\theta \times \sec^2\theta + \tan\theta \times \sec\theta\tan\theta)$

$$-b[\text{cosec}\,\theta(-\text{cosec}^2\theta) + \cot\theta(-\text{cosec}\,\theta\cot\theta)]$$

$$= a\sec\theta(\sec^2\theta + \tan^2\theta) + b\,\text{cosec}\,\theta(\text{cosec}^2\theta + \cot^2\theta)$$

Since, $0 < \theta < \frac{\pi}{2}$, so, trigonometric ratios are positive.

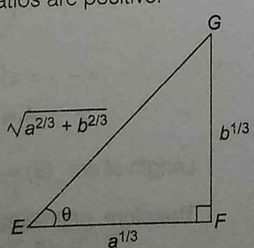
Also, $a > 0$ and $b > 0$.

$\therefore \frac{d^2l}{d\theta^2}$ is positive.

$\Rightarrow l$ is least when $\tan\theta = \left(\frac{b}{a}\right)^{1/3}$

$\therefore$ Least value of

$$l = a\sec\theta + b\,\text{cosec}\,\theta$$



$$= a\frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$$

$$= \sqrt{a^{2/3} + b^{2/3}}\,(a^{2/3} + b^{2/3}) = (a^{2/3} + b^{2/3})^{3/2}$$

$$\left(\because \text{ in } \Delta EFG,\ \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and cosec}\,\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}\right)$$

18. Let the coordinate of $R(x, 0)$.

Now, $PR + RQ = \sqrt{(x-1)^2 + (0-1)^2} + \sqrt{(x-3)^2 + (0-2)^2}$

$$= \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 6x + 13}$$

For minimum value of $PR + RQ$,

$$\frac{d}{dx}(PR + RQ) = 0$$

$$\Rightarrow \frac{d}{dx}(\sqrt{x^2 - 2x + 2}) + \frac{d}{dx}(\sqrt{x^2 - 6x + 13}) = 0$$

$$\Rightarrow \quad \frac{(x-1)}{\sqrt{x^2 - 2x + 2}} = -\frac{(x-3)}{\sqrt{x^2 - 6x + 13}}$$

On squaring both sides, we get

$$\frac{(x-1)^2}{(x^2 - 2x + 2)} = \frac{(x-3)^2}{x^2 - 6x + 13}$$

$$\Rightarrow \quad 3x^2 - 2x - 5 = 0$$

$$\Rightarrow \quad (3x - 5)(x + 1) = 0$$

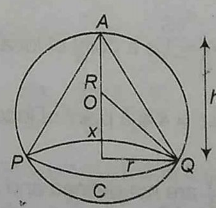
$$\Rightarrow \quad x = \frac{5}{3}, -1$$

Also, $1 < x < 3$

$\therefore$ $R = (5/3, 0)$

19. Let $OC = x$, $CQ = r$

Now, $OA = R$ (given)



Height of the cone $= h = x + R$

$\therefore$ Volume of the cone $= V = \frac{1}{3}\pi r^2 h$ ...(i)

Also, in right angled ΔOCQ,

$$OC^2 + CQ^2 = OQ^2 \Rightarrow x^2 + r^2 = R^2 \Rightarrow r^2 = R^2 - x^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$V = \frac{1}{3}\pi(R^2 - x^2)(x + R) \quad \text{...(iii)}$$

$(\because h = x + R)$

On differentiating Eq. (iii) w.r.t x, we get

$$\frac{dV}{dx} = \frac{1}{3}\pi[(R^2 - x^2) - 2x(x + R)]$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R^2 - x^2 - 2x^2 - 2xR)$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2xR - 3x^2)$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R - 3x)(R + x) \qquad \text{...(iv)}$$

For maxima, put $\frac{dv}{dx} = 0$

$$\Rightarrow \quad \frac{\pi}{3}(R - 3x)(R + x) = 0 \Rightarrow x = \frac{R}{3} \text{ or } x = -R$$

$$\Rightarrow \quad x = \frac{R}{3} \quad \text{(since, } x \text{ cannot be negative)}$$

On differentiating Eq. (iv) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}[(-3)(R + x) + (R - 3x)]$$

$$= \frac{\pi}{3}(-2R - 6x) = -\frac{\pi}{3}(2R + 6x)$$

At $x = \frac{R}{3}$, $\frac{d^2V}{dx^2} = \frac{-x\pi}{3}\left(2R + \frac{6R}{3}\right) = -\frac{4\pi}{3}R < 0$

So, V has a local maxima at $x = R/3$.

Now, substituting the value of x in Eq. (iii), we get

$$V = \frac{\pi}{3}\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right) = \frac{\pi}{3} \cdot \frac{8R^2}{9} \cdot \frac{4R}{3} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$\Rightarrow \quad V = \frac{8}{27} \times \text{Volume of sphere}$$

20. Let $f(x) = 4e^{2x} + 9e^{-2x}$

$\therefore \quad f'(x) = 8e^{2x} - 18e^{-2x}$

Put $f'(x) = 0$

$$\Rightarrow \quad 8e^{2x} - 18e^{-2x} = 0 \Rightarrow e^{2x} = 3/2$$

$$\Rightarrow \quad x = \log(3/2)^{1/2}$$

Again, $\quad f''(x) = 16e^{2x} + 36e^{-2x} > 0$

Now, $\quad f(\log(3/2)^{1/2}) = 4e^{2[\log(3/2)]^{1/2}} + 9e^{-2[\log(3/2)]^{1/2}}$

$$= 4 \times \frac{3}{2} + 9 \times \frac{2}{3} = 6 + 6 = 12$$

Hence, the minimum value is 12.

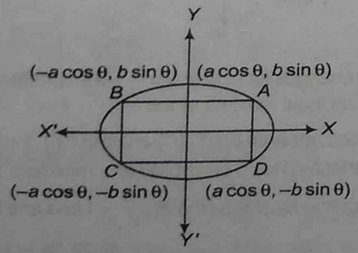
21. Given, $f(x) = 1 + 2x^2 + 2^2x^4 + 2^3x^6 + \ldots + 2^{10}x^{20}$

$$f'(x) = x(4 + 4 \cdot 2^2x^2 + \ldots + 20 \cdot 2^{10}x^{18})$$

Put $\quad f'(x) = 0 \Rightarrow x = 0$ only

Also, $f''(0) > 0$

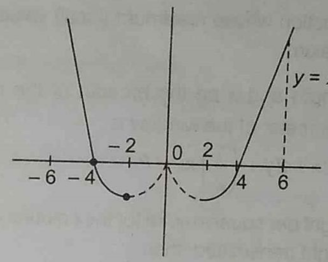
22.



Area of rectangle $ABCD = (2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

Hence, area of greatest rectangle is equal to $2ab$ when $\sin 2\theta = 1$.

23. Bold line represents the graph of $y = g(x)$, clearly $g(x)$ has neither a point of local maxima nor a point of local minima.



24. For $y = \frac{x^2 - 2}{x^2 - 4} \Rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2 - 4)^2}$

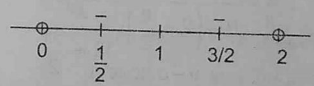
$$\Rightarrow \quad \frac{dy}{dx} > 0 \text{ for } x < 0$$

and $\quad \frac{dy}{dx} < 0$ for $x > 0$

Thus, $x = 0$ is the point of local maxima for y. Now, $(y)_{x=0} = \frac{1}{2}$ (positive). Thus, $x = 0$ is also the point of local maximum for $y = \left|\frac{x^2 - 2}{x^2 - 4}\right|$.

25. $f'(x) = (x^2 - 1)\cos x$

Sign scheme of $f'(x)$, clearly $x = \frac{\pi}{2}$ is the point of local maxima.



26. Since, $f'(x) = 4x^3 - 4 = 0 \Rightarrow x = 1$

So, there is only extrema which is minima.

Hence, 1 is a point of absolute minima.

27. Since, $f'(x) = \frac{1 + 2x}{(x^2 + 1)^{3/2}}$

Thus, $f(x)$ is increasing in $\left(-\frac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right)$.

Hence, absolute minimum occurs at $x = -\frac{1}{2}$, we have value

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

28. $\because \quad yx^2 + yx + y = x^2 - x + 1$

$\Rightarrow \quad x^2(1 - y) - x(1 + y) + (1 - y) = 0$

$\because \quad D \geq 0$

$\therefore \quad (1 + y)^2 - 4(1 - y)^2 \geq 0$

$\Rightarrow \quad -(3y - 1)(y - 3) \geq 0$

$\therefore \quad 1/3 \leq y \leq 3$

29. Should be the correct choice (we can prove by using monotonically that f cannot lie between $\left(\frac{-25}{2}\text{ and }\frac{-1}{2}\right)$. This is an example of a function whose maximum (local) value is smaller than minimum value.

30. Let y be the length and x be the breadth of the rectangular portion. Total perimeter of the window is

$$2x + 2y + \left(\frac{1}{2}\right)\pi y = P \quad \text{(say)}$$

Let amount of light per square metre for the coloured glass be μ. If L is the total light transmitted, then

$$L = 3\mu \times \text{Area of rectangular portion} + \mu \times \text{Area of semi-circular portion}$$

$$= 3\mu xy + \frac{1}{8}\mu\pi y^2$$

$$= \mu\left[\frac{3}{2}y\left\{P - \left(2 + \frac{\pi}{2}\right)y\right\} + \frac{1}{8}\pi y^2\right]$$

$$\Rightarrow \quad \frac{dL}{dy} = \frac{\mu}{2}\left[3P - 6\left(2 + \frac{\pi}{2}\right)y + \frac{\pi}{2}y\right]$$

Put $\frac{dL}{dy} = 0$, then $y = \frac{3P}{12 + \frac{5\pi}{2}}$

$$\frac{d^2L}{dy^2} = \frac{\mu}{2}\left(-12 - \frac{5\pi}{2}\right) < 0$$

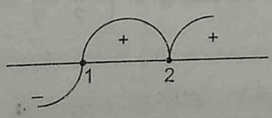
$$\therefore \quad \frac{y}{x} = \frac{2\cdot\frac{3P}{12 + 5\pi/2}}{P - \left(2 + \frac{\pi}{2}\right)\cdot\frac{3P}{12 + \frac{5\pi}{2}}}$$

$$= \frac{6}{(12 + 5\pi/2) - 3(2 + \pi/2)} = \frac{6}{6 + \pi}$$

$$\therefore \quad y : x = 6 : 6 + \pi$$

31. $\left(\frac{dL}{dy}\right)_{(y=1)} = \frac{\mu}{2}\left(3P - 12 - \frac{5\pi}{2}\right)$

32. It is clear from figure, $f'(x)$ has no sign change at $x = 2$.



Hence, $f(x)$ is neither maximum nor minimum at $x = 2$.

33. For no extremum,

$$\frac{dy}{dx} > 0 \text{ or } \frac{dy}{dx} < 0, \forall x \in R$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 + 2ax + b > 0$$

$$\Rightarrow \quad 3x^2 + 2ax + b > 0$$

$$\therefore \quad D < 0$$

$$\Rightarrow \quad 4a^2 - 4\cdot 3\cdot b < 0$$

$$\Rightarrow \quad a^2 < 3b$$

34. If y have no critical points for all $x \in R$, then

$$\frac{dy}{dx} > 0 \text{ or } \frac{dy}{dx} < 0, \forall x \in R$$

But $\frac{dy}{dx} = 3x^2 + 2ax + b > 0$

$$\Rightarrow \quad 3x^2 + 2ax + b > 0 \Rightarrow D < 0$$

$$\Rightarrow \quad 4a^2 - 4\cdot 3\cdot b < 0$$

$$\Rightarrow \quad a^2 < 3b$$

35. $\because \quad f(x) = -x^2 + 4x + 1 + \sin^{-1}\left(\frac{x}{2}\right)$

$$\Rightarrow \quad f'(x) = -2x + 4 + \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}$$

$$= 2(2 - x) + \frac{1}{\sqrt{4 - x^2}} > 0, \forall x \in [-1, 1]$$

Therefore, $f(x)$ increasing in interval $[-1, 1]$.

$\therefore$ Minimum value of $f(x)$, $f(-1) = -4 - \frac{\pi}{6}$

36. Let the point on the parabola be (t, t^2). Let d be the distance between (t, t^2) and $(0, y_0)$, then

$$d^2 = t^2 + (t^2 - y_0)^2 = t^4 + (1 - 2y_0)t^2 + y_0^2$$

$$= z^2 + (1 - 2y_0)z + y_0^2, z \ge 0 \quad (\text{let } t^2 = z)$$

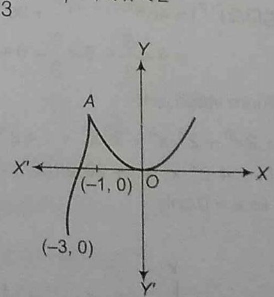
Its vertex is at $x = y_0 - \frac{1}{2} < 0$.

So the minimum value of d^2 is at $z = 0$ *i.e.*, $t^2 = 0$

$$\therefore \quad d = y_0$$

Hence, Statement I is true. Statement II is false because extremum can occur at a point, where $f'(x)$ does not exist.

37. $f'(x) = \begin{cases} 3(2 + x)^2, & -3 < x \le -1 \\ \frac{2}{3}x^{-1/3}, & -1 < x < 2 \end{cases}$



Clearly, $f'(x)$ changes its sign at $x = -1$ from positive to negative and so $f(x)$ has local maxima at $x = -1$.

Also, $f'(0)$ does not exist but $f'(0^-) < 0$ and $f'(0^+) < 0$. It can only be inferred that $f(x)$ has a possibility of a minimum at $x = 0$.

Hence, it has one local maxima at $x = -1$ and one local minima at $x = 0$

So, total number of local maxima and local minima is 2.

38. Minimum of $f(x) = -\dfrac{D}{4a} = \dfrac{-(4b^2 - 8c^2)}{4} = 2c^2 - b^2$

and maximum of $g(x) = -\dfrac{(4c^2 + 4b^2)}{4(-1)} = b^2 + c^2$

Since, $\min f(x) > \max g(x)$

$\Rightarrow \quad 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2$

$\Rightarrow \quad |c| > \sqrt{2}\,|b|$

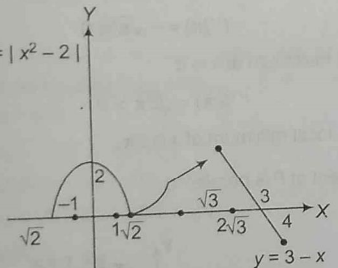
39. The given curve is $\dfrac{4}{x^2} + \dfrac{9}{y^2} = 1$

Put $x = r\cos\theta$, $y = r\sin\theta$, we get $r^2 = (2\sec\theta)^2 + (3\operatorname{cosec}\theta)^2$

So, r^2 will have minimum value $(2+3)^2$.

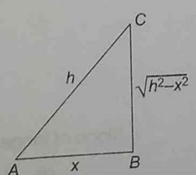
or r have minimum value equal to 5.

40. $f(x) = \begin{cases} |x^2 - 2|, & -1 \le x < \sqrt{3} \\ \dfrac{x}{\sqrt{3}}, & \sqrt{3} \le x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \le x \le 4 \end{cases}$



From the above graph,
Maximum occurs at $x = 0$ and minimum at $x = 4$.

41. Area of triangle, $\Delta = \dfrac{1}{2}x\sqrt{h^2 - x^2}$



$\dfrac{d\Delta}{dx} = \dfrac{1}{2}\left[\sqrt{h^2 - x^2} + \dfrac{x(-2x)}{2\sqrt{h^2 - x^2}}\right] = 0$

$\Rightarrow \quad x = \dfrac{h}{\sqrt{2}}$

$\dfrac{d^2\Delta}{dx^2} < 0$ at $x = \dfrac{h}{\sqrt{2}}$

$\therefore \quad \Delta = \dfrac{1}{2} + \dfrac{h}{\sqrt{2}}\sqrt{h^2 - \dfrac{h^2}{2}} = \dfrac{h^2}{4}$

42. Let $c = av + \dfrac{b}{v}$ ...(i)

When $v = 30$ Km/h, then $c =$ ₹ 75

$\therefore \quad 75 = 30a + \dfrac{b}{30}$ ...(ii)

When $v = 40$ Km/h, then $c =$ ₹ 65

$\therefore \quad 65 = 40a + \dfrac{b}{40}$ ...(iii)

On solving Eqs. (ii) and (iii), we get

$a = \dfrac{1}{2}$

and $\quad b = 1800$

On different w.r.t. v in Eq. (i),

$\dfrac{dc}{dv} = a\dfrac{-b}{v^2}$

For max of minimum c,

$\dfrac{dc}{dv} = 0$

$\Rightarrow \quad v = \pm\sqrt{\dfrac{b}{a}}$

$\Rightarrow \quad \dfrac{d^2c}{dv^2} = \dfrac{2b}{v^3}$

at $\quad v = \sqrt{\dfrac{b}{a}}, \dfrac{dx}{dv^2} > 0$

So, at $v = \sqrt{\dfrac{b}{a}}$ the speed is most economical.

$\therefore$ Most economical speed is

$c = a\sqrt{\dfrac{b}{a}} + b\sqrt{\dfrac{a}{b}} = 2\sqrt{ab}$

$c = 2\sqrt{\dfrac{1}{2} \times 1800} = 2 \times 30$

$c = 60$

43. $f(x) = \log|x| + bx^2 + ax$

$\Rightarrow \quad f'(x) = \dfrac{1}{x} + 2bx + a \Rightarrow f''(x) = \dfrac{-1}{x^2} + 2b$

Since, f has extrema at $x = -1$ and $x = 2$.

Hence, $\quad f'(-1) = 0 = f'(2)$

$f'(-1) = 0$

$\Rightarrow \quad a - 2b = 1$ ...(i)

and $f'(2) = 0 \Rightarrow a + 4b = \dfrac{-1}{2}$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a = \dfrac{1}{2}$ and $b = \dfrac{-1}{4}$

$\Rightarrow \quad f''(x) = \dfrac{-1}{x^2} + \dfrac{-1}{2}$

$= -\left(\dfrac{x^2 + 2}{2x^2}\right)$

$\Rightarrow \quad f''(-1) < 0$ and $f''(2) < 0$

Thus, f has local maxima at both $x = -1$ and $x = 2$.

Hence, obviously Statement I is correct.

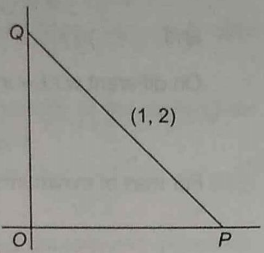
Also, while solving for Statement I, we found the values of a and b which justify that Statement II is also correct.

However, Statement II does not explain Statement I in any way.

44. Let m be the slope of the line PQ, then the equation of PQ is

$$y - 2 = m(x - 1)$$

Now, PQ meets X-axis at



$$P\left(1 - \frac{2}{m}, 0\right)$$

and Y-axis at $Q\,(0, 2 - m)$.

$\therefore$ $$OP = 1 - \frac{2}{m}$$

and $$OQ = 2 - m$$

Also, area of $\Delta OPQ = \frac{1}{2}(OP)(OQ)$

$$= \frac{1}{2}\left|\left(1 - \frac{2}{m}\right)(2 - m)\right|$$

$$= \frac{1}{2}\left|2 - m - \frac{4}{m} + 2\right|$$

$$= \frac{1}{2}\left|4 - \left(m + \frac{4}{m}\right)\right|$$

Let $$f(m) = 4 - \left(m + \frac{4}{m}\right)$$

$\Rightarrow$ $$f'(m) = -1 + \frac{4}{m^2}$$

Now, $$f'(m) = 0$$

$\Rightarrow$ $$m^2 = 4$$

$\Rightarrow$ $$m = \pm 2$$

$\Rightarrow$ $$f(2) = 0$$

and $$f(-2) = 8$$

Since, the area cannot be zero, hence the required value of m is -2.

45. $f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

As $$\frac{\tan x}{x} > 1, \forall x \neq 0$$

$\therefore$ $$f(0 + h) > f(0)$$

and $$f(0 - h) > f(0)$$

At $x = 0$, $f(x)$ attains minima.

Now, $$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\tan h}{h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\tan h - h}{h^2} \quad \text{(using L' Hospital's rule)}$$

$$= \lim_{h \to 0} \frac{\sec^2 h - 1}{2h} \quad (\because \tan^2\theta = \sec^2\theta - 1)$$

$$= \lim_{h \to 0} \frac{\tan^2 h}{2h^2} \cdot h$$

$$= \frac{1}{2} \cdot 0 = 0$$

Therefore, Statement II is true.

Hence, both statements are true but Statement II is not the correct explanation of Statement I.

46. Here, $f(x) = \int_0^x \sqrt{t}\, \sin t\, dt$,

where $x \in \left(0, \frac{5}{2}\right)$

$$f'(x) = \{\sqrt{x}\sin x - 0\} \quad \text{...(i)}$$

(using Newton-Leibnitz formula)

$\therefore$ $$f'(x) = \sqrt{x}\sin x = 0$$

$\Rightarrow$ $$\sin x = 0$$

$\therefore$ $$x = \pi, 2\pi$$

$$f''(x) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$$

$$f''(\pi) = -\sqrt{\pi} < 0$$

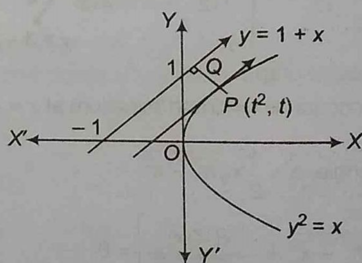
$\therefore$ Local maximum at $x = \pi$

$$f''(2\pi) = \sqrt{2\pi} > 0$$

Hence, local minimum at x is 2π.

47. $\therefore$ Tangent at P is parallel to

$$y = x + 1 \quad \text{...(i)}$$



$\therefore$ Slope of tangent at $P(t^2, t)$,

$$\frac{dy}{dx} = \left(\frac{1}{2y}\right)_{(t^2, t)} = \frac{1}{2t} \quad \text{...(ii)}$$

$\Rightarrow$ $$\frac{1}{2t} = 1 \quad \text{[since, Eqs. (i) and (ii) are parallel]}$$

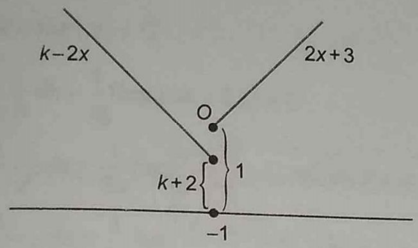
$\Rightarrow$ $$t = \frac{1}{2}$$

$\therefore$ $$P\left(\frac{1}{4}, \frac{1}{2}\right)$$

Shortest distance $= |PQ| = \dfrac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1 + 1}} = \dfrac{3}{4\sqrt{2}}$

$\therefore$ Shortest distance $= \dfrac{3\sqrt{2}}{8}$

48.



$\because \quad k + 2 \le 1$

$\therefore \quad k \le -1$

49. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow \quad P'(x) = 4x^3 + 3ax^2 + 2bx + c$

Since, $x = 0$ is a solution for

$$P'(x) = 0 \Rightarrow c = 0$$

$\therefore \quad P(x) = x^4 + ax^3 + bx^2 + d \quad$ ...(i)

Also, we have $\quad P(-1) < P(1)$

$\Rightarrow \quad 1 - a + b + d < 1 + a + b + d$

$\Rightarrow \quad a > 0$

Since, $P'(x) = 0$ only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1$, 0 and 1 only.

Also, we have $\quad P(-1) < P(1)$

$\therefore$ Maximum of $P(x) = \max\{P(0), P(1)\}$

and minimum of $P(x) = \min\{P(-1), P(0)\}$

In the interval $[0, 1]$,

$$P'(x) = 4x^3 + 3ax^2 + 2bx$$

$$= x(4x^2 + 3ax + 2b)$$

Since, $P'(x)$ has only one root $x = 0$, then

$4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore \quad (3a)^2 - 32b < 0$

$\Rightarrow \quad \dfrac{3a^2}{32} < b$

$\therefore \quad b > 0$

Thus, we have $a > 0$ and $b > 0$.

$\therefore \quad P'(x) = 4x^3 + 4ax^2 + 2bx > 0, \forall x \in (0, 1)$

Hence, $P(x)$ is increasing in $[0, 1]$.

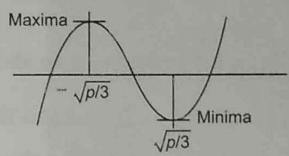
$\therefore$ Maximum of $P(x) = P(1)$

Similarly, $P(X)$ is decreasing in $[-1, 0]$.

Therefore, minimum $P(x)$ does not occur at $x = -1$.

50. Let $\quad f(x) = x^3 - px + q$

Then, $f'(x) = 3x^2 - p$



Put $\ f'(x) = 0$

$\Rightarrow \quad x = \sqrt{\dfrac{p}{3}}, -\sqrt{\dfrac{p}{3}}$

Now, $\quad f''(x) = 6x$

At $\ x = \sqrt{\dfrac{p}{3}}, f''(x) = 6\sqrt{\dfrac{p}{3}} > 0 \quad$ (minima)

and at $x = -\sqrt{\dfrac{p}{3}}, f''(x) < 0 \quad$ (maxima)

51. Let $f(x) = 1 + \dfrac{10}{3\left(x^2 + 3x + \dfrac{7}{3}\right)}$

$$= 1 + \frac{10}{3\left[\left(x + \frac{3}{2}\right)^2 + \frac{1}{12}\right]}$$

So, the maximum value of $f(x)$ at $x = -\dfrac{3}{2}$ is

$$f\left(-\frac{3}{2}\right) = 1 + \frac{10}{3\left(\frac{1}{12}\right)}$$

$$= 1 + 40 = 41$$

Day 15

Indefinite Integrals

Day 15
Outlines ...

- Integral as an Anti-Derivative
- Integration by Substitution
- Integration by Parts
- Integration by Partial Fractions
- Integration using Trigonometric Identities

Integral as an Anti-Derivative

A function $\phi(x)$ is called a **primitive** or **anti-derivative** of a function, if $\phi'(x) = f(x)$. Suppose, $f_1(x)$ and $f_2(x)$ are two anti-derivatives of $f(x)$ on $[a,b]$, then $f_1(x)$ and $f_2(x)$ differ by a constant. The collection of all its anti-derivatives is called **indefinite integral** of $f(x)$ and is denoted by $\int f(x)\, dx$.

Thus, $\frac{d}{dx}\{\phi(x)+c\} = f(x) \quad \Rightarrow \quad \int f(x)\, dx = \phi(x) + C$

where, $\phi(x)$ is anti-derivative of $f(x)$, $f(x)$ is an integrand and C is an arbitrary constant known as the **constant of integration**.

- *Anti-derivative of signum exists in that interval in which $x = 0$ is not included.*
- *Anti-derivative of odd function is always even and of even function is always odd.*

Fundamental Integration Formulae

1. Algebraic Formulae

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int (ax+b)^n dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1} + C, \ n \neq -1$

(iii) $\int \frac{1}{x}\,dx = \log|x| + C$

(iv) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}(\log|ax+b|) + C$

(v) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$, when $x < a$

(vi) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$, when $x > a$

(vii) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

(viii) $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + C$

(ix) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(x) $\int \frac{-1}{a^2 + x^2}\,dx = \frac{1}{a}\cot^{-1}\frac{x}{a} + C$

(xi) $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(xii) $\int \frac{-1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

(xiii) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log|x + \sqrt{x^2 - a^2}| + C$

(xiv) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log|x + \sqrt{x^2 + a^2}| + C$

(xv) $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\log|x + \sqrt{x^2 - a^2}| + C$

(xvii) $\int \sqrt{x^2 + a^2}\,dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\log|x + \sqrt{x^2 + a^2}| + C$

(xviii) $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

2. Trigonometric Formulae

(i) $\int \sin x\,dx = -\cos x + C$

(ii) $\int \cos x = \sin x + C$

(iii) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(iv) $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(v) $\int \sec x\,dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(vi) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(vii) $\int \sec^2 x\,dx = \tan x + C$

(viii) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(ix) $\int \sec x \cdot \tan x\,dx = \sec x + C$

(x) $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + C$

3. Exponential Formulae

(i) $\int e^x dx = e^x + C$

(ii) $\int e^{(ax+b)}\,dx = \frac{1}{a}\cdot e^{(ax+b)} + C$

(iii) $\int a^x\,dx = \frac{a^x}{\log_e a} + C$

(iv) $\int a^{(bx+c)}dx = \frac{1}{b}\cdot\frac{a^{(bx+c)}}{\log_e a} + C$

4. Logarithmic Formulae

(i) $\int \log x\,dx = x\log x - x + C$

(ii) $\int \log(ax+b)\,dx = x\log|ax+b| - x + \frac{b}{a}\log|ax+b| + C$

Some Special Integrals

1. $\int \frac{dx}{ax^2 + bx + c}$

we write, $\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dx}{x^2 + \frac{b}{a}x + \frac{c}{a}}$

$= \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4c}}$, which is of the form

$\int \frac{dx}{X^2 - A^2}, \int \frac{dx}{X^2 + A^2}$ or $\int \frac{dx}{A^2 - X^2}$

2. $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

or $\int \sqrt{ax^2 + bx + c}\,dx$

This can be reduced to one of the form of

$\int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{\sqrt{x^2 - a^2}}, \int \frac{dx}{\sqrt{x^2 + a^2}}$ or

$\int \sqrt{a^2 - x^2}\,dx, \int \sqrt{x^2 - a^2}\,dx, \int \sqrt{a^2 + x^2}\,dx$

3. $\int \frac{(px+q)}{ax^2+bx+c}dx, \int \frac{(px+q)}{\sqrt{ax^2+bx+c}}dx,$

$\int (px+q)\sqrt{ax^2+bx+c}\,dx$

Put $px+q = A$ {differentiation of (ax^2+bx+c)} + B. Find A and B by comparing the coefficients of like powers of x on the two sides.

4. $\int \frac{ax^2+bx+c}{(px^2+qx+r)}dx, \int \frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}dx,$

$\int (ax^2+bx+c)\sqrt{px^2+qx+r}\,dx$

Substitute, $ax^2+bx+c = \lambda\,(px^2+qx+r)$

$$+\mu\left\{\frac{d}{dx}(px^2+qx+r)\right\}+\gamma$$

Find λ, μ and γ. These integrations reduce to integration of three independent functions.

5. $\int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x},$

$\int \frac{dx}{(a\sin x+b\cos x)^2}, \int \frac{dx}{a+b\sin^2 x+c\cos^2 x}$

(i) Divide both the numerator and denominator by $\cos^2 x$.

(ii) Replace $\sec^2 x$ by $1+\tan^2 x$ in the denominator, if any.

(iii) Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Further integrate it.

6. $\int \frac{1}{a\sin x+b\cos x}dx, \int \frac{1}{a+b\sin x}dx,$

$\int \frac{1}{a+b\cos x}dx, \int \frac{1}{a\sin x+b\cos x+c}dx$

(i) Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

(ii) Replace $\tan\frac{x}{2} = t$

(iii) Hence, integral reduces to the form $\int \frac{dt}{at^2+bt+c}$.

7. $\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}dx$ express numerator as,

λ (denominator) + μ (differentiation of denominator) + γ

Find λ, μ and γ by comparing coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals.

8. $\int \frac{x^2+1}{x^4+kx^2+1}dx, \int \frac{x^2-1}{x^4+kx^2+1}dx, \int \frac{dx}{x^4+kx^2+1},$

where $k \in R$

Here, we divide the numerator and denominator by x^2 and put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as required.

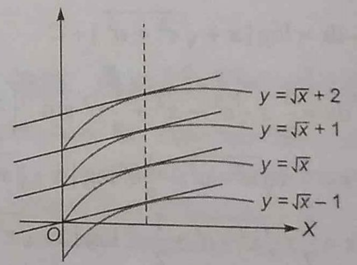
9. $\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx, \int \frac{x^2-a^2}{x^4+kx^2+a^4}dx,$

where k is constant, negative or zero.

Here, we divide numerator and denominator by x^2 and put $x-\frac{a^2}{x}=t$ or $x+\frac{a^2}{x}=t$, respectively.

Geometrical Interpretation of Indefinite Integral

If $\frac{d}{dx}\{\phi(x)\} = f(x)$, then $\int f(x)dx = \phi(x)+C$. For different values of C, we get different functions, differing only by a constant. The graphs of these functions give us an infinite family of curves such that at the points on these curves with the same x-coordinate, the tangents are parallel as they have the same slope $\phi'(x) = f(x)$.



Consider the integral of $\frac{1}{2\sqrt{x}}$, i.e., $\int \frac{1}{2\sqrt{x}}dx = \sqrt{x}+C, C \in R$

Above figure shows some members of the family of curves given by $y=\sqrt{x}+C$ for different $C \in R$.

Integration by Substitutions

The method of reducing a given integral into one on other standard integral by changing the independent variable is called **method of substitution**.

Thus, $\int f(x)dx = \int f\{g(t)\}\cdot g'(t)\,dt$, if we substitute $x = g(t)$ such that $dx = g'(t)\,dt$.

Substitution for Some Irrational Functions

1. $\int \frac{dx}{(x-\alpha)(x-\beta)}, \int \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\, dx$
   $\int \sqrt{(x-\alpha)(\beta-x)}\, dx$, put $x = \alpha \cos^2\theta + \beta \sin^2\theta$
2. $\int \frac{dx}{(px+q)\sqrt{ax+b}}$, put $ax+b=t^2$
3. $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$, put $px+q=t^2$
4. $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, put $px+q=\frac{1}{t}$
5. $\int \frac{dx}{(px^2+r)\sqrt{(ax^2+c)}}$ at first $x=\frac{1}{t}$ and then $a+ct^2=z^2$
6. $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$, where $r \ge 2$ and $r \in I$
   Put $x-k=\frac{1}{t}$
7. $\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}}\, dx$
   Substitute,
   $$ax^2+bx+c = A_1(dx+e)(2fx+g) + B_1(dx+e)+C_1$$

Comparison between Differentiation *and* Integration

- Both differentiation and integration are linear operator on functions as $\frac{d}{dx}\{af(x) \pm bg(x)\} = a\frac{d}{dx}\{f(x)\} \pm b\frac{d}{dx}\{g(x)\}$ and $\int [a\cdot f(x) \pm b\cdot g(x)]dx = a\int f(x)dx \pm b\int g(x)dx$.
- All functions are not differentiable, similarly there are some functions which are not integrable.
- Integral of a function is always discussed in an interval but derivative of a function can be discussed in a interval as well as on a point.
- Geometrically derivative of a function represents slope of the tangent to the graph of function at the point. On the other hand, integral of a function represents an infinite family of curves placed parallel to each other having parallel tangents at points of intersection of the curves with a line parallel to Y-axis.

Integration by Parts

1. If u and v are two functions of x, then
   $$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\, dx = u\int v\, dx - \int\left(\frac{du}{dx}\cdot\int v\, dx\right)dx$$

i.e., The integral of the product of two functions = (First function) × (Integral of second function) − Integral of {(differentiation of first function) × (Integral of second function)}.

Important Substitutions

S.No.	Integrand form	Substitution
(i)	$\sqrt{a^2-x^2}, \frac{1}{\sqrt{a^2-x^2}}, a^2-x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
(ii)	$\sqrt{x^2+a^2}, \frac{1}{\sqrt{x^2+a^2}}, x^2+a^2$	$x = a\tan\theta$ or $x = a\cot\theta$
(iii)	$\sqrt{x^2-a^2}, \frac{1}{\sqrt{x^2-a^2}}, x^2-a^2$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
(iv)	$\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}, \frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$
(v)	$\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}, \frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
(vi)	$\sqrt{\frac{x}{x-a}}, \sqrt{\frac{x-a}{x}}, \sqrt{x(x-a)}, \frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$ or $x = a\,\text{cosec}^2\,\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}, \sqrt{(x-\alpha)(\beta-x)}, (\beta > \alpha)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

We use the following preference order to select a second function

I	→	Inverse function
L	→	Logarithmic function
A	→	Algebraic function
T	→	Trigonometric function
E	→	Exponential function

2. If one of the function is not directly integrable, then we take it as the first function.
3. If only one function is there, *i.e.*, $\int \log x\, dx$, then 1 (unity) is taken as second function.
4. If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

Integral of the Form $\int e^x \{f(x) + f'(x)\}\, dx$

1. $\int e^x \{f(x) + f'(x)\}dx = \int \underset{\text{I}}{e^x}\, f(x)dx + \int \underset{\text{II}}{e^x} f'(x)dx$

$$= f(x)\int e^x dx - \int \{f'(x)\int e^x dx\}dx + \int e^x f'(x)dx$$

$$= f(x)e^x - \int f'(x)e^x dx + \int e^x f'(x)dx = f(x)e^x$$

2. $\int e^{ax} \sin(bx + c)\, dx = \dfrac{e^{ax}}{a^2 + b^2}\{a \sin(bx + c) - b\cos(bx + c)\} + k$

3. $\int e^{ax} \cos(bx + c)dx = \dfrac{e^{ax}}{a^2 + b^2}\{a\cos(bx + c) + b\sin(bx + c)\} + k$

- $\int e^x\{f(x) + f'(x)\}\, dx = e^x f(x) + C$
- $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}}{a^2 + b^2}(a \sin bx - b\cos bx) + C$
- $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$

Integration by Partial Fractions

Before understanding integration by partial fractions, you should know some basic definitions, which are given below

Polynomial of Degree *n*

An expression of the type

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \dots, a_n$ are real numbers $a_0 \neq 0$ and n is positive integer is called a polynomial of degree n.

A function of the form $\dfrac{P}{Q}$, where P and Q are polynomials is called **rational function.** Consider the rational function

$$\frac{x + 7}{(2x - 3)(3x + 4)} = \frac{1}{2x - 3} - \frac{1}{3x + 4}$$

The two fractions on the RHS are called the partial fractions.

Proper and Improper Functions

Any rational algebraic function is called a **proper fraction**, if the degree of numerator is less than degree of its denominator otherwise, it is called an **improper fraction.**

e.g., $\dfrac{x^2 + x + 2}{x^3 + 4x^2 - 7x + 1}$ is a proper fraction,

whereas, $\dfrac{x^4 - 9x^2 - 10x + 7}{x^2 + 4x + 5} = \left\{(x^2 - 4x + 2) + \dfrac{2x - 3}{x^2 + 4x + 5}\right\}$

is an improper fraction.

To integrate the rational function on the LHS, it is enough to integrate the two fractions on the RHS which is easy. This is known as the **method of partial fractions**. Here, we assume that the denominator can be factorised into linear or quadratic factors.

Factorisation of Denominator

The partial fractions depend on the nature of the factors of $Q(x)$. *We have to deal with the following different cases when the factors of* $Q(x)$ *are*

1. Linear and non-repeated.
2. Linear and repeated.
3. Quadratic and non-repeated.
4. Quadratic and different cases can be discussed as repeated.

Integration of Determinant

Let

$$\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\therefore \int \Delta f(x)\, dx = \begin{vmatrix} \int a_{11}(x)\, dx & \int a_{12}(x)\, dx & \int a_{13}(x)\, dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- *The integration of determinant is only applicable, if their is variable only one row or one column, otherwise we expand the determinant and integrate it.*

Different cases can be discussed as

Case I When denominator is expressible as the product of non-repeated linear factors.

Let $Q(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$

Then, we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \frac{A_3}{(x - a_3)} + \ldots + \frac{A_n}{(x - a_n)},$$

where $A_1, A_2, \ldots, A_n$ are constants and can be determined by equating the numerator on RHS to numerator on LHS and then substituting $x = a_1, a_2, \ldots, a_n$.

Case II When the denominator $Q(x)$ is expressible as the product of the linear factors such that some of them are repeating (Linear and Repeated)

Let $Q(x) = (x - a)^k (x - a_1)(x - a_2) \ldots (x - a_r)$.

Then, we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_k}{(x - a)^k} + \frac{B_1}{(x - a_1)} + \frac{B_2}{(x - a_2)} + \ldots + \frac{B_r}{(x - a_r)}$$

Case III When some of the factors in denominator are quadratic and non-repeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction of the type $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in numerator of both sides.

Case IV When some of the factors of the denominator are quadratic and repeating.

For every quadratic repeating factor of the type $(ax^2 + bx + c)^k$, we assume

$$\frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3 x + A_4}{(ax^2 + bx + c)^2} + \ldots + \frac{A_{2k-1} x + A_{2k}}{(ax^2 + bx + c)^k}$$

Integration using Trigonometric Identities

To evaluate integrals of the form $\int \sin mx \cdot \cos nx \, dx$, $\int \sin mx \cdot \sin nx \, dx$, $\int \cos mx \cdot \cos nx \, dx$ and $\int \cos mx \cdot \sin nx \, dx$, we use the following trigonometrical identities

- $2 \sin A \cdot \cos B = \sin (A + B) + \sin (A - B)$
- $2 \cos A \cdot \sin B = \sin (A + B) - \sin (A - B)$
- $2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)$
- $2 \sin A \cdot \sin B = \cos (A - B) - \cos (A + B)$
- $2 \sin A \cdot \cos A = \sin 2A$
- $\cos^2 A = \left(\frac{1 + \cos 2A}{2}\right)$
- $\sin^2 A = \left(\frac{1 - \cos 2A}{2}\right)$
- $\cos^2 A - \sin^2 A = \cos 2A$
- $\sin^2 A + \cos^2 A = 1$

Practice Zone

DAY 15

1. If $\int f(x)\,dx = F(x)$, then $\int x^3 f(x^2)\,dx$ is equal to

(a) $\frac{1}{2}[x^2 \{F(x)\}^2 - \int \{F(x)\}^2\,dx]$

(b) $\frac{1}{2}[x^2 F(x^2) - \int F(x^2)\,d(x^2)]$

(c) $\frac{1}{2}[x^2 F(x) - \frac{1}{2}\int \{F(x)\}^2\,dx]$

(d) None of the above

2. $\int \frac{dx}{1+x+x^2+x^3}$ is equal to

(a) $\log\sqrt{1+x} - \frac{1}{2}\log\sqrt{1+x^2} + \frac{1}{2}\tan^{-1}x + C$

(b) $\log\sqrt{1+x} - \log\sqrt{1+x^2} + \tan^{-1}x + C$

(c) $\log\sqrt{1+x^2} - \log\sqrt{1+x} + \frac{1}{2}\tan^{-1}x + C$

(d) $\log\sqrt{1+x} + \tan^{-1}x + \log\sqrt{1+x^2} + C$

3. $\int\sqrt{x^2+4x+1}\,dx$ is equal to **[NCERT]**

(a) $(x+2)\sqrt{x^2+4x+1} - \frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}| + C$

(b) $\frac{(x+2)}{2}\sqrt{x^2+4x+1} + \frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}| + C$

(c) $\frac{(x+2)}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}| + C$

(d) None of the above

4. If an anti-derivative of $f(x)$ is e^x and that of $g(x)$ is $\cos x$, then $\int f(x)\cos x\,dx + \int g(x)e^x\,dx$ is equal to

(a) $f(x)\cdot g(x) + C$
(b) $f(x) + g(x) + C$
(c) $e^x\cos x + C$
(d) $f(x) - g(x) + C$

5. If $\frac{d}{dx}[f(x)] = x\cos x + \sin x$ and $f(0) = 2$, then $f(x)$ is equal to

(a) $x\sin x$
(b) $x\cos x + \sin x + 2$
(c) $x\sin x + 2$
(d) $x\cos x + 2$

6. If $I_n = \int(\log x)^n\,dx$, then $I_n + nI_{n-1}$ is equal to

(a) $x(\log x)^n$ (b) $(x\log x)^n$ (c) $(\log x)^{n-1}$ (d) $n(\log x)^n$

7. $\int\frac{(x+3)e^x}{(x+4)^2}\,dx$ is equal to

(a) $\frac{1}{(x+4)^2} + C$
(b) $\frac{e^x}{(x+4)^2} + C$
(c) $\frac{e^x}{x+4} + C$
(d) $\frac{e^x}{x+3} + C$

8. If $\int\frac{f(x)}{\log\sin x}\,dx = \log\log\sin x$, then $f(x)$ is equal to

(a) $\sin x$
(b) $\cos x$
(c) $\log\sin x$
(d) $\cot x$

9. If $f(x) = \begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$, then $\int f(x)\,dx$ is equal to

(a) $\frac{x^3}{3} - x^2\sin x + \sin 2x + C$
(b) $\frac{x^3}{3} - x^2\sin x - \cos 2x + C$
(c) $\frac{x^3}{3} - x^2\cos x - \cos 2x + C$
(d) None of the above

10. $\int\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ is equal to

(a) $\sin 2x + C$
(b) $-\frac{1}{2}\sin 2x + C$
(c) $\frac{1}{2}\sin 2x + C$
(d) $-\sin 2x + C$

11. If $f(x) = \int\frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0) = 0$, then the value of $f(1)$ be

(a) $\log(1+\sqrt{2})$
(b) $\log(1+\sqrt{2}) - \frac{\pi}{4}$
(c) $\log(1+\sqrt{2}) + \frac{\pi}{2}$
(d) None of these

12. $\int\frac{(\sin\theta+\cos\theta)}{\sqrt{\sin 2\theta}}\,d\theta$ is equal to

(a) $\log|\cos\theta - \sin\theta + \sqrt{\sin 2\theta}|$
(b) $\log|\sin\theta - \cos\theta + \sqrt{\sin 2\theta}|$
(c) $\sin^{-1}(\sin\theta - \cos\theta) + C$
(d) $\sin^{-1}(\sin\theta + \cos\theta) + C$

13. If $\int \frac{\tan x}{1+\tan x+\tan^2 x}dx = x - \frac{2}{\sqrt{A}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{A}}\right)+C$, then the value of A is

(a) 1 (b) 2
(c) 3 (d) None of these

14. If $g(x)$ be a differentiable function satisfying $\frac{d}{dx}\{g(x)\}=g(x)$ and $g(0)=1$, then $\int g(x)\left(\frac{2-\sin 2x}{1-\cos 2x}\right)dx$ is equal to

(a) $g(x)\cot x + C$ (b) $-g(x)\cot x + C$
(c) $\frac{g(x)}{1-\cos 2x}+C$ (d) None of these

15. $\int e^{2ax}\frac{1-\cos 2ax}{1+\sin 2ax}dx$ is equal to

(a) $-\frac{1}{a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$
(b) $-\frac{1}{2a}e^{2ax}\cot\left(\frac{\pi}{4}+ax\right)+C$
(c) $-\frac{1}{2a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$
(d) $-\frac{1}{a}e^{2ax}\operatorname{cosec}\left(\frac{\pi}{4}+ax\right)+C$

16. If $x^2 \neq n\pi - 1, n \in N$. Then, the value of

$$\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$$ is equal to

(a) $\log\left|\frac{1}{2}\sec(x^2+1)\right|+C$ (b) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$
(c) $\frac{1}{2}\log|\sec(x^2+1)|+C$ (d) None of these

17. If $\int f(x)dx = f(x)$, then $\int \cdot\{f(x)\}^2 dx$ is equal to

(a) $\frac{1}{2}\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $\frac{\{f(x)\}^3}{3}$ (d) $\{f(x)\}^2$

18. $\int \cos^{-\frac{3}{7}} x \sin^{-\frac{11}{7}} x\, dx$ is equal to

(a) $\log|\sin^{\frac{4}{7}} x|+C$ (b) $\frac{4}{7}\tan^{\frac{4}{7}} x+C$
(c) $\frac{-7}{4}\tan^{-\frac{4}{7}} x+C$ (d) $\log|\cos^{\frac{3}{7}} x|+C$

19. $\int \frac{x^2-1}{x^4+x^2+1}dx$ is equal to

(a) $\frac{1}{2}\log\left(\frac{x^2+x+1}{x^2-x+1}\right)+C$ (b) $\frac{1}{2}\log\left(\frac{x^2-x-1}{x^2+x+1}\right)+C$
(c) $\log\left(\frac{x^2-x+1}{x^2+x+1}\right)+C$ (d) $\frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right)+C$

20. $\int \frac{dx}{(1+x^2)\sqrt{p^2+q^2(\tan^{-1}x)^2}}$ is equal to

(a) $\frac{1}{q}\log[q\tan^{-1}x+\sqrt{p^2+q^2(\tan^{-1}x)^2}]+C$
(b) $\log[q\tan^{-1}x+\sqrt{p^2+q^2(\tan^{-1}x)^2}]+C$
(c) $\frac{2}{3q}(p^2+q^2\tan^{-1}x)^{3/2}+C$
(d) None of the above

21. If $x\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$, then $\int\frac{\sin x-\cos x}{\sqrt{1-\sin 2x}}e^{\sin x}\cos x\,dx$ is equal to

(a) $e^{\sin x}+C$
(b) $e^{\sin x-\cos x}+C$
(c) $e^{\sin x+\cos x}+C$
(d) $e^{\cos x-\sin x}+C$

22. If $I=\int e^x\sin 2x\,dx$, then for what value of k, $kI=e^x(\sin 2x-2\cos 2x)+C$?

(a) 1 (b) 3
(c) 5 (d) 7

23. $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ is equal to

(a) $(\cos\theta-\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$
(b) $(\cos\theta+\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$
(c) $\frac{(\cos\theta-\sin\theta)^2}{2}\log\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)+C$
(d) $\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta+C$

24. $\int\frac{x^2}{(x\sin x+\cos x)^2}dx$ is equal to

(a) $\frac{\sin x+\cos x}{x\sin x+\cos x}+C$ (b) $\frac{x\sin x-\cos x}{x\sin x+\cos x}+C$
(c) $\frac{\sin x-x\cos x}{x\sin x+\cos x}+C$ (d) None of these

25. $\int\frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx$ is equal to

(a) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$ (b) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4}+C$
(c) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$ (d) $\frac{1}{3}\left(\frac{x+2}{x-1}\right)^{1/4}+C$

26. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to

(a) $xe^{\tan^{-1}x}+C$ (b) $x^2e^{\tan^{-1}x}+C$
(c) $\frac{1}{x}e^{\tan^{-1}x}+C$ (d) None of these

Directions (Q. Nos. 27 and 28) *If the integrand is a rational function of x and fractional powers of a linear fractional function of the form* $\frac{ax+b}{cx+d}$, *then rationalisation of the integral is affected by the substitution* $\frac{ax+b}{cx+d}=t^m$, *where m is LCM of fractional powers of* $\frac{ax+b}{cx+d}$.

27. If $I=\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = A\sqrt[4]{\frac{x-1}{x+2}}+C$, then A is equal to

(a) 1/3 (b) 2/3 (c) 3/4 (d) 4/3

28. If $I=\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = k\sqrt[3]{\frac{1+x}{1-x}}+C$, then k is equal to

(a) 2/3 (b) 3/2 (c) 1/3 (d) 1/2

Directions (Q. Nos. 29 to 32) *Each of these questions contains two statements: Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true; Statement II is false.
(d) Statement I is false; Statement II is true.

29. Statement I $\int \frac{(3-2x)}{\sqrt{(4+2x-x^2)}}dx = 2\sqrt{(4+2x-x^2)} + \sin^{-1}\left(\frac{x-1}{\sqrt5}\right)+C$

Statement II $\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)$

30. If $I_n=\int \tan^n x\,dx$, then

Statement I $5(I_6+I_4)=\tan^5 x$

Statement II $(n-1)(I_n+I_{n-2})=\tan^{n-1}x$

31. Statement I $\int \frac{d\,(x^2+1)}{\sqrt{x^2+2}}$ is equal to $2\sqrt{x^2+2}+C$.

Statement II $\int \frac{x^{9/2}}{\sqrt{1+x^{11}}}dx$ is $\frac{2}{11}\log|x+\sqrt{1+x^{11}}|+C$.

32. If $a>0$ and $b^2-4ac<0$, then

Statement I The value of the integral $\int \frac{dx}{ax^2+bx+c}$ will be of the type of $\mu\tan^{-1}\frac{x+A}{B}+C$, where A, B, C and μ are constants.

Statement II ax^2+bx+c can be written as sum of two squares.

33. $\int \frac{dx}{(\sin x+2)(\sin x-1)}$ is equal to

(a) $\frac{2}{3\left(\tan\frac{x}{2}-1\right)}-\frac{2}{3\sqrt3}\tan^{-1}\left[\frac{2\left(\tan\frac{x}{2}+\frac12\right)}{\sqrt3}\right]+C$

(b) $\frac{2}{\left(\tan\frac{x}{2}+1\right)}+C$

(c) $-\frac{2}{3}\frac{1}{\left(\tan\frac{x}{2}-1\right)}+\frac{2}{3\sqrt3}\tan^{-1}\left[\frac{2\tan\frac{x}{2}-1}{\sqrt3}\right]+C$

(d) $\frac{2}{3\left(\tan\frac{x}{2}-1\right)}+\frac{2}{3\sqrt3}\tan^{-1}\left[\frac{2\tan\frac{x}{2}-1}{\sqrt3}\right]+C$

34. $\int \frac{x^2}{(2+3x^2)^{5/2}}dx$ is equal to

(a) $\frac15\left[\frac{x^2}{2+3x^2}\right]^{3/2}+C$

(b) $\frac16\left[\frac{x^2}{2+3x^2}\right]^{3/2}+C$

(c) $\frac16\left[\frac{x^2}{2+3x^2}\right]^{7/2}+C$

(d) None of these

35. $\int \frac{5x-2}{1+2x+3x^2}dx$ is equal to [NCERT]

(a) $\frac56\log|1+2x+3x^2|-\frac{11}{3\sqrt2}\tan^{-1}\left(\frac{3x+1}{\sqrt2}\right)+C$

(b) $\frac56\log|1+2x+3x^2|-\frac{11}{3}\tan^{-1}\left(\frac{3x+1}{\sqrt2}\right)+C$

(c) $\frac13\log|1+2x+3x^2|-\frac{11}{3\sqrt2}\tan^{-1}\left(\frac{3x+1}{\sqrt2}\right)+C$

(d) None of the above

36. $\int \frac{dx}{2+\sin x+\cos x}$ is equal to [NCERT Exemplar]

(a) $\sqrt2\tan^{-1}\left(\frac{\tan x/2+1}{\sqrt2}\right)+C$

(b) $\tan^{-1}\left(\frac{\tan x/2+1}{\sqrt2}\right)+C$

(c) $\sqrt2\tan^{-1}\left(\frac{\tan x/2}{\sqrt2}\right)+C$

(d) None of the above

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37. If $\int \frac{dx}{x + x^7} = p(x)$, then $\int \frac{x^6}{x + x^7} dx$ is equal to [JEE Main 2013]

(a) $\log|x| - p(x) + C$
(b) $\log|x| + p(x) + C$
(c) $x - p(x) + C$
(d) $x + p(x) + C$

38. If $\int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1} x} dx = A(x) e^{\cot^{-1} x} + C$, then $A(x)$ is equal to [JEE Main 2013]

(a) $-x$ (b) x
(c) $\sqrt{1 - x}$ (d) $\sqrt{1 + x}$

39. The integral $\int \frac{x\,dx}{2 - x^2 + \sqrt{2 - x^2}}$ is equals to [JEE Main 2013]

(a) $\log|1 + \sqrt{2 + x^2}| + C$
(b) $-\log|1 + \sqrt{2 - x^2}| + C$
(c) $-x\log|1 - \sqrt{2 - x^2}| + C$
(d) $x\log|1 - \sqrt{2 + x^2}| + C$

40. In the integral

$\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A\cos 8x + k$, where k is an arbitrary constant, then A is equal to [JEE Main 2013]

(a) $-\frac{1}{16}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

41. If $\int f(x)dx = \Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to

(a) $\frac{1}{3}[x^3\Psi(x^3)] - \int x^2\Psi(x^3)dx] + C$
(b) $\frac{1}{3}[x^3\Psi(x^3)] - 3\int x^3\Psi(x^3)dx + C$
(c) $\frac{1}{3}[x^3\Psi(x^3)] - \int x^2\Psi(x^3)dx] + C$
(d) $\frac{1}{3}[x^3\Psi(x^3)] - \int x^3\Psi(x^3)dx] + C$

42. If the integral

$\int \frac{5\tan x}{\tan x - 2} dx = x + a\ln|\sin x - 2\cos x| + k$, then a is equal to [AIEEE 2012]

(a) -1 (b) -2
(c) 1 (d) 2

43. The value of $\sqrt{2}\int \frac{\sin x\,dx}{\sin\left(x - \frac{\pi}{4}\right)}$ is [AIEEE 2008]

(a) $x - \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$
(b) $x + \log\left|\cos\left(x - \frac{\pi}{4}\right)\right| + C$
(c) $x - \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$
(d) $x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

44. $\int \frac{dx}{\cos x + \sqrt{3}\sin x}$ is equal to [AIEEE 2007]

(a) $\frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$ (b) $\frac{1}{2}\log\tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$
(c) $\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$ (d) $\log\tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

45. $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is equal to [AIEEE 2005]

(a) $\frac{xe^x}{1 + x^2} + C$ (b) $\frac{x}{(\log x)^2 + 1} + C$
(c) $\frac{\log x}{(\log x)^2 + C}$ (d) $\frac{x}{x^2 + 1} + C$

46. If $\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B\log\sin(x - \alpha) + C$, then the value of (A, B) is [AIEEE 2004]

(a) $(\sin\alpha, \cos\alpha)$ (b) $(\cos\alpha, \sin\alpha)$
(c) $(-\sin\alpha, \cos\alpha)$ (d) $(-\cos\alpha, \sin\alpha)$

47. $\int \frac{dx}{\cos x - \sin x}$ is equal to [AIEEE 2004]

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$
(b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right| + C$
(c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + C$
(d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$

Answers

1. (b)	2. (a)	3. (c)	4. (c)	5. (c)	6. (a)	7. (c)	8. (d)	9. (d)	10. (b)
11. (c)	12. (c)	13. (c)	14. (b)	15. (b)	16. (b)	17. (a)	18. (c)	19. (d)	20. (a)
21. (a)	22. (c)	23. (d)	24. (c)	25. (a)	26. (a)	27. (d)	28. (b)	29. (c)	30. (a)
31. (c)	32. (a)	33. (a)	34. (b)	35. (a)	36. (a)	37. (a)	38. (b)	39. (b)	40. (a)
41. (c)	42. (d)	43. (d)	44. (a)	45. (b)	46. (b)	47. (d)			

Hints & Solutions

1. We have, $\int f(x)\,dx = F(x)$

$\therefore \quad \int x^3 f(x^2)\,dx = \frac{1}{2}\int x^2 f(x^2)\,d(x^2)$

$= \frac{1}{2}[x^2 F(x^2) - \int F(x^2)\,d(x^2)]$

$= \frac{1}{2}[x^2 F(x^2) - \int F(x^2)\,d(x^2)]$

2. $\int \frac{dx}{(1+x)(1+x^2)} = \frac{1}{2}\int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{1}{1+x}dx - \frac{1}{2}\int \frac{x}{1+x^2}dx$

$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log(1+x) - \frac{1}{4}\log(1+x^2) + C$

$= \frac{1}{2}\tan^{-1}x + \log\sqrt{1+x} - \frac{1}{2}\log\sqrt{1+x^2} + C$

3. Let $I = \int \sqrt{x^2+4x+1}\,dx = \int \sqrt{x^2+4x+1-2^2+2^2}\,dx$

$= \int \sqrt{(x+2)^2 - (\sqrt{3})^2}\,dx$

$\left(\because \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right)$

$I = \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log|(x+2)+\sqrt{x^2+4x+1}| + C$

4. $\int f(x)\cos x\,dx + \int g(x)e^x dx$

$= \frac{e^x}{2}(\cos x + \sin x) - \frac{e^x}{2}(\sin x - \cos x) + C$

$= \frac{e^x}{2}(2\cos x) + C = e^x\cos x + C$

5. $f(x) = \int (x\cos x + \sin x)dx = x\sin x + C$

$f(0) = 2 \Rightarrow C = 2$

$\therefore \quad f(x) = x\sin x + 2$

6. $I_n = \int (\log x)^n dx$

$= x(\log x)^n - n\int (\log x)^{n-1}\cdot\frac{1}{x}\cdot x\,dx$

$\therefore \quad I_n + nI_{n-1} = x(\log x)^n$

7. Let $I = \int \frac{(x+3)e^x}{(x+4)^2}dx = \int \frac{(x+4-1)e^x}{(x+4)^2}dx$

$= \int e^x\left(\frac{1}{x+4} - \frac{1}{(x+4)^2}\right)dx = \frac{e^x}{x+4} + C$

8. $\int \frac{f(x)}{\log\sin x}dx = \log\log\sin x$

On differentiating both sides w.r.t. x, we get

$\frac{f(x)}{\log\sin x} = \frac{\cot x}{\log\sin x}$

$\therefore \quad f(x) = \cot x$

9. We have, $f(x) = \begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$

$\Rightarrow \quad f(x) = \begin{vmatrix} 0 & \sin x - x^2 & 2-\cos x \\ x^2-\sin x & 0 & 2x-1 \\ \cos x - 2 & 1-2x & 0 \end{vmatrix}$

(interchanging rows and columns)

$\Rightarrow \quad f(x) = (-1)^3\begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$

[taking (–1) common from each column]

$\Rightarrow \quad f(x) = -f(x) \Rightarrow f(x) = 0$

$\therefore \int f(x)\,dx = 0$

10. $\int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}dx$

$= \int (\sin^4 x - \cos^4 x)dx$

$= \int (\sin^2 x - \cos^2 x)dx$

$= \int -\cos 2x\,dx = -\frac{\sin 2x}{2} + C$

11. $f(x) = \int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta = (1+x^2)\,d\theta$

$\therefore \quad f(x) = \int \frac{\tan^2\theta\sec^2\theta}{\sec^2\theta(1+\sec\theta)}d\theta$

$= \int \frac{1-\cos^2\theta}{\cos\theta\,(1+\cos\theta)}d\theta$

$= \int \sec\theta\,d\theta - \int d\theta$

$$= \log(\sec\theta + \tan\theta) - \theta + C$$
$$= \log(x + \sqrt{1+x^2}) - \tan^{-1}x + C$$
$$\Rightarrow \quad f(0) = \log(0 + \sqrt{1+0}) - \tan^{-1}(0) + C \Rightarrow C = 0$$
$$\therefore \quad f(1) = \log(1+\sqrt{2}) - \frac{\pi}{4} + 0$$

12. Let $I = \int \frac{\sin\theta + \cos\theta}{\sqrt{1-(1-2\sin\theta\cos\theta)}} d\theta$

$$= \int \frac{\sin\theta + \cos\theta}{\sqrt{1-(\sin\theta - \cos\theta)^2}} d\theta$$

Put $\sin\theta - \cos\theta = t$

$\Rightarrow (\cos\theta + \sin\theta)\, d\theta = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C$$
$$= \sin^{-1}(\sin\theta - \cos\theta) + C$$

13. $I = \int \frac{\tan x}{1 + \tan x + \tan^2 x} dx = \int \frac{\frac{\sin x}{\cos x}}{1 + \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x}} dx$

$$= \int \frac{\sin 2x}{2 + \sin 2x} dx$$
$$= \int dx - 2\int \frac{dx}{2 + \sin 2x}$$
$$= x - 2\int \frac{\sec^2 x}{2\sec^2 x + 2\tan x} dx$$

Let $\tan x = t \quad \sec^2 x\, dx = dt$

$$\therefore \quad = x - \frac{2}{2}\int \frac{dt}{t^2 + t + 1}$$
$$= x - \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$\Rightarrow \quad I = x - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C$$

Hence, we get $A = 3$.

14. We have, $\frac{d}{dx}\{g(x)\} = g(x)$

$$\Rightarrow \quad g'(x) = g(x)$$
$$\Rightarrow \quad \int \frac{g'(x)}{g(x)} dx = \int 1\, dx$$
$$\Rightarrow \quad \log_e\{g(x)\} = x + \log C_1$$
$$g(x) = C_1 e^x$$

Now, $g(0) = 1$

$$\Rightarrow \quad C_1 = 1$$
$$\therefore \quad g(x) = e^x$$
$$\therefore \quad \int g(x)\left(\frac{2 - \sin 2x}{1 - \cos 2x}\right) dx = \int e^x(\operatorname{cosec}^2 x - \cot x)\, dx$$
$$= -e^x \cot x + C$$
$$= -g(x)\cot x + C$$

15. Let $I = \int e^{2ax} \frac{1 - \cos 2ax}{1 + \sin 2ax} dx$

$$\Rightarrow I = \frac{1}{a}\int e^{2t} \frac{1 - \cos 2t}{1 + \sin 2t} dt, \text{ where } ax = t$$
$$\Rightarrow I = \frac{1}{a}\int e^{2t} \frac{1 - 2\sin\left(\frac{\pi}{4} + t\right)\cdot\cos\left(\frac{\pi}{4} + t\right)}{2\sin^2\left(\frac{\pi}{4} + t\right)} dt$$
$$\Rightarrow I = \frac{1}{a}\int e^{2t}\left\{\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4} + t\right) - \cot\left(\frac{\pi}{4} + t\right)\right\} dt$$
$$\Rightarrow I = \frac{1}{2a}\int e^{2t} \operatorname{cosec}^2\left(\frac{\pi}{4} + t\right) dt - \frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4} + t\right) dt$$
$$\Rightarrow I = -\frac{1}{2a} e^{2t}\cot\left(\frac{\pi}{4} + t\right) + \frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4} + t\right) dt$$
$$- \frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4} + t\right) dt + C$$
$$\Rightarrow \quad I = -\frac{1}{2a} e^{2t}\cot\left(\frac{\pi}{4} + t\right) + C$$
$$\therefore \quad I = -\frac{1}{2a} e^{2ax}\cot\left(\frac{\pi}{4} + ax\right) + C$$

16. We have, $\int x\sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}} dx$

$$= \int x\sqrt{\frac{2\sin(x^2+1) - 2\sin(x^2+1)\cdot\cos(x^2+1)}{2\sin(x^2+1) + 2\sin(x^2+1)\cdot\cos(x^2+1)}} dx$$
$$= \int x\sqrt{\frac{1 - \cos(x^2+1)}{1 + \cos(x^2+1)}} dx$$
$$= \int x\tan\left(\frac{x^2+1}{2}\right) dx$$
$$\therefore \quad \int \tan\left(\frac{x^2+1}{2}\right) d\left(\frac{x^2+1}{2}\right)$$
$$= \log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$$

17. We have, $\int f(x)\, dx = f(x)$

$$\Rightarrow \quad \frac{d}{dx}\{f(x)\} = f(x)$$
$$\Rightarrow \quad \frac{1}{f(x)} d\,[f(x)] = dx$$
$$\Rightarrow \quad \log\{f(x)\} = x + \log C$$
$$\Rightarrow \quad f(x) = Ce^x$$
$$\Rightarrow \quad \{f(x)\}^2 = C^2 e^{2x}$$
$$\therefore \quad \int \{f(x)\}^2 dx = \int C^2 e^{2x}$$
$$= \frac{C^2 e^{2x}}{2}$$
$$= \frac{1}{2}\{f(x)\}^2$$

18. Here, $m+n=\frac{-3}{7}+\left(\frac{-11}{7}\right)=-2$

$$I=\int\cos^{-3/7}x(\sin^{(-2+3/7)}x)dx$$
$$=\int\cos^{-3/7}x\sin^{-2}x\sin^{3/7}x\,dx$$
$$=\int\frac{\text{cosec}^2x}{\left(\frac{\cos^{3/7}x}{\sin^{3/7}x}\right)}dx=\int\frac{\text{cosec}^2x}{\cot^{3/7}x}dx$$

Put $\cot x=t$

$\Rightarrow\ -\text{cosec}^2x\,dx=dt$

$$\therefore\quad I=-\int\frac{dt}{t^{3/7}}=\frac{-7}{4}t^{4/7}+C$$
$$=-\frac{7}{4}\tan^{-4/7}x+C$$

19. $\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-1}dx$

Put $\quad x+\frac{1}{x}=t\Rightarrow\left(1-\frac{1}{x^2}\right)dx=dt$

$$\therefore\quad\int\frac{dt}{t^2-1}=\frac{1}{2}\log\left(\frac{t-1}{t+1}\right)+C$$
$$=\frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right)+C$$

20. Put $q\tan^{-1}x=t$

$$\Rightarrow\quad\frac{q}{1+x^2}dx=dt$$
$$\Rightarrow\quad\frac{1}{1+x^2}dx=\frac{dt}{q}$$
$$\therefore\quad\int\frac{dt}{q\sqrt{p^2+t^2}}=\frac{1}{q}\log\left[t+\sqrt{p^2+t^2}\right]$$
$$=\frac{1}{q}\log\left[q\tan^{-1}x+\sqrt{p^2+q^2(\tan^{-1}x)^2}\right]+C$$

21. $\int\frac{\sin x-\cos x}{\sin x-\cos x}e^{\sin x}\cdot\cos x\,dx$

$$=\int e^{\sin x}\cdot\cos x\,dx=e^{\sin x}+C$$

22. $I=\sin 2x\cdot e^x-2\int\cos 2x\cdot e^x dx$

$=\sin 2x\cdot e^x-2\cos 2x\cdot e^x-4\int e^x\sin 2x\,dx+C$

$\Rightarrow 5I=e^x(\sin 2x-2\cos 2x)+C$

$\therefore\ k=5$

23. Since, $\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\log\tan\left(\frac{\pi}{4}+\theta\right)$

and $\quad\int\sec\theta\,d\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

or $\quad\int\sec 2\theta\,d\theta=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\theta\right)$

$$2\sec 2\theta=\frac{d}{d\theta}\log\tan\left(\frac{\pi}{4}+\theta\right)$$
$$\therefore\quad I=\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\int\tan 2\theta\,d\theta$$
$$=\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta+C$$

24. Let $I=\int\frac{x^2}{(x\sin x+\cos x)^2}dx$

$$=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\cdot\frac{x}{\cos x}dx$$
$$\left[\because\frac{d}{dx}(x\sin x+\cos x)=x\cos x\right]$$
$$\therefore\quad I=\frac{-1}{(x\sin x+\cos x)}\cdot\frac{x}{\cos x}+\int\frac{1}{(x\sin x+\cos x)}\frac{\cos x-x(-\sin x)}{\cos^2x}dx$$
$$=\frac{-x}{(x\sin x+\cos x)\cos x}+\tan x+C$$
$$=\frac{-x+x\sin^2x+\sin x\cdot\cos x}{(x\sin x+\cos x)\cdot\cos x}+C$$
$$=\frac{\sin x-x\cos x}{x\sin x+\cos x}+C$$

25. $\int\frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx$

$$=\int\frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}dx$$
$$=\frac{1}{3}\int\frac{1}{t^{3/4}}dt\quad\left(\text{put }\frac{x-1}{x+2}=t\Rightarrow\frac{3}{(x+2)^2}dx=dt\right)$$
$$=\frac{1}{3}\left(\frac{t^{1/4}}{\frac{1}{4}}\right)+C=\frac{4}{3}t^{1/4}+C$$
$$=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$$

26. Put $\tan^{-1}x=t\Rightarrow\frac{dx}{1+x^2}=dt$

$$\therefore\quad\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx=\int e^t(\tan t+\sec^2 t)dt$$
$$=e^t\tan t+C=xe^{\tan^{-1}x}+C$$

27. Given, $I=\int\frac{dx}{(x-1)^2\sqrt[4]{\left(\frac{x+2}{x-1}\right)^5}}$

Put $\quad\frac{x+2}{x-1}=t$

$$\Rightarrow\quad\frac{-3}{(x-1)^2}dx=dt$$

$\therefore \quad I = -\frac{1}{3}\int \frac{dt}{x^{5/4}} = \frac{4}{3}\left[\frac{1}{t^{-1/4}}\right] + C$

$= \frac{4}{3}\left[\sqrt[4]{\frac{x-1}{x+2}}\right] + C$

$\therefore \quad A = \frac{4}{3}$

28. $I = \int \frac{dx}{(1-x)^2 \sqrt[3]{\left(\frac{x+1}{1-x}\right)^2}}$

Put $\frac{1+x}{1-x} = t \Rightarrow \frac{2}{(1-x)^2}dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t^{2/3}} = \frac{3}{2}[t^{1/3}] + C$

$= \frac{3}{2}\left[\sqrt{\frac{1+x}{1-x}} + C\right]$

$\therefore \quad k = \frac{3}{2}$

29. Let $I = \int \frac{(2-2x)}{\sqrt{(4+2x-x^2)}}dx + \int \frac{dx}{\sqrt{(4+2x-x^2)}}$

$= 2\sqrt{4+2x-x^2} + \int \frac{dx}{\sqrt{5-(x-1)^2}}$

$= 2\sqrt{4+2x-x^2} + \sin^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C$

30. Given, $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \cdot \tan^2 x\, dx$

$= \int \tan^{n-2} x \cdot (\sec^2 x - 1)\, dx$

$= \int \tan^{n-2} x \cdot \sec^2 x\, dx - \int \tan^{n-2} x\, dx$

$= \int \tan^{n-2} x \cdot \sec^2 x\, dx - I_{n-2}$

$\Rightarrow \quad I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$

Hence, $(n-1)(I_n + I_{n-2}) = \tan^{n-1} x$

Put $\quad n = 6, 5(I_6 + I_4) = \tan^5 x$

31. **Statement I** $I = \int \frac{dt}{\sqrt{1+t}}$, where $x^2 + 1 = t$

$= 2\sqrt{1+t} + C = 2\sqrt{x^2+2} + C$

Statement II $\quad I = \int \frac{x^{9/2}}{\sqrt{1+x^{11}}}dx$

Put $\quad x^{11/2} = t$, then

$t = \frac{2}{11}\log(x^{11/2} + \sqrt{1+x^{11}}) + C$

32. If $a > 0$ and $b^2 - 4ac < 0$, then

$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

$\Rightarrow \int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a\left(x + \frac{b}{2a}\right)^2 + k^2}$

where, $k^2 = \frac{4ac - b^2}{4a} > 0$

which will have an answer of the type

$\frac{1}{a} \cdot \frac{1}{\frac{k}{\sqrt{a}}} \tan^{-1} \frac{x + \frac{b}{2a}}{\frac{k}{\sqrt{a}}} + C$

$\therefore \quad \mu \tan^{-1} \frac{x + A}{B} + C$

33. $\int \frac{dx}{(\sin x + 2)(\sin x - 1)} = \frac{1}{3}\int \frac{dx}{(\sin x - 1)} - \frac{1}{3}\int \frac{dx}{(\sin x + 2)}$

$= \frac{1}{3}\int \frac{dx}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} - 1\right)} - \frac{1}{3}\int \frac{dx}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 2\right)}$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$\therefore \quad \frac{1}{3}\int \frac{2dt}{2t - 1 - t^2} - \frac{1}{3}\int \frac{2dt}{2t + 2t^2 + 2}$

$= -\frac{2}{3}\int \frac{dt}{(t-1)^2} - \frac{1}{3}\int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{3}\frac{1}{(t-1)} - \frac{1}{3}\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(t + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C$

$= \frac{2}{3}\frac{1}{\left(\tan\frac{x}{2} - 1\right)} - \frac{2}{3\sqrt{3}}\tan^{-1}\frac{\left(\tan\frac{x}{2} + \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C$

$= \frac{2}{3\left(\tan\frac{x}{2} - 1\right)} - \frac{2}{3\sqrt{3}}\tan^{-1}\left[\frac{2\left(\tan\frac{x}{2} + \frac{1}{2}\right)}{\sqrt{3}}\right] + C$

34. $\int \frac{x^2}{(2 + 3x^2)^{5/2}}dx$

Substitute $\; 2 + 3x^2 = t^2x^2$

$\Rightarrow \quad x^2 = \frac{2}{(t^2 - 3)}$

$\therefore \quad dx = -\frac{2t}{x(t^2 - 3)^2}dt$

$\therefore \quad \int \frac{x^2}{(tx)^5} \cdot \left(\frac{-2t}{x(t^2 - 3)^2}\right)dt = -2\int \frac{dt}{4t^4}$

$= -\frac{1}{2}\int \frac{dt}{t^4}$

$= \frac{1}{6t^3} + C$

$= \frac{1}{6}\left(\frac{x^2}{2 + 3x^2}\right)^{3/2} + C$

35. $\int \frac{5x-2}{1+2x+3x^2}dx$

Let $5x-2 = A\frac{d}{dx}(1+2x+3x^2)+B$

$\Rightarrow 5x-2 = A(2+6x)+B$

$\Rightarrow 5x-2 = 6Ax+(2A+B)$

On equating the coefficient of x and constant on both sides, we get

$$5 = 6A \Rightarrow A = \frac{5}{6} \text{ and } 2A+B = -2 \Rightarrow B = \frac{-11}{3}$$

$\therefore \quad I = \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$

$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{dx}{1+2x+3x^2}$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}$

and $I_2 = \int \frac{dx}{1+2x+3x^2}$

$\therefore \quad I = \frac{5}{6}I_1 - \frac{11}{3}I_2$ ...(i)

Now, $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$

Put $1+2x+3x^2 = t$

$\Rightarrow (2+6x)\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| + C_1$

$\therefore I_1 = \log|1+2x+3x^2| + C_1$ ...(ii)

Also, $I_2 = \int \frac{dx}{1+2x+3x^2}$

$\therefore \quad I_2 = \frac{1}{3}\int \frac{dx}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}$

$= \frac{1}{3}\left[\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right)\right] + C_2$

$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C_2$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C_2$ ...(iii)

On substituting the values of I_1 and I_2 from Eqs. (ii) and (i), we get

$I = \frac{5}{6}[\log|1+2x+3x^2|] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$

$\left(\because \frac{5}{6}C_1 = \frac{11}{3}C_2 = C\right)$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

36. Let $I = \int \frac{dx}{2+\sin x+\cos x}$

$\Rightarrow I = \int \frac{dx}{2+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$= \int \frac{\sec^2\frac{x}{2}dx}{2+2\tan^2\frac{x}{2}+2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$

$I = \int \frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+3}$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$\therefore \quad I = \int \frac{2dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$

$= 2\int \frac{2dt}{(t+1)^2+(\sqrt{2})^2} = 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + C$

$\Rightarrow \quad I = \sqrt{2}\tan^{-1}\left(\frac{\tan x/2+1}{\sqrt{2}}\right) + C$

37. Let $I = \int \frac{x^6}{x+x^7}dx = \int \frac{x^6}{x(1+x^6)}dx$

$= \int \frac{(1+x^6)-1}{x(1+x^6)}dx$

$\Rightarrow \quad I = \int \frac{dx}{x} - \int \frac{dx}{x+x^7} = \log|x| - P(x) + C$

38. LHS $= \int \left[\frac{x^2+1}{x^2+1} - \frac{x}{x^2+1}\right]e^{\cot^{-1}x}dx$

$= \int 1\cdot e^{\cot^{-1}x}dx - \int \frac{x}{x^2+1}e^{\cot^{-1}x}dx$

$= xe^{\cot^{-1}x} - \int x\cdot e^{\cot^{-1}x}\left(-\frac{1}{1+x^2}\right)dx$

$- \int \frac{x}{1+x^2}e^{\cot^{-1}x}dx + c = xe^{\cot^{-1}x} + C$

39. Put $t^2 = 2-x^2$

$t\,dt = -x\,dx$

$\int \frac{dt}{t+1} = -\log|1+t| + C = -\log|1+\sqrt{2-x^2}| + C$

40. LHS $= \int \frac{2\cos^2 4x}{\frac{\cos^2 2x - \sin^2 2x}{\cos 2x \sin 2x}}dx$

$= \int \frac{2\cos^2 4x \times \cos 2x \sin 2x}{\cos 4x}dx$

$= \int \cos 4x \times \sin 4x dx = \frac{1}{2}\int \sin 8x dx$

$= \frac{-1}{2}\frac{\cos 8x}{8} + k$

41. Given, $\int f(x)\,dx = \Psi(x)$

Let $I = \int x^5 f(x^3)\,dx$

Put $x^3 = t \Rightarrow x^2 dx = \dfrac{dt}{3}$ ...(i)

$$\therefore\quad I = \frac{1}{3}\int t\ f(t)\,dt$$
$$= \frac{1}{3}[t\,\Psi(t) - \int \Psi(t)\,dt]$$
$$= \frac{1}{3}[x^3\Psi(x^3) - 3\int x^2\Psi(x^3)\,dx] + C \quad \text{[from Eq. (i)]}$$
$$= \frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)\,dx + C$$

42. Given,

$$\int \frac{5\tan x}{\tan x - 2}\,dx = x + a\ln|\sin x - 2\cos x| + k \quad \text{...(i)}$$

Now, let us assume that

$$I = \int \frac{5\tan x}{\tan x - 2}\,dx$$

On multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5\sin x}{\sin x - 2\cos x}\,dx$$

Let $5\sin x = A(\sin x - 2\cos x) + B(\cos x + 2\sin x)$

$\Rightarrow$ $0\cos x + 5\sin x = (A + 2B)\sin x + (B - 2A)\cos x$

On comparing the coefficients of $\sin x$ and $\cos x$, we get

$$A + 2B = 5 \text{ and } B - 2A = 0$$

$\Rightarrow$ $A = 1$ and $B = 2$

$\Rightarrow$ $5\sin x = (\sin x - 2\cos x) + 2(\cos x + 2\sin x)$

$$\Rightarrow\quad I = \int \frac{5\sin x}{\sin x - 2\cos x}\,dx$$
$$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)}\,dx$$
$$\Rightarrow\quad I = \int \frac{\sin x - 2\cos x}{\sin x - 2\cos x}\,dx + 2\int \frac{(\cos x + 2\sin x)}{(\sin x - 2\cos x)}\,dx$$
$$\Rightarrow\quad I = \int 1\,dx + 2\int \frac{d(\sin x - 2\cos x)}{(\sin x - 2\cos x)}$$
$$I = x + 2\log|(\sin x - 2\cos x)| + k \quad \text{...(ii)}$$

where, k is the constant of integration.

By comparing the value of I in Eqs. (i) and (ii), we get

$$a = 2$$

43. Let $I = \sqrt{2}\int \dfrac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)}\,dx$

Put $x - \dfrac{\pi}{4} = t \Rightarrow dx = dt$

$$\therefore\quad I = \sqrt{2}\int \frac{\sin\left(\frac{\pi}{4} + t\right)dt}{\sin t}$$
$$= \sqrt{2}\int\left(\frac{1}{\sqrt{2}}\cot t + \frac{1}{\sqrt{2}}\right)dt$$
$$= \log|\sin t| + t + C$$
$$= x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$$

44. Let $I = \displaystyle\int \frac{dx}{2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)} = \frac{1}{2}\int \sec\left(x - \frac{\pi}{3}\right)dx$

$$= \frac{1}{2}\log\tan\left(\frac{x}{2} - \frac{\pi}{6} + \frac{\pi}{4}\right) + C$$
$$= \frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$$

45. Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$[\because \int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C]$$

$$\therefore\ \int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt = \frac{e^t}{1+t^2} + C = \frac{x}{1 + (\log x)^2} + C$$

46. Put $x - \alpha = t \Rightarrow dx = dt$

$$\therefore\quad I = \int \frac{\sin(t+\alpha)}{\sin t}\,dt$$
$$= \int \cos\alpha\,dt + \int \sin\alpha\cdot\frac{\cos t}{\sin t}\,dt$$
$$= \cos\alpha\cdot t + \sin\alpha\log\sin t + C$$
$$= x\cos\alpha + \sin\alpha\log\{\sin(x-\alpha)\} + C$$

(let $C = -\alpha\cos\alpha$)

$\therefore$ $A = \cos\alpha,\ B = \sin\alpha$

47. Let $I = \dfrac{1}{\sqrt{2}}\displaystyle\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)} = \frac{1}{\sqrt{2}}\int \sec\left(x + \frac{\pi}{4}\right)dx$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{4} + \frac{x}{2} + \frac{\pi}{8}\right)\right| + C$$
$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$$