Trishna's

IIT JEE

Super Course in

Physics

Mechanics II

ALWAYS LEARNING

PEARSON

Super Course in Physics

MECHANICS II

for the IIT-JEE

[Volume 2]

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Preface

The IIT-JEE, the most challenging amongst national level engineering entrance examinations, remains on the top of the priority list of several lakhs of students every year. The brand value of the IITs attracts more and more students every year, but the challenge posed by the IIT-JEE ensures that only the *best* of the aspirants get into the IITs. Students require thorough understanding of the fundamental concepts, reasoning skills, ability to comprehend the presented situation and exceptional problem-solving skills to come on top in this highly demanding entrance examination.

The pattern of the IIT-JEE has been changing over the years. Hence an aspiring student requires a step-by-step study plan to master the fundamentals and to get adequate practice in the various types of questions that have appeared in the IIT-JEE over the last several years. Irrespective of the branch of engineering study the student chooses later, it is important to have a sound conceptual grounding in Mathematics, Physics and Chemistry. A lack of proper understanding of these subjects limits the capacity of students to solve complex problems thereby lessening his/her chances of making it to the top-notch institutes which provide quality training.

This series of books serves as a source of learning that goes beyond the school curriculum of Class XI and Class XII and is intended to form the backbone of the preparation of an aspiring student. These books have been designed with the objective of guiding an aspirant to his/her goal in a clearly defined step-by-step approach.

- **Master the Concepts and Concept Strands!**
  This series covers all the concepts in the latest IIT-JEE syllabus by segregating them into appropriate units. The theories are explained in detail and are illustrated using solved examples detailing the different applications of the concepts.
- **Let us First Solve the Examples—Concept Connectors!**
  At the end of the theory content in each unit, a good number of "Solved Examples" are provided and they are designed to give the aspirant a comprehensive exposure to the application of the concepts at the problem-solving level.
- **Do Your Exercise—Daily!**
  Over 200 unsolved problems are presented for practice at the end of every chapter. Hints and solutions for the same are also provided. These problems are designed to sharpen the aspirant's problem-solving skills in a step-by-step manner.
- **Remember, Practice Makes You Perfect!**
  We recommend you work out ALL the problems on your own – both solved and unsolved – to enhance the effectiveness of your preparation.

A distinct feature of this series is that unlike most other reference books in the market, this is not authored by an individual. It is put together by a team of highly qualified faculty members that includes IITians, PhDs etc from some of the best institutes in India and abroad. This team of academic experts has vast experience in teaching the fundamentals and their application and in developing high quality study material for IIT-JEE at T.I.M.E. (Triumphant Institute of Management Education Pvt. Ltd), the number 1 coaching institute in India. The essence of the combined knowledge of such an experienced team is what is presented in this self-preparatory series. While the contents of these books have been organized keeping in mind the specific requirements of IIT-JEE, we are sure that you will find these useful in your preparation for various other engineering entrance exams also.

We wish you the very best!

CHAPTER

1

WORK, POWER AND ENERGY

CHAPTER OUTLINE

WORK

Work, in the context of physics, has a special meaning. The meaning of 'work' in physics is very different from the literary meaning of work. For example, if we push hard against a wall, we know we are doing work, but according to definition of work in physics, the work done is zero. *Work is done by a force on a body when the point of application of the force undergoes a displacement in the direction of the force.*

Measurement of work

Consider a constant force $\bar{F}$ acting on a body. During the time of action of this force, let the point of application of the force $\bar{F}$ undergo a displacement $\bar{S}$ as shown in Fig 1.1. The work done by $\bar{F}$ on the body is defined as the scalar or dot product of force and displacement.

$$W = \bar{F}.\bar{S} \qquad W = \left|\bar{F}\right| \left|\bar{S}\right| \cos\theta$$

where θ is the angle between $\bar{F}$ and $\bar{S}$. So, W can be written as $\left|\bar{F}\right|\cos\theta \times \left|\bar{S}\right| = \left|\bar{S}\right|\cos\theta \times \left|\bar{F}\right|$. i.e., work is the product of displacement and the component of force along the direction of displacement or, work is the product of force and the component of displacement along the direction of force. Being the scalar product of force and displacement, work can be negative when $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ or zero when $\theta = \frac{\pi}{2}$.

In fact, this is the important difference between the physicist's definition of work and the layman's definition of work. Thus, *any force acting perpendicular to the direction of displacement does no work on the body.*

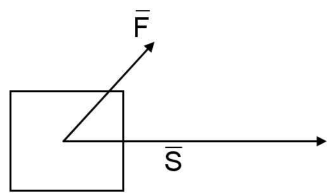


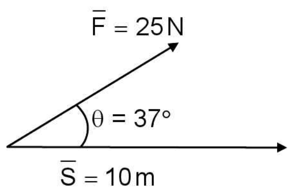
Fig. 1.1

Work is a scalar. It has magnitude, but no direction associated with it. Being the product of force and displacement, it has the dimensional formula ML^2T^{-2}. The SI unit of work is joule (J). The work done is one joule (1 J), when the point of application of a constant force of one Newton (1 N) is displaced through one metre (1 m) in the direction of the force. i.e., 1 J = 1 N × 1 m = 1 kg m s^{-2}. The CGS unit of work is erg and 1 erg = 1 dyne × 1 cm = 10^{-5} N × 10^{-2}m = 10^{-7} J.

CONCEPT STRANDS

Concept Strand 1

A force of 25 N is acting on a body at an angle of 37° to the direction of motion of a body. Calculate the work done by the force during a displacement of 10 m of the body.



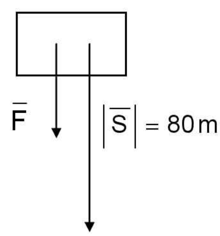
Solution

Work done by 25 N force is $W = \bar{F}.\bar{S} = FS\cos\theta$. Here F = 25 N, S = 10 m and cos θ = cos 37° = 0.8

W = 25 × 10 × cos 37° = 200 N m = 200 J

Concept Strand 2

A 100 kg block is dropped from the top of a building 80 m tall. Assume g = 10 m s^{-2}. Find the work done by the force of gravity as it falls vertically to hit the ground.



Solution

F = mg = 100 x 10 = 1000 N; S = 80 m

W = FS Cos θ = FS Cos 0° = FS = 1000 x 80 = 80,000 J.

CONCEPT STRANDS

Concept Strand 3

A 5 kg ball, thrown vertically upward, reaches a maximum height of 25 m. Assume $g = 10\ m\ s^{-2}$. Find the work done by the force of gravity on the ball till it reaches the maximum height and till it falls back to the ground.

Solution

Upward:

$F = mg = 5 \times 10 = 50$ N, vertically downward

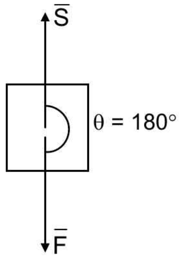
$S = h = 25$ m, vertically upward

$\theta = 180°$ between S and F

$W_1 = \bar{F} . \bar{S} = FS\cos\theta = 50 \times 25 \times \cos 180^{\circ}$

$= 5 \times 10 \times 25 \times (-1) = -1250$ J

$\therefore\ W_1 = FS\cos 180° = -1250$ J.



Downward:

$\theta = 0°$ between $\bar{F}$ and $\bar{S}$

$W_2 = |\bar{F}||\bar{S}|\cos 0°$

$= (mg)\ (h)\cos 0°$

$= 5 \times 10 \times 25 \times 1$

$= +1250$ J

$W_1 = -1250$ J.

$\therefore$ work done for whole journey 'W' is given by

$W = W_1 + W_2 = -1250 + 1250 = 0$

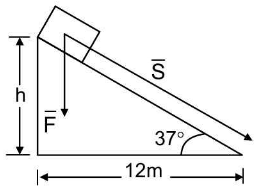
(or) Displacement $\bar{S} = 0$ for whole journey

$\therefore\ W = \overline{F.S.} = mg \times 0 = 0$

Concept Strand 4

A body of mass 25 kg slides down a smooth incline as shown. Find the work done by force of gravity till the body reaches the bottom of the inclined plane ($g = 10\ m\ s^{-2}$).

What is the work done by the normal reaction N exerted by the inclined surface on the body?



Solution

$F = mg = 25 \times 10 = 250$ N

$S = \dfrac{12}{\cos 37^{\circ}} = \dfrac{12 \times 5}{4}\ 15$ m

$W = FS\cos 53° = 250 \times 15 \times \dfrac{3}{5} = 2250$ J

Note that $W = F \times$ component of displacement along $\bar{F}$.

$= mgh = 25 \times 10 \times (12 \times \tan 37°) = 2250$ J.

θ, the angle between $\bar{N}$ and $\bar{S}$ is 90°. Therefore, W = 0 for normal reaction N.

Concept Strand 5

A body of mass 8 kg slides on a horizontal surface for a distance of 6 m. Find the work done on the body by the force of gravity.

Solution

Work done is zero, because the angle between $\bar{F}$ and $\bar{S}$ is 90°.

Work done by many forces

Consider a body, subject to three concurrent forces, $\bar{F}_1, \bar{F}_2$ and $\bar{F}_3$ as shown in Fig. 1.2. The point of application of the forces has a displacement $\bar{S}$ as shown.

If each of the three forces acts on the body while it undergoes this displacement, then, each force is said to have done some work on the body, say $\bar{F}_1$ doing work W_1, $\bar{F}_2$ doing work W_2 and $\bar{F}_3$ doing work W_3. Though there are multiple forces acting simultaneously, the work done by a

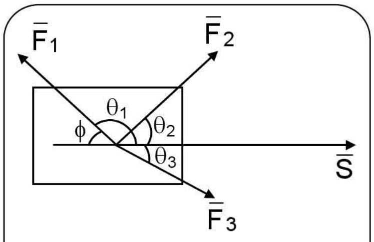


Fig. 1.2

particular force on a body is independent of all the other forces acting on the body, i.e.,

W_1 depends only on $\overline{F_1}$, not on $\overline{F_2}$ and $\overline{F_3}$.

$$W_1 = \overline{F_1}.\overline{S} = \left|\overline{F_1}\right|\left|\overline{S}\right|\cos\theta_1$$

$$= \left|\overline{F_1}\right|\left|\overline{S}\right|\cos(\pi - \phi) = -\left|\overline{F_1}\right|\left|\overline{S}\right|\cos\phi$$

W_2 depends only on $\overline{F_2}$, not on $\overline{F_1}$ and $\overline{F_3}$.

$$W_2 = \overline{F_2}.\overline{S} = \left|\overline{F_2}\right|\left|\overline{S}\right|\cos(\theta_2)$$

W_3 depends only on $\overline{F_3}$, not on $\overline{F_1}$ and $\overline{F_2}$

$$W_3 = \overline{F_3}.\overline{S} = \left|\overline{F_3}\right|\left|\overline{S}\right|\cos(-\theta_3) = \left|\overline{F_3}\right|\left|\overline{S}\right|\cos(\theta_3)$$

Note that if the displacement of the point of application of the applied force is opposite to the direction of applied force, work done by the applied force is negative.

$$W = W_1 + W_2 + W_3$$

$$= \overline{F_1}.\overline{S} + \overline{F_2}.\overline{S} + \overline{F_3}.\overline{S}$$

Since dot product is distributive

$$W = (\overline{F_1} + \overline{F_2} + \overline{F_3}).\overline{S}$$

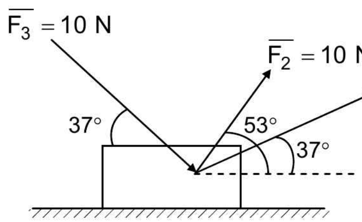
$= \overline{F_R}.\overline{S}$, where $\overline{F_R}$ is the resultant of $\overline{F_1}, \overline{F_2}$ and $\overline{F_3}$

(i.e., $\overline{F_R} = \overline{F_1} + \overline{F_2} + \overline{F_3}$)

CONCEPT STRANDS

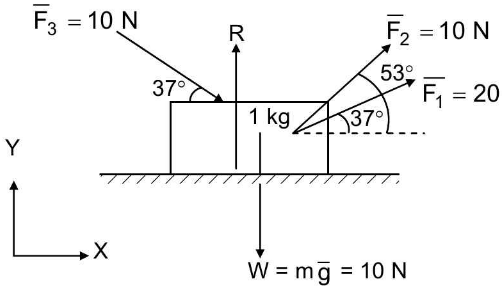
Concept Strand 6

A 1 kg block is at rest on a smooth horizontal plane. From t = 0, three concurrent forces $\overline{F_1}, \overline{F_2}, \overline{F_3}$ are applied as shown. Determine the total work done on the body by all the forces acting on the body during the period t = 0 to t = 1 second.



Solution

FBD:



Express all forces in $\hat{i}, \hat{j}$ form:

$$\overline{F_1} = 20\cos 37^\circ \hat{i} + 20\sin 37^\circ \hat{j} = 16\hat{i} + 12\hat{j}$$

$$\overline{F_2} = 10\cos 53^\circ \hat{i} + 10\sin 53^\circ \hat{j} = 6\hat{i} + 8\hat{j}$$

$$\overline{F_3} = 10\cos(-37^\circ)\hat{i} + 10\sin(-37^\circ)\hat{j} = 8\hat{i} - 6\hat{j}$$

Weight $W = m\overline{g} = -10\hat{j}$

$R = R\hat{j}; \overline{F_a} = \overline{F_1} + \overline{F_2} + \overline{F_3}$

$\therefore$ Resultant $\overline{F_R} = \left[30\hat{i} + (4 + R)\hat{j}\right]N$.

The body will not remain on the plane as sum of the 'j' components (vertical components) of the other forces is 4 Newton upward $\Rightarrow R = 0$.

$\Rightarrow \overline{F_R} = (30\hat{i} + 4\hat{j})N$

$\therefore$ Acceleration $\overline{a} = \dfrac{\overline{F}}{m}$

$$= \frac{(30\hat{i} + 4\hat{j})N}{1\ kg} = (30\hat{i} + 4\hat{j})\ ms^{-2}$$

$\therefore$ Displacement from t = 0 till t = 1s is $\overline{S} = \dfrac{1}{2}\overline{a}t^2$

$$= \frac{1}{2}(30\hat{i} + 4\hat{j}) = (15\hat{i} + 2\hat{j})\ m$$

$$W = \overline{F_R}.\overline{S} = (30\hat{i}+4\hat{j}).(15\hat{i}+2\hat{j})$$

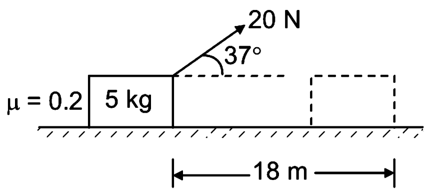
$= (30 \times 15) + (4 \times 2)$

$= 450 + 8 = 458$ J

Concept Strand 7

Consider a force of 20 N acting at an angle of 37° to the horizontal on a body of mass 5 kg, displacing it through 18 m. What is the work done by the force and by friction, if $\mu = 0.2$?

Solution



$$\overline{F} = 20\cos 37^\circ\hat{i} + 20\sin 37^\circ\hat{j} = (16\hat{i} + 12\hat{j})\text{ N}$$

$$\overline{f} = -\mu(mg - 20\sin 37^\circ)\hat{i}$$

$$= -0.2(50-12)\hat{i} = -7.6\hat{i}\text{ N}$$

$$\overline{S} = 18\hat{i}\text{ m}$$

$$W_F = (16\hat{i} + 12\hat{j}).18\hat{i} = (16 \times 18) + 0 = 288\text{ J}$$

$$W_f = (-7.6\hat{i}).18\hat{i} = -136.8\text{ J}$$

Total work done on the body, $W = W_F + W_f$

$= 288\text{ J} - 136.8\text{ J} = 151.2\text{ J}$

Concept Strand 8

A force F at an angle of 37° to the horizontal pulling a body of mass 10 kg moves it at a constant speed on a horizontal surface. The coefficient of friction between the surfaces is 0.5. Find the work done by the force in moving the body through a distance of 5 m.

Solution

For motion with constant velocity the body is in dynamic equilibrium.

i.e., $F\cos 37^\circ = \mu R = \mu(mg - F\sin 37^\circ)$ — (1)

$$\therefore F = \frac{\mu mg}{\cos 37^\circ + \mu\sin 37^\circ}$$

$\therefore$ Work done by the force $= F\cos 37^\circ \times 5$

$$= \frac{\mu mg\cos 37^\circ}{(\cos 37^\circ + \mu\sin 37^\circ)} \times 5 = 181.8\text{ J} \quad \text{— (2)}$$

Work done by friction $= -\mu(mg - F\sin 37^\circ) \times 5$

$= -F\cos 37^\circ \times 5$ (from equation (1))

$= -181.8$ J (from equation (2))

Work done by a variable force

The force applied on a body need not always be constant. Imagine a man pushing a cart along the road. At times when he gets tired, he would push the cart gently; when he recovers he would push it hard. The force varies with the distance. A plot of a variable force as a function of the distance could be like the one in Fig. 1.3. How can we calculate the work done?

Consider two neighbouring points in the path of the body separated by the distance Δx. We can assume that, when the separation Δx is very small, the force $F(x)$ will have approximately the same value over the distance Δx. Therefore, the work done during the motion of the body through the distance Δx is $F(x)\,\Delta x$.

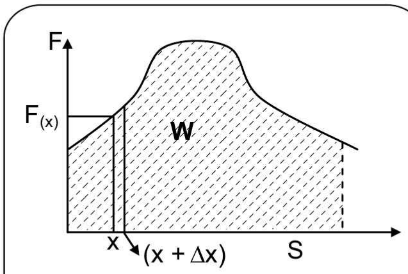


Fig. 1.3

Now, we may imagine the whole path of the body from point x_1 to x_2 to be divided into n such small elements Δx_n

where n is very large. In each of these intervals, the work done is the product of force and displacement. Hence, the total work done by the force while the body travels from x_1 to x_2 is

$$W = \sum_n F(x)\Delta x_n \Rightarrow \int_{x_1}^{x_2} F(x)dx$$

The work done by a variable force acting on a body is given by the area under the force vs displacement (F vs S) graph.

CONCEPT STRANDS

Concept Strand 9

The gravitational force of attraction of the earth (Mass M) on a body of mass m is given by $F = \frac{GMm}{r^2}$ where r is the distance of the body from the centre of the earth. Find the work done by this variable force on the body as it is lifted from the surface of the earth to a height above the surface equal to the radius R of the earth.

Solution

At any arbitrary distance r from centre of the earth

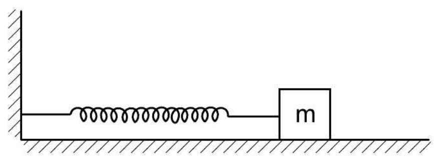
$$F = \frac{GMm}{r^2}$$

$$W = \int_R^{2R} \bar{F}.\bar{dr} = \int_R^{2R} Fdr\cos 180° = -\int_R^{2R} \frac{GMm}{r^2}dr$$

$$= \frac{^-GMm}{2R} \text{ (negative work)}$$

Concept Strand 10

Consider the block of mass m connected to a light spring of constant k as shown below. The other end of the spring is firmly secured to the wall. In the position shown, the spring will be at its natural length and, therefore, it exerts no force on the block and the block is at rest. If the block is displaced from this position to the left or right, the spring exerts a 're-storing' force on the block, the direction of this force on the block being towards right or left respectively, i.e., opposite to the direction of the displacement of the block, so as to restore it to its initial position (and the spring to its natural length).

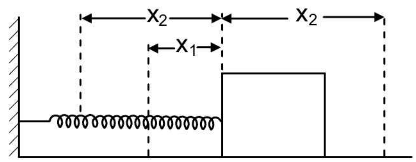


The magnitude of this force F = kx where k is a constant called spring constant and x is the displacement of the block from its equilibrium position i.e., x = compression or elongation of the spring. We note that F is a varying force, i.e., varies with instantaneous magnitude of x.

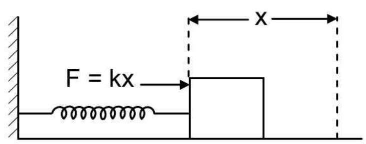
Find the work done by the spring on the block.

(i) as the block is displaced by x_1 to the left.
(ii) as the block is displaced from the above (x_1) position further to the left by ($x_2 - x_1$).
(iii) as the block is displaced from the above (x_2) position to x_2 to the right of initial position (Equilibrium position)

Solution



At any arbitrary compression x,



Infinitesimal work done by F during a further displacement dx towards left is

$$dW = kx.dx.\cos 180° = -kxdx$$

(i) $W = -\int_0^{-x_1} kx\,dx = -\frac{1}{2}kx_1^2$ (negative work)

(ii) $W = -\int_{-x_1}^{-x_2} kx\,dx = \frac{-1}{2}k\left((-x_2)^2 - (-x_1)^2\right) = -\frac{1}{2}k(x_2^2 - x_1^2)$

(negative work)

(iii) $W = -\int_{+x_2}^{+x_2} kx\,dx = -\frac{1}{2}k\left[x_2^2 - (-x_2)^2\right] = 0$

POWER

It is often necessary to know the rate at which work is done. We say, for example, the car engine delivers 15 horsepower or the rating of an electric motor is 30 kW etc. These are Statements, which denote the rate at which work is done. *Power is defined as the time rate at which a force does work.* If W is the work done in a time interval Δt by a force, the average power is

$$P_{av} = \frac{W}{\Delta t}.$$

Also, power $P = \bar{F}.\bar{v}$, where $\bar{F}$ is the applied force and $\bar{v}$ is the velocity of the point of application of the force. Hence power is also a scalar quantity. If the applied force and velocity are at right angles to each other, the power is zero. The instantaneous power is given by

$$P = \frac{dW}{dt}.$$

The SI unit of power is watt (W). 1 watt = 1 joule per second

1 W = 1J s^{-1}. Its dimensional formula is ML^2T^{-3}.

The larger units of power are as follows.

1 kW = 1000 W

1 MW = 10^6 W

1 GW = 10^9 W

1 horse power (hp) = 746 W or 1 kW = 1.34 hp

CONCEPT STRANDS

Concept Strand 11

A truck of mass 5000 kg has an engine, which delivers a constant power of 100 kW. If the resistance to the motion is assumed to be constant and equal to 5000 N, find

(i) the maximum speed with which the vehicle can travel along a straight road.

(ii) the acceleration of the truck when it moves with a speed of 10 m s^{-1}.

Solution

(i) When the vehicle moves with maximum speed (no acceleration) the entire power is utilized to overcome friction.

$P = 100$ kW $= 10^5$ W, F = 5000 N

i.e., Power = F × velocity

i.e., $10^5 = 5000 \times$ velocity

$\therefore$ Velocity $= \dfrac{10^5}{5000} = 20$ m s^{-1}

(ii) At 10 m s^{-1}, the available power is utilized for overcoming friction as well as providing acceleration.

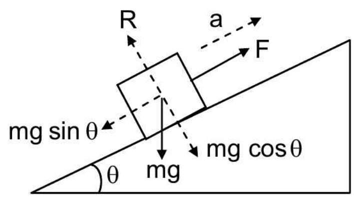
$\therefore$ (F – f) = ma But P = F × v

$$\therefore \left(\frac{P}{v} - f\right) = ma \Rightarrow \left(\frac{10^5}{10} - 5000\right) = 5000 \times a$$

$\Rightarrow$ a = 1 m s^{-2}

Concept Strand 12

A body is pulled up with a constant acceleration 'a', along an inclined plane of inclination 'θ'



(i) Find the power delivered by the pulling force at time t second after starting from rest.

(ii) Compute the average power delivered in t second.

Solution

Force equation gives

$$ma\hat{i} = F\hat{i} + mg\sin\theta(-\hat{i}) + R\hat{j} + mg\cos\theta(-\hat{j})$$

$$= (F - mg\sin\theta)\hat{i} + (R - mg\cos\theta)\hat{j}$$

Equating components, F = m(a + g sinθ)

R = mg cos θ

Velocity at the end of t seconds, v = at

Displacement at the end of t seconds, $s = \frac{1}{2}at^2$

(i) Power delivered by the force at the end of t seconds $= Fv = m(a + g\sin\theta)\,at$

(ii) Average power delivered during t seconds

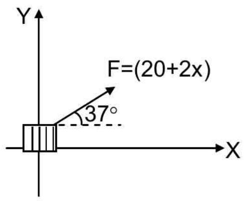
$$= \frac{W}{t} = \frac{F\frac{1}{2}at^2}{t} = \frac{1}{2}m(a + g\sin\theta)\,at$$

Concept Strand 13

A body of mass 1.6 kg, initially at rest, is moved on a horizontal surface along X-axis by applying a force F making an angle 37° with X-axis. The magnitude of the force varies as per the law F = (20 + 2x) N where x is in metre. Determine

(i) Work done by the force from x = 0 to x = 10 m

(ii) Instantaneous power at x = 5 m



Solution

(i) $W = \int_0^W dW = \int_0^{x=10} \overline{F}.\overline{dx} = \int_0^{10} (20+2x)\cos 37^\circ\,\hat{i}.dx\hat{i}$

$$= \int_0^{10} 16dx + \int_0^{10} 1.6xdx = 16x\Big|_0^{10} + \frac{1.6x^2}{2}\Bigg|_0^{10}$$

$= 160 + 80 = 240$ J

(ii) To find the instantaneous power, we must find the instantaneous force and the instantaneous velocity at x = 5 m

$$F_x = (20 + 2x)\cos 37^\circ = (16 + 1.6x)$$

To find v_x, first find a_x

$$a_x = \frac{F_x}{m} = \frac{(16 + 1.6x)}{1.6} = 10 + x$$

i.e., $\frac{dv_x}{dt} = 10 + x \Rightarrow \frac{dv_x}{dx}\frac{dx}{dt} = 10 + x$

$\Rightarrow \frac{dv_x}{dx}v_x = 10 + x \Rightarrow v_x dv_x = (10 + x)dx$

Integrating $\frac{{v_x}^2}{2} = 10x + \frac{x^2}{2} + C$ where C is the constant of integration to be determined by initial conditions, at x = 0, $v_x = 0$ (initially at rest) $\Rightarrow C = 0$

$\therefore {v_x}^2 = 20x + x^2 \Rightarrow v_x = \sqrt{20x + x^2}$

At $x = 5$ m, $v_x = \sqrt{125} = 5\sqrt{5}$ m s^{-1} and

$F_x = 16 + 1.6 \times 5 = 24$ N

$\Rightarrow P = F_x v_x = 24 \times 5\sqrt{5} = 120\sqrt{5}$ W

ENERGY

Energy is the capability of a body to do work. It is a scalar quantity and is measured in the same units as work. When work is done on a body, the energy of the body increases. When a body does work, the energy of the body decreases. Energy of a body is dependent on the frame of reference. Hence, it has different magnitudes in different frames of references. Energy is the fundamental cause of all motion. Sun's energy sustains all life forms on the Earth. A volcano releases stored energy beneath the surface of the Earth in the form of hot lava shooting up to several metres in height. Water stored in a reservoir falls through several metres to a powerhouse and converts the stored energy into mechanical motion of the turbine blades. The energy stored in an electrolytic cell drives electric current in a circuit.

Units of energy

The SI unit of energy is joule (J)

1 joule (J) = 1 newton metre (N m)
= 1 watt second (W s)

Larger units of energy are

1 kilo watt hour (kWh) = 1000 W × 1 h
= 1000 W × 3600 s
= 3.6×10^6 W s
= 3.6×10^6 J

1 mega watt hour (MWh) = 3.6×10^9 J

Smallest unit of energy is electron volt (eV)

1 electron volt (eV) = 1.6×10^{-19} J

1 kilo electron volt (keV) = 1.6×10^{-16} J

1 mega electron volt (MeV) = 1.6×10^{-13} J

The energy of subatomic particles like electrons, protons, neutrons etc., are measured in eV, keV, MeV etc.

Another such unit is rydberg

$$1 \text{ rydberg} = 13.6 \text{ eV}$$

In SI, all forms of energy are measured in joule (J). In CGS system, mechanical energy is measured is erg while heat energy is measured in calorie (cal)

1 cal = 4.18 J

Kinetic energy

A moving body possesses energy called kinetic energy. Naturally, we know that the greater its velocity the larger is its energy. In fact, this can be shown quite simply by considering the application of a force F on the body. By Newton's second law, the force produces acceleration, given by F = ma which results in a change in velocity of the body. We have the equation of motion

$$v^2 - u^2 = 2aS$$

where, u and v are the velocities of the body at the instants of application and removal of the force, respectively. Combining the two equations we get

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = FS$$

The expression $\frac{1}{2}mv^2$ and $\frac{1}{2}mu^2$ are attributes of the body having the dimension of energy, and hence are called the kinetic energy of the body. We may interpret them as the energy of motion. KE is a scalar quantity and has the same unit as work, joule (J) = N m. If p is the linear momentum of a particle of mass m, moving with a speed v and having a kinetic energy KE, then the relation between p, m, v and KE is given by

$$p^2 = (mv)^2 = 2\,m \times \frac{1}{2}mv^2 = 2m\,(KE)$$

Kinetic energy of a body can never be negative. It is again frame dependent as velocity and linear momentum of a body are frame dependent.

Some interesting cases are discussed below:

Consider a heavy particle of mass m_1, speed v_1, kinetic energy KE_1 and linear momentum p_1 and a light particle of mass m_2, speed v_2, kinetic energy KE_2 and linear momentum p_2 respectively.

(i) *If the linear momenta of both particles are same*

$$p_1 = p_2 \Rightarrow m_1v_1 = m_2v_2$$

$$\Rightarrow \frac{v_1}{v_2} = \frac{m_2}{m_1} < 1$$

(a) The lighter particle has larger speed. ($v_2 > v_1$)

$$p_1^2 = p_2^2 \Rightarrow 2m_1KE_1 = 2m_2KE_2$$

$$\therefore \frac{KE_1}{KE_2} = \frac{m_2}{m_1} < 1$$

(b) The lighter particle has more kinetic energy ($KE_2 > KE_1$)

(c) If a constant force F acts on each of these particles, and the particles are finally stopped, then the change in kinetic energy of the lighter particle is more, so that the lighter particle is stopped at a larger distance.

(ii) *If the kinetic energy of both particle are same*

$$KE_1 = KE_2 \Rightarrow$$

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2 \Rightarrow \frac{v_1}{v_2} = \sqrt{\frac{m_2}{m_1}} < 1$$

(a) A lighter particle has more speed ($v_2 > v_1$)

(b) The heavier particle has larger linear momentum.

(c) If a constant force F acts on each particle, both particles are stopped in the same distance.

Work-energy theorem I

In the equation $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = FS$, FS is the work done on the body by the external force F. Work done is positive when F is in the direction of the displacement, causing increase in KE. If F is in the direction opposite to the displacement, FS is negative and the kinetic energy decreases. We have, therefore, a direct relationship between work done on the body and the change in its kinetic energy.

"Work done by the net force on a particle equals the change in the particle's kinetic energy"

$$W = \Delta KE$$

This is known as work-energy theorem I.

CONCEPT STRANDS

Concept Strand 14

A 6 kg block falls from rest from top of a building 20 m high. Find its

(i) Kinetic energy
(ii) Velocity when it hits ground.
(acceleration due to gravity, $g = 10\ ms^{-2}$)

Solution

The only force acting is gravity.

$$F = mg = 6 \times 10 = 60\ N$$
$$S = h = 20\ m$$
$$W = \overline{F}.\overline{S} = FS\cos 0^\circ = 60 \times 20 \times 1 = 1200\ J$$
$$\Delta KE = 1200 J \Rightarrow K = 1200 - 0 = 1200\ J$$
$$\Rightarrow \frac{1}{2} \times 6 \times v^2 = 1200$$
$$\Rightarrow v^2 = \frac{2 \times 1200}{6} = 400\ m^2 s^{-2}$$
$$\Rightarrow v = \sqrt{400} = 20\ m s^{-1}$$

Concept Strand 15

A porter pushes a trolley of mass 50 kg, initially at rest, horizontally along a straight line on a smooth floor with a force of 100 N for 20 m, takes his hands off (i.e., force = 0 N) for the next 10 m, applies 25 N for the next 20 m. What is the trolley's final velocity?

Solution

$$W = \overline{F}_1 . \overline{S}_1 + \overline{F}_2 . \overline{S}_2 + \overline{F}_3 . \overline{S}_3$$
$$= F_1S_1\cos\theta_1 + F_2S_2\cos\theta_2 + F_3S_3\cos\theta_3$$

Here $F_1 = 100\ N, S_1 = 20\ m, \theta_1 = 0°$
$F_2 = 0, S_2 = 10\ m, \theta_2 = 0°$
$F_3 = 25\ N, S_3 = 20\ m, \theta_3 = 0°$

$\therefore \quad W = 100 \times 20 + 0 \times 10 + 25 \times 20 = 2500\ J$

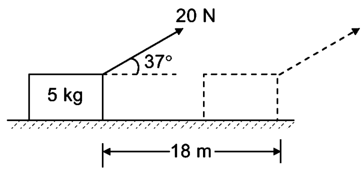
$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}mv^2\ (\because u = 0) = W$$

$$\therefore \quad \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}mv^2\ (\because u = 0) = W$$

$$\Rightarrow \quad v = \sqrt{100} = 10\ m\ s^{-1}$$

Concept Strand 16

A block of mass 5 kg is moved through 18 m on a horizontal plane by the application of a force of 20 N at 37° to the horizontal. Show that the work done by the force is equal to the KE of the block. Neglect friction between contact surfaces.



Solution

$$F = 20\ N, \theta = 37°, F_v = F\sin\theta = 20 \times \frac{3}{5} = 12\ N,$$

$$F_H = F\cos\theta = 20 \times \frac{4}{5} = 16\ N$$

For vertical equilibrium, $mg - F_v - R = 0$

$\therefore \quad 5 \times 10 - 12 - R = 0 \quad \therefore\ R = 38\ N$

Work done by 20 N on the 5 kg block during the displacement of 18 m is

$$W_F = 20\cos 37° \times 18 = 288\ J$$

Acceleration imparted to the body is

$$a = \frac{20\cos 37^\circ}{5} = \frac{16}{5}\ m\ s^{-2}$$

($\because$ Net vertical force on block is zero; acceleration is only in the horizontal direction)

KE (initial) = $\frac{1}{2}mu^2$ KE (final) = $\frac{1}{2}mv^2$

Using the relation $v^2 - u^2 = 2aS$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = maS$$

The gain in kinetic energy of the body is

$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = maS$$
$$= 5 \times \frac{16}{5} \times 18 = 288\ J$$

$\therefore \quad \Delta KE = W_F$

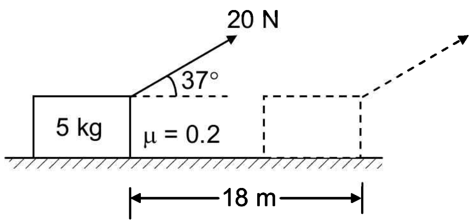
Concept Strand 17

In the previous example if the coefficient of friction between the block and the surface is $\mu = 0.2$, verify that the work energy theorem is satisfied.

Solution

$$\overline{F} = 20\cos 37^\circ\hat{i} + 20\sin 37^\circ\hat{j} = 16\hat{i} + 12\hat{j}$$

$$R = mg - F\sin\theta = 5\times 10 - 20\times \sin 37^\circ = 38\text{ N}$$



$$f = \mu R = 0.2\times 38 = 7.6\text{ N},$$

$$\overline{f} = -7.6\hat{i}\ (\because\ \overline{f}\text{ is opposite to }\overline{S})$$

$$\overline{S} = 18\hat{i}$$

$$W_F = (16\hat{i} + 12\hat{j}).18\hat{i} = 288\text{ J}$$

$$W_f = -7.6\hat{i}.18\hat{i} = -136.8\text{ J}$$

Work done by the normal reaction and the downward force due to the body's weight are zero, because the displacement is normal to both.

$\therefore$ Total work done $W = W_F + W_f = 288\text{ J} - 136.8\text{ J} = 151.2\text{ J}$

Gain in kinetic energy of the body is

$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$= \frac{1}{2}m(v^2 - u^2). = \frac{1}{2}m.2as = maS$$

But $$a = \frac{F_{|x|} - f}{m} = \frac{(16-7.6)}{5}$$

$$\Rightarrow \Delta KE = 5\times\frac{(16-7.6)}{5}\times 18 = 8.4\times 18$$

$$= 151.2\text{ J}.$$

$\therefore$ Total work done = gain in KE

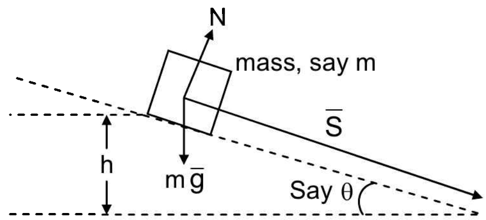
Concept Strand 18

A body is released from top of a smooth inclined plane of height h. Find its velocity when it is

(i) at the bottom of the inclined plane.
(ii) at the mid-point of the inclined plane.
(iii) and (iv) Repeat (i) and (ii) if the plane is rough with co-efficient of friction μ, given that the angle of inclination of the inclined plane with horizontal is θ.

Solution

FBD:



(i) $$W_{m\bar{g}} = m\bar{g}.\overline{S} = mg\frac{h}{\sin\theta}.\cos\left(\frac{\pi}{2} - \theta\right)$$

$$= mg\frac{h}{\sin\theta}\times\sin\theta = mgh$$

$$W_N = \overline{N}.\overline{S} = NS\cos 90^\circ = 0$$

$$\therefore\quad W = W_{mg} + W_N = mgh$$

$$\Delta KE = W$$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mgh \Rightarrow \frac{1}{2}mv^2 = mgh\ (\because\ u = 0)$$

$\Rightarrow v = \sqrt{2gh}$ at the bottom (same as free fall through h)

(ii) $$v = \sqrt{2g\frac{h}{2}} = \sqrt{gh}$$

(iii) $$\left|\overline{f}\right| = \mu mg\cos\theta$$

$$W_f = \mu mg\cos\theta.S.\cos\pi = -\mu mg\cos\theta.\frac{h}{\sin\theta}$$

$$= -\mu mgh\cot\theta$$

$$\therefore \Delta KE = W_{mg} + W_f + W_N = mgh - \mu mgh\cot\theta + 0$$

$$= mgh(1 - \mu\cot\theta)$$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mgh(1 - \mu\cot\theta)$$

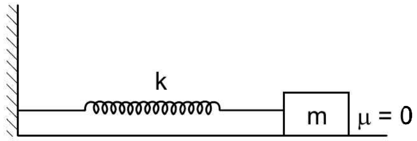
$$\Rightarrow v = \sqrt{2gh(1-\mu\cot\theta)}\ (\because\ u = 0)$$

(iv) $$v = \sqrt{gh(1-\mu\cot\theta)}$$

Concept Strand 19

Consider a spring of constant k, with one end fixed to a rigid wall and the other end connected to a block of mass m. Assume friction is absent everywhere.

Now the body is displaced from its equilibrium position by x_1 to the left and released.



Position at equilibrium

(i) Determine the velocity of the body when it reaches the equilibrium position.
(ii) Will it stop at this position?
(iii) If not, how far to the right will it move?
(iv) Suppose from the equilibrium position, we had applied a constant force F towards the left, what is the maximum displacement from equilibrium position of the block? What is the work we do for the block to reach that position?

Solution

Take motion towards the right as positive.

(i) When the body is displaced by x_1 to the left and released, it is released from rest (i.e., u = 0)

$\Rightarrow$ initial kinetic energy $KE_i = 0$.

When it reaches the equilibrium position, displacement = $+x_1$; force on the block is variable: and is kx, where x = displacement from equilibrium position.
So work done by this force,

$$W = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2;$$

$\Delta KE = W$ (Work energy theorem)

$$\therefore \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = W$$

$$\frac{1}{2}mv^2 = W = \frac{1}{2}kx_1^2, (\because u = 0)$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx_1^2$$

$$\Rightarrow v = \sqrt{\frac{k}{m}}.x_1$$

(ii) No, because velocity is non-zero.
(iii) Let it move by x' to right from equilibrium position. Force on the body is variable and at a distance +x, it is

$F = -kx. \quad \therefore W = \int_0^{x'} F dx = -\int_0^{x'} kx dx = -\frac{1}{2}kx'^2$

$\therefore \Delta KE = -\frac{1}{2}kx'^2$ as per work energy theorem

$KE_{initial} = \frac{1}{2}kx_1^2$ (from (i)) for the journey from equilibrium position to extreme position on right side.

KE_{final} should be zero. ($\because$ maximum position)

$\therefore \Delta KE = 0 - \frac{1}{2}kx_1^2 = -\frac{1}{2}kx'^2 \Rightarrow x' = x_1$

(iv) Let x" be maximum displacement to left.

$W_F = (-F)(-x") = Fx"$

$W_{spring} = -\frac{1}{2}kx"^2$

$KE_{initial} = 0, KE_{final} = 0$

$\Rightarrow \Delta KE = 0 \Rightarrow Fx" - \frac{1}{2}kx"^2 = 0$

$\Rightarrow x" = \frac{2F}{k}$ and $W_F = Fx" = F.\frac{2F}{k} = \frac{2F^2}{k}$

Conservative forces and non-conservative forces

(a) Conservative force

If the quantity of work done by a force on a body (or the work done by a body against a force) depends only on the initial and final positions of the body and not on the path through which the body has changed positions, then such a force is called a conservative force. The amount of work done in raising a body of mass m through a height is given by mgh, which is irrespective of the path followed. Also the net work done by a conservative force when a body returns to its initial point after a round trip is zero i.e., $\oint \overline{F}.d\overline{S} = 0$

For example, Gravitational force, electrostatic force, elastic forces and spring forces within elastic limit are conservative forces.

(b) Non-conservative force

If the work done by a force on a body (or by a body against a force), in moving it from one position to another position, depends upon the actual path through which the change of position has taken place, such forces are called non-conservative forces.

For example, friction force, air resistance and viscous force are non-conservation forces. When non-conservative forces are involved, the work done in moving a body through any closed path is not zero.

i.e., $\oint \overline{F}.\, d\overline{r} \neq 0,$

Potential energy

Potential energy is associated with the configuration of a system of interacting bodies, i.e., bodies, *exerting conservative forces* on one another. The Earth attracts a body lifted above the surface of the Earth by the gravitational force. The Earth-body system possesses gravitational potential energy due to their separation. A spring when compressed or extended, possesses potential energy due to the deformation. The chemical reaction in a voltaic cell separates positive and negative charges onto the terminals of the cell thereby storing potential energy in the cell to drive an electric current in a circuit. In the examples cited above, we find that work done on the system by a conservative force is converted into potential energy. A labourer carries a bag of cement to the top of a multi-storeyed building through the winding stairs. He is working against the gravitational force of the Earth, which is acting straight down.

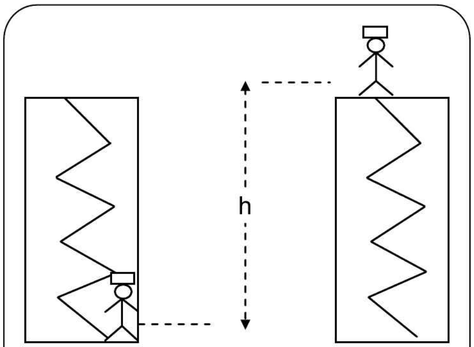


Fig. 1.4

Taking the surface of the Earth as the origin of the coordinate system, the displacement of the man and the bag of cement is h and the force acting is F = – (m + m')g, where m and m' are the mass of the man and the bag of cement respectively and g the acceleration due to gravity. Hence the work done by the man is

$$W = -F \,.\, h = -(m + m')gh$$

The configuration of the system has changed as shown; the centre of mass (explained in later chapters) of the worker and bag of cement has been lifted up through a height h above the ground. In changing the configuration of the system, the work done by the man has not vanished. It is stored as the potential energy of the new configuration of the system – Earth + man and bag of cement at a height h. So long as the man and the bag of cement are at the top of the building, the potential energy continues to be preserved. The potential energy is defined as the negative of the work done by conservative force. In this case,

$$\Delta U = -W = (m + m')\, gh$$

Another form of potential energy is associated with/stored within a spring. As we had seen in example 19 when the spring is compressed by x_1, the system released and the body reaches initial position (equilibrium position when spring's deformation is zero, i.e., spring at its natural length), the body has KE = $\frac{1}{2}kx_1^2$, i.e., the system's KE has increased by $\frac{1}{2}kx_1^2$; this means that the system's potential energy = $\frac{1}{2}kx_1^2$ when the spring is compressed by x_1. This is equally so when spring is elongated by x_1; therefore, the spring potential energy = $\frac{1}{2}kx^2$, where x is the deformation of the spring at that instant. Here PE is relative. We treat PE at natural length as zero. The potential energy of the spring gets converted into kinetic energy of the body (KE of spring = 0, because even though it has velocity, we assume no or negligible mass). When the spring + mass is oscillating, conversion from PE to KE and from KE to PE takes place cyclically.

Work-energy theorem II

"Work done by a conservative force on a body decreases the potential energy of the system by the same amount".

The above Statement is called Work-Energy Theorem II. When a body of mass m is dropped from height h, neglecting air resistance, the only force acting is $m\overline{g}$, doing work of magnitude mgh increasing KE by mgh, and decreasing PE by mgh. Hence the work done by the conservative force (gravitational force) is

$W_{CF} = -\Delta U$ (CF stands for conservative force). Similarly, in a spring mass system

$$W_{CF} = \frac{1}{2}kx^2 = -\Delta U.$$

A conservative force is one which, when acting on a body does work which does not depend on the actual path of the body, but depends only on the initial and final positions of the body.

"Work done by the conservative force is path-independent".

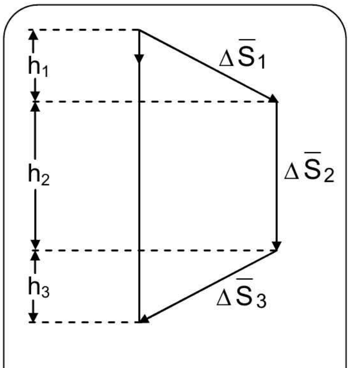


Fig. 1.5

Consider a body taken by the path $\Delta\bar{S}_1 + \Delta\bar{S}_2 + \Delta\bar{S}_3$ from initial height $(h_1+h_2+h_3)$ to ground, as shown in Fig. 1.5. The force of gravity does work

$$= m\bar{g}.\Delta\bar{S}_1 + m\bar{g}.\Delta\bar{S}_2 + m\bar{g}.\Delta\bar{S}_3$$

$= mg\ (h_1 + h_2 + h_3)$ which depends only on initial and final positions.

"Work done by a conservative force along a closed path is zero".

Clearly, if the body is further taken up to the initial point, total work done by gravity is zero.

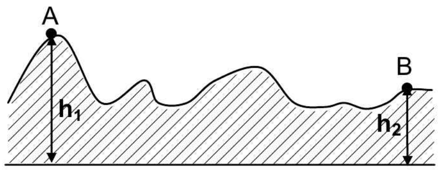
Conservation of mechanical energy

The sum of the potential energy and kinetic energy of a body together is known as the mechanical energy of the body. The total mechanical energy of a system is conserved when only conservative forces are acting on the system.

CONCEPT STRANDS

Concept Strand 20

A body is released from a point A at height h_1 on an uneven frictionless terrain. Find the velocity of the body at a point B of height h_2.



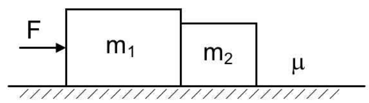
Solution

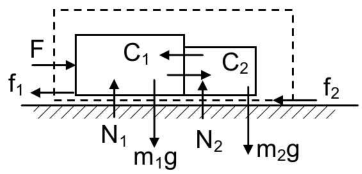
Since friction is absent, the only non-conservative force present is the normal reaction which is perpendicular to the displacement at every point. Hence, work done by it = 0. Therefore, mechanical energy is conserved:

$$mgh_1 + 0 = mgh_2 + \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2g(h_1 - h_2)}$$

Concept Strand 21

Two masses m_1 and m_2 in contact are placed on a horizontal surface. The coefficient of friction between the bodies and the surface is μ. A force F is applied as shown. Mark the conservative and non-conservative forces acting in the system and the work done by them. The forces may be represented as in the diagram. There is no friction between the blocks.





Solution

Non-conservative forces:

F	-	applied force
N_1, N_2	-	normal reaction on bodies m_1 and m_2
f_1, f_2	-	frictional forces
C_1, C_2	-	contact forces

Conservative forces m_1g, m_2g – gravity forces. Work done by the non-conservative forces can be written as

$W_{N_1} = W_{N_2} = 0$ ($\because$ forces perpendicular to displacement)

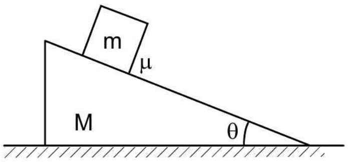
$W_{C_1} + W_{C_2} = 0$ ($\because$ internal contact forces add up to zero)

$\therefore W_F + W_{f_1} + W_{f_2} = \Delta E_{\text{system}}$

Work done by external force and frictional forces is equal to the change in the mechanical energy of the system.

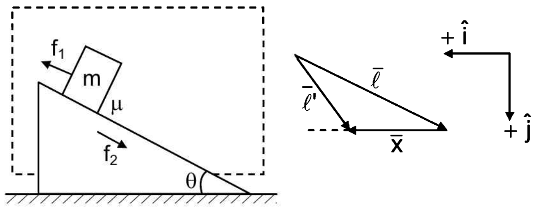
Concept Strand 22

A body of mass m slides down a wedge of mass M and inclination θ through a distance 'ℓ'. The coefficient of friction between the mass and wedge is μ and that between the wedge and floor is zero. Calculate the work done by frictional forces.



Solution

Normal reactions need not be considered since work done by them is zero. We also have $|f_1| = |f_2| = f$.



Further, when m moves down the incline, M moves towards the left taking m along. The displacements are represented in the diagram.

$\bar{\ell}$: displacement of m relative to M

$\bar{x}$: displacement of M

$\bar{\ell}'$: actual displacement of m

Note the directions of $\hat{i}$ and $\hat{j}$ marked in the sketch above.

Now, $W_{f_1} = \bar{f}_1.(\bar{\ell} + \bar{x})$

$= \left[f\cos\theta\hat{i} - f\sin\theta\hat{j}\right].\left[(x - \ell\cos\theta)\hat{i} + \ell\sin\theta\hat{j}\right];$

$= (f\cos\theta)(x - \ell\cos\theta) - (f\sin\theta)\,\ell\sin\theta$

$= fx\cos\theta - f\ell(\cos^2\theta + \sin^2\theta) = fx\cos\theta - f\ell$

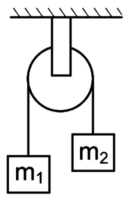
$W_{f_2} = \bar{f}_2.\bar{x} = -f\cos\theta.\,x\hat{i} = -f\,x\cos\theta$

$\therefore W_{f_1} + W_{f_2} = +fx\cos\theta - f\ell - fx\cos\theta = -f\ell$

$\Rightarrow$ The net work done by frictional forces is equal to the negative of the product of the magnitude of the frictional forces and the relative displacement. Alternatively we may look upon this as *"work done by a force is the product of the force and the displacement of the point of application of the force."*

Concept Strand 23

Two bodies of masses m_1 and m_2 connected by a string passing over a fixed, smooth pulley of negligible mass are released from rest at t = 0. Determine the



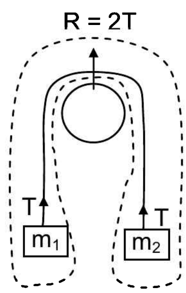
(i) velocities of the bodies as a function of time t.
(ii) acceleration of the bodies at any time t.
(iii) tension in the string at any time t.

Solution

This is a problem we have solved before. Here, we will solve it using work energy theorems. $\bar{T}$ is internal non-conservative force

$$\sum W_T = 0$$

$\bar{R}$ is the reaction force applied by pulley on the string. The point of application of $\bar{R}$ does not move



$\therefore$ $W_R = 0$

$\therefore$ Energy is conserved.

After the system is released at t = 0, let the displacement of m_2 at time t be h downwards. m_1 goes up by h. Conservation of energy requires

$$m_2gh - m_1gh = \frac{1}{2}(m_1 + m_2)v^2$$

$$\Rightarrow v^2 = \frac{2(m_2 - m_1)}{(m_2 + m_1)}gh \qquad \text{— (i)}$$

Differentiating equation (i) with respect to t,

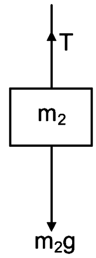
$$2v\frac{dv}{dt} = 2\frac{(m_2 - m_1)}{(m_2 + m_1)}g\frac{dh}{dt} \qquad \text{— (ii)}$$

But $\frac{dv}{dt} = a$ and $\frac{dh}{dt} = v$

Substituting in (ii), $\therefore a = \frac{(m_2 - m_1)}{(m_2 + m_1)}g$ — (iii)

$$v = at = \frac{(m_2 - m_1)}{(m_2 + m_1)}gt \qquad \text{— (iv)}$$

To calculate the tension in the string, we use work energy theorem.



$$W_T + W_{m_2g} = \Delta KE_{m_2} \Rightarrow -T.h + m_2gh = \frac{1}{2}m_2v^2$$

using (i), $-Th + m_2gh = \frac{1}{2}m_2\frac{(m_2 - m_1)}{(m_2 + m_1)}2gh$

$$\Rightarrow T = \frac{2m_1m_2}{(m_1 + m_2)}g \qquad \text{— (v)}$$

SUMMARY

(i) Dynamical problems can be solved by either
   (a) dynamical methods, i.e., by solving force equations or
   (b) energy methods, i.e., by writing energy equations.

(ii) When it is required to find accelerations, force method is easier, but when velocities are required, energy method is preferred.

(a) When only conservative forces are present, initial energy = final energy

(b) When non-conservative forces are also present, Initial energy + work done by external applied forces + work done by frictional forces (with negative sign) = final energy

CONCEPT STRAND

Concept Strand 24

An object is attached to the lower end of a vertical spring, fixed at the upper end and free at the lower end, and slowly lowered to its equilibrium position. This stretches the spring by an amount d. If the same object is attached to the same vertical spring but permitted to fall instead, through what distance does it stretch the spring?

Solution

When lowered to equilibrium position,

$kd = mg$ — (1),

PE = 0, KE = 0 so that total energy PE + KE = 0

when allowed to fall,
when at natural length ℓ_0,

PE = 0, KE ≠ 0

When at maximum stretch,

$$PE = -mgx_{max} + \frac{1}{2}kx_{max}^2;$$

KE = 0

∴ Total energy = PE + KE

$$= -mgx_{max} + \frac{1}{2}kx_{max}^2 = 0$$

(∵ mechanical energy is conserved)

∴ From Energy equation:

$$\frac{1}{2}kx_{max}^2 = mg\,x_{max}$$

$$= kd\,x_{max} \text{ (using (1))}$$

$$\Rightarrow \quad x_{max} = 2d.$$

Potential energy function

If x is the displacement variable, and if a conservative force is a function of x and the potential energy is a function of x, then both are related by

$$\int F \cdot dx = -\Delta U = -(U_2 - U_1)$$

In terms of infinitesimals, F. dx = –dU $\Rightarrow F = -\frac{dU}{dx}$

$$F(x) = -\frac{dU(x)}{dx}$$

Therefore, the conservative force as a function of x can be obtained if we know the potential energy function U(x), by differentiating U(x) with respect to x.

Let us try it out for the cases of springs and gravity.

Spring:

$$U(x) = \frac{1}{2}kx^2$$

$$-\frac{dU}{dx} = -\frac{1}{2}k \cdot 2x = -kx$$

Gravity:

$$U(h) = mgh$$

$$-\frac{dU}{dh} = -mg.$$

In both cases, negative sign shows restoring nature of the force, i.e., opposing displacement direction.

Positions of equilibrium

Consider a track in a vertical plane consisting of peaks, valleys and plateau, as shown in Fig. 1.6.

If a small body is kept at B it will be at rest and therefore at equilibrium. If the body is disturbed from B by a small displacement towards either side of B, it will slide back towards B. In other words, B is a position of stable equilibrium. If U is the potential energy function, and we assume U at y = 0 is zero then U = mgy

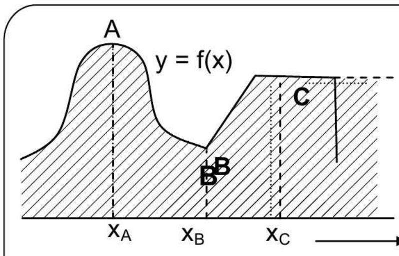


Fig. 1.6

$\Rightarrow$ U(x) = mg [f(x)]

At x = x_B, U is minimum

$\Rightarrow \frac{dU}{dx} = 0$ at $x = x_B$ and $\frac{d^2U}{dx^2} > 0$ at $x = x_B$ because the slope $\frac{dU}{dx}$ is an increasing function around x = x_B. Therefore, the condition for stable equilibrium is $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} > 0$ at that point. If the body is at A, it is again at rest (equilibrium). But if disturbed, it will slide away from the point of equilibrium $\Rightarrow$ A is a position of unstable equilibrium.

$\Rightarrow$ U is maximum $\Rightarrow \frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} < 0$ at $x = x_A$.

Point C is neutral equilibrium, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} = 0$.

CIRCULAR MOTION OF A PARTICLE IN A VERTICAL PLANE

Consider a particle of mass m tied to a string of length r and whirled in a vertical circle of radius r as in Fig. 1.7.

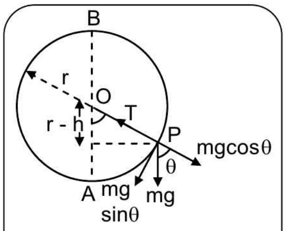


Fig. 1.7

When the particle is at point P, at a vertical height of 'h' above the lowest point A in its circular path, its speed is v and the line OP makes and angle θ with the vertical. The forces acting on the particle at P are (i) Tension T in the direction PO and (ii) weight mg acting vertically downwards. The weight mg can be again resolved into tangential and normal components namely

$$F_T = mg \sin\theta \qquad — (1)$$

and

$$F_N = mg \cos\theta \qquad — (2)$$

Here T - F_N provides the necessary centripetal force of $\frac{mv^2}{r}$ required for circular motion at P. i.e., T − mg $\cos\theta = \frac{mv^2}{r}$

$$\text{or } T = m\left(g\cos\theta + \frac{v^2}{r}\right) \qquad — (3)$$

$$\text{From Figure 1.7, } \cos\theta = \frac{r-h}{r} \qquad — (4)$$

Apply the principle of conservation of energy between the bottom most point and P, we have,

$$mgh + \frac{1}{2}mv^2 = \frac{1}{2}mv_A^2$$

$$\text{i.e., } \quad v^2 = v_A^2 - 2gh \qquad — (5)$$

Using (4) and (5) in (3) we have

$$T = \frac{m}{r}(v_A^2 - 3gh + gr) \qquad — (6)$$

Now in Fig. 1.8, v_A and v_B be the velocities at bottom and topmost points.

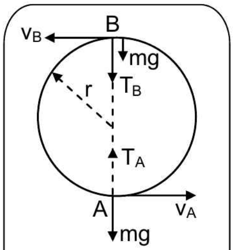


Fig. 1.8

At point A, θ = 0°, T = T_A and v = v_A.

$$\therefore \quad T_A = \frac{mv_A^2}{r} + mg \qquad — (7)$$

i.e., the tension at A should be large enough not only to overcome weight mg but also to provide the centripetal force $\frac{mv_A^2}{r}$

At point B,

$$H = 2r,\ T = T_B \text{ and } v = v_B$$

$$\therefore \quad T_B = \frac{m}{r}(v_A^2 - 5gr) \qquad — (8)$$

$$\text{Also} \quad v_B^2 = v_A^2 - 2gh = v_A^2 - 4gr \qquad — (9)$$

Using (9), (8) becomes

$$T_B = \frac{m}{r}v_B^2 - mg \qquad — (10)$$

Now to complete the vertical circular motion $T_B > 0$. For minimum velocity v_0, $T_B = 0$

$$\text{i.e.,} \quad 0 = \frac{m}{r}(v_0^2 - gr)$$

$$\text{or} \quad v_0 = \sqrt{gr} \qquad — (11)$$

The velocity at A is

$$v_0^2 = v_B^2 = v_A^2 - 4gr$$

$$v_A^2 = gr + 4gr = 5gr$$

$$\text{or} \quad v_A = \sqrt{5gr} \qquad — (12)$$

The tension at A for this velocity is

$$T_{A(min)} = \frac{m}{r}(5gr + gr) = 6\text{ mg} \qquad —(13)$$

Hence to describe a complete vertical circular motion, the minimum velocities at the bottom and topmost points are $v_A = \sqrt{5gr}$ and $v_B = \sqrt{gr}$ respectively.

Note

If the horizontal speed of the mass 'm' at its lowest position is less than $\sqrt{5gr}$, it will move along the circular part only upto certain height, where the tension in the string becomes zero (but velocity of particle is not zero). From that location, it moves in a parabolic path, like a projectile. See example 1.25 for details.

CONCEPT STRAND

Concept Strand 25

If the particle described above is given velocities

(i) $v_A = \sqrt{2gr}$

(ii) $v_A < \sqrt{2gr}$

(iii) $\sqrt{2gr} < v_A < \sqrt{5gr}$, describe its subsequent motions

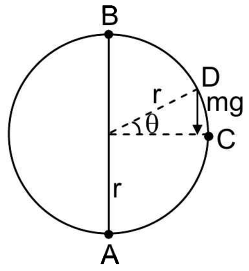
Solution

(i) When $v_A = \sqrt{2gr}$, the energy balance equation is

$$\frac{1}{2}mv_A^2 = mgh + \frac{1}{2}mv_c^2$$

Where h is the height of point C above A, to which the particle will rise. Since $v_A = \sqrt{2gr}$, substituting in the above equation

$$mgr = mgh + \frac{1}{2}mv_c^2$$



Therefore, the particle will rise to a height h = r where its velocity v_c becomes zero. The particle will stop, turn back and oscillate

(ii) when $v_A < \sqrt{2gr}$, the particle wire rise to a height h < r where it will stop, turn back and oscillate.

(iii) when $\sqrt{2gr} < v_A < \sqrt{5gr}$, the particle will rise to aheight r < h < 2r and stop. At this point the tension in the string will become zero and the force balance equation is

$$\frac{mv_D^2}{r} = mg\cos\theta$$

Since the tension is zero, the string will become slack and the particle will not continue its circular motion. Its velocity v_D however, is not zero. It is given by the energy balance equation

$$mgr(1 + \sin\theta) + \frac{1}{2}mv_D^2 = \frac{1}{2}mv_A^2$$

Thereafter, the particle will become a projectile.
It is important to remember that in this case, the tension in the string is the constraint for its circular motion.

LAW OF CONSERVATION OF LINEAR MOMENTUM

The law of conservation of linear momentum states that the total linear momentum of a system is constant if the net external force on the system is zero. Alternatively, it can be stated as "the total linear momentum of an isolated system remains constant". This law is an invariant law and holds good in all inertial frames of references, even though linear momentum is different in different inertial frames of references. In non-inertial frames of references also, the total linear momentum of a system can remain constant, if the net external force on the system (including the inertial forces) is zero. Consider an isolated system of two particles 1 and 2, of mass m_1 and m_2 respectively, moving along the same line in the same direction with constant velocities $\vec{u}_1$ and $\vec{u}_2$ respectively, such that $u_1 > u_2$. This is schematically shown in Fig. 1.9.

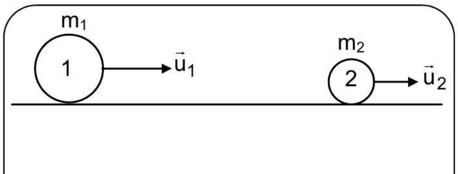


Fig. 1.9

After some time particle 1 catches up with particle 2 and collides with it. This results in an interaction between 1 and 2 for a short duration of time 't'. During this interaction (or collision), 1 exerts a force on 2 which is $\vec{F}_{21}$ and 2 exerts a force on 1 which is $\vec{F}_{12}$. Though $\vec{F}_{21}$ and $\vec{F}_{12}$ are the internal forces of the system of two particles, for particle 1, $\vec{F}_{12}$ is an external force on it and for particle 2, $\vec{F}_{21}$ is also an external force on it. The interaction between these particles is as shown in Fig. 1.10.

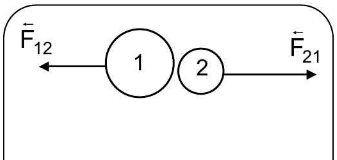


Fig. 1.10

Due to the external forces on 1 and 2, they are accelerated (as per Newton's second law) and their velocities keep on changing during the time interval of interaction. Finally, when the velocities of 1 and 2 become $\vec{v}_1$ and $\vec{v}_2$, their contact is lost and interaction stops. Now 1 and 2 move with constant velocities $\vec{v}_1$ and $\vec{v}_2$ along the same line (not necessarily the same direction as before). This is schematically shown below in Fig. 1.11.

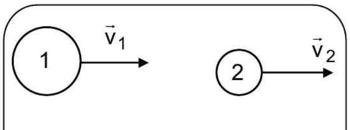


Fig. 1.11

Before collision

$\vec{p}_1$ = Linear momentum of body 1 = $m_1\vec{u}_1$

$\vec{p}_2$ = Linear momentum of body 2 = $m_2\vec{u}_2$

$\vec{p}_i = \vec{p}_1 + \vec{p}_2 = m_1\vec{u}_1 + m_1\vec{u}_2$, is the linear momentum of the two particle isolated system before collision — (A)

After collision

$\vec{p}_1'$ = linear momentum of body 1 = $m_1\vec{v}_1$

$\vec{p}_2'$ = Linear momentum of body 2 = $m_2\vec{v}_2$

$\vec{p}_f = \vec{p}_1' + \vec{p}_2' = m_1\vec{v}_1 + m_1\vec{v}_2$, is the linear momentum of the two particle system after collision — (B)

During the collision, body 1 exerts a force $\vec{F}_{21}$ on body 2 and body 2 exerts a force $\vec{F}_{12}$ on body 1. As per Newton's third law of motion, $\vec{F}_{12} = -\vec{F}_{21}$ (forces are equal and opposite). Since these forces act for the same duration of time 't' on each body, impulse of force $\vec{F}_{12}$ on body 1 is equal to the impulse of force $\vec{F}_{21}$ on body 2, but opposite in direction.

i.e., $\vec{F}_{12}t = -\vec{F}_{21}t$. Since impulse of a force in a time interval = change in momentum of the body during the same time interval

$$(m_1\vec{v}_1 - m_1\vec{u}_1 = -(m_2\vec{v}_2 - m_2\vec{u}_2)$$

$$\therefore \quad m_1\vec{v}_1 - m_1\vec{v}_2 = m_2\vec{u}_1 - m_2\vec{u}_2$$

Thus total momentum after collision = total momentum before collision i.e., total momentum of the isolated system is constant. Thus the law of conservation of linear momentum is verified.

Applications of conservation of linear momentum

(a) Recoil of a gun

When a gun is fired the bullet moves forward with high velocity and the gun moves back with a small velocity. This is called recoil of the gun. This is again based on the principle of conservation of linear momentum in the absence of any external force. Initially the gun and bullet inside are at rest; Initial linear momentum of system is zero. On releasing the bullet from rest, the bullet (mass m) leaves the muzzle with what is known as muzzle velocity of magnitude v_r (velocity relative to muzzle i.e., gun) and the gun recoils with a velocity of magnitude v. Therefore, the magnitude of the absolute velocity of bullet is $(v_r + v)$ in the forward direction. If M = Mass of gun, then

$$m\,(v_r + v) = Mv$$

Such recoil processes will not conserve kinetic energy. So you have to deal with the equation with adequate data to solve it.

(b) Rocket propulsion (Systems with variable mass)

In the systems we have dealt with so far, we have assumed that the total mass of the system remains constant. Sometimes, as in a rocket, it does not. Most of the mass of the rocket on its launch pad is fuel, all of which will eventually be burned and ejected out from the nozzle of the rocket engine. We had seen the recoil velocity of a gun when one bullet is fired. The gun initially at rest attains a velocity (backwards) v on firing a bullet. If you imagine an automatic machine gun is firing n bullets within a small time interval Δt, the recoil velocity of the gun increases for every bullet leaving muzzle. If you extend the logic and instead of discrete number of bullets leaving, but a continuous outflow of exhaust gases out of the rocket, you can see that there is a continuous increases in the velocity of the rocket; in other words an acceleration of the rocket. Assume that we are at rest relative to an inertial reference frame, watching a rocket accelerate through space with no gravitational or atmospheric friction forces.

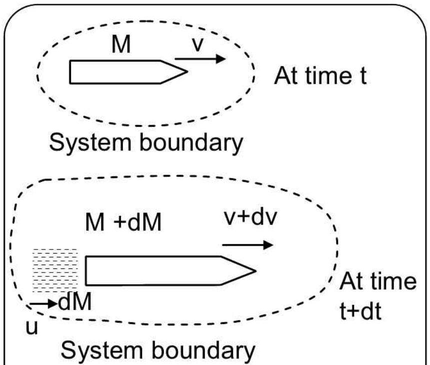


Fig. 1.12

For this one-dimensional motion, let M be the mass of the rocket and v its velocity at an arbitrary time t. After a time interval dt, the rocket now has a velocity v + dv and mass M – dM where the change in mass is dM. The exhaust products released by the rocket during the time interval dt have mass dM and velocity U relative to our inertial frame. It means that for an observer in an inertial frame such as the earth's surface, the mass ejected dM will appear to be moving in the same direction as the rocket. Our system consists of the rocket and the exhaust gases released during dt. The system is closed and isolated, therefore no external force. Therefore linear momentum is conserved during dt.

∴ Initial linear momentum of system = Mv

Final linear momentum = (M – dM) (v + dv) + dM. U

∴ Mv = (M – dM) (v + dv) + dM. U (from conservation of linear momentum)

We can simplify this equation by using relative speed

If v_{rel} = velocity of rocket relative to exhaust gases,

$\Rightarrow$ U = (v+ dv) + v_{rel}

Substituting in the momentum conservation equation,

$$Mv = (M - dM)(v + dv) + dM[(v + dv) + v_{rel}]$$

$$\Rightarrow Mv = (v + dv)[M + dM - dM] + dM.\, v_{rel}$$

$$\Rightarrow Mv = Mv + Mdv + dM\, v_{rel}$$

$$\Rightarrow -dM\,.\,v_{rel} = Mdv$$

$$\Rightarrow \frac{-dM}{dt}\cdot v_{rel} = \frac{Mdv}{dt}.$$

We replace $\frac{dM}{dt}$ (the rate at which the rocket loses mass) by –R where R is the (positive) rate of fuel mass consumption, and we recognize, $\frac{dv}{dt}$ as acceleration of the rocket.

$Rv_{rel} = Ma$ (first rocket equation).

R = rate at which fuel is consumed

v_{rel} = speed with which mass is ejected out relative to rocket

The term Rv_{rel} *is called the thrust of the rocket engine* *(F_{thrust}).*

How will the velocity of a rocket change as it consumes fuel?

Take the equation.

$$-dM\,.\,v_{rel} = Mdv \Rightarrow dv = -\upsilon_{rel}\cdot\frac{dM}{M}$$

Integrate:

$$\int_{V_i}^{V_f} dv = -v_{rel}\int_{M_i}^{M_f}\frac{dM}{M} \quad \text{(i initial, f final)} \Rightarrow$$

$$v_f - v_i = v_{rel}\ln\frac{M_i}{M_f} \quad \text{(second rocket equation)}$$

Note:

If an external force F_{ext} is acting on the rocket (or the system), in addition to the thrust given by the first rocket equation, then the force equation will be as follows. $\vec{F}ext + \vec{F}_{thrust} = m\frac{d\vec{v}}{dt}$, where $\vec{v}$ is the velocity of the rocket at the instant its mass is m and $\vec{F}_{thrust} = \vec{v}_{rel}\frac{dm}{dt}$, where $\vec{v}_{rel}$ is the relative velocity, with which the mass has been removed from the rocket.

COLLISIONS

Consider two solid spheres of masses m_1 and m_2 travelling along the same straight line with velocities u_1 and u_2 in the same direction, as shown in Fig. 1.13 (a). If $u_1 > u_2$, they collide. During the brief time Δt during which they are in contact, an impulsive force of magnitude F is applied by m_1 on m_2 in forward direction. Due to Newton's third law, equal and opposite force is applied by m_2 on m_1 as shown in Fig. 1.13 (b). Immediately on coming apart from contact, let the velocities be v_1 and v_2 in the same direction as shown in Fig 1.13 (c).

Let us see in detail what exactly happens to the two bodies in collision, when they are in contact for a very brief time interval Δt, as shown in Fig. 1.13. The time of contact Δt will be extremely small and during this time, the impulsive force F given by each on the other will be very high and also varying within the time Δt, slowly increasing, reaching maximum and then dropping slowly to zero, when the bodies lose contact and move apart with their final velocities as shown in Fig. 1.14 (deformations are shown exaggerated)

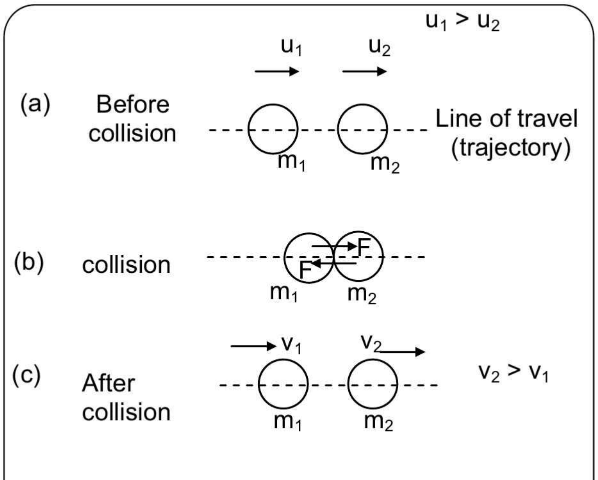


Fig. 1.13

The bodies get deformed temporarily during Δt but at the end of Δt, if they regain their original shape, no energy is lost. In the deformed condition, a body possesses elastic strain energy which is potential energy, very much similar to the potential energy $\frac{1}{2}kx^2$ that a spring possesses. In fact, a body is no different from a spring. Both possess PE when deformed and zero PE when in its natural shape. When the deformation is maximum, the velocities of both bodies are equal, U_1 reduces to the common value, U_2 increases to the common value and the difference in total KE is stored as elastic PE in both bodies. As the bodies spring back to their original shape, the reverse process takes place, strain energy is converted back into KE. When both bodies fully regain their original shape, we can apply KE conservation equation.

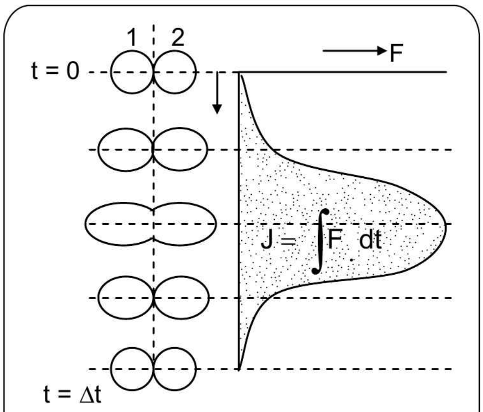


Fig. 1.14

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$

(Total energy before collision)

$$= \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx^2$$

(Total energy at the time of collision i.e., when the bodies are deformed)

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

(Total energy after collision)

where v is the common velocity at the time of collision and k is the spring constant and x is the compression (degree of deformation). Such a collision is called *'Elastic Collision'.*

Linear momentum of the system is conserved, since there is no external force on the system, during the collision process.

$\Rightarrow$ Momentum of system before collision = Momentum of system after collision.

Thus using linear momentum conservation principle, we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

If they do not regain shape fully, then some energy is lost, partly stored as strain energy in the deformed bodies, partly lost as heat and as sound. Such a collision is called 'Inelastic Collision' and we need one more data because we cannot apply energy equation, but we need one more equation to solve for v_1 and v_2. Such additional data is provided by the parameter 'coefficient of restitution' (symbol: e)

$$e = -\frac{(v_2 - v_1)}{(u_2 - u_1)} = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

This formula for e is known as Newton's empirical formula. It should be noted that e is defined only along the line of impact (means the line along which the interaction force acts). Momentum transfer takes place along the line of impact. Hence e is defined only for one-dimensional collisions.

If $v_2 = v_1$

$\Rightarrow$ both travelling with same velocity after collision

$\Rightarrow$ they are stuck to each other

$\Rightarrow$ $e = \dfrac{v_2 - v_1}{u_1 - u_2} = 0$

Such a collision is called 'completely inelastic' collision. The loss of kinetic energy in a collision is given by the formula

$$\text{Loss in KE} = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2(1-e^2)$$

Derivation of the above formula:

$$e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow v_2 = v_1 + e\,(u_1 - u_2) \qquad \text{— (1)}$$

Using (1) in momentum equation,

$$m_1 u_1 + m_2 u_2 = m_1v_1 + m_2v_1 + m_2\, e(u_1 - u_2)$$

$$\Rightarrow v_1 = \frac{m_1u_1 + m_2u_2 - m_2\, e(u_1 - u_2)}{m_1 + m_2} \qquad \text{— (2)}$$

$$\text{i.e., } v_1 = \frac{(m_1 - em_2)u_1 + (1+e)m_2u_2}{(m_1+m_2)};$$

$$v_2 = \frac{(m_2 - em_1)u_2 + (1+e)m_1u_1}{(m_1+m_2)};$$

$$\text{Loss of KE} = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_2v_2^2 \qquad \text{— (3)}$$

Substituting (1) and (2) in (3) and simplifying, we get

$$\text{Loss of KE} = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1 - u_2)^2(1 - e^2)$$

It will be useful to remember the formula. To summarize

$\therefore$ $e = 1 \Rightarrow$ energy loss is zero $\Rightarrow$ elastic collision
$e = 0 \Rightarrow$ loss is maximum $\Rightarrow$ completely inelastic
$0 < e < 1 \Rightarrow$ inelastic;

$$v_1 = \frac{(m_1 - em_2)u_1 + (1+e)m_2u_2}{(m_1+m_2)};$$

$$v_2 = \frac{(m_2 - em_1)u_2 + (1+e)m_1u_1}{(m_1+m_2)}$$

For the sake of simplicity, we have taken all velocities in the same direction. This need not be true always. In general, in terms of vectors, the equations are,

$$-\bar{J} = m_1\bar{v}_1 - m_1\bar{u}_1;\ \bar{J} = m_2\bar{v}_2 - m_2\bar{u}_2$$

$$\Rightarrow m_1\bar{u}_1 + m_2\bar{u}_2 = m_1\bar{v}_1 + m_2\bar{v}_2$$

Collisions between two bodies can be classified into:

(i) one-dimensional collision
(ii) two-dimensional collision

One dimensional collision (Head-on collision)

One-dimensional collision means the velocities of the bodies before and after collision are along the same straight line. The change of momentum vector for each body will be along the same line, which means final momentum vector of each body will be along the same line.

This is also referred to as head on collision. After collision, let $v_2 > v_1$.

Throughout our discussions on collision, we will be using solid spheres as the colliding objects. But what we discuss holds good for all types of objects. We take solid spheres for explaining only because it is easier to visualize lines of travel, line of centres and what happens during the collision process. The momentum conservation equation is

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

If masses and initial velocities are known, in order to determine the final velocities v_1 and v_2, we require one more equation. That equation will be energy equation.

If during the collision process mechanical energy is conserved, then we can use the equation, Initial energy = Final energy

We can see that mechanical energy equation will involve only kinetic energy terms, because, during collision, the bodies are at same potential energy level and therefore PE terms can be neglected. Therefore the equation is (if the collision process conserves energy).

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Therefore now we have two equations and hence the two unknowns v_1 and v_2 can be determined.

For elastic collision, we have already derived

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{— (1)}$$

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2 \qquad \text{— (2)}$$

$$(1) \Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad \text{— (3)}$$

$$(2) \Rightarrow m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \qquad \text{— (4)}$$

$$\frac{(4)}{(3)} \Rightarrow u_1 + v_1 = u_2 + v_2$$

$$\Rightarrow v_2 - v_1 = u_1 - u_2 \Rightarrow e = 1.$$

However, in all types of collisions, linear momentum of the system (of two bodies) is conserved.

i.e., $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ as we derived in our earlier discussions.

We can show the velocities of particle 1 (v_1) and particle 2 (v_2) after collision are respectively given by the following equations

$$v_1 = \frac{(m_1 - em_2)u_1 + (1+e)m_2u_2}{(m_1+m_2)};$$

$$v_2 = \frac{(m_2 - em_1)u_2 + (1+e)m_1u_1}{(m_1+m_2)};$$

One dimensional elastic collisions

In problems involving one-dimensional elastic collisions, use momentum equation as equation no.1 and then for equation no. 2, use $v_2 - v_1 = u_1 - u_2$ which is derived based on momentum and energy equations and is available readily in this shorter form.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{— (1)}$$

$$u_1 - u_2 = v_2 - v_1 \quad \text{—(2)}$$

(1) – $m_2 \times$ (2) $\Rightarrow$ (after simplification)

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} \cdot u_1 + \frac{2m_2}{m_1 + m_2} \cdot u_2$$

$$v_2 = \frac{m_2 - m_1}{m_1 + m_2} \cdot u_2 + \frac{2m_1}{m_1 + m_2} \cdot u_1$$

It will be very useful to remember this formula. You get the same result by using e = 1 in previously derived formula.

Special cases

(i) $m_1 = m_2 = m$ and $u_2 = 0$, $u_1 = u$
(A body collides with body of equal mass at rest)

$$v_1 = \frac{m-m}{2m} \cdot u + \frac{2m}{2m} \cdot 0 = 0$$

$$v_2 = \frac{m-m}{2m} \cdot 0 + \frac{2m}{2m} \cdot u = u$$

Ball 1 comes to rest; Ball 2 takes off at the same initial speed of 1. If you imagine 2 billiards balls, it is as if the second ball did not exist at all and as if the first ball continues to travel with same speed.

(ii) $m_1 = m_2 = m$ and $u_1 = u$, $u_2 = -u$
(Two bodies of equal mass collide head on at equal speeds)

$$v_1 = \frac{m - m}{2m} \cdot u + \frac{2m}{2m} \cdot (-u) = -u$$

$$v_2 = \frac{m - m}{2m} \cdot (-u) + \frac{2m}{2m}\, u = u$$

Both turn back with whatever velocities they came.
If you imagine billiard balls of same colour, it is as if no collision took place, as if each ball moved through the other.

(iii) $m_1 = m_2 = m$, $u_1 = u_1$, $u_2 = u_2$ and negative
(Two bodies of equal mass collide head on at different speeds).

$$v_1 = \frac{m-m}{2m} \cdot u_1 + \frac{2m}{2m} \cdot u_2 = u_2$$

$$v_2 = \frac{m-m}{2m} \cdot u_2 + \frac{2m}{2m} \cdot u_1 = u_1$$

Both turn back but with exchanged velocities.
Again, with billiards balls, as if each ball moved through the other.

(iv) $m_1 = m_2 = m$, $u_1 = u_1$, $u_2 = u_2$ (same sign)
(Ball 1 hits ball 2 from behind)

$$v_1 = \frac{m-m}{2m} \cdot u_1 + \frac{2m}{2m} \cdot u_2 = u_2$$

$$v_2 = \frac{m-m}{2m} \cdot u_2 + \frac{2m}{2m} \cdot u_1 = u_1$$

Simple exchange of velocities.
As if ball (1) overtook ball (2) by moving through it.

(v) $m_1 = m$, $m_2 = M >> m$
$u_1 = u$, $u_2 = 0$

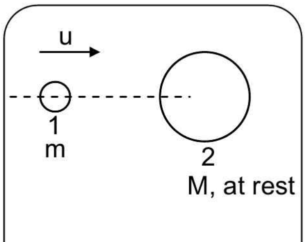


Fig. 1.15

$$v_1 = \frac{m-M}{m+M} \cdot u + \frac{2M}{m+M} \cdot 0 \approx \frac{-M}{M} \cdot u + 0$$

$\approx -u \Rightarrow$ turned back with approximately same u.

$$v_2 = \frac{M-m}{M+m} \cdot 0 + \frac{2m}{M+m} \cdot u = 0 + \text{small velocity}$$

= small velocity

A ball colliding elastically with a fixed wall is an example of (v) case

(vi) $m_1 = m$, $m_2 = M >> m$,
$u_1 = u$, $u_2 = -u$

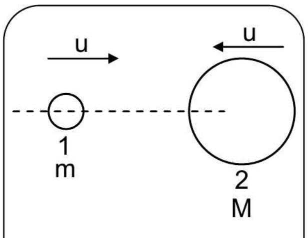


Fig. 1.16

$$v_1 = \frac{m - M}{m + M} \cdot u + \frac{2M}{m + M} \cdot (-u) \approx -3u$$

(turned back 3-times faster)

$$v_2 = \frac{M - m}{M + m} \cdot (-u) + \frac{2m}{M + m} \cdot u \approx -u + \text{small velocity}$$

(as if nothing has happened)

A heavy truck moving on a straight road, colliding with a light cyclist coming from opposite direction, is an example of this situation

Two dimensional collisions

Here, initial velocities are themselves two dimensional, i.e., along different directions.

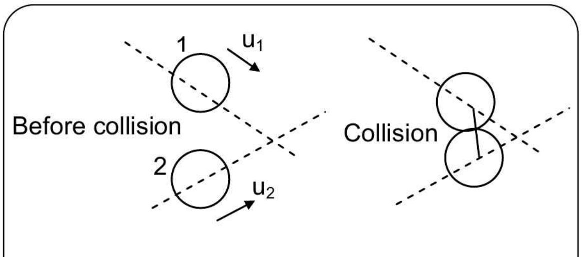


Fig. 1.17

Line of centres will be at an angle to both lines of initial trajectory. Hence resultant velocity vectors again will be two-dimensional. The trajectories of colliding bodies before and after collision are shown in Fig. 1.17 and Fig. 1.18, respectively.

Motion of the bodies after collision is shown in Fig. 1.18.

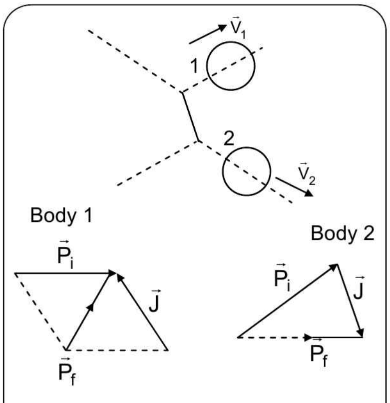


Fig. 1.18

In problems involving two-dimensional collisions there are two momentum equations (one for X direction and one for Y direction) and one energy equation if it is elastic; but there are 4 unknowns. Therefore, additional data should be provided. But if it is not an elastic collision, then two more additional data are required. For two-dimensional collisions, coefficient of restitution is not applicable. e is an empirical parameter, applicable only for one dimensional collision (because e for all collisions whether one dimensional or not, is defined only along the line of impact).

$\therefore$ Energy equation is to be used.

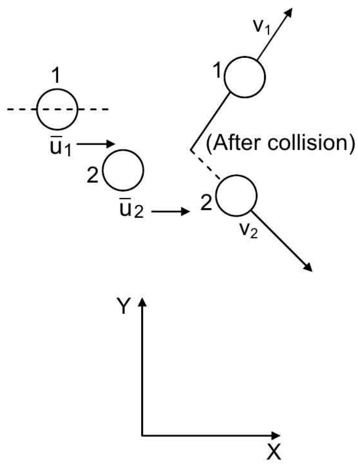
Note:

In case, a ball collides with a wall at an incident angle, the coefficient of restitution can be used for the normal components of velocities, which is along the line of impact.

CONCEPT STRAND

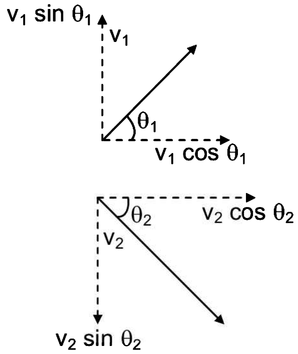
Concept Strand 26

In a glancing elastic collision between two bodies of equal mass if u_1, u_2 are initial velocities, v_1, v_2 are final velocities and θ is the angle between the directions of v_1 and v_2, show that $\cos\theta = \frac{u_1 u_2}{v_1 v_2}$ and study the special case if one of the bodies was initially at rest.



Solution

Resolve v_1, v_2 into components



Momentum equations:

X-axis : $mu_1 + mu_2 = mv_1 \cos\theta_1 + mv_2 \cos\theta_2$

$$\Rightarrow \quad u_1 + u_2 = v_1 \cos\theta_1 + v_2 \cos\theta_2 \qquad \text{— (1)}$$

Y-axis: $mv_1 \sin\theta_1 = mv_2 \sin\theta_2$

$$\Rightarrow \quad v_1 \sin\theta_1 = v_2 \sin\theta_2 \qquad \text{— (2)}$$

Energy equation:

$$\frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

$$\Rightarrow \quad u_1^2 + u_2^2 = v_1^2 + v_2^2 \qquad \text{— (3)}$$

Squaring (1),

$$\Rightarrow \quad u_1^2 + u_2^2 + 2u_1 u_2$$
$$= v_1^2 \cos^2\theta_1 + v_2^2 \cos^2\theta_2 + 2v_1v_2 \cos\theta_1 \cos\theta_2$$
$$= v_1^2 (1 - \sin^2\theta) + v_2^2(1 - \sin^2\theta_2) + 2v_1v_2\cos\theta_1 \cos\theta_2$$
$$= v_1^2 + v_2^2 - v_1^2 \sin^2\theta_1 - v_2^2\sin\theta_2 + 2v_1v_2 \cos\theta_1 \cos\theta_2 \qquad \text{— (4)}$$

(4) – (3)

$$\Rightarrow \quad 2u_1u_2 = -v_1^2\sin^2\theta_1 - v_2^2 \sin^2\theta_2 + 2v_1v_2\cos\theta_1 \cos\theta_2$$
$$= -[v_1^2 \sin^2\theta_1 + v_2^2\sin^2\theta_2 - 2v_1v_2\sin\theta_1 \sin\theta_2 + 2v_1v_2 \sin\theta_1 \sin\theta_2] + 2v_1v_2 \cos\theta_1 \cos\theta_2$$
$$= -[(v_1 \sin\theta_1 - v_2 \sin\theta_2)^2 + 2v_1v_2 \sin\theta_1 \sin\theta_2] + 2v_1v_2 \cos\theta_1 \cos\theta_2$$
$$= -2v_1v_2 \sin\theta_1 \sin\theta_2 + 2v_1v_2 \cos\theta_1 \cos\theta_2 \text{ (using (2))}$$
$$= 2v_1v_2 \cos(\theta_1 + \theta_2)$$

$$\Rightarrow \quad 2u_1u_2 = 2v_1v_2 \cos(\theta_1 + \theta_2)$$
$$= 2v_1v_2\cos\theta \; (\because \theta_1 + \theta_2 = \theta)$$

$$\therefore \quad \cos\theta = \frac{u_1u_2}{v_1v_2}$$

If $u_2 = 0$, $\cos\theta = 0 \Rightarrow \theta = 90°$

EXPLOSION

We will now deal with the phenomenon of explosion—a bomb exploding into several fragments, a solid object splitting into two or more parts when in flight and so on. This phenomenon can be considered as a reverse process of a completely inelastic collision. Two separate pieces moving independently, collide, get stuck together and move as a single piece. Now, a single moving piece, due to internal forces split into two and the pieces move independently. What is common to both systems is absence of any external force and therefore conservation of linear momentum. So we have momentum equation.

$$M\bar{u} = m\bar{v}_1 + (M - m)\bar{v}_2$$

The above vector equation is actually two equations, one along X direction and other along Y-direction. Energy equation is not available because explosions/splitting does not conserve kinetic energy. In fact, the explosion of a bomb/firecracker/mortar, etc., is caused by conversion of chemical energy into kinetic energy and therefore results in increase in KE due to explosion rather than loss of energy as in the case of an inelastic collision.

So with only two equations available and 4 unknowns, (v_{1x}, v_{1y}, v_{2x}, v_{2y}), additional data has to be provided for solving.

SUMMARY

$W = \overline{F}.\overline{S}$ $= F\,S\cos\theta$	$W \rightarrow$ Work done $F \rightarrow$ Force applied to do work $S \rightarrow$Displacement θ is the angle between $\overline{F}$ and $\overline{S}$
$W = \int_{x_1}^{x_2} \overline{F}.\overline{dx}$	$W \rightarrow$ work done by the variable force $F \rightarrow$ variable force applied to do work
$P = \frac{dW}{dt} = \frac{\overline{F}.\overline{dx}}{dt} = \overline{F}.\overline{v}$	P =Power, W = work done $\vec{v} \rightarrow$ velocity, $\overline{F}$ = applied force
$KE = \frac{1}{2}Mv^2$.	$M \rightarrow$ the mass of the body and $v \rightarrow$ its speed,
$p^2 = 2m(KE)$	$KE \rightarrow$ kinetic energy, $p \rightarrow$ linear momentum $m \rightarrow$ mass $KE \rightarrow$ kinetic energy of the body
$F = -k$ $W = \frac{1}{2}kx^2$	F = restoring force on spring x = extension/compression of spring k = spring constant W = work done by the stretching/compression force/ PE in stretched/compressed spring.
$e = \frac{\lvert \vec{v}_2 - \vec{v}_1 \rvert}{\lvert \vec{u}_1 - \vec{u}_2 \rvert}$	e = the co-efficient of restitution for one dimensional collision. (or along line of impact) $\lvert \overline{v}_2 - \overline{v}_1 \rvert$ = magnitude of velocity of separation $\lvert \overline{u}_1 - \overline{u}_2 \rvert$ = magnitude of velocity of approach
$e = \left(\frac{H_1}{H_0}\right)^{1/2}$	If a body falls from a height 'H_0' vertically downwards and rebounds to a height 'H_1' after colliding with the ground, e = the co-efficient of restitution
$h_n = e^{2n}H_0$	If a body falls from a height 'H_0' vertically downward and rebounds to a height 'h_n' after n collisions, e = coefficient of restitution of colliding bodies.
$U_{lost} = \frac{M_1M_2}{2(M_1 + M_2)}(u_1 - u_2)^2(1 - e^2)$	If a body of mass 'M_1' moving with a uniform velocity 'u_1' and another body of mass 'M_2' moving with a uniform velocity 'u_2' collide inelastically and the co-efficient of restitution is 'e'; U_{lost}=the kinetic energy lost in the collision. (for one dimensional collision)
$v_1 = \frac{(m_1 - em_2)u_1 + (1 + e)u_2m_2}{(m_1 + m_2)}$ $v_2 = \frac{(m_2 - em_1)u_2 + (1 + e)u_1m_1}{(m_1 + m_2)}$	u_1 and u_2 are the initial velocities of particles of masses m_1 and m_2. v_1 and v_2 are their velocities after collision for one dimensional collision. e = coefficient of restitution. e = 1 for perfectly elastic collision e = 0 for perfectly inelastic collision

CONCEPT CONNECTORS

Connector 1: When a horizontal force F is applied on a block of mass 10 kg kept on a horizontal surface, it is found that the block moves with constant speed. If the coefficient of sliding friction = 0.5. Find

(a) the applied force and

(b) the work done by the applied force in moving the block through a distance 5 m.

Solution: (a) Since the body moves with constant speed we can infer that body is in horizontal equilibrium.

$\therefore$ $F = \mu N = 0.5 \times 10 \times 9.8 = 49$ N

(b) $\therefore$ Work done = $W = F.x = 49 \times 5 = 245$ J.

Connector 2: A body of mass 200 g is thrown vertically upward with a speed of 10 m s^{-1}. Find the work done by the force of gravity during its time of ascent.

Solution: Max. height attained by the body will be

$$H = \frac{u^2}{2g} \quad [\because 0 - u^2 = 2gH]$$

The work done by the force of gravity $= F.H = mg\,H \cos 180° = -\,mgH$

($\because$ direction of displacement is opposite to that of the force of gravity)

$$= -mg \times \frac{u^2}{2g} = -\frac{1}{2}mu^2$$

$$= -\frac{1}{2}(0.2) \times 100 = 0.1 \times 100 = -10 \text{ J}$$

Connector 3: A uniform square iron sheet of mass 200 kg has a side 10 m and is lying on a flat horizontal ground. Find the work required to make it stand on one side.

Solution: We can consider that the entire mass of the slab is concentrated at the center of mass (Here it will be at the geometric center), which is 5 m away from any of the sides. So here the work done by external agency is equivalent to work done in lifting this mass through a distance 5 m

$\therefore$ $W = mgh = 200 \times 9.8 \times 5 = 9800$ J = 9.8 kJ

Connector 4: A block of mass M is pulled along a horizontal surface by applying a force at an angle θ to the horizontal. Coefficient of friction between the block and the surface is μ. The block travels with a uniform speed. Find the work done by the applied force during the displacement x of the block in the horizontal direction.

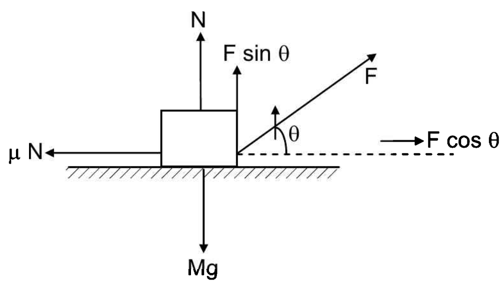
Solution: Here $F\cos\theta = \mu N$ and $F\sin\theta + N = Mg$

$\therefore$ $F\cos\theta = \mu(Mg - F\sin\theta)$, eliminating N

$$\therefore \quad F = \frac{\mu Mg}{(\cos\theta + \mu\sin\theta)}$$

$\therefore$ The work done = F x cos θ

$$= \frac{\mu Mgx\cos\theta}{(\cos\theta + \mu\sin\theta)}$$



Connector 5: An elevator weighing 500 kg is to be lifted up at a constant velocity of 0.20 m s^{-1}. What is the minimum horsepower of the motor to be used?

Solution: Since the elevator goes up with constant velocity, the total work done at any instant will be zero. Therefore the work done by the gravity should be equal to the work done by the motor.

∴ The power of the motor to be used P = F.v

= mg.v = 500 × 9.8 × 0.2 = 980 W

∴ Required horsepower = $\dfrac{980\,\text{W}}{746}$ = 1.3 hp

Connector 6: A bullet of mass 10 g is fired with a velocity of 800 m s^{-1}. It passes through a wooden block of thickness 10 cm and its velocity decreases to 100 m s^{-1}. Find average resistance offered by the wooden block.

Solution: Let F be the average resistance force offered by the wooden block.

Work done by F = change in kinetic energy

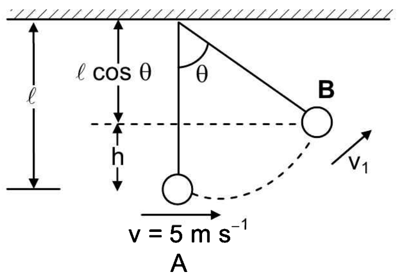
Change in kinetic energy = $\frac{1}{2}M\left(v_1^2 - v_2^2\right) = \frac{1}{2} \times 0.01 \times (800^2 - 100^2) = 3150$ J

Work done by resisting force = F × 0.1 = 3150 J

$$F = \frac{3150}{0.1} = 31.5 \text{ kN}$$

Connector 7: A pendulum bob has a speed of 5 m s^{-1} while passing through its lowest position. What is its speed when it makes an angle of 60° with the vertical? [Length of the pendulum is 1 m. Take g = 10 m s^{-2}]

Solution:



When bob moves from A to B, increase in potential energy = mgh = mg ($\ell - \ell\cos\theta$)

= mgℓ (1 – cos θ)

which is equal to decrease in kinetic energy = $\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2$

Or $\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 = mg\ell(1 - \cos\theta)$

$v_1^2 = v^2 - 2g\ell\,(1-\cos\theta)$

Substituting values v_1 = 3.9 m s^{-1}.

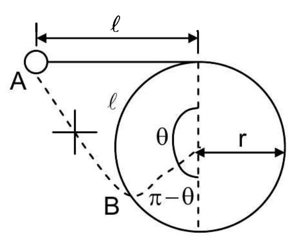
Connector 8: A body is released from a position A from the end of a string attached to a cylinder whose axis is horizontal. If $\ell < \pi r$, with what velocity will it strike the cylinder?

Solution: P.E at A = KE at B (∵ When the body strikes the cylinder at B, the string will lie along the surface of the eylinder ⇒ rθ = l) — (1)

$$\theta = \frac{\ell}{r}$$

ΔPE = mg(r + r cos (π – θ)) = mgr (1 – cos θ)

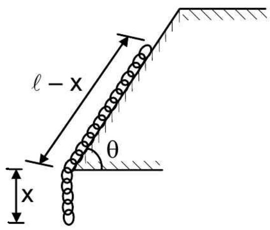
ΔKE = $\frac{1}{2}mv^2$



Substituting in (1)

$$\Rightarrow \quad v = \sqrt{2gr(1-\cos\theta)} = \sqrt{2gr\left[1-\cos\left(\frac{\ell}{r}\right)\right]}$$

Connector 9: A chain of length ℓ is released from rest on the smooth incline when x = 0. Determine the velocity v of the chain in terms of x.



Solution: Let μ be linear density

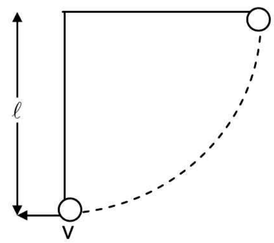
$$PE_{\text{initial}} = \mu.\ell.g\frac{\ell}{2}\sin\theta \ (x = 0)$$

For x < 0,

$$PE = \mu(\ell - x)g\frac{(\ell - x)}{2}\sin\theta - \mu xg.\frac{x}{2} \text{ (with respect to the edge of the in ehine)}$$

$$E_{\text{final}} = \frac{1}{2}\mu\ell.v^2 \Rightarrow v = \sqrt{2gx\left(\sin\theta + \frac{x}{2\ell}(1 + \sin\theta)\right)}$$

Connector 10: The bob of a pendulum is released from horizontal position. If the length of the pendulum is 1.5 m, what is the speed with which the bob arrives at the lowermost point, given that it dissipates 5% if its initial energy against air resistance?



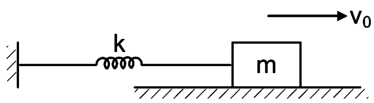
Solution: Initial energy of bob = $mg\ell$

Energy converted into kinetic energy = $0.95 \times mg\ell$

If v is the velocity at lower most point,

$$\frac{1}{2}mv^2 = 0.95 \times mg\ell \Rightarrow v = \sqrt{2g\ell \times 0.95} = 5.28 \text{ m s}^{-1}.$$

Connector 11:



A spring of constant k is connected to a mass m which is resting on a rough floor of friction coefficient μ. The spring is initially undeformed. What is the largest velocity v_0 that can be given to the mass so that it travels only in one direction?

Solution: The speed falls as KE is converted to spring energy and dissipated against friction. Let the position where it comes to rest be x.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx^2 + \mu mgx$$

The spring force =kx

This should be equal to the friction force, in the limiting case

$$kx = \mu mg$$

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}.k\left(\frac{\mu mg}{k}\right)^2 + \frac{(\mu mg)^2}{k}$$

$$= (\mu mg)^2\left(\frac{1}{2k} + \frac{1}{k}\right)$$

$$\frac{1}{2}mv_0^2 = \frac{3}{2}.\frac{(\mu mg)^2}{k}$$

$$v_0^2 = (\mu g)^2.\frac{3m}{k}$$

$$v_0 = \mu g\sqrt{\frac{3m}{k}}$$

Connector 12: A body falling from a height H hits the ground and loses y% of its energy in this impact. Find the height up to which the body will rise.

Solution: When the body comes down its PE is converted as KE At the time of collision y% of its energy is lost and during its rise again this KE get converted as PE

$\therefore \quad mgH\dfrac{(100-y)}{100} = mgx$, where x is the height up to which it rises.

$$\therefore \quad x = \frac{H(100-y)}{100}$$

Connector 13: A nut of mass 0.1 kg falls from the ceiling of an elevator moving down with a uniform speed of 5 m s^{-1}. It hits the floor and does not rebound. What is the heat produced by the impact if the height of the elevator is 3 m? If the elevator is stationary, what will the heat produced be?

Solution: Whether the elevator is moving with uniform speed or is stationary, the relative velocity with which the nut strikes the floor of the elevator is same.

Potential energy of the nut at the ceiling = $0.1 \times 9.8 \times 3 = 2.94$ J

As the nut does not rebound, P.E is fully converted in to heat energy

$\therefore \quad$ Heat produced = 2.94 J

Connector 14: A block of mass 2 kg collides with a horizontal massless spring of force constant 4 N m^{-1} compressing the spring by 3 m from its initial position. The coefficient of friction is 0.2. What was the velocity of the block just before the collision? $\left(g = 10 \text{ m s}^{-2}\right)$

Solution: Frictional force $f = \mu mg = (0.2)(2)(10) = 4$ N

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2 + f.x \Rightarrow \frac{1}{2}.2.u^2 = \frac{1}{2}\times 4\times 3^2 + 4\times 3$$

$$= 30 \Rightarrow u = \sqrt{30} \text{ m s}^{-1}$$

Connector 15: A rocket in outer space, at t = 0, has mass 1000 kg, fuel consumption rate R = 2.5 kg s^{-1} and acceleration 7.5 m s^{-2}. When all the fuel got burned out, the mass of the rocket was 250 kg.

(i) What was its speed relative to the exhaust gases at t = 0?

(ii) What is the thrust of the rocket at t = 0?

(iii) What was the rocket's gain in speed when all the fuel got burned out?

Solution: (i) First rocket equation Rate × velocity = Ma

Acceleration at $t = 0$, $\left.\dfrac{dv}{dt}\right|_0 = \dfrac{R \times v_{rel}}{M}$

$$7.5 = \frac{2.5 \times v_{rel}}{1000}$$

$$\Rightarrow \quad v_{rel} = 3000 \text{ m s}^{-1}$$

(ii) Thrust T = $R \times v_{rel} = 2.5 \times 3000 = 7500$ N

(iii) Second rocket equation gives the final speed of the rocket $v_f = v_0 + v_{rel}\,\ell n\dfrac{M_{initial}}{M_{final}}$

$$\therefore \quad v_f - v_0 = 3000\,\ell n\left(\frac{1000}{250}\right) = 3000 \ln 4 = 4159 \text{ m s}^{-1}$$

Connector 16: A body of initial mass m_0 ejects matter with a relative speed $c\sqrt{m}$ (c is a constant). Find its speed when it has lost half its mass (neglect gravity)

Solution: Conservation of momentum gives

$$(m - dm)(v + dv) + (v + \upsilon' \to \upsilon_{ral})\, dm - mv = 0 \qquad \text{(where, } v' = c\sqrt{m}\text{)}$$

$$mv + mdv - v\, dm - dm\, dv + vdm + \upsilon' \to \upsilon_{ral}\, dm - mv = 0$$

Now $\quad dm\, dv \approx 0$

$$\Rightarrow mdv = -\upsilon' \to \upsilon_{ral}\, dm = -c\sqrt{m}dm \Rightarrow dv = \frac{-c}{\sqrt{m}}dm$$

$$v_{1/2} - 0 = -c.2\left(\sqrt{\frac{m_0}{2}} - \sqrt{m_0}\right) \Rightarrow v_{1/2} = \sqrt{2}\left(\sqrt{2} - 1\right)c\sqrt{m_0}$$

Connector 17: A sphere of mass m_1 and initial velocity u_1 strikes another sphere of mass m_2 travelling in the same direction with velocity u_2. If, after an elastic impact, the first sphere rebounds and the second sphere have equal and opposite velocity, determine the ratio $\frac{m_2}{m_1}$.

Solution: Applying conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$\Rightarrow m_1u_1 + m_2u_2 = m_1(-v) + m_2(v) \quad — (1)$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{v - (-v)}{u_1 - u_2} = 1 \text{ (e = 1 for elastic collision)}$$

$$\Rightarrow u_1 - u_2 = 2v \Rightarrow v = \frac{u_1 - u_2}{2}$$

$$\therefore (1) \Rightarrow m_1u_1 + m_2u_2 = (m_2 - m_1)\frac{(u_1 - u_2)}{2}$$

$$\Rightarrow m_1\left(u_1 + \frac{u_1}{2} - \frac{u_2}{2}\right) = m_2\left(+\frac{u_1}{2} - u_2 - \frac{u_2}{2}\right)$$

$$\Rightarrow \frac{m_1}{2}(3u_1 - u_2) = \frac{m_2}{2}(u_1 - 3u_2) \Rightarrow \frac{m_2}{m_1} = \frac{3u_1 - u_2}{u_1 - 3u_2}$$

Connector 18: Show that when a heavy body collides elastically with a light body at rest, the light body moves with twice the velocity of the heavy body.

Solution: Applying conservation of momentum,

$$Mu + 0 = Mv_1 + mv_2$$

$$\Rightarrow \quad M(u - v_1) = mv_2 \quad — (1)$$

Applying conservation of energy

$$\frac{1}{2}Mu^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2$$

or $M(u^2 - v_1^2) = mv_2^2 \quad — (2)$

$$\frac{(2)}{(1)} \Rightarrow u + v_1 = v_2 \quad — (3)$$

substituting for v_2 in (1)

$Mu = Mv_1 + mu + mv_1$

$\Rightarrow$ $(M - m)\,u = (M + m)\,v_1$

If $M >> m,\ M - m = M + m \approx M$

$\therefore$ $u = v_1$ — (4)

From (3) $v_2 = 2u = 2v_1$

Alternate solution: use formula

$$v_2^1 = \frac{2m_1v_1}{m_1+m_2} - \frac{m_1-m_2}{(m_1+m_2)}v_2;\ \text{given: } v_2 = 0\ m_1 >> m_2$$

$$\therefore\quad v_2^1 = \frac{2m_1v_1}{m_1} = 2v_1$$

Connector 19: Two bodies of masses m and 3m moving with equal momenta, collide elastically. Show that the smaller body is brought to rest after collision.

Solution: p_1, p_2 be the final momenta. Then

$p + p = p_1 + p_2$ (momentum conservation)

$$\frac{p^2}{2m} + \frac{p^2}{2(3m)} = \frac{p_1^2}{2m} + \frac{p_2^2}{2\times 3m}\ \text{(energy conservation)} \Rightarrow 3p_1^2 + p_2^2 = 4p^2$$

also $p_1^2 + p_2^2 + 2p_1p_2 = 4p^2 = 3p_1^2 + p_2^2 \Rightarrow 2p_1^2 - 2p_1p_2 = 0$

$p_1 = 0$ — (i)

or $p_1 = p_2 = p$ — (ii)

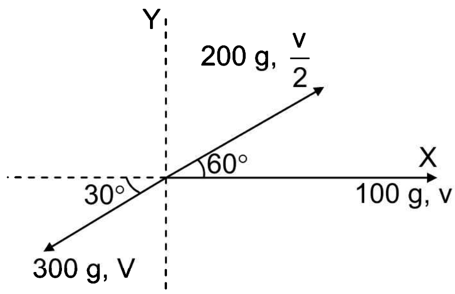
(ii) $\Rightarrow$ no collision $\Rightarrow$ (ii) is not true. $\therefore$ (i) is true $\Rightarrow$

First body comes to rest.

Connector 20: A particle of mass 600 g, initially at rest, explodes into three fragments. One of them with mass 100 g and velocity v flies off at an angle 60° to the second fragment of mass 200 g which has velocity $\frac{v}{2}$. Find the magnitude and direction of the velocity of the third fragment and the energy lost by the particle.

Solution: Take the direction of 100 g fragment along x-axis.

Equating x-components of momenta



$$100\,v + 200\,\frac{v}{2}\cos 60 = 300\,V_x$$

$$\therefore\quad V_x = \frac{v}{2}$$

Equating y components of momenta

$$200\,.\,\frac{v}{2}\sin 60 = 300\,V_y$$

$$\therefore\quad V_y = \frac{v}{2\sqrt{3}}$$

$$\therefore\quad V = \sqrt{V_x^2 + V_y^2} = \frac{v}{\sqrt{3}} \qquad \theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

Initial energy of particle = 0

$$\text{Final energy of fragments} = \frac{1}{2}\times 0.1\,v^2 + \frac{1}{2}\times 0.2\,\frac{v^2}{4} + \frac{1}{2}\times 0.3\,\frac{v^2}{3} = 0.125\,v^2\ \text{J}$$

Loss of energy = $0.15\,v^2$ J

TOPIC GRIP

Subjective Questions

1. A 20 kg box is pulled through a distance of 25 m along the floor, by a force, which makes angle 30° with the horizontal. If the friction coefficient $\mu = 0.3$, find the work done by the force for

   (i) a constant speed 5 m s^{-1}

   (ii) variable speed $v = 5 + \frac{S}{2}$, where S is the displacement. (Take $\sqrt{3} = 1.7$, g = 10 m s^{-2})

2. Man A pushes a block up a smooth incline upto certain height with uniform velocity v, using power P_1. Man B raises the same block vertically up with uniform velocity 2v, through the same height, using power $3P_1$. In which case is the

   (i) force applied less?

   (ii) work done less?

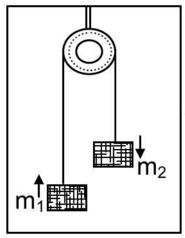
   (iii) time of travel less?

   (iv) What is the ratio of times, $\frac{t_A}{t_B}$?

3. A 1000 kg aircraft accelerates uniformly on a 50 m runway to acquire a take off speed 80 km/hour. If the frictional force between the tyres and runway surface is 2000 N, calculate:

   (i) The force applied by the propeller on the plane.

   (ii) Minimum engine power required for take off. (g =10 m s^{-2})

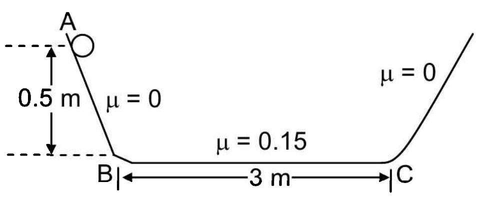


4. Masses m_1 and m_2 are connected by a light inextensible string running over a light frictionless pulley attached to an elevator. The elevator starts to move up with an acceleration a. Find the work done by tension force on m_1 and m_2, in time t with respect to

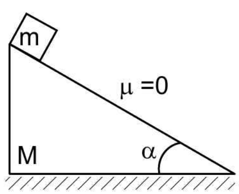
   (i) the elevator.

   (ii) the ground.

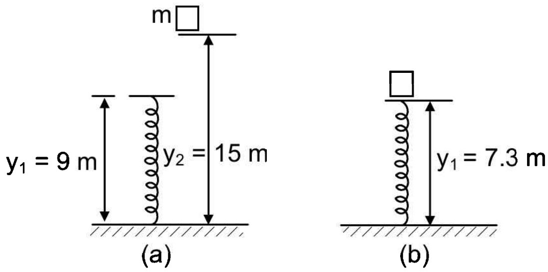
5. A point mass is released from rest from point A of the track in the form of a trough as shown. BC = 3 m. Height of A from BC is 0.5 m. The path BC is rough with $\mu = 0.15$. Other parts of the path are smooth. Determine the point on BC where the body comes to rest. $(g = 10 \text{ m s}^{-2})$



6. A mass m is released on a stationary wedge of mass M Determine velocity v acquired by the mass m sliding through a distance S on the incline

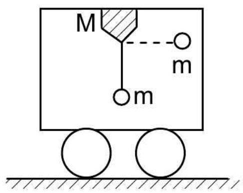


7. A ball of mass 9 kg is dropped on to a vertical light spring of force constant $k = 320\ N\ m^{-1}$. The block compresses the spring 1.7 m as shown in figure. Calculate



(i) Elastic potential energy of the spring
(ii) The amount of mechanical energy converted to thermal and sound energy due to the collision of the block with the spring.

8. A simple pendulum of length ℓ and mass m is suspended from ceiling of a trolley of mass M. The system is released from rest when the string is horizontal. Find the velocity of trolley with respect to ground, when string is vertical.



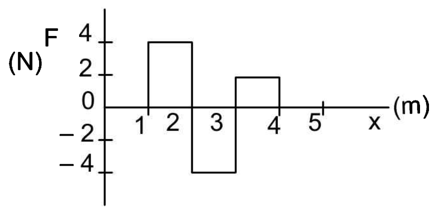
9. A ball is dropped from a height of 10m above a point A on a fixed inclined plane inclined at an angle of 30° upward with horizontal. If coefficient of restitution is $\frac{1}{\sqrt{3}}$, and if the ball hits the plane again at a point B, determine the distance AB and time between collisions. ($g = 10\ m\ s^{-2}$)

10. A particle of mass 4m, initially at rest, explodes into three fragments. Two of the fragments have mass m and 2m respectively. They are found to move with speeds v and $\frac{v}{2}$, respectively, at an angle of 60°. Find the velocity of the third fragment.

Straight Objective Type Questions

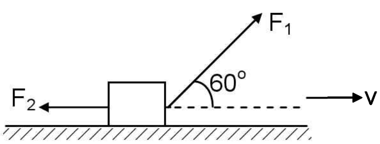
Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. The figure shows F–x graph for a body moving along a straight line. The work done by the force is



(a) 10 J (b) 6 J
(c) 4 J (d) 2 J

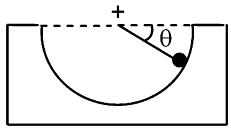
12. Two forces F_1 and F_2 act simultaneously on a body and it moves towards right with a velocity $2ms^{-1}$. If $F_1 = 6$ N and $F_2 = 3$ N, the net power is equal to



(a) 3 W (b) 6 W (c) 8 W (d) Zero

13. If the KE and momentum of a body are 100 J and 20 kg m s^{-1} respectively, the mass of the body is
(a) 1 kg (b) 2 kg (c) 3 kg (d) 4 kg

14. A particle is released from the edge of a smooth semicircular track fixed in the vertical plane. The force on the wire when the angle θ is $\frac{\pi}{4}$ is



(a) mg (b) $\frac{mg}{\sqrt{2}}$
(c) $\frac{3mg}{\sqrt{2}}$ (d) $\frac{\sqrt{5}}{2}mg$

15. A particle slides down from rest from the top of a smooth inclined plane of length ℓ making angle θ with the horizontal. At the same instant, another particle is shot up the plane with velocity v_0. If the bodies came to rest with respect to each other after a perfectly inelastic collision, v_0 is
(a) $\sqrt{2g\ell}$ (b) $\sqrt{g\ell\sin\theta}$ (c) $\sqrt{2g\ell\sin\theta}$ (d) $\sqrt{g\ell\cos\theta}$

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. **Statement 1**
When a ball is gently placed on the free end of a vertically fixed spring, its kinetic energy first increases and then decreases to zero.
and
Statement 2
Positive work done by gravity is proportional to displacement x and negative work done by spring is proportional to x^2 and constants of proportionality differ.

17. **Statement 1**
A spring vertically fixed at the bottom and a weight placed gently on top which is slowly lowered quasistatically till maximum compression, will not have mechanical energy conserved.
and
Statement 2
Mechanical energy is conserved for conservative systems.

18. **Statement 1**
A particle attached to the end of a string is in vertical circular motion with least possible kinetic energy at topmost point P. If the string breaks, the particle cannot rise above the level of P irrespective of wherever the string breaks.
and
Statement 2
Both vertical circular motion and projectile motion are processes conserving mechanical energy since tension does no work.

19. **Statement 1**
If the exhaust gases out of a rocket are stationary, it means the rocket is in uniform motion.
and
Statement 2
Relative speed of exhaust gases is one of the factors determining acceleration of the rocket.

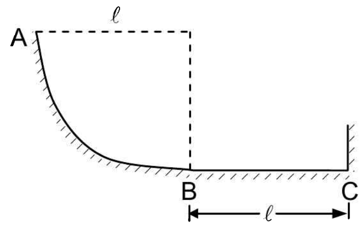
20. **Statement 1**
In a 1D collision between two bodies moving in same direction, the final kinetic energy of the trailing body can be more in inelastic collision, than what it would be if the collision were elastic.
and
Statement 2
Impulse J in inelastic collision is less than the impulse in elastic collision.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

ABC is a track on vertical plane, AB is a quarter circle of radius ℓ, BC is horizontal ℓ long. BC has friction, coefficient μ. A particle is released at A.

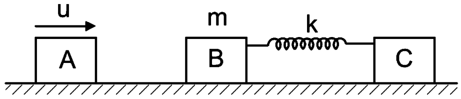


21. If the collision of the body with the wall at C is elastic, the successive heights upto which the particle rises on AB form
(a) Arithmetic progression
(b) Geometric progression
(c) Harmonic progression
(d) None of the above

22. If after several successive elastic collisions, the particle comes to rest just at the wall at C, then μ can possibly be
(a) 0.1 (b) 0.2 (c) 0.5 (d) 0.4

23. If the collision at wall C is inelastic with coefficient at restitution e, velocity of the particle at C after the first collision is
(a) $\sqrt{\ell(1-e)\mu}$ (b) $\sqrt{\ell(1-\mu e)}$ (c) $\sqrt{\ell(\mu-e)}$ (d) $e\sqrt{2g\ell(1-\mu)}$

Passage II



Blocks B (of mass m) and C, attached to spring of constant k as shown are at rest. Block A moves with velocity u, collides and comes to rest. Subsequently at some instant, C's velocity is $\frac{u}{2}$ with no relative velocity with respect to B. The loss of mechanical energy in the whole process is 25%. No friction anywhere.

24. Mass of A is
(a) $\frac{m}{2}$ (b) $\frac{3}{4}m$ (c) m (d) $\frac{3}{2}m$

25. Mass of C is

(a) $\frac{m}{2}$ (b) $\frac{3}{4}m$ (c) m (d) $\frac{3}{2}m$

26. When C's velocity is $\frac{u}{2}$, the length of the spring differs from its natural length by

(a) $\frac{u}{2}\sqrt{\frac{3m}{2k}}$ (b) $\frac{u}{2}\sqrt{\frac{3m}{k}}$

(c) $u\sqrt{\frac{3m}{k}}$ (d) $u\frac{3}{2}\sqrt{\frac{m}{k}}$

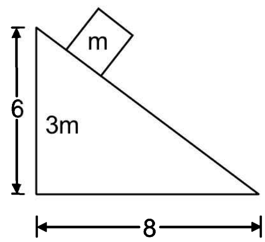
Multiple Correct Objective Type Questions

***Directions*: Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

27. The displacement of a body of mass m, acted upon by a force F(t) at any instant t is given by $x = kt^{\frac{3}{2}} + C$. Where k and C are constants. Then

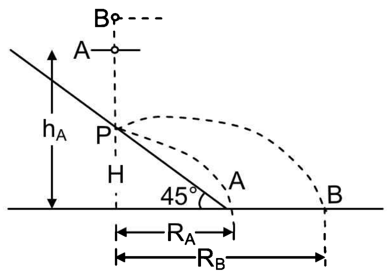
(a) Its KE is proportional to time
(b) Its acceleration is proportional to time
(c) The power delivered by the force is constant
(d) The power delivered by the force is proportional to time.

28. A mass m is released on the top of the smooth surface of a wedge of mass 3m, kept on a smooth floor and having dimensions as shown.



(a) When the mass reaches the bottom position, the wedge would have moved 2m.
(b) When the mass reaches the bottom position its velocity is $\sqrt{\frac{720}{7}}\ ms^{-1}$
(c) The velocity of the mass in the reference frame of the wedge is less than its velocity as observed from ground
(d) If the mass hits the floor elastically and rebounds it has a projectile range of $\frac{72}{7}$ m on ground for its motion after impact.

29. Two balls A and B having mass m_A and m_B are dropped on an inclined a plane of angle $\theta = 45°$, so that they fall on the same point P on the inclined plane with same K.E and rebound after perfectly elastic collision. B clears the inclined plane and reaches ground 2 s after collision, A falls just at the foot of the inclined plane.



(a) The ratio of their ranges on the ground level $\frac{R_A}{R_B} = \sqrt{m_B/m_A}$

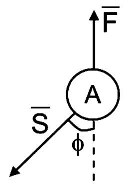
(b) The ratio of their ranges on the ground level $= m_A/m_B$

(c) Height h from where A is dropped is 25 m above ground level
(d) The velocity with which A reaches P is v = 10 m s^{-1}

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30.

Column I	Column II
(a) Work done by a lifter in bringing down a weight from a height	(p) Work done is negative
(b) Work done by the external force $\overline{F}$, A undergoing displacement $\overline{S}$ while force $\overline{F}$ is acting 	(q) Work done is positive
(c) Work done by a compressed spring while it relaxes and pushes a connected body	(r) Mechanical energy is conserved
(d) Work done by a boy in catching a cricket ball	(s) Mechanical energy is not conserved

IIT ASSIGNMENT EXERCISE

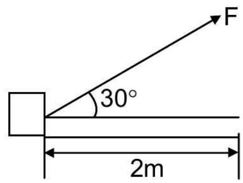
Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. Which of the following is not a unit of energy

(a) joule (b) g cm^2s^{-2} (c) kg (d) erg

32. A force of 10 N acting on a body at an angle 30° to the horizontal produces a displacement of 2 m. The work done is



(a) $10\sqrt{2}$ J (b) $10\sqrt{3}$ J

(c) $\frac{10}{\sqrt{2}}$ J (d) $\frac{10}{\sqrt{3}}$ J

33. A man pushes a wall and fails to displace it. He does

(a) negative work (b) positive but not maximum work

(c) no work at all (d) maximum work

34. The incorrect relation is

(a) $1 \text{ W} = 1.34 \times 10^{-3}$ hp (b) 1kWh = 1000 watt hour

(c) $1 \text{ kWh} = 3.6 \times 10^3$ J (d) 1 W = 1 J/s

35. A force $\overline{F} = (14\hat{i} + 5\hat{j} + 3\hat{k})$ N is acting on a body and moves it through a distance $S = (2\hat{i} + 4\hat{j} + 4\hat{k})$ m. The work done by the force is (in J)

(a) Zero (b) 60 (c) 220 (d) 160

36. The amount of water a 1 hp electric motor can pump from a well 10 m deep is (in kg/s)

(a) 1.5 (b) 3.7 (c) 5.8 (d) 7.6

37. From a waterfall, water is pouring down at the rate of 100 kg/s on the blades of a turbine. If the height of the fall is 100 m, the power delivered to the turbine is approximately equal to

(a) 100 kW (b) 10 kW (c) 1 kW (d) 100 W

38. When the speed of a body is doubled, its KE is

(a) Doubled (b) One-fourth (c) Halved (d) Quadrupled

39. If the momentum of a body is numerically equal to four times its KE, then the velocity of the body is

(a) 2 (b) 4 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

40. A body of mass m and momentum p has KE

(a) $\frac{p}{m}$ (b) $\frac{p^2}{m}$ (c) p^2m (d) $\frac{p^2}{2m}$

41. Two masses 4 m and 9 m move with equal KE. The ratio of the magnitude of their momenta is

(a) 1 : 2 (b) 2 : 3 (c) 1 : 3 (d) 3 : 1

42. A body of mass 2 kg and momentum 4 kg m s^{-1} has KE (in J)

(a) 4 (b) 2 (c) 8 (d) 6

43. If the momentum of a body is increased 4 times, its KE will increase
(a) 4 times (b) 8 times (c) 12 times (d) 16 times

44. A 300 g mass has a velocity of $(3\hat{i} + 4\hat{j})$ m s^{-1} at certain instant. Its kinetic energy is
(a) 9 J (b) 2.4 J (c) 3.75 J (d) 7.35 J

45. If the linear momentum is increased by 50% the K.E will be increased by
(a) 25% (b) 100% (c) 50% (d) 125%

46. A light and a heavy body have equal momenta. Which one has greater KE?
(a) The light body (b) Both have equal KE
(c) The heavy body (d) Data given is insufficient

47. If the KE of a body changes from E to 4E by the application of a force 4N, the displacement produced is (in the direction of the force)
(a) $\frac{2E}{3}$ (b) $\frac{\sqrt{3}}{2}E$ (c) $\frac{15}{16}E$ (d) $\frac{3E}{4}$

48. A body of mass 10 kg at rest is acted upon simultaneously by two forces 4 N and 3 N at right angles to each other. The KE at the end of 10 second is
(a) 100 J (b) 300 J (c) 125 J (d) 20 J

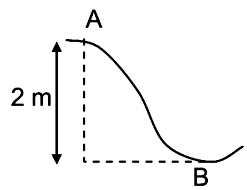
49. A bullet of mass 8 g hits a fixed body at a speed of 500 m s^{-1} and pierces it through 20 cm before coming to rest. The average energy loss in the body is (J/m)
(a) 8×10^2 (b) $50.\times 10^3$ (c) 20×10^{-3} (d) 5×10^3

50. A body of mass 2 kg is being rotated in a horizontal circle of radius 1 m with the help of a string. The energy spent by centripetal force is
(a) 20 J (b) 2 J (c) 5 J (d) Zero

51. A stationary body is slowly lowered on to a massive platform moving horizontally at a speed of 4 m s^{-1}. The distance the body slides relative to the platform, if coefficient of friction is 0.2, (g = 10 m s^{-2}) is
(a) 16 m (b) 8 m (c) 4 m (d) 2 m

52. A constant retarding force is applied to stop a train. If the speed of the train is doubled and the same retarding force is applied, distance will be
(a) the same (b) doubled (c) three times (d) four times

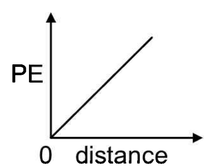
53. A body slides on a frictionless surface from A to B. If the velocity of a ball at A is 5 m s^{-1}, its velocity at B (g = 10 m s^{-2}) is

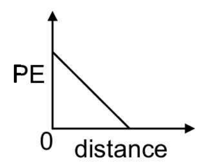


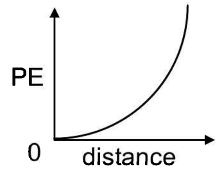
(a) $\sqrt{25}$ m s^{-1} (b) $\sqrt{40}$ m s^{-1}
(c) $\sqrt{50}$ m s^{-1} (d) $\sqrt{65}$ m s^{-1}

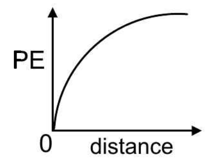
54. A spring of original length x is compressed to half its length. If the force constant of the spring is 8 N/m, its potential energy is
(a) 8x (b) $8x^2$ (c) x^2 (d) $4x^2$

55. When a body is dropped from a certain height its P.E changes with distance as

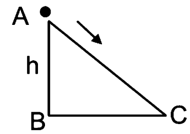
(a) 

(b) 

(c) 

(d) 

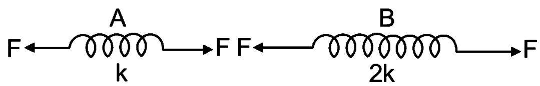
56. A body released from rest slides along a frictionless track AC. Its speed at C is



(a) mgh (b) gh

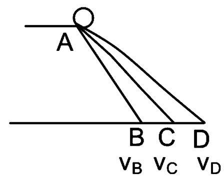
(c) $\sqrt{gh}$ (d) $\sqrt{2gh}$

57. If the energy stored in spring A = 10 J, that stored in B is (under the same stretching force)



(a) 10 J (b) 20 J (c) 5 J (d) 15 J

58. A ball is dropped along smooth inclines AB, AC and AD. The velocity acquired at B, C, D be v_B, v_C and v_D respectively. Then



(a) $v_B > v_C > v_D$ (b) $v_B < v_C < v_D$

(c) $v_B > v_C < v_D$ (d) $v_B = v_C = v_D$

59. A body of mass 500 g, moving with uniform speed on the floor goes up along a smooth incline and stops at a height 10 cm with respect to the floor. Its speed on the floor was ($g = 10$ m s^{-2})

(a) 1 m s^{-1} (b) $\sqrt{5}$ m s^{-1} (c) $\sqrt{2}$ m s^{-1} (d) $\frac{1}{\sqrt{50}}$ m s^{-1}

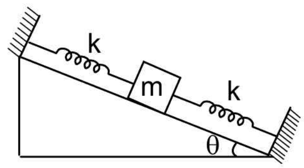
60. A ball is being hit from the ground level at an angle of 45° with the horizontal and has an initial kinetic energy E. The kinetic energy at the highest point of the path is

(a) 0 (b) $\frac{E}{\sqrt{2}}$ (c) E (d) $\frac{E}{2}$

61. A person holds a bucket of weight 50 N. He walks 5 m along the horizontal and he climbs up a vertical distance of 5 m. The work done by the person is

(a) 500 J (b) 250 J (c) Zero (d) 5000 J

62. In the arrangement shown both springs are in unstretched condition. If body m is released from rest, the maximum vertical distance that the body will come down is



(a) $\frac{2mg\sin\theta}{k}$ (b) $\sqrt{\frac{4mg\sin\theta}{k^2}}$

(c) $\sqrt{\frac{mg\cos\theta}{2k}}$ (d) $\frac{mg\sin^2\theta}{k}$

63. A spring of constant 80 N m^{-1} fixed at one end is having a mass of 0.05 kg at the other end. It is stretched by 0.1 m and released. The maximum velocity attained by the mass is

(a) 2 m s^{-1} (b) $2\sqrt{2}$ms^{-1} (c) 4 m s^{-1} (d) 1 m s^{-1}

64. Potential energy of a particle as a function of its displacement from origin is given by $U = x^4 - bx^2$. The body is in stable equilibrium at origin, if b is

(a) −1 (b) 2 (c) −3 (d) both a and c

65. A pendulum of length 1 m and bob mass equal to 0.5 kg is projected from the bottom position with an initial horizontal velocity of $v < \sqrt{5g\ell}$. If the bob loses the circular path when its velocity is 2 m s^{-1} (take this instant as t = 0), then the time after this the bob will reach vertical position above the point of suspension is approximately.

(a) $1\frac{1}{8}$ s (b) $1\frac{3}{8}$ s (c) 1.5 s (d) $1\frac{3}{4}$ s

66. In the above, its loss in potential energy at t = 0 is

(a) 5 J (b) 6 J (c) 7 J (d) 8 J

67. A rocket is held tight on a test pad and fuel is burnt at the rate of 10 kg s^{-1} and the exhaust speed is 1000 m s^{-1}. The force required to hold he rocket stationary is

(a) 10,000 N (b) 12,000 N (c) 13,160 N (d) 14,820 N

68. In the above case the power lost in the exhaust gas (exclude heat) is

(a) 2×10^6 W (b) 3×10^5 W (c) 4×10^6 W (d) 5×10^6 W

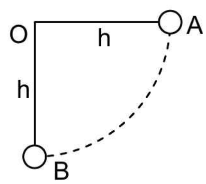
69. A shell is fired from a cannon with a velocity v at an angle θ to the horizontal. At the highest point of its path, it explodes into two pieces of equal masses; one of the pieces retraces its path to the point of projection. The speed of the second piece immediately after the explosion is

(a) $3v\cos\theta$ (b) $2v\cos\theta$ (c) $v\cos\theta$ (d) $\frac{2}{3}v\cos\theta$

70. In elastic collisions

(a) Momentum is conserved
(b) KE is conserved
(c) Both momentum and KE are conserved
(d) Momentum not conserved but KE is conserved

71. A and B are two identical balls attached to strings as shown. A is released from position OA. A collides with B elastically. Then,



(a) A and B move with different non-zero velocities
(b) B moves forward and A rebounds with the same velocity
(c) A and B stick together and move with same velocity
(d) A comes to rest and B moves with the velocity of A

72. An inelastic ball falls from a height of 100 m. It loses 20% energy due to impact. The ball will again rise to a height of

(a) 80 m (b) 98 m (c) 60 m (d) 40 m

73. A ball hits the floor and rebounds after inelastic collision. In this case

(a) the momentum of the ball just after the collision is the same as that just before the collision
(b) the mechanical energy of the ball remains the same in the collision
(c) the total momentum of the ball and the earth is conserved
(d) the total energy of the ball and the earth is conserved

74. Which one of the following Statements is true?

(a) Momentum is conserved in elastic collisions but not in inelastic collisions
(b) Total KE is conserved in elastic collisions but momentum is not conserved in elastic collisions
(c) Total KE is not conserved but momentum is conserved in inelastic collisions
(d) KE and momentum both are conserved in all types of collisions

75. A particle A suffers an oblique elastic collision with a particle B that is at rest. If their masses are the same, then, after the collision

(a) They will move in opposite directions
(b) A continues to move in the original direction while B remains at rest
(c) They will move in mutually perpendicular directions
(d) A comes to rest and B starts moving in the direction of the original motion of A

76. Which one of the following Statements does not hold good when two balls of masses m_1 and m_2 undergo elastic collision?

(a) When $m_1 < m_2$ and m_2 at rest, there will be maximum transfer of momentum
(b) When $m_1 > m_2$ and m_2 at rest, after collision the ball of mass m_2 can move with two times the velocity of m_1
(c) When $m_1 = m_2$ and m_2 at rest, there will be maximum transfer of KE
(d) When collision is oblique and m_2 at rest with $m_1 = m_2$, after collisions the ball moves in opposite directions

77. A body is dropped from height h while another identical body is thrown up with velocity $\sqrt{2gh}$. If they have a completely inelastic collision, they will reach ground in

(a) $\sqrt{\frac{h}{2g}}$ (b) $\sqrt{\frac{2h}{g}}$ (c) $\sqrt{\frac{3h}{2g}}$ (d) $\sqrt{\frac{h}{g}}$

78. A stationary bomb explodes into two parts, 4 kg and 8 kg. The velocity of the 8 kg mass is 6 m s^{-1}. The KE of the other body is

(a) 48 J (b) 24 J (c) 288 J (d) 16 J

79. A stationary body of mass m explodes into 3 parts with mass ratio of 1:3:3. The two fragments with equal mass move at right angles to each other with velocity of 15 m s^{-1}. The velocity of the third fragment is (m s^{-1})

(a) $\sqrt{2}$ (b) 5 (c) $20\sqrt{2}$ (d) $45\sqrt{2}$

80. A space craft of mass M is moving with velocity v in free space when it explodes and breaks in two. After the explosion, a mass m of the spacecraft is left stationary. The velocity of other part is

(a) $\frac{Mv}{(M-m)}$ (b) $\frac{Mv}{(M+m)}$ (c) $\left(\frac{m}{M+m}\right)\times\frac{1}{v}$ (d) $\frac{(M+m)v}{M}$

81. A particle of mass m moves in the positive direction with speed v_0 at the origin. If a force $F = -kx^3$ acts on the particle, the distance from the origin where the particle stops is

(a) $\left(\frac{mv_0^2}{k}\right)^{1/4}$ (b) $\left(\frac{2mv_0^2}{k}\right)^{1/4}$ (c) $\left(\frac{mv_0^2}{k}\right)^{1/3}$ (d) $\left(\frac{mv_0^2}{2k}\right)^{1/4}$

82. A body moves a distance of 10 m along a straight line under the action of a force of 5 N. If the work done is 25 J, the angle, which the force makes with the direction of motion of the body, is

(a) 0° (b) 30° (c) 60° (d) 45°

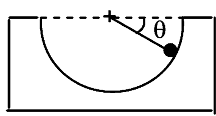
83. A car climbs up a gradient of 1 in 20 at a speed 5 m s^{-1}. The car weighs 6000 kg and the coefficient of friction is 0.01. The power required (in kW) is ($g = 10$ m s^{-2})

(a) 10 (b) 18 (c) 24 (d) 32

84. A 25000 kg airplane takes off from rest on the run away, and reaches an altitude of 5000 m and cruises at 900 km/hour in 8 minute. Assuming no viscous losses, the average power generated by the engines during this period (in MW) is

(a) 10.5 (b) 12.8 (c) 3.4 (d) 4.2

85. A particle of mass m released from top position on one side of a smooth semicircular surface of radius r in the vertical plane. Maximum power generated by gravity is when θ is:



(a) 0 (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\sin^{-1}\frac{1}{\sqrt{3}}$

86. If the displacement of a body of mass m is given by $x = a\sin\omega t$, then its kinetic energy is

(a) $\frac{1}{2}ma^2\omega^2[1+\cos\omega t]$ (b) $\frac{1}{4}ma^2\omega^2[1+\sin 2\omega t]$ (c) $\frac{1}{4}ma^2\omega^2[1+\cos 2\omega t]$ (d) $\frac{1}{4}ma^2\omega^2\sin^2\omega t$

87. A rocket with mass m and flow rate $\dot{m}$ at velocity u travels on a horizontal circular wire of radius r and friction coefficient μ. The constant velocity with which it travels is (neglect gravity)

(a) $\frac{\dot{m}\,u}{m\,\mu}$ (b) μu (c) $\frac{m\,\mu^2}{\dot{m}\,r}\mu$ (d) $\sqrt{\frac{\dot{m}ru}{m\mu}}$

88. A body of mass 1 kg has velocity 1 m s^{-1}, up an inclined plane of angle of 30° to the horizontal. The friction coefficient is $\frac{1}{\sqrt{3}}$. The distance the body travels before stopping is (g = 10 m s^2)

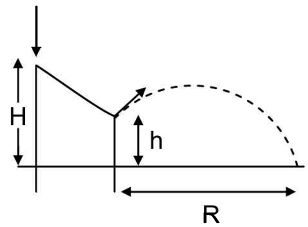
(a) 5 cm (b) 7.5 cm (c) 10 cm (d) 6.7 cm

89. A spring is held between two supports and it is in its normal length. If the center is displaced by δx normal to its length, the energy stored is the spring in proportional to $(\delta x)^n$ where n is

(a) 1 (b) 2 (c) 3 (d) 4

90. Water flows from the top of an inclined plane of height H and is leaves at the end of a tap arranged for maximum range. The maximum value of R is obtained for h of

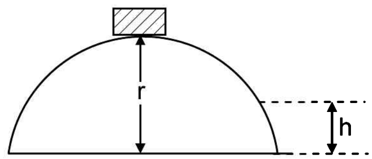
$$\left(R = \frac{u\sqrt{u^2+2gh}}{g}\right)$$



(a) H (b) $\frac{H}{2}$

(c) $\frac{H}{3}$ (d) 0

91. A small body of mass m slides down from the top of a hemisphere of radius r. The surface of block and hemisphere are frictionless. The height at which the body loses contact with the surface of the sphere is



(a) $\frac{3}{2}r$ (b) $\frac{2}{3}r$

(c) $\frac{1}{2}gt^2$ (d) $\frac{1}{2}r$

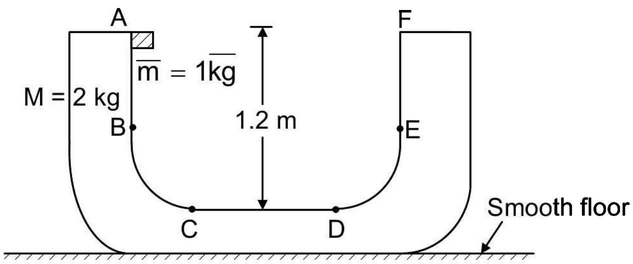
92. A person standing on a vehicle moving with speed v is thrown against a wall when it comes to rest suddenly. Assuming the mass of the person is m and the wall acts as a spring of constant k, the maximum force experienced is

(a) $\sqrt{mk}\ v$ (b) $\sqrt{2mk}\ v$ (c) $\sqrt{\frac{mk}{2}}\ v$ (d) $\sqrt{3mk}\ v$

93. A body of mass 1 kg is whirled in a vertical circle of radius 0.5 m. What is the velocity of the body when the string makes an angle of 37° with the vertical, if the tension in the string in this position is 10 N? (tan 37° = $\frac{3}{4}$, g = 10 m s^{-2})

(a) 1 m s^{-1} (b) 2 m s^{-1} (c) $\sqrt{6}$ m s^{-1} (d) $\sqrt{3}$ m s^{-1}

94.



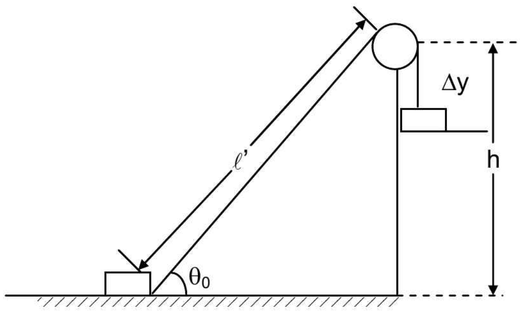
A trough shaped body M of mass 2 kg and having smooth surface is kept on a smooth floor. Sections AB and EF of the inner surface are perfectly vertical while section CD is horizontal and the inside curvature is smooth. A body m = 1 kg is released at the top most point A inside M. Neglect friction everywhere.

Kinetic energy of m when it is at mid position of CD is

(a) 12 J (b) 10 J (c) 8 J (d) 7.1 J

95. A mass m resting on a smooth horizontal slot is connected to another equal mass over a smooth pulley. The system is at rest when $\theta = \theta_0 (\Delta y << \ell = \ell' + \Delta y)$. The velocity of the mass on the floor at $\theta = 90°$ is

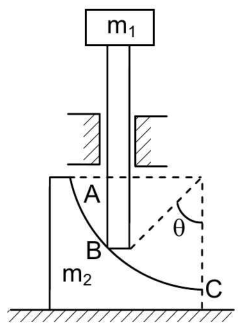
(a) $\sqrt{2gh}$ (b) $\sqrt{2gh(1-\cos\theta_0)}$ (c) $\sqrt{2gh(1-\sin\theta_0)}$ (d) $\sqrt{2gh(\operatorname{cosec}\theta_0 - 1)}$



96. A block of mass m at the end of a string is whirled round in a vertical circle of radius R. The critical speed of the block at the top of its string below which the string would slacken before the block reaches the top is

(a) Rg (b) $\frac{1}{\sqrt{Rg}}$ (c) $\frac{R}{g}$ (d) $\sqrt{Rg}$

97. A body of mass m_2 has one of its surfaces in the form of a quarter circle of radius R. A mass m_1 is placed on top of a plunger assembly. Assume all surfaces are frictionless. The final velocity of m_2, if plunger starts to descend from initial position B as shown, is



(a) $\sqrt{2gR.\frac{m_1}{m_2}}$ (b) $\sqrt{2gR(1-\sin\theta)\frac{m_1}{m_2}}$

(c) $\sqrt{2gR(1-\cos\theta)\frac{m_1}{m_2}}$ (d) $\sqrt{2gR.\frac{m_2}{m_1}}$

98. Potential energy of a particle free to move along x-axis is given by $\left(x^3 - \frac{x^2}{2}\right)$ J, where x is in metre. The particle is initially kept at x = 0 and then given a slight displacement Δx in +x direction. The forces acting on the particle will displace it to:

(a) x = 0 (b) $x = +\infty$ (c) $x = -\infty$ (d) $x = +\frac{1}{3}$m

99. Two springs P and Q are identical except that P is stiffer than Q. If they are stretched by the same amount and W_P and W_Q represent the work expended on the springs then

(a) $W_P > W_Q$ (b) $W_P < W_Q$ (c) $W_P = W_Q$ (d) $W_P = \frac{W_Q}{2}$

100. Two equal masses are sent down on two inclined planes as shown. Both surfaces have same coefficient of friction μ. The loss in kinetic energy is

(a) more in (1) (b) more in (2) (c) same in both cases (d) data insufficient

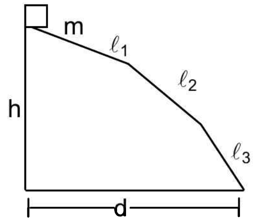


Figure (1)

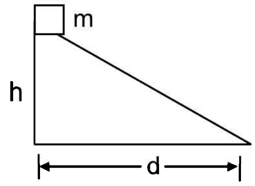


Figure (2)

101. Two masses m each are connected by a spring. An identical system of masses moving with a speed v_0 collides with this system along the axis. The ratio of maximum compression of spring in moving system to that in rest system is(Assume both springs are initially in their normal lengths)

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) 1

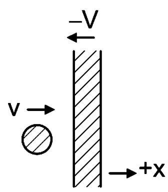
102. A and B are two points on the edge of the floor of a circular room of radius R. If the collisions with the walls are elastic, and if the particle has exactly two collisions with the wall before reaching B from A, AB is a side of a square inscribed with the circumference of the room

(a) $\frac{R}{2}$ (b) $\frac{\sqrt{3}R}{2}$ (c) $\sqrt{2}R$ (d) R

103. Three identical balls of mass m are placed on a straight line on a smooth table separated by a distance between each pair. The critical condition that a striker ball of mass m' (= k times m where k is a constant), knocks all three balls off the table is

(a) $k > 3$ (b) $k > 6$ (c) $k > 1$ (d) $k > 0.5$

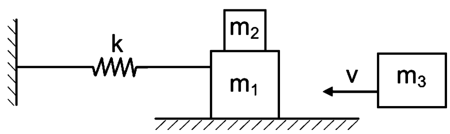
104. A particle moving with a velocity v in +x direction has an inelastic collision with a wall moving in opposite direction with a speed V and coefficient of restitution is e. If the speed of the ball remains same after collision then V is (modulus value)



(a) $\frac{1-e}{1+e}v$ (b) $\frac{1-e}{1+2e}v$

(c) $\frac{1-e}{1+e}v^2$ (d) $\frac{1-2e}{1+e}v$

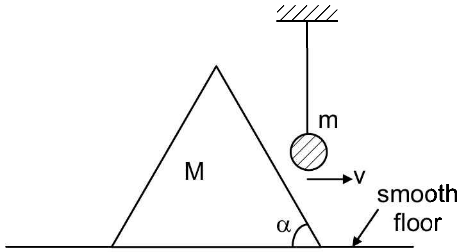
105. A block m_1 rests on a smooth floor and is connected to a spring of constant k. A mass m_2 is placed on top of m_1 and coefficient of friction between them is μ. The minimum velocity of a body of mass m_3 which strikes m_1 and sticks to it so that m_2 slips is



(a) $\mu g\sqrt{\frac{m_1+m_2+m_3}{k}}$ (b) $\mu g\sqrt{\frac{(m_1+m_2)m_3}{m_1 k}}$

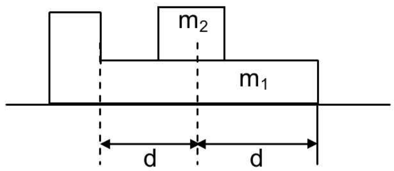
(c) $\mu g\sqrt{\frac{(m_1+m_3)m_1}{m_3 k}}$ (d) slips for all values

106. A large wedge of base angle α and mass M, moving with a velocity v on a smooth floor has an elastic collision with the bob of mass m, of a pendulum hung from the ceiling and m << M. The velocity of the bob after collision is:



(a) $v\sin^2\frac{\alpha}{2}$ (b) $v\cos^2\frac{\alpha}{2}$ (c) $2v\cos\alpha$ (d) $2v\sin\alpha$

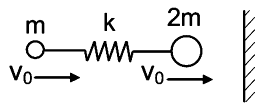
107. A body m_1 rests on the smooth floor while another body m_2 is placed on top. All surfaces are smooth. If a velocity v_0 is given to m_1 towards the right, and the collision of m_2 with the side of m_1 is elastic the time taken for m_2 to slide off is $\left(\frac{m_2}{m_1}=k\right)$



(a) $\left(\frac{2k+3}{k+1}\right)\left(\frac{d}{v_0}\right)$ (b) $\frac{2d}{v_0}\cdot\frac{1}{k+1}$

(c) $\frac{3d}{v_0}$ (d) $\frac{1-k}{1+k}\cdot\frac{d}{v_0}$

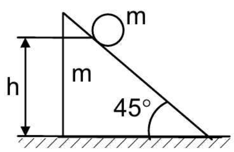
108. The system shown (the spring is without tension) travels towards the wall and has an elastic collision. The maximum compression of the spring is



(a) $v_0\sqrt{\frac{m}{k}}$ (b) $2v_0\sqrt{\frac{m}{k}}$

(c) $2v_0\sqrt{\frac{2m}{3k}}$ (d) $2v_0\sqrt{\frac{m}{3k}}$

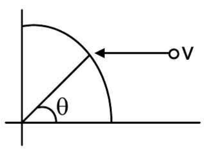
109. A mass m is released from a height h on a block of mass m, which rests on a smooth floor. After elastic collision with the surface the mass will rise to a height of



(a) h (b) $\frac{2h}{3}$

(c) $\frac{4h}{9}$ (d) $\frac{h}{2}$

110. A particle strikes a quarter circular disc and rebounds elastically with a velocity as shown. The angle by which its velocity vector is rotated is



(a) θ (b) 2θ (c) $\pi-2\theta$ (d) $\frac{\pi}{2}+\theta$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

If the distance s moved by the body is not in the direction of force but makes an angle θ with it, then the work done is given by $W = F(s\cos\theta) = \overline{F}.\overline{s}$.

and

Statement 2

If the force and displacement are at right angles to each other the work done is zero.

112. Statement 1

The amount of work that we must do in order to bring a moving body to rest is equal to the negative value of kinetic energy of the body.

and

Statement 2

The kinetic energy of a moving body is equal to the total work that is done on the body starting from rest.

113. Statement 1

A negative mechanical energy implies that its potential energy is negative.

and

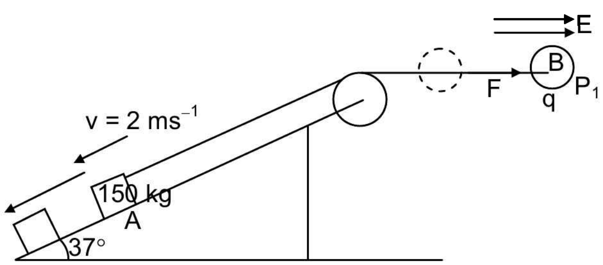
Statement 2

The potential energy value for the reference state is arbitrary.

Linked Comprehension Type Questions

Directions: **This section contains a paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Concept of 'work' in physics is different from the usual meanings of work, when we use the word work in day-to-day language. When a body under the influence of a constant force $\overline{F}$ moves a distance $\overline{d}$, work done = $\overline{F}\cdot\overline{d}$. Work is not a vector but it can be positive or negative. If the body moves in the direction of force applied, positive work is done on the body. If the body moves in the opposite direction of the force applied, we say negative work is done. The body on which work is done gains or loses energy depending on whether positive or negative work is done.



Force doing the work can be conservative or non-conservative. If from state A to state B a body is taken through different paths and work done is same, then the forces are conservative. An example is gravitational force. On the other hand frictional force is non-conservative and the work done depends on the path taken.

If a body is in motion under a system of conservative forces, the sum of kinetic and potential energies is constant. Now consider a system shown in the figure. A body of 150 kg mass is sliding down the inclined plane having a smooth surface, with velocity $v = 2\ \text{m s}^{-1}$. A force of $F = 1000$ N is applied as shown by switching on a uniform electric field $\vec{E}$ on the body 'B', with charge q at time $t = 0$. The body stops after time t.

114. How much does the body move on the inclined plane before it comes to a halt?
(a) 1 m (b) 2 m (c) 3 m (d) 3.2 m

115. The acceleration of the body during the motion is
(a) $0.5\ \text{m s}^{-2}$ (b) $\frac{2}{3}\text{m s}^{-2}$ (c) $2\ \text{m s}^{-2}$ (d) $2.3\ \text{m s}^{-2}$

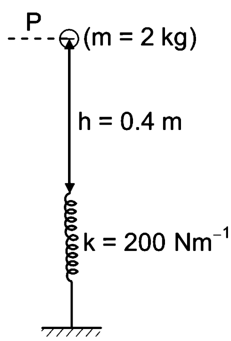
116. The time it takes for the body to stop is
(a) 1 s (b) 2 s (c) 3 s (d) 4 s

Multiple Correct Objective Type Questions

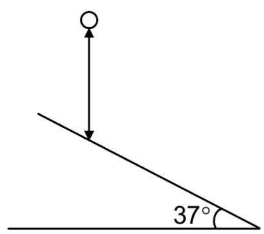
Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. In which of the following is work done by the external agent described?
(a) The motor of a lift going up with the same acceleration as the acceleration due to gravity
(b) A man carrying a bucket of water on level ground.
(c) A crane lowering a load vertically with a constant velocity
(d) A man carrying a bucket of water on a stair case

118. A ball of mass 2 kg is dropped from a height of 0.4 m above the free end of a vertically fixed spring of constant $k = 200\ \text{N m}^{-1}$
(a) The maximum K.E attained by the ball is 9 J
(b) The maximum K.E is attained 0.5 m below original position P.
(c) The maximum potential energy attained by the spring is 16 J
(d) The maximum potential energy attained by the spring is 9 J



119. A ball falls on an inclined plane of angle of inclination 37°, from a height of $\frac{20}{9}$m above the point of impact. The coefficient of restitution of the impact is, $e = \frac{9}{16}$. Then,



(a) The maximum vertical height reached by the ball above the point of impact is 0.8 m
(b) The maximum vertical height reached by the ball above the point of impact is 0.92 m

(c) The velocity of the ball after the impact is 5 m s^{-1}

(d) The ball will fall back on the inclined plane after $\frac{3}{4}$ s after the impact

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

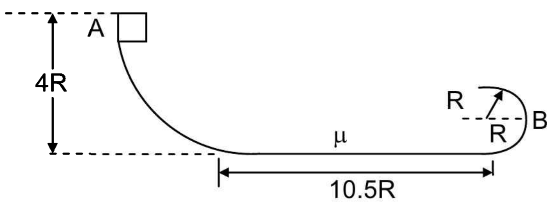
120. Match the physical quantities, which are unchanged in the process in column II

Column I	Column II
(a) Momentum	(p) elastic collision between two bodies
(b) Total energy	(q) inelastic collision between two bodies
(c) Kinetic energy	(r) explosion
(d) Relative velocity	(s) recoil

ADDITIONAL PRACTICE EXERCISE

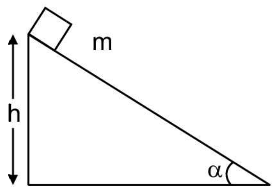
Subjective Questions

121. The track in the vertical plane has a curved portion of total height 4R, a straight level portion of length 10.5R and a circular portion of radius R. The straight portion alone is rough. An object starts at A from rest. When it is at B, the resultant force acting on it makes an angle $\tan^{-1}\frac{3}{4}$ with the vertical. Determine



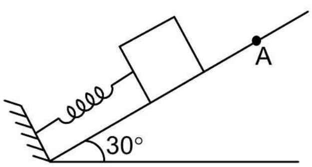
(i) the velocity of the body at B in terms of R
(ii) the coefficient of friction μ in the straight portion. ($g = 10$ m s^{-2})

122. A block of mass $m = 5$ kg slides from the top of a smooth fixed inclined plane of altitude $h = 9$ m and angle of inclination $\tan^{-1}0.75$ with the horizontal. Calculate



(i) the work done by each force acting on the block, and the total work done by all forces, till it reaches the bottom,
(ii) the velocity of the block as it reaches the bottom and
(iii) the time taken for the block to reach the bottom.

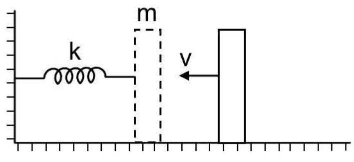
123. If a mass of 10 kg is kept on the inclined plane at the end of a spring and slowly allowed to come to rest, the spring has a maximum compression of 25 cm. The friction coefficient between plane and mass is $\mu = \frac{1}{2\sqrt{3}}$.



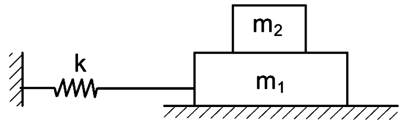
(i) What is the spring constant k?
(ii) Now the mass is kept at a farther point A on the plane and released from rest. The spring has a maximum compression of 2 m. What is the height descended by the mass? ($g = 10$ m s^{-2})

124. (i) A block of mass m moving at a speed v_0 compresses a spring through a distance x_0 before its speed is halved. Find the spring constant.

(ii) Suppose this solid block is pushed against a spring of same spring constant as above. The natural length of the spring is L and it is now held compressed to half its natural length and then the block is released. Find the velocity of the block when the spring relaxes to its original length for the first time.



125. A block m_1 rests on a smooth floor and is connected to a spring of constant k. A block of mass m_2 rests on top. The coefficient of friction between the masses is μ. The mass m_2 is suddenly given a velocity v_0. If it stops slipping when the compression of spring is maximum, find the maximum compression of the spring



126. A 10 H.P pump, working at 80% efficiency is used for drawing water from a well. The water level in the well is 6 m below the location of the pump. The water outlet from pump is horizontal and the water flow rate through the pump is 50 litre per second.

(i) What is the energy spent by the pump in 10 hour?

(ii) What is the speed with which water will come out of the pump?

127. For a projectile, launched from the ground at an angle 'θ' with the horizontal, the ratio of its kinetic energy to its potential energy at the maximum height is 3:7. What is the corresponding ratio when the projectile is at.

(i) $\frac{3}{4}$ of its maximum height?

(ii) $\frac{1}{2}$ of its maximum height?

(iii) $\frac{1}{4}$ of its maximum height?

(iv) At what height, expressed as fraction of maximum height, will its KE and PE be equal?

128. A block of mass m = 12.25 kg is placed on a table of mass M = 37.5 kg which can move without friction on a level floor. A particle of mass m_0 = 0.25 kg moving horizontally strikes the block totally inelastically with velocity 300 m s^{-1} (μ between block and table = 0.25). Calculate

(i) the final velocity of the combined mass.

(ii) the kinetic energy acquired by the combined mass.

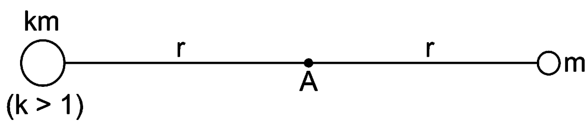
(iii) the relative retardation of the block.

129. A stone is attached to a light inextensible string of length 2m and rotated in a vertical circle with the axis of rotation passing through the other end of string.

(i) If the speed of the stone at the top most position is 10 m s^{-1}, what is the speed of the stone at the lower most position? Take g = 10 m s^{-2}

(ii) If the speed of the stone at the lowest position is 8 m s^{-1}, will it be possible to complete the vertical circular motion?

130.



Two particles of mass m and km respectively are connected to the ends of a light, inextensible string of length 2r and tied to a nail at A. Initially the strings are held horizontal as shown. When released, the particles collide in an inelastic collision (e = 0.5).

(i) What is the speed of the heavy and light particles immediately after collision?

(ii) What is the value of k for which the lighter particle reaches its initial position after collision?

(iii) For the value of k in (ii) above, what is the speed of the heavy particle after collision?

Straight Objective Type Questions

***Directions:* This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. A travelling bullet is brought to rest by a wooden block. Let F' be the time average of the resistance force and F" be the distance average of the resistance force, Then

(a) F' = F"
(b) F' > F"
(c) F' < F"
(d) cannot be concluded with given data

132. Work done in time t ($\le t_0$) on a body of mass m, which is accelerated from rest to speed u in time t_0, as a function of time t is given by

(a) $\frac{1}{2}m\frac{u^2}{t_0^2}t^2$
(b) $\frac{1}{2}m\frac{u^2}{t_0}t$
(c) $m\frac{u}{t_0}t^2$
(d) $\frac{1}{2}m\frac{u}{t_0}t^2$

133. A particle is in motion along a straight line under the action of a force F which varies with velocity v as per law $F = \frac{A}{v}$ where A is a constant. The work done by the force in time t is

(a) At
(b) At^2
(c) $\frac{A}{t}$
(d) $\frac{A}{t^2}$

134. A force $(3\hat{i}-2\hat{j})$ N acting on a particle, does zero work when the particle is displaced from point (1, −1) to a point (2, a). Then a is (position co-ordinates are in metre)

(a) $\frac{1}{2}$ m
(b) 1 m
(c) $\frac{3}{2}$ m
(d) $-\frac{3}{2}$ m

135. A loco-engine of mass 40 ton moves on a straight track having $\mu = 0.01$. When its speed is 72 km h^{-1}, power developed by engine is 880 kW. Its acceleration at that instant is (in m s^{-2}) (g = 10 m s^{-2})

(a) 0.1
(b) 1
(c) 0.2
(d) 2

136. A body of mass 4 kg is projected at 20 m s^{-1} at an angle 57° to horizontal. Power of the gravitational force on the block at its highest point is

(a) 480 W
(b) 240 W
(c) 640 W
(d) zero

137. Power of frictional force on a body of mass m as a function of time t, if the body is released at t = 0 on a rough inclined plane of angle θ and coefficient of friction μ (< tanθ) is

(a) $\mu mg^2 t\cos\theta$
(b) $\mu mg^2 t\sin\theta$
(c) $\mu mg^2 t\sin\theta(\sin\theta - \mu\cos\theta)$
(d) $\mu mg^2 t\cos\theta(\sin\theta - \mu\cos\theta)$

138. If a body moves from rest along a straight line under constant power, its displacement is proportional to time raised to power

(a) $\frac{1}{2}$
(b) 1
(c) $\frac{3}{2}$
(d) 2

139. A particle of mass m is subjected to constant power P. Its displacement when velocity increases from u to v is

(a) $\frac{(v^2-u^2)m}{2P}$
(b) $\frac{(v^3-u^3)m}{3P}$
(c) $\frac{2m(v^2-u^2)}{P}$
(d) $\frac{3m(v^3-u^3)}{P}$

140. Power of a force acting on a particle of mass 2 kg varies with time as per $P = \frac{2t^2}{3}$. where P is in watt and t is in second. At t = 0, the particle is at rest. Its velocity at t = 3 s is (in m s^{-1})

(a) 2
(b) $\sqrt{6}$
(c) $2\sqrt{2}$
(d) 4

141. A block is moved from rest by constant power P along a rough horizontal plane (coefficient of friction μ). Then the maximum velocity attained by the block is

(a) $\frac{P}{\mu mg}$ (b) $\frac{\mu mg}{P}$ (c) μPmg (d) $\sqrt{\frac{P}{\mu mg}}$

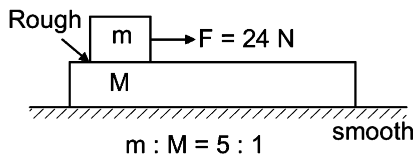
142. If the kinetic energy of a body increases by 800%, its momentum increases by

(a) 400% (b) 200% (c) 141% (d) 100%

143. Ratio of a particle's momentum to kinetic energy is inversely proportional to time. Then the particle executes

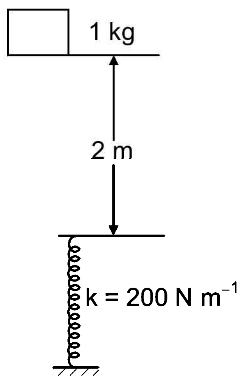
(a) uniform motion
(b) uniformly accelerated motion
(c) simple harmonic motion
(d) none of the above

144. Initially, blocks are at rest. Work done by friction in the first 5 s is zero. Work done by F = 24 N in the next 5 s is 1800 J. How many among 0.05, 0.15, 0.25 are acceptable values for coefficient of friction? (Friction exists between m and M only)



(a) zero
(b) one
(c) two
(d) three

145. A 1 kg stone is dropped down from a height of 2m on a vertically fixed spring of spring constant k = 200 N m^{-1}. The maximum energy stored in the spring subsequently is. (in joule). Take g = 10 m s^{-2}.

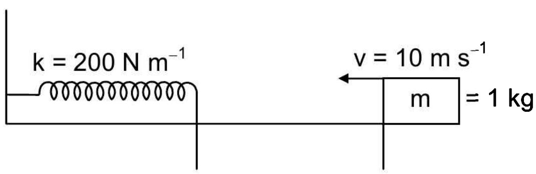


(a) 20.05
(b) 25
(c) 26.2
(d) 28

146. A body of mass m is projected along a rough inclined plane (having an angle of inclination with horizontal θ, equal to angle of repose) with a velocity v. It travels up a maximum distance s before it comes to a halt. Then v is

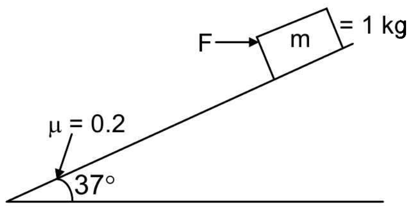
(a) $\sqrt{g\,s\cos\theta}$
(b) $2\sqrt{g\,s\sin\theta}$
(c) $2\sqrt{g s \tan\theta}$
(d) $\sqrt{g\,s\tan^2\theta}$

147. In the set up shown the mass m = 1 kg, kept on a rough floor is projected towards the spring with an initial velocity of v = 10 m s^{-1}. The final energy stored in the spring is 0.04 J. Then the μ of the floor can be



(a) 0.45 (b) 0.35 (c) 0.3 (d) 0.25

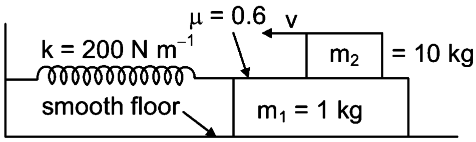
148. A mass of 1 kg is sliding down on a rough inclined plane inclined up at 37° to horizontal and μ = 0.2. When the velocity of the mass is 12 m s^{-1} a horizontal force F = 10 N starts acting on the body and brings it to halt. The work done by the force is



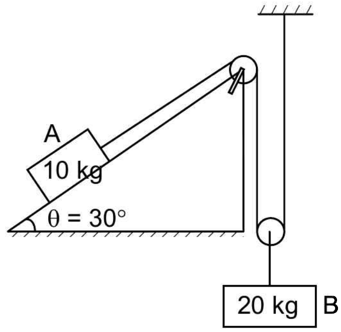
(a) +120 J
(b) −10 J
(c) −80 J
(d) −120 J

149. In the set up shown, the floor is smooth, m_1 = 1 kg, m_2 = 10 kg and the coefficient of friction between their surfaces is μ = 0.6. Assume the surface of m_1 is large enough so that when m_2 is given a velocity v as shown, it slips on m_1 till the spring is compressed to maximum i.e., till m_1 stops. The spring constant $k = 200\ N\ m^{-1}$. The maximum velocity achieved by m_1 is

(a) $3\sqrt{2}\ m\ s^{-1}$ (b) $5\sqrt{2}\ m\ s^{-1}$ (c) $3\sqrt{3}\ m\ s^{-1}$ (d) $\sqrt{7}\ m\ s^{-1}$



150. Block A (mass = 10 kg) and block B (mass = 20 kg) are connected as shown. The pulleys are light and smooth. The strings are light and inextensible. Initially the system is at rest. When released, the work done by the tension in string on block A, in moving it trough 3 m upwards along the smooth inclined plane (angle of inclination with horizontal = 30°) is ($g = 10\ m\ s^{-2}$)



(a) 450 J (b) 500 J
(c) 300 J (d) 250 J

151. Work of W is performed to compress a spring and then an additional 2W is performed to elongate it. The ratio of elongation to compression is

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$

152. A simple pendulum oscillates so that θ is the maximum angle with vertical. Its maximum speed is $\sqrt{\frac{7}{5}}$ times its speed at $\frac{\theta}{2}$. Then, θ is

(a) $\cos^{-1}\frac{1}{2}$ (b) $\cos^{-1}\frac{1}{4}$ (c) $\cos^{-1}\frac{1}{6}$ (d) $\cos^{-1}\frac{1}{8}$

153. A block dropped from a height h over a vertically held spring, compresses the spring to maximum of x at which instant its acceleration is 3g. Then, $\frac{h}{x}$ is

(a) 1 (b) 2 (c) 3 (d) 4

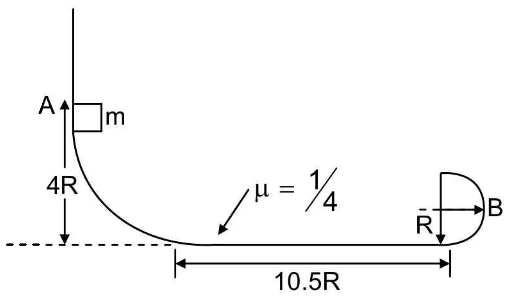
154. A block released from top of an inclined plane, of angle θ with the horizontal, with an initial velocity u down the plane has a speed v after travelling a distance s down the plane. Coefficient of friction is μ. Then which of the following will be constant?

(a) $v^2 - 2gs\sin\theta - 2\mu gs\cos\theta$
(b) $v^2 - 2gs\sin\theta + 2\mu gs\cos\theta$
(c) $v^2 + 2gs\sin\theta - 2\mu gs\cos\theta$
(d) $v^2 + 2g\sin\theta + 2\mu gs\cos\theta$

155. The potential energy of a particle as a function of its position is given by $U = x^3 - 6x^2$ (in SI unit). Then

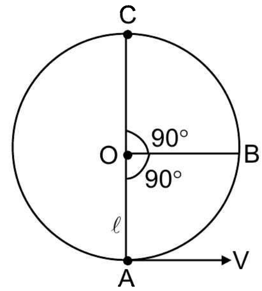
(a) at x = 3 m it is at stable equilibrium
(b) at x = 4 m it is unstable equilibrium
(c) at x = 4 m it is at stable equilibrium
(d) at x = 1 m the force on it is +8 N

156. The track in the vertical plane has a curved portion of total height 4R, a straight portion of length 10.5 R and a semi circular portion of radius R. The straight line portion is rough $\mu = \frac{1}{4}$, curved portions are smooth Mass m = 1 kg released at A, when it reaches B; the normal reaction on it is:



(a) 7.5 N (b) 8.2 N (c) 9.1 N (d) 10.8 N

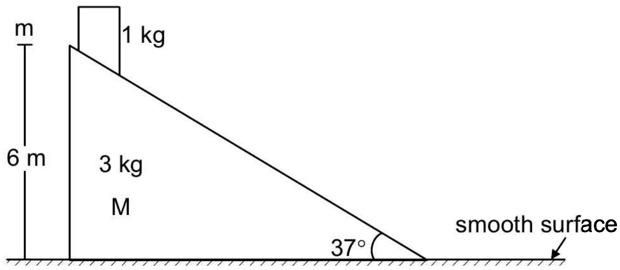
157. A pendulum consists of a bob of mass 'm' suspended by a light, inextensible string of length 'ℓ' fixed at O. What is the horizontal velocity 'V' to be given to the bob at position A, so that it passes position B but does not reach position C?



(a) $\sqrt{2g\ell} < V < 2\sqrt{g\ell}$ (b) $\sqrt{g\ell} < V < \sqrt{2g\ell}$

(c) $\sqrt{g\ell} < V < \sqrt{5g\ell}$ (d) $\sqrt{2g\ell} < V < \sqrt{5g\ell}$

158. A 1 kg mass is released from the top surface of a smooth surface of wedge kept on another smooth surface, as shown. The kinetic energy of the wedge when the mass reaches bottom is approximately



(a) 8.6 J (b) 10.1 J (c) 12 J (d) 15 J

159. Two masses are attached, each to either end of a spring. The spring is compressed till its potential energy is U and then released. Their velocity of separation at the instant the heavier mass has a momentum p is less than or equal to

(a) $\frac{U}{2p}$ (b) $\frac{U}{p}$ (c) $\frac{2U}{p}$ (d) $\frac{4U}{p}$

160. From a hopper, cement is falling vertically on to a horizontal conveyor belt at a constant rate of 10 kg/s. If the belt is moving at a constant speed of 15 m s^{-1}, what is the minimum power required to keep the belt moving?

(a) zero (b) 1.125 kW (c) 0.15 kW (d) 2.25 kW

161. A rocket moving in free space at 70 m s^{-1} comes to a halt by burning fuel equal to half of its original mass. The relative velocity of the exhaust must be equal to

(a) 120 m s^{-1} (b) 100 m s^{-1} (c) 90 m s^{-1} (d) 80 m s^{-1}

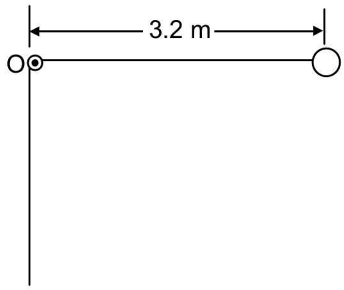
162. A body A is dropped from a height h. At that instant, another body B is projected up from ground below with a velocity of +25 m s^{-1}. [Take upward direction as +ve] The masses of the bodies are equal and the bodies have an elastic collision in mid-air. After collision the velocity of A is equal to −7 m s^{-1}. Then velocity of A before collision is (in m s^{-1})

(a) −25 (b) −28 (c) −32 (d) −35

163. A body of mass M moving with kinetic energy K collides elastically with a mass m at rest and kinetic energy of M after collision is $\frac{K}{4}$ and it moves in the same direction. Then the ratio $\frac{M}{m}$ is

(a) 2 (b) 3 (c) 3.5 (d) 4.2

164. The bob of a pendulum, 3.2 m long and fixed at O as shown is released from a horizontal position as shown. The coefficient of restitution for collision of the bob with the wall is 0.5. After the fourth collision, the bob will rise to a height of (g = 10 m s^{-2})



(a) 0.125 m (b) 0.5 m (c) 0.75 m (d) 1.25 cm

165. Two bodies are projected vertically upward from the same point at the same instant with velocities u and 1.3 u in m s^{-1}. The coefficient of restitution of the floor is e = 0.5. The bodies will meet each other after time t equal to (in second) (g = 10 m s^{-2})

(a) 1.2 u (b) 0.8 u (c) 0.73 u (d) 0.25 u

166. A block of mass M is suspended by a string. Bullets, each of mass m, are fired vertically upward towards the centre of the block at speed u. If the coefficient of restitution is e, how many bullets per second are to be fired so that the string is just taut?

(a) $\frac{Mu(1+e)}{mg}$ (b) $\frac{Mg(1+e)}{mu}$ (c) $\frac{Mg}{mu(1+e)}$ (d) $\frac{Mg}{mue}$

167. A ball falling from height H rebounds from floor to height $\frac{H}{2}$. The time of contact is proportional to the time of fall. The average force exerted by the floor is proportional to H^n where n is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) none of these

168. After an elastic collision between a particle of speed v and a particle at rest, the lighter particle moves at $\frac{v}{2}$. The ratio of their masses (heavier : lighter) is

(a) $\frac{3}{2}$ (b) 2 (c) 3 (d) 4

169. A heavy body and a light body travel toward each other along a straight path, with speed 5 m s^{-1} and 10 m s^{-1} respectively. If the coefficient of restitution is 2/3, then the velocity of the light body after collision is

(a) 5 m s^{-1} (b) 10 m s^{-1} (c) 15 m s^{-1} (d) 20 m s^{-1}

170. A bomb explodes into two parts, after which one part is at rest. The kinetic energy of the other part of mass 2 kg forms a ratio 3 : 4 or 4 : 3 with the (chemical) energy of explosion. The mass of the bomb is (in kg) [Assume that the chemical energy is fully released as mechanical energy only]

(a) 5 (b) 8 (c) 10 (d) 6

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

A force $\overline{F}$ acting on a particle is given by the expression $\overline{F} = q(\overline{v} \times \overline{B})$, where $\overline{v}$ is the velocity of the particle and q and $\overline{B}$ are constants. The work done by the force on the particle is zero.

and

Statement 2

The force $\overline{F}$ is perpendicular to $\overline{v}$.

172. Statement 1

If the speed of a particle is 2 m s^{-1}, the magnitude of its linear momentum (in kg m s^{-1}) is numerically equal to its kinetic energy (in joule), irrespective of the mass of the particle.

and

Statement 2

Mass of a particle is equivalent to energy (in joule).

173. Statement 1

A block released from top of a rough inclined plane will reach bottom with the same kinetic energy whatever be the angle of inclination and height as long as the horizontal displacement remains same

and

Statement 2

In an inclined plane situation, for a given horizontal dimension both the magnitudes of friction force and displacement change as angle of inclination changes.

174. Statement 1

A body is projected upward along an inclined plane from its bottom with a speed v, when another identical body is projected down along on the same inclined plane with same speed, v. When they travel the same distance on the inclined plane, the loss in KE is same for both.

and

Statement 2

Same magnitude of frictional force in the opposite direction of motion develop in each case and does same negative work.

175. Statement 1

If there are conservative and non-conservative forces acting, the mechanical energy, is conserved.

and

Statement 2

Energy may be transformed from one kind to another but it cannot be created or destroyed.

176. Statement 1

The conservative force is the negative gradient of potential energy $\left(F = -\dfrac{dU}{dx}\right)$

and

Statement 2

Potential energy can be defined only for conservative forces.

177. Statement 1

If the work done by a force on a particle does not depend on path followed by the particle, then the force is called conservative.

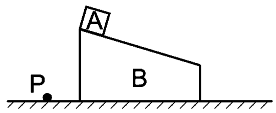
and

Statement 2

The work done by conservative forces acting on a particle depend only on the initial and final positions of the particle.

178. Statement 1

A block A is released from rest on top of a movable wedge B as shown. No friction anywhere. At the end of the journey, the block A cannot be found at a point such as P shown, whatever may be the masses, angle, height etc.



and

Statement 2

Momentum is conserved in the horizontal direction due to absence of external forces.

179. Statement 1

Linear momentum of a body at rest in a moving train is zero relative to a man sitting in the train.

and

Statement 2

Linear momentum of a body at rest on a moving train is not zero for a man standing on the ground.

180. Statement 1

A ball dropped on an inclined plane ($\theta = 30°$ with horizontal) from a height h, rebounds and reaches ground. Smaller the value of co-efficient of restitution e, upto the limit that the body clears the inclined plane and reaches ground, slower it reaches ground after rebounding.

and

Statement 2

Smaller the value of e, smaller the velocity after impact.

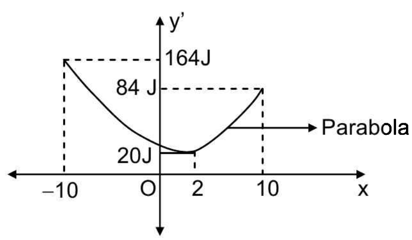
Linked Comprehension Type Questions

Directions: **This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

A single conservative force F(x) acts on a 1 kg particle that moves along the X-axis. The variation of potential energy with x is given as

Given at x = 5 m, the particle has a kinetic energy of 20J and its potential energy is related to position 'x' as per $U = 20 + (x - 2)^2$J where x is in m.



181. The mechanical energy of the system is

(a) 20 J (b) 29 J (c) 40 J (d) 49 J

182. The value of x for which the kinetic energy is maximum is

(a) 0 (b) 2 m (c) 1 m (d) 4 m

183. The maximum value of kinetic energy is

(a) 20 J (b) 49 J (c) 29 J (d) 40 J

Passage II

Collision of two bodies when kinetic energy is conserved is called elastic collision and otherwise it is called inelastic collision. Whenever two bodies collide, with both of them moving or only one of them moving initially, the total momentum in any axis will be conserved as per Newton's 2nd law. But kinetic energy will be conserved only if the collision is elastic. When two bodies collide and after collision they stick together, we call it perfectly inelastic collision and obviously kinetic energy is not conserved. In normal case kinetic energy lost is used up as heat or for distractive work done. But in the case of atoms colliding inelastically, the energy difference can be used up for excitation of the atom to higher level. Now consider the problem given below. A moving hydrogen atom makes a head-on inelastic collision with a hydrogen atom at rest. Before collision both were in ground state and after collision they move together and one atom goes to minimum excitation state.

[Given that $m_H = 1.67 \times 10^{-27}$ kg. Minimum excitation energy of Hydrogen atom = 10.2, eV; 1 eV = 1.6×10^{-19} J]

184. If u is the initial velocity of moving hydrogen atom, after collision, the combined velocity is

(a) u (b) $\frac{u}{2}$ (c) $\frac{u}{4}$ (d) 0

185. Kinetic energy lost after collision is

(a) $m_H \frac{u^2}{2}$ (b) $m_H \frac{u^2}{4}$ (c) $m_H \frac{u^2}{8}$ (d) $2m_H u^2$

186. The value of initial velocity of the hydrogen atom is

(a) 6.25×10^4 m s^{-1} (b) 5.6×10^4 m s^{-1} (c) 6.8×10^5 m s^{-1} (d) 7.2×10^4m s^{-1}

Passage III

A ball moving with a velocity of 20 m s^{-1} in +X direction collides with a heavy wall moving in front of it in the same direction with a constant speed of 10 m s^{-1}. The coefficient of restitution of the collision is 0.8

187. The modulus value of relative velocity of the ball with respect to the wall after collision is

(a) 6 m s^{-1} (b) 8 m s^{-1} (c) 9 m s^{-1} (d) 10 m s^{-1}

188. The actual velocity of the ball after collision is (take +X as positive direction

(a) −8 m s^{-1} (b) −18 m s^{-1} (c) −2 m s^{-1} (d) 2 m s^{-1}

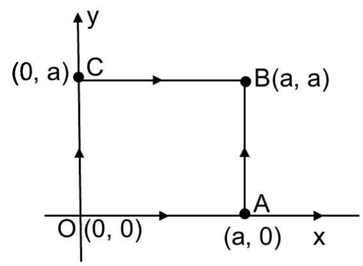
189. Loss in kinetic energy of the ball in percentage is:

(a) 90 (b) 85 (c) 96 (d) 99

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers outof which ONE OR MORE answers will be correct.**

190. A force $\overline{F} = k(y^2\hat{i} + x\hat{j})$, where k is a positive constant, acts on a particle when it is at position (x, y). First, the particle is taken through path OAB and the work done by the force is W_{OAB}. Then the particle is taken on through path OCB and the work done by the force is W_{OCB}. Then



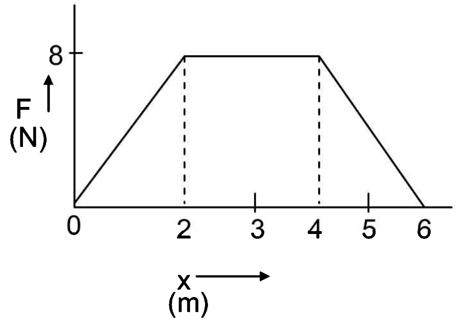
(a) $W_{OAB} = ka^2$ (b) $W_{OCB} = ka^2$

(c) the force is conservative (d) the force is non-conservative

191. A force $\overline{F} = (4\hat{i} + 5\hat{j})$ N acts on a particle located at the origin O

(a) The work done in taking the particle parallel to the axes to a point B(2 m, 2.4 m) is 15 J
(b) The work done in taking the particle from O along the X-axis to a point A(3 m, 0) is 15 J
(c) The work done in taking the particle from O to (3, 0) and then to (2, 2.4) is 20 J
(d) The work done in taking the particle from O directly to (2, 2.4) is 20 J

192. Figure shows the force F (in newton) acting on a body as a function of x. The work done in moving the body.



(a) from x = 0 to x = 2 m is 8 J.
(b) from x = 2 m to x = 4 m is 16 J
(c) from x = 0 to x = 6 m is 30 J
(d) from x = 0 to x = 6 m is 32 N

193. A block of mass 5 kg, initially at rest on a horizontal floor, moves under the action of a horizontal force of 20 N. The coefficient of friction between the block and the floor is 0.2. If g = 10 m s^{-2}

(a) The work done by the applied force in 8 s is 1280 J.
(b) The work done by the frictional force in 8 s in 640 J
(c) The work done by the net force in 8 s in 720 J.
(d) The change in K.E of the block in 10 s is 1000 J.

194. A block of mass 4 kg is hanging over a smooth and light pulley through a light and inextensible string. The other end of the string is pulled by a constant force F = 60 N. The kinetic energy of the particle increases by 60 J in a given interval of time. Then (g = 10 m s^{-2}).

(a) The tension in the string is 60 N
(b) The displacement of the block in the given interval of time is 3 metre.
(c) Work done by gravity is 120 J
(d) Work done by the force is 180 J

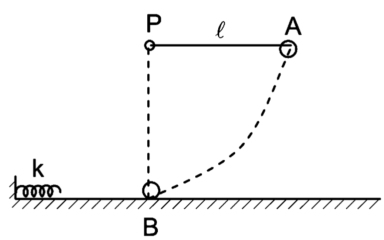
195. A body of mass 1 kg moving along a straight line with a velocity of 4 m s^{-1}, collides head on with a body of mass 2 kg moving along the same line with a velocity of 3 m s^{-1}. After collision the two bodies stick together and move with a common velocity of magnitude (m s^{-1})

(a) $\frac{3}{2}$ (b) $\frac{10}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

196. A body of mass 5 kg moving a velocity of 4 m s^{-1} along a straight line, collides with a body of 2 kg moving along the same line with a velocity of 8 m s^{-1}. If the collision is perfectly inelastic, the magnitude of the velocity of composite mass after the collision is

(a) $\frac{30}{7}$ m s^{-1} (b) $\frac{36}{7}$ m s^{-1} (c) $\frac{7}{4}$ m s^{-1} (d) $\frac{4}{7}$ m s^{-1}

197. A 0.8 kg ball is attached to a light, inextensible string of length ℓ and the other end of the string is fixed at P at a height ℓ above a smooth floor. B is a ball of 1 kg mass kept on the smooth floor vertically below P. A spring of spring constant 1000 N m^{-1} is kept as shown with one end fixed. A is held in a horizontal position as shown and released, and has a collision with B, with co-efficient of restitution, e = 0.8. A comes to a stop and B moves ahead and compresses spring and the maximum acceleration the spring produces on B is 80 m s^{-2}. Then

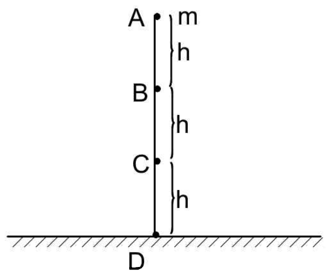


(a) length $\ell = \frac{1}{2}$ m (b) $\ell = 1$ m
(c) Maximum P.E attained by the spring is 3.2 J. (d) After B returns and hits A, B will come to a halt.

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198. A particle of mass m is projected vertically upwards from point D on ground with a kinetic energy K. PE_A, PE_B, PE_C and PE_D are the potential energies at positions A, B, C and D respectively. Given $PE_D = 0$ and maximum height is at A, $g = 10\ m\ s^{-2}$. Match the columns (KE_A, KE_B, KE_C = kinetic energies at A, B and C)



Column I	Column II
(a) KE_B	(p) Half of kinetic energy at C
(b) PE_C	(q) One third of total mechanical energy at B
(c) $\frac{PE_B}{KE_B}$	(r) Greater than 1
(d) $\frac{KE_C}{KE_B}$	(s) $\frac{KE_C}{PE_C}$

199. Two identical spherical balls, each of mass m, travelling along the same line with speeds u_1 and u_2 undergo one dimensional elastic collision. The resulting speeds are v_1 and v_2 respectively

Column I	Column II
(a) $\vec{u}_2 = 0$	(p) speeds exchanged after collision.
(b) $\vec{u}_2 = -\vec{u}_1$	(q) motion of balls after collision is as if no collision took place.
(c) $\vec{u}_1$ and $\vec{u}_2$ are in opposite directions and $u_1 > u_2$	(r) $\vec{v}_2 = \vec{u}_1$
(d) $\vec{u}_1$ and $\vec{u}_2$ are in the same direction and $u_1 > u_2$	(s) both balls turn back with exchanged speeds.

200. A body of mass m_1 collides one dimensionally with another stationary body of mass m_2. The initial and final velocities of mass m_1 are u_1 and v_1 and the final velocity of mass m_2 is v_2. The coefficient of restitution is e. After the collision

Column I	Column II
(a) Velocity of the second body is maximum when	(p) $e = 1$
(b) Momentum of the second body is maximum when	(q) $m_1 >> m_2$
(c) Kinetic energy of the second body is maximum when	(r) $e = 0$
(d) Velocity of the two bodies are equal when	(s) $m_1 << m_2$

ANSWERS KEYS

1. (i) 1275 J
   (ii) 3666 J
2. (i) A
   (ii) $W_A = W_B$
   (iii) B
   (iv) 3; $-\frac{1}{9} < \sin\phi < 1$
3. 6.94 KN; 0.154 MW
4. (i) 0
   (ii) Tat^2
5. $\frac{1}{3}m$
6. $v = \frac{2m^2g\left(h - s\sin\alpha\right)\cos^2\alpha}{\left(M+m\right)\left(M+m\sin^2\alpha\right)}$
7. 462 J; 217 J
8. $v = \sqrt{2\frac{m}{M}\frac{g\ell}{\left(1+\frac{M}{m}\right)}}$
9. $\frac{20}{3}\left(\sqrt{3}+1\right)$
10. $\sqrt{3}\,v$ at 150° to $x-$axis

11. (d) **12.** (d) **13.** (b)
14. (c) **15.** (c) **16.** (a)
17. (b) **18.** (d) **19.** (d)
20. (a) **21.** (a) **22.** (b)
23. (d) **24.** (b) **25.** (a)
26. (a)
27. (a), (c)
28. (a), (b), (d)
29. (a), (c), (d)
30. (a) → (p), (s)
(b) → (p), (r)
(c) → (q), (r)
(d) → (p), (s)

31. (c) **32.** (b) **33.** (c)
34. (c) **35.** (b) **36.** (d)
37. (a) **38.** (d) **39.** (c)
40. (d) **41.** (b) **42.** (a)
43. (d) **44.** (c) **45.** (d)
46. (a) **47.** (d) **48.** (c)
49. (d) **50.** (d) **51.** (c)
52. (d) **53.** (d) **54.** (c)
55. (b) **56.** (d) **57.** (c)
58. (d) **59.** (c) **60.** (d)
61. (b) **62.** (d) **63.** (b)
64. (d) **65.** (a) **66.** (c)
67. (a) **68.** (d) **69.** (a)
70. (c) **71.** (d) **72.** (a)
73. (c) **74.** (c) **75.** (c)
76. (d) **77.** (c) **78.** (c)
79. (d) **80.** (a) **81.** (b)
82. (c) **83.** (b) **84.** (d)
85. (d) **86.** (c) **87.** (d)
88. (a) **89.** (d) **90.** (d)
91. (b) **92.** (a) **93.** (a)
94. (c) **95.** (d) **96.** (d)
97. (c) **98.** (d) **99.** (a)
100. (c) **101.** (d) **102.** (c)
103. (d) **104.** (a) **105.** (d)
106. (d) **107.** (c) **108.** (c)
109. (b) **110.** (c) **111.** (b)
112. (d) **113.** (b) **114.** (c)
115. (b) **116.** (c)
117. (a), (c), (d)
118. (a), (b), (c)
119. (c), (d)
120. (a) → (p), (q), (r), (s)
(b) → (p), (q), (r), (s)
(c) → (p)
(d) → (p)

121. (i) $\frac{\sqrt{3}}{2}gr$
(ii) $\frac{1}{4}$

122. (i) 450 J
(ii) $\sqrt{180}$ m s^{-1}
(iii) $\sqrt{5}$ s

123. (i) 100 N m^{-1}
(ii) 1.75 m

124. (i) $\frac{3mv_0^2}{4x_0^2}$
(ii) $\sqrt{\frac{k}{m}}\frac{L}{2}$

125. $x = \frac{2\mu m_2 g}{k}$

126. (a) 2.6856×10^8J
(b) 10.9 m s^{-1}

127. (a) $\frac{19}{21}$ (b) $\frac{13}{7}$
(c) $\frac{33}{7}$ (d) $\frac{5}{7}H_{max}$

128. (i) 1.5 m s^{-1}
(ii) 56.25 J
(iii) 3.33 m s^{-2}

129. (a) 13.42 m s^{-1}
(b) 3.14 m

130 (i) $V_{heavy} = \frac{\left(k-2\right)}{\left(k+1\right)}\sqrt{2gr};$

$V_{ligh(t)} = \frac{\left(2k-1\right)\sqrt{2gr}}{\left(k+1\right)}$

(ii) k = 2
(iii) $V_{heavy} = 0$ af(t)e(r) collision

131. (a) **132.** (a) **133.** (a)
134. (a) **135.** (b) **136.** (d)
137. (d) **138.** (c) **139.** (b)
140. (b) **141.** (a) **142.** (b)
143. (b) **144.** (d) **145.** (b)
146. (b) **147.** (a) **148.** (d)
149. (a) **150.** (d) **151.** (d)
152. (d) **153.** (a) **154.** (b)
155. (c) **156.** (a) **157.** (a)
158. (a) **159.** (c) **160.** (d)
161. (b) **162.** (c) **163.** (b)
164. (d) **165.** (d) **166.** (c)
167. (a) **168.** (c) **169.** (c)
170. (b) **171.** (a) **172.** (b)
173. (d) **174.** (d) **175.** (b)
176. (a) **177.** (a) **178.** (a)
179. (b) **180** (d) **181.** (d)
182. (b) **183.** (c) **184.** (b)
185. (b) **186.** (a) **187.** (b)
188. (d) **189.** (d)
190. (a), (d)
191. (c), (d)

192. (a), (b), (d)
193. (a), (d)
194. (a), (b), (d)
195. (b), (c)
196. (b), (d)
197. (a),(c)
198. (a) $\rightarrow$ (p), (q)
(b) $\rightarrow$ (p), (q)
(c) $\rightarrow$ (r), (s)
(d) $\rightarrow$ (r), (s)
199. (a) $\rightarrow$ (p), (q), (r)
(b) $\rightarrow$ (p), (q), (r), (s)
(c) $\rightarrow$ (p), (q), (r), (s)
(d) $\rightarrow$ (p), (r)
200. (a) $\rightarrow$ (p), (q)
(b) $\rightarrow$ (p), (s)
(c) $\rightarrow$ (p)
(d) $\rightarrow$ (r)

HINTS AND EXPLANATIONS

Topic Grip

1. Let F be the applied force, f the frictional force, m the mass and μ the coefficient of friction

(i) $f = \mu\,(mg - F \sin 30)$

$F \cos 30 = f = \mu(mg - F \sin 30)$

$$\Rightarrow F = \frac{\mu mg}{\cos 30 + \mu \sin 30} = \frac{(0.3)20 \times 10}{\frac{\sqrt{3}}{2} + \frac{0.3}{2}}$$

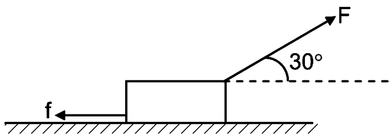
$= 60$ (Using $\sqrt{3} = 1.70$)

$\Rightarrow$ $F \cos 30 = 60 \times 0.85 = 51$ N.

Work done, $W = F (\cos 30)\, 25 = 51 \times 25$

$= 1275$ J

(ii)



$f = \mu(mg - F \sin 30)$

$$F\cos 30 - f = ma = m\frac{dv}{dt} \quad \text{—(1)}$$

$$F\cos 30 - \mu\left(mg - F\sin 30\right) = m\frac{dv}{dt}$$

$$F\left(\cos 30 + \mu\sin 30\right) = \mu mg + m\frac{dv}{dt}$$

But $\cos 30 + \mu \sin 30 = \frac{\sqrt{3}}{2} + 0.3 \times \frac{1}{2}$

$= \frac{1.7}{2} + 0.15 = 1 \quad v = 5 + \frac{S}{2}$

$$\Rightarrow \frac{dv}{dt} = \frac{1}{2}\frac{dS}{dt} = \frac{1}{2}v = \frac{1}{2}\left(5 + \frac{S}{2}\right)$$

$$\Rightarrow F = 0.3 \times 10 \times 20 + 20 \times \frac{1}{2} \times \left(5 + \frac{S}{2}\right)$$

$= 60 + 50 + 5S = 110 + 5S$

$$\text{Work done, } W = \int_0^{25} F\cos 30.dS$$

$$= \int_0^{25}\left(110 + 5S\right)\frac{\sqrt{3}}{2}dS$$

$$= 0.85\left[110S + \frac{5S^2}{2}\right]_0^{25}$$

$= 0.85[110 \times 25 + 2.5 \times 625] = 3666$ J

2. Part 1:

(i) Let θ be the angle of the incline, F the force applied by A, F' the force applied by B and m the mass of the block

$F = mg \sin \theta$ — (1)

$P_1 = mg \sin \theta.v$ — (2)

$F' = mg$ — (3)

$3P_1 = mg.2v$ — (4)

$$F = \frac{P_1}{v} \quad F' = \frac{3P_1}{2v}$$

$\Rightarrow$ $F < F'$

(ii) From (4) and (2) $\sin \theta = \frac{2}{3}$

Work done by A,

$$W_A = mg\sin\theta.\frac{h}{\sin\theta} = mgh = W_B$$

(Work done by B)

(iii) $t_A = \frac{h}{\sin\theta}\Big/ v \;; t_B = \frac{h}{2v}$

(iv) (a) $\therefore \frac{t_A}{t_B} = \frac{h}{v.\sin\theta}.\frac{2v}{h} = \frac{2}{\sin\theta} = 3$

3. (i) The propeller must apply a constant force F throughout. $F - F_f = ma$, where F_f is the frictional force.

$$v = 80 \times \frac{5}{18} = \frac{200}{9}\ \text{m s}^{-1}$$

But constant acceleration,

$$a = \frac{v^2}{2s} = \left(\frac{200}{9}\right)^2.\frac{1}{2 \times 50} = \frac{400}{81}$$

$$\Rightarrow F = 2000 + 1000\left(\frac{400}{81}\right) = 6.94\ \text{kN}$$

(ii) Minimum power $= F \times v$

$= 6.94 \times 10^3 \times \frac{200}{9}$

$= 0.154$ MW

4. (i) Distance covered in time $t = \frac{1}{2}a't^2 = s'$

$\Rightarrow$ Work done $= Ts' - Ts' = 0$

(ii) Acceleration for $m_1 = a + a'$

Acceleration for $m_2 = a - a'$

$$\text{Work} = T \times \left[\frac{1}{2}\left(a + a'\right)t^2 + \frac{1}{2}\left(a - a'\right)t^2\right] = Tat^2$$

5. Total $Energy_{initial}$ = PE at A = m.g.(0.5) = 5 m
= KE at B

Frictional force $f = \mu mg = 1.5$ m

Body comes to rest when KE = 0 $\Rightarrow W_{friction}$
$= -\Delta KE \Rightarrow W_{friction} = 5m$

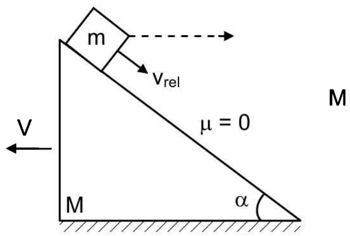
$\Rightarrow$ 1.5 m. d = 5 m where d is the total distance covered along BC, d > 3 $\Rightarrow$ not possible

$\Rightarrow W_{friction} = 1.5\text{ m} \times 3 = 4.5$ m

$\Rightarrow$ K.E at C = 5 m – 4.5 m = 0.5 m. Body goes up the wall and returns to C with KE = 0.5 m

$\Rightarrow$ Distance $= \dfrac{0.5\text{ m}}{1.5\text{ m}} = \dfrac{1}{3}$ metre from C.

6. (i)



Applying conservation of momentum in the x-direction. If v_{rel} = velocity of m with respect to wedge when its displacement is S relative to wedge, and V = velocity of wedge towards left at that instant,

Then, $MV = m(v_{rel}\cos\alpha - V)$

$$\Rightarrow |v_{rel}| = \frac{V(M+m)}{m\cos\alpha} \qquad \text{— (1)}$$

Energy equation:

Let height of m initially be h

Initial PE = mgh

Loss in PE = mgS sinα

$$K.E = \frac{1}{2}MV^2 + \frac{1}{2}m(v_{rel}\cos\alpha - V)^2 + \frac{1}{2}m(v_{rel}\sin\alpha)^2$$

Equating energies, simplifying, and substituting for v_{rel} from (1) we get,

KE:

$$\frac{1}{2}MV^2 + \frac{1}{2}m\left(\frac{V(M+m)}{m\cos\alpha}\cos\alpha - V\right)^2$$

$$+\frac{1}{2}m\left(\frac{V(M+m)}{m\cos\alpha}\sin\alpha\right)^2 = \Delta PE$$

$$\frac{1}{2}MV^2 + \frac{1}{2}m\left[\frac{VM+Vm-mV}{m}\right]^2$$

$$+\frac{1}{2}m\left[\frac{(VM+Vm)^2}{m^2}\tan^2\alpha\right] = \Delta PE$$

$$\frac{1}{2}\left[MV^2 + m\frac{V^2M^2}{m^2} + m\frac{V^2}{m^2}(M+m)^2\tan^2\alpha\right]$$

$= \Delta PE$

$$\frac{V^2}{2}\left[\frac{Mm + M^2 + (M+m)^2\tan^2\alpha}{m}\right] = \Delta PE$$

$$\frac{V^2}{2}\left[\frac{M(m+M) + (M+m)^2\tan^2\alpha}{m}\right] = mgS\sin\alpha$$

$$\therefore \quad V^2 = \frac{2m^2 gS\sin\alpha}{(M+m)\left[M + (M+m)\tan^2\alpha\right]}$$

Multiply and divide by $\cos^2\alpha$ on the R. H. S.,

$$V^2 = \frac{2m^2gS\sin\alpha\cos^2\alpha}{(M+m)\left[M\cos^2\alpha + (M+m)\sin^2\alpha\right]}$$

$$= \frac{2m^2gS\sin\alpha\cos^2\alpha}{(M+m)\left[M(\cos^2\alpha + \sin^2\alpha) + m\sin^2\alpha\right]}$$

$$= \frac{2m^2gS\sin\alpha\cos^2\alpha}{(M+m)\left[M + m\sin^2\alpha\right]}$$

$$V^2 = \frac{2m^2g(S\sin\alpha)\cos^2\alpha}{\left[(M+m)(M+m\sin^2\alpha)\right]}$$

7. (i) Energy of the spring

$$= \frac{1}{2}kx^2 = \frac{1}{2} \text{ x } 320 \text{ x } (1.7)^2 = 462\text{J}$$

(ii) $\Delta PE = mg(y_2 - y_1) = 9 \text{ x } 9.8 \text{ x } (15 - 7.3) = 679\text{J}$

$\Rightarrow$ Heat energy = 679 J – 462 J = 217 J

8. $PE_{initial} = mg\ell$

$$KE_{final} = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$$

Momentum conservation:

mu = Mv

$$\Rightarrow u = \frac{M}{m}v \;\therefore \frac{1}{2}m.\frac{M^2}{m^2}v^2 + \frac{1}{2}Mv^2 = mg\ell$$

$$\Rightarrow v^2\left(1 + \frac{M}{m}\right) = \frac{2mg\ell}{M} \Rightarrow v = \sqrt{2\frac{m}{M}\frac{g\ell}{\left(1+\frac{M}{m}\right)}}$$

9. $u = \sqrt{2gh} = \sqrt{200} = 10\sqrt{2}\ m\ s^{-1}$

Perpendicular component of v = e u cos α

Parallel component of

$v = u \sin\alpha$

$\therefore v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$

$= u\sqrt{\frac{1}{4} + \frac{1}{3}.\frac{3}{4}} = \frac{u}{\sqrt{2}} = 10\ m\ s^{-1}$

Acceleration normal to the plane = g cos α

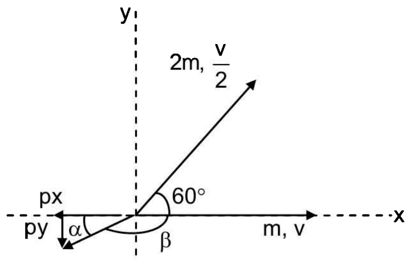
$\Rightarrow$ Time of flight $= \frac{2eu\cos\alpha}{g\cos\alpha} = \frac{2eu}{g} = 2\sqrt{\frac{2}{3}}\ s$

Acceleration parallel to the plane = g sin α

$\Rightarrow$ Range $= (u\sin\alpha)t + \frac{1}{2}(g\sin\alpha)t^2$

$= \frac{20}{3}(\sqrt{3}+1)\ m$

10.



Let fragment m have velocity $\overline{v}$ in the x-direction and fragment 2m have velocity $\frac{v}{2}$ at 60° to the x-axis.

Conservation of momentum in the x-direction gives

$mv + 2m.\frac{v}{2}\cos 60 + p_x = 0$ where $px = \frac{-3}{2}mv.$

Along the y-direction $2m.\frac{v}{2}.\frac{\sqrt{3}}{2} + p_y = 0$

$P_y = \frac{-\sqrt{3}}{2}mv$

$\therefore \quad |\overline{p}| = \sqrt{p_x^2 + p_y^2} = \sqrt{3}\ mv$

speed $= \sqrt{3}\ v$

$\tan\alpha = \frac{P_y}{P_x} = \frac{1}{\sqrt{3}}$

$\therefore \quad \alpha = 30° \therefore \beta = 150°$

11. Work done = 4 × 1 + (−4) × 1 + 2 × 1 = 2 J

12. $F' = F_1\cos 60 - F_2 = 6 \times \frac{1}{2} - 3 = 0$

$\Rightarrow$ Power = F. v = 0

13. KE = 100 J; p = 20 kg m s^{-1}

$KE = \frac{p^2}{2m}$

$\therefore \quad 100 = \frac{(20)^2}{2m} \Rightarrow m = 2\ kg$

14. $\frac{1}{2}mv^2 = mgh$

$h = R\sin\theta$

$\frac{v^2}{R} = 2g\sin\theta$

$F_R = m\left(\frac{v^2}{R} + g\sin\theta\right)$

$= 3mg.\sin\theta = \frac{3}{\sqrt{2}}mg$

15. $s_1 = \frac{1}{2}at^2;\ s_2 = v_0t - \frac{1}{2}at^2$

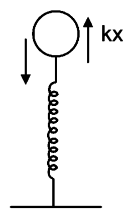
$s_1 + s_2 = \ell = v_0t \Rightarrow t = \frac{\ell}{v_0}$

$v_1 = v_2$ (as per data)

$at = v_0 - at$

$a = \frac{v_0}{2t} = \frac{v_0^2}{2\ell} = g\sin\theta \Rightarrow v_0 = \sqrt{2g\ell\sin\theta}$

16. (a)



Work done by gravity is positive as the ball moves in the direction of the force: $W_g = mgx$, (where x is the displacement from vertical position)

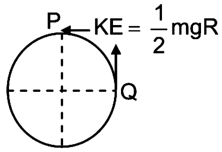
Work done by spring is negative as the ball moves opposite direction of spring force

$W_s = \int_0^x kx.dx = -\frac{1}{2}kx^2$

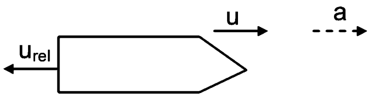
$K.E = W_g + W_s = mgx - \frac{1}{2}kx^2 = x\left(mg - \frac{kx}{2}\right)$

$\therefore$ KE first increases and then comes to zero

17. External force applied on the body is non-conservative. So mechanical energy is not conserved. Only for conservative systems, mechanical energy is conserved. ΔPE = ΔKE.

18. 

Minimum K.E at top point P is $\frac{1}{2}$mgR. PE at P is 2 mgR. Total mechanical energy is $\frac{5}{2}$ mgR. Assume string breaks at Q. The whole of KE at Q can be converted to P.E. So it will rise $\frac{R}{2}$ over P, because K.E at Q is $\frac{3}{2}$mgR

19. 

$$a = \left(-\frac{dm}{dt}\right)u_{rel}$$

But here given $u_{rel} = u$

($\therefore$ exhaust's absolute velocity is zero!)

$\therefore$ Rocket is accelerated.

Statement 2 is of course true

20. The elastic collision will have more impulse than inelastic collision for the same initial conditions of two bodies moving in the same direction. A typical example is a body coming to a halt in elastic collision will have some velocity and hence KE if the collision is inelastic.

21. Initial energy = $mg\ell$

After one to and fro journey, energy at B

$= mg\ell - \mu mg\ell \times 2$

$\therefore$ height of rise $= \ell(1 - 2\mu)$

Progression: $\ell, \ell(1-2\mu), \ell(1-4\mu)\ldots \Rightarrow$ AP

22. $mg\ell = (2n+1)\,\mu mg\ell$ where n = 0, 1, ……

$\Rightarrow \quad \mu = 1, \frac{1}{3}, \frac{1}{5}, \ldots\ldots$

23. When the particle first comes down to B its velocity is $\sqrt{2g\ell}$. When it reaches C, its velocity is $\sqrt{2g\ell(1-\mu)}$. After collision at C, the particle returns with velocity $e\sqrt{2g\ell(1-\mu)}$

24. Loss of energy can be only in collision between A and B

Collision A and B:

Let $p = m_A u$

Given $\frac{p^2}{2m} = (75\%)\frac{p^2}{2m_A} \Rightarrow m_A = \frac{3}{4}m$

25. Given at some instant B and C have equal velocity of $\frac{u}{2} \Rightarrow$

$$p = m_A u = (m + m_C)\frac{u}{2}$$

$$\Rightarrow \quad m_C = 2m_A - m = \frac{3}{2}m - m = \frac{m}{2}$$

26. Clearly CM of B and C moves at $\frac{u}{2} \Rightarrow$ When velocity of C is $\frac{u}{2}$, the spring is at maximum compression/elongation

Energy equation:

$$\frac{1}{2}\left(\frac{3}{4}m\right)u^2 = \frac{1}{2}\left(\frac{3}{2}m\right)\left(\frac{u}{2}\right)^2 + \frac{1}{2}kx^2$$

$$\Rightarrow \quad x^2 = \frac{3}{8}\frac{mu^2}{k}$$

$$\Rightarrow \quad x = \sqrt{\frac{3m}{2K}}.\frac{u}{2}$$

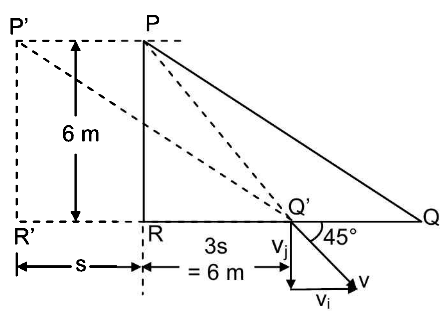
27. $\Rightarrow \quad x = kt^{\frac{3}{2}} + C$

$$v = \frac{dx}{dt} = k.\frac{3}{2}t^{\frac{1}{2}} \Rightarrow \therefore \text{KE} \propto v^2 \propto t$$

$$a = \frac{dv}{dt} = k\frac{3}{2}.\frac{1}{2}t^{-\frac{1}{2}} = \frac{3}{4}k\frac{1}{t^{1/2}}$$

$$\Rightarrow F = ma = \frac{3}{4}\frac{km}{t^{1/2}}$$

$$\text{Power} = F \times v = \frac{3}{4}\frac{km}{t^{1/2}}.k\frac{3}{2}t^{1/2} = \frac{9}{8}k^2 m$$

28. 

When the mass m, reaches bottom position, since the center of mass does not move horizontally, if the wedge moves s backward from original position, m will move 3s, forward.

$s + 3s = R'Q' = 8 \text{ m} \Rightarrow s = 2 \text{ m}\ ;\ 3s = 6 \text{ m}$

$\therefore$ (a) is correct.

Path of m is P to Q', RQ' = 3s = 6 m.

Hence m moves at 45° with horizontal/vertical. Its final velocity v

$\therefore\quad v_i = v_j \Rightarrow v = \sqrt{v_i^2 + v_j^2} = \sqrt{2}v_i$

$\Rightarrow$ For conservation of horizontal momentum: $3\text{ mv}' = \text{mv}_i \Rightarrow v' = \dfrac{v_i}{3}$

Energy equation :

$$\frac{1}{2}(3m)\left(\frac{v_i}{3}\right)^2 + \frac{1}{2}m\left(\sqrt{2}\,v_i\right)^2$$

$$= mgh = m \times 10 \times 6$$

$$\Rightarrow v_i^2 = \frac{360}{7}$$

$$v^2 = 2v_i^2 = \frac{720}{7} \Rightarrow v = \sqrt{\frac{720}{7}}\ \text{ms}^{-1}$$

$\Rightarrow$ (b) is correct

On the reference frame of the wedge the velocity of the mass $v_r > v \Rightarrow$ (c) is wrong.

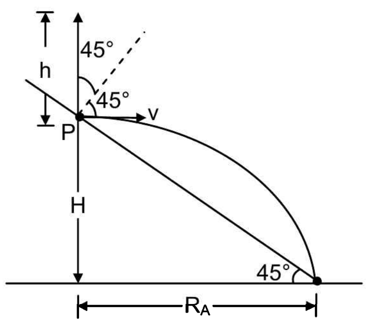
Since the mass hits ground elastically at 45° to horizontal, its range $= \dfrac{v^2}{g}$

$$= \frac{720}{7 \times 10} = \frac{72}{7}\text{m}$$

$\Rightarrow$ (d) is correct

$\therefore$ (a), (b) and (d) are correct

29.



For elastic collision on the inclined plane rebound velocity v will be horizontal for A and B therefore the time to fall to ground will be same for both; $t_A = t_B = 2$ s. Since they reach P with same KE

$$\frac{1}{2}m_A v_A^2 = \frac{1}{2}m_B v_B^2 \Rightarrow \frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}}$$

$$\therefore\quad \frac{R_A}{R_B} = \frac{v_A t_A}{v_B t_B} = \sqrt{\frac{m_B}{m_A}} \Rightarrow \text{(a) is correct}$$

$$H = R_A \Rightarrow \frac{1}{2}gt^2 = v_A t$$

$$\Rightarrow\quad v_A = \frac{gt}{2} = \frac{10 \times 2}{2} = 10\ \text{m s}^{-1}$$

$\Rightarrow$ (d) is correct

For A, h' above P is given by : $v^2 = 2gh'$

$\Rightarrow\quad 100 = 2 \times 10 \times h'$

$h' = 5$ m

$\Rightarrow\quad \therefore h_A = H + h' = 25$ m

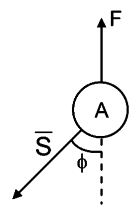
$\Rightarrow$ (c) is correct

$\therefore$ (a), (c) and (d) are correct

30. (a) Work done is obviously negative since final configuration has less potential energy. Energy is not conserved since a part of the energy is used in overcoming friction between muscles

$\therefore$ (a) $\rightarrow$ p, s

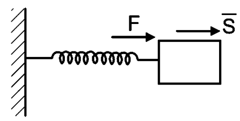
(b)



Work done is negative because the component of the displacement along the line of force is opposite to the force. Mechanical energy is conserved

$\therefore$ (b) $\rightarrow$ p, r

(c)



Force $\overline{F}$ and displacement $\overline{S}$ are in the same direction. Work done by spring is positive and mechanical energy is conserved

$\therefore$ (c) $\rightarrow$ q, r

(d) Reaction force of the palm is opposite in direction to the recoil of the palm. Hence work done by the boy is negative and mechanical energy is not conserved since a part of the energy is spent to counteract frictional losses between muscles

$\therefore$ (d) $\rightarrow$ p, s

IIT Assignment Exercise

31. kg is a unit of mass

32. $W = \bar{F}.\bar{S} = \bar{F}\,\bar{S}\cos\theta = 10 \times 2 \times \cos 30$

$= 10\sqrt{3}J$

33. Since there is no displacement, no work is done.

34. $1kWh = 3.6 \times 10^3$ J is incorrect

$1W = 1J/s \Rightarrow 1kWh = \dfrac{1000 \times 1J}{1s}$

$= \dfrac{1000 \times 1J}{\left(\dfrac{1}{60 \times 60}\right)} = 3.6 \times 10^6 J / h$

$\Rightarrow 1$ kWh $= 3.6 \times 10^6$ J

35. $W = \bar{F}.\bar{S} = 14 \times 2 + 5 \times 4 + 3 \times 4 = 60$ J

36. 1 HP = 746 W

Power = 746 W, S = 10m, t = 1s

$\text{Power} = \dfrac{mg \times S}{t} \Rightarrow m = \dfrac{P \times t}{g \times S} = \dfrac{746 \times 1}{9.8 \times 10} = 7.6$ kg

$\Rightarrow$ Rate = 7.6 kg s^{-1}

37. The energy per second delivered to the turbine blades = mgh =100 × 10 × 100

= 10^2 kW

38. $KE = \dfrac{1}{2}mv^2 \qquad KE' = \dfrac{1}{2}mv'^2$

$v' = 2v \Rightarrow KE' = \dfrac{1}{2}m(2v)^2 = 4$ KE.

39. $p = mv = 4\ KE = 4\dfrac{1}{2}mv^2 = 2mv^2$

$\Rightarrow mv = 2mv^2 \Rightarrow v = \dfrac{1}{2}$

40. $KE = \dfrac{1}{2}mv^2, p = mv \Rightarrow KE = \dfrac{1}{2}\dfrac{(mv)^2}{m} = \dfrac{p^2}{2m}$

41. $\dfrac{KE_1}{KE_2} = \dfrac{m_1 v_1^2}{m_2 v_2^2} = 1 \quad \Rightarrow \dfrac{v_1}{v_2} = \sqrt{\dfrac{m_2}{m_1}} = \dfrac{3}{2}$

$\dfrac{p_1}{p_2} = \dfrac{m_1 v_1}{m_2 v_2} = \dfrac{4}{9} \times \dfrac{3}{2} = \dfrac{2}{3}$

42. $KE = \dfrac{p^2}{2\,m} = \dfrac{4^2}{2 \times 2} = 4$ J

43. $p = mv, KE = \dfrac{1}{2}mv^2 = \dfrac{p^2}{2m}$

$p' = 4p \Rightarrow KE'$

$= \dfrac{(p')^2}{2m} = \dfrac{(4p)^2}{2m} = \dfrac{p^2}{2m} \times 16 = 16$ KE

44. $KE = \dfrac{1}{2}m\left|\bar{v}\right|^2 = \dfrac{1}{2}m\bar{v}.\bar{v} = \dfrac{1}{2} \times 0.3 \times 5^2$

= 3.75 J

45. $mv \xrightarrow{\text{increased to}} m.\dfrac{3}{2}v$

$\therefore E \xrightarrow{\text{increased to}} \dfrac{9}{4}E$

$\therefore$ % increase in $E = \dfrac{5}{4}E \times \dfrac{100}{E} \Rightarrow 125\%$

46. Let m < M — (1)

$m\bar{v} = M\bar{V}$ (given)

$e = \dfrac{p^2}{2m}; \quad E = \dfrac{P^2}{2M}$ — (2)

$\Rightarrow$ $p^2 = 2me; P^2 = 2ME$

But $p^2 = P^2$ (given) — (3)

$\Rightarrow$ 2me = 2ME

Since m < M; e > E

47. $W = F.S = \Delta K.E$

$\Rightarrow 4S = 4E - E = 3E \Rightarrow S = \dfrac{3E}{4}$

48. $F_{Net} = \sqrt{F_x^2 + F_y^2 + 2F_xF_y\cos 90}$

$= \sqrt{4^2 + 3^2} = 5$

$a = \dfrac{F_{Net}}{m} = \dfrac{5}{10} = 0.5$

$v = u + at = 0 + 0.5 \times 10 = 5$

$KE = \dfrac{1}{2}mv^2 = \dfrac{1}{2} \times 10 \times 5^2 = 125$ J

49. $E = \dfrac{1}{2}mv^2 = \dfrac{1}{2} \times 8 \times 10^{-3} \times 500 \times 500\ J = 1000$ J

Rate of energy loss $= \dfrac{1000}{20 \times 10^{-2}} = 5 \times 10^3\ J\ m^{-1}$.

50. There is no radial displacement. Since centripetal force is radial, no work is done.

51. Considering frame attached to the massive body.

Frictional work = F × S = ΔKE ⇒ $\mu\, mgS = \frac{1}{2}mv^2$

$$S = \frac{v^2}{2\mu g} = \frac{4^2}{2 \times 0.2 \times 10} = 4\text{ m}$$

52. $\frac{1}{2}mv^2 = -F.d \Rightarrow \frac{d_1}{d_2} = \left(\frac{v_1}{v_2}\right)^2 \Rightarrow 2v_1 = v_2$

⇒ $d_2 = 4\, d_1$

53. ΔKE = ΔPE, = mgh

⇒ $v^2 = u^2 + 2gh = 5^2 + 2 \times 10 \times 2 = 25 + 40 = 65$

$v = \sqrt{65}\text{ ms}^{-1}$

54. $PE = \frac{1}{2}kx'^2 = \frac{8}{2}x'^2$

Now $x' = \frac{x}{2} \Rightarrow PE = 4\left(\frac{x}{2}\right)^2 = x^2$

55. P.E = mgh, where h is the height from the ground.

⇒ PE = mg $(\ell_0 - x)$,

where x is the distance travelled and ℓ_0 is the original height.

56. ΔP.E = ΔK.E , $\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh}$

57. $E = \frac{1}{2}k\,x^2 = \frac{1}{2}k\left(\frac{F}{k}\right)^2 = \frac{1}{2}\frac{F^2}{k}$

$$\Rightarrow \frac{E_1}{E_2} = \frac{k_2}{k_1} = 2 \Rightarrow E_2 = \frac{E_1}{2} = 5\text{ J}$$

58. Velocity acquired by the ball will be the same at B, C and D as the difference in heights is the same.

59. $\frac{1}{2}mv^2 = mgh$

$\therefore v = (2gh)^{1/2} = (2 \times 10 \times 10 \times 10^{-2})^{1/2} = \sqrt{2}\text{ m s}^{-1}$

60. At the highest point the ball has only horizontal velocity, which is v cos 45 $= \frac{v}{\sqrt{2}}$. Hence energy is

$$\frac{1}{2}m\left(\frac{v}{\sqrt{2}}\right)^2 = \frac{E}{2}$$

61. While walking, the weight of the bucket acts downwards perpendicular to the direction of motion. Therefore, no work is done; in climbing up a distance of 5 m, the work done is 50 × 5 = 250 J

62. $mgy = \frac{1}{2}(2k)\left(\frac{y}{\sin\theta}\right)^2 \Rightarrow y = \frac{mg}{k}\sin^2\theta$

63. $\frac{1}{2}mv^2 = \frac{1}{2}.\frac{1}{2}kx^2$

⇒ Spring constant for half length

$= 2k\; x' = \frac{x}{2}$

$\therefore\; E = \frac{1}{2}(2k)\left(\frac{x}{2}\right)^2$

$$v^2 = \frac{kx^2}{2m} = \frac{80 \times (0.1)^2}{2 \times 0.05} \Rightarrow v = 2\sqrt{2}\text{ms}^{-1}$$

64. $U = x^4 - bx^2 \Rightarrow \frac{dU}{dx} = 4x^3 - 2bx$

= 0 for equilibrium

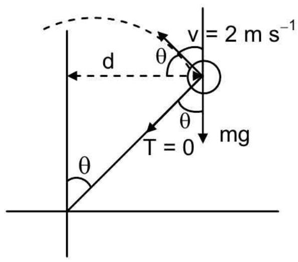
$4x^2 - 2b = 0 \Rightarrow x = \pm\sqrt{\frac{b}{2}}$

Now for stable equilibrium at origin:

$\frac{d^2U}{dx^2} > 0;\ \frac{d}{dx}(4x^3 - 2bx) > 0 \Rightarrow 12x^2 - 2b > 0$

At x = 0; −2b > 0; b → negative

65.



When it just loses circular path; t = 0

$\therefore\; mg\cos\theta = \frac{mv^2}{\ell} \Rightarrow v^2 = g\ell\cos\theta$

$4 = 10 \times 1 \times \cos\theta \Rightarrow \cos\theta = 0.4$

After this it moves in projectile motion

Time to cross vertical position

$$= \frac{d}{v\cos\theta} = \frac{\ell\sin\theta}{v\cos\theta} = \frac{1 \times \sqrt{1 - (0.4)^2}}{2 \times 0.4}$$

$$\simeq \frac{0.92}{0.8} \simeq \frac{9}{8}\text{ s}$$

66. Loss in PE = mg($\ell + \ell\cos\theta$)

= 0.5 × 10 × (1 + 1 × 0.4)

= 5 × 1.4 = 7 J

67. Force exerted = $Rv_{rel} = 10 \times 10^3\text{ N} = 10^4\text{ N}$

68. Power $P = \frac{1}{2}\Delta mv^2 = \frac{1}{2} \times 10 \times (10^3)^2$

$= 5 \times 10^6$ W

69. Just before exploding, the momentum of the body was $mv\cos\theta$. Just after explosion, momentum of the body retracing the path

$$= -\frac{1}{2} mv\cos\theta$$

Momentum of the other half of the body $= \frac{1}{2} mv'$

Conservation of momentum requires

$M v\cos\theta = -\frac{1}{2} mv\cos\theta + \frac{1}{2} mv' \Rightarrow v' = 3v\cos\theta$

70. In elastic collisions, both momentum and KE are conserved. (Definition of elastic collision)

71. When a body suffers an elastic collision with another body of the same mass at rest, it comes to rest after collision while the second body starts moving with the same velocity as that of the first.

72. Kinetic energy at the moment of impact = potential energy = mg × 100J.

Energy available for the ball to rebound

$= mg \times 100 \times \frac{80}{100} = mg \times 80$ J.

The height to which the ball rebounds = 80m

73. The total momentum of the ball and the earth is conserved.

74. By definition the choice is (c)

75. $\bar{p} = \bar{p}_1 + \bar{p}_2$ — (1)

$$\frac{p^2}{2m_1} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$$

$m_1 = m_2 \Rightarrow p^2 = p_1^2 + p_2^2$ — (2)

(1) and (2) $\Rightarrow \bar{p}_1 \perp \bar{p}_2$

76. In (d), they will move in mutually perpendicular directions.

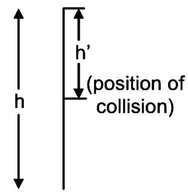
77. Let t be the time for collision. Change in momentum up to t = impulse

$\Delta p = 2mgt$, Let h be the total height.

$v_{rel} = \sqrt{2gh}$ = (data)

$$t = \frac{h}{v_{rel}} = \sqrt{\frac{h}{2g}} \Rightarrow h' = \frac{1}{2}gt^2 = \frac{h}{4}$$

$p = m\sqrt{2gh} - 2mgt = 0$



Hence they come to rest at the point of collision, at $h' = \frac{h}{4}$

Balance height to fall $= h - \frac{h}{4} = \frac{3h}{4}$

$$\Rightarrow \text{Time to fall} = \sqrt{\frac{2.\frac{3h}{4}}{g}} = \sqrt{\frac{3h}{2g}}$$

78. Using conservation of momentum: 8 × 6 = 4 × v. Velocity of 4 kg part is 12 m s^{-1};

K.E is $\frac{1}{2}nv^2 = 288j$

79. $p_1 = 15 \times 3m = p_2; -\bar{p}_3 = \bar{p}_1 + \bar{p}_2 \Rightarrow mv$

$p_3 = 1 \times v = \sqrt{(15\times 3)^2 \times 2} = 45\sqrt{2}$ m s^{-1}

80. $Mv = m \times 0 + (M-m)v' \Rightarrow v' = \frac{Mv}{M-m}$

81. Δ KE = work done by force

$$= \int_0^s F.dx = -\int_0^s kx^3 dx = -\frac{ks^4}{4} \text{ (–ve work)}$$

$$\Rightarrow \frac{ks^4}{4} = \frac{mv_0^2}{2} \Rightarrow s = \left(\frac{2mv_0^2}{k}\right)^{1/4}$$

82. $W = |\vec{F}|\,|\vec{L}| \cos\theta = 5 \times 10 \times \cos\theta = 25$J (given)

$\therefore \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$

83. $F = mg\sin\theta + F$

$= 6000 \text{ x } 10 \text{ x } \frac{1}{20} + 0.01 \times 6000 \times 10$

$= 3000 + 600 = 3600$N

Velocity of the car = 5 m s^{-1}

Power = 3600 × 5 = 18 kW

84. $v = 900 \times \frac{5}{18} = 250$ m s^{-1}, H = 5000 m

$$\Delta E = \frac{1}{2}mv^2 + mgh = \left(\frac{250^2}{2} + 10 \times 5000\right) \times m$$

$$= (81250) \times 25 \times 10^3$$

$$P_{av} = \frac{\Delta E}{t} = \frac{8.125}{4.8} \times 10^2 \times 25 \times 10^3 \simeq 4.2 \text{ MW}$$

85. The centripetal force is $F = mg\cos\theta$, and speed v is given by $\frac{1}{2}mv^2 = mg R\sin\theta$

Power = F.v

$$= mg\cos\theta.\sqrt{2gR\sin\theta}$$

$$= m\sqrt{2g^3R}.\sqrt{\sin\theta\left(1-\sin^2\theta\right)}$$

$\Rightarrow$ Maximum when $\frac{d}{d\theta}\left[\sin\theta - \sin^3\theta\right]^{1/2} = 0 \Rightarrow$

$\cos\theta - 3\sin^2\theta\cos\theta = 0$

$\sin\theta = \frac{1}{\sqrt{3}}$

Maximum at $\sin\theta = \frac{1}{\sqrt{3}}$

86. $x = a\sin\omega t$

$$v = \frac{dx}{dt} = a\omega\cos\omega t$$

$$\frac{1}{2}mv^2 = \frac{1}{2}ma^2\omega^2\cos^2\omega t$$

$$= \frac{1}{2}ma^2\omega^2\left[\frac{1+\cos 2\omega t}{2}\right]$$

$$= \frac{1}{4}ma^2\omega^2\left[1+\cos 2\omega t\right]$$

87. $\dot{m}u = \frac{\mu mv^2}{r} \Rightarrow v^2 = \frac{\dot{m}ru}{\mu m} \Rightarrow v = \sqrt{\frac{\dot{m}ur}{\mu m}}$

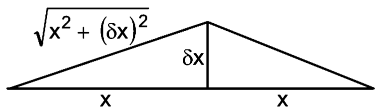
88. Work done by forces = ΔKE

$$-mgs\sin\alpha - \mu mgs\cos\alpha = 0 - \frac{1}{2}mv_0^2$$

$$\therefore s = \frac{v_0^2}{2g\left(\sin\alpha + \mu\cos\alpha\right)}$$

$$= \frac{1^2}{2 \times 10\left(\frac{1}{2} + \frac{1}{\sqrt{3}}.\frac{\sqrt{3}}{2}\right)} = 5\text{cm}$$

89.



$$\text{Energy} = \left[\frac{1}{2}K\left(\sqrt{x^2+\delta x^2} - x\right)\right] \times 2$$

where k is spring constant of $\frac{1}{2}$ section

$$= kx^2\left(\sqrt{1+\left(\frac{\delta x}{x}\right)^2} - 1\right)^2$$

$$= kx^2\left[1 + \frac{1}{2}\left(\frac{\Delta x}{x}\right)^2 - 1\right]^2 \left[\because \frac{\Delta x}{x} << 1\right]$$

$$= kx^2 \times \frac{1}{4}\left(\frac{\delta x}{x}\right)^4$$

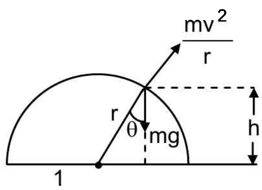
i.e., $E \propto (\delta x)^4 \Rightarrow n = 4$

90. $mg(H - h) = \frac{1}{2}mv^2$

$v^2 = 2g(H - h) \Rightarrow$ using the formula given to R:

$$R = \frac{\sqrt{2g\left(H-h\right)}.\sqrt{2gH}}{g} \text{ max for } h = 0$$

91.



$$mg(r - h) = \frac{1}{2}mv^2 \qquad \text{— (1)}$$

when it is just about to lose contact normal reaction = 0

$$\therefore \frac{mv^2}{r} = mg\cos\theta \text{ (centripetal force)}$$

$$\cos\theta = \frac{h}{r}$$

$$\therefore \frac{mv^2}{r} = mg\frac{h}{r} \qquad \text{— (2)}$$

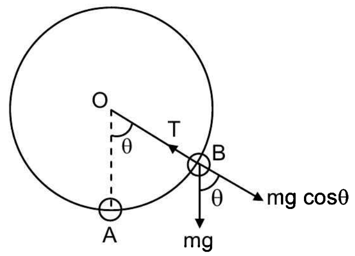
From equations (1) and (2)

$$r - h = \frac{h}{2} \Rightarrow h = \frac{2}{3}r.$$

92. $\frac{1}{2}mv^2 = \frac{1}{2}kx^2, x = \sqrt{\frac{m}{k}}.v$

$F = kx = \sqrt{mk}v$

93.



At B we have, $T - mg\cos\theta = \frac{mv^2}{r}$

$$v^2 = \frac{r(T - mg\cos\theta)}{m}$$

$$\Rightarrow \quad v^2 = \sqrt{\frac{r(T - mg\cos\theta)}{m}}$$

Given m = 1 kg, r = 0.5 m, θ = 37°, T = 10 N

$$v^2 = \sqrt{\frac{0.5(10 - 1 \times 10\cos 37^\circ)}{1}}$$

$$= \sqrt{0.5\left(10 - 10 \times \frac{4}{5}\right)} = \sqrt{0.5 \times 2} = 1 \text{ m s}^{-1}$$

94. (c)

At position CD if m is moving to right M should move to left[conservation of horizontal momentum since no horizontal force acts on the system]

$$mv = MV \Rightarrow 1 \times v = 2V \Rightarrow V = \frac{v}{2}$$

For conservation of energy

$$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = mgh$$

$$\Rightarrow \quad \frac{1}{2} \times 2 \times \frac{v^2}{4} + \frac{1}{2} \times 1 \times v^2 = 1 \times 10 \times 1.2$$

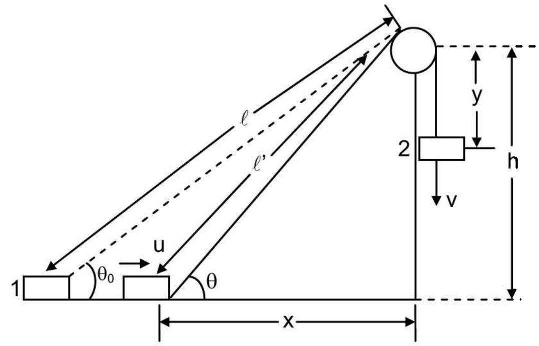
$$\frac{v^2}{2}.\frac{3}{2} = 12 \Rightarrow v^2 = 16$$

$$\frac{1}{2}mv^2 = \frac{16}{2} = 8\text{ J}$$

95. The height through which the 2nd mass drops

$= \ell - h = h \operatorname{cosec}\theta_0 - h$

$= h(\operatorname{cosec}\theta_0 - 1)$



At any intermediate position:

$(\ell')^2 = (\ell - y)^2 = x^2 + h^2$

Differentiate

$$2(\ell - y)\left(-\frac{dy}{dt}\right) = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = v; \ \frac{dx}{dt} = -u[\because x \text{ shrinks}]$$

$$-v = -u\frac{x}{(\ell - y)} = -u\cos\theta$$

when θ = 90° ⇒ cos θ = 0;

∴ v = 0; y = (ℓ – h)

K E of mass 1 = change in P.E of mass

$$2\text{ mg}(\ell - h) = \frac{1}{2}mu^2$$

(the velocity of the other mass is zero)

$$u = \sqrt{2gh(\operatorname{cosec}\theta_0 - 1)}$$

96. String will slacken when tension is zero.

$$\frac{mv^2}{R} = mg \Rightarrow v = \sqrt{gR}$$

97. The final vertical velocity of $m_1 = 0$

$$\Rightarrow \frac{1}{2}m_2v^2 = m_1gR(1 - \cos\theta)$$

$$\Rightarrow v = \sqrt{2gR(1 - \cos\theta)\frac{m_1}{m_2}}$$

98. First to ascertain equilibrium positions:

$$\frac{dU}{dx} = \frac{d}{dx}\left(x^3 - \frac{x^2}{2}\right) = 0$$

$$\Rightarrow \quad 3x^2 - \frac{2x}{2} = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{3}$$

To ascertain stable or unstable: $\dfrac{d^2U}{dx^2} = 6x - 1$

at $x = 0$; $\dfrac{d^2U}{dx^2} = -1$, hence unstable equilibrium

at $x = \dfrac{1}{3} \Rightarrow \dfrac{d^2U}{dx^2} = 2 - 1 = +1$ stable equilibrium

Hence it will be in stable equilibrium at $x = +\dfrac{1}{3}m$

you can check on the force acting on the particle and confirm this

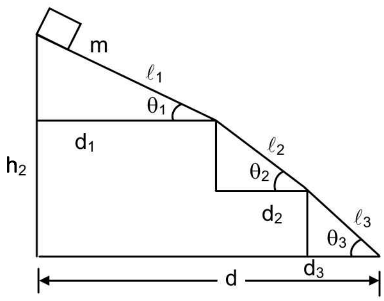
$$F = -\frac{dU}{dx} = -(3x^2 - x)$$

$\Rightarrow$ F is positive if $-(3x^2 - x) > 0$

$x - 3x^2 > 0$; $x < \dfrac{1}{3}m$, Hence when displaced slightly by $+\Delta x$ the force will be in $-x$ direction till it reaches $x = \dfrac{1}{3}$ m

99. $W \propto k$. Since $k_P > k_Q$, $W_P > W_Q$

100.



Loss in K.E = work done by frictional forces

In case 1

$$W_1 = f_{r1} \times \ell_1 + f_{r2} \times \ell_2 + f_{r3} \times \ell_3$$

$$= \mu mg \cos\theta_1\, \ell_1 + \mu mg \cos\theta_2\, \ell_2 + \mu mg \cos\theta_3 \ell_3$$

$$\mu mg \frac{d_1}{\ell_1}.\ell_1 + \mu mg \frac{d_2}{\ell_2}.\ell_2 + \mu mg \frac{d_3}{\ell_3}.\ell_3$$

$$= \mu mg\left(d_1 + d_2 + d_3\right) = \mu mgd$$

Similarly for second case $W_2 = \mu mgd$.

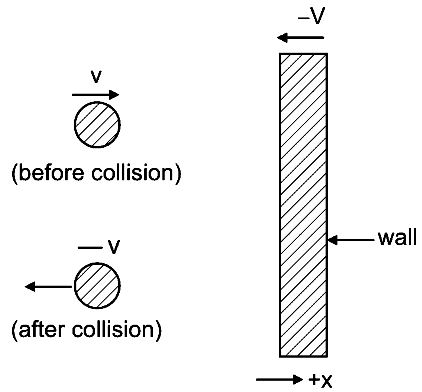
101. When the two systems collide, the moving mass in one system gets velocity zero, while one mass in the other system gets velocity v_0. At this point, both systems are identical, and they will have the same maximum compression.

102. This can only happen if the path forms 3 sides of square.

$$\Rightarrow \quad AB = 2\sin\frac{\pi}{4}R = \sqrt{2}R$$

103. $k > 1$ is sufficient as it will still retain some forward velocity after each collision with mass m. If $k < 1$, then only one of the masses m will fall off.

104.



velocity of approach (with respect to wall)

$= (v + V)$

since the speed remain same v,

velocity of separation(with respect to wall)

$= (-v + V)$

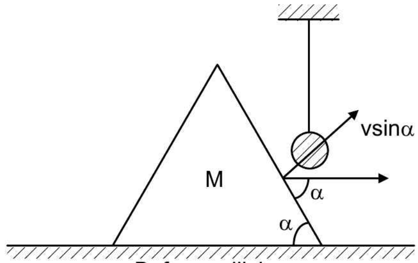
$\therefore \quad -v + V = -e(v + V)$

$V(1 + e) = v(1 - e)$

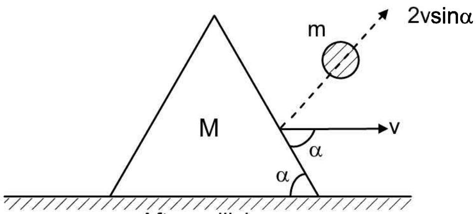
$$V = \left(\frac{1-e}{1+e}\right)v$$

105. $(m_1 + m_3)$ acquire an instantaneous velocity and it would acquire very large force to move m_2 the same way

106.



Before collision



After collision

Between the bob and the surface of the wedge, the velocity of approach along the line of contact is v sinα. Velocity of approach = velocity of separation ($\because$ elastic collision). Since M >> m, after collision also v' ≃ v. Therefore velocity of the bob (to achieve velocity of separation along the line of collision equal to v sinα) must be 2vsinα, along the line of collision, so that

$$e = \frac{-(0 - v\sin\alpha)}{(2v\sin\alpha - v\sin\alpha)} = 1$$

107. Time till collision $= \dfrac{d}{v_0}$

Velocity of separation = velocity of approach

$$\therefore \text{Time} = \frac{2d}{v_0} \quad \therefore \text{Total time} = \frac{d}{v_0} + \frac{2d}{v_0} = \frac{3d}{v_0}$$

108. $KE_1 = \dfrac{1}{2}.(3m)v_0^2;$

After collision

Hence for conservation of momentum

$$3m\, v_{final} = -2mv_0 + mv_0$$

$$\Rightarrow v_{final} = \frac{v_0}{3}$$

$$KE_2 = \frac{1}{2}.(3m)\left(\frac{v_0}{3}\right)^2$$

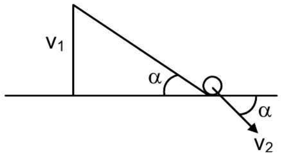
Maximum compression occurs when the bodies are stationary with respect to the centre of mass.

$$\Delta KE = \frac{1}{2}.(3m)v_0^2\left(1 - \frac{1}{9}\right) = \frac{1}{2}kx^2$$

$$\Rightarrow x^2 = \frac{8mv_0^2}{3k}$$

$$\Rightarrow \quad x = 2v_0\sqrt{\frac{2m}{3k}}$$

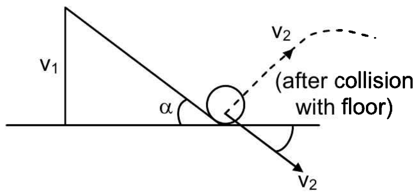
109.



Conservation of energy gives

$$\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = mgh$$

$$\Rightarrow \quad v_1^2 + v_2^2 = 2gh \qquad \text{— (1)}$$



Conservation of momentum in the x – direction gives

$v_2 \cos\alpha = v_1$

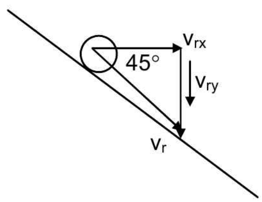
$\Rightarrow \quad v_{2x} = v_1 \Rightarrow$ On the reference frame of the wedge

$$v_{rx} = v_{2x} + v_1 = 2v_1$$

$$v_{ry} = v_{2y} \quad \therefore \tan 45 = \frac{v_{rx}}{v_{ry}} = 1$$

$$\Rightarrow \quad \therefore v_{rx} = v_{ry} = 2v_1$$

$$\therefore \quad v_2^2 = v_{2x}^2 + v_{2y}^2 = v_1^2 + (2v_1)^2 = 5v_1^2$$



From equation — (1)

$$v_1^2 + 5v_1^2 = 2gh$$

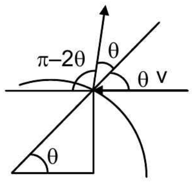
$$\therefore \quad \frac{v_{2y}}{2v_1} = 1$$

$$v_1 = \sqrt{\frac{gh}{3}} \Rightarrow v_{2y} = 2v_1 \Rightarrow v_{2y}^2 = 2gh'$$

$$h' = \frac{v_{2y}^2}{2g} = \frac{4 \times \frac{gh}{3}}{2g}$$

$$h' = \frac{2h}{3}$$

110.



$\Rightarrow \quad \alpha = \pi - 2\theta$

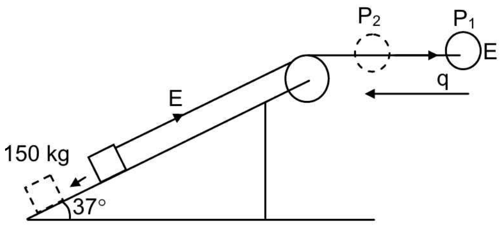
111. (b)

112. (d)

Frictional losses + work done by us should be equal to negative of K.E.

113. (b), when the reference chosen is such that potential energy is negative and its modulus value exceeds the kinetic energy the total mechanical energy is negative

114.



Initial KE = $\frac{1}{2}mv^2 = \frac{1}{2} \times 150 \times (2)^2 = 300J$

work done by the resultant force = –300 J

(b) total force = F – mg sinθ

$= 1000 - 150 \times 10 \times 0.6$

$= 100$ N

Work done = KE = $|100 \times S| = 300 \Rightarrow S = 3$ m

Note the work done is negative, because the body moves opposite to the resultant force applied.

115. $v^2 = 2\,as \Rightarrow a = \frac{v^2}{2s} = \frac{4}{2 \times 3} = \frac{2}{3}\,m\,s^{-2}$

116. $v = at \Rightarrow t = \frac{v}{a} = 2 \times \frac{3}{2} = 3$ s

Change in momentum = impulsive force. When the body has stopped, change in momentum = mv = Fδt.

$150 \times 2 = 100\ \delta t;\ \delta t = 3$ s

117. (a), (c), (d)

118. The ball moves with acceleration, till the neutral position, where mg = kx

$\Rightarrow$ $2 \times 10 = 200 \times x \Rightarrow x = 0.1$ m. This is the maximum velocity (KE) position.

(b) is correct

Energy equation at maximum KE:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = mg(h + x)$$

$$\Rightarrow \quad KE_{max} + \frac{1}{2} \times 200 \times (0.1)^2 = 2 \times 10\ (0.5)$$

$\Rightarrow$ $KE_{max} = 10 - 1 = 9$ J

(a) is correct

Energy equation at maximum spring energy (maximum compression)

$$mg\ (h + x_m) = \frac{1}{2}kx_m^2$$

$$\Rightarrow 20(0.4 + x_m) = \frac{1}{2} \times 200\ (x_m)^2$$

$$\Rightarrow \quad 100\ x_m^2 - 20\ x_m - 8 = 0$$

$$\Rightarrow \quad 25x_m^2 - 5x_m - 2 = 0$$

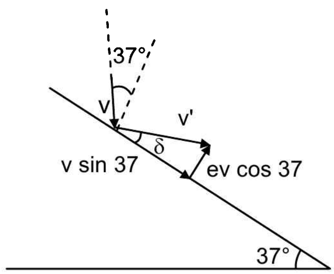
$$x_m = \frac{5 \pm \sqrt{25 + 200}}{50} = \frac{20}{50} = 0.4\,m$$

$$PE_{max} = \frac{1}{2} \times 200 \times 0.16 = 16\ J$$

(c) is correct

∴ (a), (b) and (c) are correct

119.



At the impact point, velocity of approach : v

$$v^2 = 2gh = 2 \times 10 \times \frac{20}{9} \Rightarrow v = \frac{20}{3}\,m\,s^{-1}$$

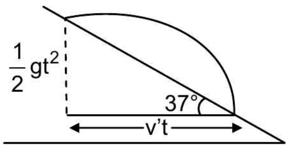
After impact : $v_p \rightarrow$ parallel to the plane : $v_p = v \sin 37$

$$= \frac{20}{3} \times \frac{3}{5} = 4\ m\ s^{-1}$$

$v_x \to$ normal to the plane

$$v_x = e\,v\cos 37 = \frac{9}{16}\times\frac{20}{3}\times\frac{4}{5} = 3\,\text{m s}^{-1}$$

$$\tan\ \delta = \frac{3}{4} \Rightarrow \delta = 37^\circ \ \Rightarrow v' \text{ parallel to horizontal.}$$



Hence height the ball moves above point of impact is zero.

$$v' = \sqrt{3^2 + 4^2} = 5\,\text{m s}^{-1}$$

(c) is correct

Time to reach back on the plane:

$$\text{(see fig.)}: \frac{\frac{1}{2}gt^2}{v't} = \tan 37^\circ = \frac{3}{4}$$

$$t = \frac{3}{4}\frac{v'\times 2}{g} = \frac{3\times 5\times 2}{4\times 10} = \frac{3}{4}\text{s}$$

(d) is correct

$\therefore$ (c) and (d) are correct

120. (a) $\to$ p, q, r, s
(b) $\to$ p, q, r ,s
(c) $\to$ p
(d) $\to$ p

Additional Practice Exercise

121. (i) Let velocity at B be v.

$$\Rightarrow mgR + \frac{1}{2}mv^2 = mg.4R - \mu mg(10.5R)$$

$$\Rightarrow v^2 = gR(6 - 21\mu)$$

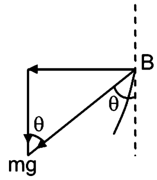
(ii) Forces at B are $N = \dfrac{mv^2}{R}$ and mg;

Angle with vertical by resultant is

$$mgR + \frac{1}{2}mv^2 = mg.4R - \mu mg(10.5R)$$

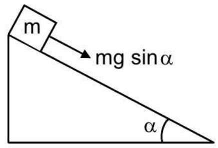
$$= \tan^{-1}\frac{3}{4}$$

$$\therefore \quad 6 - 21\mu = \frac{3}{4} \Rightarrow \mu = \frac{1}{4}$$



$$\therefore \qquad v = \frac{\sqrt{3}}{2}\sqrt{gR}$$

122. (i)



The only forces acting on the block are its weight mg and the normal force N.

Component of weight along the incline

$= mg\sin\alpha$

Work done = Force $\times$ distance

$$= mg\sin\alpha\frac{h}{\sin\alpha} = mgh$$

$\Rightarrow 5 \times 10 \times 9 = 450$ J

work done by normal force = 0

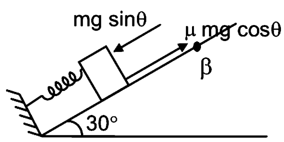
(ii) $a = g\sin\alpha$;

$$s = \frac{h}{\sin\alpha};\quad v = \sqrt{2as} = \sqrt{2gh} = \sqrt{180}\ \text{m s}^{-1}$$

(iii) $$t = \frac{v}{a} = \frac{\sqrt{180}}{10\times 0.6} = \sqrt{5}\ \text{s}$$

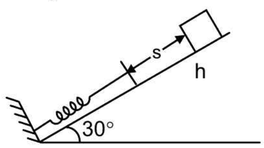
$$(\because \sin\alpha = \frac{3}{5} \text{ as } \tan\alpha = \frac{3}{4})$$

123. (i)



$$kx_0 + \mu mg\cos\theta = mg\sin\theta$$

$$\Rightarrow k = \frac{mg}{x_0}(\sin\theta - \mu\cos\theta)$$



$$\frac{100}{0.25}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}.\frac{\sqrt{3}}{2}\right)$$

$\Rightarrow$ k = 100 N m^{-1}

(ii) Loss of P.E. = mg (s+2) sin θ = 50 (s+2)

$$W_f = \mu mg\cos\theta\,(s+2)$$

$$= \frac{1}{2\sqrt{3}}.100.\frac{\sqrt{3}}{2}(s+2) = 25(s+2)$$

$$\therefore \frac{1}{2}k.2^2 = 50(s+2) - 25(s+2)$$

$\Rightarrow$ 200 = 25 (s+2) $\Rightarrow$ s = 6 m

$\therefore$ Total height descended by mass

= (s + 2) sin θ = 4 m

124. (i) Change in K.E when the block compresses the spring through x_0

$$\frac{1}{2}mv_0^2 - \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 = \frac{1}{2}m\left[v_0^2 - \frac{v_0^2}{4}\right] = \frac{3}{8}mv_0^2$$

By Work-Energy theorem

$$\frac{1}{2}kx_0^2 = \frac{3}{8}mv_0^2$$

(i) $\therefore k = \dfrac{3mv_0^2}{4x_0^2}$

(ii) Initially the compression of the spring is $\dfrac{L}{2}$. using the principle of conservation of energy

$$\frac{1}{2}k\left(\frac{L}{2}\right)^2 = +\frac{1}{2}mv^2$$

Solving the above equation:

$$v = \sqrt{\frac{k}{m}}\,\frac{L}{2}$$

125. The force on $m_1 = -\mu\, m_2g$

$\therefore$ work done = F. distance = $(-\mu m_2 g) \times (-x)$

$= \mu m_2 g.x$

work done = energy in spring = $\dfrac{1}{2}kx^2$

$$\frac{1}{2}kx^2 = \mu m_2 gx \Rightarrow x = \frac{2\mu m_2 g}{k}$$

126. The power rating of the pump is 10 HP

= 10 × 746 W = 7460 W

(a) Hence the energy spent by the pump in 10 hour is

U = power × time

= 7460W × 10 h

= 74600Wh (1h = 3600 s)

= 74600 × 3600 W s

= 74600 × 3600 J (∵ W s = J)

= 2.6856×10^8 J

Note:

Only 80% of this energy spent by the pump was useful in lifting the water from the well.

(b) m = mass of water flowing per second

= (volume of water per second) × density

$$= 50\frac{\text{litre}}{\text{second}} \times \frac{1\text{ kg}}{\text{litre}} = 50 \text{ kg s}^{-1}$$

Energy spend by pump per second

= power × time

= 10 H P × 1 second

= 10 × 746 W × 1 second

= 7460 W s

= 7460 J

Out of this only 80% is available for lifting water because the efficiency of pump is 80%.

Hence energy available per second for lifting water

U = 7460 × 0.8

= 5968 J — (1)

This energy is used for increasing the potential energy and kinetic energy of the well water.

$\therefore$ PE/ second + KE/second = U

PE/second = mgh

= 50 × 10 × 6 = 3000 J — (2)

$\therefore$ KE/second = U – PE/second

= 5968 – 3000 = 2968 — (3)

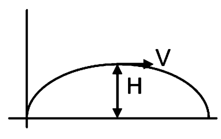
$\therefore$ $\dfrac{1}{2}mv^2 = 2968$

$\therefore$ $v^2 = \dfrac{2 \times 2968}{m} = \dfrac{2 \times 2968}{50}$

$\therefore$ $v = \sqrt{118.72} = 10.9 \text{ m s}^{-1}$

Hence the speed with which water flows out of the pump is 10.9 m s^{-1}.

127.



At maximum height H:$PE_{(max)} = mgH \Rightarrow$

$$\frac{PE_{(m)}}{KE} = \frac{7}{3} \Rightarrow \frac{PE_{(m)}}{\text{Total E}} = \frac{7}{10}$$

$\Rightarrow$ $PE_{(m)} = \dfrac{7}{10}T$

[T → Total energy which is always conserved]

(i) $h_1 = \frac{3H}{4} \Rightarrow PE = \frac{3}{4} \times \frac{7}{10} T = \frac{21}{40} T$

$\therefore$ $KE = \frac{19}{40} T$ [$\because$ T is conserved]

$\therefore$ $\frac{KE}{PE} = \frac{19}{21}$

(ii) $PE = \frac{1}{2} \times \frac{7}{10} T = \frac{7}{20} T \Rightarrow KE = \frac{13}{20} T$

$\Rightarrow$ $\frac{KE}{PE} = \frac{13}{7}$

(iii) $PE = \frac{1}{4}\frac{7}{10} T = \frac{7}{40} T \Rightarrow KE = \frac{33}{40} T$

$\Rightarrow$ $\frac{KE}{PE} = \frac{33}{7}$

(iv) at h = (x H) when PE = KE

$PE = x \frac{7}{10} T$

$\Rightarrow$ $KE \frac{10 - 7x}{10} T \Rightarrow \frac{KE}{PE} = \frac{10 - 7x}{7x} = 1$ (given)

$x = \frac{10}{14} = \frac{5}{7}$

128. (i) $0.25 \times 300 = (12.25 + 37.5 + 0.25)v = 50v$

$\Rightarrow$ $v = 1.5\ m\ s^{-1}$

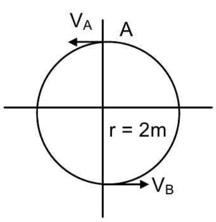
(ii) $K.E = \frac{1}{2} mv^2 = \frac{1}{2} \times 50 \times 1.5^2 = 56.25\ J$

(iii) $F_f = \mu m_2 g = 0.25 \times 12.5 \times 10 = 31.25\ N$

$a_{rel} = a_1 + a_2 = \frac{F_f}{m_1} + \frac{F_f}{m_2}$

$= \frac{31.25}{12.5} + \frac{31.25}{37.5} = 3.33\ m\ s^{-2}$

129. (a)



Let 'm' be the mass of stone. V_A = speed at highest position V_B = speed at lowest position.

At A

$PE_A = mgh = mg\ (2r) = mg \times 2\times2$

$= 4\ mg = 40\ m$ (taking $g = 10\ m\ s^{-2}$)

$KE_A = \frac{1}{2} mV_A^2 = \frac{1}{2} \times m \times 10^2 = 50m$

Total energy = TE = $PE_A + KE_A$ = 40m + 50 m

= 90 m

At B

$PE_B = 0$; h = 0

$KE_B = \frac{1}{2} V_B^2$

$\therefore$ $TE = PE_B + KE_B = 0 + \frac{1}{2} mV_B^2 = \frac{1}{2} mV_B^2$

$\therefore$ $90\ m = \frac{1}{2} mV_B^2$

$90 \times 2 = V_B^2$

$\therefore$ $V_B = \sqrt{90 \times 2} = \sqrt{180} = 13.42\ m\ s^{-1}$

$\therefore$ The speed of stone at

B = $13.42\ m\ s^{-1}$

(b) If $V_B = 8\ m\ s^{-1}$,

$KE_B = \frac{1}{2} mV_B^2 = \frac{1}{2} m \times 8^2 = 32\ m$

$PE_B = 0$

$\therefore$ $TE = KE_B + PE_B = = 32\ m + 0 = 32\ m$

The PE at A is 40 m, which means

KE at A = TE − PE at A = 32 m − 40 m = −8 m

Kinetic energy cannot be negative

$\therefore$ The stone will not be able to reach A. Hence it will not be able to complete the circular motion.

130. $m_1 = km,\ m_2 = m$

$u_1 = \sqrt{2gr}$ (just before collision)

$u_2 = -\sqrt{2gr}$ (just before collision)

e = 0.5

(i) $v_1 = \frac{(m_1 - em_2)u_1 + (1 + e)m_2u_2}{(m_1 + m_2)}$

$= \frac{(km - 0{,}5m)\sqrt{2gr} - 1.5m\sqrt{2gr}}{(km + m)}$

$= \frac{(k - 0.5)\sqrt{2gr} - 1.5\sqrt{2gr}}{(k + 1)}$

$= \frac{(k - 2)}{(k + 1)}\sqrt{2gr}$

$v_2 = \frac{(m_2 - em_1)u_2 + (1 + e)m_1u_1}{(m_1 + m_2)}$

$$= \frac{-\left(m - \frac{km}{2}\right)\sqrt{2gr} + 1.5\ km\sqrt{2gr}}{(km + m)}$$

$$= \frac{\left(\frac{k}{2} - 1\right)\sqrt{2gr} + \frac{3k}{2}\sqrt{2gr}}{(k+1)} = \frac{(2k-1)}{(k+1)}\sqrt{2gr}$$

(ii) For m to reach its starting position, $v_2 = \sqrt{2gr}$ (from energy conservation for m_2)

$$\Rightarrow \frac{(2k-1)}{(k+1)}\sqrt{2gr} = \sqrt{2gr} \Rightarrow \frac{2k-1}{k+1}$$

$= 1 \Rightarrow 2k - 1 = k + 1 \Rightarrow k = 2$

(iii) when $k = 2, v_1 = 0$ ($\therefore k - 2 = 0$). Hence the heavier block stops.

131. The meaning of an 'average quantity' means 'if it were constant' what is that value?

If a force were to be 'constant' at all times, it will be the same constant value at all positions and vice versa

Let us prove mathematically,

$$F' = \frac{\int F dt}{\int dt} = \frac{\Delta p}{\Delta t} = \frac{mu}{\Delta t} = m\left(\frac{u}{\Delta t}\right)$$

$$\left(\frac{u}{\Delta t} \text{ is the average acceleration}\right)$$

$$F'' = \frac{\int F ds}{\int ds} = \frac{\Delta K}{\Delta S} = \frac{\frac{1}{2}mu^2}{\Delta S} = \frac{\frac{1}{2}mu^2}{\frac{1}{2}\left(\frac{u}{\Delta t}\right)(\Delta t)^2}$$

$$= m\left(\frac{u}{\Delta t}\right) = F'$$

132. (a)

$$a = \frac{u}{t_0} \quad v = at = \frac{u}{t_0}t$$

$$\Delta KE = \frac{1}{2}mv^2 = \frac{1}{2}m.\frac{u^2}{t_0^2}t^2 = \text{Work done.}$$

133. $W = \int Power.dt = \int F.v.dt$

$$= \int \frac{A}{v} v dt = \int_0^t A dt = At$$

134. Since the work done is zero, $\overline{F}.\overline{S} = 0$

$$\overline{S} = (2-1)\,\hat{i} + (a - (-1))\,\hat{j} = 1\hat{i} + (a+1)\,\hat{j}$$

$$\therefore \quad (3\hat{i} - 2\hat{j}).\left[1\hat{i} + (a+1)\hat{j}\right] = 0$$

$3 - 2(a+1) = 0$

$$a = \frac{1}{2}$$

135. F.v = P

$$\Rightarrow F = \frac{P}{v} = \frac{880 \times 10^3}{20} = 44 \times 10^3\ N$$

$$\frac{F}{m} = \frac{44 \times 10^3}{40 \times 10^3} = 1.1\ m\ s^{-2}$$

$$a = \frac{F}{m} - \mu g = 1.1 - 0.1 = 1\ m\ s^{-2}$$

136. $\overline{F}.\overline{v} = 0$ at highest point. ($\because$ $\overline{g}$ is normal to $\overline{v}$)

137. Power = F.v

$F = \mu mg\cos\theta$

$v = at = g(\sin\theta - \mu\cos\theta)t$

$\therefore \quad P = \mu mg^2 \cos\theta\,(\sin\theta - \mu\cos\theta)$

138. At $t = 0$, K.E = 0

$$P = \frac{dW}{dt} = \frac{d(KE)}{dt}$$

P constant $\Rightarrow$

$KE \propto t$

$v^2 \propto t$

$v \propto \sqrt{t}$

$$\frac{ds}{dt} \propto \sqrt{t}$$

$ds \propto \sqrt{t}\,dt$

At $t = 0, S = 0$

$S \propto t^{3/2}$

Aliter:

Power $P = ML^2T^{-3}$ = constant

Since mass (m) is constant, L^2T^{-3} = constant

L^2T^{-3} = constant

$L^2 \propto T^3 \rightarrow L \propto T^{3/2}$

$L \rightarrow$ displacement, $T \rightarrow$ time $\therefore S \propto t^{3/2}$

139. $P = Fv = mav \Rightarrow a = \dfrac{P}{mv}$

$\Rightarrow \dfrac{dv}{dt} = \dfrac{P}{mv} \Rightarrow \dfrac{dv}{ds}.\dfrac{ds}{dt} = \dfrac{P}{mv}$

$\Rightarrow \dfrac{dv}{ds}.v = \dfrac{P}{mv} \Rightarrow v^2dv = \dfrac{P}{m}ds$

Integrating:

$\Rightarrow \left.\dfrac{v^3}{3}\right|_u^v = \dfrac{Ps}{m} \Rightarrow s = \dfrac{m(v^3 - u^3)}{3P}$

140. $\dfrac{1}{2}mv^2 = \int_0^t Pdt = \int \dfrac{2}{3}t^2dt = \dfrac{2}{9}t^3 = 6\text{ J at } t = 3\text{ s}$

$\Rightarrow \dfrac{1}{2} \times 2 \times v^2 = 6$

$\Rightarrow v^2 = 6 \quad v = \sqrt{6}\ \text{m s}^{-1}$

Aliter:

$P = \dfrac{2t^2}{3} \Rightarrow Fv = \dfrac{2t^2}{3}$

$\Rightarrow mav = \dfrac{2t^2}{3}$

$\Rightarrow av = \dfrac{2t^2}{3m} = \dfrac{2t^2}{3 \times 2} = \dfrac{t^2}{3}; a = \dfrac{dv}{dt}$

$\Rightarrow v\dfrac{dv}{dt} = \dfrac{t^2}{3} \Rightarrow \left.\dfrac{v^2}{2}\right|_0^v = \dfrac{t^3}{9} \Rightarrow \dfrac{v^2}{2} = \dfrac{t^3}{9}$

$v = \sqrt{\dfrac{2}{9}t^3} = \dfrac{\sqrt{2}}{3} \times t\sqrt{t}$; when $t = 3$ s

$v = \dfrac{\sqrt{2}}{3} \times 3 \times \sqrt{3} = \sqrt{6}\ \text{m s}^{-1}$

141. (a)

Fv = constant = P

For maximum v, F is minimum

Minimum $F = \mu mg$

$\Rightarrow v = \dfrac{P}{\mu mg}$

142. (b)

$P^2 = 2m\ KE \Rightarrow P_1^2 = 2m\ \ KE_1$ — (1)

$KE_2 = KE_1 + \Delta KE = KE_1 + 8\ KE_1 = 9\ KE_1$

$P_2^2 = 2m\ \ KE_2 = 2m\ .\ 9\ KE_1 = 18\ m\ KE_1$ — (2)

$\dfrac{P_2^2}{P_1^2} = 9 \rightarrow P_2 = 3P_1 \Rightarrow \Delta P = P_2 - P_1$

$= 3P_1 - P_1 = 2P_1 = 200\%$

143. (b) $\dfrac{mv}{\frac{1}{2}mv^2} \propto \dfrac{1}{t} \Rightarrow v \propto t$

$\therefore$ a is constant

Uniformly acceleration motion

144. (d) During first 5s work done by fractional force is zero, hence the mass m has not slipped over M and obviously it will never slip again. They will move together. For a body in motion under uniform acceleration, starting from rest, the displacement in equal intervals of time (not necessarily 1 s) is in the ratio 1 : 3 : 5 etc. Here take interval of 5s. Power = F.S $\Rightarrow P \propto S$ ($\therefore$ F constant).

$\therefore\ P_1\ (\text{I}^{\text{st}}\text{ interval}) = \dfrac{P_2(\text{of }2^{\text{nd}}\text{ interval})}{3}$

$= \dfrac{1800}{3} = 600\text{ J}$

Let the masses be x.M and x.m

Since bodies move together acceleration a is:

$a = \dfrac{F}{x(M+m)} = \dfrac{f}{Mx}$ ($f \rightarrow$ frictional force between M and m)

$\dfrac{24}{(5+1)} = \dfrac{f}{1} \Rightarrow f = 4n$ —(1)

At the end of first 5s, momentum P is:

$P = F \times t = 24 \times 5 = 120\text{ kg ms}^{-1}$

$P = \sqrt{2m'kE} \Rightarrow \therefore \sqrt{2(M+m) \times 600} = 120$

$\Rightarrow (M + m) = 12\text{ kg} \Rightarrow M = 10\text{ kg}; m = 2\text{ kg}$

$\mu\ mg = f \Rightarrow$ (From (1)): $\mu \times 10 \times 10 \geq 4$

$\Rightarrow \mu \geq 0.04$

Aliter:

Since the work done by the frictional forces is zero during any interval, there is no slippage between M and m.

$a = \dfrac{24}{(M+m)} = \dfrac{24}{6M}$ ($\because$ m = 5M $\rightarrow$ data)

$f = \mu mg = Ma = \dfrac{24}{6} \Rightarrow \mu \geq \dfrac{0.4}{m}$ —(1)

At any instant t,

momentum p = F.t = and KE = $\dfrac{p^2}{2(M+m)}$

$= \dfrac{p^2}{2 \times 6M}$

$$KE_{(t=5)} = \frac{(F \times 5)^2}{2(6M)}; KE_{(t=10)} = \frac{(F \times 10)^2}{2(6M)}$$

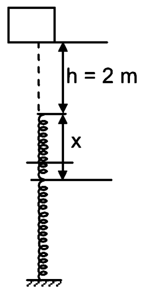
Given: $\dfrac{(F \times 10)^2 - (F \times 5)^2}{2(6M)} = 1800$

$\Rightarrow \dfrac{(240)^2 - (120)^2}{12M} = 1800$

$M = 2\text{ kg} \Rightarrow m = 10\text{ kg}$

From eqn (1) $\Rightarrow \mu \geq \dfrac{0.4}{10} \geq 0.04$

145. (b) When the body comes to rest and compresses the spring maximum by x, the energy equation is



$$mg\,(h + x) = \frac{1}{2}kx^2$$

$$1 \times 10(2 + x) = \frac{1}{2}200x^2$$

$$100x^2 - 10x - 20 = 0$$

$$10x^2 - x - 2 = 0$$

$x = \dfrac{1 \pm \sqrt{1+80}}{20} \Rightarrow$ take only positive value

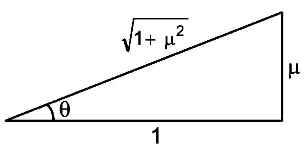
$x = 0.5 \Rightarrow$ spring energy

$$= \frac{1}{2}kx^2 = \frac{1}{2} \times 200 \times 0.25 = 25\text{ J}$$

or: spring energy = $mgh' = 1 \times 10 \times 2.5 = 25$ J

146. (b)

$\mu = \tan\theta$



Energy equation:

$$\frac{1}{2}mv^2 = mgh + fs\left[f \rightarrow \text{frictional force}\right]$$

$$\frac{1}{2}mv^2 = \left[mg\sin\theta + \mu mg\cos\theta\right]s$$

since θ is the angle of repose $\Rightarrow mg\sin\theta = \mu mg\cos\theta$

$\therefore \dfrac{1}{2}mv^2 = 2\,mg\sin\theta \,.\, s \Rightarrow v^2 = 4gs\,.\,\sin\theta$

$$v = 2\sqrt{gs\sin\theta}$$

Aliter:

Work done by force perpendicular to the plane is zero. Applying work energy theorem I, we have the friction force $f = mg\sin\theta$ (since $\mu = \tan\theta$)

$\Rightarrow \dfrac{1}{2}mv^2 = (f + mg\sin\theta)s = 2\,mg\,s\sin\theta$

$$v = 2\sqrt{gs\sin\theta}$$

147. (a)

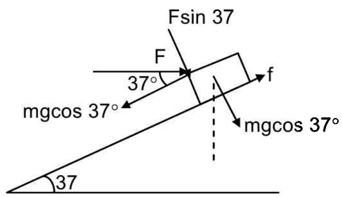
The mass after compressing the spring maximum will return and will stop where $kx \leq \mu mg$,

$200x = \mu \times 1 \times 10;\ \dfrac{1}{2}kx^2 = 0.04$ (data)

$\Rightarrow 100x^2 = 0.04 \Rightarrow x = 0.02$ m

$200 \times 0.02 = \mu \times 10 \Rightarrow \mu \geq 0.4$

148. (d)



Force acting up the plane F'

$= F\cos37° + f(\text{frictional})$

$f = \mu(F\sin37° + mg\cos37°)$

$= 0.2(10 \times 0.6 + 1 \times 10 \times 0.8)$

$= 0.2(6 + 8) = 2.8$ N

$F' = (10 \times 0.8) + 2.8 = 10.8$ N

$mg\sin\theta = 1 \times 10 \times 0.6 = 6$N

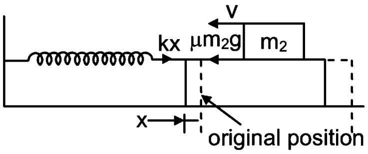
$\therefore$ acceleration up $= \dfrac{10.8 - 6}{1} = 4.8\text{ m s}^{-2}$

$v^2 = 2as \Rightarrow s = \dfrac{v^2}{2a} = \dfrac{144}{2 \times 4.8} = 15$ m

Work done is negative since the body moves in the opposite direction of Fcos37° and zero work done by Fsin 37°

$\therefore$ W = $-10\cos37° \times 15 = -120$ J

149. (a)



m_2 is exerting a frictional force $\mu\, m_2 g$ on m_1 and the force by spring is kx in the opposite direction. Let x_1 be the neutral point so that $kx_1 = \mu m_2 g$. Because of K.E acquired by m_1 it will continue to compress the spring till $x_2 = 2x_1$

$\therefore$ x_1 it will have maximum velocity

Energy equation: $\frac{1}{2}m_1 v_1^2 = (\mu m_2 g)x_1 - \frac{1}{2}kx_1^2$

$kx_1 = \mu m_2 g \Rightarrow 200x_1 = 0.6 \times 10 \times 10$

$\Rightarrow$ $x_1 = 0.3$ m

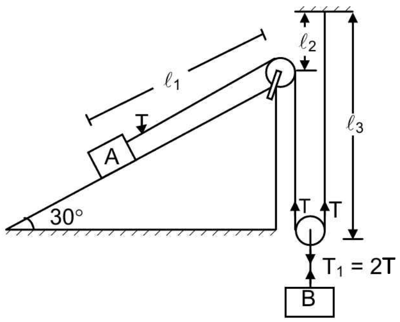
$\therefore$ $\frac{1}{2} \times 1 \times v_1^2$

$= 0.6 \times 10 \times 10 \times 0.3 - \frac{1}{2} \times 200 \times 0.09$

$= 18 - 9 = 9$

$v_1^2 = 2 \times 9 \Rightarrow v_1 = 3\sqrt{2}$ m s^{-1}

150.



$\ell_1 + (\ell_3 - \ell_2) + \ell_3$ = constant ($\therefore$ length of string)

$\Rightarrow$ $\ell_1 + 2\ell_3$ = constant ($\therefore \ell_2$ = constant)

$$\frac{d^2\ell_1}{dt^2} + \frac{2d^2\ell_3}{dt^2} = 0$$

$$|\bar{a}_A| = |-2\bar{a}_B|$$

For A

$T - m_A g \sin\theta = m_A a_A$ — (i)

For B

$m_B g - 2T = m_a a_B$

$= m_B \frac{a_A}{2}$ — (ii)

(i) × 2 + (ii) $\Rightarrow m_B g - 2m_A g\sin\theta = 2m_A a_A + \frac{m_B a_A}{2}$

$\theta = 30°$, $m_B = 20$ kg, $m_A = 10$ kg

$\therefore$ $20 \times 10 - 2 \times 10 \times 10 \times \frac{1}{2}$

$= 2 \times 10 \times a_A + \frac{20 \times a_A}{2}$

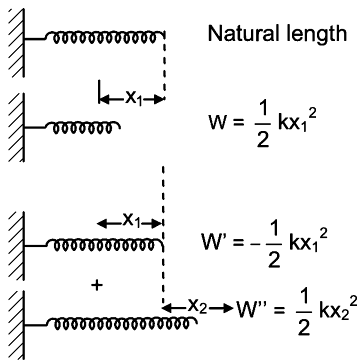
$200 - 100 = 30\, a_A \rightarrow a_A = \frac{100}{30}$ ms^{-2}

substituting in (i), we get $T = m_A a_A + m_A g \sin\theta$

$$= \frac{10 \times 100}{30} + 10 \times 10 \times \frac{1}{2} = \frac{250}{3}\text{N}$$

$\therefore W = TS = \frac{250}{3} \times 3 = 250$J

151. (d)

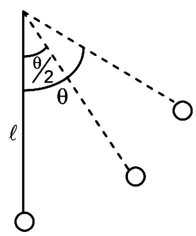


Given, W' + W" = 2W $\Rightarrow$ –W + W" = 2W

$\Rightarrow$ W" = 3W

$\Rightarrow$ $\frac{1}{2}kx_2^2 = 3.\frac{1}{2}kx_1^2 \Rightarrow \frac{x_2}{x_1} = \sqrt{3}$

152. (d)



At θ: P.E = $mg\ell(1 - \cos\theta)$, K.E = 0 — (1)

At $\frac{\theta}{2}$: P.E = $mg\ell\left[1 - \cos\left(\frac{\theta}{2}\right)\right]$,

K.E = $\frac{1}{2}mu^2$

Total energy = $\frac{1}{2}$ mu² + mgl $(1 - \cos\frac{\theta}{2})$ —(2)

At θ = 0: P.E = 0, K.E = $\frac{1}{2}m\frac{7}{5}u^2$ — (3)

(3) and (1) ⇒ $\frac{1}{2}m\frac{7}{5}u^2 = mg\ell(1-\cos\theta)$

(2) and (1)

$$\Rightarrow \frac{1}{2}mu^2 + mg\ell\left(1-\cos\frac{\theta}{2}\right) = mg\ell(1-\cos\theta)$$

$$\Rightarrow \frac{1}{2}mu^2 = mg\ell\left(\cos\frac{\theta}{2}-\cos\theta\right)$$

$$\therefore \frac{7}{5} = \frac{(1-\cos\theta)}{\left(\cos\frac{\theta}{2}-\cos\theta\right)}$$

$$\Rightarrow 7\cos\frac{\theta}{2} - 7\cos\theta = 5 - 5\cos\theta$$

$$\Rightarrow 2\cos\theta - 7\cos\frac{\theta}{2} + 5 = 0$$

$$[\because \cos\theta = 2\cos^2\frac{\theta}{2} - 1]$$

$$\Rightarrow 4\cos^2\frac{\theta}{2} - 7\cos\frac{\theta}{2} + 3 = 0$$

$$\Rightarrow \cos\frac{\theta}{2} = \frac{7 \pm \sqrt{49-48}}{8} = 1 \text{ or } \frac{3}{4}$$

(Ignore 1, $\because \frac{\theta}{2} = 0$) $\therefore \cos\frac{\theta}{2} = \frac{3}{4}$

$$\Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$= 2\times\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$$

$$\theta = \cos^{-1}\left(\frac{1}{8}\right)$$

153. (a)

$\frac{1}{2}kx^2 = mg(h+x)$ — (1)

$kx - mg = 3mg \Rightarrow kx = 4\,mg$

$\therefore$ (1) ⇒ $\frac{1}{2}\times 4\,mgx = mg(h+x)$

$\Rightarrow 2x = h + x \Rightarrow x = h$

154. (b)

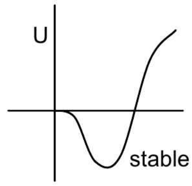
$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh - \mu mg\cos\theta.s$$

$\Rightarrow v^2 = u^2 + 2gh - 2\mu g\cos\theta.s$

$= u^2 + 2gs\sin\theta - 2\mu g\cos\theta s$ ($\therefore h = s\sin\theta$)

$\Rightarrow v^2 - 2gs\sin\theta + 2\mu gs\cos\theta = u^2$ = constant

155.



$$U = x^3 - 6x^2 \Rightarrow F = \frac{-dU}{dx} = -[3x^2 - 6\times 2x]$$

$F = -3x^2 + 12x \Rightarrow$ At x = 1, F = −3 + 12 = 9 N

It is in stable equilibrium when $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2}$ is positive

$$\frac{dU}{dx} = 3x^2 - 12x = 0 \Rightarrow x = 4\text{ m}$$

$$\frac{d^2U}{dx^2} = 6x - 12 \text{ at } x = 4;\ \frac{d^2U}{dx^2} = 6\times 4 - 12 > 0$$

Hence stable equilibrium

156. (a)

When it reaches B, the energy equation is

$$mgR + \frac{1}{2}mv^2 = mg4R - \mu mg(10.5R)$$

$$v^2 = gR(6-21\mu) = gR\left[6 - \frac{21}{4}\right] = gR\frac{3}{4}$$

Normal reaction $N = \frac{mv^2}{R} = 1\times\frac{3}{4}gR\times\frac{1}{R}$

$= 7.5$ N

157. (a)

C is point on the circular path and hence to just reach C the initial velocity v is $\sqrt{5g\ell}$

To just reach B,

PE at B = KE at A

$$mg\ell = \frac{1}{2}mv^2 \rightarrow v^2 = 2g\ell$$

$v = \sqrt{2g\ell}$, the bob will pass above B.

$\therefore$ If $\sqrt{2g\ell} < v < \sqrt{5g\ell}$, the bob will pass B but will not reach C.

158. (a)

Let v_x and v_y represent the X and Y components of velocity of mass m and V be the velocity of the wedge, when m is at the bottom-most position on the wedge

Since momentum in X–direction is conserved $v_x = 3V$

On the reference frame of the wedge relative velocity $v_x' = v_x + V = 4V$ and $v_y = v_y$

$$\frac{v_y'}{v_x'} = \tan 37° = \frac{3}{4}$$

$$\therefore \quad v_y = v_y' = v_x'\frac{3}{4} = 4V\frac{3}{4} = 3V$$

Energy equation:

$$mgh = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}MV^2$$

$$1 \times 10 \times 6 = \frac{1}{2} \times 1 \times (3V)^2 + \frac{1}{2} \times 1 \times (3V)^2 + \frac{1}{2} \times 3V^2$$

$$120 = 9V^2 + 9V^2 + 3V^2 \Rightarrow V^2 = \frac{120}{21}$$

$$\text{Energy of the wedge} = \frac{1}{2}MV^2 = \frac{1}{2} \times 3 \times \frac{120}{21} = \frac{60}{7} \approx 8.6 \text{ J}$$

Aliter:

Let p_x and p_y be the X and Y momenta and v_x and v_y be the velocity components. For the wedge, total momentum = p_x,

$$\Rightarrow \quad v = \frac{v_x}{3} \text{(conservation of momentum)}$$

$$\frac{v_y}{v_x + {}^{v_x}\!/_3} = \tan 37° = \frac{3}{4} \Rightarrow \frac{v_x}{v_y} = 1$$

$\therefore$ for the mass m:

$$\Rightarrow \quad \frac{p_x}{p_y} = 1$$

conservation of energy gives

$$\frac{p_x^{\ 2}}{2\times 1} + \frac{p_y^{\ 2}}{2\times 1} + \frac{p_x^{\ 2}}{2\times 3} = 1 \times 10 \times 6$$

$$(\because p_x = p_y) \Rightarrow p_x^{\ 2}\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{6}\right] = 60$$

$$\Rightarrow \quad p_x^{\ 2} = 60 \times \frac{6}{7} S$$

$$KE_{wedge} = \frac{p_x^2}{2M} = \left(60 \times \frac{6}{7}\right) \times \frac{1}{2} \times \frac{1}{3} \simeq 8.6 \text{ J}$$

159. (c)

p equal for both, (considering magnitudes only)

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \leq U$$

$$\Rightarrow \quad \frac{1}{2}p(v_1 + v_2) \leq U \Rightarrow v_1 + v_2 \leq \frac{2U}{p}$$

160. (d)

$$P = \overline{F}_{thurst} \cdot \overline{v}_{rel}$$

$$v_{rel} = |\overline{v}_{rel}| = 15 - 0 = 15 \text{ ms}^{-1}$$

$$\overline{F}_{thrust} = \left(\frac{dm}{dt}\right)\overline{v}_{rel}$$

$$\therefore \quad F_{thrust} = \frac{10 \text{ kg}}{s} \times \frac{15 m}{s} = 150 \text{ N}$$

$\therefore$ Power P = 150 × 15 = 2250 W = 2.25 kW

161. (b)

Change in velocity of the rocket = v = 70 m s^{-1}

Relative velocity of exhaust = v_{rel}

$$v = v_{rel}\ell n\frac{M_0}{M_f} \Rightarrow 70 = v_{rel}\,\ell n\frac{M_0}{\left({}^{M_0}\!/_2\right)}$$

$$= v_{rel}\ell n(2)$$

$$v_{rel} = \frac{70}{\ell n(2)} = \frac{70}{0.7} = 100 \text{ m s}^{-1}$$

162. (c)

$v_{A/B} = v_A - v_B = 0 - (+25) = -25$ m s^{-1} remains constant throughout the journey. Let the velocities just before and after collision be as shown and note that since it is an elastic collision between equal masses, velocities will interchange

Before collision After collision

A $\downarrow v_A'$ $\downarrow v_A'' = v_B' = -7 \text{ m s}^{-1}$(data)

[$\because$ elastic collision, velocities inter change]

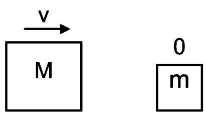
Before collision $v_A' - v_B' = -25 \text{ m s}^{-1}$

$v_A' = -25 + v_B' = -25 - 7 = -32 \text{ m s}^{-1}$

163. (b)

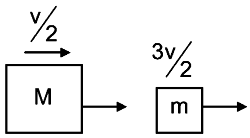
K reduces to $\frac{K}{4} \Rightarrow v \rightarrow \frac{v}{2}$

Before collision:



$\rightarrow$ since elastic collision;

velocity of approach = velocity of separation. Hence after collision



$\Rightarrow$ for conservation of momentum

$$Mv = M\frac{v}{2} + m\frac{3v}{2} \Rightarrow M = 3\text{ m}$$

$$\frac{M}{m} = 3$$

Aliter

$$P = \sqrt{2 \times \text{mass} \times \text{kinetic energy}}$$

K is conserved.

$\therefore$ kinetic energy of m = $k - \frac{k}{4} = \frac{3k}{4}$

$$P_i = P_M\ (\because P_m = 0) = \sqrt{2Mk}$$

$$P_f = P'_M + P'_M = \sqrt{2M\frac{k}{4}} + \sqrt{2m\,.\frac{3k}{4}}$$

$P_i = P_f = $ ($\because$ conservation of linear momentum)

$$\therefore \sqrt{2Mk} = \sqrt{2M\frac{k}{4}} + \sqrt{2m\frac{3k}{4}}$$

$$\therefore \sqrt{2M} = \sqrt{\frac{M}{2}} + \sqrt{\frac{3m}{2}}$$

$$\Rightarrow \quad 2\sqrt{M} = \sqrt{M} + \sqrt{3}\,m$$

$$\Rightarrow \quad \sqrt{M} = \sqrt{3m} = \sqrt{\frac{M}{m}} = \sqrt{3} \Rightarrow \frac{M}{m} = 3$$

164. (d)

$v_0 \rightarrow$ velocity with which it reaches the wall first time

$$\frac{1}{2}mv_0^2 = mgL \Rightarrow v_0$$

$$= \sqrt{2gL} = \sqrt{2 \times 10 \times 3.2} = 8 \text{ m s}^{-1}$$

After first collision velocity = ev_0

After 2nd collision velocity = $e.ev_0 = e^2v_0$

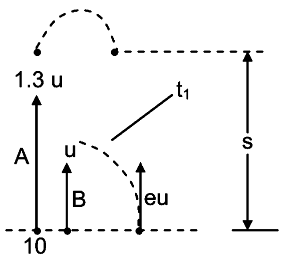
$\therefore$ 4th collision velocity = $e^4v_0 = 0.5^4 \times 8$

$$= \frac{5^4}{10^4}.8 = \frac{625 \times 8}{10000} = 0.5 \text{ m s}^{-1}$$

$$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} = \frac{(0.5)^2}{20}$$

$$= 0.0125 \text{ m} = 1.25 \text{ cm}$$

165. (d)



Let the bodies A and B have initial velocities 1.3 u and u upwards. Initial relative velocity = 1.3 u − u = 0.3 u which remains the same till B rebounds at time

$$t_1 = \frac{2u}{g}$$

$= 0.2u$

[$\because$ g = 10 m s^{-2}]

At that time their separation is : $S = t_1 v_r = 0.2\text{ u} \times 0.3\text{ u} = 0.06\text{ u}^2$(m)

Their relative velocity before rebound of

B = 0.3 u (same as initial)

After rebound, $v_B = e.u = 0.5u$ (up positive)

$v_A = 1.3\ u - gt = 1.3\ u - 2\ u = -0.7\ u$

$\therefore\ v_r' = v_A - v'_B = -0.7u - \left(\frac{u}{2}\right) = -\ 1.2u$

Time to meet $t_2 = \frac{|s|}{|v_r^1} = \frac{0.06u^2}{1.2u} = 0.05\ u$

Hence total time = $t_1 + t_2 = 0.2u + 0.05u$
$= 0.25u$

166. (c)

Just taut ⇒ Tension is just zero

⇒ Mg to be balanced by upward impulsive force

Impulse, Fdt = change in momentum, dp

= mu(1 + e) per bullet

× n bullets per second

⇒ Mg = F = nmu(1 + e)

⇒ $n = \frac{M}{m} \cdot \frac{g}{u} \frac{1}{(1+e)}$

167. (a)

$u = \sqrt{2gh}\quad v = \sqrt{2g\frac{h}{2}} = \sqrt{gh}$

$\therefore\ F\Delta t \propto \sqrt{gh}\,(1+\sqrt{2})$

($\therefore$ Impulse = change in momentum)

As per data,

$\Delta t \propto \sqrt{\frac{2h}{g}} \Rightarrow \Delta t = K\sqrt{\frac{2h}{g}}$

$\therefore\quad F = \frac{g\left(1+\sqrt{2}\right)}{\sqrt{2}K}$, constant and independent of 'h'.

168. (c)

'particle' ⇒ 1 D collision

well known formula

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2$$

can be used to conclude

But logic is faster

Let M and m, M > m

If m was at rest:

After collision m will move at > v

($\because$ If M = m, m will move at v. Since M > m, m will move at > v)

$\therefore$ M was at rest:

If m = M, m will come to rest

since m < M, m will go back

$\therefore\quad \frac{m - M}{m + M}v = -\frac{v}{2}$

⇒ M = 3m

169. (c) Initial Relative velocity = 10 + 5 = 15 m s^{-1}

Final relative velocity = $\frac{2}{3} \times 15 = 10$ m s^{-1}

The small mass (m) << the large mass (M), hence the total momentum is virtually due to large mass only. Hence its velocity is almost same after collision, i.e., 5 ms^{-1}.

⇒ Final velocity of small mass = $v_x + v_M$
$= 10 + 5 = 15$ m s^{-1}

Aliter:

M and m are the masses (M >> m)

u_1 and u_2 are + 5 m s^{-1} and – 10 m s^{-1} $e = \frac{2}{3}$

$$v_2 = \frac{(m - eM)u_2 + (1+e)Mu_1}{(M + m)}$$

$$= \frac{-eMu_2 + (1+e)Mu_1}{M,}$$

[$\because$ m << M, m – eM ≈ –eM and M + m ≈ M)

$= -eu_2 + (1 + e)\,u_1$

$= -\frac{2}{3}\times(-10) + \left(1+\frac{2}{3}\right)5$

$= \frac{20}{3} + \frac{25}{3} = \frac{45}{3} = 15\ ms^{-1}$

170. (b)

Clearly, the bomb was in motion (otherwise the 2 kg part also will be at rest $\because$ momentum = 0)

$\therefore$ momentum of bomb = p = momentum of 2 kg after explosion

$$\frac{1}{2}\frac{p^2}{m_1 + m_2} + E_{explosion} = \frac{1}{2}\frac{p^2}{m_2}\ (m_1 \text{ at rest})$$

$\therefore\quad E_{(explosion)} < \frac{p^2}{2m_2}$. Hence from data given:

$\therefore\quad E_{explosion} = \frac{3}{4}\left(\frac{1}{2}\frac{p^2}{m_2}\right)$

$$\Rightarrow \therefore \frac{1}{2}\frac{p^2}{(m_1+m_2)}+\left(\frac{3}{4}\right)\frac{p^2}{2m_2}=\frac{p^2}{2m_2}$$

$$\therefore \quad \frac{1}{m_1+m_2}=\frac{1}{4m_2}\Rightarrow m_1+m_2=4m_2=8\text{ kg}$$

171. (a)

172. (b) $mv=\frac{1}{2}mv^2\rightarrow v=2\text{ m s}^{-1}$

$E = mc^2$ relates mass and energy.

173. (d)

friction force : $\mu mg\cos\theta$

displacement : $\frac{x}{\cos\theta}$, where x is horizontal displacement

Product is constant . However, if height is raised, initial PE will change, so statement 1 is false

174. (d)

The work done by gravity also counts, negative in first case and positive in second case

175. (d)

176. (b)

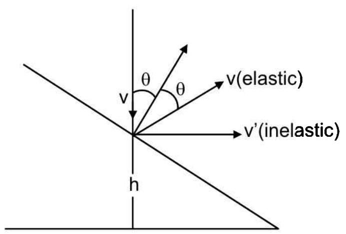
177. (a)

178. (a)

So when wedge moves to left, block has to move to right

179. (b)

180. (d)



Obviously statement 2 is true. But Statement 1 is not Time to fall will depend on the vertical component of velocity. Smaller the e, larger the θ, smaller the vertical component, hence it reaches faster

$$-h=u\sin\theta t-\frac{1}{2}gt^2\Rightarrow\frac{1}{2}gt^2=h+(u\sin\theta t)$$

The term in bracket is positive if u has vertical component and when it vanishes (i.e., v' is horizontal) or become negative, the value of t reduces

181. At $x = 5$ m: kinetic energy = 20 J

$x = 5$ m: potential energy

$= 20 + (5 - 2)^2 = 29$J

Total energy = kinetic energy = potential energy

$= 20 + 29 = 49$ J

182. Kinetic energy is maximum at x where potential energy is minimum, i.e., at $x = 2$ m.

183. $K.E._{max} = T.E. - P.E_{min} = 49 - 20 = 29$ J

184. By conservation of momentum, if v is the velocity of the combined mass:

$$m_H u = 2\,m_H v$$

$$\Rightarrow \quad v=\frac{u}{2}$$

185. The energy difference Δk is given by

$$\frac{1}{2}m_H u^2=2.\ \frac{1}{2}m_H\left(\frac{u}{2}\right)^2+\Delta k=m_H\frac{u^2}{4}+\Delta k$$

$$\Delta k=m_H\frac{u^2}{4}$$

186. Energy lost = excitation energy

$$m_H\frac{u^2}{4}=10.2\times1.6\times10^{-19}\text{ J}$$

$$u^2=\frac{10.2\times1.6\times10^{-19}\times4}{1.67\times10^{-27}}=39.08\times10^8$$

$$u=6.25\times10^4\text{ m s}^{-1}.$$

187. velocity of approach = $(u - v) = 20 - 10$

$= 10\text{ m s}^{-1}$

velocity of separation = $-e$(velocity of approach)

$= -0.8 \times 10 = -8\text{ m s}^{-1}$

Hence modulus value of relative velocity = 8 m s^{-1}

188. $u' - v = -8 \Rightarrow u' = v - 8 = 10 - 8 = 2\text{ m s}^{-1}$

It is in the +X direction

189. Percentage loss in

$$KE=\left(\frac{\frac{1}{2}mv_0^2-\frac{1}{2}mv^2}{\frac{1}{2}mv_0^2}\right)\times100$$

$$=\left[1-\left(\frac{2}{20}\right)^2\right]\times100=99\%$$

where [$v_0 \rightarrow$ initial velocity of the ball, $v \rightarrow$ final velocity of the ball]

190. (a), (d)

$$W_{OA} = 0\left(\because y = 0 \text{ along OA and } \vec{F}.\vec{x} \to kx\hat{j}.\, x\hat{i} = 0\right)$$

$$W_{AB} = \int_0^a k(y^2\hat{i} + a\hat{j}).\, dy\hat{j} = k\int_0^a a\,dy = ka^2\ ;$$

$\therefore$ $W_{OAB} = ka^2$

$\Rightarrow$ (a) is correct

W_{OCB}: path OC $\Rightarrow x = 0$

$\Rightarrow$ $\overline{F}\,.\,\overline{ds} = k(y^2\hat{i})\,.\,dy\hat{j} = 0$

path CB $\Rightarrow y = a \Rightarrow F = k(a^2\hat{i} + x\hat{j})$

$$\therefore\quad W_{CB} = \int_0^a \overline{F}\,.\,\overline{ds} = \int_b^a k(a^2\hat{i} + x\hat{j}).dx\hat{i}$$

$$= k\int_0^a a^2 dx = ka^3$$

Since the work done is path dependent, the force is non-conservative.

(d) is correct

$\therefore$ (a) and (d) are correct

191. (a) $\overline{F} = (4\hat{i} + 5\hat{j})\,\text{N}$

$W_x = 4\,\hat{i}\,.\,2\hat{i} = 8\text{J}$

$W_y = 5\hat{j}\,.\,2.4\,\hat{j} = 12\ \text{J}$

$W = W_x + W_y = 8 + 12 = 20\ \text{J}$

(a) is wrong

(b) $W_x = 4\hat{i}\,.\,3\hat{i} = 12\ \text{J}$

(b) is wrong

(c) $W_x = 4\hat{i}\,.\,3\text{i} = 12\text{J}$

$\overline{S} = (2\hat{i} + 2.4\,\hat{j}) - (3\hat{i} + 0\hat{j})$

$= (-\hat{i} + 2.4\,\hat{j})$ m

$W_2 = \overline{F}\,.\,\overline{S} = (4\hat{i} + 5\hat{j})\,.(-\hat{i} + 2.4\,\hat{j})$

$= -4 + 12 = 8\ \text{J}$

$\therefore$ $W = W_x + W_2 = 12 + 8 = 20\ \text{J}$

(c) is correct

(d) $\overline{S} = (2\hat{i} + 2.4\hat{j})$

$W = \overline{F}\,.\,\overline{S} = (4\hat{i} + 5\hat{j})\,.\,(2\hat{i} + 2.4\,\hat{j})$

$= (4 \times 2) + (5 \times 2.4) = 8 + 12 = 20\ \text{J}$

(d) is correct.

192. Work done = area under the F – x graph

(a) work done by the force in moving the body from $x = 0$ to $x = 2$ m $W_1 = \frac{1}{2}\text{AD} \times \text{OD}$

$= \frac{1}{2} 8\ \text{N} \times 2\ \text{m} = 8\text{J}$

(a) is correct.

(b) Work done in moving the body from $x = 2$ m to $x = 4$ m

W_2 = area of the rectangle $= 2 \times 8 = 16$ J

(b) is correct.

(c) work done in moving the body from $x = 0$ to $x = 6$ m

W_3 = area of the triangle OAD + area of the rectangle ABD + area of the triangle BEC

$= 8 + 16 + 8 = 32$ J.

(c) is wrong.

(d) is correct.

193. Force of friction (f) $= \mu mg = 0.2 \times 5 \times 10 = 10$ N

Applied force = 20 N

Since friction opposes motion, the net force acting on the body when it is moving is

$F' = F - f = 20 - 10 = 10$ N

$\therefore$ Acceleration $a = \dfrac{F'}{m} = \dfrac{10\ \text{N}}{5\ \text{kg}} = 2\ \text{m s}^{-1}$

The distance travelled by block in 8 s is

$S = ut + \frac{1}{2}at^2 = \frac{1}{2} \times 2 \times (8)^2 = 64\,\text{m}$

$\therefore$ Work done by applied force in 8 s is

(a) W = applied force × distance moved in 8 s

$W = fs = 20\ \text{N} \times 64 = 1280$ J correct.

(b) Work done by the force of friction in 8s is

$W = -f \times s = -10\ \text{N} \times 64 = -640$ J wrong.

(c) Work done by the net force in 8 s is

$W = F's = 10 \times 64 = 640$ J wrong

(d) Velocity acquired by the block in 10 s is

$v = u + at = 0 + 2 \times 10 = 20\ \text{m s}^{-1}$

Kinetic energy of the block at t = 10 s is

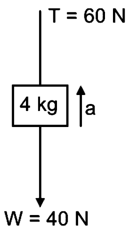
$\text{K.E} = \frac{1}{2} \times 5 \times (20)^2 = \frac{1}{2} \times 5 \times 400 = 1000$ J

Since the initial KE = 0

Change in kinetic energy = 1000 J. Correct

194. Mass of the block = 4 kg

constant force = 60 N



Kinetic energy = 60 J

Free body diagram is shown here

From work energy theorem

$W_{net} = \Delta KE$

$\Rightarrow$ $(60 - 40)\, S = 60$

$20\, S = 60$

$S = \dfrac{60}{20} = 3$ m

Work done by gravity = i.e. $-40 \times 3 = -120$ J

Work done by tension is 60 N $\times$ 3 = 180 J

195. $m_1 = 1$ kg, $u_1 = 4$ m s^{-1}

$m_2 = 2$ kg, $u_2 = +3$ m s^{-1} (or -3 m s^{-1})

$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\, v$

$\therefore \quad v = \dfrac{(m_1 u_1 + m_2 u_2)}{(m_1 + m_2)}$

$\therefore \quad v = \dfrac{(1 \times 4) + (2 \times 3)}{(1 + 2)} = \dfrac{4 + 6}{3} = \dfrac{10}{3}\ \text{ms}^{-1}$

$\Rightarrow$ (b) is correct

or $v = \dfrac{(1 \times 4) - (2 \times 3)}{(1 + 2)} = \dfrac{4 - 6}{3} = \dfrac{-2}{3}$ m s^{-1}

$\Rightarrow$ (c) is correct

Hence magnitude of $v = \dfrac{2}{3}$ ms^{-1}

196. mass = 5 kg velocity = 4 m s^{-1} $m_2 = 2$ kg

$v_2 = 8$ m s^{-1}

$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\, V$

$5 \times 4 + 2 \times 8 = (5 + 2)\, V$

$20 + 16 = 7\, V$

$V = \dfrac{36}{7}$ m s^{-1}

If two bodies moving in opposite directions, collides inelastically then $20 - 16 = 4$

$4 = 7\, V$ i.e., $V = \dfrac{4}{7}$ ms^{-1}

so options (b) and (d) are correct

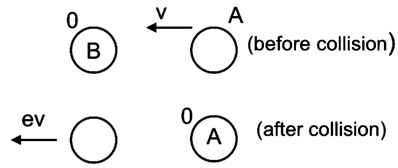
197. Max acceleration, when maximum compression, say x

$F = kx = ma = 1 \times 80 = 80$ N

$E = \dfrac{1}{2}\dfrac{F^2}{K} = \dfrac{1}{2}.\dfrac{80^2}{1000} = 3.2$ J

$\Rightarrow$ (c) is correct

For the collision between A and B the velocities will be as shown.



Since A halts, for conservation of momentum

$0.8\, v = 1 v' \Rightarrow v' = 0.8\, v$

$\dfrac{1}{2} m (v')^2 = \dfrac{1}{2} kx^2 = 3.2$ J

$\therefore \quad \dfrac{1}{2} \times 1 \times (0.8)^2\, v^2 = 3.2$

$\Rightarrow v^2 = \dfrac{6.4}{0.64} = 10$

$\Rightarrow v = \sqrt{10}$ m s^{-1}

$\dfrac{1}{2} mv^2 = mg\ell \Rightarrow \dfrac{1}{2} \times 0.8 \times 10$

$= 0.8 \times 10 \times \ell \Rightarrow$

$\Rightarrow \quad \ell = \dfrac{1}{2}$ m

$\Rightarrow$ (a) is correct

After B returns and hits A, let B come to a stop. Then,

Velocity of separation = ev" = $(0.8 \times 0.8 \times v)$

For conservation of momentum

$1 \times 0.8v = 0.8 \times (0.8)^2\, v \rightarrow$ not possible, hence B will not stop after collision.

$\therefore$ (a) and (c) are correct

198. Total mechanical energy = K

$PE_A = mg(3h) = KE_D = K$

$\therefore \quad mgh = \frac{K}{3}$

$PE_C = mgh = \frac{K}{3}; KE_C$

$= K - \frac{K}{3} = \frac{2}{3}K$

$PE_B = mg(2h) = \frac{2K}{3}; KE_B = K - \frac{2K}{3} = \frac{K}{3}$

$KE_A = 0 = PE_D : \frac{KE_C}{PE_C} = 2$

(a) $KE_B \to p, q$

(b) $PE_C \to p, q$

(c) $\frac{PE_B}{KE_B} = 2 \to r, s$

(d) $\frac{KE_C}{KE_B} = 2 \to r, s$

199. General expression for the velocities v_1 and v_2 after collision are given by

$$v_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1 + \frac{2m_2}{(m_1 + m_2)}u_2$$

$$v_2 = \frac{2m_1}{(m_1 + m_2)}u_1 - \frac{(m_1 - m_2)}{(m_1 + m_2)}u_2$$

In the present case, $m_1 = m_2$. Therefore

$v_1 = u_2$ and $v_2 = u_1$

(a) $\to$ p, q, r

(b) $\to$ p, q, r, s

(c) $\to$ p, q, r, s

(d) $\to$ p, r

Aliter:

$$v_1 = \frac{(m_1 - em_2)u_1 + (1 + e)m_2u_2}{(m_1 + m_2)}$$

$$v_2 = \frac{(m_2 - em_1)u_2 + (1 + e)m_1u_1}{(m_1 + m_2)}$$

e = 1 (for perfectly elastic collision)

if $m_1 = m_2$, $v_1 = v_2$ and $v_2 = u_1$

200. (a) $m_1u_1 = m_1v_1 + m_2v_2$ — (1)

$e = \frac{v_2 - v_1}{u_1}$ — (2)

Combining the two equations

$m_1u_1 = m_1(v_2 - eu_1) + m_2v_2$

$\therefore \quad m_1u_1(1 + e) = (m_1 + m_2)v_2$

$$v_2 = \frac{m_1u_1(1 + e)}{m_1 + m_2}$$

$$= \frac{u_1(1 + e)}{1 + \frac{m_2}{m_1}} \quad \text{— (3)}$$

v_2 is maximum when $m_1 >> m_2$ and e = 1

$\therefore$ (a) $\to$ p, q

(b) From (3),

$$m_2v_2 = \frac{m_1m_2u_1(1 + e)}{m_1 + m_2}$$

$$= \frac{m_1u_1(1 + e)}{1 + \frac{m_1}{m_2}}$$

Given the momentum m_1u_1

Momentum transfer is maximum when

$m_1 << m_2$ and e = 1

$\therefore$ (b) $\to$ p, s

(c) Kinetic energy of the second body is

$$K = \frac{1}{2}m_2\frac{(1+e)^2m_1^2u_1^2}{(m_1 + m_2)^2}$$

$$= \frac{1}{2}m_2(1 + e)^2\frac{u_1^2}{\left(1 + \frac{m_2}{m_1}\right)^2}$$

K is maximum when e = 1, $m_1 >> m_2$

$\therefore$ (c) $\to$ p, q

(d) velocities are equal when

$eu = v_2 - v_1 = 0$

i.e., $\quad e = 0$

$\therefore$ (d) $\to$ r

for m_2:

(velocity maximum = K.E.$_{\text{maximum}}$

CHAPTER 2

ROTATIONAL DYNAMICS

CHAPTER OUTLINE

SYSTEM OF PARTICLES AND CENTRE OF MASS

If a system consists of a large number of particles, having different types of motion, it would be quite complicated and laborious to describe the motion of particles of the system. Rather, it will be more convenient to describe the motion of the entire system, by reducing it to an equivalent single particle, located at a characteristic geometric point of the system, called the 'centre of mass' of the system of particles.

For example, a car has a large number of components like engine, gear box, clutch, axle, wheels, chassis etc. When we say a car is moving on the road at 40 kmph, which part of the car are we referring to? We are reducing the system of particles (or components) of a car into a single particle, located at a geometric point called the 'centre of mass of the car' and describing the motion of that point.

Centre of mass of a system of particles is a geometric point, where the entire mass of the system of particles is assumed to be concentrated and all external forces on the system of particles appear to be applied at that point.

Locating the centre of mass of a two particle system

Consider a system consisting of 2 particles (1 and 2) of mass m_1 and m_2, located at position vectors $\bar{r}_1$ and $\bar{r}_2$ respectively as shown in Fig. 2.1.

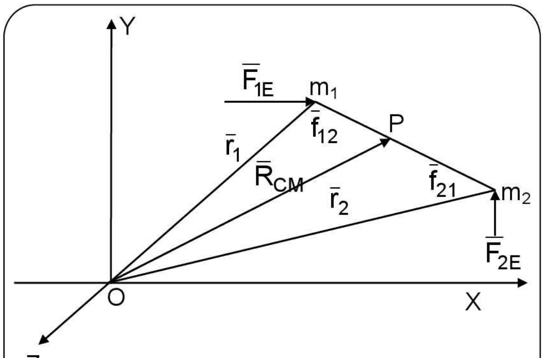


Fig. 2.1

$\bar{f}_{12}$ is the internal force exerted by particle 2 on particle 1 and $\bar{f}_{21}$ is the internal force exerted by particle 1 on particle 2. As $\bar{f}_{12} = -\bar{f}_{21}$ by Newton's third law of motion,

$$\bar{f}_{12} + \bar{f}_{21} = 0 \qquad \text{—(1)}$$

Hence the net internal forces of the system is zero. $\bar{F}_{1E}$ and $\bar{F}_{2E}$ are the external forces on the system, applied at 1 and 2 respectively.

The net external force on system, $\bar{F} = \bar{F}_{1E} + \bar{F}_{2E}$ —(2)

If $\bar{v}_1$ and $\bar{v}_2$ are the velocities of 1 and 2, then

$$\bar{v}_1 = \frac{d\bar{r}_1}{dt} \text{ and } \bar{v}_2 = \frac{d\bar{r}_2}{dt}$$

If $\bar{p}_1$ and $\bar{p}_2$ are the linear momenta of 1 and 2, respectively, then $\bar{p}_1 = m_1 v_1 = m_1 \frac{d\bar{r}_1}{dt} = \frac{d}{dt} m_1 \bar{r}_1$ —(3)

($\because m_1$ = constant)

$$\bar{p}_2 = m_2 v_2 = m_2 \frac{d\bar{r}_2}{dt} = \frac{d}{dt} m_2 \bar{r}_2 \qquad \text{—(4)}$$

($\because m_2$ = constant)

The total force on particle 1, is $\bar{F}_{1E} + \bar{f}_{12}$ while the total force on particle 2, is $\bar{F}_{2E} + \bar{f}_{21}$. As per Newton's second law of motion,

$$\frac{d\bar{p}_1}{dt} = \bar{F}_{1E} + \bar{f}_{12} \text{ and}$$

$$\frac{d\bar{p}_2}{dt} = \bar{F}_{2E} + \bar{f}_{21}$$

$$\therefore \quad \frac{d\bar{p}_1}{dt} + \frac{d\bar{p}_2}{dt} = \bar{F}_{1E} + \bar{f}_{12} + \bar{F}_{2E} + \bar{f}_{21}$$

$$= \bar{F}_{1E} + \bar{F}_{2E} + \left(\bar{f}_{12} + \bar{f}_{21}\right)$$

$$= \bar{F}_{1E} + \bar{F}_{2E} \left(\because \bar{f}_{12} + \bar{f}_{21} = 0 \text{ from } (1)\right)$$

$= \bar{F}$ (from (2)), where $\bar{F}$ = net external force on system of 2 particles

From (3) and (4), we get

$$\frac{d}{dt}(m_1\bar{r}_1) + \frac{d}{dt}(m_2\bar{r}_2) = \bar{p}_1 + \bar{p}_2$$

$$\text{i.e., } \frac{d^2}{dt^2}\left(m_1\bar{r}_1 + m_2\bar{r}_2\right) = \frac{d\bar{p}_1}{dt} + \frac{d\bar{p}_2}{dt} = \bar{F}$$

Multiplying and dividing LHS with $(m_1 + m_2)$, we get

$$\left(m_1 + m_2\right)\frac{d^2}{dt^2}\frac{\left(m_1\bar{r}_1 + m_2\bar{r}_2\right)}{\left(m_1 + m_2\right)} = \bar{F} \qquad \text{—(5)}$$

If we consider that the entire mass of the system $m = (m_1 + m_2)$ is located at a point P, called the centre of mass of the system, whose position vector is $\bar{R}_{cm}$ and the entire external force $\bar{F}$ is assumed to be acting at the centre of mass, then

$$\overline{F} = m\overline{a}_{CM} = m\frac{d^2\overline{R}_{cm}}{dt^2}$$

$$(\because\ \overline{a}_{cm} = \frac{d\overline{v}_{cm}}{dt} = \frac{d^2\overline{R}_{cm}}{dt^2}) \qquad \text{— (6)}$$

Comparing (5) and (6), we get

$$\left(m_1 + m_2\right)\frac{d^2\overline{R}_{cm}}{dt^2} = \left(m_1 + m_2\right)\frac{d^2}{dt^2}\frac{\left(m_1\overline{r}_1 + m_2\overline{r}_2\right)}{\left(m_1 + m_2\right)}$$

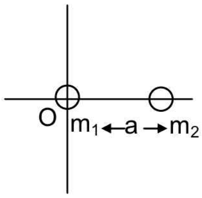
$$\Rightarrow\ \overline{R}_{cm} = \frac{m_1\overline{r}_1 + m_2\overline{r}_2}{\left(m_1 + m_2\right)}$$

Thus the system of two particles can be reduced to a single particle of mass $m = m_1 + m_2$, located at P, whose position is $\overline{R}_{cm} = \frac{m_1\overline{r}_1 + m_2\overline{r}_2}{\left(m_1 + m_2\right)}$. The motion of this system can be described by the motion of the centre of mass P.

CONCEPT STRANDS

Concept Strand 1

Consider a system of two particles, as shown. Find the centre of mass.



Solution

Choosing the origin of the coordinate system at the location of mass m_1.

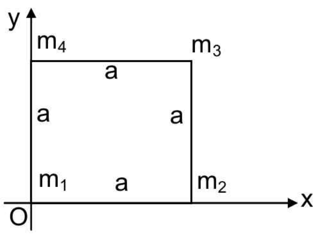
$$X_{cm} = \frac{m_2 a}{m_1 + m_2}$$

Special case: If $m_1 = m_2 = m$,

$$X_{cm} = \frac{a}{2}$$

Concept Strand 2

Consider the following system of four particles having mass m_1, m_2, m_3 and m_4 located at the corners of a square of side a. Find the center of mass of the system with respect to the origin.



Solution

Choosing the origin of the coordinate system at the location of mass m, form a table of coordinates of the four particles as below:

	x	y
m_1	0	0
m_2	a	0
m_3	a	a
m_4	0	a

Then

$$X_{cm} = \frac{m_1 0 + m_2 a + m_3 a + m_4 0}{m_1 + m_2 + m_3 + m_4} = \frac{(m_2 + m_3)a}{m_1 + m_2 + m_3 + m_4}$$

$$Y_{cm} = \frac{m_1 0 + m_2 0 + m_3 a + m_4 a}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(m_3 + m_4)a}{m_1 + m_2 + m_3 + m_4}$$

$\therefore$ Position vector of the center of mass can be written as $\overline{R}_{cm} = X_{cm}\hat{i} + Y_{cm}\hat{j}$

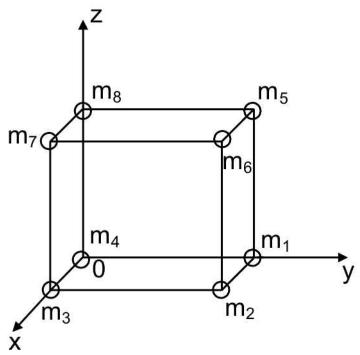
$$\text{i.e., } \overline{R}_{cm} = \left[\frac{(m_2 + m_3)a}{m_1 + m_2 + m_3 + m_4}\right]\hat{i} + \left[\frac{(m_3 + m_4)a}{m_1 + m_2 + m_3 + m_4}\right]\hat{j}$$

If $m_1 = m_2 = m_3 = m_4 = m$, say, $X_{cm} = \frac{a}{2}$ and $Y_{cm} = \frac{a}{2}$.

Hence the centre of mass lies at the geometric centre of the square.

Concept Strand 3

Consider eight particles at the corners of a cube of side 'a'. Find the coordinates of the CM



Solution

Choosing the origin of the coordinate system at the location of particle m_4.

$$X_{cm} = \frac{(m_2 + m_3 + m_6 + m_7)a}{m_1 + m_2 + \dots\dots\dots + m_8}$$

	x	y	z
m_1	0	a	0
m_2	a	a	0
m_3	a	0	0
m_4	0	0	0
m_5	0	a	a
m_6	a	a	a
m_7	a	0	a
m_8	0	0	a

Similarly, $Y_{cm} = \dfrac{(m_1 + m_2 + m_5 + m_6)a}{m_1 + m_2 + \dots\dots + m_8}$

and $Z_{cm} = \dfrac{(m_5 + m_6 + m_7 + m_8)a}{m_1 + m_2 + \dots\dots + m_8}$

As a special case, when all the masses are equal to 'm', we can see that

$X_{cm} = \dfrac{a}{2}$, $Y_{cm} = \dfrac{a}{2}$ and $Z_{cm} = \dfrac{a}{2}$

i.e., the centre of mass will be the geometric centre of the cube.

We have now located the centre of mass of a system of particles in three different cases. The first one is a two particle system which is a one dimensional case. The second one is a four particle system which is a two dimensional case and the last one is an eight particle system which is a three dimensional case. In all these cases, we have seen that when the masses are identical, geometric centre of the mass distributed will be the centre of mass. This is applicable for a continuous distribution of mass also. i.e., if the mass distribution of a regularly shaped body is uniform, the geometric centre will be the centre of mass.

Notes:

(i) The centre of mass of a two particle system lies on the straight line joining the two particles at a location between the two particles.

(ii) The centre of mass of a two particle system is located near the heavier particle, on the line joining the two particles.

(iii) If the particles are of same mass (i.e., $m_1 = m_2 = m$), then $\overline{R}_{cm} = \dfrac{\overline{r}_1 + \overline{r}_2}{2}$. i.e., the centre of mass is located at the centre of the line joining the two masses.

(iv) The term $m_1\overline{r}_1$ is called the moment of mass of particle 1 about the origin of co-ordinate system and $m_2\,\overline{r}_2$ is the moment of mass of particle 2 about the origin.

(v) The geometric locations of the centre of mass (point P) does not depend upon the choice of co-ordinate system. Even if we select any other point as the origin, we will obtain the same point P as the centre of mass of the 2 particle system.

(vi) There may be no mass at the centre of mass of a system. Clearly, there is no physical mass located at P in the example.

(vii) If the origin O of the co-ordinate system is shifted to the centre of mass (i.e., to point P) then $\overline{R}_{cm} = 0 \Rightarrow m_1\overline{r}_1 + m_2\overline{r}_2 = 0$

The sum of moments of all masses of a system, about the centre of mass of a system, is zero.

(viii) Clearly, the expression for centre of mass of the system does not contain any external forces or internal force of the system. Thus, the centre of mass of a system is independent of the forces acting on the particles of the system.

(ix) The centre of mass of a system of particles is 'a mass weighted average of the positions of the particles of the system'.

(x) If the origin of the co-ordinate system coincides with the position of one of the masses, say m_1, there $\bar{r}_1 = 0$. Then $\bar{R}_{cm} = \dfrac{m_2\bar{r}_2}{(m_1 + m_2)}$

Locating the centre of mass of a general multiparticle system

Consider a system of particles, m_1, m_2, m_3...... whose position vectors are $\bar{r}_1, \bar{r}_2, \bar{r}_3, \ldots$ as shown in Fig. 2.2. Then the center of mass of this system of particles is given by the relation

$$\bar{R}_{cm} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + m_3\bar{r}_3 + \ldots}{m_1 + m_2 + m_3 \ldots}$$

$$= \frac{\text{moments of the masses with respect to the origin}}{\text{total mass}}$$

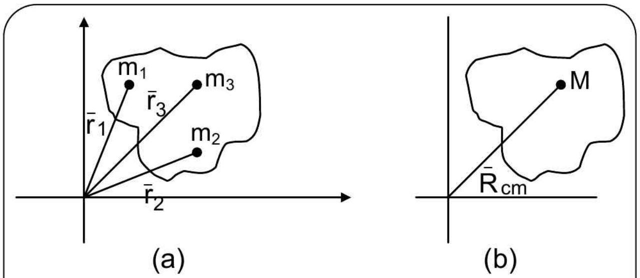


Fig. 2.2

$\bar{r}$ is the position vector of each particle which has x, y and z components. Similarly, the CM also has three components X_{cm}, Y_{cm} and Z_{cm}

$$R_{cm} = |\bar{R}_{cm}| = \sqrt{X_{cm}^2 + Y_{cm}^2 + Z_{cm}^2}, \text{ where}$$

$$X_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{i=N} m_ix_i}{M}$$

Where $M = m_1 + m_2 + \ldots + m_n$

$$Y_{cm} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{(m_1 + m_2 + \ldots + m_n)} = \frac{\sum_{i=1}^{i=n} m_iy_i}{M}$$

$$Z_{cm} = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{(m_1 + m_2 + \ldots + m_n)} = \frac{\sum_{i=1}^{i=N} m_iz_i}{M}$$

Notes:

(i) The centre of gravity of a rigid body is a point, on the body or outside the body, at which the entire weight of the body is considered to be concentrated

(ii) In a region of uniform gravitational field, the centre of mass and centre of gravity of a uniform rigid body will coincide

(iii) If the gravitational field is not uniform, for a uniform rigid body its centre of mass and centre of gravity may not coincide

(iv) The centre of gravity of a rigid body has no meaning in a region where there is no effective gravitational field. However, centre of mass of the rigid body has a definite meaning even in such regions

NEWTON'S LAWS OF MOTION FOR A SYSTEM OF PARTICLES

Consider a system of n particles of masses m_1, m_2, m_3 ......, m_n at position vectors $\bar{r}_1, \bar{r}_2, \bar{r}_3, \ldots$ and $\bar{r}_n$ respectively. Let $\bar{F}_{1ext}, \bar{F}_{2ext}, \ldots \bar{F}_{n\,ext}$ be the external forces of system and $\bar{f}_{1\,int}, \bar{f}_{2\,int}, \ldots$ and $\bar{f}_{n\,int}$ be the internal forces of system on particles 1, 2, ...and n respectively. The velocities of particles are $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$, their linear momentum are $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n$ and accelerations are $\bar{a}_1, \bar{a}_2, \ldots$ and $\bar{a}_n$ respectively.

If we consider that the entire mass of the system $M = m_1 + m_2 + \ldots m_n$ is concentrated at a point, called the centre of mass of the system, then the position vector of the centre of mass

$$\bar{R}_{cm} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + \ldots + m_n\bar{r}_n}{(m_1 + m_2 + \ldots + m_n)}$$

$$= \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + \ldots + m_n\bar{r}_n}{M}$$

$$\therefore \quad M\bar{R}_{cm} = m_1\bar{r}_1 + m_2\bar{r}_2 + \ldots + m_n\bar{r}_2 \qquad \text{— (1)}$$

Hence the sum of the moments of masses about the origin is equal to the moment of the total mass of the system, placed at the centre of mass.

Taking the first derivative of (1) with respect to time, we get

$$\frac{d}{dt}\left(M\overline{R}_{cm}\right)=\frac{d}{dt}\left(m_1\overline{r}_1+m_2\overline{r}_2+m_n\overline{r}_n\right)$$

$$\Rightarrow \quad M\frac{d}{dt}\overline{R}_{cm}=m_1\frac{d\overline{r}_1}{dt}+m_2\frac{d\overline{r}_2}{dt}+\ldots\ldots\ldots\ldots+m_n\frac{d\overline{r}_n}{dt}$$

$$\Rightarrow \quad M\overline{v}_{cm}=m_1\overline{v}_1+m_2\overline{v}_2+\ldots\ldots\ldots\ldots+m_n\overline{v}_n \qquad \text{— (2)}$$

$$\overline{v}_{cm}=\frac{\left(m_1\overline{v}_1+m_2\overline{v}_2+\ldots\ldots\ldots+m_n\overline{v}_n\right)}{M} \qquad \text{— (3)}$$

where $\overline{v}_{cm}$ is the velocity of the centre of mass of the system.

$$v_{cm}=\left|\overline{v}_{cm}\right|=\sqrt{v_{X\,cm}^{\;2}+v_{Y\,cm}^{\;2}+v_{Z\,cm}^{\;2}}\text{, where}$$

$$V\rightarrow \upsilon=\frac{m_1v_{1x}+m_2v_{2x}+\ldots\ldots\ldots+m_nv_{nx}}{M}=\frac{\sum_{i=1}^{i=n}m_iv_{ix}}{M}$$

$$V\rightarrow \upsilon=\frac{m_1v_{1y}+m_2v_{2y}+\ldots\ldots\ldots\ldots\ldots+m_nv_{ny}}{M}=\frac{\sum_{i=1}^{i=n}m_iv_{iy}}{M}$$

and

$$V\rightarrow \upsilon=\frac{m_1v_{1z}+m_2v_{2z}+\ldots\ldots\ldots\ldots\ldots+m_nv_{nz}}{m}=\frac{\sum_{i=1}^{i=n}m_iv_{iz}}{M}$$

If $\overline{p}_{system}$ is the total linear momentum of the system, then

$$\overline{p}_{system}=\overline{p}_1+\overline{p}_2+\ldots\ldots\ldots\ldots\ldots+\overline{p}_n$$

$$=m_1\overline{v}_1+m_2\overline{v}_2+m_n\overline{v}_n \qquad \text{— (4)}$$

comparing (2) and (4) we get

$$M\overline{v}_{cm}=\overline{p}_{system} \qquad \text{— (5)}$$

Taking the first derivative of (2) with respect to time, we get

$$\frac{d}{dt}\left(M\overline{v}_{cm}\right)=\frac{d}{dt}\left(m_1\overline{v}_1+m_2\overline{v}_2+\ldots\ldots\ldots\ldots+m_n\overline{v}_n\right)$$

$$\Rightarrow \quad M\frac{d}{dt}\overline{v}_{cm}=m_1\frac{d}{dt}\overline{v}_1+m_2\frac{d}{dt}\overline{v}_2+\ldots\ldots\ldots+m_n\frac{d}{dt}\overline{v}_n$$

$$\Rightarrow \quad M\overline{a}_{cm}=m_1\overline{a}_1+m_2\overline{a}_2+m_n\overline{a}_n \qquad \text{— (6)}$$

$$\overline{a}_{cm}=\frac{m_1\overline{a}_1+m_2\overline{a}_2+\ldots\ldots\ldots m_n\overline{a}_n}{M}=\frac{\sum_{i=1}^{i=n}m_ia_i}{M} \qquad \text{— (7)}$$

But $m_1\overline{a}_1$ = total force on particle 1 = $\overline{F}_{1\,ext}+\overline{f}_{1int}$

$m_2\overline{a}_2$ = total force on particle 2 = $\overline{F}_{2\,ext}+\overline{f}_{2\,int}$

$m_n\overline{a}_n$ = total force on particle n = $\overline{F}_{n\,ext}+\overline{f}_{2\,int}$

$$\therefore m_1\overline{a}_1+m_2\overline{a}_2+\ldots\ldots\ldots\ldots+m_n\overline{a}_n$$

$$=\overline{F}_{1\,ext}+\overline{f}_{1\,int}+\overline{F}_{2\,ext}+\overline{f}_{2\,int}+\ldots\ldots\ldots+\overline{F}_{n\,ext}+\overline{f}_{n\,int}$$

$$=\left(\overline{F}_{1\,ext}+\overline{F}_{2\,ext}+\overline{F}_{n\,ext}\right)+0=\overline{F}$$

(Net external force on system)

$(\because\ \overline{f}_{1\,int}+\overline{f}_2\ \text{int}+\ldots\ldots\ldots\overline{f}_{n\,int}=0)$

Substituting this value in (6) above, we get

$M\overline{a}_{cm}=\overline{F}$, where $\overline{F}$ is the net external force on the system.

$$M\overline{a}_{cm}=\overline{F} \qquad \text{— (8)}$$

This is *Newton's second law of motion for the system of particles*. The following points are to be noted.

(i) For a system of particles, the centre of mass represents the effective point at which the total mass of the system is considered to be concentrated whose motion is governed by Newton's laws of motion.

(ii) The above law for the system of particles is stated for the acceleration of the centre of mass of the system and not the individual particles of the system. In fact, equation (8) gives no idea about the accelerations of the individual particles of the system.

(iii) The above law is stated for the net external forces on the system and does not include the internal forces.

(iv) The above law holds good only for a closed system (i.e., a system of invariant mass), when there is no addition or removal of mass from the system of particles.

Taking the first derivative of (4) with respect to time, we get

$$\frac{d}{dt}\left(\overline{p}_{system}\right)=\frac{d}{dt}\left(\overline{p}_1+\overline{p}_2+\ldots\ldots\ldots\ldots+\overline{p}_n\right)$$

$$=\frac{d\overline{p}_1}{dt}+\frac{d\overline{p}_2}{dt}+\ldots\ldots\ldots\ldots\ldots+\frac{d\overline{p}_n}{dt}$$

$$=m_1\overline{a}_1+m_2\overline{a}_2+\ldots\ldots\ldots+m_n\overline{a}_n$$

$$=\overline{F}$$

So, rate of change of linear momentum of the centre of mass of the system of particles is equal to the net external force on the system.

If $\overline{F}=0$

Then $\frac{d}{dt}(\overline{p}_{system})=0$

$\Rightarrow \quad \overline{p}_{system}$ = constant (or independent of time)

This is called as the *law of conservation of linear momentum* for a system of particles. Hence, if the net external force on a system of particles is zero, the centre of mass of the system is either at rest $\left(\overline{p}_{system} = 0\right)$ or moves with constant velocity $\left(\overline{p}_{system} = \text{constant}\right)$.

The expression $p^2 = 2 \times \text{mass} \times \text{kinetic energy}$ can be applied to individual particles of a system.

If $KE_1 = \frac{1}{2} m_1 v_1^2, KE_2 = \frac{1}{2} m_2 v_2^2, \ldots\ldots\ldots\ldots$

$KE_n = \frac{1}{2} m_n v_n^2$

are the kinetic energies of the particles of the system, total kinetic energy of the system, $KE = KE_1 + KE_2 + \ldots\ldots KE_n$. Thus only if all the particles of a system are not in motion (i.e., v = 0 for all particles), then only the kinetic energy of the system will be zero. Even if one particle of a system is in motion, the kinetic energy of the system of particles cannot be zero. If all particles of a system are at rest, total kinetic energy and total linear momentum of the system are both zero. However, if the total linear momentum of a system is zero, it is not necessary that the kinetic energy of the system is zero.

For example,

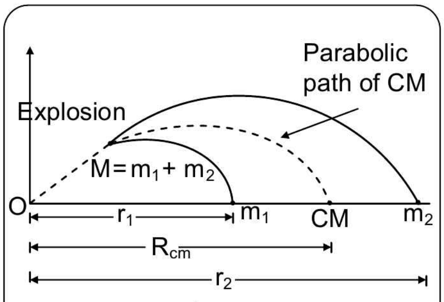


Fig. 2.3

In the explosion of a bomb at rest, for conservation of linear momentum (no external forces are involved in explosion), the total linear momentum of the shrapnels after explosion is zero but their kinetic energy is not zero as they are in motion. The trajectory of a particle is actually the path of its centre of mass. When a projectile explodes, the trajectory of the centre of mass of the fragments remains the same as the original parabolic trajectory of the projectile.

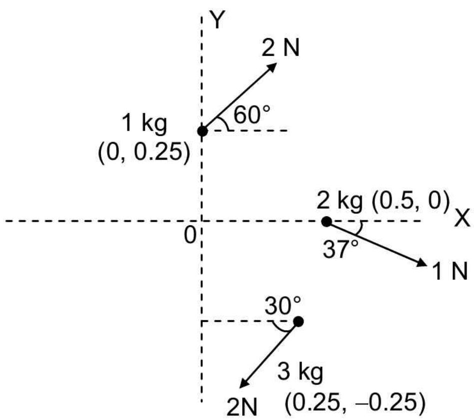
CONCEPT STRAND

Concept Strand 4

Three particles of masses 1 kg, 2 kg and 3 kg are at the points (0, 0.25 m), (0.5 m, 0) and (0.25 m, –0.25 m) respectively in the XY plane at t = 0, and are subject to forces 2 N, 1 N and 2 N as shown in the figure. Discuss the motion of the centre of mass

Solution

The coordinates of the C.M. is



$$X_{cm} = \frac{0 + 0.5 + 0.25}{6} = 0.13\,\text{m}$$

$$Y_{cm} = \frac{0.25 + 0 - 0.25}{6} = 0$$

$\therefore$ C.M is along the X-axis at 0.13 m from the origin. Taking components of forces,

$$F_X = F_{X1} + F_{X2} + F_{X3} = 2\cos 60^\circ + \cos 37^\circ - 2\cos 30^\circ$$
$$= 0.068\text{ N}$$

$$F_Y = F_{Y1} + F_{Y_2} + F_{Y3} = 2\sin 60^\circ - \sin 37^\circ - 2\sin 30^\circ$$
$$= 0.132\text{ N}$$

$$\therefore \quad a_{Xcm} = \frac{F_X}{M} = \frac{0.068}{6} = 0.011\text{ m s}^{-2}$$

$$a_{Ycm} = \frac{F_Y}{M} = \frac{0.132}{6} = 0.22\text{ m s}^{-2}$$

Therefore, the motion of the system at t = 0 is equivalent to a mass 6 kg at (0.13, 0) subject to acceleration as given above.

Resultant acceleration of CM, $a = \sqrt{a_{Xcm}^2 + a_{Ycm}^2}$

$$= \sqrt{(0.011)^2 + (0.22)^2}$$

$$= \sqrt{0.0485} = 0.220\text{ m s}^{-2}$$

C-FRAME AND ITS SIGNIFICANCE

A frame of reference, whose origin is at the centre of mass of a system, is called a C–frame or centre of mass frame of reference. In this frame, $\overline{R}_{cm} = 0$ and hence $\overline{v}_{cm} = 0$. Hence, the centre of mass is at rest in this frame and consequently $\overline{p}_{cm} = M\overline{v}_{cm} = 0$ in this frame i.e., the total linear momentum of the system in the C–frame is zero.

The concept of C–frame is ideal in solving problems in which the relative motion of particles within a system are required to be analysed and not the motion of the entire system. If the net external force acting on a system of particles is zero with respect to any inertial frame, then the C–frame also will be inertial. However, if the net external force acting on a system of particles is not zero with respect to an inertial frame, then the C–frame will be non-inertial

If K = Kinetic energy of a system of particles with respect to an inertial frame.

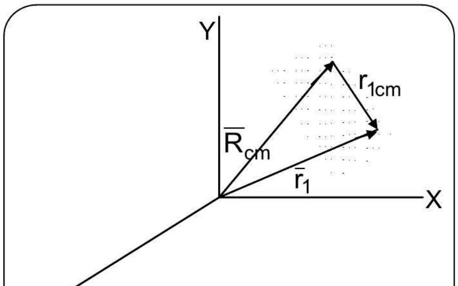


Fig. 2.4

From Fig. 2.4, $\overline{r}_1 = \overline{R}_{cm} + \overline{r}_{1cm}$

$\therefore \quad \overline{v}_1 = \overline{v}_{cm} + \overline{v}_{1cm}$

$\therefore \quad \overline{v}_{1cm} = \overline{v}_1 - \overline{v}_{cm}$

$$K = \frac{1}{2}m_1|\overline{v}_1|^2 + \frac{1}{2}m_2|\overline{v}_2|^2 + ... + \frac{1}{2}m_n|\overline{v}_n|^2,$$

K_C = Kinetic energy of a system of particles with respect to C–frame

$$= \frac{1}{2}m_1|\overline{v}_{1cm}|^2 + \frac{1}{2}m_2|\overline{v}_{2cm}|^2 + ...\frac{1}{2}m_n|\overline{v}_{ncm}|^2$$

$$= \frac{1}{2}m_1|(\overline{v}_1 - \overline{v}_{cm})|^2 + \frac{1}{2}m_2|\overline{v}_2 - \overline{v}_{cm}|^2 + ... + \frac{1}{2}m_n|(\overline{v}_n - \overline{v}_{cm})|^2$$

$\overline{p}_{system}$ = total linear momentum of the system about an inertial frame

M = total mass of system, then

$$K = K_C + \frac{p^2_{system}}{2M}$$

Notes:

(i) If total linear momentum of a system about an inertial frame is zero (i.e., $\overline{p}_{system} = 0$), then $K = K_C$ i.e., the total kinetic energy of the system of particles about an inertial frame is equal to the kinetic energy of system of particles about the C–frame.

(ii) From the above, it is clear that the kinetic energy of a system of particles will be minimum with respect to C–frame.

(iii) Kinetic energy of a system of particles about an inertial frame is zero, only if the kinetic energy of the system of particles about C–frame is zero and the total linear momentum of system about the inertial frame is zero i.e., K = 0, only if $K_C = 0$ and $\overline{p}_{system} = 0$.

(iv) If the total linear momentum of a system of particles about an inertial frame is zero (i.e., $\overline{p}_{system} = 0$), then it is not necessary that the total kinetic energy of the system about the inertial frame is zero because K_C may or may not be zero. i.e., $\overline{p}_{system} = 0$ does not imply K = 0, unless $K_C = 0$. or K_C to become zero, all the particles of the system must be at rest.

Reduced mass of a two particle system

Consider a two particle system of masses m_1 and m_2 having velocities $\overline{v}_1$ and $\overline{v}_2$ respectively with respect to an inertial frame. Then, the velocity of the centre of mass of the system, as seen earlier, is

$$\overline{v}_{cm} = \frac{m_1\overline{v}_1 + m_2\overline{v}_2}{m_1 + m_2} \qquad \text{— (1)}$$

Velocity of 1 with respect to CM = $\overline{v}_{1cm} = \overline{v}_1 - \overline{v}_{cm}$

$$= \overline{v}_1 - \left[\frac{m_1\overline{v}_1 + m_2\overline{v}_2}{(m_1 + m_2)}\right] = \frac{m_2(\overline{v}_1 - \overline{v}_2)}{(m_1 + m_2)}$$

Velocity of 2 with respect to CM = $\overline{v}_{2cm} = \overline{v}_2 - \overline{v}_{cm}$

$$= \bar{v}_2 - \left[\frac{m_1\bar{v}_1 + m_2\bar{v}_2}{(m_1 + m_2)}\right] = \frac{m_1(\bar{v}_2 - \bar{v}_1)}{(m_1 + m_2)}$$

Linear momentum of 1 with respect to CM,

$$\bar{p}_{1\,cm} = m_1\bar{v}_{1\,cm} = \frac{m_1m_2(\bar{v}_1 - \bar{v}_2)}{(m_1 + m_2)}$$

We now define a quantity

$$\mu = \frac{m_1m_2}{(m_1 + m_2)}$$

which is called the reduced mass of the system

$\bar{p}_{1\,cm} = |\bar{p}_{1\,cm}| = \mu v_{rel}$, when $v_{rel} = |\bar{v}_1 - \bar{v}_2|$ is the magnitude of the relative velocity of the particles

Similarly, linear momentum of particle 2 with respect to CM,

$$\bar{p}_{2cm} = m_2\bar{v}_{2cm}$$

$$= \frac{m_1m_2(\bar{v}_2 - \bar{v}_1)}{(m_1 + m_2)} = \mu(\bar{v}_2 - \bar{v}_1)$$

$$p_{2cm} = |\bar{p}_{2cm}| = \mu v_{rel}, \text{ where, } v_{rel} = |\bar{v}_2 - \bar{v}_1|$$

$\therefore \quad \bar{p}_{1cm} = -\bar{p}_{2cm}$ and

$p_{1cm} = p_{2cm} = \mu v_{rel}$

Kinetic energy of particle 1 with respect to CM,

$$K_{1cm} = \frac{1}{2}m_1|\bar{v}_{1cm}|^2 = \frac{p^2_{1cm}}{2m_1}$$

Kinetic energy of particle 2 with respect to CM,

$$K_{2cm} = \frac{1}{2}m_2|\bar{v}_{2cm}|^2 = \frac{p^2_{2cm}}{2m_2}$$

Hence total kinetic energy of the particles in C–frame

$$K_C = K_{1\,cm} + K_{2\,cm}$$

$$= \frac{p^2_{1cm}}{2m_1} + \frac{p^2_{2cm}}{2m_2} = \frac{(\mu v_{rel})^2}{2m_1} + \frac{(\mu v_{rel})^2}{2m_2}$$

$$= \frac{m_2\mu^2v^2_{rel} + m_1\mu^2v^2_{rel}}{2m_1m_2}$$

$$= \frac{\mu^2v^2_{rel}[m_1 + m_2]}{2m_1m_2} = \frac{1}{2}\frac{\mu^2v^2_{rel}}{\mu} = \frac{1}{2}\mu v^2_{rel}$$

$$\therefore \quad K_C = \frac{1}{2}\mu v^2_{rel}$$

Hence the motion of the two particle system is equivalent to the motion of a single particle of mass equal to the reduced mass of the system μ, having a velocity equal to $\bar{v}_{rel}$. If the two particles of the system interact (or exert force on each other) the total mechanical energy of the system (E_C) in the C–frame will be $E_C = U + K_C$, where U = potential energy due to interaction of the particles. Recall the expression for the kinetic energy lost in a one-dimensional collision.

$\Delta KE = \frac{1}{2}\frac{m_1m_2}{(m_1 + m_2)}|\bar{u}_1 - \bar{u}_2|^2(1-e)^2$ can be written as

$$\Delta KE = \frac{1}{2}\mu|\bar{v}_{rel}|^2(1-e)^2$$

RIGID BODY

A rigid body can be considered as a continuous distribution of a system of particles. In a rigid body, the particles are arranged so that the distance between the particles is fixed, A rigid body does not undergo changes in shape or size under the action of external forces. However, there is no such thing as a perfectly rigid body; when the forces are small, they do not change the distances between particles. Therefore, they do not cause any macroscopic change in shape or size of the rigid body. The entire mass of the body may be considered to be concentrated at the centre of mass of the body.

Locating the centre of mass of a rigid body

For a rigid body, if it is given that the mass distribution is uniform, by locating its geometric centre, we can locate its centre of mass. The centre of mass of some rigid bodies having uniform mass distribution is given in the table below.

Centre of mass of rigid bodies

Table 2.1

Sl. No.	Shape of body	Position of centre of mass
1	Uniform rod	Mid point of the rod
2	Circular ring	Centre of ring
3	Circular disc	Centre of disc
4	Cubical block	Point of intersection of diagonals

Sl. No.	Shape of body	Position of centre of mass
5	Cylinder	Centre of cylinder
6	Solid cone	3/4th of height of cone from apex, on its axis
7	Triangular lamina	Point of intersection of the medians

For a continuous distribution of mass (i.e., for a rigid body) the position of the centre of mass can be determined by integration method.

$$\bar{R}_{cm} = \frac{\text{moments of the masses with respect to the origin}}{\text{total mass}}$$

$$= \frac{\int \bar{r}\,dm}{M}$$, where M = mass of rigid body, $\bar{r}$ is the position vector of an elemental mass 'dm'. The co-ordinates of the centre of mass can be written as

$$X_{cm} = \frac{1}{M}\int x\,dm,\ Y_{cm} = \frac{1}{M}\int y\,dm \text{ and } Z_{cm} = \frac{1}{m}\int z\,dm$$

In certain cases, we may have to use both the ideas i.e., the geometric centre and the expression for the position co-ordinates of the centre of mass to locate the centre of mass. The centre of mass of a rigid body may be within the body, outside the body or on its surface. Also there may be no mass at the centre of mass of a rigid body.

(Example, Hollow sphere, ring etc)

CONCEPT STRANDS

Concept Strand 5

Locate the centre of mass of a uniform ring of radius R.

Solution

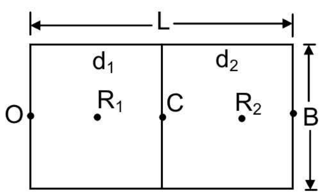
The ring being a symmetric body, for each mass element at one point there is a corresponding mass element diametrically opposite to it. Therefore, the centre of mass will be the geometric centre, although it is outside the body.

Concept Strand 6

A uniform thin lamina of length L and breadth B is made up of two equal parts of different densities d_1 and d_2 ($d_1 > d_2$). Find the distance of its centre of mass from its geometric centre.

Solution

Each half will have a length $\frac{L}{2}$. The centre of mass of each half will be its geometric centre. Let these be R_1 and R_2. Then, from the point O, the distances of R_1 and R_2 are $\frac{L}{4}$ and $\frac{3L}{4}$, respectively.



Let M_1 and M_2 be the masses of the two halves. Then the CM is at

$$\frac{M_1\frac{L}{4} + M_2\frac{3L}{4}}{M_1 + M_2} = \frac{Ad_1\frac{L}{4} + Ad_2\frac{3L}{4}}{Ad_1 + Ad_2} = \frac{L(d_1 + 3d_2)}{4(d_1 + d_2)}$$

The distance of the CM from the geometric centre is

$$D = \frac{L}{2} - \frac{L}{4}\left(\frac{d_1 + 3d_2}{d_1 + d_2}\right)$$

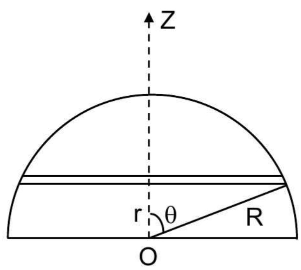
and it is closer to O than the geometric centre.

Concept Strand 7

Locate the C.M of a uniform semicircular disc.

Solution

Let R be the radius of the semicircular disc and ρ the uniform density of the material of the disc. Since the disc is symmetric about the Z–axis, the C.M will lie along the Z-axis near to the point O as more mass is concentrated towards the base. Consider a thin strip at a distance r from the point O. Then the C.M is given by



$$H = \frac{\int r dm}{\int dm} = \frac{\int_0^R \rho.2\sqrt{R^2 - r^2}.rdr}{\int_0^R \rho.2\sqrt{R^2 - r^2}\,dr}$$

We now make the substitution

$r = R\cos\theta$

$dr = -R\sin\theta d\theta$

when $r = 0, \theta = \frac{\pi}{2}$ and when $r = R, \theta = 0$

$$\therefore H = \frac{\int_{\pi/2}^{0} R\sin\theta.R\cos\theta(-R\sin\theta)d\theta}{\int_{\pi/2}^{0} R\sin\theta(-R\sin\theta)d\theta} = R$$

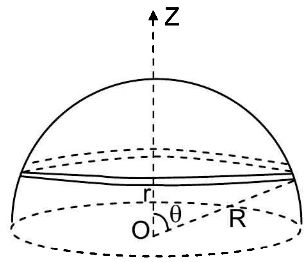
$$\frac{\int_{\pi/2}^{0} \sin^2\theta\cos\theta d\theta}{\int_{\pi/2}^{0} \sin^2\theta d\theta} = \frac{4R}{3\pi}$$

Concept Strand 8

Locate the centre of mass of a uniform hemisphere.

Solution

Let R be the radius of the hemisphere and ρ the uniform density of the material of the hemisphere. Consider a thin annular disc of thickness dr at a height r from the centre of the base.



Since the hemisphere is symmetric about the Z-axis, it is obvious that the C.M will lie along the Z-axis. And since more mass is concentrated towards the lower part of the hemisphere, it is clear that the C.M will be near the origin along the Z-axis. We now have, for the height H at which the C.M is located,

$$H\int dm = \int r dm$$

We now make the substitution

$dm = \rho.\pi(R^2 - r^2)dr$

$$\therefore \quad H = \frac{\int_0^R (R^2 - r^2)\,r.dr}{\int_0^R (R^2 - r^2)dr}$$

$$= \frac{R^2\int_0^R rdr - \int_0^R r^3dr}{R^2\int_0^R dr - \int_0^R r^2dr} = \frac{3}{8}R$$

Motion of a rigid body

For translational motion of a rigid body, we can assume that the entire mass of the body is concentrated at the centre of mass.

When a force is applied on a rigid body, it may either move as a whole in any direction, or may turn or rotate, or may undergo both motions simultaneously. For translational motion to occur, the force must be applied at the centre of mass of the body. If the force acts at a point other than the centre of mass, the resulting motion will be either rotational or both translational and rotational.

When a rigid body is in translational motion, all elements (or particles) of the body move with the same velocity. The paths of the particles are parallel lines, either straight or curved but their paths are not closed paths.

When a rigid body is in rotational motion, different elements (particles) of the body move with different speeds but all elements move with the same angular velocity. The paths of the particles are concentric circles (i.e., closed lines).

Rotation of a rigid body

(i) Axis of rotation

Consider a light rectangular block lying on a smooth table. A force F is applied on the block (as shown in the figure) such that its line of action is away from the centre of mass. It is common experience, then, that the body will rotate such that all the particles of the body move along circles whose centres lie on a straight vertical line. This line is called the "axis of rotation". It is also obvious that a given point in the body moves in a plane perpendicular to the axis of rotation. This plane is called the plane of rotation.

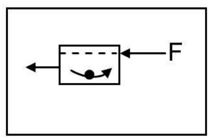


Fig. 2.5

(ii) Moment of inertia of a rigid body

The opposition of a particle/system of particles/rigid body to its state of rotation is called the rotational inertia (I) of that particle/system of particles/rigid body. While the opposition to the translational motion of a particle/body depends on the mass of the particle/body, its rotational inertia (I) depends on the following:

(a) Axis of rotation
The same particle/body can have different rotational inertia about different axes of rotation.

(b) Mass/masses of the particles which comprise the system of rotating body.

(c) The distribution of the masses of the particles of the system with respect to the axis of rotation. (i.e., the shape of the rotating body)

The rotational inertia of a particle/body about an axis of rotation is called the *moment of inertia (I)* of that particle/body about that axis of rotation.

For a particle of mass 'm' the moment of inertia is given by $I = mr^2$, where r = distance of particle from the axis of rotation. For a system of particles,

$$I = \sum_{i=1}^{i=n} m_i r_i^2$$

and for a rigid body or continuous mass distribution,

$$I = \int r^2 dm,$$

when dm is a small mass element located at a perpendicular distance 'r' from the axis of rotation. The SI unit of moment of inertia (I) is kg m^2 and its dimensional formula is ML^2. Moment of inertia is neither a vector nor a scalar but a tensor. The moment of inertia (I) in rotational motion is analogous to mass (m) in translational motion.

(iii) Moment of inertia and angular momentum

Just as for translational motion we define the linear momentum as $\overline{p} = m\dfrac{d\overline{r}}{dt}$, the angular momentum of the body about the axis of rotation as for rotational motion we define

$$L = I\frac{d\theta}{dt} = I\omega = \sum_i m_i r_i^2 \omega$$

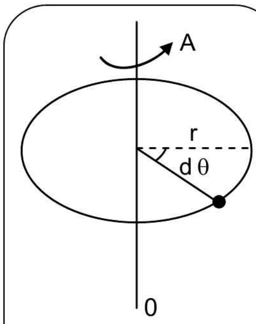


Fig. 2.6

Moment of inertia (M.I.) of some regular bodies

(i) M.I. of a ring about the axis passing through its centre and perpendicular to its plane:

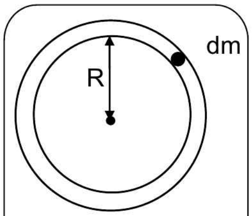


Fig. 2.7

Consider a mass element dm on the ring of radius R.

$$M.I = \int R^2 dm = R^2 \int dm = R^2 m$$

$$\therefore \quad I = mR^2$$

(ii) M.I. of a thin hollow cylinder about its axis.
Consider a ring of width $d\ell$ at a distance ℓ from one end of a thin hollow cylinder of mass m and length L. Its mass is $\dfrac{m}{L}d\ell$.

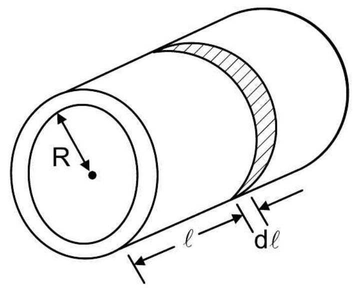


Fig. 2.8

$$\text{M. I.} = \int \frac{m}{L} d\ell \; R^2 = \frac{mR^2}{L}\int d\ell$$

$$= \frac{mR^2}{L}.L$$

$\therefore \quad I = mR^2$

The M.I. is the same as that for the ring.

(iii) M.I. of a thin disc about an axis perpendicular to the plane of the disc and passing through its centre. Consider a ring of width dr at a radius r of a disc of mass m and radius R.

Its mass is $\frac{m}{\pi R^2}(2\pi r dr)$

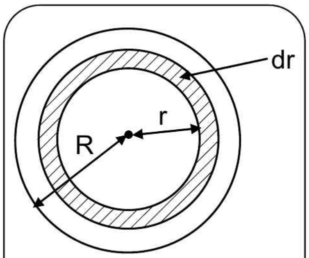


Fig. 2.9

$$\text{M.I.} = \int \frac{m}{\pi R^2}\,(2\pi r dr).\, r^2$$

$$= \frac{2m}{R^2}\int r^3 dr = \frac{2m}{R^2}\frac{R^4}{4}$$

$$\therefore \quad I = \frac{mR^2}{2}$$

(iv) M.I. of a solid cylinder about its axis:
Consider a solid cylinder of mass m length L and radius R, respectively. It can be thought as made up of a large number of discs as shown:

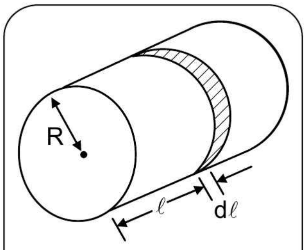


Fig. 2.10

$$\text{M. I.} = \frac{mR^2}{2}\int \frac{d\ell}{L}$$

$$\therefore \quad I = \frac{mR^2}{2}$$

The M.I. is the same as that for the disc.

(v) M.I. of a thin rod about an axis perpendicular to its length and passing through its mid point. Consider a rod of mass m and length ℓ. Consider an element of length dx at a distance x from the axis. Its mass is $\frac{m}{\ell}$ dx.

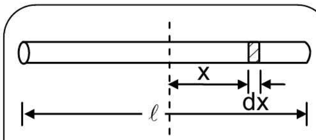


Fig. 2.11

$$\text{M.I.} = \int_{-\ell/2}^{+\ell/2} \frac{m\,dx}{\ell}\, x^2 = \left.\frac{mx^3}{3\ell}\right|_{-\ell/2}^{\ell/2} \quad \therefore I = \frac{m\ell^2}{12}$$

(vi) M.I. of a square plate about an axis perpendicular to its edge and passing through the centre. A square plate has sides a and thickness b. Consider an element of mass $\frac{m}{a^2 b}$ ab dx as shown.

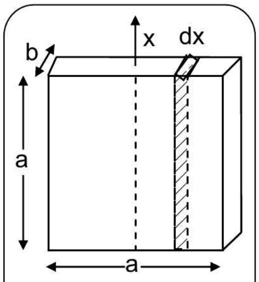


Fig. 2.12

$$\text{M.I.} = \int_{-a/2}^{+a/2} \frac{m}{a^2 b}.ab\,dx.x^2 = \left.\frac{mx^3}{3a}\right|_{-a/2}^{+a/2}$$

$$\therefore I = \frac{ma^2}{12}$$

The M.I is the same as that for a rod.

(vii) M.I of a solid sphere about a diameter.

Consider a solid sphere of radius R and mass m.

Consider an elemental disc subtending an angle $d\theta$ at the centre.

Its volume is π (R sin θ)^2. R sinθ dθ.

Its mass is $\dfrac{m}{\frac{4}{3}\pi R^3}$. $\pi R^3 \sin^3\theta\, d\theta$.

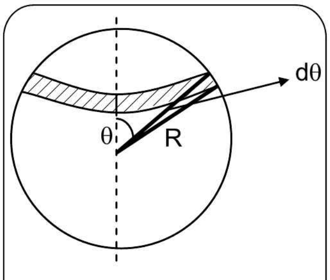


Fig. 2.13

$$\therefore \quad \text{M.I.} = \int_0^{\pi} \frac{3m}{4}.\sin^3\theta.\frac{(R\sin\theta)^2}{2}d\theta$$

$$= \frac{3}{8}mR^2 \int_0^{\pi} \sin^5\theta\, d\theta = \frac{3}{8}mR^2.\frac{16}{15}$$

$$\therefore \quad I = \frac{2}{5}mR^2$$

(viii) M.I. of a thin spherical shell about a diameter.

Consider a thin shell of mass m and radius R. Consider a thin ring subtending an angle $d\theta$ at the centre. Its surface area is $2\pi R \sin\theta$. R dθ and the mass is $\dfrac{m}{4\pi R^2}$ $2\pi R^2 \sin\theta\, d\theta$.

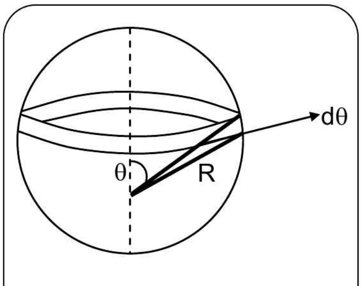


Fig. 2.14

$$\therefore \quad \text{M.I.} = \int_0^{\pi} \frac{m}{2}.\sin\theta\,d\theta.(R\sin\theta)^2 = \frac{mR^2}{2}\int_0^{\pi}\sin^3\theta\,d\theta$$

$$\therefore \quad I = \frac{2}{3}mR^2$$

Parallel axes theorem

In the calculation of the M. I. certain theorems have been found to be extremely useful. *The moment of inertia of a body about any axis is equal to the sum of its moments of inertia about a parallel axis passing through the centre of mass of the body and the product of the mass of the body and the square of the distance between the axes.*

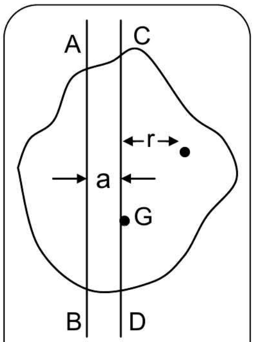


Fig. 2.15

Let G be the centre of mass of a body of mass m. Let AB be an axis and CD an axis parallel to AB, passing through G. Let a be the distance between the axes (Fig. 2.15). If I and I_0 are the moments of inertia of the body about the axes AB and CD respectively, by the theorem of parallel axes, $I = I_o + ma^2$

The significance of centre of mass of a body for rotational motion is clear from the parallel axes theorem. The moment of inertia of a body about an axis of rotation will be minimum only if that axis passes through the centre of mass of the body. It should also be rotated that for a rigid body, the moment of inertia about an axis of rotation that passes through its centre of mass is not zero.

Perpendicular axes theorem

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about two axes at right angles to each other in the plane of the lamina and passing through the point where the perpendicular axis intersects the lamina.

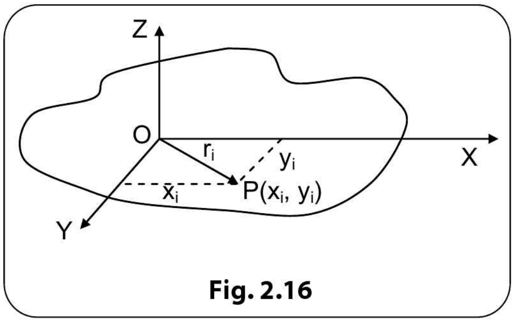


Fig. 2.16

Proof:

Consider a particle P of mass m_i at a distance r_i from the origin of a co-ordinate system set up on the plane lamina with the axes OX and OY in the plane of the lamina and OZ in a direction perpendicular to the plane of the lamina (Fig. 2.16).

M.I. of the particle about OX = $m_i' y_i^2$. M.I. of the lamina about OX = $\sum m_i' y_i^2$.

Similarly, M.I. of the lamina about OY = $\sum m_i' x_i^2$. Now, M.I. of the lamina about an axis through O and perpendicular to the plane of the lamina = $\sum m_i' r_i^2$

$$= \sum m_i' (x_i^2 + y_i^2) = \sum m_i' x_i^2 + \sum m_i' y_i^2$$

i.e., $I = I_x + I_y$

Note that the three axes considered do not have to pass through the center of mass of the body. It must be noted that perpendicular axes theorem is valid only for plane lamina.

CONCEPT STRANDS

Concept Strand 9

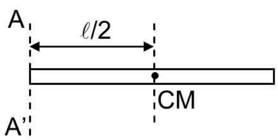
Calculate the M.I. of a thin rod about an axis passing through one end of the rod and perpendicular to its length.

Solution

$$I_{AA'} = I_{CM} + m\left(\frac{\ell}{2}\right)^2$$

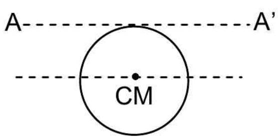
$$= m\frac{\ell^2}{12} + m\frac{\ell^2}{4}$$

$$\text{ie., } I_{AA'} = \frac{m\ell^2}{3}$$



Concept Strand 10

Calculate the M.I. of a solid sphere about a tangent.



Solution

$$I_{AA''} = I_{CM} + mR^2$$

$$= \frac{2}{5} mR^2 + mR^2$$

$$I_{AA''} = \frac{7}{5} mR^2$$

Concept Strand 11

M.I. of a thin ring about a diameter can be found from the expression for M.I. about an axis perpendicular to its plane by using the perpendicular axis theorem.

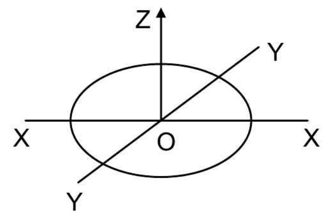
Solution

Since, by symmetry, $I_{xx} = I_{yy} = I$

$I_{zz} = I_{xx} + I_{yy}$

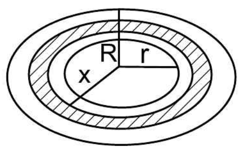
$mR^2 = I + I$

$$\therefore I = \frac{mR^2}{2}$$



Concept Strand 12

Calculate the M.I. of an annular disc about an axis passing through its centre and perpendicular to its plane.



Solution

Mass of the annular disc = $\frac{m}{\pi(R^2 - r^2)}.2\pi x.\,dx$

M.I. of the disc about an axis through the centre and perpendicular to the plane of the disc

$$I = \int_r^R \frac{2m}{(R^2 - r^2)} x.\, x^2 dx = \frac{2m}{2(R^2 - r^2)}(R^4 - r^4)$$

$$\therefore I = \frac{m(R^2 + r^2)}{2}$$

Concept Strand 13

Calculate the MI of an annular disc about a diameter.

Solution

MI of an angular disc can be obtained from the expression for MI about the perpendicular axis passing through the centre.

By the perpendicular axis theorem and by using circular symmetry,

$$I_{AA} + I_{BB} = I_{CC} \quad \text{i.e., } 2I_D = \frac{m(R^2 + r^2)}{2}$$

$$\therefore \quad \text{MI about a diameter } I_D = \frac{m(R^2 + r^2)}{4}$$

Radius of gyration

If 'I' is the moment of inertia of a body of mass M about an axis of rotation such that $I = MK^2$, then K is called the radius of gyration of the body about that axis of rotation

i.e., $I = \Sigma m_i r_i^2 = MK^2$

The SI unit of radius of gyration is metre (m). The following points shall be noted:

1. Radius of gyration of a body is dependent on the axis of rotation. Hence for different axes of rotation, the radius of gyration of the body will be different
2. The radius of gyration of a rigid body is independent of the mass of the body but it depends upon the distribution of the mass about the axis of rotation

Consider a rigid body of mass M to be made of n identical particles, each of mass m

$\therefore \quad M = nm$

Let $r_1, r_2, \ldots r_n$ be the distances of these particles from the axis of rotation

Then $I_1 = mr_1^2, I_2 = mr_2^2, \ldots I_n = mr_n^2$ are the moment of inertia of these particles about the axis of rotation.

The total moment of inertia 'I' of the body about the axis of rotation is given by

$I = I_1 + I_2 + \ldots + I_n$

$= mr_1^2 + mr_2^2 + \ldots + mr_n^2$

$= m[r_1^2 + r_2^2 + \ldots + r_n^2]$ — (i)

But $I = MK^2$ ($\because M = nm$)

$= nmK^2$ — (ii) where K = radius of gyration

From (i) and (ii), we get

$nmK^2 = m[r_1^2 + r_2^2 + \ldots + r_n^2]$

$$\therefore K^2 = \frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}$$

$K = \sqrt{\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}}$, independent of m or M. Thus radius of gyration of a rigid body about an axis of rotation is the square root of the mean of the sum of the square distances of the particles of the body.

CONCEPT STRANDS

Concept Strand 14

Calculate the radius of gyration of a solid sphere of mass M and radius R, about its diameter.

Solution

$$I = \frac{2}{5}MR^2 \text{ and } I = MK^2$$

$$\Rightarrow \quad K^2 = \frac{\frac{2}{5}MR^2}{M} = \frac{2}{5}R^2$$

$$\Rightarrow \quad K = \sqrt{\frac{2}{5}}R$$

Concept Strand 15

A circular disc of mass 6 kg has a M.I of 6×10^{-2} kg m^2 about a diameter. What is its radius of gyration about an axis of rotation perpendicular to the plane of the disc and passing through its centre?

Solution

By using the perpendicular axes theorem

$$I = \frac{MR^2}{2} = 2I_d = 2 \times \frac{MR^2}{4} = 12 \times 10^{-2} \text{ kg m}^2$$

$$K = \sqrt{\frac{I}{M}} = \sqrt{\frac{12 \times 10^{-2}}{6}} = \frac{\sqrt{2}}{10} \text{ m}$$

TORQUE

The rotating effect of a force about a point (or axis of rotation) is called the torque. It is the rotational analogue of force in translational motion.

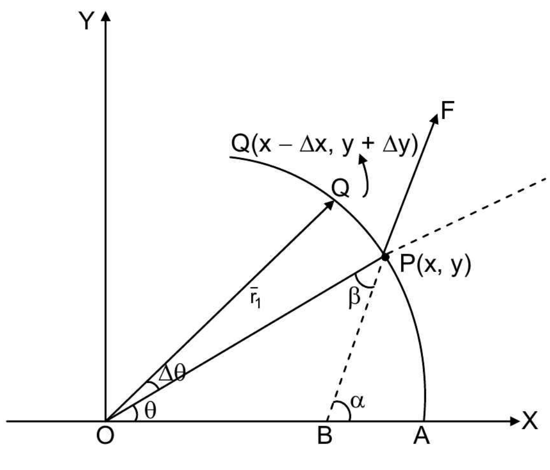


Fig. 2.17

Consider a particle of mass 'm' at a point P(x, y) in the XY plane. Its position vector $\overline{OP} = \bar{r}_1$ makes an angle θ with the X-axis.

$$\bar{r} = x\hat{i} + y\hat{j} \text{ and } |\bar{r}_1| = r$$

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Let a force $\bar{F} = F_x\hat{i} + F_y\hat{j}$ act on the particle at P,

so that the line of action of $\bar{F}$ makes on angle α with the X-axis as shown in Fig. 2.17. The angle between $\bar{r}$ and $\bar{F}$ is β so that θ + β = α (from ΔOPB).

Let this force $\bar{F}$ rotate the particle in the XY plane with radius r so that the particle undergoes a small angular displacement of Δθ and moves to new position.

Q(x – Δx, y + Δy). Its position vector $\overline{OQ} = \bar{r}_2$ and $|\bar{r}_2| = |\bar{r}_1| = r$.

$\therefore$ $x - \Delta x = r\cos(\theta + \Delta\theta) = r\cos\theta\cos\Delta\theta - r\sin\theta\sin\Delta\theta$

$= r\cos\theta - r\sin\theta\,\Delta\theta$ ($\because \cos\Delta\theta \simeq 1$ and $\sin\Delta\theta \simeq \Delta\theta$)

Similarly, $y + \Delta y = r\sin(\theta + \Delta\theta) = r\sin\theta\cos\Delta\theta + r\cos\theta\sin\Delta\theta$

$= r\sin\theta + r\cos\theta\,\Delta\theta$

$\therefore$ $\Delta x = x - (x - \Delta x) = r\cos\theta - (r\cos\theta - r\sin\theta\,\Delta\theta)$

$= r\sin\theta\,\Delta\theta$

$= y\,\Delta\theta$ ($\because r\sin\theta = y$)

Similarly, $\Delta y = (y + \Delta y) - y = (r\sin\theta + r\cos\theta\Delta\theta) - r\sin\theta$

$= r\cos\theta\,\Delta\theta = x\Delta\theta$

Displacement

$$\overline{\Delta r} = \bar{r}_2 - \bar{r}_1 = \left[(x - \Delta x)\hat{i} + (y + \Delta y)\hat{j}\right] - (x\hat{i} + y\hat{j})$$

$$= -\Delta x\hat{i} + \Delta y\hat{j}$$

Small amount of work done by the applied force

$$\Delta W = \bar{F}\,.\,\overline{\Delta r}$$

$$= (F_x\hat{i} + F_y\hat{j}).(-\Delta x\bar{i} + \Delta y\hat{j}) = -F_x\Delta x + F_y\Delta y$$

$$= -F_x y\Delta\theta + F_y x\Delta\theta \text{ (using values of } \Delta x \text{ and } \Delta y)$$

$$= (F_y x - F_x y)\,\Delta\theta = \tau\Delta\theta$$

The term $(F_y x - F_x y)$ is called as the torque (τ) acting on the particle about Z axis, which is the axis of rotation

$\therefore$ $\tau = F_y x - F_x y$

$= rF\sin\alpha\cos\theta - rF\cos\alpha\sin\theta$

($\because F_x = F\cos\alpha$, $F_y = F\sin\alpha$; $x = r\cos\theta$, $y = r\sin\theta$)

$= rF\sin(\alpha - \theta)$

$= rF\sin\beta$, where β is the angle between $\bar{r}$ and $\bar{F}$ clearly

$$\bar{\tau} = \bar{r} \times \bar{F}$$

where $\overline{\tau}$ is along the axis of rotation (axial vector). Also r sin β = d, is the perpendicular distance of line of action of force $\overline{F}$ from the Z-axis.

$\overline{d}$ is also called the *moment arm of the force.* The force $\overline{F}$ can be resolved into a radial component (along the position vector) equal to $F_r = F\cos\beta$ and a tangential component (perpendicular to the position vector). $F_t = F\sin\beta$

$$\overline{\tau} = \overline{r} \times \overline{F}_t$$

Thus torque is the moment of a force about a point (or axis of rotation). If the point is on the line of action of force, the torque exerted by the force about that point will be zero. *The radial component of a force cannot produce any torque.* Torque is a vector and its SI unit is newton metre (N m). Its dimensional formula is ML^2T^{-2} (same as work but work is a scalar quantity).

Note:

In Fig. 2.17, the rotation axis is the z-axis and the force $\overline{F}$ is in the xy plane. That is why the torque turned out to be $(F_y x - F_x y)$. In general,

if $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ for the point of application of force from axis of rotation and

$\overline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, then torque ($\overline{\tau}$) about axis of rotation is given by

$$\begin{aligned}\overline{\tau} &= \overline{r} \times \overline{F}\\ &= \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \times \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)\\ &= \left(F_z y - F_y z\right)\hat{i} + \left[F_x z - F_z x\right]\hat{j} + \left[F_y x - F_x y\right]\hat{k}\end{aligned}$$

It must be kept in mind that only that component of the torque which is along the axis of rotation produces the rotation of the body/particle.

In the mathematical form, torque can also be written as

$$\overline{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(F_z y - F_y z\right)\hat{i} + \left(F_x z - F_z x\right)\hat{j} + \left[F_y x - F_x y\right]\hat{k}$$

It also must be kept in mind that the torque due to a force about any point on the axis of rotation is same.

Work done in rotational motion

We have seen earlier that when a torque $\overline{\tau}$ acts on a particle, producing a small angular displacement of Δθ, the small amount of work done (dW) is given by dW = τΔθ. Since dW is scalar and τ is vector, the correct expression will be

$$dW = \overline{\tau}.\overline{d\theta}$$

Hence the work done by a torque in rotating a body from an initial angular position θ, to final angular position θ_2 is given by

$$W = \int_0^W dW = \int_{\theta_1}^{\theta_2} \overline{\tau}.\overline{d\theta}$$

Power in Rotational Motion

$$\text{Instantaneous power } P = \frac{dW}{dt} = \frac{\overline{\tau}.\overline{d\theta}}{dt}$$

$$= \overline{\tau}.\frac{\overline{d\theta}}{dt} = \overline{\tau}.\overline{\omega}\left(\because \frac{\overline{d\theta}}{dt} = \overline{\omega}\right)$$

$$P = \overline{\tau}.\overline{\omega}$$

where, $\overline{\omega} = \dfrac{d\overline{\theta}}{dt}$ is called the angular velocity.

Relation between torque ($\overline{\tau}$) and angular acceleration ($\overline{\alpha}$)

We know that a force $\overline{F}$ acting on a particle produces linear acceleration $(\overline{a})$. Similarly, a torque acting on a particle, about an axis of rotation, produces an angular acceleration $(\overline{\alpha})$. Referring to Fig. 2.16.

$\overline{\tau} = \overline{r} \times \overline{F} \Rightarrow \tau = rF\sin\beta$ (β = angle between $\overline{r}$ and $\overline{F}$)

∴ $\tau = rF_t$ (F_t = tangential component of F = F sin β)

= r ma_t (∵ $F_t = ma_t$ as per Newton's second law, a_t being the tangential acceleration along the tangent to the path at P.) = rm(αr) (∵ a_t = αr) = $(mr^2)\alpha$

= Iα (∵ moment of inertia of particle about axis of rotation. I = mr^2)

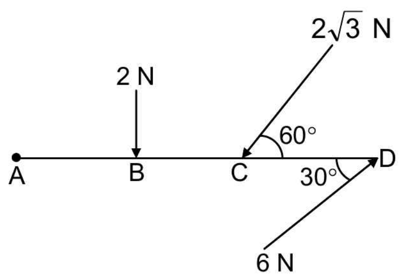
$$\overline{\tau} = I\overline{\alpha}$$

holds good for particle or rigid body, where I = moment of inertia of particle/body about the axis of rotation. Please note that this relation holds good only in an inertial frame of reference. Also $\overline{\tau} = I\overline{\alpha}$, only when $\overline{\tau}$, I and $\overline{\alpha}$ are all about the same axis of rotation. This expression is known as the consequence of Newton's second law of motion applied to rotation.

CONCEPT STRANDS

Concept Strand 16

On a light, rigid rod AD, hinged at A and free to rotate in a vertical plane, forces of 2 N, $2\sqrt{3}$ N and 6 N are applied as shown. If AB = BC = CD = 1 m, calculate the net torque about hinge A.



Solution

Torque (τ_1) due to F_1 (= 2 N) is

$\tau_1 = r_1F_1 \sin 90°$ (r_1 = AB = 1 m) = 1 m × 2 N × 1
= 2 N m (clockwise) = – 2 N m (taking clockwise torque as negative)

$F_2 = 2\sqrt{3}$ N can be resolved as

$F_{2r} = 2\sqrt{3}\cos 60^\circ$

$= \dfrac{2\sqrt{3}\times 1}{2} = \sqrt{3}$ N, along the rod AD, the radial component of force which does not produce torque and

$F_{2t} = 2\sqrt{3}\times\dfrac{\sqrt{3}}{2} = 3$ N is perpendicular to rod AD, acting downwards.

$\therefore \tau_2 = r_2F_{2t}$ (r_2 = AC = 1 + 1 = 2 m) = 2 × 3
= 6 N m (clockwise) = –6 N m

6 N force produces F_{3r} = 6 cos 30° = $3\sqrt{3}$ N radial component, not producing torque and

F_{3t} = 6 sin 30° = 3N, acting vertically upwards

$\therefore \quad \tau_3 = r_3F_{3t}$ (r_3 = AD = 3 m)
= 3 × 3
= 9 N m (anticlockwise)

$\therefore$ Net torque $\tau = \tau_1 + \tau_2 + \tau_3$ (because all are along the same axis of rotation)

= –2 N m – 6 N m + 9 N m = 1 N m

(anticlockwise)

Concept Strand 17

In the above question, what is the angular acceleration produced on the rigid rod, if its moment of inertia about the axis of rotation is 0.1 kg m²?

Solution

τ = +1.0 N m (anticlockwise)
I = 0.1 kg m²; $\tau = I\alpha$

$$\Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{1.0}{0.1} = +10 \text{ rad s}^{-2}$$

Concept Strand 18

A flywheel of M.I = 0.1 kg m² rotates at a rate of 5 revolution/s. What is the torque required to bring it to a stop in 2 second?

Solution

Angular velocity = 10π rad/s

Angular deceleration α required is obtained from the equation, $0 = 10\pi - \alpha.2$ ($\omega = \omega_0 - \alpha t$)

$\therefore \quad \alpha = 5\pi$ rad/s²

Torque required is $\tau = I\alpha$
= 0.1 × 5π N m

Connected bodies

Consider a fixed pulley that is free to rotate about its axis. If the string that passes over it does not slip, then

$$m_1g - T_1 = m_1a$$
$$T_2 - m_2g = m_2a$$

$$(T_1 - T_2).r = I\alpha = \frac{Ia}{r}\left(\because a = r\alpha\right)$$

$$\Rightarrow \quad a = \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{r^2}}\right)g$$

Note that the above analysis is valid only if the pulley is not smooth (so that $T_1 \neq T_2$) and the string does not slip on the pulley.

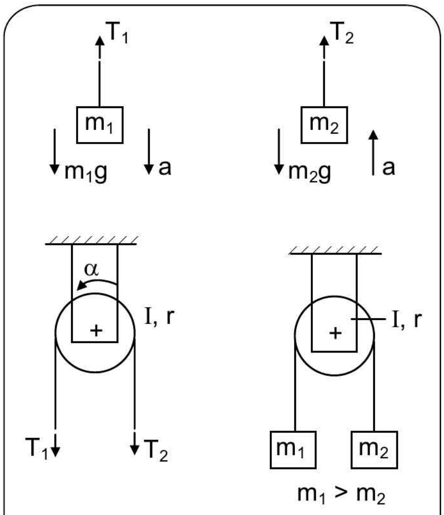
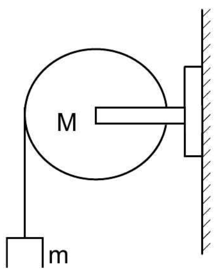


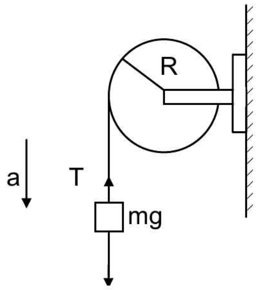
Fig. 2.18

CONCEPT STRAND

Concept Strand 19

A light inextensible rope is wound over a solid cylinder of mass M and radius R. A mass m is tied to the free end of the rope. The mass m is released from a height h with no initial velocity. As the mass falls down, the rope unwinds with the cylinder rotating with negligible friction. Find the acceleration of the falling mass.





Solution

Let the tension in the string be T.

$$\tau = RT = I\alpha_C = \frac{1}{2}MR^2\alpha_C$$

$$= \frac{1}{2}MR^2\frac{a}{R} = \frac{1}{2}MRa \left(\because a = \alpha_C R\right) \Rightarrow T = \frac{1}{2}Ma$$

$$\text{Also, } mg - T = mg - \frac{1}{2}Ma = ma$$

$$\therefore \ a = \frac{g}{1 + \frac{M}{2m}}$$

Kinetic energy of rotation

The kinetic energy of a particle in the rigid body under rotation is $\frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2$

Therefore, the kinetic energy of the whole body is

$$K = \frac{1}{2}\left(\sum_i m_i.r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

WORK-ENERGY THEOREM

(i) Work done by a torque on a body increases the rotational kinetic energy by the same magnitude

$$W_\tau = \int \tau \, d\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

(ii) When no external torque acts on a body, the mechanical energy is conserved.

CONCEPT STRAND

Concept Strand 20

A flywheel of mass 2 kg and radius 10 cm mounted on the central axis is given a constant torque of 0.25 N m. If the flywheel starts from rest, find the change in kinetic energy of the flywheel after 8 second. What was the average power delivered by the torque?

Solution

angular acceleration $\alpha = \frac{\tau}{I} = \frac{0.25}{\frac{1}{2}\times 2\times (0.1)^2} = 25 \text{rad s}^{-2}$

total angle of rotation $\Delta\theta$ =

$\frac{1}{2}\alpha t^2 = \frac{1}{2}\times 25\times 8^2 = 800$ rad

change in K.E = $\frac{1}{2}I\left(\omega^2 - \omega_0^2\right) = W = \tau\Delta\theta$

$= 0.25 \times 800 = 200$ J

average power delivered = $\frac{W}{t} = \frac{200}{8} = 25$ W

ANGULAR MOMENTUM

Angular momentum of a particle

Since the term contains the word angular, it might seem that angular momentum refers to rotational motion. On the other hand, it is associated with any particle in motion, which need not be rotational motion. The angular momentum of a particle is measured with respect to a fixed reference point and it is defined as a vector $\overline{\ell}$, defined as $\overline{\ell} = \overline{r} \times \overline{p}$, where $\overline{r}$ is the vector connecting the position of the particle with the reference point about which angular momentum is measured and $\overline{p}$ is the linear momentum vector of the particle.

CONCEPT STRAND

Concept Strand 21

A particle is moving parallel to the X-axis with a velocity 4 m s^{-1}. What is its angular momentum about the origin when it is at a point with coordinates $\left(5\hat{i} + 3\hat{j}\right)$ m ?

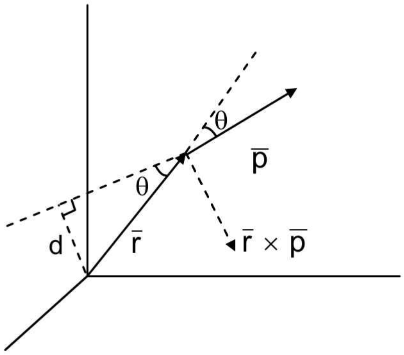
Solution

$$\overline{\ell} = \overline{r}\times\overline{p} = \left(5\hat{i} + 3\hat{j}\right) \times \left(4\hat{i}\right) = -12\,\hat{k} \text{ kg m}^2 \text{ s}^{-1}$$

$\therefore$ $\overline{\ell} = \overline{r} \times \overline{p} = \overline{r} \times m\overline{v}$ (m = mass of particle, $\overline{v}$ = velocity of particle)

$\ell = |\overline{\ell}| = mvr\sin\theta$, where θ = angle between $\overline{r}$ and $\overline{p}$ (or angle between $\overline{r}$ and $\overline{v}$)

The SI unit of angular momentum is kg m^2 s^{-1}. It is sometimes expressed is joule second (J s) and its dimensional formula is ML^2T^{-1}.



If $\overline{d}$ represents the perpendicular distance from a point to the direction of linear momentum $(\overline{p})$ or velocity $(\overline{v})$ of a particle of mass m, the angular momentum of that particle $(\overline{\ell})$ about that point is $|\overline{\ell}| = |\overline{r}\times\overline{p}| = r\sin\theta\, p = dp$.

Angular momentum is also called 'moment of linear momentum'.

If $\overline{r} = x\hat{i} + y\hat{i} + z\hat{k}$, $\overline{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$

and $\overline{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$,

then $\overline{\ell} = \overline{r}\times\overline{p} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\times\left(p_x\hat{i} + p_y\hat{j} + p_z\hat{k}\right)$

$= \left(p_z y - p_y z\right)\hat{i} - \left(p_z x - p_x z\right)\hat{j} + \left(p_y x - p_x y\right)\hat{k}$

This can be expressed as $\overline{\ell}$ =

$m\left[\left(v_z y - v_y z\right)\hat{i} - \left(v_z x - v_x z\right)\hat{j} + \left(v_y x - v_x y\right)\hat{k}\right]$

$$\overline{\ell} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

Angular momentum is zero for any point along the direction of linear momentum (or velocity)

Angular momentum of rotating particle about the centre of rotation

For a particle rotating about a point O with radius r and angular velocity ω, its velocity $\overline{v}$ is always perpendicular to position $\overline{r}$, so that $\theta = 90°$ and $\sin\theta = 1$. Also $v = r\omega$

$\therefore$ Angular momentum of a rotating particle is given by $\ell = rp$ ($\because \sin\theta = 1$)

$= rmv = rm(r\omega)$ ($\because v = r\omega$) $= mr^2\omega$

For a particle since $I = mr^2$, where I = momentum of inertia of particle about axis of rotation.

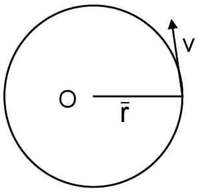
$$\ell = mr^2\omega = I\omega$$

CONCEPT STRAND

Concept Strand 22

A particle moves in a circular orbit in the X-Y plane. Calculate its angular momentum.

Solution



$\overline{\ell} = \overline{r}\times m\overline{v}$

Since $\overline{v}$ is always perpendicular to $\overline{r}$, $\overline{\ell}$ has the magnitude $\ell = mvr$ in a direction coming out of the plane of the paper (positive Z axis)

Physical meaning of angular momentum of a particle

Let a particle of mass m be at position A at time t = 0. OA = r = radius of circular path. In a small interval of time Δt, the particle is at A', after undergoing an angular displacement of $\Delta\theta$.

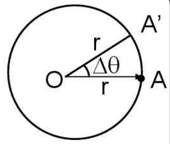


Fig. 2.19

$$\underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta\theta}{\Delta t} = \omega, \text{ the angular velocity of particle at A.}$$

$$\bar{v} = r\bar{\omega} \text{ and } |\bar{v}| = v = r\omega$$

$$\Delta A = \text{Area of sector A'OA} = \frac{1}{2} r^2 \Delta\theta$$

$$\therefore \quad \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} \qquad \text{— (1)}$$

Areal velocity of the particle is defined as the area swept by the radius vector in unit time.

$$\Rightarrow \quad \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$$

$$\frac{1}{2} r^2 \left(\underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta\theta}{\Delta t} \right) = \frac{1}{2} r^2 \omega = \frac{1}{2} rv \qquad \text{— (2)}$$

$$\ell = mvr = 2m \times \frac{1}{2} vr = 2m \times \frac{dA}{dt} \text{ from (ii)}$$

$$\Rightarrow \quad \ell = 2m \frac{dA}{dt}$$

Angular momentum = 2 × mass × areal velocity

Angular Momentum of a system of particles

The angular momentum of a system of particles about a point is the vector sum of the angular momentum of the individual particles of the system about the same point.

$$\therefore \quad \bar{L} = \bar{\ell}_1 + \bar{\ell}_2 + \dots\dots\dots\dots + \bar{\ell}_n$$

$$= (\bar{r}_1 \times \bar{p}_1) + (\bar{r}_2 \times \bar{p}_2) + \dots\dots\dots + (\bar{r}_n \times \bar{p}_n)$$

$$\therefore \quad \bar{L} = \sum_{i=1}^{i=n} m_i (\bar{r}_i \times \bar{v}_i)$$

The velocities of the particles of the system may change due to internal forces of the system (resulting from collisions among particles) or due to external forces on the system. Consequently, angular momentum of the system may change with time. *The physical quantity which can change the angular momentum of a system of particles is the net external torque on the system.*

Angular momentum of a rigid body in rotational motion

A rigid body is a system of closely packed particles in which the distribution of the mass about the axis of rotation is fixed. Hence the particles of the rigid body rotate about an axis instead of a point. There are only two directions for the angular momentum of a rigid body. Hence the angular momentum of a rigid body can be written in scalar form with appropriate signs.

For a rigid body rotating about a fixed axis of rotation, its angular momentum $\bar{L}$ about that axis of rotation is given by

$$\bar{L} = I\bar{\omega}$$

where, I is the moment of inertia of the body about that fixed axis of rotation.

$$\bar{L} = I\bar{\omega} = MK^2\bar{\omega},$$

where K = radius of gyration of the body about the fixed axis of rotation.

Relation between torque ($\bar{\tau}$) and angular momentum ($\bar{L}$)

We have seen earlier that for a rigid body $\bar{\tau} = I\bar{\alpha}$

$$\therefore \bar{\tau} = I\frac{d\bar{\omega}}{dt} \left(\because \bar{\alpha} = \frac{d\bar{\omega}}{dt} \right)$$

$$= \frac{d}{dt}(I\bar{\omega}) \;\; (\because \text{I = constant about axis of rotation for a rigid body})$$

$$= \frac{d\bar{L}}{dt} (\because \bar{L} = I\bar{\omega})$$

Hence for a rigid body, net torque acting on a body about an axis is equal to the rate of change of angular momentum about that axis.

Law of conservation of angular momentum

If the net external torque acting on a rigid body or system of particles is zero, the total angular momentum of the rigid body/system of particles is conserved.

i.e., $L = I\omega$ = constant, where $\tau = 0$.

When a ballet dancer stretches her arms, her moment of inertia about the axis of rotation increases and when she brings her arms towards her body, the moment of inertia decreases. Since no torque is applied (stretching takes place radially) angular momentum is conserved.

$\therefore \quad L = I_1\omega_1 = I_2\omega_2$

$\Rightarrow \quad \omega_2 = \dfrac{I_1\omega_1}{I_2}$

The speed of rotation decreases on stretching arms and increases on pulling arms towards the body.

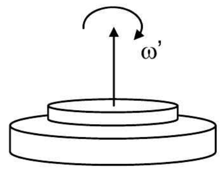
CONCEPT STRAND

Concept Strand 23

A circular disc of mass m and radius R is rotating in a horizontal plane with constant angular velocity ω about the vertical axis passing through the centre of the disc. A smaller disc of the same material and thickness and of radius $\dfrac{R}{2}$ is gently placed on top of the larger disc concentrically. The two discs start rotating together. Assuming there is no slip between the discs, calculate the angular velocity of the combined system.

Solution

Since there is no external torque on the system, the angular momentum is conserved.



$I\omega = I'\omega'$

where I is the M.I. of the larger disc, I' that of the combined system, ω the angular velocity of the larger disc and ω' the angular velocity of the combined system.

$$\therefore \quad \frac{mR^2}{2}\omega = \left[\frac{1}{2}\left(\frac{m}{4}\left(\frac{R}{2}\right)^2 + mR^2\right)\omega'\right]$$

$\left(\because \text{the smaller disc has mass } \dfrac{m}{4}\right)$

$$\therefore \quad \omega' = \frac{16}{17}\omega$$

Relation between angular momentum (L) and kinetic energy (KE) of a rotating body

In translational motion, we have $p^2 = 2m$ KE. In rotational motion, we have

$L^2 = 2I\,(KE)_R$,

where

L = angular momentum of the body about the fixed axis of rotation.

I = moment of inertia of the body about the fixed axis of rotation and

$(KE)_R$ = rotational kinetic energy of the body about the axis of rotation.

Angular Impulse

We have $\bar{\tau} = \dfrac{d\bar{L}}{dt} \Rightarrow \bar{\tau}dt = d\bar{L}$

The term $\bar{\tau}dt$ is called angular impulse (analogous to impulse $\bar{J} = \bar{F}dt$)

$d\bar{L}$ = change in angular momentum

$$\therefore \quad \int_{t_1}^{t_2}\bar{\tau}dt = \int_{t_1}^{t_2}d\bar{L} = \bar{L}_2 - \bar{L}_1$$

Hence angular impulse = change in angular momentum. This is known as *Angular Impulse – Angular momentum Theorem.*

Newton's laws of rotation

The laws governing rotational motion can be written by analogy with the laws of linear motion:

(i) A body will continue to be at rest ($\omega = 0$) or in a state of uniform rotational motion (ω = constant) unless it is acted upon by a torque to change that state.

(ii) The rate of change of angular momentum is equal to the net torque acting on the body.

$$\bar{\tau} = \frac{d}{dt}(I\bar{\omega}) = I\frac{d\bar{\omega}}{dt} = I\bar{\alpha}$$

Comparison of linear and rotational motion

Table 2.2

Sl. N0	Term	Linear motion representation	Term	Rotational motion representation
1.	Position	x	Angle	θ
2.	Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
3.	Acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$
4.	Mass	M	Moment of inertia	I
5.	Linear momentum	p = mv	Angular Momentum	$L = I\omega$
6.	Force	F = ma	Torque	$\tau = I\alpha$
7.	Net force	Produces acceleration	Net torque	Produces angular acceleration
8.	Newton's law	$F = \frac{dp}{dt}$	Consequence of Newton's law	$\tau = \frac{dL}{dt}$
9.	Work	$W = \int \bar{F}.\overline{ds}$	Work	$W = \int \bar{\tau}.\overline{d\theta}$
10.	Power	$P = \bar{F}.\bar{v}$	Power	$P = \bar{\tau}.\bar{\omega}$
11.	Kinetic energy	$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m}$	Kinetic energy	$KE = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$
12.	Kinematic equation, applicable for uniform acceleration or deceleration	$v = u + at$ $\left(\frac{v+u}{2}\right)t = S$ $S = ut + \frac{1}{2}at^2$ $v^2 - u^2 = 2aS$ $S_n = u + \frac{a}{2}(2n-1)$	Kinematic equation, applicable for uniform angular acceleration or deceleration	$\omega_f = \omega_i + \alpha t$ $\frac{(\omega_f + \omega_i)}{2}t = \theta$ $\theta = \omega_i t + \frac{1}{2}\alpha t^2$ $\omega_f^2 - \omega_i^2 = 2\alpha\theta$ $\theta_n = \omega_i + \frac{\alpha}{2}(2n-1)$

CONCEPT STRAND

Concept Strand 24

A disc rotates in a horizontal plane about the vertical axis passing through its centre at a rate of 30 rev/minute. A body of mass 1 gram placed at a distance r from the centre rotates with the disc without slipping or falling. If the coefficient of friction between the coin and disc is 0.15, what is the distance r of the coin from the centre?

Solution

For coin to rotate without slipping or falling, the frictional force must provide the centripetal force.

$$\mu\, mg \geq m\, r\omega^2 \text{ or } r \leq \frac{\mu g}{\omega^2} = \frac{0.15 \times 10}{\pi^2}\text{ m} \sim 15 \text{ cm.}$$

Equilibrium of rigid bodies

Consider a rigid body in static equilibrium under the action of several forces $\overline{F}_1, \overline{F}_2, \ldots\ldots$ Then,

$\Sigma\overline{F}_i = 0$ (for translational equilibrium) and applying Newton's law of rotation for the body, $\Sigma\overline{\tau}_i = 0$ (for rotational equilibrium) where, $\overline{\tau}_i$ are the torques of the forces $\overline{F}_i$ with respect to an arbitrary point.

When a body is in rotational equilibrium but not in translational equilibrium, the net torque will be zero only about the centre of mass of the body and not about any other point.

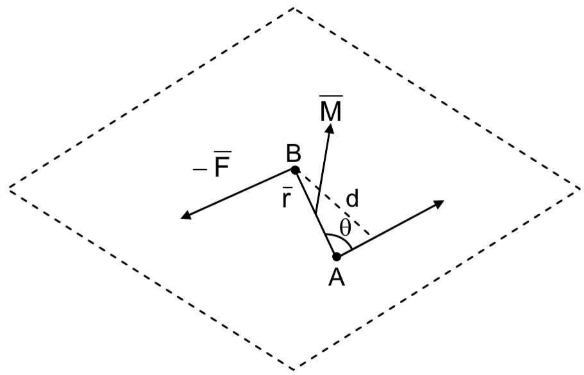


Fig. 2.20

Couple

Two forces $\overline{F}$ and $-\overline{F}$ having the same magnitude, parallel lines of action and opposite sense are said to form a couple.

A couple does not exert a net force on a body but exerts a torque.

An example of a couple is the operation of the wheel spanner of a car wheel. By applying equal and opposite forces at the ends of the spanner arms, the nut advanced or recedes.

The moment of a couple is defined as

$$\overline{M} = \overline{r} \times \overline{F}$$

Where $\overline{r}$ is the vector joining the points of application of the two forces constituting the couple. The sense of $\overline{M}$ is defined by the right hand rule.

$\overline{M}$ is a free vector in the sense that the vector $\overline{r}$ between the points of application of the forces is independent of the choice of coordinate system, that is, wherever we fix the origin $\overline{r}$ is invariant.

The magnitude of $\overline{M}$ is M = Fd

Where, d is the perpendicular distance between the force vectors.

As is obvious, couples are additive, just like forces. Hence the necessary and sufficient condition for the equilibrium of a rigid body is

$$\Sigma\overline{F} = 0;\ \Sigma\overline{M} = \Sigma\overline{r} \times \overline{F} = 0$$

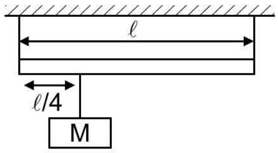
Where, Σ stands for sum.

Just as a simple force can only be balanced by an equal and opposite force, a couple of moment $\overline{M}$ can be balanced only by another couple of moment – M.

CONCEPT STRAND

Concept Strand 25

A mass M is hung from a point at a distance $\frac{\ell}{4}$ from one end of a massless, rigid rod of length ℓ, hung horizontally by two massless inextensible strings. Find the tensions in each string.

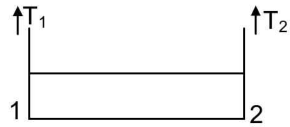


Solution

$T_1 + T_2 = Mg$

Taking moments about 1, $Mg.\frac{\ell}{4} - T_2.\ell = 0$

$\Rightarrow T_2 = \frac{Mg}{4} \Rightarrow T_1 = \frac{3Mg}{4}$



TYPES OF ROTATIONAL MOTION OF RIGID BODIES

Rotational motion of rigid bodies can be classified into two categories:

(i) Pure rotation

In pure rotation, the axis of rotation is fixed and the system does not have any translational motion. The angular velocity and angular acceleration of all particles of the system are same. However, the velocity and acceleration of different particles of the systems are different (because their distances from axis of rotation are different). For example, rotation of a ceiling fan, rotation of the blades of a wind mill, rotation of fixed pulleys, rotation of hinged doors and windows a bicycle on its stand with the rear wheel rotating, etc., can be analysed as pure rotation. The problems involving pure rotation can be analysed using the kinematic equations for rotational motion and applying the equation $\tau = I\alpha$, where τ is the net torque about the axis of rotation, I = moment of inertia of the rigid body about the axis of rotation and α = angular acceleration of the rigid body about the axis of rotation. Conventionally anticlockwise torque is taken as positive and clockwise torque as negative. The figure below shows a uniform disc of mass M and radius R, in pure rotation about a fixed axis through its centre, with a constant angular velocity 'ω'

The velocity of points P, S and Q on its rim have same magnitude = $R\omega$ but their directions are different. The centre C of the disc is at rest.

If the brakes of a bicycle are applied and the bicycle is moved forward, the wheels of the bicycle will be in pure translation. This situation is shown below. If the centre C is moving with a velocity "v" and acceleration "a" then all points P, Q and S also have same magnitude and direction of velocity as well as acceleration as that of point C.

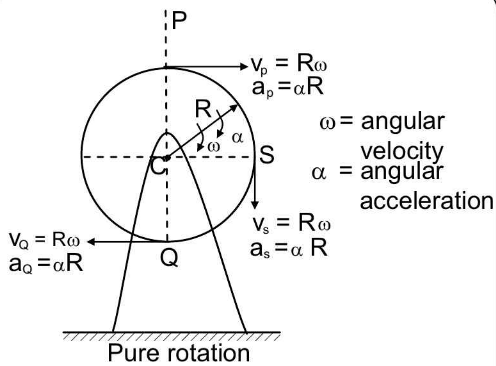


Fig. 2.21

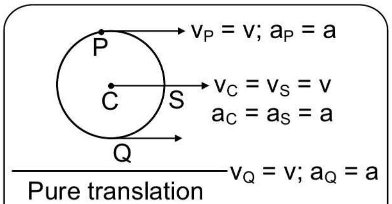


Fig. 2.22

(ii) Rolling Motion (or plane motion)

The motion of a rigid body undergoing rotation about an axis, with the axis of rotation having a translational motion, is called rolling motion or plane motion. Thus rolling motion is a combination of translational and rotational motion. For example, a spinning ball thrown in air, wheels of vehicles moving on road etc. Rolling motion can further be classified as fallows:

(a) Pure rolling (or rolling without slipping/sliding)
In this case, there is no relative motion between the contact points, i.e., the contact point between the body and the surface is at relative rest.

(b) Impure rolling (or rolling with slipping/sliding)
In this case, there is relative motion between the contact points.
Consider a disc of mass, M and radius R, rolling on horizontal ground.

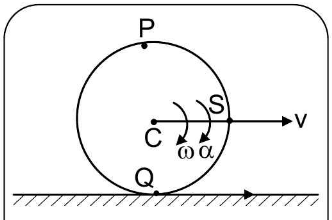


Fig. 2.23

Its angular velocity and angular acceleration are ω and α respectively with respect to an axis through the centre of mass and the velocity of its centre of mass C is $\overline{v}$. Rolling motion can be analysed separately for the translational motion of the centre of mass of the body and for the pure rotation of the body about the axis of rotation and combining the two to obtain the resultant motion.

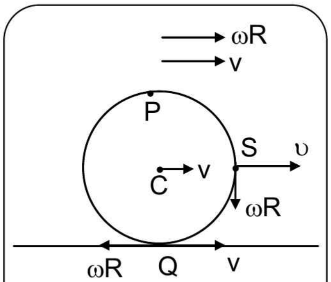


Fig. 2.24

Resultant velocity of P is $v_P = v + \omega R$
Resultant velocity of Q is $v_Q = v - \omega R$
Resultant velocity of S is $v_S = \sqrt{v^2 + \omega^2 R^2}$

If 'a' is the translational acceleration of centre of mass C, then tangential acceleration are given by

$a_P = a + \alpha R$

$a_S = \sqrt{a^2 + \alpha^2 R^2}$

$a_Q = a - \alpha R$

If ω = 0, the accelerations are the total accelerations of point P, Q and S as given above. But if ω ≠ 0, then the centripetal acceleration $\omega^2 R$ will also have to be considered (both magnitude and direction) to find the resultant acceleration of each point. Kinetic energy of a body in rolling is the sum of KE due to rotation and KE due to translation.

Pure Rolling (or rolling without slipping/sliding)

In pure rolling motion, the relative velocity of the point of contact (between the body and the surface on which it rolls) is zero. Pure rolling could be either uniform or accelerated. In uniform pure rolling, the velocity of the centre of mass and the angular velocity of the body about an axis passing through the centre of mass are constant. However, in accelerated pure rolling (or non-uniform pure rolling), the velocity of centre of mass and/or the angular velocity of the body about an axis passing through the centre of mass are not constant.

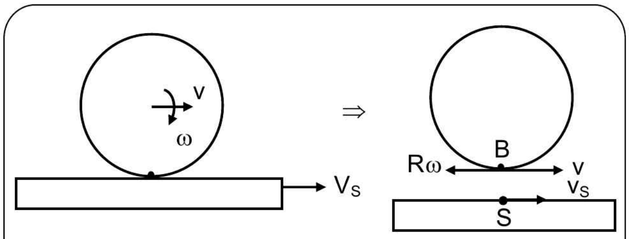


Fig. 2.25

The velocity of point of contact B on the body is $(v - R\omega)$. The velocity of the point S on the surface is v_S

Velocity of contact point B with respect to contact point S is $v_{BS} = (v - R\omega) - v_S$ — (i)

For pure rolling, $v_{BS} = 0$ — (ii)

From (i) and (ii), we get $(v - R\omega) - v_S = 0$. The condition for pure rolling is

$$v - R\omega = v_S$$

(if the point S is also moving)

$$v = R\omega$$

(if the point S is at rest, i.e., the surface on which the body rolls, is at rest)

The friction in pure rolling is static in nature and can vary from zero to a maximum value of $\mu_S N$, where μ_S = coefficient of static friction between the contact surfaces and N = normal reaction at the contact surface. Since the point of contact is at relative rest, no work is done by the friction in pure rolling. Hence in pure rolling, friction is non-dissipative in nature

If a round body of mass M and radius 'R' is in pure rolling on ground ($v_S = 0$), with a velocity v for its centre of mass and uniform angular velocity 'ω' about an axis of rotation through its centre of mass, then its total kinetic energy is given by

KE(Total) = KE(translation) + KE(rotation)

$= \frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\omega^2$, I_{cm} = moment of inertia about axis through centre of mass)

$= \frac{1}{2}M(R\omega)^2 + \frac{1}{2}I_{cm}\omega^2$ ($\because$ $v = R\omega$ for pure rolling on ground) $= \frac{1}{2}\omega^2\left[MR^2 + I_{cm}\right]$

$= I_p\omega^2$ (by parallel axes theorem]

where I_p = moment of inertia of the body about an axis passing through the point of contact. Hence pure rolling can also be analysed as a pure rotation about an axis of rotation through the point of contact with the same angular velocity 'ω' (which is the angular velocity of the body about an axis through its centre of mass)

CONCEPT STRAND

Concept Strand 26

Calculate the K.E. of a disc of mass m, radius R as it rolls on a horizontal surface with angular velocity ω.

Solution

K = K.E. of translational motion + K.E. of rotational motion $= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

But $v_{cm} = R\omega$

$\therefore$ K.E. $= \frac{1}{2}m(R\omega)^2 + \frac{1}{2}\frac{mR^2}{2}\omega^2 = \frac{3}{4}mR^2\omega^2$

Alternatively, the K.E. can be calculated by considering rolling as a pure rotation of the point of contact

K.E. $= \frac{1}{2}I_p\omega^2$ where I_p is the M.I. about an axis passing through the point of contact. By parallel axis theorem,

$$I_p = I_{CM} + mR^2 = \frac{3}{2}mR^2$$

$$\therefore \quad K..E. = \frac{1}{2}.\frac{3}{2}mR^2.\omega^2 = \frac{3}{4}mR^2\omega^2$$

Comparison of kinetic energy of bodies rolling without slipping (Mass of body = M translational speed of centre of mass = v)

Table 2.3

Sl. No.	Rolling body	$KE_{rotational}$	$KE_{translational}$	KE_{TOTAL}
1.	Circular Ring	$\frac{1}{2}Mv^2$	$\frac{1}{2}Mv^2$	Mv^2
2.	Circular disc	$\frac{Mv^2}{4}$	$\frac{1}{2}Mv^2$	$\frac{3}{4}Mv^2$
3.	Solid cylinder	$\frac{Mv^2}{4}$	$\frac{1}{2}Mv^2$	$\frac{3}{4}Mv^2$
4.	Solid sphere	$\frac{Mv^2}{5}$	$\frac{1}{2}Mv^2$	$\frac{7}{10}Mv^2$
5.	Hollow sphere	$\frac{Mv^2}{3}$	$\frac{1}{2}Mv^2$	$\frac{5}{6}Mv^2$

Accelerated pure rolling (Rolling on a ramp)

A round object rolls down a ramp without slipping. The tendency for the body to slide down is prevented by the force of friction f_s acting up the ramp. But f_s is not f_{smax}; it is just enough to prevent sliding. The kinematic equations can be written as follows:

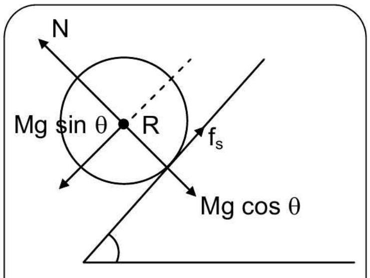


Fig. 2.26

Translational motion : $f_s - Mg\sin\theta = Ma_{cm}$

Rotational motion : $Rf_s = I_{cm}\alpha$

Forces Mg cosθ and N produce no torques. The force of friction provides the torque for rotation. Now,

$a_{CM} = -\alpha R$,

The negative sign being due to the fact that a_{cm} is down the ramp and taken negative while the rotation being anti-clockwise is taken as positive.

Combining the above equations we get the following general results: $a_{cm} = \dfrac{-g\sin\theta}{\left(1+\dfrac{K^2}{R^2}\right)}$

$$\alpha = \frac{g\sin\theta}{\left(1+\dfrac{K^2}{R^2}\right)R}$$

$$f = \frac{Mg\sin\theta}{\left(1+\dfrac{R^2}{K^2}\right)}$$

$$\mu_s = \frac{f}{N} = \frac{\tan\theta}{\left(1+\dfrac{R^2}{K^2}\right)}$$

where μ_s is the coefficient of static friction and K, the radius of gyration

Friction maintains accelerated pure rolling.

The table below gives the values of the dynamical parameters related to round bodies rolling down a ramp as per the formulae above:

Table 2.4

Body	K^2	α	$\lvert a_{CM}\rvert$	f	$\mu_s \geq$
ring/ hollow cylinder	R^2	$\dfrac{g\sin\theta}{2R}$	$\dfrac{1}{2}g\sin\theta$	$\dfrac{1}{2}Mg\sin\theta$	$\dfrac{1}{2}\tan\theta$
disc/ solid cylinder	$\dfrac{R^2}{2}$	$\dfrac{2g\sin\theta}{3R}$	$\dfrac{2}{3}g\sin\theta$	$\dfrac{1}{3}Mg\sin\theta$	$\dfrac{1}{3}\tan\theta$
sphere	$\dfrac{2}{5}R^2$	$\dfrac{5g\sin\theta}{7R}$	$\dfrac{5}{7}g\sin\theta$	$\dfrac{2}{7}Mg\sin\theta$	$\dfrac{2}{7}\tan\theta$
spherical shell	$\dfrac{2}{3}R^2$	$\dfrac{3g\sin\theta}{5R}$	$\dfrac{3}{5}g\sin\theta$	$\dfrac{2}{5}Mg\sin\theta$	$\dfrac{2}{5}\tan\theta$

General expression for the angular momentum of a rigid body

The motion of a rigid body comprises of both translational and rotational motions. Such a motion may or may not be pure rolling. The angular momentum of such a body about a point O (which can be conveniently assumed as the origin of a co-ordinate system) is given by the expression.

$$\bar{L} = \bar{L}_1 + \bar{L}_{cm}$$

where $\bar{L}$ = angular momentum of the rigid body about a stationary axis through the origin O,

$\bar{L}_1$ = angular momentum of the centre of mass of the rigid body with respect to the stationary axis through O = $M\left(\bar{r}_{cm}\times\bar{v}_{cm}\right)$

($\bar{v}_{cm}$ = velocity of centre of mass with respect to stationary axis through O, $\bar{r}_{cm}$ = position of centre of mass with respect to O, M = mass of rigid body.)

$\bar{L}_{cm}$ = angular momentum of the body with respect to a frame of reference attached to the centre of mass and about the centre of mass (a point)

$$\therefore \quad \bar{L} = M\left(\bar{r}_{cm}\times\bar{v}_{cm}\right) + \bar{L}_{cm}$$

The angular momentum about the centre of mass (point) is not the same as about an axis of rotation. However, in cases where the point O (origin) and centre of mass of the rigid body are in the plane of motion, the angular momentum about the centre of mass $(\overline{L}_{cm})$ is the same as the angular momentum about the axis of rotation passing through the centre of mass $(I_{CM}\overline{\omega})$. In such case,

$$\overline{L} = M(\overline{r}_{cm} \times \overline{v}_{cm}) + I_{cm}\overline{\omega}$$

where I_{cm} = moment of inertia of the rigid body about an axis of rotation passing through the centre of mass of the body.

CONCEPT STRANDS

Concept Strand 27

A disc of radius R and mass M is released from a point O on a rough inclined plane of angle θ with the horizontal so that is rolls without slipping. Find the angular momentum of the disc about this point after time t.

Solution

Let f be the magnitude of the frictional force. From Tabel 2.4.

$$a = \frac{2}{3} g \sin\theta$$

$$\Rightarrow \quad v = at = \frac{2}{3} gt \sin\theta$$

$$\omega = \alpha.t = \frac{at}{r} = \frac{2}{3}\frac{gt\sin\theta}{r}$$

$$\Rightarrow \quad L = mrv_{cm} + I_{cm}\,\omega = mr \times$$

$$\frac{2}{3}gt\sin\theta + \frac{mr^2}{2}.\frac{2}{3}\frac{gt\sin\theta}{r}$$

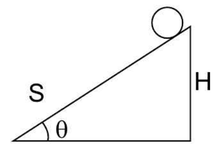
$$= mgrt \sin\theta$$

Concept Strand 28

A disc of radius R and mass M is released from a point O at a height H on a rough inclined plane of angle θ with the horizontal. Find the velocity of its centre of mass when it reaches the base of the inclined plane by rolling without slipping.

Solution

Let f be the magnitude of the frictional force. From Tabel 2.4.



$$a = \frac{2}{3} g \sin\theta$$

$$\frac{H}{S} = \sin\theta \rightarrow S = \frac{H}{\sin\theta}$$

$$v^2 - u^2 = 2as$$

$$\Rightarrow \quad v^2 - 0 = 2 \times \frac{2}{3} g\sin\theta \times \frac{H}{\sin\theta} = \frac{4}{3} gH$$

$$\therefore \quad v = \sqrt{\frac{4}{3}gH} = 2\sqrt{\frac{gH}{3}}$$

Hence the velocity of C.M of the disc at the base will be $2\sqrt{\frac{gH}{3}}$.

Impure rolling (Rolling with slipping/sliding)

We have seen that if the relative velocity of the point of contact of the body with the surface on which it rolls is not zero, then the rolling is called impure rolling. From Fig. 2.27,

$$v_B = v - R\omega$$

$$v_{BS} = v_B - v_S = (v - R\omega) - v_S$$

If $(v - R\omega) - v_S \neq 0$, it is impure rolling (i.e., sliding or slipping occurs between the contact surfaces)

$\therefore$ $v - R\omega \neq v_S$, rolling is impure

If $V_S \rightarrow \upsilon_S = 0$ (like rolling on ground), then

$v \neq R\omega$ for impure rolling

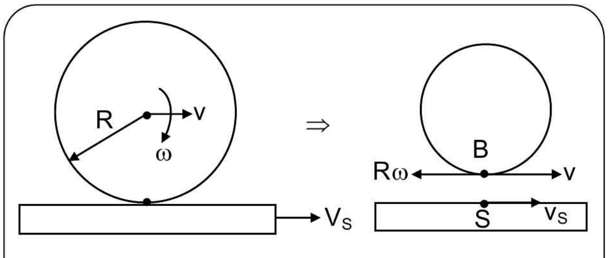


Fig. 2.27

Since slipping occurs in impure rolling, the nature of friction is kinetic. The frictional force is $f = \mu_K N$ for impure rolling, where μ_K = co-efficient of sliding friction between contact surfaces and N = normal reaction at the contact surface. Since the point of application of friction gets displaced, friction is dissipative in nature (c.e. work is done by friction in impure rolling).

Since kinetic friction opposes relative motion between the contact surfaces, the friction on the rolling body will be in the direction opposite to $(v - R\omega) - v_S$

If $v_S = 0$ and $v > R\omega$, then $v - R\omega > 0$

i.e., relative velocity of the rolling body is in the forward direction; hence friction f is in the backward direction

If $v < R\omega$ so that $v - R\omega < 0$, then the relative velocity of the body is in the backward direction; hence friction f acts in the forward direction

In impure rolling, $v_{cm} \neq \omega R$ and $a_{cm} \neq \alpha R$

Role of friction in rolling

(i) In pure rolling with uniform velocity for the centre of mass, there is no friction involved i.e., friction is zero.

(ii) In pure rolling with acceleration (like pure rolling on inclined plane), an external force or torque alters the rolling motion of the body, which can occur in one of the following two ways.

(a) The line of action of the external force passes through centre of mass of the rolling body.

(b) The line of action of the external force does not pass through the centre of mass of the rolling body.

Depending on the line of action, the external force causes linear acceleration or angular acceleration or both. The effect of the external force or torque is moderated by friction between rolling body and surface. Thus friction maintains accelerated pure rolling. This friction is self adjusting and static in nature. If the line of action of the external force passes through the centre of mass of the rolling body, it tries to make the body move forward and friction is in the backward direction of the external force. If the external force does not pass through the centre of mass of the rolling body, it produces a torque on the body. There is a sliding tendency due to torque in the backward direction of applied force. The cumulative result is that either there is no friction or there is friction in the forward direction.

(iii) In the case of uniform rolling (rolling with slipping/sliding), the friction is kinetic and dissipative in nature. The friction is directed opposite to the direction in which the body is slipping. If slip is in forward direction, friction is in the backward direction and vice versa.

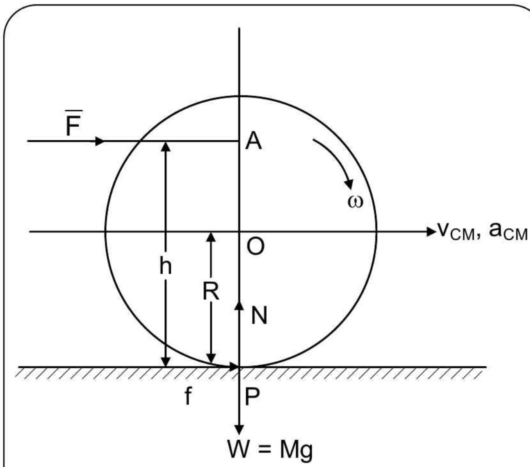


Fig. 2.28

A round body of mass M and radius R is rolling on a horizontal floor as shown. A horizontal force $\overline{F}$ is applied at point A on the body at a height 'h' above the floor. The free body diagram of the body is as shown.

We have $|\overline{f}| \leq \mu N$ — (i)

$\overline{W} + \overline{N} + \overline{F} + \overline{f} = M\overline{a}_{CM}$ — (ii)

from Newton's 2nd law of motion

$N = mg$ — (iii)

for vertical equilibrium

Hence the equation for translational motion of CM becomes

$F + f = Ma_{CM}$ — (iv)

For rotational motion of the body, we have

$F(h - R) - fR = I_{CM}\alpha = \chi MR^2 \alpha$ — (v)

where χ is a number defined by $I_{CM} = MK^2 = \chi MR^2$

Equation (i), (ii), (iii), (iv) and (v) are applicable to either pure rolling or rolling with slipping

For pure rolling, we have $a_{CM} = \alpha R$ — (vi)

$v_{CM} = \omega R$ — (vii)

and $x_{CM} = \theta R$ — (viii)

Substituting (vi) in (v), we get

$F(h - R) - fR = \chi MRa_{CM}$ — (ix)

$(\because a_{CM} = R\alpha)$

$\Rightarrow F(h - R) - fR = \chi R(F + f)$ [from (iv)]

$$\therefore \quad f = F\left[\frac{h}{R(1+\chi)} - 1\right]$$

Hence frictional force f depends on $\frac{h}{R}$ and χ (shape of rolling body)

If $h = R(1 + \chi)$, $f = 0$ i.e., no frictional force and pure rolling occurs.

If $h > R(1 + \chi)$, $f > 0$ i.e., friction is in the same direction as applied force $\overline{F}$

If $h < R(1 + \chi)$, $f < 0$ i.e., friction is in the opposite direction of applied force $\overline{F}$. For almost all shapes of rolling bodies, centre of mass is at a height of R from contact surface. If $\overline{F}$ is applied at the centre of mass, $\frac{h}{R} < (1 + \chi)$ for almost all bodies $\Rightarrow$ friction will be in a direction opposite to the direction of $\overline{F}$, when $\overline{F}$ is applied at the centre of mass. The relation between $\frac{f}{F}$ and $\frac{h}{R}$ is shown below.

It can be seen that the direction of frictional force in pure rolling depends upon $\frac{h}{R}$, and on the shape of the rolling body. The condition for static friction for the body to roll without slipping is obtained from equations (i), (iii) and (x) and can be expressed as

$$\mu_s \geq \frac{F}{Mg}\left[\frac{h}{R(1+\chi)} - 1\right] \qquad \text{—(xi)}$$

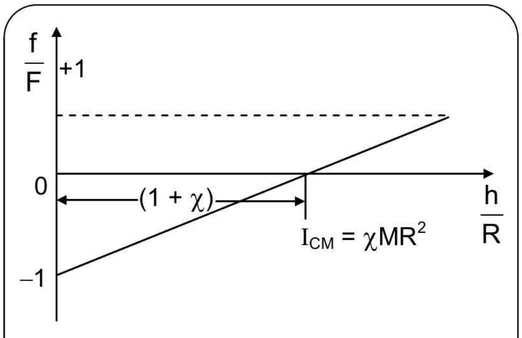


Fig. 2.29

where, μ_s = coefficient of static friction for contact surface. For a given value of F and h on a given body, if μ_s is less than the right hand side of the inequality in (xi), generally the body slips and rolls under the action of $\overline{F}$.

Note:

The equation $f = F\left[\frac{h}{R(1+\chi)} - 1\right]$ can also be used for rolling on a ramp. In that case, $F = Mg\sin\theta$ at the centre of mass and $h = R$. The limiting value of friction will be $\mu Mg\cos\theta$ for rolling on ramp. θ is the angle made by the ramp surface with the horizontal.

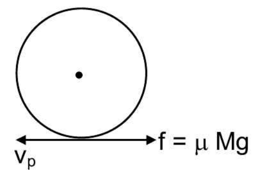
CONCEPT STRAND

Concept Strand 29

A disc of mass M and radius R spinning with an angular velocity ω_0 is gently placed on a horizontal surface with a coefficient of friction μ. Discuss the subsequent motion.

Solution

Let the initial angular velocity be ω_0 which is larger than that for pure rolling given by $\omega = \frac{v_{cm}}{R}$. Hence the point of contact has a backward velocity v_p. Frictional force $f = \mu Mg$ begins to act in the forward dirction and provides angular deceleration, gradually decreasing ω_0. In this phase, the disc's motion is a combination of slipping in the backward direction and rolling in the forward direction. Eventually, whenthefrictionalforcereduces$\omega_0$to $\omega = \frac{v_{cm}}{R}$,purerotation results.



The dynamical equations are:

for the linear motion : $f = \mu Mg = M a_{cm}$

$$\Rightarrow a_{cm} = \mu g \qquad \text{—(16)}$$

for the angular motion:

$$\tau = fR = \mu MgR = I_{cm}\alpha \Rightarrow \alpha = \frac{2\mu g}{R} \qquad \text{—(17)}$$

'α' is the angular retardation. If we write 'v' for the velocity of the centre of mass when rolling starts,

$$v = a_{cm} t = \mu g t \text{ (using (16))} \qquad \text{—(18)}$$

$$\text{and } \frac{v}{R} = \omega_0 - \alpha t = \omega_0 - \frac{2\mu g t}{R} \text{ (using (17))}$$

$$= \omega_0 - \frac{2v}{R} \text{ (using (18))}$$

$$\therefore \quad v = \frac{\omega_0 R}{3} \qquad \text{— (19)}$$

Rolling starts in a time

$$t = \frac{v}{\mu g} \text{ (from (18))}$$

$$= \frac{\omega_0 R}{3\mu g} \text{ (from (19))} \qquad \text{— (20)}$$

We may also calculate the energy expended during the time when the disc was slipping (no work is done during pure rolling). Initial velocity of the point of contact was $\omega_0 R$ and final velocity when rolling started was 0. Therefore, the displacement of the point of contact is $-\frac{\omega_0 R}{2} \times t$, and substituting for t from equation (20)

$$w = \mu Mg\left(-\frac{\omega_o R}{2} \times \frac{\omega_o R}{3\mu g}\right) = -\frac{1}{6} M\omega_0{}^2 R^2 \qquad \text{— (21)}$$

COLLISION OF A POINT OBJECT WITH A RIGID BODY

The collision of a point object with a rigid body can be analyzed using Newton's law of rotation.

Elastic collision

Consider a body of mass 'm' traveling with velocity $\bar{v}$, which undergoes elastic collision with a rigid body of mass M at rest, at a point where position vector is given by $\bar{r}$ with respect to the centre of mass of the rigid body. Let the velocity of the point mass after collision be $\bar{v}_1$ and the velocity of the centre of mass and angular velocity of M be $\bar{v}_2$ and $\bar{\omega}$ respectively.

Applying conservation of momentum,

$$m\bar{v} = m\bar{v}_1 + M\bar{v}_2$$

Applying conservation of angular momentum

$$\bar{r} \times m\bar{v} = \bar{r} \times m\bar{v}_1 + I\bar{\omega}$$

For elastic collision, applying conservation of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1{}^2 + \frac{1}{2}Mv_2{}^2 + \frac{1}{2}I\omega^2$$

Solving the equations above, we can get $\bar{v}_1$, $\bar{v}_2$ and $\bar{\omega}$

Inelastic collision

Consider a mass m moving with velocity $\bar{v}$, undergoing a perfectly inelastic collision with a rigid body of mass M, initially at rest. Let O be the centre of mass of the rigid body and O' be the centre of mass of the system after collision. Let $\bar{v}_1$ be the velocity of the new centre of mass after collision and $\bar{\omega}$ the angular velocity of the new system. Conservation of linear momentum gives $m\bar{v} = (M+m)\bar{v}_1$

Conservation of angular momentum gives

$$\bar{r} \times m\bar{v} = \left(I + m.\bar{r}.\bar{r}\right).\bar{\omega}$$

where, F is the position vector of the point of impact with respect to the new centre of mass O'.

CONDITIONS FOR SLIDING AND/OR TOPPLING OF A RIGID BODY ON ROUGH SURFACE

Consider a rigid, uniform block of mass M, height H and width B placed on a rough horizontal floor as shown in the figure 2.26

Let an external force F be applied on one face of the block, at a height of h from the base as shown in Fig. 2.26. Since the line of action of F does not pass through the centre of mass O of the block, the normal reaction N will be shifted from the vertical through centre of mass by a distance x, towards the far face of the block (away from the face on which F is applied)

The static friction force developed at the contact surface is f.

When the block is in equilibrium, we have

F = f (for horizontal translational equilibrium)

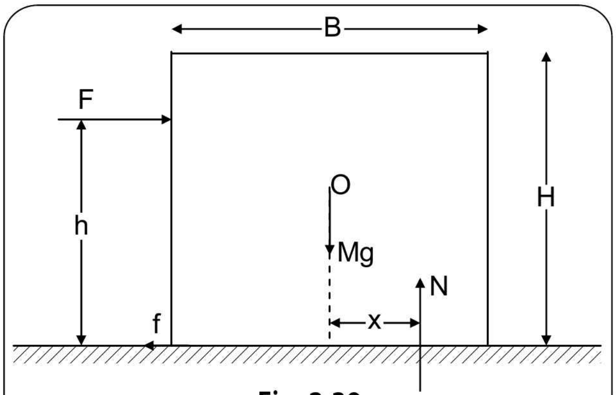


Fig. 2.30

N = Mg (for vertical translational equilibrium)

For rotational equilibrium, taking moments of all forces about the centre of mass O of the block, we get $F\left(h-\frac{H}{2}\right)+f\frac{H}{2}-Nx=0$

$$\Rightarrow x=\frac{Fh}{Mg} \qquad (\because f=F)$$

For sliding to occur, $f \geq f_{limit}$

$\Rightarrow \quad F \geq \mu N$

$\Rightarrow \quad F \geq \mu Mg$

For no sliding, the applied forde $F \leq \mu Mg$.

For no toppling, $x \leq \frac{B}{2} \Rightarrow F \leq \frac{BMg}{2h}$ ($\because$ N = 0, for $x \geq \frac{B}{2}$)

For toppling to occur before sliding (i.e., j toppling occurs first), the condition is $\frac{BMg}{2h} < \mu Mg$

i.e., $h > \frac{B}{2\mu}$ is the condition for toppling to occur first

For sliding to occur before toppling, the condition is $\frac{BMg}{2h} > \mu Mg$

i.e., $h < \frac{B}{2\mu}$ is the condition for sliding to occur before toppling

Moment of Inertia of some regular shaped bodies

Table 2.5

Sl. No.	Name of body	Axis of rotation	Moment of inertia (I)	Radius of gyration (K)
1.	Thin uniform rod of length ℓ and mass M	Through its centre and perpendicular to its length	$\frac{1}{12}M\ell^2$	$\frac{\ell}{\sqrt{12}}$
		Transverse axis through one end of rod	$\frac{1}{3}M\ell^2$	$\frac{\ell}{\sqrt{3}}$
2.	Thin rectangular lamina of sides 'a' and 'b', mass M of lamina	Through its centre of mass and perpendicular to its plane.	$\frac{M}{12}\left(a^2+b^2\right)$	$\sqrt{\frac{a^2+b^2}{12}}$
		Through any of its sides	$\frac{M}{3}\left(a^2+b^2\right)$	$\sqrt{\frac{a^2+b^2}{3}}$
3.	Thin circular ring of radius R and mass M	Through its centre of mass and perpendicular to its plane	MR^2	R
		About any diameter	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
		About a tangent to the ring, parallel to its diameter	$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
4.	Circular disc of mass M and radius R	Through its centre of mass and perpendicular to its plane	$\frac{1}{2}MR^2$	$\frac{R}{2}$
		Through any diameter	$\frac{1}{4}MR^2$	$\frac{R}{2}$
		Through a tangent to the disc and parallel to any diameter	$\frac{5}{4}MR^2$	$\frac{\sqrt{5}R}{2}$

Sl. No.	Name of body	Axis of rotation	Moment of inertia (I)	Radius of gyration (K)
5.	Solid sphere of mass M and radius R	About its diameter	$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
		About any tangent to the sphere	$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
6.	Hollow sphere of mass M and radius R	About its diameter	$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
		About a tangent to the sphere	$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
7.	Solid cylinder of mass M and radius R and length L	About the axis of the cylinder	$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
		About a transverse axis passing through its centre of mass	$M\left[\frac{L^2}{12}+\frac{R^2}{4}\right]$	$\sqrt{\left(\frac{L^2}{12}+\frac{R^2}{4}\right)}$
		About a transverse axis passing through one of its ends	$M\left[\frac{L^2}{3}+\frac{R^2}{4}\right]$	$\sqrt{\left(\frac{L^2}{3}+\frac{R^2}{4}\right)}$
8.	Hollow cylinder of mass M and outer radius R and inner radius 'r'.	About its axis	$\frac{1}{2}M\left[R^2+r^2\right]$	$\sqrt{\frac{R^2+r^2}{2}}$

SUMMARY

$$\vec{R}_{cm} = \frac{M_1\vec{r}_1 + M_2\vec{r}_2 ....M_n\vec{r}_n}{(M_1 + M_2 + ... + M_n)} = \sum_{i=1}^{i=N}\frac{M_i\vec{r}_i}{M}$$

$$\vec{R}_{cm} = X_{cm}\hat{i} + Y_{cm}\hat{j} + Z_{cm}\hat{k}$$

$$\left|\vec{R}_{cm}\right| = \sqrt{{X_{cm}}^2 + Y^2_{cm} + Z^2_{cm}}$$

$$\vec{X}_{cm} = \frac{M_1\vec{x}_1 + M_2\vec{x}_2 ....M_x\vec{x}_n}{(M_1 + M_2 + ... + M_n)} = \sum_{i=1}^{i=N}\frac{M_i\vec{x}_i}{M}$$

$$\vec{Y}_{cm} = \sum_{i=1}^{i=N}\frac{M_i\vec{y}_i}{M} \text{ and } \quad \vec{Z}_{cm} = \sum_{i=1}^{i=N}\frac{M_i\vec{z}_i}{M}$$

$\vec{R}_{cm} \rightarrow$ position vector of the centre of mass

$M \rightarrow$ total mass of a system of N particles of masses M_1, M_2, M_3,..............M_n and position vectors $\vec{r}_1$, $\vec{r}_2$ .... $\vec{r}_n$

$\vec{X}_{cm} \rightarrow$ X-coordinate of center of mass.

$\vec{Y}_{cm} \rightarrow$ Y-coordinate of center of mass

$\vec{Z}_{cm} \rightarrow$ Z-coordinate of center of mass

$$\vec{R}_{cm} = \frac{M_1\vec{r}_1 + M_2\vec{r}_2}{(M_1 + M_2)} \text{ (for a two particle system)}$$

$$\vec{R}_{cm} = \frac{\vec{r}_1 + \vec{r}_2}{2} \text{ if } M_1 = M_2$$

$\vec{R}_{cm} \rightarrow$ position vector of the centre of system of two masses having position vectors $\vec{r}_1$ and $\vec{r}_2$

$$\vec{v}_{cm} = \sum_{i=1}^{i=N}\frac{M_i\vec{v}_i}{M}, \vec{a}_{cm} = \sum_{i=1}^{i=N}\frac{M_1\vec{a}_i}{M}$$

$$\vec{p}_{cm} = M\vec{v}_{cm} = \sum_{i=1}^{i=N}M_1\vec{v}_i$$

$\vec{v}_{cm} \rightarrow$ velocity of center of mass

$\vec{a}_{cm} \rightarrow$ acceleration of center of mass

$\vec{p}_{cm} \rightarrow$ momentum of center of mass

$$F_{ext} = M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d}{dt}M\vec{v}_{cm} = \frac{d\vec{p}_{cm}}{dt}$$

$F_{ext} \rightarrow F_1+F_2+.........+F_N$ is the total external force acting on the system.

$$\vec{\tau} = \vec{r}\times\vec{F} \text{ .;} \quad \left|\vec{\tau}\right| = rF\sin\theta$$

$$\left|\vec{\tau}\right| = \left|\vec{F}\right|d$$

Torque about the origin.

$\vec{r} \rightarrow$ position vector

$\vec{F} \rightarrow$ Force acting on the particle.

$\theta \rightarrow$ angle between force and position vector.

d = perpendicular distance to line of action of force from axis of rotation

$$\vec{L} = \vec{r}\times\vec{p} \text{ ,} \quad \left|\vec{L}\right| = rp\sin\theta = mvr = mr^2\omega$$

$$\vec{L} = I\vec{\omega}$$

$$\left|\vec{L}\right| = 2m\left(\frac{dA}{dt}\right)$$

$L \rightarrow$ Angular momentum of a particle

$\vec{p} \rightarrow$ momentum of a particle.

$\vec{r} \rightarrow$ position vector of particle.

$\omega \rightarrow$ Angular velocity.

$L \rightarrow$ Angular momentum

$m \rightarrow$ mass of the body

$\frac{dA}{dt} \rightarrow$ areal velocity of position vector

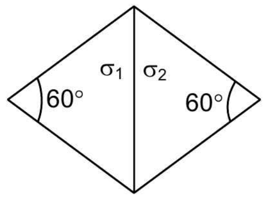
$L = \sqrt{2I(KE)_R}$ or $L^2 = 2I\,(KE)_R$	$L \rightarrow$ Angular momentum, I = moment of inertia KE $\rightarrow$ Kinetic energy
$I = mr^2$ $I = \sum_{i=1}^{i=N} m_i r_i^2$;$I = MK^2$	$I \rightarrow$ moment of inertia $K \rightarrow$ radius of gyration from the axis of rotation $M \rightarrow$ mass of the body $I \rightarrow$ moment of inertia of the body about the axis of rotation.
$K = \left(\dfrac{r_1^2 + r_2^2 + ------ + r_n^2}{n}\right)^{\frac{1}{2}}$	$K \rightarrow$ Radius of gyration of 'n' identical particles having position vectors $r_1, r_2, \ldots r_n$
$\vec{\tau} = I\vec{\alpha}$; $\dfrac{d\vec{L}}{dt} = \vec{\tau}$	$\vec{\tau} \rightarrow$ Torque $\vec{\alpha} \rightarrow$ Angular acceleration. $I \rightarrow$ Moment of inertia.
$W = \tau\theta$, $P = \tau\omega$	$W \rightarrow$ Work done in rotating a body through an angle θ. $P \rightarrow$ Power.
$(KE)_R = \dfrac{1}{2} I\omega^2$	$(KE)_R \rightarrow$ Rotational kinetic energy.
$K.E_{TOTAL} = K.E_{Translational} + KE_{rotational}$ $= \dfrac{1}{2}Mv^2 + \dfrac{1}{2}I\omega^2 = \dfrac{1}{2}M\left[r^2\omega^2 + K^2\omega^2\right]$ $= \dfrac{1}{2}Mr^2\omega^2\left[1 + \dfrac{K^2}{r^2}\right] = \dfrac{1}{2}Mv^2\left[1 + \dfrac{K^2}{r^2}\right]$	Kinetic energies of bodies in both translational and rotational motions $M \rightarrow$ Mass of the body. $r \rightarrow$ radius of the body. $\omega \rightarrow$ Angular velocity. $I \rightarrow$ Moment of inertia
$= \dfrac{g\sin\theta}{\left(1 + \dfrac{K^2}{r^2}\right)} = \dfrac{g\sin\theta}{1 + \dfrac{I}{mr^2}}$ $v = \sqrt{\dfrac{2gh}{1 + \dfrac{K^2}{r^2}}}$	$a \rightarrow$ acceleration of a body which is rolling without slipping on an inclined plane. $v \rightarrow$ velocity of a body at the bottom of the inclined plane. $h \rightarrow$ height of inclined plane. $\theta \rightarrow$ angle of inclination $m \rightarrow$ mass of body $r \rightarrow$ radius of body $K \rightarrow$ radius of gyration of body about axis of rotation
$a = \dfrac{g}{\left(1 + \dfrac{M}{2m}\right)}$ $\alpha = \dfrac{a}{R} = \dfrac{g}{R\left[1 + \dfrac{M}{2m}\right]}$ $T = \dfrac{mg}{\left[1 + \dfrac{2m}{M}\right]}$	$a \rightarrow$ acceleration of a point mass attached to a string wound on a cylinder or disc of Moment of Inertia 'I'. $\alpha \rightarrow$ Angular acceleration of disc or cylinder $T \rightarrow$ Tension in the string, M = mass of cylinder M = point mass at the end of string
$L = I_1\omega_1 = I_2\omega_2$ = constant if $\tau = 0$	L = angular momentum I_1, I_2...... are moment of inertia ω_1, ω_2 .... are angular velocities

Acceleration of different bodies, rolling without slipping, on the same inclined plane

Body	$\frac{K^2}{r^2}$	$a = \frac{g\sin\theta}{1+\frac{K^2}{r^2}}$
Ring	1	$\frac{g\sin\theta}{2}$
Disc or cylinder	$\frac{1}{2}$	$\frac{2g\sin\theta}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5g\sin\theta}{7}$
Hollow sphere	$\frac{2}{3}$	$\frac{3g\sin\theta}{5}$

CONCEPT CONNECTORS

Connector 1: Find the location of the center of mass of the rhombus of side a and densities σ_1 and σ_2 as shown.



Solution: Since the rhombus has two of its angles as 60°, by symmetry, the two constituent triangles are equilateral. The center of mass for a uniform equilateral triangular lamina is its geometric center.

If x is the distance of the C.M from the base (i.e., the vertical diagonal),

$$\frac{x}{a/2} = \tan 30 \Rightarrow x = \frac{a}{2} \cdot \frac{1}{\sqrt{3}}$$

The mass of each of the constituent triangles is proportional to its area. Let the area of each triangle be A.

The masses are given by $M_1 = \sigma_1 A$ and $M_2 = \sigma_2 A$

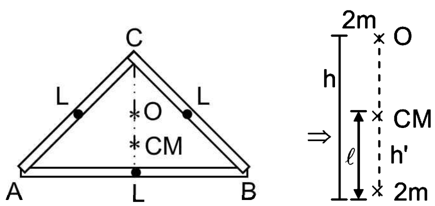
Then,

$$(M_1 + M_2)\, x_{cm} = M_1(-x) + M_2 x$$

$$x_{cm} = \frac{(-M_1 + M_2)}{(M_1 + M_2)} \frac{a}{2\sqrt{3}}$$

$$= \left(\frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}\right) \frac{a}{2\sqrt{3}}$$ from the vertical diagonal, located on the horizontal diagonal.

Connector 2: Three rods of equal lengths are joined to form an equilateral triangle. The mass of one rod is double the mass of each of the other two rods. Locate its centre of mass with respect to its geometric centre. (Assume that the mass distribution of each rod is uniform)



Solution: Let mass of the rods AC and CB = m each and that of AB = 2m. Let the length of each rod be L. The geometric centre of each rod will be the centre of mass of the respective rods. Now AC and BC have their combined centre of mass at O. The center of mass of the system will be at the mid point of the line joining O and mid point of AB.

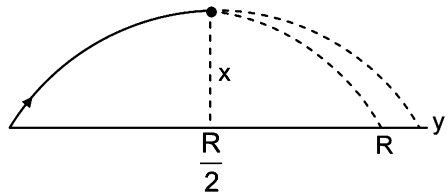
$$h = \text{Height of geometric center above AB} = \frac{L}{2}\tan 30° = \frac{L}{2\sqrt{3}} (CO = OL)$$

$$h' = \text{Height of the center of mass} = \frac{h}{4} = \frac{1}{4}\frac{\sqrt{3}}{2}L = \frac{\sqrt{3}L}{8}$$

$$\therefore \quad \text{Distance from the geometric centre} = \frac{L}{2\sqrt{3}} - \frac{\sqrt{3}L}{8} = \frac{L}{8\sqrt{3}}$$

Connector 3: A body of mass m projected with a velocity u at an angle θ with the horizontal breaks into two fragments at the maximum height attained, the mass ratio being x : y with the smaller mass x coming to rest and falling to a point vertically down. Find the distance from the point of projection where the heavier mass will land.

Solution: When a projectile breaks into fragments, the C.M of the fragments will be the same as that of the original projectile.



Position of mass x = R/2

Position of mass y = X = ?

$$\text{Here, } R = \frac{\frac{xR}{2} + yX}{x+y}$$

$$\therefore \quad R(x+y) = \frac{xR}{2} + yX$$

$$\therefore \quad Rx + Ry - \frac{Rx}{2} = yX$$

$$\therefore \quad X = \frac{Rx + Ry - \frac{Rx}{2}}{y} = R\left(\frac{\frac{x}{2}+y}{y}\right) = \left(\frac{x+2y}{2y}\right)R = \left(\frac{x+2y}{2y}\right)\frac{u^2 \sin 2\theta}{g}$$

$$\left(\because R = \frac{u^2 \sin^2 2\theta}{g}\right)$$

Connector 4: A wheel of moment of inertia 2 kg m² about an axis through its centre and perpendicular to its plane, rotates at 50 rpm about this axis. Find the average torque required to stop the wheel in one minute.

Solution: The initial angular velocity = 50 rpm = $\frac{5\pi}{3}$ rad s^{-1}

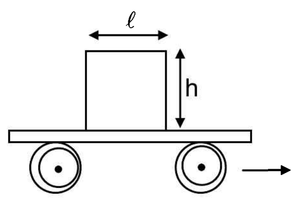
$\omega = \omega_0 + \alpha t$;

Since the average torque is required, we find the uniform torque which will cause the same change in angular velocity in the same time. Since this torque is constant, the value of angular acceleration is also constant.

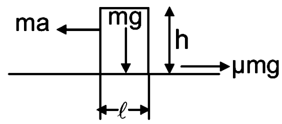
$$\alpha = \frac{|\omega - \omega_0|}{t} = \frac{\left|0 - \frac{5\pi}{3}\right|}{60} \text{rad s}^{-2} = \frac{\pi}{36} \text{ rad s}^{-2}$$

$$\text{Torque } \tau = I\alpha = (2 \text{ kg m}^2)\left(\frac{\pi}{36} \text{ rad s}^{-2}\right) = \frac{\pi}{18} \text{ N m}$$

Connector 5: A uniform solid rectangular block of length ℓ, breadth b, height h is kept on the rough floor of a truck as shown. (coefficient of friction: μ)The truck gradually increases its acceleration. When the acceleration is a_1, the block topples once, when acceleration is a_2 the block slides. Determine a_1 and a_2 in terms ℓ, b, h, μ and g.



Solution: For toppling (comparing torques)

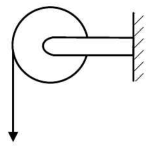


$$ma\frac{h}{2} \geq mg\frac{\ell}{2}$$

$$\Rightarrow \quad a \geq \frac{\ell}{h}g \Rightarrow a_1 = \frac{\ell}{h}g$$ (And since no sliding at this stage $ma_1 < \mu mg$)

Now when it slides $a \geq \mu g \Rightarrow a_2 = \mu g$.

Connector 6: A wheel of radius 20 cm can rotate freely about its centre as shown in the figure. When a light string is wrapped over the rim and it is pulled down with a force of 5 N, it is found that an angular acceleration of 2 rad s^{-2} is produced on the wheel. Find the moment of inertia of the wheel about its axis of rotation.



Solution: Torque $\tau = |\bar{r} \times \bar{F}| = 5 \times 0.2 = 1$ N m

Also torque $\tau = I\alpha$

$$\therefore \quad I = \tau/\alpha = \frac{1}{2} = 0.5 \text{ kg m}^2$$

Connector 7: A cylinder of mass M is suspended by means of two light, inextensible strings wrapped around it as shown. Find the tension in the string and the velocity of the cylinder as it falls through a height h.

Solution:

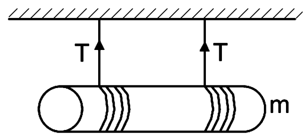
$$mg - 2T = ma \qquad \text{— (1)}$$

$$2Tr = \frac{1}{2}mr^2\alpha = \frac{1}{2}mra$$

or

$$2T = \frac{1}{2}ma \qquad \text{— (2)}$$

Solving (1) and (2)

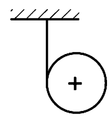


$$a = \frac{2}{3}g \text{ and } T = \frac{mg}{6}$$

$$v^2 = 2\left[\frac{2}{3}g\right]h \quad (\therefore v^2 = 2ah)$$

$$\therefore \; v = \sqrt{\frac{4gh}{3}}$$

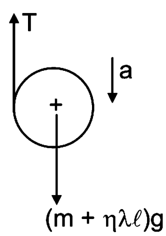
Connector 8: A thin ring of mass m has a long, thin, inextensible string of mass m' wrapped around it. If the system is released as shown, find its linear acceleration.



Solution: Let ℓ be the length, λ the linear density and η the fraction of length remaining on the ring.

Mass = $m + \eta\lambda\ell$

$$\Rightarrow (m + \eta\lambda\ell)\,g - T = (m + \eta\lambda\ell)\,a \qquad \text{—(1)}$$



$$Tr = I\alpha = (m + \eta\lambda\ell)\,r^2\,\frac{a}{r}$$

$$\therefore T = (m + \eta\lambda\ell)a \qquad \text{— (2)}$$

substitute (2) in equation (1)

$$(m + \eta\lambda\ell)g - (m + \eta\lambda\ell)a = (m + \eta\lambda\ell)a$$

$$a = \frac{g}{2}$$, vertically downwards.

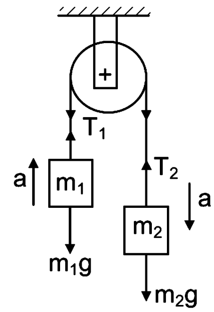
Connector 9: Two masses of 0.5 kg and 0.8 kg are connected by an inextensible light thread passing over a pulley of mass 1 kg and radius 0.1 m. Calculate (no slipping between pulley and the thread) ($g = 10$ m s^{-2})

(i) the acceleration of the masses and the pulley.

(ii) the tensions in the vertical portions of the string, if it does not slip.

Solution:

(i) Let the two masses connected by an inextensible light thread be m_1 and m_2.



Equations of motion,

$T_1 - m_1 g = m_1 a$ — (1)

$m_2 g - T_2 = m_2 a$ — (2)

Considering rotational motion of the pulley (a disc),

$\tau = T_2R - T_1R = \frac{1}{2}MR^2 \times \alpha \ \ (\therefore \tau = I\alpha)$ — (3)

where α = angular acceleration of the pulley. But $a = \alpha R$ for no slip

$\therefore (T_2 - T_1)R = \frac{1}{2}MR^2 \times \frac{a}{R} \Rightarrow T_2 - T_1 = \frac{1}{2}Ma$ Solving (1), (2) and (3),

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{M}{2}} = \frac{+0.300\times 10}{1.8} = \frac{+10}{6}\ m\ s^{-2}$$

(ii) $m_2 = 800$ gram, $m_1 = 500$ gram, $M = 1$ kg

From equation —(1)

$T_1 = m_1(g + a) = 0.5 \times \left(10 + \frac{10}{6}\right) = 5.83$ N

From equation —(2)

$T_2 = m_2(g - a) = 0.8 \times \frac{50}{6} = 6.67$ N

Connector 10: A uniform sphere of mass m, and radius R starts rolling without slipping down an inclined plane of angle θ after a time t from start. Find the angular momentum of the sphere relative to the initial point of contact, after time t.

Solution: $mg\sin\theta - F_f = ma$

$F_f R = I\alpha = \frac{Ia}{R}$

$mg\sin\theta = a\left(m + \frac{I}{R^2}\right) = a\left(1 + \frac{2}{5}\right)m$

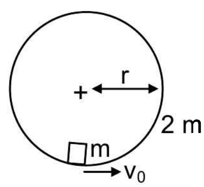
$a = \frac{5}{7}g\sin\theta$

$\omega = \frac{v}{R} = \frac{at}{R} \Rightarrow v = at$

$L = mvR + I_{CM}\omega = mRat + I_{CM}\frac{at}{R} = maRt\left(1 + \frac{2}{5}\right) = maRt\frac{7}{5} = mg\sin\theta.\ Rt\ \frac{5}{7}.\frac{7}{5}$

$L = mgR\sin\theta.\ t$

Connector 11: A thin ring, of mass 2 m and radius r, in the horizontal plane can rotate freely about a vertical axis through its centre. A body of mass m is placed inside the ring and given a velocity v_0 as shown. If the inside of the ring has coefficient of friction μ, find the final angular velocity.



Solution: The angular momentum about the centre of the ring is conserved.

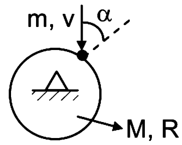
Initial angular momentum = $mv_0 . r$

Let the final angular velocity be ω and the final velocity of m be v $\Rightarrow v = \omega r$

Final angular momentum = $I\omega + mvr = 2m.r^2.\omega + mvr = mvr\,(2+1) = 3\,mvr = mv_0r$

$\Rightarrow \quad v = \dfrac{v_0}{3} \Rightarrow \omega = \dfrac{v}{r} = \dfrac{v_0}{3r}$

Connector 12: A bullet of mass m travelling with a speed v hits a disc of mass M and radius R as shown and gets stuck to it. Find the angular velocity of the system.



Solution: By conservation of angular momentum about 'C'

$$mv \sin\alpha \,.\, R = \left(m + \frac{M}{2}\right)R^2.\omega$$

$$\omega = \left(\frac{2m}{2m+M}\right)\frac{v\sin\alpha}{R}$$

Connector 13: A uniform circular horizontal rough platform, free to rotate about a vertical axis through its centre is at rest along with a man whose mass is $\frac{1}{4}$th of that of platform. The man walks round the edge completing one round and comes back to his starting point. Determine the angle through which the platform has turned.

Solution:. L about axis through O is zero and conserved.

Let mass of man = m and mass of platform = 4 m.

Radius = R

At any instant let ω_{rel} be angular velocity of man relative to platform and let ω be angular velocity of platform.

Then, $4\dfrac{mR^2}{2}\omega + mR^2(\omega + \omega_{rel}) = 0$

$\Rightarrow \omega = \dfrac{-\omega_{rel}}{3} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{d\theta_{rel}}{3dt}$

$\Rightarrow d\theta = -\dfrac{d\theta_{rel}}{3} \Rightarrow \theta = -\theta_{rel}/3 \quad (\because \text{at } \theta = 0, \theta_{rel} = 0)$

Putting $\theta_{rel} = 2\pi \Rightarrow \theta = \dfrac{-2\pi}{3}$

(negative sign shows platform rotates in the opposite direction)

Connector 14: A uniform circular disc of mass 100 g and radius, 2 cm is rotated about one of its diameters at an angular speed of 10 rad s^{-1}. Find the kinetic energy of the disc and angular momentum about the axis of rotation.

Solution: Moment of inertia of the disc about one of its diameters = $\dfrac{1}{4}MR^2$

$= \dfrac{1}{4} \times 0.1 \text{ kg} \times (0.02)^2 = 1 \times 10^{-5} \text{ kg m}^2$

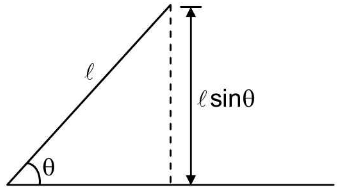
K.E. = $\dfrac{1}{2}I\omega^2 = \dfrac{1}{2} \times 1 \times 10^{-5} \times 100 = 5 \times 10^{-4}$ J

Angular momentum about the axis of rotation = $L = I\omega = 1 \times 10^{-5} \times 10$ rad s^{-2}

$= 1 \times 10^{-4}$ Js

Connector 15: A cylinder is released from the top of an incline of inclination θ with horizontal and length ℓ. If the cylinder rolls without slipping, find the speed with which it reaches the bottom.

Solution: Gain in KE = loss in PE



$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mg\ell\sin\theta$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{2}mR^2\right]\omega^2 + \frac{1}{2}mv^2 = \frac{3}{4}mv^2 = mg\ell\sin\theta$$

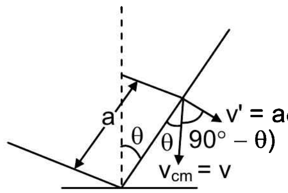
$$\therefore \quad v = \sqrt{\frac{4}{3}g\ell\sin\theta}$$

Connector 16: A uniform rod of length 1 m and mass 0.5 kg is held with one end resting on a smooth horizontal table making 30° with the vertical. Find its angular velocity when the rod is released and makes 60° with the vertical. Calculate the angular momentum about the center of mass.

Solution: C.M of the rod will fall vertically. The rod has rotational motion about the vertical velocity axis. Let v be the vertical velocity of the C.M of the rod when it is inclined at θ with the vertical. Let α be the initial angle and length 2 a. From conservation of energy

$$mg\,a\cos\alpha = mg\,a\cos\theta + \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}mv^2$$

But $I = \frac{1}{12}m(2a)^2 = \frac{1}{3}ma^2$ and v = v' cos (90° – θ)



= v' sin θ = ωa sinθ

$$\therefore\ mg\,a\cos\alpha = mg\,a\cos\theta + \frac{1}{2}\left(\frac{1}{3}ma^2\right)\omega^2 + \frac{1}{2}m\omega^2a^2\sin^2\theta$$

$$\text{or } ga\cos\alpha = ga\cos\theta + \frac{1}{6}a^2\omega^2 + \frac{1}{2}a^2\omega^2\sin^2\theta$$

$$ga(\cos\alpha - \cos\theta) = a^2\omega^2\left(\frac{1}{6} + \frac{1}{2}\sin^2\theta\right) \Rightarrow \left[\frac{6g(\cos\alpha - \cos\theta)}{a(1+3\sin^2\theta)}\right]^{\frac{1}{2}} = \omega$$

$$2a = 1\text{ m}, \therefore a = 0.5\text{ m};\ g = 10\text{ m s}^{-2};\ \alpha = 30°, \theta = 60° \Rightarrow \omega = \left(\frac{60(0.866-0.5)}{0.5(1+3\times0.75)}\right)^{\frac{1}{2}}$$

ω = 3.68 rad s^{-1}

$$\text{Angular momentum of the rod } = I_{CM}\omega = \frac{1}{3}ma^2\omega = \frac{1}{3}\times0.5\times0.25\times3.68$$

= 0.153 kg m^2 s^{-1}

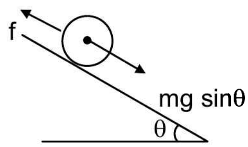
Connector 17: A uniform round object with radius R and radius of gyration K, is placed on an inclined plane of inclination θ to horizontal. What should be the minimum coefficient of friction so that the object rolls (without slipping)?

Solution: Condition for pure rolling:

Rω = v

$\Rightarrow$ R(ω_0 + α t) = v_0 + at

$\Rightarrow$ Rα = a ($\because$ ω_0 and v_0 are zero)



mgsinθ – f = ma —(1)

f. R = I α = mK^2 α —(2)

$$(2) \Rightarrow f = \frac{mK^2\alpha}{R}$$

Substitute in (1)

$$\Rightarrow \quad a = \frac{mg\sin\theta - f}{m} = g\sin\theta - \frac{K^2\alpha}{R}$$

substituting condition for rolling viz. $R\alpha = a$

$$\alpha\left(1+\frac{K^2}{R^2}\right) = \frac{g\sin\theta}{R} \Rightarrow \alpha = \frac{g\sin\theta}{R\left(1+\frac{K^2}{R^2}\right)}$$

substituting in (2)

$$f = \frac{mk^2}{R^2}\cdot\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

But $f \le f_{max} = \mu mg\cos\theta$

$$\Rightarrow \frac{mK^2}{R^2}\cdot\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} \le \mu mg\cos\theta \Rightarrow \mu \ge \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)}$$

$$\therefore \mu_{min} = \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)}$$

Connector 18: A uniform rod of mass M and length L is lying on a smooth table. A small particle of mass m moving on the surface of the table with velocity v_0 hits the rod normally at a distance d from the centre. The particle m comes to rest after a perfect elastic collision.

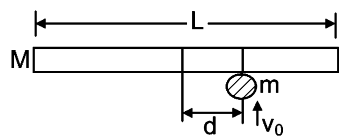
(i) Find an expression for ratio $\frac{m}{M}$

(ii) If the collision is at one end of the rod, what is the value of ω and the velocity of the end point?

Solution:

(i) Assuming m comes to rest after collision, using conservation of linear momentum:

$$mv_0 = Mv \Rightarrow v = \frac{mv_0}{M} \qquad \text{— (1)}$$



From conservation of angular momentum:

$$mv_0 d = \frac{ML^2}{12}\omega \Rightarrow \omega = mv_0\frac{12d}{ML^2} \qquad \text{— (2)}$$

conservation of energy: $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv_0^2$ — (3)

Substitute from (1) and (2)

$$\frac{1}{2}mv_0^2 = \frac{1}{2}M\frac{m^2v_0^2}{M^2} + \frac{1}{2}\frac{ML^2}{12}\frac{m^2v_0^2(12)^2d^2}{M^2L^4}$$

$$\Rightarrow 1 = \frac{m}{M} + \frac{m}{M}\frac{12d^2}{L^2}$$

$$\frac{m}{M} = \frac{L^2}{L^2+12d^2}$$

(ii) From equations (1) and (2) above,

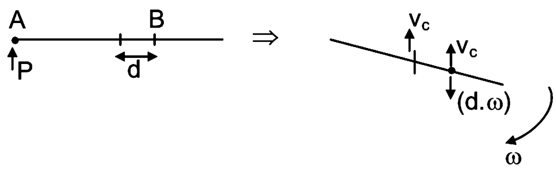
$$\omega = \frac{m}{M}\frac{v_0 12d}{L^2} = \frac{1}{4}\frac{v_0 12\frac{L}{2}}{L^2} = \frac{3v_0}{2L} \quad \left(\therefore d = \frac{L}{2}, \frac{m}{M} = \frac{1}{4}\right)$$

velocity of end point:

$$v' = v + \omega.\frac{L}{2} = \frac{m}{M}v_0 + \frac{L}{2}\frac{3v_0}{2L} = \frac{v_0}{4} + \frac{3v_0}{4} = v_0 = \text{velocity of approach (elastic collision)}$$

Connector 19: A uniform rod of length 'ℓ' is resting on a smooth table. A sharp impulse is given at one end of the rod, normal to the length of rod. Locate a point B on the rod where the velocity is zero immediately after the blow.

Solution: Let ℓ be the length of the rod, M its mass and $\frac{\ell}{2}$ its C.M. Let v_c be the velocity of rod resulting from the impulse.



Let P be the impulse: $P = Mv_C \Rightarrow v_C = \frac{P}{M}$ —(1)

Angular impulse $= P\frac{\ell}{2} = I_C\omega \Rightarrow P\frac{\ell}{2} = \frac{M\ell^2}{12}\omega$

$\therefore \quad \frac{P}{M} = \frac{\omega\ell}{6}$ —(2)

From (1) and (2) $v_C = \frac{\omega\ell}{6} \Rightarrow \frac{v_C}{\omega} = \frac{\ell}{6}$ —(3)

For B to be at rest, $d.\omega = v_C$

$d = \frac{v_c}{\omega} = \frac{\ell}{6}$ from the centre. (towards the other ends of rod, when impulse was not applied)

This point B is called the centre of percussion.

Connector 20: A body of mass m and speed v hits a rod of length ℓ and mass m at a point $\frac{\ell}{3}$ from its fixed support as shown and sticks to it. Find the angular velocity of the rod.

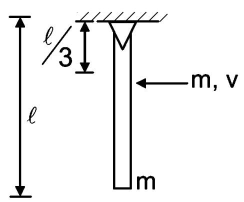
Solution: Angular momentum of system is conserved about the fixed axis of rotation, as the net external torque on system is zero.

$\therefore \quad L_i = L_f$

$mvr_1 = mr_1v_1 + I_0\omega \; (v_1 = r_1\omega)$

$= mr_1^2\omega + \frac{1}{3}m\ell^2\omega$

$$\Rightarrow \frac{mv\ell}{3} = \left(m\left(\frac{\ell}{3}\right)^2 + \frac{1}{3}m\ell^2\right)\omega$$

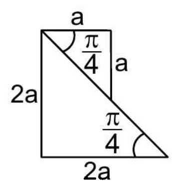


$$\omega = \frac{v/3}{\left(\frac{1}{3}+\frac{1}{9}\right)\ell} = \frac{3v}{4\ell}$$

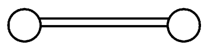
TOPIC GRIP

Subjective Questions

1. A small wedge of two sides a each is placed on top of a wedge of sides 2a. If the system is placed on a smooth floor, find the distance moved by the large wedge when the small wedge just reaches the ground. Ratio of mass of large wedge to small wedge is 4:1. Neglect friction.



2. Two uniform spherical balls of masses 20 kg, 10 kg and radii 0.2 m, 0.1 m respectively, are attached to the ends of a rod of mass 10 kg, and length 1 m. The balls and rod are homogenous and of uniform composition. Determine the moment of inertia of this system about an axis passing through its centre of mass and perpendicular to the rod.

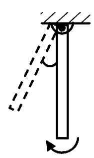


3. Determine moment of inertia of a solid cylinder of mass M, radius R, length ℓ about an axis YY passing through its centre of mass and perpendicular to the axis of cylinder.

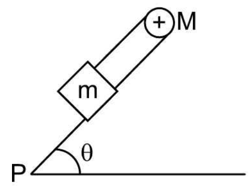
4. A uniform rod is rotating on a horizontal plane about a vertical axis through one end with a constant angular velocity ω_0. Due to internal shear forces, the rod breaks at the mid point of its length, and the part with the fixed axis has an angular velocity $1.5\omega_0$ in the same sense. Determine the angular velocity of the other half.

5. A uniform disc of mass M and radius R is free to rotate about a horizontal axis through its centre perpendicular to its plane. A particle of mass m is attached to a point on the edge of the disc. If the motion starts when the radius to the particle makes an angle β with the upward vertical, determine the angular velocity when the particle is in its lowest position.

6. A uniform thin rod of length ℓ is suspended freely at its end and given an angular velocity ω_0. What is the maximum angle it makes with the vertical?

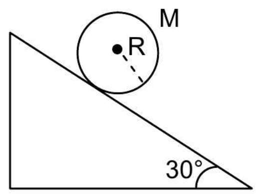


7. A wheel of radius r and mass M about its axis is fixed at the top of a smooth inclined plane of angle of inclination θ, as in figure. A string wrapped round the wheel has at its free end, supporting a block of mass m, which can slide on the plane. Initially, the wheel is rotating at a speed ω_0 in a direction such that the block slides up the plane.
   (i) How far will the block move up before stopping?
   (ii) Calculate the velocity and acceleration of the block at time 't' after start but before stopping.



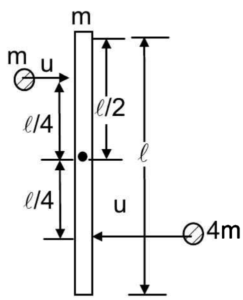
8. A uniform disc of mass 2 kg and radius 0.46 m rolls without slipping down an inclined plane of length 36 m and slope 30° with horizontal. The disc starts from rest at the top of the incline. Find

(i) the angular acceleration and the linear acceleration of the disc.
(ii) The time for the disc to reach the bottom of the incline
(iii) The angular velocity of the disc at the bottom of the incline
(iv) Torque, about the center
(v) Frictional force on the disc.



9. A sphere starting from rest rolls down without slipping along an incline of 1in 100 in 5.3 s. The radius of the sphere is 5 cm and mass is 250 g. If the distance moved by the sphere is 1 m, calculate the acceleration due to gravity.

10. A rod of length ℓ and mass m is lying on a smooth horizontal surface. Two particles of masses m and 4 m respectively, each with speed u, strike the rod perpendicular from opposite directions, at $\frac{\ell}{4}$ each from center as shown and get stuck to it. Determine the translational and angular velocities of the rod.



Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

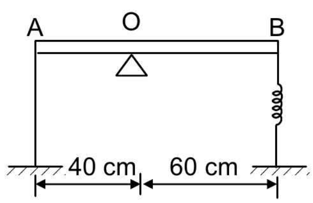
11. A rope of length L is placed along an inclined plane of angle α with the horizontal such that its one end just touches the horizontal floor. The final velocity of the rope is (neglect friction)

(a) $\sqrt{gL}$ (b) $\sqrt{2gL\sin\alpha}$ (c) $\sqrt{\frac{gL\sin\alpha}{2}}$ (d) $\sqrt{gL\sin\alpha}$

12. Moment of inertia of a rod about an axis through the center is 600 g cm^2. Its mass is 30 g. The moment of inertia about another parallel axis at a distance of 5 cm from its centre is

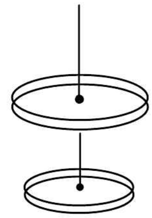
(a) 1350 g cm^2 (b) 1150 g cm^2 (c) 1250 g cm^2 (d) 1475 g cm^2

13. A uniform rod AB of mass 2 kg and length 1 m is placed on a wedge O To keep the rod horizontal, its end A is tied with a thread and the spring has tension 6 N. The reaction of support O on the rod when the thread is burnt is ($g = 10$ m s^{-2})



(a) 15 N (b) 40 N (c) 20 N (d) 25 N

14. Two identical disks are positioned on a vertical axis as shown in the figure. The bottom disc is rotating at angular velocity ω_0 and has rotational kinetic energy K_0. The top disk is allowed to fall slowly, from its initial rest position and finally it sticks to the bottom disk. The angular momentum of the system after the collision is



(a) $\frac{K_0}{4\omega_0}$ (b) $\frac{K_0\omega_0}{4}$

(c) $\frac{2K_0}{\omega_0}$ (d) $\frac{K_0}{2\omega_0}$

15. A bobbin with inner radius r_1 and outer radius r_2 is rolling down the inclined plane without slipping and reaches the bottom with velocity v_0. Now the same body slides down without rolling on a smooth inclined plane of same angle of inclination and from same height and reaches with velocity pv_0. Then the radius of gyration of the bobbin is

(a) $\frac{r_2}{r_1}p^2$ (b) r_2p^2 (c) $r_2\sqrt{p^2-1}$ (d) $\frac{r_1r_2}{(p^2-1)^{1/2}}$

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. **Statement 1**

For a system of isolated mutually interacting particles, both linear momentum and angular momentum remain constant.

and

Statement 2

Resultant force on any particle will be zero, hence resultant torque on it also is zero.

17. **Statement 1**

Moment of inertia of a uniform square plate about any axis lying on its plane and dividing the plate into equal areas is same irrespective of the orientation and location of the axis.

and

Statement 2

Equal areas of a uniform plate will have equal mass.

18. **Statement 1**

A cyclist going around a curve of radius R with speed v leans inward at $\tan^{-1}\frac{v^2}{Rg}$ to the vertical.

and

Statement 2

For dynamic and rotational equilibria of a body both resultant force and resultant torque are to be zero.

19. **Statement 1**

Angular velocity of a body, in pure rolling, about the instantaneous axis is the same as its angular velocity about the centre of mass.

and

Statement 2

Velocity of the point of contact is zero while the velocity of the centre of mass is v.

20. **Statement 1**

When a round body is rolling down on an inclined plane acceleration is always independent of coefficient of friction, μ.

and

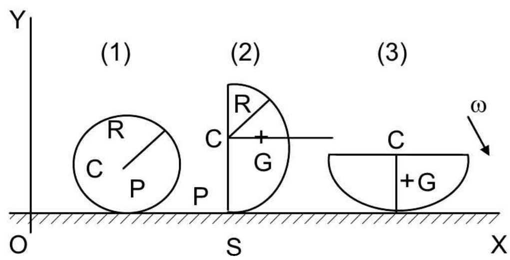
Statement 2

In case of pure rolling down the inclined plane, $a = \dfrac{g\sin\theta}{1+\dfrac{I}{mr^2}}$.

Linked Comprehension Type Questions

***Directions:* This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I



The figures 1, 2, 3 show a solid sphere and a solid hemisphere, having same mass m and radius R, kept on a horizontal surface S. Obviously, the hemisphere (2) is not in equilibrium. The two bodies differ in an important aspect. That is, the centre of mass G of the hemisphere does not coincide with the geometric centre C. The sphere will remain at rest in equilibrium where as the hemisphere will roll clockwise and reach position 3 acquiring an angular velocity ω as shown in Fig 3. It is assumed that the static friction is sufficient to prevent any slipping at the contact points with the surface S at any stage.

21. The ratio of the moments of inertia of the hemisphere about the axis OX in position 2 and 3 is $\left(CG = \dfrac{3R}{8}\right)$

(a) 1 (b) $\dfrac{28}{13}$ (c) $\dfrac{13}{28}$ (d) $\dfrac{32}{13}$

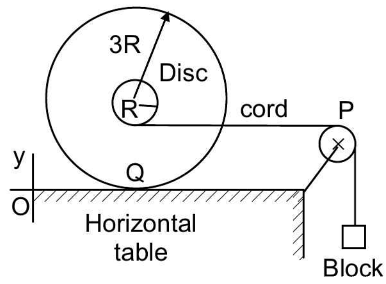
22. The instantaneous acceleration of the centre C when the hemisphere is released in position (2) is

(a) $\dfrac{5g}{7}\hat{i}$ (b) $\dfrac{15g}{56}\hat{i}$ (c) $\dfrac{10g}{13}\hat{i}$ (d) Zero

23. The instantaneous acceleration of the centre of the hemisphere when it reaches position (3) is

(a) $\dfrac{20g}{13}\hat{i}$ (b) $\dfrac{5g}{7}\hat{i}$ (c) $\dfrac{15g}{26}\hat{i}$ (d) Zero

Passage II



In a toy mechanism, a light and inextensible cord is tightly wound on a stepped disc of mass m and radius 3R. The other end of the cord is connected to a block of mass m after passing over a smooth pulley P. The disc is free to roll on the horizontal table S. The disc has a radius of gyration 2R about an axis perpendicular to its plane and passing through its centre. The system is released from rest at time t = 0. It is assumed that the disc rolls on the table without slipping. The radius of the step on which the cord is wound is R.

24. As the block accelerates downward, the disc rolls rightward. Let the acceleration of the centre of the disc be $a_1\hat{i}$ and that of the block $-a_2\hat{j}$. The correct relation between a_1 and a_2 is

(a) $a_1 = a_2$ (b) $a_1 = 3a_2$ (c) $a_1 = \frac{3a_2}{2}$ (d) $a_2 = 3a_1$

25. The tension in the cord is

(a) mg (b) $\frac{9\,mg}{17}$ (c) $\frac{11\,mg}{17}$ (d) $\frac{13\,mg}{17}$

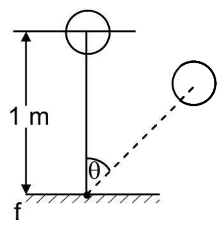
26. The moment of inertia of the disc about the axis passing through the point of contact Q and parallel to OZ axis is

(a) $9\,mR^2$ (b) $4\,mR^2$ (c) $13\,mR^2$ (d) $10\,mR^2$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers of which ONE OR MORE answers will be correct.**

27. A small sphere of mass m, is attached to a rod of negligible mass and length 1 m (upto centre of sphere) and kept on a rough floor in a vertical position. The sphere is given a slight push to right side so that it starts falling



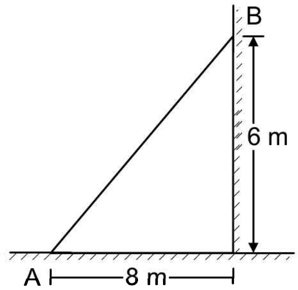
(a) The frictional forces developed on the rod initially will be directed to left.

(b) The frictional forces developed on the rod initially will be directed to right

(c) The normal reaction N will be zero when the rod has fallen through an angle $\theta = \cos^{-1}\frac{2}{3}$ with the vertical.

(d) The normal reaction N will be zero when the rod has fallen through an angle $\theta = \sin^{-1}\frac{3}{5}$ with the vertical.

28. A 10 m long pole, of mass 10 kg is kept slanting on a smooth wall as shown in figure and is stable. The floor is rough. ($g = 10\ m\ s^{-2}$)



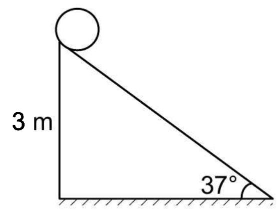
(a) Minimum value of μ of the floor is $\frac{2}{3}$

(b) Minimum value of μ of the floor is $\frac{1}{3}$

(c) Normal reaction at floor is 100 N

(d) Normal reaction at wall is 200 N

29. A sphere of mass 1 kg and radius 0.1 m is released from a height h = 3 m on a fixed inclined plane of angle 37° with horizontal and the contact surfaces on the inclined plane have a $\mu = 0.2$ ($g = 10\ m\ s^{-2}$)

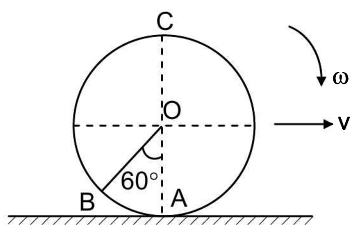


(a) Acceleration of the sphere is $4.4\ m\ s^{-2}$
(b) When the sphere reaches bottom, its KE is 22 J
(c) When the sphere reaches bottom, its KE is 30 J
(d) If μ is increased to 0.22 the KE energy of the sphere at the bottom will be less than the KE it had attained in the previous case.

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30. A disc of radius R rolls on a rough surface with constant angular velocity ω. v is the linear speed of the centre of mass of the disc. Then the speed of points is column I are



Column I	Column II
(a) A	(p) $2R\omega$
(b) B	(q) v
(c) C	(r) 2v
(d) O	(s) 0

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

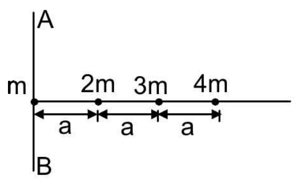
31. The separation between carbon and oxygen atoms in CO is 1.2 Å. The distance of centre of mass from carbon atom is

(a) 0.21 Å (b) 0.69 Å (c) 0.91 Å (d) 0.43 Å

32. A projectile is fired upwards at a speed of 100 m s^{-1} at an angle of 37° with the horizontal. It explodes into two parts at the highest point in the mass ratio 1 : 3 and the lighter one comes to rest. The distance from the point of projection where the heavier mass lands is (Take g = 10 m s^{-2})

(a) 110 m (b) 1120 m (c) 960 m (d) 1960 m

33. Four point masses of values as shown are connected by a thin massless rod as shown in figure. The radius of gyration of the system about AB is



(a) $\sqrt{2}a$ (b) $\sqrt{3}a$ (c) 2a (d) $\sqrt{5}a$

34. A uniform solid sphere and a uniform hollow sphere of the same mass have the same moment of inertia about their diameters. Then the radii of solid and hollow sphere are in the ratio

(a) $\sqrt{\frac{3}{5}}$ (b) $\sqrt{\frac{5}{3}}$ (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

35. A ring and a disc of the same radius have the same moment of inertia about an axis of rotation passing through their centers and perpendicular to their planes. The mass of the ring and disc are in the ratio

(a) $\frac{1}{2}$ (b) 2 (c) $\sqrt{\frac{1}{2}}$ (d) $\sqrt{2}$

36. The moment of inertia of a thin uniform disc of mass M and radius R about a chord of length R is

(a) $\frac{MR^2}{2}$ (b) $\frac{MR^2}{4}$ (c) $\frac{5}{4}MR^2$ (d) MR^2

37. Moment of inertia of a disc of mass M and radius r about its tangent and perpendicular to the plane, is

(a) $\frac{2}{3}Mr^2$ (b) $2\,Mr^2$ (c) Mr^2 (d) $\frac{3}{2}Mr^2$

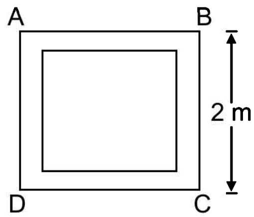
38. Moment of inertia of a rod of length L, mass M about an axis perpendicular to its length and passing through its center is $\frac{ML^2}{12}$. The moment of inertia about a parallel axis through one end is

(a) $\frac{ML^2}{4}$ (b) $\frac{ML^2}{2}$ (c) $2ML^2$ (d) $\frac{ML^2}{3}$

39. During rotation, the diameter of a flywheel increases by 1%. The percentage increase in its moment of inertia about the central axis is

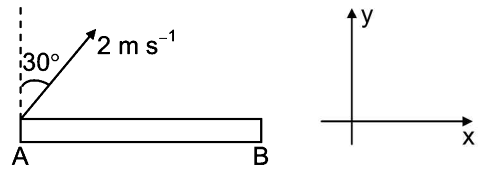
(a) 1% (b) 2% (c) 4% (d) 0.5%

40. ABCD is a framework of 4 thin rods, each of length 2 m. The moment of inertia of the frame about AC is 2 kg m^2. The mass of the frame is



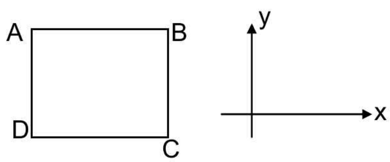
(a) 2 kg (b) 2.5 kg (c) 3 kg (d) 4 kg

41. The velocity of the end A of a rigid rod moving in the XY plane makes an angle of 30° with the Y axis. Then v_{xB} is



(a) 2 m s^{-1} (b) 1 m s^{-1} (c) 0 m s^{-1} (d) cannot evaluate

42. The rigid square plate ABCD moves in the xy plane. If $\overline{v}_A = v(\hat{i}+\hat{j})$ and $\overline{v}_D = v\hat{j}$, v_C/v_B is



(a) 1 (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

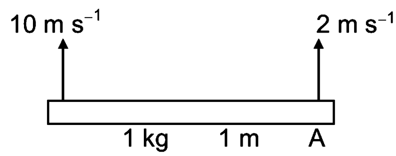
43. Dimensional formula of angular momentum is

(a) ML^2T^2 (b) ML^2T^{-2} (c) $ML^2\,T^{-1}$ (d) $ML^{-1}T^2$

44. A particle of mass 2 kg is at a point (4 m, 1 m) and has velocity $(3\hat{i}+6\hat{j})$ m s^{-1}. Magnitude of the angular momentum of the particle about the point (3 m, 2 m) is (in kg m^2 s^{-1})

(a) 0 (b) $6\sqrt{10}$ (c) 18 (d) 30

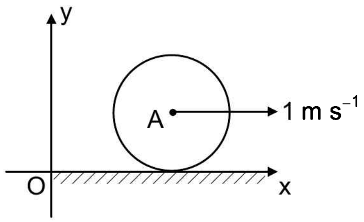
45. A rigid rod moves in a plane such that the ends have speeds 10 m s^{-1} and 2 m s^{-1} as shown. Its angular momentum about the end A is (in kg m^2 s^{-1})



(a) $\frac{2}{3}$ (b) 3

(c) $\frac{11}{3}$ (d) $\frac{8}{3}$

46. A disc of radius 1 m rolls without slipping as shown and its center is at (3 m, 1 m) at one instant. Its angular momentum is zero about

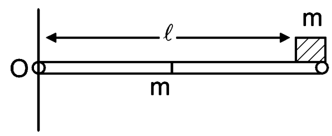


(a) (3 m, 0) (b) (0, 0) (c) (0, 1.5 m) (d) (b) and (c)

47. Two equal and unlike parallel forces constitute a
(a) Torque (b) Couple (c) Moment (d) Impulse

48. The power required to maintain a rotor at a uniform speed of 300 rad s^{-1} to transmit a torque of 150 N m is
(a) 50 kW (b) 60 kW (c) 45 kW (d) 2 kW

49. The rod of mass m and length ℓ can rotate freely about O in a vertical plane. A point mass m is attached to the other end of the rod and the body is released in the horizontal position. Then the initial angular acceleration is

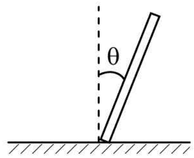


(a) $\frac{2}{3}\frac{g}{\ell}$ (b) $\frac{9}{8}\cdot\frac{g}{\ell}$

(c) $\frac{g}{\ell}$ (d) $\frac{g}{2\ell}$

50. A disc of mass 100 kg and radius 1 m, free to rotate about an axis through its centre and perpendicular to its plane, is acted upon by a torque of 100 N m. Its angular acceleration in rad s^{-2} is
(a) 2 (b) 1 (c) 0.5 (d) 4

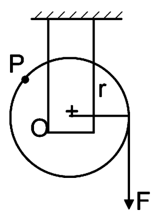
51. A rigid rod of length ℓ is falling down, without slipping. When the rod makes an angle θ with the vertical, its angular acceleration is



(a) $\frac{g\sin\theta}{\ell}$ (b) $\frac{3g\sin\theta}{\ell^2}$

(c) $\frac{3g\sin\theta}{2\ell}$ (d) $\frac{g\cos\theta}{\ell}$

52. A disc-like pulley of mass 1 kg, rotating about a horizontal axis through its centre O, is wound with a weightless thread and a force F is applied. If the tangential acceleration of the point P on the pulley is 1 m s^{-2}, the force F is of magnitude



(a) $\frac{1}{2}$ N (b) $\frac{1}{4}$ N

(c) $\frac{1}{8}$ N (d) 1 N

53. A horizontal circular table rotates at a constant angular velocity about a vertical axis through its centre. There is no friction and no driving torque. A concentric circular pan rests on it and rotates with it. The bottom of the pan is covered with a layer of uniform thickness of ice, which is also rotating with the pan on the table. The ice melts and the water doesn't escape. Then the angular velocity of the turn table will
(a) Increase (b) Becomes double (c) Becomes half (d) Decrease

54. A horizontal platform of mass 100 kg in the form of a circular disc rotates at 10 rpm along a vertical axis passing through its center .A man weighing 60 kg is standing on its edge. The angular velocity with which the platform rotates, if the man moves to its center is

(a) 22 rpm (b) 11 rpm (c) 44 rpm (d) 66 rpm

55. What will be the duration of a day if the Earth shrinks to 1/3rd of its original volume, mass remaining unchanged? (Take $3^{1/3} = \sqrt{2}$)

(a) Remain unchanged (b) Becomes half (c) Doubled (d) Triples

56. A gymnast spinning on one leg with his arms out stretched, suddenly lowers his arms. Then his

(a) angular velocity decreases. (b) angular momentum increases.
(c) the moment of inertia decreases. (d) angular velocity remains constant.

57. A uniform disc of radius R rotates about an axis through its centre and perpendicular to its plane with angular velocity ω. A stationary disc of the same material and thickness but half the radius is placed on it axially. The final angular velocity of the system is

(a) $\frac{4}{5}\omega$ (b) $\frac{16}{17}\omega$ (c) $\frac{\omega}{2}$ (d) $\frac{2\omega}{3}$

58. Rotational kinetic energy (K) and angular momentum (L) are related as

(a) $K^2 = 2\,IL$ (b) $L = 2\frac{I}{K}$ (c) $L^2 = 2IK$ (d) $L^2 = 2\frac{I}{K}$

59. A disc of mass 20 kg rotates about an axis through its centre and perpendicular to its plane with angular speed of 100 rad s^{-1}. The radius of the disc is 0.25 m. The kinetic energy associated with the rotation of the disc is

(a) 3152 J (b) 3512 J (c) 3215 J (d) 3125 J

60. 500 J is spent in increasing the speed of a flywheel from 60 rpm to 360 rpm. The moment of inertia of the wheel is

(a) 0.62 kg m^2 (b) 0.72 kg m^2 (c) 0.92 kg m^2 (d) 0.52 kg m^2

61. Two monkeys of mass m sit at either end of a uniform horizontal disc of mass 4m and angular speed ω about a vertical axis through its centre. The work done by one monkey to pull the other one to the center is

(a) $\frac{2}{3}mr^2\omega^2$ (b) $mr^2\omega^2$ (c) $2mr^2\omega^2$ (d) 0

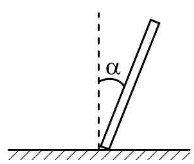
62. The rotational energy of a body for a given angular speed depends on its

(a) mass only (b) size only
(c) mass and size (d) mass as well as distribution of mass about the axis of rotation

63. A uniform disc of mass m and radius r can rotate freely about its axis, which is horizontal and passing through its centre. A point mass m is attached at the edge of the disc at the same level as the center. The maximum angular velocity is

(a) $\sqrt{\frac{g}{r}}$ (b) $\sqrt{\frac{g}{2r}}$ (c) $2\sqrt{\frac{g}{3r}}$ (d) $\sqrt{\frac{g}{5r}}$

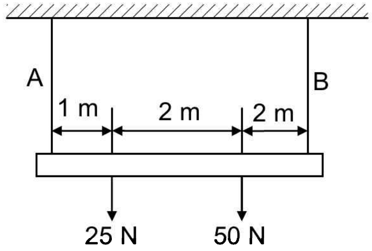
64. A vertical rod of length ℓ falls without slipping on a rough surface. Its angular velocity after it falls through angle α with the vertical is



(a) $\frac{g}{\ell}\sin\alpha$ (b) $\sqrt{\frac{6g}{\ell}}\sin\frac{\alpha}{2}$

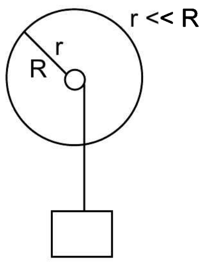
(c) $\sqrt{\frac{3g}{\ell}}\cos\frac{\alpha}{2}$ (d) $\frac{g}{\ell}\cos\alpha$

65. A rigid massless rod is suspended horizontally by two wires A and B. The force in wire B is



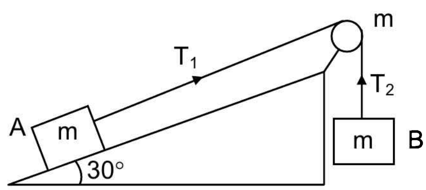
(a) 50 N (b) 37.5 N (c) 35 N (d) 30 N

66. A flywheel rotating at 150 rpm can just raise a load of 50 kg through 4 m before coming to rest. Calculate the moment of inertia of the flywheel. (Neglect friction) ($g = 10\ m\ s^{-2}$)



(a) $16\ kg\ m^2$ (b) $8\ kg\ m^2$ (c) $32\ kg\ m^2$ (d) $20\ g\ m^2$

67. The light, inextensible string does not slip over the pulley (disc) of mass m. The inclined plane is smooth. The acceleration of the body B when the system is released from rest is



(a) $\frac{g}{3}$ (b) $\frac{g}{4}$

(c) $\frac{g}{5}$ (d) $\frac{g}{6}$

68. The ratio of rotational kinetic energy to total kinetic energy of a uniform rolling body of radius R and radius of gyration K is

(a) $\frac{K^2}{K^2+R^2}$ (b) $K^2(K^2+R^2)$ (c) $\frac{K^2+R^2}{K^2}$ (d) $K^2+\left(\frac{R^2}{K^2}\right)$

69. A uniform disc weighing 2 kg rolls without slipping over a horizontal plane with a velocity of $4\ m\ s^{-1}$. The kinetic energy of disc is (in joule)

(a) 20 (b) 24 (c) 16 (d) 8

70. A solid sphere of mass 1 kg rolls without sliding with uniform velocity $0.1\ m\ s^{-1}$ along a horizontal table. The total energy of the sphere is

(a) 0.0007 J (b) 0.07 J (c) 0.007J (d) 0.7 J

71. A smooth body of mass M slides down an inclined plane and reaches the bottom with velocity v. If the same mass were in the form of a ring which rolls down without slipping the velocity at the bottom would be

(a) $\sqrt{2}v$ (b) $\frac{v}{\sqrt{2}}$ (c) 2v (d) $\frac{v}{2}$

72. The ratio of the total K.E. of a rolling disc to its rotational K. E. is
(a) 1:3 (b) 1:2 (c) 2:1 (d) 3:1

73. A thin uniform circular ring is rolling down without slipping, on an inclined plane of inclination 30° with horizontal. Its linear acceleration is
(a) g/2 (b) g (c) g/4 (d) 2/3g

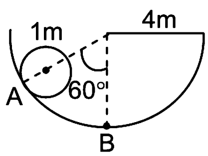
74. A uniform solid sphere and a uniform solid cylinder having the same mass and radius roll down without slipping along the same inclined plane. The ratio of the magnitudes of the linear acceleration of sphere and cylinder will be
(a) 5 : 4 (b) 15 : 14 (c) 4 : 5 (d) 14 : 15

75. A uniform hollow cylinder and a uniform solid cylinder of same mass and radius are made to roll without slipping down a rough inclined plane. Which one reaches the bottom first?
(a) solid cylinder (b) hollow cylinder (c) both together (d) angle dependant

76. A uniform cylinder, a uniform solid sphere and a uniform hollow spherical shell, all roll down on a rough inclined plane. If only one of them slip, it is
(a) solid sphere (b) hollow sphere
(c) cylinder (d) depending on the mass of the body.

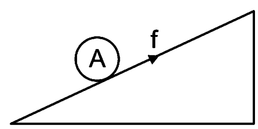
77. A uniform sphere and a uniform spherical shell of mass m each and radius r_1 and r_2 respectively, in pure rolling on a horizontal surface, are approaching a rough inclined plane with the same velocity. The maximum distances moved by these bodies on the inclined plane are in the ratio
(a) $\frac{r_1}{r_2}$ (b) $\frac{3\,r_2^2}{5\,r_1^2}$ (c) $\frac{21}{25}$ (d) 1 : 1

78.



A uniform disc of mass m = 1 kg and radius 1 m starts rolling down from A without slipping, on the rough inner surface of a semicircular container of radius 4 m. The angular velocity of the disc at B is:
(a) g rad s^{-1} (b) 2.5 g rad s^{-1} (c) $\sqrt{2g}$ rad s^{-1} (d) $\sqrt{3g}$ rad s^{-1}

79. The friction force developed between the uniform disc A and inclined plane is as shown.



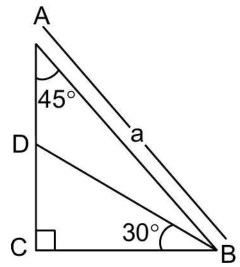
(a) the body should be rolling up.
(b) the body should be rolling down.
(c) the body should be at instantaneous rest.
(d) It could either be moving up or moving down or at instantaneous rest.

80. A uniform rod of mass m and length ℓ is pivoted about one end and hung vertically. Another mass m hits it perpendicular to its length with a velocity v at its midpoint and sticks to it. The initial angular velocity of the rod is
(a) $\frac{v}{\ell}$ (b) $\frac{v}{2\ell}$ (c) $\frac{6v}{7\ell}$ (d) $\frac{v}{3\ell}$

81. A 12 m tall flagpost of uniform linear density 50 kg m^{-1} collapses to the ground. The kinetic energy with which it hits the ground is($g = 10\ m\ s^{-2}$)
(a) 72000 J (b) 36000 J (c) 6000 J (d) 18000 J

82. A chain is placed on a smooth table with $1/4^{th}$ of its length hanging over the edge. If the total length is 2 m and mass is 4 kg, the energy needed to pull the overhanging part back to the surface of the table is ($g = 10\ m\ s^{-2}$)
(a) 50 J (b) 5 J (c) 2.5 J (d) 0.25 J

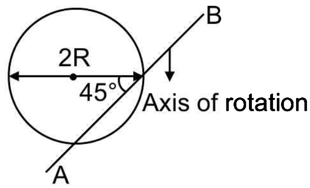
83. Four rods of equal mass m each, AB, BC, CA and DB are placed as shown. The moment of inertia of the system about B perpendicular to the plane ABC is (AB = a)



(a) $\frac{25}{18}ma^2$ (b) $\frac{103}{72}ma^2$

(c) $\frac{39ma^2}{19}$ (d) $\frac{ma^2}{3}$

84. The moment of inertia of a thin sheet of mass m and radius R about AB is



(a) $\frac{mR^2}{2}$ (b) $\frac{3}{4}mR^2$ (c) $\frac{5}{4}mR^2$ (d) mR^2

85. The moment of inertia about a transverse axis through the center of a disc radius is 20 cm, density 9 g cm^{-3} and thickness 7 cm is

(a) 1584×10^7 g cm^2 (b) 1.584×10^6 g m^2 (c) 1.584×10^7 g cm^2 (d) 1.584×10^4 g cm^2

86. A particle of mass m is projected upwards from level ground with an initial velocity v, making an angle of 45° with the horizontal. The angular momentum of the particle about the point of projection, when the particle is at its maximum height, is

(a) $\frac{mv^3}{\sqrt{2}g}$ (b) $\frac{mv^3}{g\sqrt{32}}$ (c) $\frac{m^2v}{\sqrt{2}g}$ (d) $\frac{m^2v^2}{\sqrt{2}g}$

87. A flywheel of moment of inertia 1 kg m^2 and radius 1 m starts rotating due to a constant torque 3 N m. The velocity of a point on the rim after 1s is (m s^{-1})

(a) 3 (b) $\frac{3}{2}$ (c) 6 (d) $\frac{3}{4}$

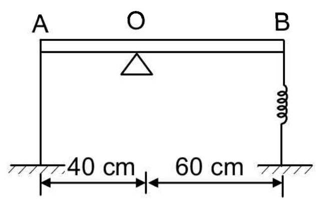
88. A uniform horizontal rod of mass m and length ℓ, initially at rest, is free to rotate about a vertical axis through its centre. It is subjected to constant horizontal force F acting on the rod at a distance of $\frac{\ell}{4}$ from the centre and always perpendicular to the rod. The angle of rotation of the rod at the end of time t after commencement of motion is

(a) $\frac{2Ft^2}{5m\ell}$ (b) $\frac{5Ft^2}{2m\ell}$ (c) $\frac{3F\ell^2}{2mt}$ (d) $\frac{3Ft^2}{2m\ell}$

89. Due to friction between ocean waters and the Earth's surface, the rotational kinetic energy of the Earth is continuously decreasing. If the Earth's angular speed decreases by 0.0016 rad/day in a century, the average torque of the friction on the Earth of radius 6400 km and mass 6×10^{24} kg is

(a) 6.8×10^{20} N m (b) 5.7×10^{20} N m (c) 6.2×10^{20} N m (d) 5.2×10^{20} N m

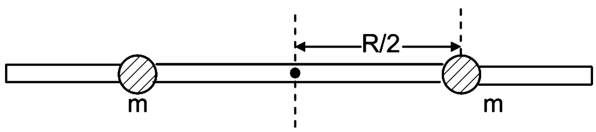
90. A uniform rod AB of mass 2 kg and length 1 m is placed on a wedge O. To keep the rod horizontal, its end A is tied with a thread and the spring has tension 6 N. The reaction of support O on the rod when the thread is burnt is



(a) 15 N (b) 40 N

(c) 20 N (d) 25 N

91.



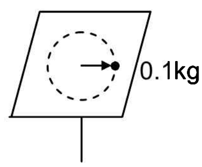
A rod of mass 2m and of length 2R can rotate about a vertical axis through its centre as shown in the figure. The system rotates at angular velocity ω, when the 2 point masses of m each are at a distance R on either side of the axis. The masses are simultaneously pulled to a distance of $\frac{R}{2}$ from the axis by a force F directed along the rod (see figure). The new angular momentum of the system is

(a) $\frac{17}{6}mR^2\omega$ (b) $4mR^2\omega$ (c) $\frac{8}{3}mR^2\omega$ (d) $\frac{9}{8}mR^2\omega$

92. A horizontal turn table in the shape of a disc of radius r and mass M rotates with angular velocity ω_0, about the vertical axis through its centre O and carries a gun of mass m_g. The gun fixed at the edge of the turn-table fires a bullet of mass m with a tangential muzzle velocity v with respect to the gun. The increase in angular velocity of the system when the gun fires the bullet is

(a) $\frac{mv}{r\left(\frac{M}{2}+m_g+m\right)}$ (b) 0 (c) $\frac{mv}{r\left(M+2m_g+m\right)}$ (d) $\frac{mv}{r\left(\frac{M}{2}+m_g\right)}$

93. A ball of mass 0.1 kg rotates in a horizontal circle of radius 1 m at a constant speed of 2 m s^{-1} on a frictionless table as shown in figure. The ball is attached to a string which passes through a hole in the table. By pulling the string at the lower end, the radius of the path is reduced to 0.5 m.



(a) new velocity of the ball is 2 m s^{-1}.
(b) new velocity of the ball is 3 m s^{-1}.
(c) final tension in the string is 4 N
(d) final tension in the string is 3.2 N

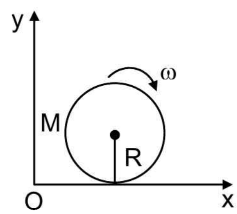
94. One quarter sector is cut from a uniform circular disc of radius 10 cm. This sector has mass 0.5 kg. It is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc at 20 rad s^{-1}. Its kinetic energy about axis of rotation is

(a) 0.7 J (b) 1.0 J (c) 0.9 J (d) 0.5 J

95. A cord 3 m long is coiled around the axle of a wheel. The cord is pulled with a constant force of 44 N. When the cord detaches itself from the axle of the wheel, the wheel rotates at 3 rev s^{-1}. The moment of inertia of the wheel and axle is

(a) 2 kg m^2 (b) 0.7 kg m^2 (c) 2.5 kg m^2 (d) 0.9 kg m^2

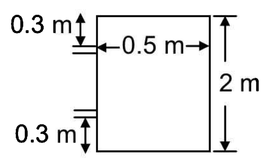
96. A disc of mass M and radius R is rolling with angular speed ω on a horizontal plane. Angular momentum about the origin O and total kinetic energy of the disc is



(a) $\frac{1}{2}MR^2\omega, \frac{1}{2}MR^2\omega^2$ (b) $MR^2\omega, MR^2\omega^2$

(c) $\frac{3}{2}MR^2\omega, \frac{3}{4}MR^2\omega^2$ (d) $\frac{3}{4}MR^2\omega, \frac{3}{2}MR^2\omega^2$

97. A door is 2 m high, 0.5 m wide and weighs 8 kg. The door is supported by 2 hinges located at 0.3 m from the ends. The horizontal force exerted by one of the hinges on the door is (N)



(a) 50 (b) 14
(c) 80 (d) 28

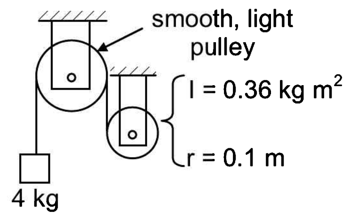
98. A wheel disc of radius 10 cm and mass 0.6 kg is fixed at the top of an inclined plane of inclination 37° with the horizontal. A string is wrapped round the wheel and its free end supports a block of mass 0.1 kg which can slide upwards on the plane. If at one instant, the wheel rotates at 20 rad s^{-1}, the time taken by the block to stop when the wheel comes to rest uniformly is (g = 10 m s^{-2}, tan 37° = $\frac{3}{4}$)

(a) $\frac{5}{9}$ s (b) 1 s (c) $\frac{4}{3}$ s (d) 1.5 s

99. A uniform cylinder of mass 900 g and radius 10 cm rotates freely about its fixed longitudinal axis which is kept horizontal. A particle of mass 100 g hangs from the end of a light inextensible string wound round the cylinder. When the system is allowed to move, the angular acceleration of the cylinder is (Take g =10 m s^{-2})

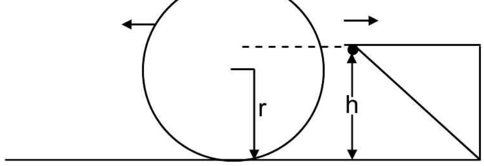
(a) 20 rad s^{-2} (b) 15.6 rad s^{-2} (c) 18.2 rad s^{-2} (d) 2.3.rad s^{-2}

100. A light, inextensible string wrapped on a fixed wheel of moment of inertia 0.36 kg m^2 and radius 10 cm, goes through a light smooth pulley supporting a block of mass 4 kg as shown in figure. Then the acceleration of the block is (g = 10 m s^{-2})



(a) 1 m s^{-2} (b) 2 m s^{-2}
(c) 1.5 m s^{-2} (d) 3 m s^{-2}

101.



A solid uniform ball of radius r rolls on a smooth horizontal surface and hits a wedge. If the ball rebounds with a pure rolling motion, the ratio of the height of the wedge to the radius of the ball is

(a) $\frac{3}{2}$ (b) $\frac{5}{3}$ (c) $\frac{7}{5}$ (d) $\frac{9}{7}$

102. A hollow spherical shell of radius 6 cm is resting on a smooth horizontal table. The height h above the table where the sphere should be hit with a cue held horizontally, such that the sphere moves without sliding on the table is

(a) 2.5 cm (b) 10 cm (c) 6 cm (d) 9 cm

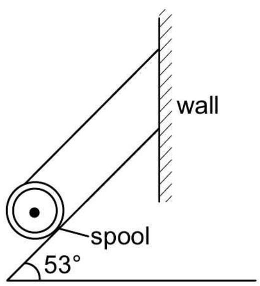
103. A solid uniform sphere rotating about its axis with rotational kinetic energy E_i and zero translational energy is placed on a rough horizontal plane. Friction coefficient μ varies at each point on the horizontal plane. If ω and ω_0 are the angular velocities of the sphere and the sphere begins pure rolling with total kinetic energy of E_r, then $\frac{E_r}{E_i}$

(a) $\frac{2}{5}$ (b) $\frac{5}{7}$ (c) $\frac{2}{7}$ (d) $\frac{3}{5}$

104. A thread is wound around the circumference of a disc of mass m, radius r. The free end of the thread is held and then the disc is released. The maximum angular velocity of the disc (g = 10 m s^{-2}), when it falls through a height h is

(a) $\sqrt{\frac{gh}{r^2}}$ (b) $\sqrt{\frac{2gh}{r^2}}$ (c) $\sqrt{\frac{4gh}{r^2}}$ (d) $\sqrt{\frac{4gh}{3r^2}}$

105. A spool with thread wound on it is placed on a smooth inclined plane, inclined at 53° with the horizontal. The free end of the thread is attached to a wall, Mass of the spool = 0.3 kg. Its moment of inertia about its axis of rotation is 6×10^{-4} kg m^2. The wound radius of the spool is 2 cm. The acceleration of the spool is (g = 10 m s^{-2})



(a) 1.8 m s^{-2} (b) 1.33 m s^{-2} (c) 2.5 m s^{-2} (d) 3.1 m s^{-2}

106. A pencil of length ℓ placed vertically on a smooth table falls down. The linear velocity of a point on the pencil at a distance $\frac{\ell}{3}$ from bottom end of pencil, when it is about to land on the table is

(a) $\sqrt{\frac{3g\ell}{2}}$ (b) $\sqrt{\frac{g\ell}{3}}$ (c) $\sqrt{2g\ell}$ (d) $\frac{1}{3}\sqrt{2g\ell}$

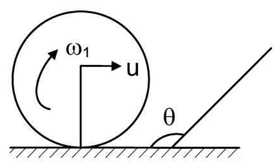
107. A cylinder rolls without slipping along a horizontal plane, with a velocity of v_0. It reaches a plane inclined at an angle 37° with the horizontal. The velocity with which the cylinder starts up the inclined plane is

(a) $\frac{11}{15}v_0$ (b) v_0 (c) $\frac{13}{15}v_0$ (d) $\frac{4v_0}{5}$

108. A uniform ball of radius 17cm rolls down from the top of a fixed, smooth sphere of radius 34 cm without slipping. If the angle with the vertical through which the point of contact of the ball had moved when it loses contact is 53°, then the angular velocity of the ball at that instant is,

(a) 14 rad s^{-1} (b) 5 rad s^{-1} (c) 8 rad s^{-1} (d) 10 rad s^{-1}

109. A uniform solid cylinder of diameter 14 cm rolls without slipping with angular velocity 25 rad s^{-1}. The cylinder suddenly contacts a plane inclined at an angle θ. The value of θ which brings the cylinder immediately to rest after impact is



(a) 45° (b) 60°
(c) 170° (d) 30°

110. A solid ball of diameter 11 cm is rotating about one of its horizontal diameters with an angular velocity of 120 rad s^{-1}. It is released from a height = 1.8 m and falls freely to collide with the horizontal floor $\left(e = \frac{5}{6}\right)$. μ between the ball and ground is 0.2. The fractional change in angular momentum after collision is nearly

(a) 0.4 (b) 0.5 (c) 0.6 (d) 0.7

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

A particle is in uniform motion on a plane. Let P_t denote the point where the particle is at time t. Let O be a fixed point on the plane. Then area of $\Delta OP_1P_2 = \frac{1}{2}$ the area of ΔOP_2P_4.

and

Statement 2

Angular momentum about any point of a particle in uniform motion is constant.

112. Statement 1

A turntable rotates without friction, with a child on its rim. When the child walks inward, the kinetic energy of the system increases.

and

Statement 2

Any work done on the system will increase its kinetic energy.

113. Statement 1

When a solid sphere purely rolls down a rough incline, friction force is non-zero, but mechanical energy is conserved.

and

Statement 2

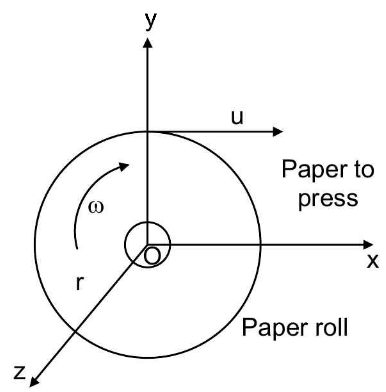
Kinetic friction is negligibly low.

Linked Comprehension Type Questions

Directions: **This section contains a paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Read the following passages and answer the questions

In a continuous printing process, paper is drawn into the press at a constant speed u. The figure shows the paper being drawn into the press from a big paper roll which is free to rotate about a fixed axis OZ without friction. Let r denote the radius of the roll at any time t and x denote the thickness of the paper.



114. The angular velocity of the paper roll at any instant is given by

(a) $\frac{u}{r}$ (b) $\frac{ux}{r^2}$ (c) $\frac{u}{x}$ (d) $\frac{ur}{x^2}$

115. The variation of radius r with time t is given by the equation.

(a) $r\frac{dr}{dt} = -xu$ (b) $r\frac{dr}{dt} = -\frac{xu}{\pi}$ (c) $r\frac{dr}{dt} = -\frac{xu}{2\pi}$ (d) $r\frac{dr}{dt} = xu$

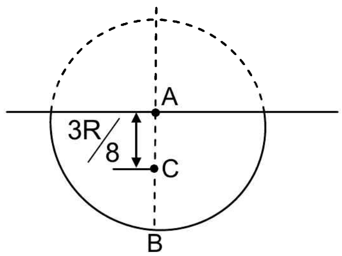
116. The instantaneous angular acceleration of the paper roll is

(a) $\dfrac{u^2}{\pi r^2}$ (b) $\dfrac{u^2 x}{\pi r^3}$ (c) $\dfrac{u^2 x}{2\pi r^3}$ (d) $\dfrac{u^2}{x^2}$

Multiple Correct Objective Type Questions

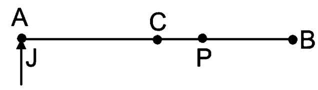
Directions: **Each question in this section has four suggested answers outof which ONE OR MORE answers will be correct.**

117. Consider a hemisphere of mass m and radius R. A is the centre of the complete sphere, (the center of curvature) C is the COM of the hemisphere and B is a point on the surface of the hemisphere collinear with AC. Consider the axes normal to the plane of the paper passing through these points as AA', CC' and BB' and moment of inertia of the hemisphere about these axes as $I_{AA'}$, $I_{CC'}$ and $I_{BB'}$ respectively. Then



(a) $I_{AA'} = \dfrac{2}{5} mR^2$ (b) $I_{AA'} = \dfrac{mR^2}{5}$ (c) $I_{CC'} = \dfrac{83}{320} mR^2$ (d) $I_{BB'} = \dfrac{13}{20} mR^2$

118. A metre stick of mass m is lying on a smooth floor. An impulse J is given at one end, normal to the stick, so that the rod turns with an initial $\omega = 8$ rad s^{-1}. Immediately after the impulse a point P is found to be at rest. Then



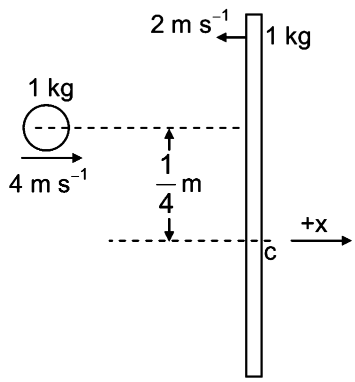
(a) The velocity of COM is $\dfrac{2}{3}$ m s^{-1}

(b) velocity of centre point C is $\dfrac{4}{3}$ m s^{-1}

(c) AP = $\dfrac{2}{3}$ m

(d) If the rod is hanging from A from a pivot and the same impulse is given at P, normal to the rod, the reaction impulse at the pivot is 1 N s.

119. A point mass of 1 kg moving with velocity 4 m s^{-1} has a perfectly elastic collision with a uniform rod of mass 1 kg and 1 m long, moving with velocity 2 m s^{-1} in the opposite direction, at a point $\dfrac{1}{4}$ m above its centre. Take +ve direction as shown. Neglect gravity. Then after collision



(a) velocity of the point mass is $-\dfrac{4}{11}$ m s^{-1}

(b) velocity of the point mass is $+\dfrac{4}{11}$ m s^{-1}

(c) translational velocity of the rod is $\dfrac{26}{11}$ m s^{-1}

(d) angular velocity of the rod is $\dfrac{145}{11}$ m s^{-1}

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120.

Column I	**Column II**
(a) A sphere undergoing a pure rolling on a horizontal surface under the action of a horizontal force F, applied on top, in the direction of motion	(p) Velocity of the point of contact of sphere with the surface is zero
(b) A sphere undergoing rolling with slipping on a horizontal surface under the action of a horizontal force F, applied on top, in the direction of motion.	(q) Friction on sphere supports forward motion
(c) A sphere undergoing pure rolling down an inclined plane, under gravity	(r) Frictional force on sphere is $F_r < \mu N$, when μ = coefficient of static friction for contact surfaces and N= Normal reaction
(d) A sphere undergoing rolling with slipping down an inclined plane, under gravity	(s) Direction of the frictional force on the sphere is opposite to the direction of motion of the sphere

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. A particle of mass 0.2 kg is projected from a point P with a speed of 30 m s^{-1} at an angle $\pi/4$ radian to the horizontal. Find the magnitude and direction of angular momentum of the particle about P after 3 second from launch.

122. The moment of inertia of a solid flywheel about its axis of rotation perpendicular to its plane and passing through its centre is 0.1 kg m^2. It is made to rotate by applying a tangential force of 20 N with a massless inextensible chord wound round the circumference. The radius of the wheel is 10 cm.

(i) Calculate the angular acceleration of the flywheel.

(ii) What would be the angular acceleration if a mass of 2 kg was hung from the end of the chord? (g = 10 m s^{-2})

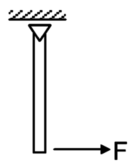
123. The angular momenta of two particles 1 and 2 with respect to a point O is time-varying and are given by $\overline{L_1} = 2t\hat{i} + 3t^2\hat{j}$ and $\overline{L_2} = 4t^2\hat{i} - 2t^3\hat{j}$. If $\overline{\tau_1}$ and $\overline{\tau_2}$ are the resultant torques about the point O acting on 1 and 2 respectively,

(i) Determine the angle between $\overline{\tau_1}$ and $\overline{\tau_2}$ as a function of time t.

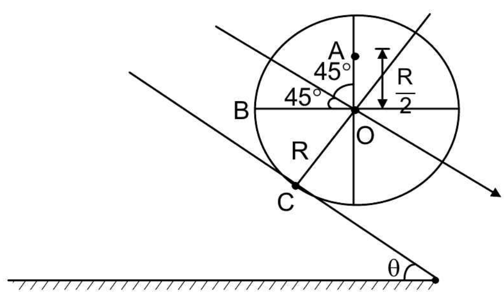
(ii) Determine the time at which $\overline{\tau_1}$ and $\overline{\tau_2}$ are perpendicular to each other.

124. A horizontal turntable in the form of a solid disc is rotating with a constant angular velocity ω_0 about a vertical axis through its centre. A man running tangential to the rim of the turntable with double the velocity of a point on the rim jumps on to the turntable. The angular velocity now is ω_1. The man walks radially a distance $\frac{R}{2}$ (R = radius of turntable) and stops. The angular velocity now is ω_2. If $\frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_1}$, determine the ratio of masses of the turntable and the man.

125. A constant force $F\left(=\frac{mg}{2}\right)$ is applied to the end of a rigid rod in the vertical position as shown. Find the maximum angle it makes with the vertical.

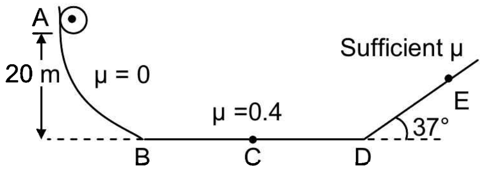


126. A bicycle wheel is freely rolling down an incline without slipping. At the instant as shown, speed of point A is $\sqrt{80 + 32\sqrt{2}}$ m s^{-1}. Determine the speeds of points O, B, and C at the same instant.



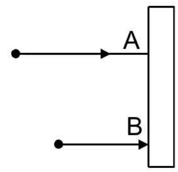
127. A rough floor is parallel to the XZ plane, 1 metre below origin O. At t = 0, a metal shell spinning at $\omega = -100\,\hat{k}$ rad s^{-1} is kept on the floor at P = (0, –1 m, 0). Radius of the shell is 0.1 m, its mass is 0.3 kg. μ of the floor is 0.4 (i) Find the time after which angular momentum of the shell about O will be constant (ii) Find the final constant value of angular momentum about O. (g = 10 m s^{-2})

128.



A round object of uniform composition starts from rest at A and slides down without rolling upto B. BD is horizontal track with $\mu = 0.4$. At C pure rolling starts. BC = 32 m. (i) What is the object? (ii) From C the object continues to roll upto E before returning. What is the length DE? (g = 10 m s^{-2})

129. A thin uniform bar lies on a frictionless horizontal surface. Its mass is 0.16 kg and length is $\sqrt{3}$ m. Two particles, each of mass 80 g are moving at right angles to the bar on the same surface and towards the bar, one with a velocity of 10 m s^{-1} and the other with 6 m s^{-1} as shown in the figure.

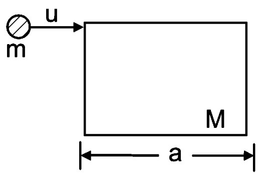


The first particle strikes the bar at point A and the second particle strikes at point B. Points A and B are at a distance of 0.5m from the bar's center. The particles strike the bar at the same instant of time and stick to the bar on collision.

(i) Calculate the loss of kinetic energy of the system in the above collision.

(ii) Find the velocity of the bar after collision.

130.

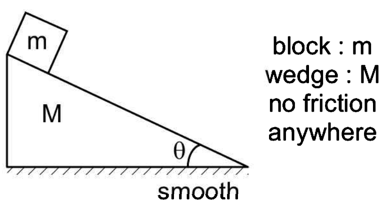


A square plate of uniform thickness and composition of mass M and side a is at rest lying on a smooth horizontal table. A particle of mass m, travels with velocity u parallel to one edge of the plate and strikes the corner and get stuck. Find the (i) translational and (ii) rotational speed of the system.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

131. When released, the condition for the distance m slides on the smooth incline to be less than the distance M slides on the smooth horizontal surface is

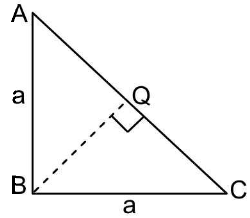


(a) $m > M$ (b) $m > \dfrac{M}{\cos\theta}$ (c) $m > \dfrac{M}{\cos^2\theta}$ (d) impossible to achieve

132. A 75 kg mass is raised to the surface of Earth from a depth of 100 m by a rope of density 3 kg/m. The work done against the gravitational force is

(a) 100 kJ (b) 175 kJ (c) 225 kJ (d) 375 kJ

133. For the right angled isosceles triangular body ABC, The ratio of moments of inertia $I_{AC} : I_{BQ}$ is



(a) $\frac{1}{2}$ (b) 2

(c) $\frac{1}{4}$ (d) 1

134. Moment of inertia of ring of radius R and mass M distributed non-uniformly, about an axis passing through the centre and perpendicular to the plane of the ring is

(a) MR^2 (b) more than MR^2 (c) less than MR^2 (d) not equal to MR^2

135. Four identical straight, thin rods, each of mass m and length ℓ, are made to form a square. Its moment of inertia about one side is

(a) $2m\ell^2$ (b) $\frac{5}{3}m\ell^2$ (c) $\frac{4}{3}m\ell^2$ (d) $\frac{3}{2}m\ell^2$

136. Moment of inertia a uniform right angled isosceles triangular plate about an axis passing through its centroid and parallel to the hypotenuse is I. Its moment of inertia about an axis passing through the centroid and perpendicular to its plane is

(a) 2I (b) 3I (c) 4I (d) 5I

137. At an instant a rigid body is in pure rotation about a point P in it with an angular velocity $\left(-2\hat{i}+\hat{k}\right)$ rad s^{-1}, its CM is at position vector $\left(\hat{i}-2\hat{j}\right)$m and point P is at position vector $a\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\sqrt{3}$ m from origin. At that instant the velocity of the rigid body is (in m s^{-1})

(a) 4 (b) $\sqrt{21}$ (c) $\sqrt{29}$ (d) 9

138. A rigid body is in uniform rotation about an axis $2\hat{i}+3\hat{j}$. Which among the following can be resultant force acting on a particle P of the body situated away from the axis?

(a) $3\hat{i}-6\hat{j}+9\hat{k}$ (b) $9\hat{i}-6\hat{j}+3\hat{k}$ (c) $-6\hat{i}+9\hat{j}-3\hat{k}$ (d) $6\hat{i}+4\hat{j}-9\hat{k}$

139. A particle is in uniform circular motion on XY plane with a speed of 1 m s^{-1}. Its angular momentum about origin is zero at t = 0 s, 1s, 3 s, ..., and maximum when at a distance from origin of (in m)

(a) $\frac{3}{\pi}$ (b) $\frac{9}{2\pi}$ (c) $\frac{4}{\pi}$ (d) $\frac{11}{2\pi}$

140. A particle in motion is subjected to a force of constant magnitude and always directed towards origin. Then

(a) Its angular momentum about origin will vary with time.
(b) Its angular momentum about origin will be constant.
(c) Its path will necessarily be a circle.
(d) both (b) and (c)

141. A particle is moving on XY plane such that its angular momentum about origin is constant. The particle may be executing

(a) uniform motion
(b) motion along straight line with constant acceleration
(c) (a) or (b) or motion along a straight line with variable acceleration
(d) (a) or (b) or (c) or uniform circular motion

142. Two antiparallel forces, 3 N and 8 N acting on a rod make it to purely translate along a direction, 45° to its length. If one of the forces is acting at an end of the rod, the ratio of the distance between their points of action to the length of the rod is

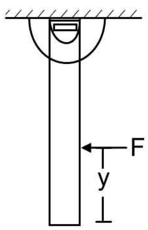
(a) $\frac{3}{11}$ (b) $\frac{5}{16}$ (c) $\frac{3}{8}$ (d) $\frac{5}{8}$

143. A uniform rectangular block is at rest on a horizontal surface. Coefficient of friction is more than 1. The minimum magnitude of a horizontal force to be applied on the block so as to disturb it is equal to its weight and to be applied at a height h above the resting surface. Then the minimum possible volume of the block is

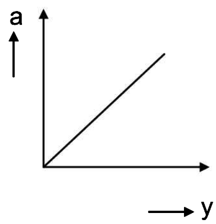
(a) h^3 (b) $2h^3$
(c) $4h^3$ (d) $8h^3$

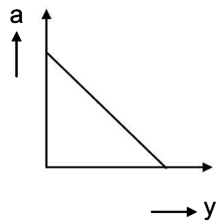
144. If a rigid body is in pure rotation under constant power, starting from rest, if θ is angular displacement and ω its instantaneous angular velocity, then ω is proportional to

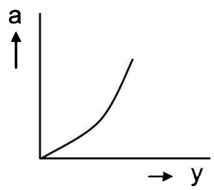
(a) θ (b) $\theta^{2/3}$ (c) $\theta^{1/3}$ (d) θ^2

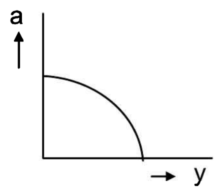
145. 

Graph of acceleration of CM (a) vs y is best represented by

(a) 

(b) 

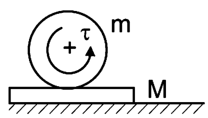
(c) 

(d) 

146. A uniform bar PQ of mass 300 g and length 1 m is pivoted about a horizontal axis through its lower end. Initially it is held vertical and allowed to fall freely down. Its angular acceleration at the instant when the angular displacement is 30° will be (rad s^{-2}) ($g = 10$ m s^{-2})

(a) 4.3 (b) 13 (c) 7.5 (d) 2.5

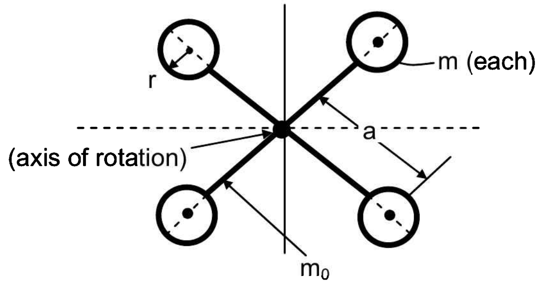
147. A fixed axis disc of mass m and radius r is used to move a plank of mass M. If the floor is smooth, the force between them when an external for a torque τ is applied on the disc is



(a) $\frac{2\tau}{r}$ (b) $\frac{\tau}{r}\frac{M}{m}$

(c) $\frac{2\tau}{r\left(2+\frac{m}{M}\right)}$ (d) $\frac{\tau}{r\left(\frac{m}{M}+1\right)}$

148. An arrangement consists of 4 identical cylindrical tubes, each of mass m at a radial distance a from the axis of rotation. Starting from rest, this attains an angular velocity ω_0 in a time t under a constant torque τ, applied to the shaft. If the radius of each tube is r and mass of the supporting arms is m_0 ($r << a$), then 't' is given by



(a) $t = \dfrac{4mr^2\omega_0}{\tau}$ (b) $t = \dfrac{2(m+m_0)(a^2+2r^2)\omega_0}{\tau}$

(c) $t = \left(\dfrac{(a^2+r^2)4m}{\tau} + \dfrac{4m_0(a-r)^2}{3\tau}\right)\omega_0$ (d) $t = \dfrac{2(m+m_0)(r^2+2a^2)\omega_0}{\tau}$

149. A wheel of mass 10 kg and diameter 0.4 m is uniformly accelerated and attains an angular velocity of 20 rad s^{-1} in 1 s after rotation begins. The torque exerted by the wheel about its rotating axis is

(a) 0.9 N m (b) 1.2 N m (c) 0.49 N m (d) 4 N m

150. A horizontal cylindrical railway tank wagon with radius r and length ℓ, 50% full of petrol is driven around a curve of radius R at a speed v. The curve converges into a straight track where the train maintains the same constant speed v. The torque experienced by the petrol (mass m) is

(a) $-mg\left(\dfrac{4r}{5\pi}\right)\dfrac{v^2}{\sqrt{v^4+R^2g^2}}$ (b) $mg\left(\dfrac{4r}{7\pi}\right)\dfrac{v^2}{\sqrt{v^2+R^2g^2}}$

(c) $-mg\left(\dfrac{4r}{3\pi}\right)\dfrac{v^2}{\sqrt{v^4+R^2g^2}}$ (d) $-mg\left(\dfrac{4\pi}{7\pi}\right)\dfrac{v^4}{\sqrt{v^4+R^2g^2}}$

151. A thin circular ring of mass M and radius R is rotating about an axis through its centre and perpendicular to its plane at a constant angular velocity ω. Two objects each of mass m are attached to the opposite ends of a diameter. Then the ring rotates with a new angular velocity

(a) $\dfrac{M+2m}{\omega}$ (b) $\dfrac{\omega M+2m}{M}$ (c) $\dfrac{m+2M}{\omega}$ (d) $\dfrac{M\omega}{M+2m}$

152. If the ice on the polar caps of the Earth melts, the duration of day will

(a) Decrease (b) Remains the same

(c) Increase (d) May increase or decrease

153. A man of mass m at the edge of a horizontal platform of mass m in the form of a uniform disc covers an angle $\dfrac{\pi}{4}$ rad on the disc. If the disc can rotate about a vertical axis through its centre, the disc rotates through an angle of

(a) $\dfrac{\pi}{4}$ rad (b) $\dfrac{\pi}{2}$ rad (c) $\dfrac{\pi}{6}$ rad (d) $\dfrac{\pi}{12}$ rad

154. A smooth uniform bar of length 1 m and mass 0.4 kg has 2 identical rings of negligible size, each of mass 0.1 kg, which can freely slide along the bar and the system is initially rotating with an angular velocity of 20 rad s^{-1} about an axis passing through its centre. The two rings are initially situated in close proximity of the axis of rotation. There are no external forces. When the rings reach the end of the bar, the new angular velocity is

(a) 10 rad s^{-1} (b) 8 rad s^{-1} (c) 12 rad s^{-1} (d) 16 rad s^{-1}

155. A uniform rod of length ℓ is free to rotate in a vertical plane about a fixed horizontal axis through its bottom end. If it is allowed to fall from rest from upright position, its angular velocity as a function of angular displacement θ is given by

(a) $\sqrt{\frac{6g}{\ell}}\sin\frac{\theta}{2}$ (b) $\sqrt{\frac{3g}{\ell}}\cos\frac{\theta}{2}$ (c) $\sqrt{\frac{6g}{\ell}}\cos\theta$ (d) $\sqrt{\frac{3g}{\ell}}\sin\theta$

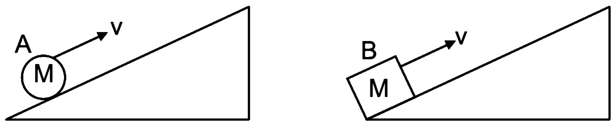
156. A thin spherical shell of radius r with translational velocity of 20 m s^{-1} on a smooth horizontal surface reaches a rough surface of $\mu = 0.4$ at $t = 0$. Its angular velocity ω after 1 s is ($g = 10$ m s^{-2})

(a) $\frac{6}{r}$ (b) $\frac{4}{r}$ (c) $\frac{3}{r}$ (d) $\frac{8}{r}$

157. In the above case, the translational velocity after 1s is

(a) 10 m s^{-1} (b) 12 m s^{-1} (c) 13 m s^{-1} (d) 16 m s^{-1}

158.



A spherical body A of mass M and a square body B of mass M are launched with the same velocity (pure translation) on identical inclined planes with rough surfaces. (Co-efficient of friction is large enough for sphere to roll). Then

(a) both will go up the same distance (b) A will move farther

(c) B will move farther (d) Depends on the value of M.

159. In the above case, if the planes are smooth and s_A and s_B are the distances travelled by A and B on the inclined planes then $s_A : s_B$ is

(a) 1 : 1 (b) 7 : 5 (c) 5 : 7 (d) 1.35 : 1.21

160. A block moves up and then down a rough inclined plane. A cylinder rolls without slipping up and down then down a rough inclined plane. The directions of the frictional forces in the two cases are same in

(a) ascent only (b) descent only

(c) both in ascent and descent (d) neither in ascent nor in descent.

161. A block moves up and down an inclined plane of angle 37° with the horizontal. A cylinder of the same mass rolls without slipping up and down the same inclined plane. In all cases frictional forces are of same magnitude. Then, the coefficient of friction is

(a) 0.25 (b) 0.33 (c) 0.6 (d) 0.75

162. A solid sphere, a hollow sphere, a solid disc and a hollow disc, all four of them were released simultaneously from the top of an inclined plane and they roll down without slipping. Two of them reached the bottom simultaneously. It is certain that one of these two is

(a) solid sphere (b) hollow sphere (c) solid disc (d) hollow disc

163. A ring and a solid sphere released simultaneously at the top of an inclined plane of angle of inclination with horizontal equal to $\tan^{-1} 0.7$ reach the bottom simultaneously. How many among the following values are acceptable for coefficient of friction? 0.1, 0.15, 0.2.

(a) none (b) one (c) two (d) three

164. A disc of radius R rolls with uniform speed v on a horizontal conveyor which moves with uniform speed u. The disc makes one revolution in time

(a) $\frac{2\pi R}{v-u}$ (b) $\frac{2\pi R}{v+u}$ (c) $\frac{2\pi R}{u-v}$ (d) Cannot be concluded

165. When a disc of mass m is given an angular velocity and released on a horizontal surface with coefficient of friction μ, it starts pure rolling after a displacement s. Then μmgs is the

(a) loss in total kinetic energy. (b) loss in rotational kinetic energy.
(c) loss in translational kinetic energy (d) gain in translational kinetic energy.

166. A solid sphere and a hollow sphere of same mass and radius are given the same angular velocity and placed over the same rough surface. Comparing their velocities at steady state,

(a) velocity of solid sphere will be higher (b) velocity of solid sphere will be lower
(c) both will be equal (d) Cannot be concluded

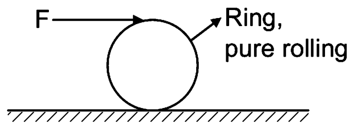
167. If a typical round rigid body, released at the top point of a semicircular vertical track, (convex downwards), rolls without slipping along the track, the ratio of the normal reaction by the track to its weight when it is at the lowest point of the track is

(a) less than 1 (b) $\geq 1, < 2$ (c) $\geq 2, < 3$ (d) ≥ 3

168. If a rigid body rolls down two different inclined planes of same height but different inclinations and different roughness, comparing speeds when reaching bottom and the times of descent in the two cases,

(a) speeds same, times same (b) speeds same, times different
(c) speeds different, times same (d) speeds different, times different

169. The friction force on the ring is



(a) $\frac{F}{3}$ towards left (b) $\frac{2F}{3}$ towardsleft
(c) $\frac{F}{3}$ towards right (d) zero

170. If a rigid body is rolling on a horizontal surface without application of any external force, its angular momentum is constant

(a) only about points lying on line of travel of CM
(b) (a) or about the instantaneous point of contact
(c) (a) or (b) or any point lying on the horizontal surface
(d) (a) or (b) or any point in space

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

Theorem of parallel axes holds even if the rigid body is non-homogeneous.

and

Statement 2

If non-homogeneous, the CM will certainly differ from geometric centre.

172. Statement 1

Theorem of perpendicular axes holds even if the plane body is non-homogeneous.

and

Statement 2

If non-homogeneous, the CM will certainty differ from the geometric centre.

173. **Statement 1**

If a rigid body is in general plane motion, no two points of the body can have the same velocity vector.

and

Statement 2

Velocity of any point of the body whose position vector with respect to CM is $\overline{r}$ is given by $\overline{a} + \overline{r} \times \overline{b}$ where $\overline{a}$ and $\overline{b}$ are constant vectors.

174. **Statement 1**

If several forces act on a rigid body such that resultant is zero, then the resultant torque on the body is independent of the origin chosen only if the number of forces is even.

and

Statement 2

Torque due to couples is independent of origin.

175. **Statement 1**

When a small ring and a very big sphere made of same material are allowed to roll down an inclined plane, the big sphere will reach the ground first.

and

Statement 2

mgsinθ value is more for the big sphere

176. **Statement 1**

When the centre of mass of a round rigid body is in uniform motion on a horizontal surface such that the point of contact of the body with the surface has zero velocity, then friction force at the contact surfaces is zero.

and

Statement 2

Friction force comes into play only when there is relative motion at the point of contact.

177. **Statement 1**

When a round body is kept on a smooth surface and given an impulse and it moves ahead and reaches a rough patch, its energy will always be reduced.

and

Statement 2

At the rough patch, if the body is not performing pure rolling, the friction changes the $\frac{v}{\omega}$ ratio and in the process does negative work.

178. **Statement 1**

Two identical balls are rolling down on two inclined planes of equal angle 30° each, one with μ large enough for pure rolling, the other with μ less than the critical value required for pure rolling. The work done by the frictional torque is positive in the first case and negative in the second case.

and

Statement 2

Losses due to friction happens only in second case.

179. Statement 1

If a smooth, rolling sphere has a glancing elastic collision with another identical sphere at rest, after collision their velocities will be at right angles to each other.

and

Statement 2

Rotational kinetic energies of the two spheres after collision will be equal.

180. Statement 1

If a sphere rolling on a horizontal surface has an elastic collision normally against a smooth wall, its return motion also will be pure rolling.

and

Statement 2

Its speed and angular speed do not change after such an elastic collision.

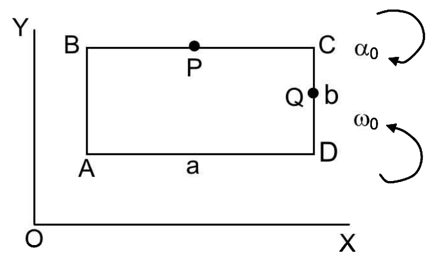
Linked Comprehension Type Questions

***Directions:* This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Read the passage and answer the questions

Passage I

A rigid body is one which does not deform no matter what forces act on it. The distance between two points in the body remains constant. When all the particles constituting the body move parallel to a fixed plane, it is called plane motion. The figure shows a rigid body ABCD moving relative to a fixed reference plane OXY in plane motion. P and Q are two points inside the body.



181. From the definition of the rigid body, we can conclude the following:

(a) Angle APQ, ABP, PQC. remain constant with respect to time.

(b) Relative velocity of P with respect to Q is zero

(c) The motion of A with respect to P is a circular motion. So is the motion of Q with respect to P, A with respect to D, and any point with respect D, and any point with respect to any other point on the body.

(d) Both (a) and (c) are correct

182. At a given instant the angular velocity ω_0 and angular acceleration α_0 for the circular motion of any point on the body relative to any other point on the body are shown. Choose the correct statement.

(a) The velocity of D with respect to O is $\overline{v}_{DO} = a\omega_0\hat{j}$

(b) The velocity of C with respect to D is $\overline{v}_{CD} = b\omega_0\hat{i}$

(c) The velocity of A with respect to D is $\overline{v}_{AD} = -a\omega_0\hat{j}$

(d) The velocity of B with respect to D is zero

183. Translational motion of the rigid body is defined as one in which any line, say PQ, remains parallel to a fixed direction. Let $\overline{R}_A, \overline{R}_B, \overline{R}_P, ... \overline{v}_A, \overline{v}_B, \overline{v}_P ...$ and $\overline{a}_A, \overline{a}_B, \overline{a}_P ...$ denote the position vectors, velocity vectors and acceleration vectors of A, B, P... with respect to O. Then, for translational motion.

(a) $\overline{R}_P - \overline{R}_A$ must be a constant vector

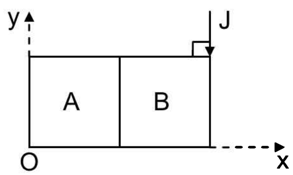
(b) $\overline{v}_A = \overline{v}_B = \overline{v}_P = ...$ and $\overline{a}_A = \overline{a}_B = \overline{a}_P = ...$ at any instant t.

(c) A, B, C, P, ... must move in straight lines that are parallel.

(d) All of the above.

Passage II

A and B are two uniform square plates of sides a joined together. Total mass is M. Density of B is twice that of A. Impulse J is provided as shown



184. x coordinate of centre of mass is

(a) $\frac{3a}{2}$ (b) $\frac{4a}{3}$

(c) $\frac{7a}{6}$ (d) $\frac{13a}{12}$

185. Moment of Inertia of the body about an axis passing through CM and parallel to y axis is

(a) $\frac{11}{36}Ma^2$ (b) $\frac{11}{48}Ma^2$

(c) $\frac{29}{36}Ma^2$ (d) $\frac{29}{48}Ma^2$

186. Immediately after impulse, x coordinate of the instantaneous centre of rotation is

(a) $\frac{7}{15}a$ (b) $\frac{7}{10}a$

(c) $\frac{a}{5}$ (d) $\frac{3}{10}a$

Passage III. Two typical round rigid bodies of the same mass are simultaneously released at the top of an inclined plane. They roll down and reach bottom at different instants such that the translational kinetic energy of one and the rotational kinetic energy of the other are in ratio 5 : 3

187. Comparing their translational kinetic energies, the one reaching earlier will have

(a) the higher value (b) the lower value

(c) the same value as that of the other (d) any of the above is possible

188. The two bodies could be

(a) ring and solid sphere (b) ring and spherical shell (c) spherical shell and disc (d) disc and solid sphere

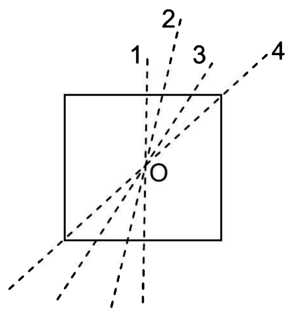
189. If the smaller of the two times of descent is 3 s, the other is

(a) $\sqrt{10}$ s (b) $\sqrt{12}$ s (c) $\sqrt{15}$ s (d) $\sqrt{21}$ s

Multiple Correct Objective Type Questions

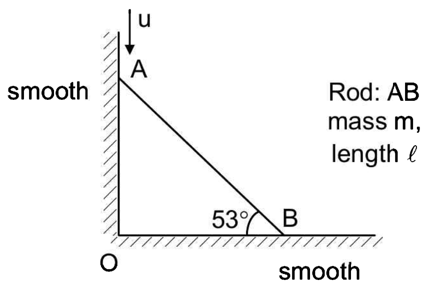
Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

190. Moment of Inertia of a thin uniform square plate about an axis through centre and perpendicular to the plate is



(a) $I_1 + I_4$ (b) $I_2 + I_3$ (c) $2I_1$ (d) $2I_2$

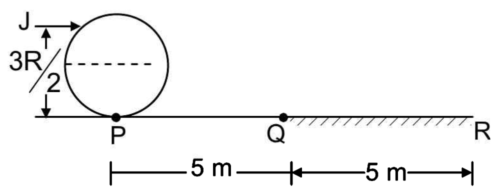
191. At the instant shown, given that speed of end A is u,



(a) speed of end B is $\frac{4}{3}u$ (b) speed of CM is $\frac{5}{6}u$

(c) angular velocity of the rod is $\frac{5u}{3\ell}$ (d) angular momentum of the rod about O is $\frac{5}{9}mu\ell$

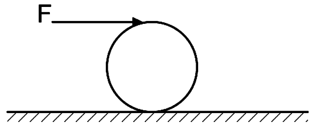
192. A disc of radius R and mass m is kept on a smooth surface P. The floor from P to Q is smooth and from Q onwards it is rough, $\mu = 0.3$. The surface of the disc is rough. It is given a horizontal impulse J at a height $h = \frac{3R}{2}$



(a) After moving through the rough patch, its translational velocity will be reduced from initial value.
(b) There will be no loss of energy for journey from P to R
(c) The final energy is $E = \frac{3}{4}\frac{J^2}{m}$
(d) Just on entering the rough patch at Q, the friction is μmg

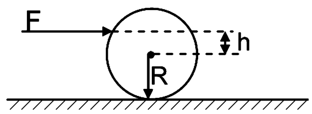
193. A disc rolls without slipping on a plane without application of an external force (other than gravity). The plane can be
(a) Smooth and horizontal (b) rough and horizontal (c) smooth and inclined (d) rough and inclined

194. A constant force is applied at the top of a ring as shown



(a) If the surface is smooth, the ring will slip
(b) Whether the surface is smooth or rough, the ring will roll
(c) The work done by friction will increase the rotational kinetic energy
(d) The work done by friction is zero

195. A round object is rolling on a horizontal surface under the action of an externally applied force F at a height h above centre as shown. Then



(a) If h = 0, there will be no frictional force
(b) If h < 0, the frictional force will oppose F
(c) If h > 0, the frictional force will reinforce F
(d) If h = +R, the frictional force will be maximum

196. A uniform solid sphere A rolling on a horizontal surface with velocity v and angular velocity ω collides elastically with an identical sphere B which is at rest. After collision their velocities are v_A and v_B respectively and angular velocities are ω_A and ω_B respectively. Then
(a) $v_A + v_B = v$ (b) $v_B = v$ (c) $\omega_A + \omega_B = \omega$ (d) $\omega_A = \omega$

197. A uniform rigid light rod AB, with centre C, lying on a smooth horizontal table has two unequal point masses attached to it, one at each end. An impulse J is applied at A in the plane of the table and perpendicular to AB. Then
(a) Point B cannot be at rest
(b) No point of the rod, in between A and C can be at rest
(c) If a point on the rod is at rest, then mass of the particle at B is more than that at A
(d) Some point on the rod has to be at rest

Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

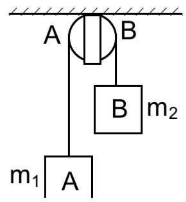
198. Column I gives a rigid body and describes an axis passing through at least one point of the body. Column II gives possible values of $\frac{K^2}{R^2}$ where K is the radius of gyration about that axis. Match the columns.

Column I	Column II
(a) disc: radius R: axis lies in the plane of the disc	(p) 0.33
(b) Ring: radius R : axis lies in the plane of the ring	(q) 0.66
(c) Square plate: Side R : axis lies in the plane of the plate	(r) 1
(d) solid sphere, radius R	(s) 1.33

199. A rigid body is acted upon by two forces as per column I. Column II gives possible motions of the body. Match the columns

Column I	Column II
(a) Equal and opposite but neither through CM	(p) translation
(b) Equal, and in same direction but neither through CM	(q) rotation
(c) Unequal and opposite, but neither through CM	(r) no translation
(d) Unequal and in same direction but neither through CM	(s) no rotation

200. Figure represents a mechanical system in which masses m_1 and m_2 ($m_1 \neq m_2$) are connected by an inextensible string running over a pulley of mass M fixed to a support. The following are the parameters of motion : T_1 and T_2 – Tension in the string on either side at A and B respectively



I – Moment of inertia of the pulley

R – Radius of the pulley

α – Angular acceleration of the pulley

Here friction refers to pulley support.

	Column I		**Column II**
(a)	frictionless smooth pulley	(p)	pulley rotates
(b)	frictionless rough pulley	(q)	pulley does not rotate
(c)	with friction smooth pulley	(r)	$T_1 \neq T_2$
(d)	with friction rough pulley	(s)	$(T_1 - T_2) R = I\alpha$

SOLUTIONS

ANSWERS KEYS

Topic Grip

1. $\frac{a}{5}$
2. 13 kg m^2
3. $\frac{M}{12}\left(3R^2 + \ell^2\right)$
4. $2\omega_0$
5. $\sqrt{\frac{8g}{R}\frac{m}{M+2m}} \cdot \cos\left(\frac{\beta}{2}\right)$
6. $\cos^{-1}\left(1 - \frac{\ell\omega_0^2}{3g}\right)$
7. (i) $\frac{\omega_0^2 r^2}{2mg\sin\theta}\left(m + \frac{M}{2}\right)$

   (ii) $v = r\omega_0 - \frac{mg\sin\theta}{m + \frac{M}{2}} . t$

   $a = \frac{mg\sin\theta}{m + \frac{M}{2}}$
8. (i) $a = 3.27\ ms^{-2}$

   $\alpha = 7.1$ rad s^{-2}

   (ii) 4.69 s

   (iii) 33.3 rad s^{-1}

   (iv) 1.50 N m

   (v) 3.26 N
9. 9.96 m s^{-2}
10. $v_{cm} = -\frac{u}{2}$

    $\omega = \frac{84}{29} . \frac{u}{\ell}$

11. (d)	12. (a)	13. (c)
14. (c)	15. (c)	16. (c)
17. (b)	18. (b)	19. (a)
20. (d)	21. (b)	22. (b)
23. (d)	24. (c)	25. (d)

26. (c)
27. (b), (c)
28. (a), (c)
29. (a)
30. (a) – (s)

    (b) – (q)

    (c) – (p), (r)

    (d) – (q)

IIT Assignment Exercise

31. (b)	32. (b)	33. (d)
34. (b)	35. (a)	36. (d)
37. (d)	38. (d)	39. (b)
40. (c)	41. (b)	42. (d)
43. (c)	44. (c)	45. (d)
46. (c)	47. (b)	48. (c)
49. (b)	50. (a)	51. (c)
52. (a)	53. (d)	54. (a)
55. (b)	56. (c)	57. (b)
58. (c)	59. (d)	60. (b)
61. (a)	62. (d)	63. (c)
64. (b)	65. (c)	66. (a)
67. (c)	68. (a)	69. (b)
70. (c)	71. (b)	72. (d)
73. (c)	74. (b)	75. (a)
76. (b)	77. (c)	78. (c)
79. (d)	80. (c)	81. (b)
82. (c)	83. (a)	84. (b)
85. (c)	86. (b)	87. (a)
88. (d)	89. (b)	90. (c)
91. (c)	92. (a)	93. (d)
94. (d)	95. (b)	96. (c)
97. (b)	98. (c)	99. (c)
100. (a)	101. (c)	102. (b)
103. (c)	104. (d)	105. (b)
106. (b)	107. (c)	108. (d)
109. (b)	110. (a)	111. (a)
112. (b)	113. (c)	114. (a)

115. (c)
116. (c)
117. (a), (c), (d)
118. (b), (c)
119. (a), (c)
120. (a) – (p), (q)

     (b) – (q), (r)

     (c) – (p), (s)

     (d) – (r), (s)

Additional Practice Exercise

121. $-191\ \hat{k}$ kg m^2 s^{-1}
122. (i) 20 rad s^{-2}

     (ii) 16.7 rad s^{-2}
123. (i) $\theta = \cos^{-1}\left(\frac{4 - 9t^2}{\sqrt{\left(9t^2+1\right)\left(9t^2+16\right)}}\right)$

     (ii) $\frac{2}{3}$ s
124. 4
125. $\frac{\pi}{2}$ rad
126. $\overline{v}_O = 8\hat{i}$ m s^{-1},

     $\overline{v}_B = 8\left[\left(1 - \frac{1}{\sqrt{2}}\right)\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right] ms^{-1}$

     $\overline{v}_c = 0$
127. (i) 1 s

     (ii) $1\hat{k}$ kg m^2 s^{-1}
128. (i) spherical shell

     (ii) 20 m
129. (i) 2.72 J

     (ii) 4 m s^{-1}
130. (i) $\frac{mu}{M + m}$

     (ii) $\frac{3mu}{\left(M + 4m\right)a}$
131. (d)

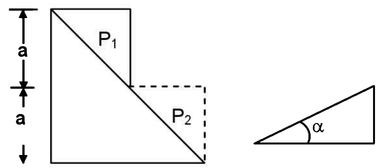
132. (c)	133. (d)	134. (a)
135. (b)	136. (c)	137. (b)
138. (b)	139. (b)	140. (b)

141. (d) **142.** (b) **143.** (c)
144. (c) **145.** (b) **146.** (c)
147. (c) **148.** (c) **149.** (d)
150. (c) **151.** (d) **152.** (c)
153. (c) **154.** (b) **155.** (a)
156. (a) **157.** (d) **158.** (b)
159. (a) **160.** (b) **161.** (a)
162. (b) **163.** (d) **164.** (d)
165. (d) **166.** (b) **167.** (c)
168. (b) **169.** (d) **170.** (d)
171. (c) **172.** (c) **173.** (a)
174. (d) **175.** (b) **176.** (c)
177. (d) **178.** (d) **179.** (c)
180. (d) **181.** (d) **182.** (c)
183. (d) **184.** (c) **185.** (a)
186. (b) **187.** (a) **188.** (c)
189. (a)
190. (a), (b), (c), (d)
191. (a), (b), (c)
192. (b), (c)
193. (a), (b), (d)
194. (b), (d)
195. (b)
196. (a), (b), (c), (d)
197. (b), (d)
198. (a) – (p), (q), (r)
(b) – (q), (r), (s)
(c) – (p)
(d) – (q), (r), (s)
199. (a) – (q), (r), (s)
(b) – (p), (q), (s)
(c) – (p), (q), (s)
(d) – (p), (q), (s)
200. (a) – (q), (s)
(b) – (p), (r), (s)
(c) – (q), (s)
(d) – (p), (q), (r), (s)

HINTS AND EXPLANATIONS

Topic Grip

1.

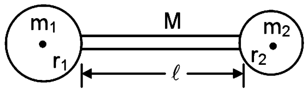


The vertical distance the small wedge comes down = a = Horizontal distance moved relative to the large wedge.

Since there are no horizontal external forces, the centre of mass is not displaced in the x – direction. If x is the distance moved by big wedge, actual distance mo ved by the small wedge = (a – x). COM remains stationary.

$\therefore \quad 4mx = m(a - x) \Rightarrow x = \frac{a}{5}$

2.



With origin at mid point of rod,

$$x_{CM} = \frac{-m_1\left(r_1 + \frac{\ell}{2}\right) + m_2\left(r_2 + \frac{\ell}{2}\right)}{M + m_1 + m_2}$$

$$= \frac{20(-0.2 - 0.5) + 10(0.1 + 0.5)}{10 + 20 + 10} = -0.2\text{ m}$$

$$I_{CM} = \frac{2}{5}m_1r_1^2 + m_1\left(r_1 + \frac{\ell}{2} - x_{CM}\right)^2$$

$$+\frac{2}{5}m_2r_2^2 + m_2\left(r_2 + \frac{\ell}{2} + x_{CM}\right)^2$$

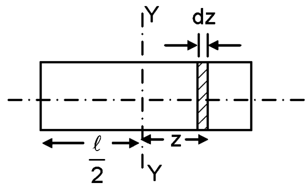
$$+\frac{M\ell^2}{12} + M\left(x_{CM}\right)^2$$

$$\Rightarrow I_{CM} = \frac{2}{5} \times 20\ (0.2)^2 + 20\ (0.2 + 0.5 - 0.2)^2$$

$$+ \frac{2}{5} \times 10 \times (0.1)^2 + 10 \times (0.1 + 0.5 + 0.2)^2$$

$$+ 10 \times \frac{1}{12} + 10(0.2)^2 = 13\text{ kg m}^2$$

3.



Consider thin disc, dz thick, radius R at z from YY axis. Moment of inertia of this thin disc about its axis parallel to YY is

$$dI = dm\frac{R^2}{4}$$

$dm = \pi R^2 \rho dz$ (where $\rho = \frac{M}{\pi R^2 \ell}$ = density of cylinder material); $dI_{yy} = dI + dmz^2$

For solid cylinder,

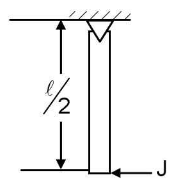
$$I_{yy} = \int_{-\ell/2}^{+\ell/2} dm\frac{R^2}{4} + \int_{-\ell/2}^{+\ell/2} dmz^2$$

$$= \frac{\pi R^4 \rho}{4}\int_{-\ell/2}^{+\ell/2} dz + \pi R^2 \rho \int_{-\ell/2}^{+\ell/2} z^2 dz$$

$$\rho.\pi R^2 \ell\left[\frac{R^2}{4} + \frac{\ell^2}{12}\right] = M\left(\frac{R^2}{4} + \frac{\ell^2}{12}\right)$$

$$= \frac{M}{12}\left(3R^2 + \ell^2\right)$$

4. For the hinged part:

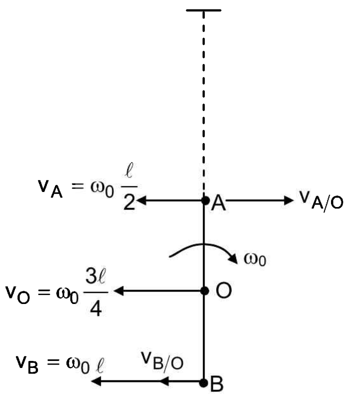


Angular impulse = change in angular momentum

$$J.\frac{\ell}{2} = (4I_0)\left(\frac{3\omega_0}{2} - \omega_0\right)$$

where I_0 is the moment of inertia of a rod of length $\frac{\ell}{2}$ and mass $\frac{m}{2}$ about its centre of mass.$\Rightarrow J\ell = 4I_0\omega_0$

If we consider the lower point AOB; the velocities of points A, O, B are as shown. Hence with respect to centre



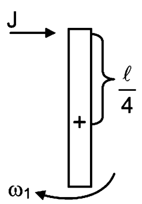
$$v_{A/O} = \omega_0 \frac{\ell}{2} - \omega_0 \frac{3\ell}{4} = -\omega_0 \frac{\ell}{4}$$

$$v_{B/O} = \omega_0 \ell - \omega_0 \frac{3\ell}{4} = +\omega_0 \frac{\ell}{4}$$

Hence the lower half portion can be considered to have translational motion of $\omega_0 \frac{3\ell}{4}$ of COM as well as a rotation of ω_0 about its centre.

For the lower part change of rotational angular momentum about its centre is:

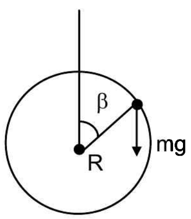
$J.\frac{\ell}{4} = I_0 (\omega_1 - \omega_0)$ (For the lower part)



$$\frac{4I_0\omega_0}{4} = I_0(\omega_1 - \omega_0)$$

$$\omega_1 = 2\omega_0$$

5.



Initial: P.E = mgR (1 + cosβ) (taking PE at lower most position as zero)

$$\text{Final : K.E} = \frac{1}{2}\frac{MR^2}{2}.\omega^2 + \frac{1}{2}.mR^2 . \omega^2$$

$$\Rightarrow mgR . 2\cos^2 \beta/2 = \frac{1}{4}. R^2 \omega^2 (M+2m)$$

$$\Rightarrow \omega = \sqrt{\frac{8g}{R}\frac{m}{(M+2m)}} . \cos \beta/2$$

6. Let mass be m

Energy Equation:

$$\text{Initial Energy} = \frac{1}{2}I\omega_0^2 = \frac{m\ell^2}{6}\omega_0^2$$

$(\because I = \frac{m\ell^2}{3}$ about the hinge)

$$\text{final P.E} = mg\frac{\ell}{2}(1 - \cos\theta_m)$$

$(\because$ COM is at a depth of $\frac{\ell}{2}$ from hinge)

$$\text{Equating, } \theta_m = \cos^{-1}\left(1 - \frac{\ell\omega_0^2}{3g}\right)$$

7. Block moves with retardation 'a'

(i) $mg\sin\theta - T = ma$

$$T.r = I\alpha = \frac{Ia}{r}$$

$$mg\sin\theta = \left(m + \frac{I}{r^2}\right)a$$

$$\Rightarrow a = \left(\frac{mg\sin\theta}{m + \frac{M}{2}}\right)$$

$$v_0 = \omega_0 r; v_0^2 = 2as$$

$$s = \frac{v_0^2}{2a} = \frac{\omega_0^2 r^2}{2mg\sin\theta}\left(m + \frac{M}{2}\right)$$

(ii) $v = v_0 - at = \omega_0 r - \left(\dfrac{mg\sin\theta}{m + \frac{M}{2}}\right)t$

$$a = \frac{mg\sin\theta}{m + \frac{M}{2}}$$

8.

(i) $a = \dfrac{g\sin\theta}{\left(1+\dfrac{K^2}{R^2}\right)}$ and $K^2 = \dfrac{R^2}{2}$ for disc

$a = \dfrac{2}{3}g\sin 30 = \dfrac{2}{3}\times 9.8\times\dfrac{1}{2} = 3.27\ \text{m s}^{-2}$

$a = R\alpha$

$\therefore\quad \alpha = \dfrac{3.27}{0.46} = 7.1\ \text{rad s}^{-2}$

(ii) Since the disc starts from rest, $S = \dfrac{1}{2}at^2$ or t

$= \sqrt{\dfrac{2S}{a}} = \sqrt{\dfrac{2\times 36}{3.27}} = 4.69\ \text{s}$

(iii) As there is no slipping between disc and incline, frictional force as well as normal force do no work. Thus, only gravity does work, so that mechanical energy is conserved. The initial Kinetic Energy is zero.

Final Kinetic Energy $= E_{kf} = \dfrac{1}{2}Mv^2 + \dfrac{1}{2}I\omega^2$

= Loss of potential energy.= MgL sin30°

Putting $I = MR^2$ and $v = \omega R$, we have,

$\dfrac{1}{2}\left(MR^2 + \dfrac{1}{2}MR^2\right)\omega^2 = \dfrac{3}{4}MR^2\omega^2$

$0.75MR^2\omega^2 = MgL\sin30°$

$\Rightarrow\quad 0.75\times(0.46)^2\omega^2 = 9.8\times 36\times 0.50$

$\Rightarrow\quad \omega = 33.3\ \text{rad s}^{-1}$

Aliter:

$\omega = \alpha t$

$= 7.1\times 4.69 = 33.3\ \text{rad s}^{-1}$

(iv) Torque $\tau = I\alpha = \dfrac{1}{2}MR^2\times\alpha = \dfrac{1}{2}\times 2\times(0.46)^2\times 7.1 = 1.50\ \text{N m}$

(v) Frictional force on the disc = f

$\tau = fR$

$f = \dfrac{\tau}{R} = \left(\dfrac{1.50}{0.46}\right) = 3.26\ \text{N}$

9. Acceleration of the sphere rolling down the incline without slipping is $a = \dfrac{5}{7}g\sin\theta$

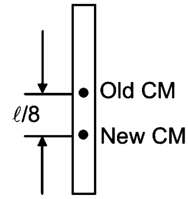
S = Distance $= \dfrac{1}{2}at^2 = \dfrac{1}{2}\times\dfrac{5}{7}g\sin\theta\times t^2$;

But S = 1 m (data)

i.e; $1 = \dfrac{1}{2}\times\dfrac{5}{7}\times\dfrac{g}{100}\times 5.3^2$ ($\because \sin\theta = \dfrac{1}{100}$ per data, t = 5.3 s per data)

$\therefore\quad g = \dfrac{280}{5.3\times 5.3} = \dfrac{280}{28.09} = 9.96\ \text{m s}^{-2}$

10.



Conservation of linear momentum:

$\dfrac{mu - 4mu}{6m} = v_{CM} = \dfrac{-u}{2}\left(\overset{+}{\rightarrow}\right)$

After collision, CM changes.

Take origin at old CM

New CM $= \dfrac{m.\dfrac{\ell}{4} - 4m.\dfrac{\ell}{4}}{6m} = \dfrac{-\ell}{8}$ (i.e., towards 4m)

Conservation of angular momentum (about new CM)

Before collision (clockwise)

$L_0 = mu\left(\dfrac{\ell}{4}+\dfrac{\ell}{8}\right) + 4mu\left(\dfrac{\ell}{4}-\dfrac{\ell}{8}\right)$

$= mu\ell.\dfrac{7}{8}$ — (1)

$I_{newCM} = \dfrac{m\ell^2}{12} + \dfrac{m\ell^2}{64} + m.\dfrac{9\ell^2}{64} + 4m.\dfrac{\ell^2}{64} = \dfrac{29}{96}m\ell^2$

Angular momentum after collision

$L_0 = \dfrac{29}{96}m\ell^2\omega'$ — (2)

Equating (1) and (2) $\omega' = \dfrac{84}{29}\dfrac{u}{\ell}$

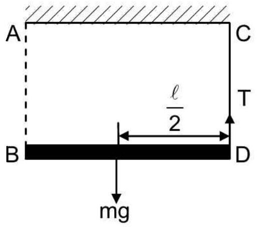
11. $PE = mg\dfrac{L\sin\alpha}{2}$

$KE = \dfrac{1}{2}mv^2 \Rightarrow v^2 = gL\sin\alpha$

$\Rightarrow\quad v = \sqrt{gL\sin\alpha}$

12. By parallel axis theorem,

$I = I_{cm} + M\times d^2 = 600 + 30\times 5^2 = 1350\ \text{g cm}^2$

13. 

Let m be the mass of the rod and ℓ be its length. COM of the rod is at its centre. When AB is cut:

$$mg - T = m\,a_{CM} \qquad \text{— (1)}$$

Taking moments about COM:

$$T\frac{\ell}{2} = I\alpha = \frac{m\ell^2}{12}\alpha \qquad \text{— (2)}$$

$$\alpha = \frac{a_{CM}}{\ell/2} \qquad \text{— (3)}$$

$$\therefore \quad T\frac{\ell}{2} = \frac{m\ell^2}{12}\,\frac{a_{CM}}{\ell/2} \times T = \frac{m}{3}a_{CM} \qquad \text{— (4)}$$

From (1) & (4) $\times\ mg - \frac{m}{3}a_{CM} = m\,a_{CM} \times a_{CM} = \frac{3g}{4}$;

$$T = \frac{m}{3}\frac{3g}{4} = \frac{mg}{4}$$

original $T = \frac{mg}{2}$

$\therefore \quad \Delta T = -50\%$

14. $K_0 = \frac{1}{2} I\,\omega_0^{\,2}$

$$L = I\omega_0 = \frac{2K_0}{\omega_0}$$

initial angular momentum is conserved

15. $\frac{1}{2}Mv_0^2 + \frac{1}{2}Mk^2\frac{v_0^2}{r_2^2} = \frac{1}{2}Mp^2v_0^2$

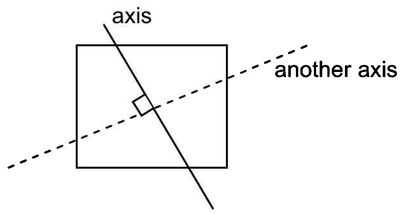
(conservation of KE)

$$\frac{1}{2}Mv_0^2\left(1 + \frac{k^2}{r_2^2}\right) = \frac{1}{2}Mv_0^2.p^2$$

$r_2^{\,2} + k^2 = p^2r_2^{\,2} \Rightarrow k^2 = p^2r_2^{\,2} - r_2^{\,2} = r_2^{\,2}(p^2 - 1)$

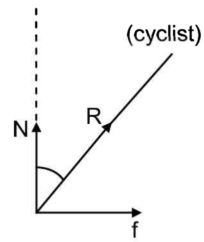
$k = r_2\sqrt{p^2 - 1}$

16. Resultant on any particle will not be zero. However sum of all resultants will be zero.

17. 

Let I_{22} be the M.I perpendicular to the plate through its centre. By perpendicular axis theorem M.I. about each axis shown $= \frac{I_{22}}{2}$ = constant

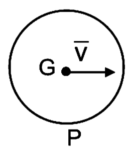
18. But the situation is not one of dynamic equilibrium, but to provide centripetal acceleration.



As you tilt, the frictional forces developed as shown : $\bar{R} = \bar{N} + \bar{f}$; R sin θ necessary centripetal force: R sin $\theta = \frac{mv^2}{r}$

(Leaning inward provides torque of normal reaction to be equal and opposite to torque by frictional force.)

19. With respect to G, $v_p = -\bar{v}$, $\omega = \frac{v}{R}$



With respect to P, $v_a + \bar{v}$, $\omega = \frac{v}{R}$

20. When μ less than the minimum required

$$a = \frac{mg\sin\theta - \mu mg\cos\theta}{m} = g(\sin\theta - \mu\cos\theta)$$

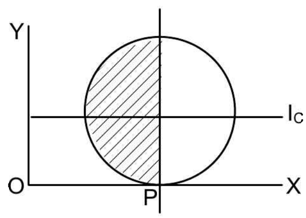
21. 

Fig 2:

Let m be the mass of given body. Considering full sphere. $I_C' = \frac{2}{5}(2m)R^2$.

$\therefore \quad I_C = \frac{1}{2} I_C'$

$I_C = \frac{2}{5}mR^2$, I_C passes through COM.

$I_{ox} = I_C + mR^2 = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$

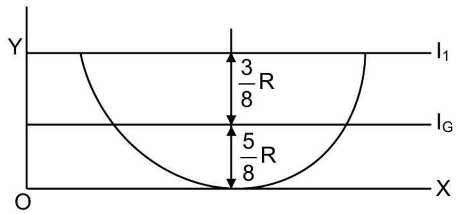


Fig 3 :

$I_1 = \frac{2}{5}mR^2 = I_G + m\left(\frac{3}{8}R\right)^2$

$\Rightarrow \quad I_G = \frac{2}{5}mR^2 - \left(\frac{3}{8}\right)^2 mR^2$

By parallel axis theorem

$I_{ox} = I_G + m\left(\frac{5}{8}R\right)^2$

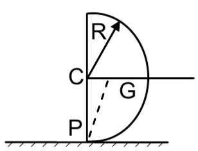
$I_{ox} = \frac{2}{5}mR^2 + mR^2\frac{(25-9)}{64} = mR^2\left(\frac{2}{5} + \frac{1}{4}\right)$

$I_{ox} = \frac{13}{20}mR^2$

The ratio is $\frac{7}{5} \times \frac{20}{13} = \frac{28}{13}$

22. $J = Wx_{CM} = mg\left(\frac{3R}{8}\right)$; Also $J = I_p\alpha$ where I_p is the M.I. about an axis at P normal to the plane of the paper.

$I_G = \frac{2}{5}mR^2 - m\left(\frac{3}{8}R\right)^2$



$(PG)^2 = R^2 + \left(\frac{3}{8}R\right)^2$

$\therefore \quad I_p = I_G + m(PG)^2$

$= \frac{2}{5}mR^2 - m\left(\frac{3}{8}R\right)^2 + mR^2 + m\left(\frac{3}{8}R\right)^2$

$= \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$

$\therefore \quad J = mg\left(\frac{3R}{8}\right) = \frac{7}{5}mR^2\alpha$

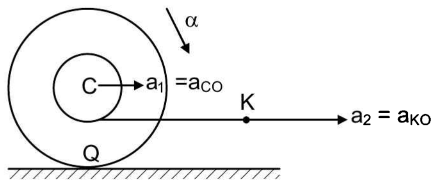
$\therefore \quad R\alpha = \frac{15}{56}g = a_{CD}$

$\bar{a}_{CD} = \frac{15}{56}g\hat{i}$

23. Torque = mg.0 =0 (about point of contact)

Angular acceleration = 0

24.



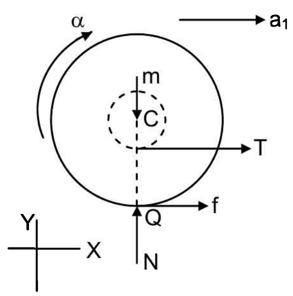
As the cord is inextensible, every point on the straight part of the cord has acceleration of magnitude a_2.

With respect to the point Q through which the axis of rotation passes,

$a_2 = 2R\alpha$; $a_1 = 3R\alpha$

$\frac{a_1}{3} = \frac{a_2}{2} = R\alpha$; $a_1 = \frac{3a_2}{2}$

25.



$f(3R) + T(R) = -4\,mR^2\alpha$

$3f + T = -4\,mR\alpha$ — (1)

$f + T = ma_1$ — (2)

$mg - T = ma_2$ — (3)

$a_1 = 3R\alpha$; $a_2 = \frac{2}{3}a_1$ (from Q # 24)

$= \frac{2}{3} \times 3R\alpha = 2R\alpha \times T = \frac{13\,mg}{17}$

Aliter:

From (Q 24): $a_2 = 2R\alpha$

$$mg - T = ma_2 = m\ 2R\alpha$$

$$\times\ \alpha = \frac{mg - T}{m2R} \qquad \text{— (1)}$$

Taking torque about Q

$$T.\ 2R = I_Q\ \alpha$$

Substitute I_Q from Q 26 and use eqn (1) above:

$$T2R = \frac{13mR^2(mg - T)}{m2R}$$

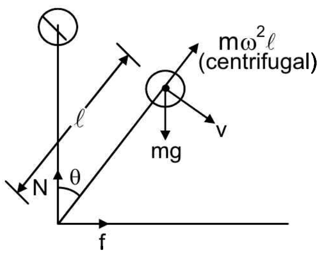
$$\Rightarrow \quad 4T = 13\ mg - 13T \Rightarrow T = \frac{13mg}{17}$$

26. $I_Q = I_C + m\ (3R)^2$

from parallel axis theorem.

$$= m\ (2R)^2 + m\ (3R)^2 = 13\ mR^2$$

27.



If there is no friction, the COM will be falling vertically down, i.e., bottom end will be moving to left. Hence, when there is friction, the direction of friction is such as to oppose the tendency of the bottom end to move to the left and so it will act as shown, i.e., towards right.

Resolving forces along radial $\Rightarrow mg\cos\theta - m\omega^2\ell$

$$= N\cos\theta + f\sin\theta \qquad \text{— (1)}$$

From energy equation $\ell\ (1 - \cos\theta)\ mg$

$$= \frac{1}{2}I\omega^2 = \frac{1}{2}m\ell^2\omega^2 \qquad \text{— (1)}$$

$$\therefore \quad \omega^2 = \frac{2g}{\ell}(1 - \cos\theta) \qquad \text{— (2)}$$

$\Rightarrow$ substitute (2) in (1)

$$mg\cos\theta - m\ \frac{2g}{\ell}(1 - \cos\theta).\ell$$

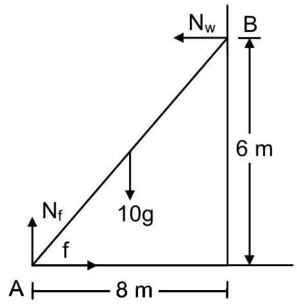
$$= N\cos\theta + f\sin\theta \qquad \text{— (3)}$$

$f = 0$, when $N = 0$

For N to be zero $\Rightarrow$ LHS = 0 $\Rightarrow mg\cos\theta = 2\ mg\ (1 - \cos\theta)$

$$\cos\theta = \frac{2}{3}$$

28.



For the horizontal stability

$$f - N_w = 0$$

For vertical stability

$$N_f - 100 = 0 \Rightarrow N_f = 100\ N$$

$\Rightarrow$ (c) is correct

Taking moments about A

$$N_w\ .\ 6 - 100 \times 4 = 0\ \times N_w = \frac{400}{6} = \frac{200}{3}\ N$$

$$\therefore \quad f = N_w = \frac{200}{3}\ \times \mu N_f \geq \frac{200}{3}\ \times \mu 100 \geq \frac{200}{3}$$

$$\mu_{min} = \frac{2}{3}\ \times \text{(a) is correct}$$

$\therefore$ (a) and (c) are correct

29. For the sphere to roll without slipping

$$\mu mg\cos\theta > \frac{mg\sin\theta}{1 + \frac{mr^2}{I}}$$

$$\Rightarrow \quad \mu > \frac{\tan\theta}{1 + \frac{mr^2}{I}} = \frac{\tan\theta}{1 + \frac{mr^2}{\frac{2}{5}mr^2}} = \tan\theta.\frac{2}{7}$$

$$> \frac{2}{7} \times 0.75 = 0.214$$

since $\mu < 0.214$ it slips $\times\ a = \dfrac{mg\sin\theta - f}{m}$

$\Rightarrow f = \mu\ mg\cos\theta \Rightarrow$

$$a = \frac{10 \times 0.6 - 0.2 \times 1 \times 10 \times 0.8}{1} = 4.4\ m\ s^{-2}$$

$\Rightarrow$ (a) is correct

when it reaches bottom $\Rightarrow v^2 = 2a\ell$

$$= 2 \times 4.4 \times 5 = 44$$

$$\therefore \quad \text{Translational KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 1 \times 44$$

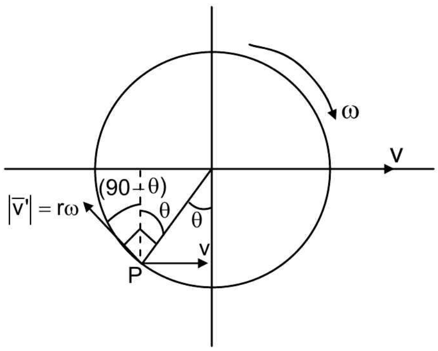
$$= 22\ J$$

$\therefore$ total KE > 22 J and KE < 30 J

When $\mu = 0.22$, it is higher than the minimum value required for rolling without slipping; the ball will roll without slipping and the KE is exactly mgh $= 1 \times 10 \times 3 = 30$ J, increases

$\therefore$ (a) is correct

30. With respect to the axis of rotation passing through A, the velocity at any point on the rim is given by $v_P = 2v \sin\left(\frac{\theta}{2}\right)$ as shown below:



$|\bar{v}'| = r\omega = v$ and the angle between $\bar{v}'$ & $\bar{v}$ is $(90 + (90 - \theta)) = 180 - \theta$

$\therefore$ The resultant velocity of P is: $\bar{v}_P = \bar{v}' + \bar{v}$

$v_P^2 = v^2 + v^2 + 2v^2 \cos(180 - \theta) = 2v^2 - 2v^2 \cos\theta$

$= 2v^2 (1 - (1 - 2\sin^2\frac{\theta}{2}) = 4v^2 \sin^2\frac{\theta}{2}$

$v_P = 2v \sin\frac{\theta}{2}$

Hence, speed of A is 0 $\quad\therefore$ (a) $\rightarrow$ (s)

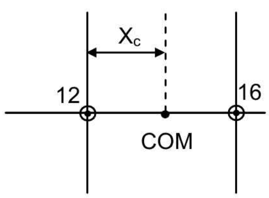
speed of B is V $\quad\therefore$ (b) $\rightarrow$ (q)

speed of C is $2V = 2R\omega$ $\quad\therefore$ (c) $\rightarrow$ (p), (r)

speed of O is V $\quad\therefore$ (d) $\rightarrow$ (q)

IIT Assignment Exercise

31.



$m_c = 12$ unit $\quad m_O = 16$ unit

Taking carbon as origin

$\Rightarrow \sum m_i x_i = 12\times0+16\times1.2$

$$X_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{16 \times 1.2}{12 + 16} = \frac{4.8}{7} \sim 0.69 \text{ Å}$$

32. Since no external forces are involved in explosion, the CM continues to move in the same parabolic path as before. Let R be the range of the original projectile. Then

$$m.\ x_{CM} = \frac{m}{4}.\frac{R}{2} + \frac{3m}{4}R'$$

$\because \quad (x_{CM})_{final} = R;$

substituting $mR = \frac{mR}{8} + \frac{3}{4}mR'$

$$\Rightarrow \quad mR\left(1 - \frac{1}{8}\right) = \frac{3}{4}mR'$$

$$R' = \frac{7}{6}R = \frac{7}{6}.u^2\ \frac{2.\sin\alpha\cos\alpha}{g}$$

$$= \frac{7}{6}\times 100^2 \times \frac{\frac{3}{5}\times\frac{4}{5}}{10} \times 2 = 1120 \text{ m.}$$

33. $K^2 = \frac{\Sigma mr^2}{\Sigma m}$

$$= \frac{m \times 0 + 2m \times a^2 + 3m(2a)^2 + 4m(3a)^2}{m + 2m + 3m + 4m}$$

$$K^2 = \frac{50ma^2}{10m} = 5a^2$$

$K = \sqrt{5}a$

34. Moment of Inertia of a solid sphere about its diameter $= \frac{2}{5}Mr_s^2$

Moment of Inertia of hollow sphere about its diameter $= \frac{2}{3}Mr_h^2$

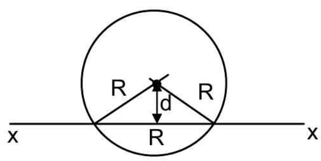
Since both their masses are given to be equal,

$$\frac{1}{5}r_s^2 = \frac{1}{3}r_h^2;\ \frac{r_s}{r_h} = \sqrt{\frac{5}{3}}$$

35. Moment of inertia of ring $= M_r R^2$.

Moment of Inertia of disc $= \frac{1}{2}M_d R^2,\ \frac{M_r}{M_d} = \frac{1}{2}$

36.



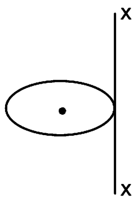
$$d^2 = R^2 - \left(\frac{R}{2}\right)^2 = \frac{3}{4}R^2$$

$$I_{xx} = I_{dia} + Md^2$$

$$I_{xx} = \frac{1}{4}MR^2 + Md^2 = \left(\frac{1}{4} + \frac{3}{4}\right)MR^2 = MR^2$$

$$= MR^2$$

37.



$$I_{xx'} = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$$

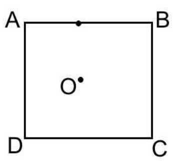
38. By parallel axes theorem,

$$I = I_{cm} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

39. $I = \frac{1}{2}mr^2 \times \frac{dI}{dr} = mr = \frac{2I}{r}$

$$\frac{dI}{I} = \frac{2dr}{r} = 2 \times 1\% = 2\%$$

40.

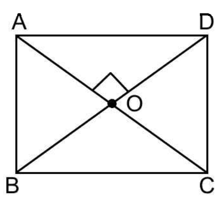


For single side of length ℓ and mass M

Moment of inertia about an axis, normal to ABCD and passing through O is

$$I_O = \frac{M\ell^2}{12} + M\left(\frac{\ell}{2}\right)^2 = \frac{M\ell^2}{3}$$

$$I_{total} = \frac{4M\ell^2}{3}$$



$$I_{AC} + I_{BD} = \frac{4M\ell^2}{3} \text{ (perpendicular axis theorem)}$$

$$\therefore \quad I_{AC} = I_{BD} = \frac{2M\ell^2}{3}$$

$$\frac{2M\ell^2}{3} = 2 \text{ kg m}^2 \text{ (data)}$$

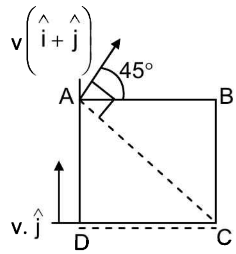
$$M = \frac{3}{\ell^2} = \frac{3}{4} = 0.75 \text{ kg}$$

$\therefore$ Total mass = 4M = 4 × 0.75 = 3 kg

41. Since the body is rigid, the velocity along the rod should be same

$\Rightarrow \quad v_{xB} = 2 \sin 30° = 1 \text{ m s}^{-1}$

42.



The motion can be considered as rotation about an instantaneous centre of rotation. To get it, draw normals to the velocities at A and D.

From the construction, C is the center of rotation, so that $v_C = 0$

$$\Rightarrow \quad \frac{v_C}{v_B} = 0$$

43. Angular momentum = $I\omega = ML^2T^{-1}$

44. $\overline{L} = \overline{r} \times \overline{p}$

$$\overline{r} = (4-3)\hat{i} + (1-2)\hat{j} = \hat{i} - \hat{j}$$

$$\overline{p} = 2(3\hat{i} + 6\hat{j})$$

$$\times \overline{L} = (\hat{i} - \hat{j}) \times 2(3\hat{i} + 6\hat{j})$$

$$= 2(6+3)\hat{k} = 18\hat{k}$$

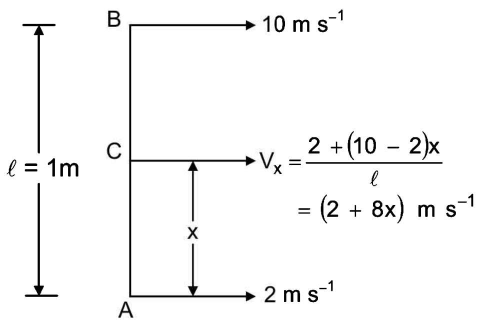
45. $\omega = \frac{10-2}{1} = 8 \text{ rad s}^{-1}$

$$v_{cm} = \left(\frac{10+2}{2}\right) = 6 \text{ m s}^{-1}$$

Angular momentum with respect to A is

$$L_A = I_A\omega = \frac{m\ell^2}{3}\omega = \frac{1 \times 1^2}{3} \times 8 = \frac{8}{3} \text{ kg m}^2 \text{ s}^{-1}$$

Aliter 1:



Consider an element of length dx at C, at a distance x from end A.

v_x = velocity of an elemental mass dm at $x = (2 + 8x)$

v = Relative velocity of C with respect to A = (2 + 8x) − 2 = 8x m s^{-1}

Angular momentum of elemental mass at C

$$dL = (dm)\,vx$$

$$= \left(\frac{M}{\ell}\right)dx\,vx$$

$$= \left(\frac{1\text{ kg}}{1\text{ m}}dx\right)8x\,.\,x = 8x^2\,dx$$

$$\therefore \quad L = \int_0^L dL = \int_0^1 8x^2\,dx = \left|8\left(\frac{x^3}{3}\right)\right|_0^1$$

$$= \frac{8}{3}\text{ kg m}^2\text{ s}^{-1}$$

Aliter 2:

$$v_{cm} = \frac{10+2}{2} = 6\text{ m s}^{-1}$$

Relative velocity of cm with respect to end A = $v'_{cm} = v_{cm} - 2 = 4$ m s^{-1}

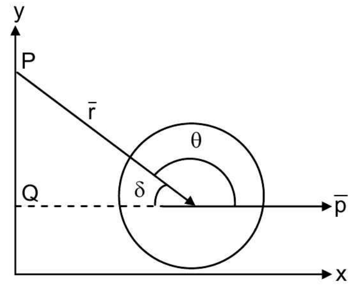
$$L_A = mv'_{CM}x_{cm} + I_{cm}\omega$$

$$= (1 \times 4 \times 0.5) + \frac{1 \times 1^2}{12} \times 8$$

$$= 2 + \frac{2}{3} = \frac{8}{3}\text{ kg m}^2\text{ s}^{-1}$$

46. $\bar{L} = I_{cm}\bar{\omega} + \bar{r} \times \bar{p}$

The angular momentum about (3, 0) and (0, 0) is non-zero as $I_{cm}\bar{\omega}$ and $\bar{r} \times \bar{p}$ add up in magnitude.



Angular momentum about

Point 'P' (0, 1.5 m) is:

$$\bar{L}_P = \bar{r} \times \bar{p} + I_{cm}\bar{\omega}$$

$$\bar{r} \times \bar{p} = r\,mv\sin\delta\left(\hat{k}\right)$$

$$= \hat{k}\ mv\,r\sin\delta$$

$$= \hat{k}\ mv\,(PQ) = \hat{k}\ (0.5)\,mv$$

[$\because$ PQ = OP − OQ = 1.5 − 1 = 0.5 m]

$$I_{cm}\,\bar{\omega} = \frac{mR^2}{2}.\frac{v}{R}\left(-\hat{k}\right) = \frac{1}{2}mv \times 1\left(-\hat{k}\right)$$

$$\therefore \quad \bar{L}_P = 0$$

47. Two equal and unlike parallel forces constitute a couple, by definition.

48. Power = Torque × angular velocity
= 150 × 300 = 45 kW

49. Torque $\tau = \left(m\frac{\ell}{2} + m\ell\right)g = m\ell.\,\frac{3}{2}g$

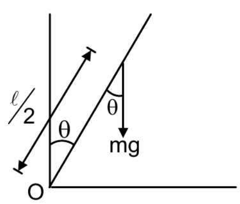
$$\tau = I\alpha \times \ell m\frac{3}{2}g = \left(\frac{m\ell^2}{3} + m\ell^2\right)\alpha$$

$$\frac{3}{2}g = \ell\alpha\left[\frac{1}{3}+1\right] \Rightarrow \alpha = \frac{9}{8}.\frac{g}{\ell}$$

50. $I = \dfrac{100 \times 1^2}{2} = 50\text{ kg m}^2$

$$\alpha = \frac{\tau}{I} = \frac{100}{50} = 2\text{ rad s}^{-2}$$

51.



Torque about O $= mg\,\frac{\ell}{2}\sin\theta = I\alpha = \frac{m\ell^2}{3}\alpha$

$$\alpha = \frac{3g\sin\theta}{2\ell}$$

52. $\tau = F \,.\, r = I\alpha = \frac{mr^2}{2}.\frac{a_T}{r} \Rightarrow F = \frac{ma_T}{2}$

$F = \frac{1\times 1}{2} = \frac{1}{2}$ N

53. No torque on system, so angular momentum $L = I\omega$ is conserved. Once water is formed, it will move to the outer edge of the round table. Thus I increases as $I \propto$ distance^2 As I increases ω decreases.

54. $I_1 = \frac{m_1 r^2}{2}$; $I_2 = m_2 r^2$; $I' = I_1 + I_2$; $I'' = I_1$, as the moment of inertia of the man when he is at the center is negligible

$$\omega'' = \frac{I'\omega'}{I''} = \left(\frac{\frac{m_1}{2} + m_2}{\frac{m_1}{2}}\right)\omega' = \frac{50+60}{50}\times 10$$

$= 22$ rpm.

55. $V_2 = \frac{1}{3}V_1$; $V_1 = 3\,V_2$ ie. $r_1^3 = 3r_2^3 \times r_1 = 3^{\frac{1}{3}} r_2$

$I_1\,\omega_1 = I_2\,\omega_2$; $I \propto r^2$; $T \propto \frac{1}{\omega}$; $\therefore \frac{I_1}{T_1} = \frac{I_2}{T_2}$

$T_2 = T_1 \times \frac{r_2^2}{r_1^2} = T_1 \times \frac{r_2^2}{3^{2/3}\times r_2^2}$

$= T_1 \times \frac{1}{3^{2/3}} = \frac{1}{2}T_1$

56. I = mass × distance^2

If distance decreases, I decreases. By conservation of angular momentum, ω increases

57. $I_1\omega = (I_1 + I_2)\,\omega_2$

$$\omega_2 = \frac{\frac{mr^2\omega}{2}}{\frac{mr^2}{2} + \frac{1}{2}\frac{m}{4}\frac{r^2}{4}} = \frac{16}{17}\omega$$

Since radius of second disc is half that of the first its area is one fourth and hence mass is one-fourth.

58. $K = \frac{1}{2}I\omega^2$: $L = I\omega$; $L^2 = I^2\omega^2 = \frac{2}{2}I^2\omega^2$

$= 2\,I\frac{1}{2}I\omega^2$, $L^2 = 2IK$

59. $\omega = 100$ rad s^{-1}, $M = 20$ kg, $r = 0.25$ m, $I = \frac{1}{2}Mr^2$

K.E $= \frac{1}{2}I\omega^2 = \frac{1}{2}\times\frac{1}{2}\times Mr^2\times\omega^2$

$= \frac{1}{4}\times 20\times(0.25)^2\times 100^2 = 3125$ J

60. Energy spent = difference in kinetic energy.

$500 = \frac{1}{2}I\left(\omega_2^2 - \omega_1^2\right)$; $\omega_1 = \frac{60\,\text{x}\,2\pi}{60} = 2\pi$;

$\omega_2 = \frac{360}{60}\times 2\pi = 12\pi$

$500 = \frac{1}{2}I(140\,\pi^2)$; $I = \frac{500\times 2}{140\times\pi^2} = 0.72$ kg m^2

61. $I_1 = \left(2m + \frac{1}{2}4m\right)r^2 = 4mr^2$;

$I_2 = \left(m + \frac{1}{2}4m\right)r^2 = 3mr^2$

$\because\ I_1\omega = I_2\omega'$

$\omega' = \frac{4}{3}\omega$

$$\Delta KE = \frac{1}{2}I_2\omega'^2 - \frac{1}{2}I_1\omega^2$$

$$= \frac{1}{2}(3m)r^2\times\left(\frac{4}{3}\omega\right)^2 - \frac{1}{2}4mr^2\omega^2$$

$$= \frac{1}{2}\left(\left(\frac{4}{3}\right)^2 3 - 4\right)mr^2\omega^2$$

$$= \frac{2}{3}mr^2\omega^2$$

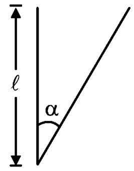
62. Rotational energy $= \frac{1}{2}I\omega^2$

I depends on the mass and its distribution about the axis of rotation.

63. PE at initial position = KE at lowest position

$$mgr = \frac{1}{2}\left(mr^2 + \frac{mr^2}{2}\right)\omega^2 \Rightarrow 2\sqrt{\frac{g}{3r}} = \omega$$

64.



Energy equation

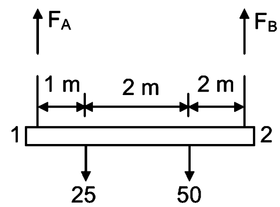
$$mg\frac{\ell}{2}(1-\cos\alpha) = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{m\ell^2}{3}\omega^2$$

$$3g(1-\cos\alpha) = \ell\,\omega^2 \Rightarrow 3g\left[1-\left\{1-2\sin^2\left(\frac{\alpha}{2}\right)\right\}\right]$$

$$= \ell\omega^2$$

$$\omega^2 = \frac{6g}{\ell}\sin^2\frac{\alpha}{2} \Rightarrow \omega = \sqrt{\frac{6g}{\ell}}\sin\frac{\alpha}{2}$$

65.



Taking moment of forces torque about 1,

$$25\times 1 + 50 \Rightarrow 3 = 5\times F_B$$

$$\Rightarrow \quad F_B = 35\text{ N}$$

66. $\omega = 150\text{ rpm} = \frac{150}{60}\times 2\pi = 5\pi\text{ rad s}^{-1}$.

Work done = 50 × 10 × 4 = 2000 J

$$= \frac{1}{2}I\omega^2 \times I = \frac{2000\text{x}2}{(5\pi)^2} = 16\text{ kg m}^2$$

67. $T_1 - mg\sin 30° = ma$

$$-T_2 + mg = ma$$

$$\text{Adding } T_1 - T_2 = 2ma - \frac{mg}{2}$$

$$T_2 - T_1 = \frac{mg}{2} - 2ma \qquad \text{— (1)}$$

$$(T_2 - T_1)R = I\alpha$$

$$= \frac{mR^2}{2}.\frac{a}{R} \Rightarrow T_2 - T_1 = \frac{ma}{2} \qquad \text{— (2)}$$

From (1) + (2)

$$\frac{mg}{2} - 2ma = \frac{ma}{2}$$

$$\Rightarrow \quad a = \frac{g}{5}$$

68. Rotational K.E. $= \frac{1}{2}I\omega^2 = \frac{1}{2}MK^2\times\frac{v^2}{R^2}$

$$= \frac{1}{2}Mv^2\times\frac{K^2}{R^2}$$

$$\text{Total K.E} = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2\times\frac{K^2}{R^2}$$

$$= \frac{1}{2}Mv^2\left[1+\frac{K^2}{R^2}\right]$$

$$\text{ratio} = \frac{\frac{K^2}{R^2}}{\left(1+\frac{K^2}{R^2}\right)} = \frac{K^2}{K^2+R^2}$$

69. $K.E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\times\frac{1}{2}MR^2\times\frac{v^2}{R^2}$$

$$= \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2 = \frac{3}{4}\times 2\times 4^2 = 24\text{ J}$$

70. $K.E. = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}\times\frac{2}{5}MR^2\times\frac{v^2}{R^2}$$

$$= Mv^2\left(\frac{1}{5}+\frac{1}{2}\right) = \frac{0.01\times 7}{10} = 0.007\text{ J}$$

71. $mgh = \frac{1}{2}mv^2$

$$= \frac{1}{2}mv'^2 + \frac{1}{2}I\omega'^2 = \frac{1}{2}mv'^2 + \frac{1}{2}mR^2\frac{v'^2}{R^2}$$

$$= \frac{1}{2}2mv'^2 \Rightarrow v' = \frac{v}{\sqrt{2}}$$

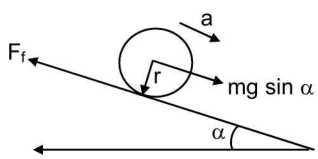
72. Total KE $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}mv^2 + \frac{1}{2}.\frac{1}{2}mr^2\frac{v^2}{r^2} = \frac{3}{4}mv^2$$

$$\text{Rotational KE} = \frac{1}{4}mv^2$$

∴ Ratio = 3 : 1.

73.



$$mg\sin\alpha - F_f = ma$$

$$r.F_f = I\alpha = mr^2 \cdot \frac{a}{r} \Rightarrow F_f = ma$$

$$2ma = mg\sin\alpha \times a = g\frac{\sin\alpha}{2} = \frac{g}{4}$$

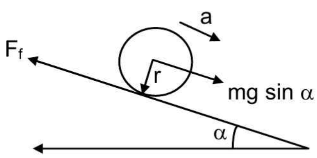
Aliter:

$$a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} = \frac{g\times\sin 30^\circ}{\left(1+\frac{R^2}{R^2}\right)}$$

($\because$ K = R for ring)

$$= \frac{g}{2}\times\frac{1}{2} = \frac{g}{4}$$

74.



$$a = \frac{g\sin\alpha}{1+\frac{I}{mr^2}}$$

$$\frac{a_{sphere}}{a_{cylinder}} = \frac{1+\frac{1}{2}}{1+\frac{2}{5}} = \frac{15}{14}$$

75. $a \Rightarrow \dfrac{1}{1+\frac{I}{mr^2}}$

$I_h > I_s \Rightarrow a_s > a_h \Rightarrow$ solid cylinder reaches the bottom first.

76. Frictional force

$$F_f = \frac{mg\sin\alpha}{1+\frac{r^2}{k^2}} = \frac{mg\sin\alpha}{1+\frac{mr^2}{I}}$$

In critical condition : $\mu mg\cos\alpha = \dfrac{mg\sin\alpha}{1+mr^2/I}$

$$\mu = \tan\alpha \frac{1}{1+\frac{mr^2}{I}}$$

(i) sphere : μ_{sp}

$$= \tan\alpha \frac{1}{1+\frac{mr^2}{\frac{2}{5}mr^2}} = \frac{2}{7}\tan\alpha = 0.28\tan\alpha$$

(ii) spherical shell : μ_{sh}

$$= \tan\alpha \frac{1}{1+\frac{mr^2}{\frac{2}{3}mr^2}} = \frac{2}{5}\tan\alpha = 0.4\tan\alpha$$

(iii) cylinder : μ_s

$$= \tan\alpha \frac{1}{1+\frac{mr^2}{mr^2/2}} = \frac{1}{3}\tan\alpha = 0.33\tan\alpha$$

Since spherical shell needs maximum μ, it will be the first to slip

Aliter:

$$a = \frac{g\sin\alpha}{1+\frac{I}{mr^2}}$$

a is min for shell for no slipping

$\Rightarrow$ It has maximum friction for given angle

$\Rightarrow$ it will slip first

77. KE sphere $= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\frac{v^2}{r^2} = \frac{7}{10}mv^2$

KE shell $= \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{3}mr^2\frac{v^2}{r^2} = \frac{5}{6}mv^2$

The height gone up $\propto$ KE

($\because$ mgh = KE)

$$\therefore \quad \frac{h_1}{h_2} = \frac{\frac{7}{10}mv^2}{\frac{5}{6}mv^2} = \frac{42}{50}$$

$$\therefore \quad \frac{s_1}{s_2} = \frac{21}{25} (\because s \propto h)$$

78.

$$\because KE = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{2}\frac{v^2}{r^2} = \frac{1}{2}mv^2\left[1+\frac{1}{2}\right] = \frac{3}{4}mv^2$$

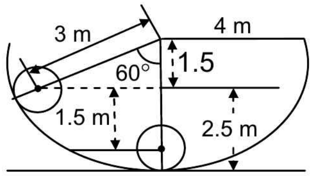
$$E = \frac{3}{4}mv^2 = mgh = mg(1.5) \quad (\because h = 1.5\text{ m})$$

$$v^2 = g\times\frac{6}{3} = 2g \Rightarrow v = \sqrt{2g}$$

$$\omega = \frac{v}{r} = \sqrt{2g}\ \text{rad s}^{-1}$$

79. For rolling up or down or at rest, the frictional force developed is as shown.

80. For angular momentum conservation



$$mv\frac{\ell}{2}=\left(\frac{m\ell^2}{4}+\frac{m\ell^2}{3}\right)\omega$$

$\Rightarrow \quad \omega = \dfrac{6}{7}\dfrac{v}{\ell}$ where v is the speed of the point mass

81. Potential energy of erect flag post

$$= \int_0^L \lambda d\ell g\ell = \frac{\lambda g L^2}{2}$$

$$= \frac{50\times 12\times 12\times 10}{2}$$

= 36000 J energy lost = 36000 J.

Aliter:

m = mass of flag post = 50 × 12 = 600 kg

The CM of flag post is at a height

$$h_{CM} = \frac{\ell}{2} = \frac{12}{2} = 6\,m$$

$\therefore \quad PE_{initial} = mgh_{CM} = 600 \times 10 \times 6 = 36000$ J

This is the KE with which the flag post hits ground.

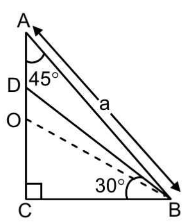
82. Mass of the hanging portion of chain = 1 kg. The height through which the CM has to be raised = 0.25 m.

Energy needed = mgh = 1 × 10 × 0.25

= 2.5 J

83. AB = a, BC = a cos45 = $\dfrac{a}{\sqrt{2}}$

$$\frac{BC}{BD} = \cos 30 = \frac{\sqrt{3}}{2}$$



$$\Rightarrow BD = BC\frac{2}{\sqrt{3}} = \frac{a}{\sqrt{2}}.\frac{2}{\sqrt{3}} = a\sqrt{\frac{2}{3}}$$

Let O be the centre of AC

$$\Rightarrow \quad AC = BC = \frac{a}{\sqrt{2}}\,;\, CO = \frac{AC}{2} = \frac{a}{2\sqrt{2}}$$

$$\Rightarrow \quad BO^2 = CO^2 + BC^2$$

$$= \frac{a^2}{8} + \frac{a^2}{2} = \frac{5a^2}{8}$$

$$AC = AB\cos 45 = \frac{a}{\sqrt{2}}$$

About an axis at B normal to the plane containing the rods, the moment of inertia of the rods are

$$I_{AB} = \frac{ma^2}{3}; I_{BD} = \frac{m}{3}(BD)^2 = \frac{ma^2}{3}.\frac{2}{3}$$

$$I_{BC} = \frac{m}{3}(BC)^2 = \frac{ma^2}{3}.\frac{1}{2}$$

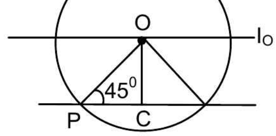
$$I_{AC} = \frac{m}{12}(AC)^2 + m(BO)^2 = \frac{ma^2}{12\times 2} + ma^2.\frac{5}{8}$$

$$= \frac{ma^2}{3}.2$$

$$I_{Total} = \frac{ma^2}{3} + \frac{ma^2}{3}.\frac{2}{3} + \frac{ma^2}{3}.\frac{1}{2} + \frac{ma^2}{3}.2$$

$$= \frac{ma^2}{3}\left[1 + \frac{2}{3} + \frac{1}{2} + 2\right] = \frac{25}{18}ma^2$$

84.



$$PC = OC = \frac{R}{\sqrt{2}}$$

$I_O \dfrac{mR^2}{4}$, using

Perpendicular axis theorm

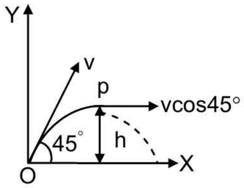
$$I = I_O + m(OC)^2 = \frac{mR^2}{4} + \frac{mR^2}{2} = \frac{3}{4}mR^2$$

85. $I = \dfrac{1}{2}MR^2$. $M = \rho V = \rho \times \pi R^2 \times t$

$$I = \frac{1}{2} \times 9 \times \pi \times 7 \times 400 \times 400$$

$= 1.584 \times 10^7$ g cm^2

86.



For projectile motion,

$$h = \frac{v^2 \sin^2 45^0}{2g} = \frac{v^2}{4g}$$

$$v' = \frac{v}{\sqrt{2}} \Rightarrow L = \frac{mv}{\sqrt{2}}.\frac{v^2}{4g} = \frac{mv^3}{\sqrt{32g}}$$

87. $\alpha = \dfrac{\tau}{I} = 3$ rad s^{-2} ;

ω after 1 second = αt = 3 rad s^{-1}.

$\times\ v = r\omega = 1\times 3 = 3$ m s^{-1}.

88. Torque about the centre due to the force

$$\tau = F \times r = \frac{F\ell}{4}$$

Now this torque of $\dfrac{F\ell}{4}$ will produce an angular acceleration

$$\therefore \quad \tau = I_c \times \alpha = \frac{m\ell^2}{12} \times \alpha \text{ as } I_c = \frac{m\ell^2}{12}$$

$$\therefore \alpha = \frac{3F}{m\ell}$$

$$\text{Now } \theta = \frac{1}{2}\alpha\, t^2 = \frac{1}{2}\left(\frac{3F}{m\ell}\right)t^2 = \left(\frac{3F}{2m\ell}\right)t^2$$

89. Torque produced by the ocean water in decreasing Earth's angular velocity

$$\tau = I\alpha = \frac{2}{5}MR^2 \times \alpha$$

Where $\alpha = \dfrac{\Delta\omega}{\Delta t} = \left(\dfrac{\Delta\theta}{\Delta t_1}\right)\dfrac{1}{\Delta t_2}$ and has the dimension of rad s^{-2}.

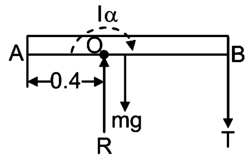
Converting the given values to SI units i.e., $\Delta\theta$ = 0.0016 rad, Δt_1 = one day = 86400 s; and Δt_2 = one century = 86400 × 365 × 100 s,

we get:

$$\tau = \frac{2}{5} \times 6 \times 10^{24} \times 64^2 \times 10^{10} \times \frac{0.0016}{86400^2 \times 100 \times 365}$$

$= 5.7 \times 10^{20}$ N m (where 1year = 365 × 86400 s)

90.



When thread is burnt, the rod experiences clockwise moment due to

(i) its own weight, mg,

(ii) tension F_s in the spring.

Let α be angular acceleration of rod,

then downward acceleration of its C.M will be $0.1\alpha = \dfrac{\alpha}{10}$ as O is the instantaneous axis of rotation.

M.I. of rod about O is

$$I = \frac{m\ell^2}{12} + m\left(\frac{\ell}{2} - 0.4\ell\right)^2 = \frac{14}{150}m\ell^2$$

$$= \frac{14}{150} \times 2 \times 1^2 = \frac{14}{75}\text{ kg m}^2$$

Considering F.B.D of rod as shown above, and taking moments about O,

$mg \times (0.10) + F_s \times 0.6 = I\alpha$

$2 \times 10 \times 0.1 + 6 \times 0.6 = I\alpha$

$\Rightarrow \quad 5.6 = \dfrac{14}{75}\alpha$

or $\alpha = 30$ rad s^{-2}

Force balance $ma = mg + F_s - R$;

$a = r\alpha = 0.1 \times 30 = 3$ m s^{-2}

$\Rightarrow \quad 2 \times 3 = 2 \times 10 + 6 - R$

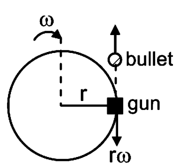
$\Rightarrow \quad R = 20$ N

91. $I = 2mR^2 + \dfrac{(2m)(2R)^2}{12} = \left(2 + \dfrac{2}{3}\right)mR^2$

$\Rightarrow \quad L = I\omega = \dfrac{8}{3}mR^2\omega$

Angular momentum is conserved, as no torque is exerted on the system.

92. Initial angular momentum of system,



$L_i = I_i\omega_0$

$= \left[\frac{Mr^2}{2} + m_g r^2 + mr^2\right]\omega_0$

$\therefore \quad L_i = \left[\frac{M}{2} + m_g + m\right]r^2\omega_0 \qquad \text{— (1)}$

Final angular momentum of system,

$L_f = I_f\omega - m(v - r\omega)r$

[v is w.r.t the rotating turntable and $v = v_b + r\omega$ where v_b is the actual velocity.]

$\because \quad v_{bullet} = v - r\omega,$

$= \left[\frac{M}{2}r^2 + m_g r^2\right]\omega - mvr + mr^2\omega$

$\Rightarrow \quad L_f = \left[\frac{M}{2} + m_g + m\right]r^2\omega - mvr \qquad \text{— (2)}$

But $L_i = L_f$ ($\therefore$ angular momentum is conserved)

$\therefore \quad \left(\frac{M}{2} + m_g + m\right)r^2\omega_0 = \left[\frac{M}{2} + m_g + m\right]r^2\omega - mvr$

$\Rightarrow \quad mv = \left[\frac{M}{2} + m_g + m\right]r\ (\omega - \omega_0)$

$(\omega - \omega_0) = \dfrac{mv}{\left(\frac{M}{2} + m_g + m\right)r}$

93. When r = 1 m, tension

$T = \frac{mv_0^2}{r} = \frac{0.1 \times 2^2}{1} = 0.4 \text{ N}$

Since no external torque acts on the particle, its angular momentum is conserved.

i.e $mv_0 r = mv\left(\frac{r}{2}\right)$

$\therefore \quad v = 2v_0 = 4 \text{ m s}^{-1}$

$T = \frac{mv^2}{r/2} = \frac{8mv_0^2}{r} = 8 \times 0.1 \times 2^2 = 3.2 \text{ N}$

94. If 4M is the mass of the full disc then MI of the quarter section for the given axis is $\frac{1}{4}$th of the original disc

$= \frac{1}{2}(4MR^2).\frac{1}{4} = \frac{1}{2}MR^2$

Kinetic Energy $= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{MR^2\omega^2}{2}$

$= \frac{1}{2} \times \frac{0.5}{2} \times 0.1 \times 0.1 \times 400 = \frac{1}{2}\text{ J}$

95. K. E. of rotation $= \frac{1}{2}I\omega^2$

Work done $= F \times S = 44 \times 3 = 132 \text{ J} = \frac{1}{2}I\omega^2$

$\Rightarrow \quad I = \frac{264}{\omega^2}$

But $\omega = 6\pi$ rad s^{-1}

$I = \frac{264}{(6\pi)^2} = 0.7 \text{ kg m}^2$

96. $\overline{L}_{Total} = \overline{L}_{C.M} + \overline{r}_{CM} \times \overline{p}$

$\Rightarrow \overline{L}_{Total} = \overline{L}_{CM} + M(\overline{r}_{CM} \times \overline{v}_{CM})$

Here, $L_{CM} = I\omega = \frac{1}{2}MR^2\omega$ and

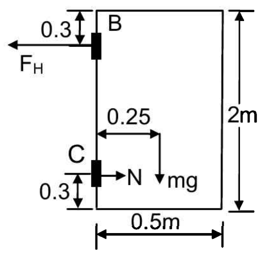
$M(\overline{r}_{CM} \times \overline{v}_{CM}) = MRv_{CM} = MR(R\omega) = MR^2\omega$

$\Rightarrow L_{Total} = \frac{1}{2}MR^2\omega + MR^2\omega = \frac{3}{2}MR^2\omega$

Kinetic Energy of rotation $= \frac{1}{2}I\omega^2 = \frac{1}{2}\omega \times I\omega$

$= \frac{1}{2}L_{Total}\,\omega = \frac{3}{4}MR^2\omega^2$

97.



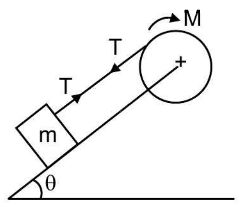
Taking moments about C,

$mg\left(\frac{w}{2}\right) = F_H \times (\ell - 2 \times 0.3)$ (w - width, ℓ - height)

$8 \times 9.8 \times 0.25 = 1.4\ F_H$

$F_H = \frac{19.6}{1.4} = 14 \text{ N}$

98.



The linear deceleration of the block is a .

The angular deceleration of the wheel is $\alpha = \frac{a}{r}$

If T is the tension in the string,

$mg\sin\theta - T = ma$

$T \times r = \frac{Mr^2}{2}.\alpha$

$T\times r = \frac{Mr^2}{2} \times \frac{a}{r} \Rightarrow T = \frac{Ma}{2}$ as $\alpha = \frac{a}{r}$

$\therefore \quad mg\sin\theta = M\frac{a}{2} + ma = a\left[m + \frac{M}{2}\right]$

$$a = \frac{2mg\sin\theta}{2m + M}$$

Initial velocity of block = ωr. Final velocity = 0

Acceleration = -a

$v = u - at \Rightarrow 0 = r\omega - at$

$$t = \frac{r\omega}{a} = \frac{r\omega(2m + M)}{2mg\sin\theta}$$

$$= \frac{0.1 \times 20(2 \times 0.1 + 0.6)}{2 \times 0.1 \times 10 \times \sin 37^0} = \frac{4}{3}s$$

99. $mg - T = ma$ —(1)

Moment of couple = T. r acting on the cylinder of mass M and radius r

$\therefore \quad Tr = I\alpha$ or $T = \frac{I\alpha}{r}; I = \frac{Mr^2}{2}; \alpha = \frac{a}{r}$

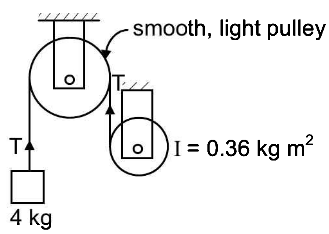
$\therefore \quad T = \frac{Mr^2}{2}\frac{\alpha}{r} = \frac{Mr}{2}\frac{a}{r} = \frac{Ma}{2}$ —(2)

substitute in (1) $\Rightarrow a = \frac{2mg}{(2m + M)}$

$\therefore \quad \alpha = \frac{a}{r} = \left(\frac{2mg}{M + 2m}\right)\frac{1}{r} = \frac{2 \times 0.1 \times 10}{1.1 \times 0.1}$

$= 18.2 \text{ rad s}^{-2}$

100.



I = 0.36(kg m²) For the wheel with r = 10 cm = 0.1m.

Smaller pulley is light.

Mass of the block, m = 4 kg

$\Rightarrow \quad mg - T = ma$ — (1)

$T \times r = I\alpha = \frac{Ia}{r}$

$\therefore \quad T = \frac{Ia}{r^2} \quad \Rightarrow mg = \left(m + \frac{I}{r^2}\right)a$

$$\Rightarrow a = \frac{mg}{\left(m + \frac{I}{r^2}\right)} = \frac{4 \times 10}{\left(4 + \frac{0.36}{0.1^2}\right)} = 1 \text{ m s}^{-2}$$

101. Change in momentum of ball = mv–(–mu) = m(v + u) = Impulse exerted on the ball. (We assume that the impulse acting on the ball is horizontal)

$\therefore$ Change in angular momentum of ball = moment of impulse

i.e. $\frac{I(u + v)}{r} = m(u + v)(h - r)$

$\therefore \quad r(h-r) = \frac{I}{m}$

For a uniform sphere, $I = \frac{2}{5}mr^2$

$\therefore \quad r(h-r) = \frac{2}{5}r^2 \Rightarrow \frac{h}{r} = \frac{7}{5}$

Aliter:

$I_{cm} = \chi mr^2, \chi = \frac{2}{5}$ for solid sphere

f = 0 ($\because$ smooth floor)

$\Rightarrow h = r[1 + \chi] = r\left[1 + \frac{2}{5}\right] = \frac{7}{5}r$

102. Impulse on the sphere = $J = m\, v_{cm}$

$J.h = \frac{2}{3}mR^2.\omega + m\, v_{cm}.R$

For angular momentum about point of contact

$\omega = \frac{v_{cm}}{R}$ (for rolling) $\quad \Rightarrow v_{CM} = R\omega \times J = mr\omega$

$m\omega R.h = \frac{2}{3}m\omega R^2 + m\omega R^2$

$\Rightarrow \quad h = \frac{5}{3}R = 10 \text{ cm}$

Aliter:

$I_{cm} = \chi mr^2$, where $\chi = \frac{2}{3}$ for hollow sphere

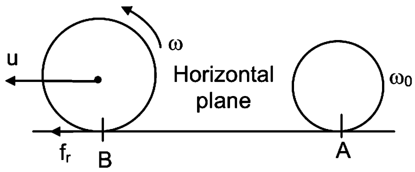
$$f = F\left[\frac{h}{R[1 + \chi]} - 1\right]$$

f = 0 ($\because$ table is smooth)

$$\Rightarrow \quad h = R[1+\chi] = R\left[1+\frac{2}{3}\right]$$

$$= \frac{5R}{3} = \frac{5}{3}\times 6 = 10 \text{ cm}$$

103.



μ is variable and at every point of contact with the horizontal plane and variation of μ is not known. So we cannot write the equation of motion.

But conservation of angular momentum about the initial point of contact is taken to find energy.

(since the frictional force acts through initial point of contract B)

$$I_{CM}\,\omega_0 = I_{CM}\,\omega + mvR \text{ where } I_{CM} = \frac{2}{5}mR^2 \qquad \text{— (1)}$$

When pure rolling commences and F_r, the friction force vanishes, $v = R\omega$ — (2)

From (1) & (2), $\omega = \dfrac{2\omega_0}{7}$

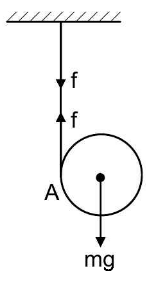
$$\therefore \quad E_r = \frac{mv^2}{2} + I_{CM}\frac{\omega^2}{2} = \frac{1}{2}\left(mR^2 + I_{CM}\right)\frac{4\omega_0^{\,2}}{49}$$

$$\left[mR^2 = \frac{5}{2}I_{CM}\right]$$

$$= I_{CM}\frac{\omega_0^{\,2}}{7} = \frac{2}{7}I_{CM}\frac{\omega_0^{\,2}}{2} = \frac{2}{7}E_i$$

$$\therefore \quad \frac{E_r}{E_i} = \frac{2}{7}$$

104. $mg - T = ma$



$$T.r = I\,\alpha$$

$$I = \frac{mr^2}{2} \;\&$$

$$mg - ma = \frac{I\alpha}{r}$$

$$g = a + \frac{r\alpha}{2} = \frac{3}{2}r\alpha$$

$$\alpha = \frac{2g}{3r}$$

$$\omega = \sqrt{2\alpha\theta}$$

$$\theta = \frac{h}{r}$$

$$\Rightarrow \omega = \sqrt{\frac{4gh}{3r^2}}$$

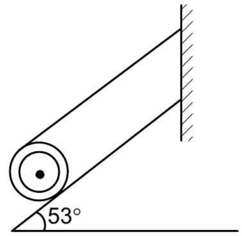
Aliter:

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}.\frac{mr^2}{2}.\omega^2 + \frac{1}{2}mr^2\omega^2$$

$$gh = r^2\omega^2\left(\frac{1}{2}+\frac{1}{4}\right) \Rightarrow \omega = \sqrt{\frac{4gh}{3r^2}}$$

105.



Let the acceleration of the spool axis be A. After descending through a height h, the velocity acquired by the spool is v.

$$\therefore \quad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I.\frac{v^2}{r^2}$$

$$= \frac{1}{2}\left(M + \frac{I}{r^2}\right)v^2$$

If θ is the inclination of the plane,

$$v^2 = \frac{2Ah}{\sin\theta}\left(\because v^2 - u^2 = 2as \text{ and } s = \frac{h}{\sin\theta}\right)$$

$$\Rightarrow \quad Mgh = \frac{1}{2}\left(M + \frac{I}{r^2}\right)\frac{2Ah}{\sin\theta}$$

$$A = \frac{Mg\sin\theta}{M + \dfrac{I}{r^2}} = \frac{0.3\times 10\times \sin 53^0}{0.3 + \dfrac{0.6\times 10^{-3}}{\left(2\times 10^{-2}\right)^2}}$$

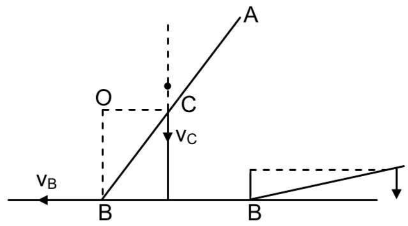
$$= 1.33 \text{ m s}^{-2}$$

106. $P.E = m.g\,\dfrac{\ell}{2}$

$$\frac{1}{2}I\omega^2 = mg\frac{\ell}{2}$$

$$\Rightarrow \quad I\omega^2 = mg\ell$$

$$I = \frac{m\ell^2}{3}$$

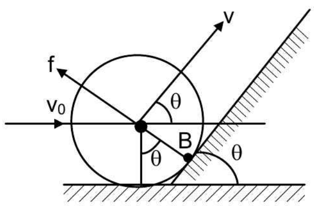


[The COM is falling vertically down since there is no horizontal force. Considering an intermediate position of fall as shown, the end B and centre C have velocities as shown. To get instantaneous centre of rotation, draw normal to these velocities which meet at O. It can easily be seen that at the final stage the instantaneous centre of rotation, will be at B, in the limit. Hence the MI in the final condition is to be taken about B.]

$$\therefore \quad \frac{1}{3}m\ell^2\omega^2 = mg\ell \Rightarrow \omega = \sqrt{\frac{3g}{\ell}}$$

$$v = \frac{\ell}{3}\omega = \sqrt{\frac{g\ell}{3}}$$

107.



Angular momentum about B (the first point of contact on the incline)

$L_i = I\omega_0 + mv_0R\cos\theta$

Angular velocity, $\omega_0 = \dfrac{v_0}{R}$

Angular momentum about B after impact = L_F

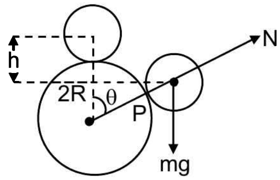
$= mv R + I\omega;\ \omega = \dfrac{v}{R}$

where v = required velocity along the inclined plane. (The normal reaction at B is passing through the centre of the cylinder and hence it does not give any torque) Since angular momentum about B remains conserved, we have

$I\omega_0 + m.v_0R\cos\theta = I\omega + mvR$

$$v = v_0\frac{\left(\frac{I}{R^2} + m\cos\theta\right)}{\frac{I}{R^2} + m} = \frac{13v_0}{15}$$

108.



Radius of the circular path of ball is 3R.

$$mg\cos\theta - N = \frac{mv^2}{3R}$$

When the ball leaves contact with the sphere, we have N = 0

$$\Rightarrow \quad mg\cos\theta = \frac{mv^2}{3R}$$

$\Rightarrow \quad v^2 = 3\,Rg\cos\theta$

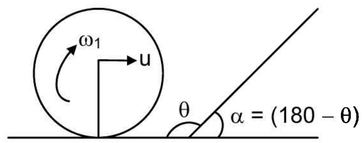
Here R = 0.17 m, g = 10 m s^{-2}

$\cos\theta = \cos 53° \ = 0.60$

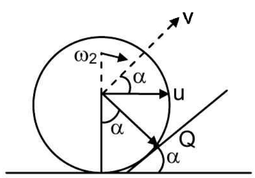
$\therefore \quad v^2 = 3 \times 0.17 \times 10 \times 0.6 = 3.06$

$$\omega = \frac{v}{r} = \frac{\sqrt{3.06}}{0.17} = 10 \text{ rad s}^{-1}$$

109. F.B.D.(i)



F.B.D.(ii)



Since the normal reaction at point of contact passes through the centre of the cylinder and hence apply no torque, its angular momentum about the point of contact is conserved.

Apply the principle of conservation of angular momentum about the point of impact. Take the above free body diagrams FBD (i) and FBD (ii).

Taking moment of momentum about contact point Q in FBD (ii)

We have, $I\omega_1 + mu\cos\alpha \,.\, R = I\omega_2 + mvR$

Putting $u = \omega_1 R$, $v = \omega_2 R$ and $I = \dfrac{mR^2}{2}$

$$\frac{1}{2}mR^2(\omega_1) + mR\cos\alpha(\omega_1 R) = \frac{1}{2}mR^2\omega_2 + mR^2\omega_2$$

$$\Rightarrow \quad \omega_2 = \omega_1\left(\frac{1}{3} + \frac{2}{3}\cos\alpha\right)$$

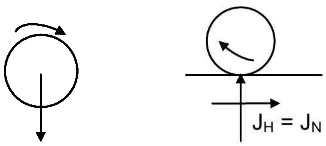
For $\omega_2 = 0$, finally and suddenly, $\alpha = \cos^{-1}\left(\frac{-1}{2}\right)$

$\therefore \quad \alpha = 120^0$

$\therefore \quad \theta = 60^0$

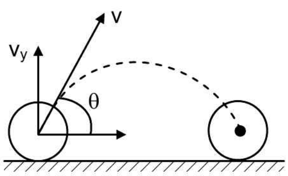
Note: Size of the cylinder and its angular velocity are immaterial.

110. When ball falls freely it accelerates translationally keeping angular velocity ω_0 constant. The ball's velocity just before collision with the floor is $v_0 = \sqrt{2gh} = 6\ \text{m}\,\text{s}^{-1}$



Since e = 5/6, the ball rebounds

vertically upwards with velocity $v_y = ev_0 = 5\ \text{m s}^{-1}$



Normal impulsive reaction exerted by the floor

= Change in vertical momentum of the ball

i.e., $J_N = m(v_0 + ev_0) = 11 \times m$

where m = mass of the ball. The ball moves like a projectile after bouncing.

Frictional Impulse, $J_H = \mu J_N = 2.2 \times m$

M.I. of the ball about the diameter is $I = \frac{2}{5}mR^2$

(R is ball's radius).

Initial angular momentum = $I\omega_0$

Retarding angular impulse produced by $J_H = RJ_H$

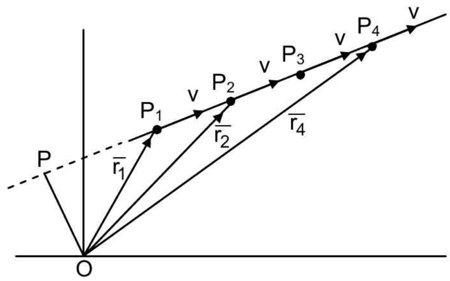
change in angular momentum = RJ_H

$$\text{Fractional change} = \frac{RJ_H}{I\omega_0} = \frac{0.055 \times 2.2 \times m}{\frac{2}{5}m(0.11)^2 \times 120}$$

$$= \left(\frac{5 \times 2.2}{2 \times 2 \times 0.11 \times 120}\right) \times 100$$

$$= \frac{5}{24} \times 100 = 20.8\%$$

111.



At all points along the trajectory, $|\overline{r} \times \overline{v}| = OP.v$

$\Rightarrow$ same value for all r_n.

$\overline{L} = \overline{r} \times \overline{p} = \overline{r} \times m\overline{v} = \text{constant}$

$\Rightarrow \quad \overline{r} \times \overline{v} = \frac{\overline{L}}{m} = \text{constant}$

(in this situation, because uniform motion $\times$ no force

$\Rightarrow \quad \overline{r} \times \overline{F} = 0$

$\Rightarrow \quad \tau = \frac{d\overline{L}}{dt} = 0 \Rightarrow \overline{L}$ constant.

Area of a triangle of sides $\overline{a}$ and $\overline{b}$ is $\frac{1}{2}\overline{a} \times \overline{b}$

Here area of $\Delta OP_1P_2 = \frac{1}{2}\left(\overrightarrow{OP_1} \times \overrightarrow{P_1P_2}\right)$

$$= \frac{1}{2}\left(\overline{r_1} \times \overline{v}\right) = \frac{\overline{L}}{2m}$$

Area of $\Delta OP_2P_4 = \frac{1}{2}\left(\overrightarrow{OP_2} \times \overrightarrow{P_2P_4}\right) = \frac{1}{2}\left(\overline{r_2} \times 2\overline{v}\right)$

$$= 2\left(\frac{(\overline{r_2} \times \overline{v})}{2}\right) = \frac{\overline{L}}{2m} \times 2$$

$= 2 \times$ area of $\Delta\, OP_1P_2$

112. K.E increases

L remains constant ($\therefore$ no external torque)

$$KE = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$$

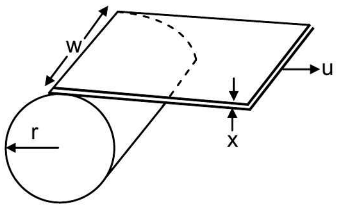
I decreases. $\therefore$ KE increases

113. Work done by friction is zero. But it exists in pure rolling on inclined plane ($\because$ POC has no displacement)

114. The tangential component of velocity at the periphery of the roll = u = rω

$$\omega = \frac{u}{r}$$

115.



If the width of the roll be w, the rate of volume of paper going out of the roll

$$= -\frac{d}{dt}\left(\pi r^2 w\right) = (xw)\,u$$

$$2\pi r\frac{dr}{dt}.w = xwu$$

$$r\frac{dr}{dt} = -\frac{xu}{2\pi}$$

116. $$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{u}{r}\right) = -\frac{u}{r^2}\frac{dr}{dt}$$

$$= -\frac{u}{r^2}\left(\frac{-xu}{2\pi r}\right) = \frac{u^2 x}{2\pi r^3}$$

117. $I_{AA}' = \frac{1}{2}I$ of total sphere

$$= \frac{1}{2}\frac{2}{5}MR^2 = \frac{2}{5}\left(\frac{M}{2}\right)R^2 = \frac{2}{5}mR^2$$

⇒ (a) is correct

(M → Mass of total sphere, m → mass of hemisphere)

$$I_{AA'} = I_{CC'} + m\left(\frac{3R}{8}\right)^2$$

$$\Rightarrow I_{cc'} = \frac{2}{5}mR^2 - \frac{9}{64}mR^2 = \frac{83}{320}mR^2$$

⇒ (c) is correct

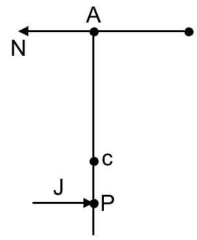
$$I_{BB'} = I_{cc}' + m\left(\frac{5R}{8}\right)^2$$

$$= mR^2\left(\frac{83}{320} + \frac{25}{64}\right) = \frac{13}{20}mR^2$$

⇒ (d) is correct

118. By symmetry COM is at C.

$$v_c = \frac{J}{m};\ I_c\omega_c = J.\frac{\ell}{2}$$



$$\Rightarrow \frac{m\ell^2}{12}.8 = J\frac{\ell}{2}$$

$$\Rightarrow \frac{J}{m} = \frac{8\ell}{6} \Rightarrow$$

$$\Rightarrow v_c = \frac{J}{m} = \frac{8\ell}{6} = \frac{4}{3}\ m\ s^{-1}$$

⇒ (b) is correct

($\because \ell = 1$ m)

Since P is at rest : ω (CP) = v_c ⇒

$$8\ .\ CP = \frac{4}{3} \Rightarrow CP = \frac{1}{6} \Rightarrow \therefore\ AP = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}\ m$$

⇒ (c) is correct

When the rod is hung from A and J is given at P.

J − N = mv_c [where N is the reaction impulse at A] — (1)

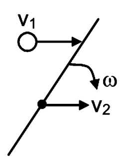
Taking moments about pivot at A :

$$J.\frac{2}{3}\ell = \frac{m\ell^2}{3}\omega \Rightarrow \omega = \frac{2J}{\ell m} = \frac{2J}{m}$$

$$v_c = \omega.\frac{\ell}{2} = \frac{2J}{m}.\frac{1}{2} \Rightarrow J = mv_c;$$

∴ from equation (1) N = 0

119.



Conservation of linear momentum : let v_1 be the velocity of the mass and v_2 be the velocity of the rod after collision. ∴ 1 × 4 − 1 × 2

$= 1v_1 + 1v_2 \Rightarrow v_1 + v_2 = 2$ — (1)

conservation of angular momentum:

$$1\times4\times\frac{1}{4} = 1\times1^2\times\frac{1}{12}\omega + v_1\times\frac{1}{4}$$

$(4 - v_1)\,3 = \omega$ — (2)

At the point of contact, velocity of approach

= velocity of separation

$$(4 + 2) = \left(v_2 + \frac{1}{4}\omega - v_1\right) \Rightarrow 4v_2 - 7v_1 = 12 \qquad —(3)$$

solve between (1) & (2) $\Rightarrow 4v_2 - 7v_1 = 12$

$$4v_2 - 4v_1 = 8$$

$$-11v_1 = 4 \Rightarrow v_1 = -\frac{4}{11}\ \text{m s}^{-1}$$

$\Rightarrow$ (a) is correct

$$v_2 = 2 + \frac{11}{4} = \frac{26}{11}\ \text{m s}^{-1}$$

$$\Rightarrow \omega = \left(4 + \frac{4}{11}\right)3 = \frac{144}{11}\text{rad s}^{-1}$$

$\Rightarrow$ (c) is correct

$\therefore$ (a) and (c) are correct

Additional chk : Energy audit (only for reference)

Initial Energy $= \frac{1}{2} \times 1 \times 4^2 + \frac{1}{2} \times 1 \times 2^2 = 10\ \text{J}$

Final energy

$$= \frac{1}{2}\times 1 \times \frac{16}{121} + \frac{1}{2}\times 1 \times \left(\frac{26}{11}\right)^2 + \frac{1}{2}\times\frac{1\times 1}{12}\times\left(\frac{144}{11}\right)^2$$

$$= 10\ \text{J}$$

120. (a) velocity of the point of contact is zero. Force of friction is just enough to prevent sliding. Friction supports forward motion but $F_f \le \mu N$. (not $F_r < \mu N$ or $F_f = \mu N$)

$\therefore$ (a) $\rightarrow$ (p), (q)

(b) $F_f = \mu N$. Directions of frictional force is opposite to the relative velocity of the point of contact but relative velocity of point of contact is opposite to the direction of motion of sphere!

(b) $\rightarrow$ (q), (r)

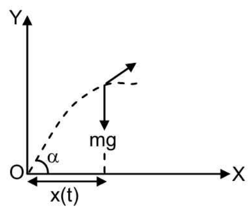
(c) velocity of point of contact is zero. Directions of frictional force is opposite to the direction of motion but $F_r \le \mu N$(not $F_f < \mu N$)

(c) $\rightarrow$ (p), (s)

(d) $\rightarrow$ (r), (s)

Additional Practice Exercise

121.



Torque about O = $\tau(t) = x(t) \times F$

$$= x(t)\,.\,mg$$

$$L = \int \tau\, dt$$

$$x(t) = u \cos \alpha\, t$$

$$\bar{L} = \left(-\hat{k}\right)\int_0^3 mg\, u \left(\cos\alpha\right) t\, dt$$

$$\bar{L} = \left(-\hat{k}\right) mg\, u \cos\alpha \left.\frac{t^2}{2}\right|_0^3$$

$$= \left(-\hat{k}\right)0.2 \times 10 \times 30 \times \frac{1}{\sqrt{2}} \times \frac{9}{2}$$

$$\Rightarrow \bar{L}(t) = -191\,\hat{k}$$

122.

(i) $T = F.r = I\alpha$

$$\alpha = \frac{F.r}{I} = \frac{20 \times 0.1}{0.1} = 20\ \text{rad s}^{-2}$$

(ii) $mg - T = ma$

$$T.\,r = I\alpha = \frac{Ia}{r}$$

$$\Rightarrow \quad T = \frac{Ia}{r^2}$$

$$mg = \left(m + \frac{I}{r^2}\right)a \Rightarrow \frac{a}{r} = \alpha = \left(\frac{mg}{mr + \frac{I}{r}}\right)$$

$$\Rightarrow \quad \alpha = \frac{2 \times 10}{2 \times 0.1 + \frac{0.1}{0.1}} = \frac{20}{0.2 + 1}$$

$$= \frac{20}{1.2} = 16.7\ \text{rad s}^{-2}$$

123. $\bar{L}_1 = 2t\hat{i} + 3t^2\hat{j}$

$$\therefore \quad \bar{\tau}_1 = \frac{d\bar{L}_1}{dt} = 2\hat{i} + 6t\hat{j}$$

$$\bar{L}_2 = 4t^2\hat{i} - 2t^3\hat{j}$$

$$\therefore \quad \bar{\tau}_2 = 8t\hat{i} - 6t^2\hat{j}$$

(i) $\cos\theta$

$$= \frac{\overline{\tau_1} \cdot \overline{\tau_2}}{|\overline{\tau_1}||\overline{\tau_2}|}$$

$$= \frac{\left(2\hat{i} + 6t\hat{j}\right) \cdot \left(8t\hat{i} - 6t^2\hat{j}\right)}{\sqrt{4+36t^2}\sqrt{64t^2+36t^4}}$$

$$= \frac{16t - 36t^3}{4t\sqrt{1+9t^2} \cdot \sqrt{16+9t^2}}$$

$$= \frac{4-9t^2}{\sqrt{(9t^2+1)(9t^2+16)}}$$

(ii) At $\theta = \frac{\pi}{2}$, $\cos\theta = 0$ Numerator = 0

$4 - 9t^2 = 0$

$\Rightarrow t = \frac{2}{3}$ s

124. No external torque. Hence L is constant. Let m be mass of man, xm be mass of turntable.

Initial angular momentum

$$= \frac{xmR^2}{2}.\omega_0 + m.2\omega_0 R.R = mR^2\omega_0\left(\frac{x}{2}+2\right)$$

When man is on the turntable at R,

$$I = x\frac{mR^2}{2} + mR^2$$

$$= mR^2\left(\frac{x}{2}+1\right)$$

and $L = mR^2\left(\frac{x}{2}+1\right)\omega_1$

$$\therefore \quad mR^2\omega_0\left(\frac{x}{2}+2\right) = mR^2\left(\frac{x}{2}+1\right).\omega_1$$

$$\Rightarrow \frac{\omega_1}{\omega_0} = \frac{\frac{x}{2}+2}{\frac{x}{2}+1} \qquad \text{— (1)}$$

When man is at R/2.

$$I' = x\frac{mR^2}{2} + m\frac{R^2}{4} = mR^2\left(\frac{x}{2}+\frac{1}{4}\right)$$

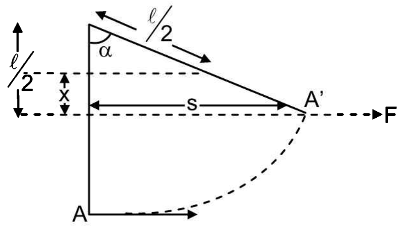
And $L = mR^2\left(\frac{x}{2}+\frac{1}{4}\right).\omega_2$

$$\therefore \quad mR^2\left(\frac{x}{2}+\frac{1}{4}\right).\omega_2 = mR^2\left(\frac{x}{2}+1\right)\omega_1$$

$$\Rightarrow \frac{\omega_2}{\omega_1} = \frac{\frac{x}{2}+1}{\frac{x}{2}+\frac{1}{4}}$$

Given $\frac{\omega_1}{\omega_0} = \frac{\omega_2}{\omega_1} \Rightarrow \frac{\frac{x}{2}+2}{\frac{x}{2}+1} = \frac{\frac{x}{2}+1}{\frac{x}{2}+\frac{1}{4}} \Rightarrow x = 4$

125.



Let ℓ be the length of the rod. When the end of the rod moves to the maximum position A', work done by F is

$$W_F = F.S = F.R\sin\alpha = \frac{mg}{2}R\sin\alpha$$

COM lifts by $x = \frac{\ell}{2} - \frac{\ell}{2}\cos\alpha = \frac{\ell}{2}(1-\cos\alpha)$

Work done by gravity :

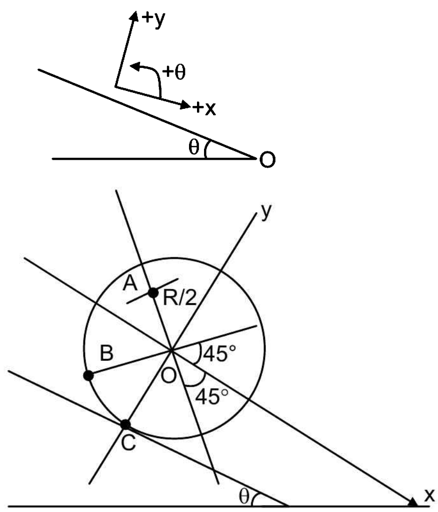
$$W_g = mgx = mg\frac{\ell}{2}(1-\cos\alpha)$$

Equating these two:

$$\frac{mg}{2}\ell\sin\alpha = mg\frac{\ell}{2}(1-\cos\alpha)$$

$\sin\alpha + \cos\alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}$

126.



Co-ordinate system assumed : Origin at O

$$\overline{r_A} = \frac{R}{2}\cos 135^\circ\,\hat{i} + \frac{R}{2}\sin 135^\circ\,\hat{j}$$

$$= \frac{-R}{2\sqrt{2}}\hat{i} + \frac{R}{2\sqrt{2}}\hat{j}$$

$$\overline{\omega} = \frac{v_{CM}}{R}\left(-\hat{k}\right)$$

$$\overline{v_A} = \overline{v_{CM}} + \overline{\omega} \times \overline{r_A}$$

$$\overline{v_A} = \overline{v_{CM}} + \overline{\omega} \times \overline{r_A}$$

$$= V_{CM}\,\hat{i} + \frac{v_{CM}}{2\sqrt{2}}\hat{j} + \frac{v_{CM}}{2\sqrt{2}}\hat{i}$$

$$= V_{CM}\left[\left(1 + \frac{1}{2\sqrt{2}}\right)\hat{i} + \frac{1}{2\sqrt{2}}\hat{j}\right]$$

$$\left|\overline{v_A}\right|^2 = V_{CM}^2\left[\left(1 + \frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2\right]$$

$$= 80 + 32\sqrt{2} \Rightarrow v_{CM} = 8\text{ m s}^{-1}$$

$$\overline{v_0} = \overline{v_{CM}} = 8\hat{i}\text{ m s}^{-1}$$

$$\left|\overline{v_0}\right| = 8\text{ m s}^{-1}$$

$$\overline{\omega} = \frac{8}{R}\left(-\hat{k}\right)\text{ rad s}^{-1}$$

$$\overline{r_B} = R\cos(-135^\circ)\,\hat{i} + R\sin(-135^\circ)\,\hat{j}$$

$$= \frac{-R}{\sqrt{2}}\hat{i} - \frac{R}{\sqrt{2}}\hat{j}$$

$$\overline{v_B} = \overline{v_{CM}} + \overline{\omega} \times \overline{r_B}$$

$$= 8\,\hat{i} + \frac{8}{R}\left(-\hat{k}\right) \times \left[\left(\frac{-R}{\sqrt{2}}\hat{i} - \frac{R}{\sqrt{2}}\hat{j}\right)\right]$$

$$= 8\,\hat{i} + \frac{8}{\sqrt{2}}\hat{j} - \frac{8}{\sqrt{2}}\hat{i} = 8\left[\left(1 - \frac{1}{\sqrt{2}}\right)\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right]$$

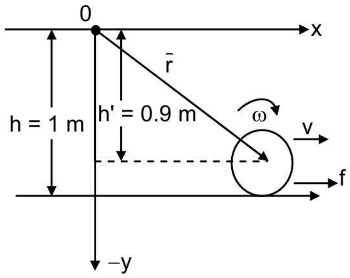
$$\overline{r_c} = -R\hat{j}$$

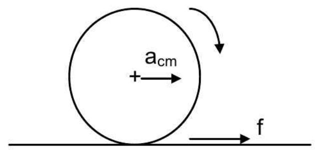
$$\overline{v_C} = \overline{v_{CM}} + \overline{\omega} \times \overline{r_C} = 8\hat{i} + \left[\frac{8}{R}\left(-\hat{k}\right) \times \left(-R\hat{j}\right)\right]$$

$$= 8\hat{i} - 8\hat{i} = 0 = 0$$

$$v_C = 0$$

127.





Since ω is in the $-\hat{k}$ direction rotation is clockwise and the shell's translatory motion is along $+\hat{i}$

Initially the angular momentum about 'O' will be only rotational angular momentum $L_0 = I\omega_0$. Now frictional force is developed and the translational velocity is increased and 'ω' is reduced till no slipping occurs, and pure rolling starts

Then

$$\overline{L} = \overline{r} \times \overline{p} + \overline{L}_{spin}$$

$$\left|\overline{r} \times \overline{p}\right| = r(mv)\sin\theta = (mv)h'$$

$\Rightarrow$ this will be constant when v is constant i.e., when pure rolling starts; so also $\left|\overline{L}_{spin}\right|$.

On the rough floor $a_{cm} = \dfrac{f}{m} = \dfrac{\mu mg}{m} = \mu g$

Let the shell start pure rolling after time t

Its translational velocity $v = at = \mu gt$

$\omega' = \omega_0 - \alpha t$

$$\Rightarrow \alpha = \frac{\tau}{I} = \frac{fr}{\frac{2}{3}mr^2} = \frac{3f}{2mr} = \frac{3\mu mg}{2mr} = \frac{3\mu g}{2r}$$

$v = at = \mu gt \Rightarrow$ when no slipping : $\dfrac{v}{r} = \omega'$

$$\frac{\mu gt}{r} = \omega' = \omega_0 - \frac{3\mu gt}{2r} \Rightarrow t\frac{\mu g}{r}\left(1 + \frac{3}{2}\right) = \omega_0$$

$$t = \frac{2r\omega_0}{5\mu g} = \frac{2 \times 0.1 \times 100}{5 \times 0.4 \times 10} = \frac{20}{20} = 1\text{ s}$$

$v = at = \mu gt = 0.4 \times 10 \times 1 = 4\text{ m s}^{-1}$

$$\omega' = \frac{v}{r} = \frac{4}{0.1} = 40\text{ rad s}^{-1}$$

$$\Rightarrow \overline{\omega}' = -40\hat{k}\text{ rad s}^{-1}$$

(i) After t = 1 s, pure rolling starts and, the angular momentum about O is constant

(ii) Final angular momentum about O:

$$\overline{r} \times m\overline{v} = \left(-r\sin\theta\,\hat{j} + r\cos\theta\,\hat{i}\right) \times 0.3\left(4\hat{i}\right)$$

$(r\sin\theta = (1 - 0.1) = 0.9\text{ m})$

$$= (0.3)\left(-0.9\hat{j}\right) \times \left(4\,\hat{i}\right) = 1.08\hat{k}$$

$$\overline{L}(\text{spin}) = I\overline{\omega} = \frac{2}{3} \times 0.3 \times 0.01 \times \left(-40\hat{k}\right) = -\,0.08\hat{k}$$

Total angular momentum about O

$= (1.08 - 0.08)\hat{k} = 1\hat{k}$ kg m² s⁻¹

This value will remain constant:

128. v_{CM} at B $= \sqrt{2gh} = 20$ m s⁻¹

From B to C, retardation a = μg = 4 m s⁻²

$\therefore \quad v_{CM}$ at C $= \sqrt{u^2 - 2as} = \sqrt{400 - 256}$

$= 12$ m s⁻¹

$\therefore \quad$ time $= \dfrac{v-u}{a}\dfrac{12-20}{-4} = 2$ s

Angular velocity ω_0 at B = 0

ω at C $= \alpha t$

$$\alpha = \frac{\mu mgR}{I} = \frac{\mu mgR}{mK^2} = \frac{4R}{K^2}$$

$$\omega \text{ at } C = \frac{4R}{K^2} \times 2 = \frac{8R}{K^2}$$

For pure rolling, $\omega R = v_{CM}$

$$\Rightarrow \frac{8R^2}{K^2} = 12 \Rightarrow K^2 = \frac{2}{3}R^2$$

$\Rightarrow$ the object is spherical shell

From C upto E, pure rolling,

$\therefore$ If E is at height h_1

$$mgh_1 = \frac{1}{2}mv^2_{CM} + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m.12^2 + \frac{1}{2}.\frac{2}{3}mR^2.\left(\frac{12}{R}\right)^2$$

$$= \frac{1}{2}m.12^2.\frac{5}{3}$$

$$\Rightarrow h_1 = 12^2.\frac{5}{6}.\frac{1}{10} = 12 \text{ m}$$

$$\therefore \quad DE = \frac{h_1}{\sin 37^\circ} = \frac{12}{(3/5)} = 20 \text{ m}$$

129. Two equal masses strike the rod symmetrically so that the centre of mass remains at the same position as before. After the collision the rod will have linear and rotational motion. So it will possess rotational kinetic energy $= \frac{1}{2}I\omega^2$ and linear kinetic energy $\left(\frac{1}{2}mv^2\right)$.

Angular momenta w.r.t CM

Clockwise $L_1 = (0.08 \times 10) \times 0.5$

Anti-clockwise $L_2 = (0.08 \times 6) \times 0.5$

Net clockwise angular momentum

$L = L_1 - L_2 = 0.08 \times 4 \times 0.5 = 0.16$ kg m² s⁻¹

Moments of Inertia of the rod and masses A and B about 'CM' will be

$$I = \frac{0.16 \times \left(\sqrt{3}\right)^2}{12} + 0.08 \times (0.5)^2 + (0.08) \times (0.5)^2$$

$= 0.08$ kg m²

But Angular Momentum L = I ω and rotational KE $= \frac{1}{2}I\omega^2$ Thus,

$$(KE)_{rot} = \frac{1}{2}\left(I\omega^2\right) = \frac{1}{2}\frac{(I\omega)^2}{I} \Rightarrow (KE)_{rot} = \frac{1}{2}\frac{(L)^2}{I}$$

$$= \frac{1}{2} \times \frac{(0.16)^2}{0.08} = 0.16 \text{ J}$$

(i) Linear Momentum and Kinetic Energy :

For conservation of linear momentum, total linear momentum of the combined mass = initial linear momentum.

Total linear momentum of the combined mass

$= (0.08 \times 10) + (0.08 \times 6) = 0.08 \times 16$

$= 1.28$ kg m s⁻¹

Now $(KE)_{linear} = \frac{1}{2}mv^2 = \frac{1}{2}\frac{(mv)^2}{m}$

(where m is the combined mass)

$$= \frac{1 \quad \times 1.28^2}{2(0.16 + 0.08 + 0.08)} = 2.56 \text{ J}$$

So total Kinetic Energy $= K.E = KE_{rot} + KE_{linear}$

$= 0.16 + 2.56 = 2.72$ J

Initial KE $= \frac{1}{2} \times 0.08 \times (10)^2 + \frac{1}{2}(0.08)(6)^2 = 5.44$J

$\therefore$ Net loss of Kinetic Energy after collision

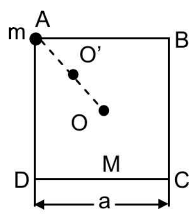
$= 5.44 - 2.72 = 2.72$ J

(ii) Let v_{CM} the velocity of the bar after collision.

$0.08 \times 10 + 0.08 \times 6 = v_{CM}(0.16 + 0.08 + 0.08)$

$v_{CM} = 4$m s⁻¹ = Velocity of the bar after collision

130.



(i) No external forces in the horizontal plane.

$\therefore$ Linear momentum conserved.

$$mu = (M + m)\,v_{CM} \Rightarrow v_{CM} = \frac{m}{M+m}.u$$

(ii) The centre of mass changes after collision.

The new centre of mass is O'

Treat O as origin. Then

$$(M+m)(OO') = m.\sqrt{2}\frac{a}{2} + M \times (0)$$

$$\Rightarrow OO' = \frac{m}{(M+m)}.\frac{a}{\sqrt{2}} \qquad \text{— (1)}$$

$$AO' = \frac{a}{\sqrt{2}} - OO' = \frac{M}{(M+m)}.\frac{a}{\sqrt{2}} \qquad \text{— (2)}$$

Since no torque about O' angular momentum is conserved.

$$L_{before} = \left|\overline{O'A} \times m\overline{u}\right|$$

$$= mu\left(\sqrt{2}\frac{a}{2} - \frac{m}{M+m}.\frac{a}{\sqrt{2}}\right)\sin 45^\circ$$

$$= \frac{Mm}{(M+m)}.\frac{a}{2}.u$$

$$L_{after} = I_{O'}\omega$$

$$\Rightarrow I_{O'} = \left[I_O + M(OO')^2\right] + m(AO')^2$$

$$I_{o'} = M\frac{a^2}{6} + M.\frac{m^2}{(M+m)^2}.\frac{a^2}{2} + m.\frac{M^2}{(M+m)^2}.\frac{a^2}{2}$$

$$= \frac{(M+4m)}{6(M+m)}.Ma^2$$

$$\therefore \omega = \frac{Mm}{(M+m)}.\frac{a}{2}.u.\frac{6(M+m)}{(M+4m)Ma^2}$$

$$= \frac{3m}{(M+4m)a}.u.$$

131. Let m slide by ℓ on incline M slide by L

Condition: CM has zero horizontal displacement

$$\Rightarrow m(\ell\cos\theta - L) = ML$$

$$\Rightarrow \frac{\ell}{L} = \frac{M+m}{m\cos\theta}, \text{ always} > 1$$

132. Set the surface as zero potential.

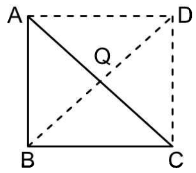
$$PE_1 = -\left[\text{mass of rope} \times g \times \frac{\ell}{2} + m_1 g\ell\right]$$

$$= -100 \times 3 \times 10 \times \frac{100}{2} - 75 \times 10 \times 100$$

$$= -225 \text{ kJ}$$

$PE_2 = 0 \Rightarrow W = 225$ kJ

133.



Completing the square

$I'_{AC} = I'_{BD}$

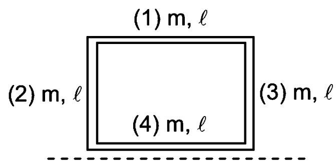
$\therefore$ For triangle : $I_{AC} = \frac{1}{2}I'_{AC}$

$I_{BQ} = \frac{1}{2}I'_{BD}$

$\therefore$ $I_{AC} = I_{BQ}$

134. Non-uniform distribution immaterial. All particles are R away

135.

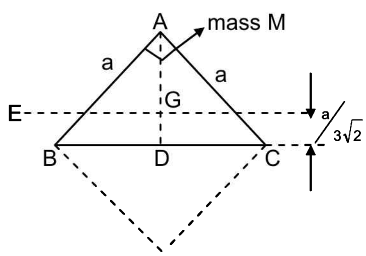


$I_1 = m\ell^2$

$I_2 = I_3 = \frac{m\ell^2}{3}$; $I_4 = 0$

$I = I_1 + I_2 + I_3 + I_4 = \frac{5}{3}m\ell^2$

136.



$I_{BC} = \frac{Ma^2}{12}$ (half of that for a square of mass 2M)

$I_{EG} = I$(Given)

By parallel axis theorem,

$$I_{BC} = I_{EG} + M\left(\frac{a}{3\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \frac{Ma^2}{12} = I + \frac{Ma^2}{18} \Rightarrow I = \frac{Ma^2}{36}$$

$$I_{AD} = \frac{Ma^2}{12} \text{ (half of that for a square of mass 2M)}$$

$$= 3I$$

Required I_Z through G = $I_{EG} + I_{AD}$

(by perpendicular axes theorem) = 4I

137. $\bar{\omega} = \left(-2\hat{i} + \hat{k}\right)$ (data)

$$\bar{r}_P = a\left(\hat{i} + \hat{j} + \hat{k}\right)m \quad \text{(data)}$$

$$\left|\bar{r}_P\right| = r_P = \sqrt{3} \text{ m (data)}$$

$$\therefore \quad \sqrt{a^2 + a^2 + a^2} = \sqrt{3} \Rightarrow \sqrt{3}\, a^2 = \sqrt{3}$$

a = ±1 m, –ve excluded.

$$\therefore \bar{r}_P = \left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$\bar{r}_{cm} = \left(\hat{i} - 2\hat{j}\right)m \text{(data)}$$

$$\therefore \quad \bar{r}_{P/cm} \;\; \bar{r}_P - \bar{r}_{CM}$$

$$= \left(\hat{i} + \hat{j} + \hat{k}\right) - \left(\hat{i} - 2\hat{j}\right) = \left(3\hat{j} + \hat{k}\right)$$

$$\bar{v}_P = 0 \Rightarrow v_{cm} + \bar{\omega} \times \bar{r}_{P/cm} = 0$$

$$\Rightarrow \quad v_{cm} = -\,\bar{\omega} \times \bar{r}_{P/cm} = \bar{r}_{P/cm} \times \bar{\omega}$$

$$\Rightarrow \quad v_{cm} = \left(3\hat{j} + \hat{k}\right) \times \left(-2\hat{i} + \hat{k}\right) = \left(+6\hat{k} + 3\hat{i} - 2\hat{j}\right)$$

$$\Rightarrow \quad v_{cm} = \sqrt{6^2 + 3^2 + (-2)^2} = \sqrt{49} = 7 \text{ m s}^{-1}$$

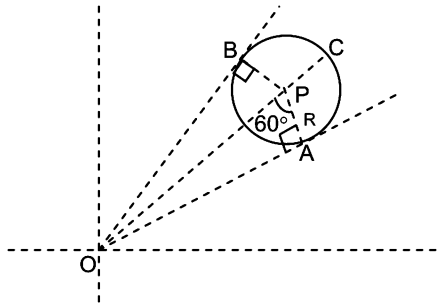
Hence (b)

138. Uniform rotation ⇒ no torque of force on P, about the axis

If $\bar{F}$ is parallel to the axis, then no torque. But options show that this is not the case(especially since all options have $\hat{k}$ components).

Let the perpendicular from P to the axis meet it at O. Let O be origin. $\bar{r} = \overline{OP}$; If $\hat{F} = \hat{r}$, no torque $\because \bar{r} \times \bar{F}$ will be zero i.e; $\bar{F}$ will be perpendicular to axis. i.e; $\bar{F}.\left(2\hat{i} + 3\hat{j}\right)$ will be zero ⇒ $\bar{F}$ is of the form $n\left(3\hat{i} - 2\hat{j}\right)$ + any $\hat{k}$ component, n any constant, positive or negative.

139.



L = 0 at A and B

($\because$ At A and B, $\bar{r}$ parallel to $\bar{p}$)

Since the shorter interval AB is equal to 1 s, hence the longer interval is BCA = 2 s and hence the total time period is 3 s

$$\text{Time period} = 3 \text{ s} \Rightarrow \frac{2\pi R}{1} = 2\pi R = 3$$

$$\Rightarrow R = \frac{3}{2\pi},\; \frac{t_{AB}}{t_{BCA}} = \frac{1}{2} \Rightarrow \therefore \text{ half angle } 60° \text{ as shown}$$

$$\therefore \quad \frac{R}{OP} = \sin 30 = \frac{1}{2} \Rightarrow OP = 2R$$

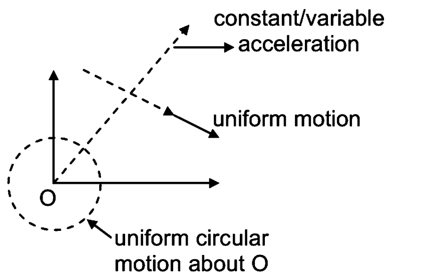
$$\Rightarrow \quad OP + R = 3R = \frac{9}{2\pi}$$

L is maximum at C

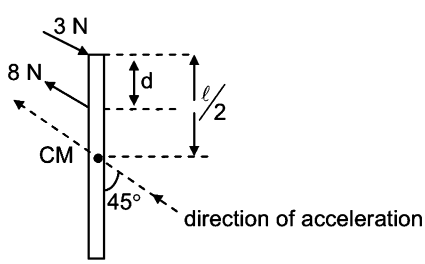
140.

Note that it can have an initial component of velocity not perpendicular to its position vector

141.



142.



It is easy to see

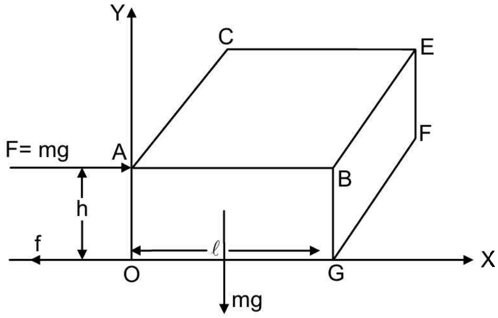
(i) smaller force should act farther from CM,

(ii) both should act on same side CM, so that torque about CM is zero

$$\Rightarrow \quad 3.\frac{\ell}{2}\sin 45° = 8\left(\frac{\ell}{2} - d\right)\sin 45°$$

$$\Rightarrow \quad \frac{d}{\ell} = \frac{5}{16}$$

143.



Frictional force $f = \mu mg > mg$ ($\because \mu > 1 \rightarrow$ data)

Hence the force applied $F = mg$, cannot move the block sliding. For the block to be toppled the maximum torque happens when F is applied at the top edge. Hence $OA = h$ (maximum size). For toppling, about the edge GF :

$$F.(OA) > mg.\frac{\ell}{2} \Rightarrow mg.h > mg\frac{\ell}{2}$$

$\ell < 2h \Rightarrow$ maximum limiting case is $\ell = 2h$

Since the force may be applied on the other edge

i.e., BE, for that side also the same condition applies. Hence $BE = 2h$

Hence maximum volume is $h.2h.2h = 4h^3$

144. $P = \tau.\omega = I\alpha.\omega = I\dfrac{d\omega}{dt}\omega$

$$= I\frac{d\omega}{d\theta}.\frac{d\theta}{dt}.\omega = I\frac{d\omega}{d\theta}.\omega^2$$

$$\Rightarrow \quad \omega^2.d\omega = \frac{P}{I}d\theta$$

$$\Rightarrow \quad \left.\frac{\omega^3}{3}\right|_0^{\omega} = \left.\frac{P}{I}\theta\right|_0^{\theta} \Rightarrow \omega \propto \theta^{1/3}$$

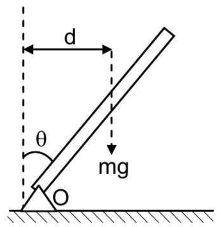
145. $F(\ell - y) = I\alpha$

$$a_{CM} = \alpha\frac{\ell}{2} = \frac{F}{I}\frac{\ell}{2}(\ell - y)$$

which is equation of a straight line in the form

$y = c - mx$

146.



Considering the rotation of the bar about O, $mg.d = I\alpha$, where d is the distance of the center of mass from the vertical passing through O.

$$d = \frac{\ell}{2}\sin\theta \,;\, I = \frac{m\ell^2}{3},$$

$$mg\frac{\ell}{2}\sin\theta = \frac{1}{3}m\ell^2\alpha$$

$$\alpha = \frac{3g\sin\theta}{2\ell} = \frac{3 \times 10 \times \frac{1}{2}}{2 \times 1} = 7.5 \text{ rad s}^{-2}$$

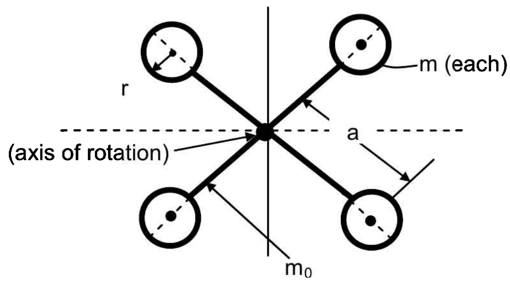
147. $\alpha = \dfrac{\tau - F_f.r}{I}$

$$F_f = Ma \Rightarrow \frac{F_f}{M} = a = r\alpha = \frac{r(\tau - F_f.r)}{I} = \frac{\tau r - F_f r^2}{\frac{mr^2}{2}}$$

$$\Rightarrow \quad \frac{F_f}{M} = \frac{2\tau}{mr} - \frac{2F_f}{m} \Rightarrow F_f = \frac{\frac{2\tau}{mr}}{\frac{1}{M} + \frac{2}{m}}$$

$$\Rightarrow \quad F_f = \frac{2\tau}{r\left(2 + \frac{m}{M}\right)}$$

148.



$$I\omega_0 = \tau t \therefore t = I\omega_0/\tau$$

$$\frac{4}{\tau}\left(mr^2 + ma^2 + \frac{m_0(a - r)^2}{3}\right)\omega_0$$

$$= \left((a^2 + r^2)\frac{4m}{\tau} + \frac{4m_0(a - r)^2}{3\tau}\right)\omega_0$$

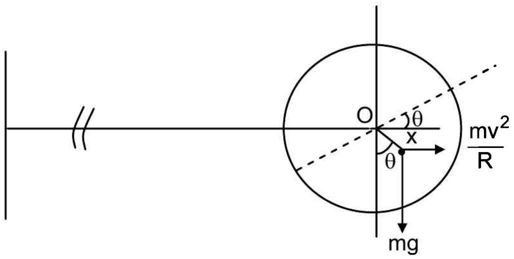
149. $\omega = \omega_0 + \alpha t$

$20 = \alpha \times 1$

$\alpha = 20$ rad s^2

$\tau = I\alpha = \frac{1}{2}MR^2\alpha = \frac{1}{2}10 \times (0.2)^2 \times 20 = 4$ N m

150.



On the curved path, the surface tilts as shown. Taking moments about O: [COM at $\times = \frac{4r}{3\pi}$]

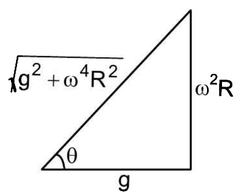
$$mg \times \sin\theta = \frac{mv^2}{R}x\cos\theta$$

$$\tan\theta = \frac{v^2}{Rg} = \frac{\omega^2 R^2}{Rg} = \frac{\omega^2 R}{g}$$

When on straight path, centrifugal force mv^2/R vanishes

$\therefore$ Torque $= mg \times \sin\theta$

$$= mg\left(\frac{4r}{3\pi}\right)\frac{v^2}{\sqrt{v^4 + R^2g^2}}$$



When the wagon enters the straight segment of track, liquid (petrol) oscillates about an axis through O with an amplitude of θ. When on straight track,

Torque = Iα = – mg x sinθ

$$= -mg\left(\frac{4r}{3\pi}\right)\left(\frac{\omega^2 R}{\sqrt{g^2 + \omega^4 R^2}}\right)$$

$$= -mg\left(\frac{4r}{3\pi}\right)\left(\frac{v^2}{\sqrt{R^2g^2 + v^4}}\right)$$

151. Conservation of angular momentum gives

$I\omega = (I + I')\,\omega'$

$$\omega' = \frac{MR^2}{MR^2 + 2mR^2}\cdot\omega = \frac{M\omega}{M + 2m}$$

152. As ice melts, the water so formed flows to equatorial region thereby increasing moment of inertia I.

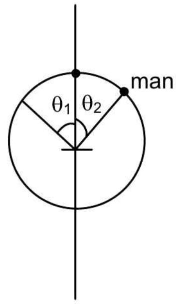
Hence ω decreases as Iω = constant.

So $T = \frac{2\pi}{\omega}$ increases.

153. By conservation of angular momentum,

$I_1\omega_1(t) = I_2\omega_2(t)$ at all times t.

$\Rightarrow I_1\theta_1 = I_2\theta_2$



Also $\theta_1 + \theta_2 = \frac{\pi}{4}$

$$\frac{I_2}{I_1} = \frac{mr^2}{\frac{mr^2}{2}} = 2$$

$$\Rightarrow \theta_2 = \frac{\theta_1}{2} \Rightarrow \frac{3\theta_1}{2} = \frac{\pi}{4}$$

$$\Rightarrow \theta_1 = \frac{\pi}{6} \text{ rad}$$

154. $I_0 = \frac{ML^2}{12} + 0$

$$I_1 = \frac{ML^2}{12} + 2m\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{mL^2}{2}$$

Angular momentum is conserved,

$\therefore I_0\omega_0 = I_1\omega_1$

$$\frac{\omega_1}{\omega_0} = \frac{I_0}{I_1} = \frac{\frac{ML^2}{12}}{\frac{ML^2}{12} + \frac{mL^2}{2}}$$

$$\omega_1 = \omega_0\left(\frac{M}{M + 6m}\right) = 20\left(\frac{0.4}{1}\right) = 8 \text{ rad s}^{-1}$$

155. Loss of P.E $= mg\frac{\ell}{2}(1 - \cos\theta)$

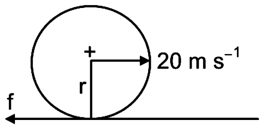
$$= mg\frac{\ell}{2}\left[1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right]$$

$$= mg\ell\sin^2\frac{\theta}{2}$$

Gain in K.E $= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{m\ell^2}{3}.\omega^2$

Equating $\omega = \sqrt{\frac{6g}{\ell}}\sin\frac{\theta}{2}$

156.



$\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{fr}{\frac{2}{3}mr^2} = \frac{0.4mgr}{\frac{2}{3}mr^2} = \frac{0.6g}{r}$$

$$\omega = 0 + \alpha t = \frac{0.6g}{r}t.$$

$\therefore$ After $t = 1$ s $\Rightarrow \omega = \dfrac{0.6g}{r} = \dfrac{6}{r}$

157. $f = \mu mg = 0.4$ mg

$$a = \frac{f}{m} = 0.4g$$

$v_C = u - at = 20 - 0.4gt = 20 - 4 = 16$ m s^{-1}

[$\because t = 1$ s, $g = 10$ m s^{-2}]

158. $KE_A = KE_B$

But friction dissipates energy for A only till it starts rolling, so final energy is more for A

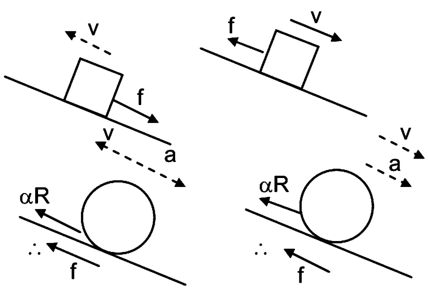
$\Rightarrow$ $mgh_A > mgh_B \Rightarrow h_A > h_B$

159. $KE_A = KE_B$, and no frictional losses

$\Rightarrow mgh_A = mgh_B \Rightarrow h_A = h_B \Rightarrow 1{:}1$

160. Block:

Cylinder :



161. In rolling, $f = \dfrac{mg\sin\theta}{1+\dfrac{R^2}{K^2}} = \dfrac{mg\sin\theta}{3}$ (for cylinder)

In sliding $f = \mu mg\cos\theta$

Equating, $\mu = \dfrac{\tan\theta}{3} = \dfrac{3/4}{3} = \dfrac{1}{4} = 0.25$

162. $t \propto \sqrt{1 + \dfrac{K^2}{R^2}}$

$\Rightarrow$ for those two $\dfrac{K^2}{R^2}$ must be same,

$\dfrac{K^2}{R^2}$ values: solid sphere : $\dfrac{2}{5} = 0.4$

hollow sphere : between the limits 0.4 and 0.67 (spherical shell)

solid disc: $\dfrac{1}{2} = 0.5$

hollow disc : 0.5 to 1(Ring)

clearly the pair can be hollow sphere and solid disc or hollow sphere and hollow disc

163. They will take same time only if both slide (and not roll)

$\therefore$ μ should be less than or just equal to $\dfrac{\tan\theta}{1 + \dfrac{R^2}{K^2}}$ for sphere

$$= \frac{0.7}{1 + \frac{5}{2}} = 0.2, \text{ the critical value.}$$

164. Let the translational velocity of the disc on the reference frame of the conveyor belt be v_r. Then this reference frame, for rolling without slippage, $v_r = r\omega$. ω is unchanged on this or outside reference frame.

When the disc rolls in the direction of the conveyor belt velocity,

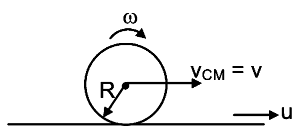
$$v_r = v - u;\ \omega = \frac{2\pi}{T} = \frac{v_r}{r} = \frac{v - u}{r}$$

$$\Rightarrow\ T = \frac{2\pi r}{v - u}$$

When the disc rolls in the opposite direction of conveyor belt velocity, $v_r = v + u$

$$\Rightarrow\ T = \frac{2\pi r}{v + u}$$

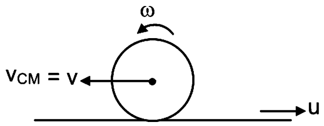
Case (i)



poc condition: $v - \omega R = u$

$$\Rightarrow\ T = \frac{2\pi R}{v - u}$$

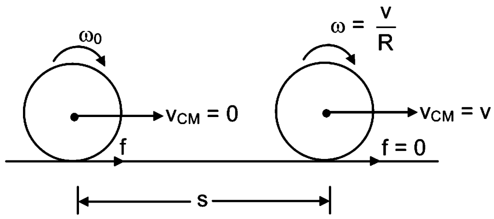
Case (ii)



poc condition : $\omega R - v = u \Rightarrow T = \dfrac{2\pi R}{v + u}$

Hence cannot be concluded from the given data.

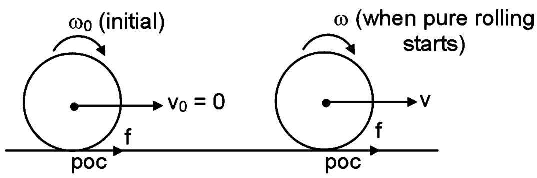
165.



Initially because of slippage friction f = μ mg and acts in the direction of its translatory motion as shown till it starts pure rolling when f becomes zero. Hence work done by friction

$= \mu\, mg\,.\, s = KE = \dfrac{1}{2}mv^2$

166.



Friction forces act along poc and hence the angular momentum about that point is unchanged. Let ω_0 and ω be the initial and final value of angular velocity. Taking angular momentum about the poc.

For sphere $\Rightarrow \dfrac{2}{5}mr^2\omega_0 = I\omega + mvr$

$$= \frac{2}{5}mr^2\omega + mr^2\omega$$

$$\frac{2}{5}\omega_0 = \omega\left[\frac{2}{5} + 1\right] \Rightarrow \omega = \frac{2}{7}\omega_0 < 0.3\omega_0$$

For thin shell $\Rightarrow \dfrac{2}{3}mr^2\omega_0 = \dfrac{2}{3}mr^2\omega + mr^2\omega$

$$\frac{2}{3}\omega_0 = \omega\left(\frac{2}{3} + 1\right) \Rightarrow \omega = \frac{2}{5}\omega_0 = 0.4\omega_0$$

For hollow sphere obviously the value of ω is $0.3\,\omega_0 < \omega < 0.4\,\omega_0$

i.e., larger than sphere.

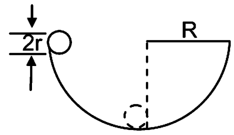
167. Energy equation:

$$mg(R - r) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\frac{v^2}{R^2}$$

$$= \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$$

$\Rightarrow$ centripetal force $\dfrac{mv^2}{(R - r)} = \dfrac{2mg}{1 + \dfrac{K^2}{R^2}}$

since $0 < \dfrac{K^2}{R^2} \le 1$



$$mg \le \frac{mv^2}{(R - r)} \le 2mg$$

$\therefore\quad N = mg + \dfrac{mv^2}{R - r} \Rightarrow 2\text{ mg} \le N \le 3\text{ mg}$

168. $a = \dfrac{g\sin\theta}{1 + K^2/R^2}$

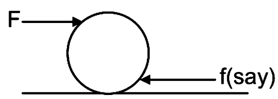
$s = \dfrac{1}{2}at^2; \sin\theta = \dfrac{h}{s}$

$$\frac{h}{\sin\theta} = \frac{1}{2}\left(\frac{g\sin\theta}{1 + \dfrac{K^2}{R^2}}\right)t^2$$

$$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}\sqrt{1 + \frac{K^2}{R^2}}$$

μ does not matter as long as it is rolling

169.



$F - f = ma$ — (1)

$(F + f)R = mR^2\alpha = mR^2\dfrac{a}{R} \Rightarrow F + f = ma$ — (2)

(1) and (2) $\Rightarrow f = 0$

170. $\bar{L} = \bar{L}_{CM} + \bar{r}_{CM} \times \bar{p}$

$L_{CM} = I\omega$ constant

$$\frac{d}{dt}\left(\overline{r}\times\overline{p}\right)=\frac{d\overline{r}}{dt}\times\overline{p}+\overline{r}\times\frac{d\overline{p}}{dt}$$

$$=\overline{v}\times\overline{p}+\overline{r}\times 0$$

$$m\overline{v}\times\overline{v}+0=0+0=0$$

$\Rightarrow$ L is constant about any origin

171. Statement 1 is correct, from the proof of the theorem
Statement 2: not necessarily

Consider a disc with density which varies as r.

CM $\equiv$ geometric centre

172. same as earlier

173. $\overline{v}=\overline{v}_{CM}+\overline{\omega}\times\overline{r}$

$\overline{v}=\overline{a}+\overline{r}\times\overline{b}$

where $\overline{a}=\overline{v}_{CM}$ and $\overline{b}=-\overline{\omega}$

174. n can be any integer
Proof: origin O :

$\overline{\tau}_O=\overline{r}_1\times\overline{F}_1+\overline{r}_2\times\overline{F}_2+\ldots.$

Consider another origin at O'

$$\overline{\tau}_{O'}=\overline{r}_1'\times\overline{F}_1+\overline{r}_2'\times F_2+\ldots.$$

$$\overline{r}_1'=\overline{r}_1+\overline{r}_{O'O}$$

$$\therefore\quad \overline{\tau}_{O'}=\overline{\tau}_O+\overline{r}_{O'O}\times\left(\overline{F}_1+\overline{F}_2+\ldots\ldots\right)$$

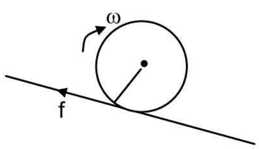
$$=\overline{\tau}_O+\overline{r}_{O'O}\times(O)$$

175. $t\propto\sqrt{1+\frac{K^2}{R^2}}$, and independent of m; (see solution of 168)

176. Under no relative motion, friction force is less than limiting but not necessarily zero

177. If the impulse is such that v = rω pure rolling from start itself, hence no loss in energy on rough patch

178.



In both cases frictional torque does positive work τ = fr. Loss is due to movement of POC against frictional force, in second case

179. Statement 1 is correct

$\because\quad \overline{p}_0=\overline{p}_1+\overline{p}_2$ — (1)

$$\frac{p_0^2}{2m}+RKE=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+RKE$$

since no internal torque, the second sphere will have no rotation.

$p_0^2=p_1^2+p_2^2$ — (2)

$\therefore$ By (1) and (2) $\overline{p}_1$ and $\overline{p}_2$ will be at right angles

180. Direction of angular velocity does not change, but direction of velocity reverses $\Rightarrow$ slipping

181. Rigid body AP, PQ, QA remain constant with time. Hence any angle must remain constant. The body may have both translational and rotational motions. Since the body is rigid all points have same transitional velocity. The remaining motion is rotational motion. The angular velocity between any pair of particles in the rigid body remains the same.

182. $\overline{\omega}=\omega_0\hat{k}$

$$\overline{v}_{AD}=\overline{\omega}\times\overline{DA}=\omega_0\hat{k}\times\left(-a\hat{i}\right)=-a\omega_0\hat{j}$$

183. In translation the vector $\overline{AP}$ must be a constant vector.

$\overline{R}_A-\overline{R}_P=\text{constant}$

$$\overline{v}_A-\overline{v}_P=\overline{v}_{AP}=\frac{d}{dt}\left(\overline{R}_A-\overline{R}_P\right)=0$$

$\overline{v}_A=\overline{v}_P$. This is true, for any two points.

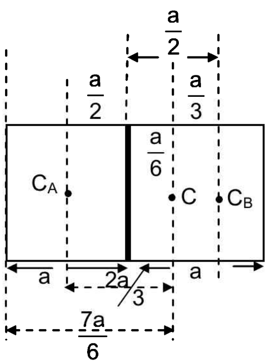
184. $y_{CM}=\frac{a}{2}$

Total mass = M

$M_A=\frac{M}{3}, M_B=\frac{2}{3}M$

$Mx_{cm}=\frac{M}{3}.\frac{a}{2}+\frac{2}{3}M.\frac{3}{2}a\qquad x_{cm}=\frac{7}{6}a$

185.



I due to A:

(i) about C_A : $\frac{M}{3}.\frac{a^2}{12}=\frac{Ma^2}{36}$ — (1)

(ii) Additional term by parallel axis theorem :

$$\frac{M}{3}.\left(\frac{2a}{3}\right)^2=\frac{4}{27}Ma^2 \quad \text{— (2)}$$

Due to B:

(iii) about C_B : $\frac{2M}{3}.\frac{a^2}{12}=\frac{Ma^2}{18}$ — (3)

(iv) Additional : $\frac{2M}{3}.\left(\frac{a}{3}\right)^2=\frac{2}{27}Ma^2$ — (4)

Adding, $\frac{11}{36}Ma^2$

186. By perpendicular axis theorem,

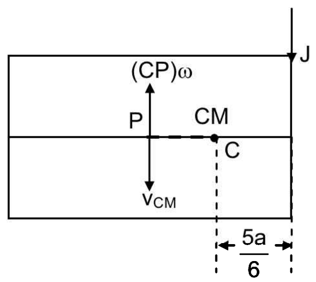
$$I_C=\frac{11}{36}Ma^2+\frac{Ma^2}{12}=\frac{7}{18}Ma^2$$

$$v_{CM}=\frac{J}{M} \quad \text{— (1)}$$

Consider the torque impulse about COM

$$J\,.\,\frac{5a}{6}=I_{cm}.\omega$$

$$\omega=\left(J.\frac{5}{6}a\right)\Big/\frac{7}{18}Ma^2=\frac{15}{7}\frac{J}{Ma} \quad \text{— (2)}$$



If P is the point which is at rest immediately after impulse, then (CP) $\omega=v_{CM}$

$$\Rightarrow \quad CP=\frac{J}{M}.\frac{7Ma}{15J}=\frac{7}{15}a$$

$\Rightarrow$ x coordinate of instantaneous centre is

$$\frac{7a}{6}-\frac{7a}{15}=\frac{7}{10}a$$

187. Time of descent $t=\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}.\sqrt{1+\frac{K^2}{r^2}}$ (see problem 168.)

Let $\frac{K^2}{r^2}=\eta$ t smaller $\Rightarrow \eta$ smaller

Energy : mgh = E = TKE(T) + RKE(R)

$$T=\frac{1}{2}mv^2,\ R=\frac{1}{2}mv^2\eta=T.\eta$$

$$\therefore \quad E=T(1+\eta)\Rightarrow T=\frac{E}{1+\eta}$$

η smaller $\Rightarrow$ T higher

Aliter:

The one reaching faster moves under larger acceleration since

$v^2=2as\Rightarrow$ s is same for both so that v^2 is larger. Hence KE is larger.

188. Let T_1, $T_1\eta_1$ and T_2, $T_2\eta_2$ be the respective energies

$$\frac{T_1}{T_2\eta_2}=\frac{5}{3}$$

$$\Rightarrow \quad \frac{E/1+\eta_1}{\left(E/1+\eta_2\right)\eta_2}=\frac{5}{3}$$

$$\Rightarrow \quad \frac{1+\frac{1}{\eta_2}}{1+\eta_1}=\frac{5}{3}$$

$$\Rightarrow \quad 3+\frac{3}{\eta_2}=5+5\eta_1$$

$$\Rightarrow \quad \frac{3}{\eta_2}=2+5\eta_1$$

$$\Rightarrow \quad \eta_2=\frac{2}{3}\text{ (spherical shell)};\ \eta_1=\frac{1}{2}\text{ (disc)}$$

189. $t \propto \sqrt{1+\eta}$

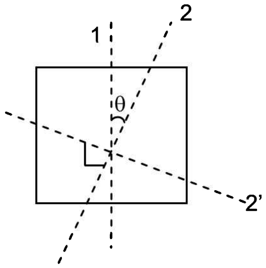
disc : $t \propto \sqrt{\frac{3}{2}}$

spherical shell $t \propto \sqrt{\frac{5}{3}}$

$$\frac{t_{shell}}{t_{disc}} = \sqrt{\frac{5}{3} \times \frac{2}{3}} = \sqrt{\frac{10}{9}}$$

$$\therefore t_{shell} = \sqrt{\frac{10}{9}} \times t_{disc} = \sqrt{\frac{10}{9}} \times 3 = \sqrt{10}\,s$$

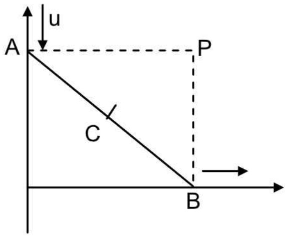
190.



For any pair of perpendicular axes for a square, mass distribution for the two halves will be the same.

$I_{zz} = I_2 + I_{2'} = 2I_2 = 2I_1$ and so on

191.



P is the center of rotation

To find the centre of rotation, draw normals to u_A and u_B at A and B. They meet at P.

$PA = 0.6\,\ell$

$PA\omega = u$

$$\Rightarrow \omega = \frac{u}{0.6\ell} = \frac{5}{3}\frac{u}{\ell}$$

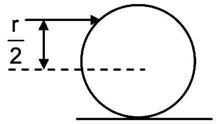
$$v_B = PB\omega = 0.8\ell\,\frac{5}{3}.\frac{u}{\ell} = \frac{4}{3}u$$

$$PC = \ell/2 \Rightarrow v_C = \frac{\ell}{2}\omega = \frac{\ell}{2}.\frac{5}{3}\frac{u}{\ell} = \frac{5}{6}u$$

$$L_P = I_P\omega = \left[\frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2\right]\omega$$

$$= \frac{5}{9}mu\ell \Rightarrow L_0 \neq \frac{5}{9}mu\ell$$

192.



$$J = mv_0,\ J\frac{r}{2} = I\omega_0 = \frac{mr^2}{2}\omega_0$$

$$\therefore\ mv_0\,.\frac{r}{2} = \frac{mr^2}{2}\omega_0 \Rightarrow v_0 = r\omega_0$$

Hence, initially, itself there is no slippage and the movement is pure rolling motion.

Hence when it reaches rough path, no frictional force will be developed and no loss of energy.

$\Rightarrow$ (b) is correct

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mr^2}{2}\frac{v^2}{r^2} = mv^2\left(\frac{1}{2}+\frac{1}{4}\right)$$

$$= \frac{3}{4}mv^2 = \frac{3}{4}.m\frac{J^2}{m^2} = \frac{3}{4}\frac{J^2}{m}$$

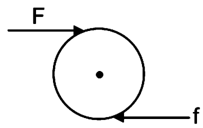
$\Rightarrow$ (c) is correct

$\therefore$ (b) and (c) are correct

193. friction = 0 on horizontal plane whether smooth or rough

friction required > 0 on inclined plane. Hence it will slip on smooth inclined plane.

194.



$F - f = ma$ — (1)

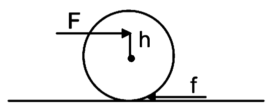
$FR + fR = mR^2\alpha$ — (2)

For rolling, $\alpha = \frac{a}{R}$

$\Rightarrow$ (2) $\Rightarrow F + f = ma$ — (2)

(1) and (2) $\Rightarrow f = 0$

195.



$F - f = ma$ — (1)

$$Fh + fR = mK^2\frac{a}{R} \Rightarrow F.\frac{h}{R} + f = m\frac{K^2}{R^2}.a \quad \text{— (2)}$$

$$(1) + (2) \Rightarrow a = \left(\frac{1+\frac{h}{R}}{1+\frac{K^2}{R^2}}\right).\frac{F}{m}$$

$$\Rightarrow \quad (1) \Rightarrow f = \left[1 - \frac{1 + \frac{h}{R}}{1 + \frac{K^2}{R^2}}\right] F \qquad \text{— (3)}$$

If $h = 0$, $f > 0$ (option a) not correct. $f > 0$ if $1 + \frac{h}{R} < 1 + \frac{K^2}{R^2}$

$\Rightarrow$ if $h < \frac{K^2}{R}$ (option b is correct)

Option c need not be always correct

Friction force reinforces F when f in (3) is negative.

i.e., $x > \frac{K^2}{R}$

f is maximum when $h = -R$ and $f = F$ in that case

196. Impulse acts through centres $\Rightarrow$ no torque on A nor B.

$\Rightarrow \quad \omega_A = \omega, \omega_B = 0$

Kinetic Energy conserved; Rot K.E remains same

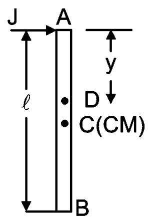
$$\left(\frac{1}{2} I\omega^2 = \frac{1}{2} I\omega_A^2 + 0\right)$$

$\therefore$ Translational K.E is conserved

$\Rightarrow \quad v_A = 0, v_B = v$

197. Let the total mass be M

Then,



$$v_{CM} = \frac{J}{M} \quad \text{— (1)}$$

$$J.\frac{\ell}{2} = I_{CM}.\omega$$

$$= M\left(\frac{\ell}{2}\right)^2 \omega$$

$$\Rightarrow \quad \omega = \frac{J}{M \ell/2} = \frac{v_{CM}}{\ell/2}$$

Let point D(AD = y) have zero velocity

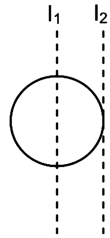
Then, $v_D = o \Rightarrow v_{CM} + \omega\left(\frac{\ell}{2} - y\right) = 0$

$$\Rightarrow \quad v_{CM} + \frac{v_{CM}}{\ell/2}\left(\ell/2 - y\right) = 0$$

$$\Rightarrow \quad 2 - \frac{2y}{\ell} = 0 \Rightarrow y = \ell$$

$\Rightarrow$ Point D coincides with B

198. Disc:



Minimum value

$$I_1 = \frac{mR^2}{4} = mK^2$$

$$\frac{K^2}{R^2} = 0.25$$

Maximum value

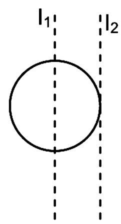
$$I_2 = m\frac{R^2}{4} + mR^2 = 1.25 mR^2$$

$$\frac{K^2}{R^2} = 1.25$$

$$0.25 \le \frac{K^2}{R^2} \le 1.25$$

$$\frac{K^2}{R^2} : (0.25, 1.25)$$

Ring :



Minimum

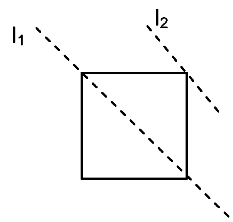
$$I_1 = \frac{MR^2}{2}$$

Maximum

$$I_2 = \frac{MR^2}{2} + MR^2 \text{ or } \frac{K^2}{R^2} = 1.5$$

$$0.5 \le \frac{K^2}{R^2} \le 1.5$$

Square:



Minimum

$$I_1 = \frac{MR^2}{12} \Rightarrow \frac{K^2}{R^2} = 0.083$$

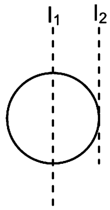
Maximum

$$I_2 = \frac{MR^2}{12} + M\left(\frac{R}{\sqrt{2}}\right)^2 = 0.583MR^2$$

$$\frac{K^2}{R^2} = 0.583$$

$$0.083 \le \frac{K^2}{R^2} \le 0.583$$

Sphere:



Minimum

$$I_1 = \frac{2}{5}MR^2 \Rightarrow \frac{K^2}{R^2} = 0.4$$

Maximum

$$I_2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2 = 1.4MR^2$$

$$\Rightarrow \frac{K^2}{R^2} = 1.4$$

$$0.4 \le \frac{K^2}{R^2} \le 1.4$$

199. Clearly, no translation only in (a)

All other cases: translation

Depending upon values of F_1 and F_2 and moment arms, all cases may lead to either rotation or no rotation:

200. (a) No torque on pulley. Pulley does not rotate

(b) because of friction pulley rotates, $T_1 \neq T_2$ and $(T_1 - T_2) R = I\alpha$

(c) pulley does not rotate

(d) $T_1 \neq T_2$; pulley may or may not rotate

Due to friction as the support, there is torque opposing the rotation so that

$(T_1 - T_2)R \neq I\alpha$

CHAPTER 3

GRAVITATION

CHAPTER OUTLINE

INTRODUCTION

Every body in the Universe attracts every other body in the Universe. This force of attraction between any pair of bodies in the Universe is called *gravitational force*. It is a fundamental force and also is the weakest force in nature. Any body which has some mass (be it a very small mass like that of an electron or a very large mass like that of Sun or stars) exerts a gravitational force on any other body having some mass. Thus gravitational force is due to the mass of the interacting bodies. The law which governs the gravitational force between any pair of bodies in the Universe is called 'Newton's Universal law of gravitation, named after Sir Isaac Newton who discovered this law.

UNIVERSAL LAW OF GRAVITATION

According to Newton's universal law of gravitation, *"every particle in the universe attracts every other particle in the universe with a force that is directly proportional to the product of the masses of those particles and inversely proportional to the square of the distance between those particles. This force acts along the line joining the two particles."*

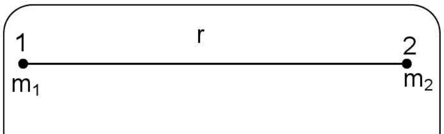


Fig. 3.1

Consider two particles 1 and 2 of mass 'm_1' and 'm_2' respectively, separated by a distance 'r'. The gravitational force of attraction between the two particles (F), as per Newton's universal law of gravitation is

$$F \propto m_1 m_2 \text{ and } F \propto \frac{1}{r^2}$$

$$\therefore \quad F \propto \frac{m_1 m_2}{r^2} \text{ or}$$

$$F = \frac{Gm_1 m_2}{r^2},$$

where 'G' is a constant of proportionality called *universal gravitational constant*. Particle 1 exerts a force on particle 2 $(\vec{F}_{21})$ and it is an attractive force directed towards 1 from 2. Similarly, particle 2 exerts a force on particle 1 $(\vec{F}_{12})$ and it is an attractive force directed towards 2 from 1.

Thus the gravitational force between 1 and 2 form an action-reaction pair.

$$\vec{F}_{12} = -\vec{F}_{21} \text{ ; } F = |\vec{F}_{12}| = |\vec{F}_{21}|\text{, then } F = \frac{Gm_1 m_2}{r^2}$$

If $m_1 = m_2 = 1$ kg and $r = 1$ m, then

$$F = \frac{Gm_1 m_2}{r^2} = \frac{G \times 1 \times 1}{1^2} = G$$

Thus the universal gravitational constant 'G' is numerically equal to the gravitational force between two particles of unit mass each, separated by unit distance. The value of 'G' was first experimentally established by Henry Cavendish. The SI unit of 'G' is N m^2 kg^{-2} and its dimensional formula is $M^{-1}L^3T^{-2}$. The value of $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

This law is called *universal law of gravitation* because it holds good irrespective of the nature of the objects (like size, shape, mass etc.) and at all places and at all times. The value of G does not depend upon the mass of the particles, the distance between the particles or the medium separating them.

Characteristics of gravitational force

(i) The gravitational force between any two bodies form an action-reaction pair. The force on each body due to the other body is of same magnitude but opposite in direction.

(ii) The gravitational force between any pair of bodies is always attractive in nature.

(iii) The gravitational force between a pair of bodies is independent of the presence or absence of any other body in their neighbourhood

(iv) The gravitational force between any pair of bodies is independent of the medium separating the bodies. Hence protecting a body (or shielding a body) from gravitational force is impossible.

(v) Gravitational force is a central force i.e., it acts along the line joining the two interacting particles.

(vi) Gravitational force is the weakest force in nature.

(vii) Gravitational force is negligibly small in case of light bodies but becomes quite significant in case of massive bodies like planets, satellites and stars.

(viii) Gravitational force is a long-range force i.e., it is effective even if the distance between the interacting particles is very large. For example, the gravitational force between Sun and planet Pluto exists even though the distance between them is large and is the cause for the motion of Pluto around the Sun.

(ix) Gravitational force is a conservative force. Hence potential energies are associated with gravitational forces

Notes:

Newton's law of Universal Gravitation is strictly applicable to particles or point masses. If the sizes of the bodies are very small compared to their distance of separation, such bodies can also be treated as particles. It can also be shown that a body having spherical symmetry of mass distribution can be treated as a particle, with mass concentrated at the centre of the sphere only for gravitational interaction at points outside the sphere. If the interacting bodies cannot be reduced to particles, integration method will have to be used for determining the gravitational force.

CONCEPT STRANDS

Concept Strand 1

Calculate the gravitational force between an electron (mass = 9.1×10^{-31} kg) and a proton(mass = 1.67×10^{-27} kg) separated by a distance of 1 m.

Solution

Since the distance of separation (r = 1 m) is very large compared to the sizes of electron and proton, we can treat them as particles and apply the law of universal gravitation.

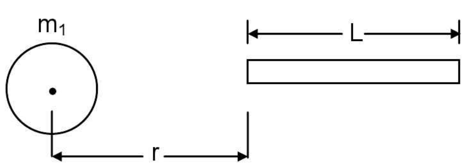
$$\therefore \quad F = \frac{GM_pM_e}{r^2} = \frac{6.67\times10^{-11}\times1.67\times10^{-27}\times9.1\times10^{-31}}{1^2}$$

$$= 1.01 \times 10^{-67}\ \text{N}$$

As this force is very small (in comparison to the electric force between an electron and a proton), gravitational force on small charged particles are usually neglected.

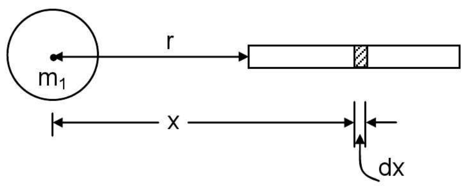
Concept Strand 2

A uniform solid sphere of mass m_1 is separated from a uniform rod of length 'L' and mass m_2. Calculate the gravitational force exerted by the sphere on the rod.



Solution

Due to spherical symmetry, the uniform sphere of mass m_1 can be considered as a particle of mass m_1 at the centre of the sphere (This is explained in later sections of this chapter). However, we cannot treat the uniform rod as a particle. Hence, consider an element of the rod, of length dx, at a distance x from the centre of the sphere. This elemental length of rod can be treated as a particle.



Mass of elemental rod, $dm = \left(\frac{m_2}{L}\right)dx$

Newton's law of universal gravitation can be applied between m_1 and dm

$\therefore$ Gravitational force on the elemental rod

$$dF = \frac{Gm_1dm}{x^2} = \frac{Gm_1m_2}{Lx^2}dx$$

$\therefore$ Total force on the rod,

$$F = \int_0^F dF = \int_{x=r}^{x=L+r} \frac{Gm_1m_2dx}{Lx^2}$$

$$= \frac{Gm_1m_2}{L}\int_{x=r}^{x=L+r}\frac{dx}{x^2}$$

$$= \frac{Gm_1m_2}{L}\int_{x=r}^{x=L+r}\frac{dx}{x^2} = \frac{Gm_1m_2}{L}\left(-\frac{1}{x}\right)_r^{L+r}$$

$$= -\frac{Gm_1m_2}{L}\left(\frac{1}{L+r}-\frac{1}{r}\right) = -\frac{Gm_1m_2}{L}\left[\frac{r-(L+r)}{r(L+r)}\right]$$

$$= \frac{-Gm_1m_2}{L}\times\frac{-L}{r(L+r)} = \frac{Gm_1m_2}{r(L+r)}$$

If $r >> L$, then $r + L \simeq r$

$$\therefore \quad F = \frac{Gm_1m_2}{r^2}$$

Hence if the sphere and rod are separated by a very large distance, much larger than the length of the rod, then the rod can also be treated as a particle.

Superposition principle of gravitational forces

The gravitational force between any two particles is independent of the presence or absence of other particles. This gives rise to the superposition principle of gravitational forces. According to the superposition principle, the gravitational force on a particle of mass m, due to a distribution of particles of masses $m_1, m_2,\ldots,m_n$ around it, is the vector sum of the gravitational forces exerted on m by each of the other particles $m_1, m_2,\ldots m_n$, the forces between each pair being independent of the other particles

For example

Consider a distribution of six particles (m_1, m_2, m_3, m_4, m_5 and m_6) around a particle of mass m as shown

$$\overline{F}_{01} = \frac{-Gmm_1}{|\overline{r}_1|^2}\hat{r}_1 = -\frac{Gmm_1}{|\overline{r}_1|^3}\overline{r}_1$$

$$\overline{F}_{02} = -\frac{Gmm_2}{|\overline{r}_2|^2}\hat{r}_2 = -\frac{Gmm_2}{|\overline{r}_2|^3}\overline{r}_2$$

....

....

$$\overline{F}_{06} = -\frac{Gmm_6}{|\overline{r}_6|^2}\hat{r}_6 = \frac{-Gmm_6}{|\overline{r}_6|^3}\overline{r}_6$$

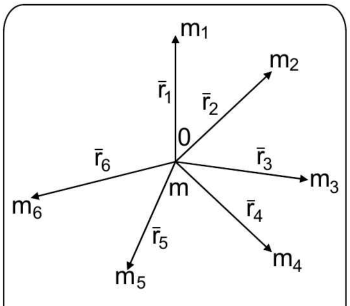


Fig. 3.2

$\overline{F}_G$ = gravitational force on m due to $m_1, m_2,\ldots,m_6$ as per superposition principle is

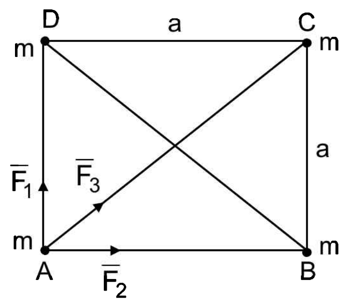
$$\overline{F}_a = \overline{F}_{01} + \overline{F}_{02} + \overline{F}_{03} + \overline{F}_{04} + \overline{F}_{05} + \overline{F}_{06}$$

$$= -Gm\sum_{i=1}^{i=6}\frac{m_i}{|\overline{r}_i|^3}\overline{r}_i$$

CONCEPT STRAND

Concept Strand 3

Four identical particles, each of mass m, are placed at the corners of a square of side a. Calculate the net gravitational force on each particle.



Solution

Let us consider a particle of mass 'm' placed at corner A of the square of side a

Gravitational force on A due to particle at D is $\overline{F}_1$ and

$$F_1 = |\overline{F}_1| = \frac{Gmm}{a^2} = \frac{Gm^2}{a^2}, \text{ along AD}$$

Gravitational force on A due to particle at B is $\overline{F}_2$ and

$$F_2 = |\overline{F}_2| = \frac{Gmm}{a^2} = \frac{Gm^2}{a^2}, \text{ along AB}$$

Gravitational force on A due to particle at C is $\overline{F}_3$ and

$$F_3 = |\overline{F}_3| = \frac{Gmm}{\left(\sqrt{2}\,a\right)^2} \left(\because AC = \sqrt{2}\,a\right) = \frac{Gm^2}{2a^2}, \text{ along AC}$$

As per superposition principle, net gravitational force on A is $\overline{F} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3$

$$= \overline{F}_R + \overline{F}_3 \qquad \left(\text{where } \overline{F}_R = \overline{F}_1 + \overline{F}_2\right)$$

$$F_R = \sqrt{F_1^2 + F_2^2} \qquad \left(\because \overline{F}_1 \text{ perpendicular to } \overline{F}_2\right)$$

$$|\overline{F}| = \frac{Gm^2}{a^2}\left[\sqrt{2} + \frac{1}{2}\right] = \frac{\left(2\sqrt{2}+1\right)Gm^2}{2a^2}, \text{ along AC}$$

The magnitude of the net gravitational force on the particles at B, C and D also will be of magnitude $\frac{\left(2\sqrt{2}+1\right)Gm^2}{2a^2}$ but their directions will be along BD (for particle at B), along CA (for particle at C) and along DB (for particle at D).

The gravitational force between larger bodies (other than particles) can be determined by using the superposition principle.

GRAVITATIONAL FIELD

The presence of a particle/body (or a mass) modifies the space surrounding the particle/body. This modified space surrounding a particle or body (or anything having mass) is called the gravitational field of that particle or body. Any other mass brought inside this gravitational field will experience a gravitational force due to its interaction with this gravitational field. The field concept is very useful in dealing with non–contact forces (or action at a distance)

We know that the net gravitational force on a particle may be due to another single particle or due to a distribution of large number of particles. The advantage of the gravitational field concept is that it helps us to measure the net gravitational force on a mass, without bothering whether it is a single particle or a distribution of particles that exert this force on the concerned mass.

Every point in a gravitational field is characterized by two properties, out of which one is a vector quantity and the other is a scalar. The vector quantity is called Gravitational Field Intensity $\overline{E}$ and the scalar quantity is called Gravitational Potential V.

GRAVITATIONAL FIELD INTENSITY ($\overline{E}$)

Gravitational field intensity $\overline{E}$ at a point in a gravitational field is a vector quantity, defined mathematically as $\overline{E} = \underset{\Delta m \to 0}{\text{Lt}} \frac{\overline{F}}{\Delta m}$, where Δm is an infinitesimally small mass (but not zero) placed at the point, where it experiences a gravitational force $\overline{F}$. To get a practical measure of the gravitational field, it is defined as the gravitational force exerted on a unit mass placed at that point. The SI unit of gravitational field intensity is newton per kilogram (N kg^{-1}) and its dimensional formula is M^0LT^{-2} (same as the dimensional formula of acceleration).

The gravitational field intensity ($\overline{E}$) is also known as *strength of the gravitational field* or simply *Gravitational Field*. If a particle of mass m is brought to a point, where the gravitational field is $\overline{E}$, the net gravitational force ($\overline{F}$) acting on that particle at that point is given by

$$\overline{F} = m\overline{E}$$

CONCEPT STRAND

Concept Strand 4

The gravitational field at a point is given by $\bar{E} = (6\hat{i} + 8\hat{j} + 10\hat{k})$ N kg^{-1}. Calculate the net gravitational force exerted on a particle of mass 5 kg placed at that point.

Solution

$$\bar{F} = m\bar{E}$$

$$= 5 \times \left[6\hat{i} + 8\hat{j} + 10\hat{k}\right] \text{ N}$$

$$= \left[30\hat{i} + 40\hat{j} + 50\hat{k}\right] \text{ N}$$

$\therefore$ Magnitude of the gravitational force $F = |\bar{F}|$

$$= \sqrt{30^2 + 40^2 + 50^2}$$

$$= \sqrt{900 + 1600 + 2500} = \sqrt{5000}$$

$$= 70.71 \text{ N}$$

This gravitational force acts in the same direction as the gravitational field intensity $(\bar{E})$ at that point.

Note:
In this problem, we do not know anything about the source/sources which exert this net gravitational force on the particle of mass 5 kg. Hence it is not correct to say that the gravitational force exerted by a field is in the direction of the source.

Gravitational field intensity due to a particle or point mass

Consider a particle of mass M, placed at a point O. We want to determine the gravitational field intensity due to M, at a point P near it. OP is the position vector $\bar{r}$ (taking position O as the origin).

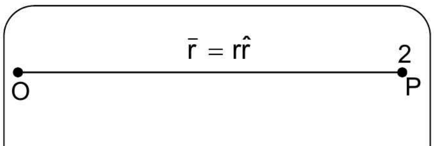


Fig. 3.3

If a point particle of mass m is placed at P, the gravitational force on it will be $\frac{GMm}{r^2}$ towards O (i.e., along PO)

$$\therefore \quad \bar{F} = -\frac{GMm}{r^2}\hat{r}$$

The gravitational field intensity at P at the location of the point mass m is given by

$$\bar{E} = \frac{\bar{F}}{m} = -\frac{GM}{r^2}\hat{r}$$

This is defined for all points, wherever the point mass is kept except at the location of the particle of mass M.

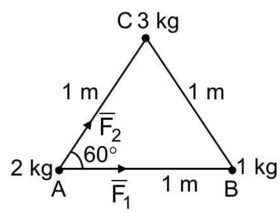
CONCEPT STRAND

Concept Strand 5

Three particles of masses 1 kg, 2 kg and 3 kg are placed at the corners of an equilateral triangle of side 1 m. Calculate the magnitude and direction of the gravitational field intensity at the location of the 2 kg mass. What is the net gravitational force on the 2 kg mass?

Solution

This problem can be solved by two methods



Method 1

Calculate the force $\bar{F}_1$ and $\bar{F}_2$ on the 2 kg mass due to 1 kg and 3 kg respectively and find out the resultant force $\overline{F} = \overline{F}_1 + \overline{F}_2$, which is the net force on 2 kg mass. Gravitational field at the location of 2 kg mass is given by $E = \dfrac{\overline{F}}{m} = \dfrac{\overline{F}}{2\text{ kg}}$.

Method 2

Let $\overline{E}_1$ be the gravitational field at the location of 2 kg mass due to 1 kg mass and $\overline{E}_2$ be the gravitational field at the same location due to 2 kg mass.

We have

$$\overline{E}_1 = \frac{-Gm_1}{r_1^2}\hat{r}_1 \ (m_1 = 1\text{ kg}, r_1 = 1\text{ m } \hat{r}_1 \text{ along BA})$$

$$\therefore \quad E_1 = \frac{G \times 1}{(1)^2} = G, \text{ along AB}$$

Similarly,

$$\overline{E}_2 = -\frac{Gm_2}{r_2^2}\hat{r}_2 \ (m_2 = 3\text{ kg}, r_2 = 1\text{ m}, \hat{r}_2 = \text{along CA})$$

$$\therefore \quad E_2 = \frac{G \times 3}{1^2} = 3G, \text{ along AC}$$

By superposition principle, the gravitational field at A(location of 2 kg mass) is given by

$$\overline{E} = \overline{E}_1 + \overline{E}_2$$

$$\therefore \quad E = \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\theta}$$

$$= \sqrt{G^2 + (3G)^2 + 2 \times G \times 3G \times \cos 60^\circ}$$

($\because \theta = 60°$ between $\overline{E}_1$ and $\overline{E}_2$)

$$= \sqrt{G^2 + 9G^2 + 3G^2}$$

$$= \sqrt{13G^2} = \sqrt{13}G \ \text{ N kg}^{-1}$$

$$= \sqrt{13} \times 6.67 \times 10^{-11}\text{ N kg}^{-1}$$

$$= 24.05 \times 10^{-11}\text{ N kg}^{-1}$$

If $\overline{E}$ makes an angle of ϕ with AB,

$$\tan\phi = \frac{E_2\sin\theta}{E_1 + E_2\cos\theta} \ \text{(From parallelogram law of vectors)}$$

$$= \frac{3G\sin 60^\circ}{G + 3G\cos 60^\circ} = \frac{3\sin 60^\circ}{1 + 3\cos 60^\circ}$$

$$= \frac{3\sqrt{3}}{5} = 1.039$$

$\therefore \phi = \tan^{-1}(1.039) = 46.1°$

Hence the gravitational field at A (location of 2 kg mass) makes an angle of 46.1° in the anti-clockwise direction with AB.

The net gravitational force on 2 kg mass,

$$F = mE$$
$$= 2 \times 24.05 \times 10^{-11}$$
$$= 48.1 \times 10^{-11}\text{ N}$$

Gravitational field intensity due to a thin uniform ring at a point on its axis

Consider a thin, uniform ring of mass M and radius a with centre at O. The gravitational field intensity $(\overline{E})$ due to this ring at a point P on the axis of the ring, distant 'r' from the centre of the ring is to be determined.

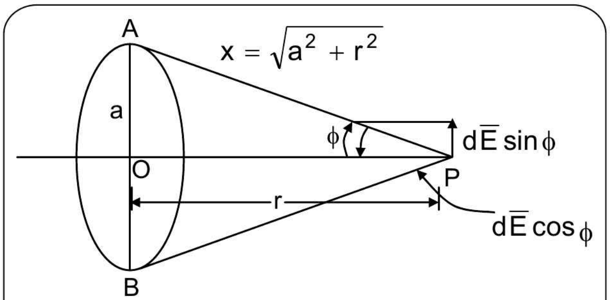


Fig. 3.4

Let λ = mass per unit length of ring = $\dfrac{\text{Mass}}{\text{circumference}}$

$$= \frac{M}{2\pi a}$$

Consider an element of the ring, of length '$d\ell$' at A. Its mass $dm = \lambda d\ell = \dfrac{Md\ell}{2\pi a}$

The gravitational field intensity at P due to this elemental ring at A is $d\overline{E} = \dfrac{Gdm}{x^2}$, along PA

$\overline{AP}$ makes an angle ϕ with OP. Now $d\overline{E}$ can be resolved as $d\overline{E}\sin\phi$ perpendicular to the axis OP and $d\overline{E}\cos\phi$ along the axis as shown.

$d\overline{E}\sin\phi$ component gets cancelled by the field of a diametrically opposite element at B. Hence the effective component of all ring elements is only $d\overline{E}\cos\phi$

$$dE\cos\phi = \frac{Gdm}{x^2}\cos\phi$$

$$= \frac{GM}{x^2 \times 2\pi a}\cos\varphi\ d\ell\ (\because dm = \frac{Md\ell}{2\pi a})$$

∴ The resultant gravitational field at P due to all ring elements is $\overline{E} = \int dE\cos\phi$, along PO

$$\therefore \quad E = \int \frac{GM}{x^2 \times 2\pi a}\cos\phi\ d\ell$$

$$= \frac{GM}{2\pi a x^2}\cos\varphi \int_{\ell=0}^{\ell=2\pi a} d\ell = \frac{GM}{x^2}\cos\varphi$$

$$= \frac{GM}{x^2}.\frac{r}{x} \qquad (\because \cos\phi = \frac{r}{x})$$

$$= \frac{GMr}{x^3}$$

$$= \frac{GMr}{(a^2+r^2)^{3/2}} \qquad \left[\because x = (a^2+x^2)^{1/2}\right]$$

$$\therefore \quad E = \frac{GMr}{(a^2+r^2)^{3/2}}$$

Notes:

(i) At the centre of the ring (point O), r = 0
$\Rightarrow$ E = 0. Hence *the gravitational field at the centre of the thin, uniform ring is zero.*

(ii) If a << r, $E = \frac{GMr}{(r^2)^{3/2}} = \frac{GM}{r^2}$. So for points on the axis which are at a very large distance from the centre of the ring, the ring can be treated as a particle (or point mass).

(iii) The position where the gravitational field $\overline{E}$ becomes maximum or minimum is determined by putting $\frac{dE}{dr} = 0$.

$$E = \frac{GMr}{(a^2+r^2)^{3/2}}$$

$$\frac{dE}{dr} = 0 \Rightarrow r = \pm\frac{a}{\sqrt{2}}$$

At $+\frac{a}{\sqrt{2}}$, E is maximum negative (minimum) and at $-\frac{a}{\sqrt{2}}$, E is maximum positive (maximum).

The variation of $\overline{E}$, along the axis of the ring on either side of the ring is as shown below in Fig.3.4.

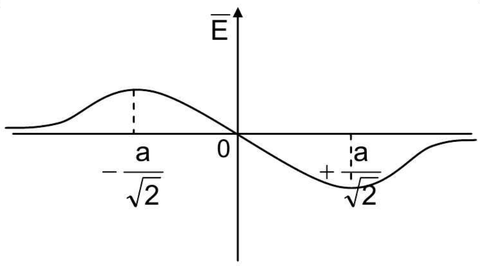


Fig. 3.5

Gravitational field intensity due to a uniform disc at a point on its axis

Consider a uniform disc of mass M and radius a with centre at O. The point P is on the axis of the disc at a distance r from centre O.

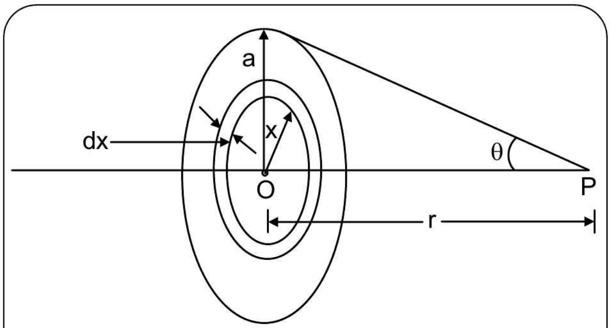


Fig. 3.6

The disc can be dividend into thin, uniform rings. Consider one such ring of radius x and width along the disc equal to dx.

$$\text{Mass of ring dm} = \frac{\text{Mass of disc}}{\text{Surface Area of disc}} \times \text{surface area of ring}$$

$$= \left(\frac{M}{\pi a^2}\right)(2\pi x dx) = \frac{2M}{a^2}xdx$$

The gravitational field at P due to this elemental ring is

$$dE = \frac{G\ dm\ r}{(x^2+r^2)^{3/2}} \text{ along PO}$$

$$\therefore \quad dE = \frac{Gr}{(x^2+r^2)^{3/2}}.\frac{2M}{a^2}xdx = \frac{2GMr}{a^2}.\frac{x}{(x^2+r^2)^{3/2}}dx$$

$\therefore$ Gravitational field at P due to disc,

$$E = \int dE = \frac{2GMr}{a^2}\int_{x=0}^{x=a}\frac{x\,dx}{(x^2+r^2)^{3/2}}$$

$$= \frac{2GMr}{a^2}\left|-\frac{1}{\sqrt{x^2+r^2}}\right|_0^a = \frac{2GMr}{a^2}\left[\frac{1}{r}-\frac{1}{\sqrt{a^2+r^2}}\right]$$

$$= \frac{2GM}{a^2}\left[1-\frac{r}{\sqrt{a^2+r^2}}\right]$$

$$= \frac{2GM}{a^2}\left[1-\cos\theta\right], \text{ where } \cos\theta = \frac{r}{\sqrt{a^2+r^2}}$$

$$E = \frac{2GM}{a^2}\left[1-\cos\theta\right]$$

Notes:

(1) When P is very near to centre O, $\theta \approx 90°$, $\cos\theta = 0$

$\Rightarrow E = \dfrac{2GM}{a^2}$ (maximum value)

When P is far away from O, $\theta = 0°$, $\cos\theta = 1 \Rightarrow E = 0$

(2) If the disc is infinitely large, $\cos\theta = \cos 90° = 0$ and $E = \dfrac{2GM}{a^2}$ for all points on the axis.

Hence *the gravitational field due to an infinitely large disc along its axis is uniform (i.e., it is independent of the distance from the disc).*

Gravitational field due to a thin, uniform shell (hollow sphere)

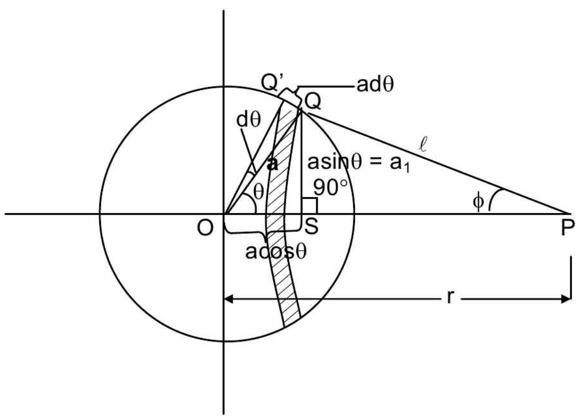


Fig. 3.7

Consider a thin, uniform spherical shell of mass M and radius a, with centre at point O. We want to calculate the gravitational field due to this shell at a point P, distant r from O.

Figure 3.7 shows a spherical shell. The shaded area represents a thin ring of radius $a_1 = a\sin\theta$ and width $Q'Q = a d\theta$

$PQ = \ell$ and angle OPQ is ϕ

From ΔPSQ, we have $PQ^2 = PS^2 + QS^2$

i.e., $\ell^2 = [r - OS]^2 + QS^2$

$= r^2 - 2r(OS) + (OS)^2 + QS^2$

$\Rightarrow \ell^2 = r^2 + a^2 - 2r(OS) \quad [\because QS^2 + OS^2 = OQ^2 = a^2]$

$= r^2 + a^2 - 2ra\cos\theta \quad (\because OS = a\cos\theta)$

$\therefore \ell^2 = a^2 + r^2 - 2ar\cos\theta$ —(i)

Area of shaded ring = circumference × width

$= 2\pi a_1 \times a d\theta$

$= 2\pi a\sin\theta . a d\theta$

$= 2\pi a^2 \sin\theta d\theta$

Mass of shaded ring, $dm = \dfrac{M}{\text{Area of shell}} \times \text{area of ring}$

$$= \frac{M}{4\pi a^2}\times 2\pi a^2 \sin\theta d\theta$$

$$= \frac{M}{2}\sin\theta d\theta \qquad \text{— (ii)}$$

The gravitational field at P due to this ring is

$dE = \dfrac{G dm}{\ell^2}\cos\phi$ ($\because$ $\sin\phi$ components of the ring get cancelled for diametrically opposite points)

$$\therefore\ dE = \frac{GM}{2}\frac{\sin\theta d\theta \cos\phi}{\ell^2} \qquad \text{— (iii)}$$

From ΔPOQ, we have $a^2 = \ell^2 + r^2 - 2\ell r\cos\phi$

$$\therefore\ \cos\phi = \frac{\ell^2 + r^2 - a^2}{2\ell r} \qquad \text{— (iv)}$$

We have $\ell^2 = a^2 + r^2 - 2ar\cos\theta$ from (i)

Differentiating (i), we get

$$2\ell d\ell = 2ar\sin\theta d\theta$$

$$\therefore\ \sin\theta d\theta = \frac{\ell d\ell}{ar} \qquad \text{— (v)}$$

Using values from (iv) and (v) is (iii), we get

$$dE = \frac{Gm}{4ar^2}\left[1-\frac{(a^2-r^2)}{\ell^2}\right]d\ell$$

i.e., $$E = \int dE = \int_{\ell_1}^{\ell_2}\frac{Gm}{4ar^2}\left[1-\frac{(a^2-r^2)}{\ell^2}\right]d\ell$$

i.e., $$E = \frac{Gm}{4ar^2}\left[\ell + \frac{(a^2 - r^2)}{\ell}\right]_{\ell_1}^{\ell_2}$$

The following cases are of particular interest:

(i) P outside the shell (r > a)

In this case, value of ℓ varies from $\ell_1 = (r - a)$ to $\ell_2 = (r + a)$

$$\therefore \quad E = \frac{Gm}{4ar^2}\left[\ell + \frac{(a^2 - r^2)}{\ell}\right]_{r-a}^{r+a}$$

$$= \frac{Gm}{4ar^2}\left[(r+a) + (a-r) - \{(r-a) - (a+r)\}\right]$$

$$= \frac{Gm}{4ar^2}\left[2a + 2a\right] = \frac{Gm}{r^2}$$

$\therefore \ E = \frac{Gm}{r^2}$ for all outside points. Hence, *for calculation of gravitational field at external points of a thin, uniform spherical shell, the shell can be treated as a point mass(particle) kept at the geometric centre of the shell.*

(ii) P inside the shell (r < a)

In this case, ℓ varies from $\ell_1 = (a - r)$ to $\ell_2 = (a + r)$

$$\therefore \ E = \frac{Gm}{4ar^2}\left[\ell + \frac{(a^2 - r^2)}{\ell}\right]_{a-r}^{a+r}$$

$$= \frac{Gm}{4ar^2}\left[(a+r) + (a-r) - \{(a-r) + (a+r)\}\right]$$

$\therefore \ E = 0$ for inside points of shell. Hence, *the gravitational field inside a thin, uniform spherical shell due to the mass of the shell, is zero.*

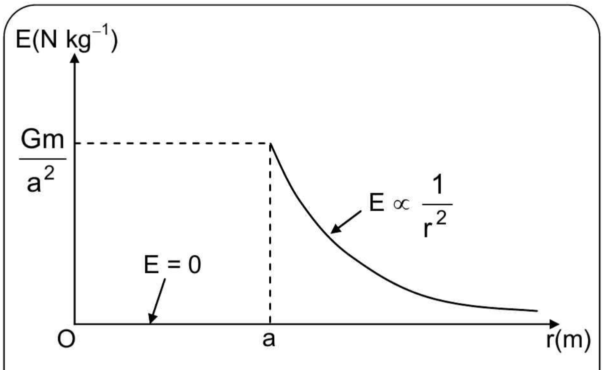


Fig. 3.8

Variation of gravitational field due to uniform, thin spherical shell of radius 'a'.

Gravitational field intensity due to a uniform solid sphere

Consider a uniform solid sphere of mass M and radius 'a' with centre at point O. Let us evaluate its gravitational field intensity at a point P, distant 'r' from O.

(i) P is outside the sphere (r > a)

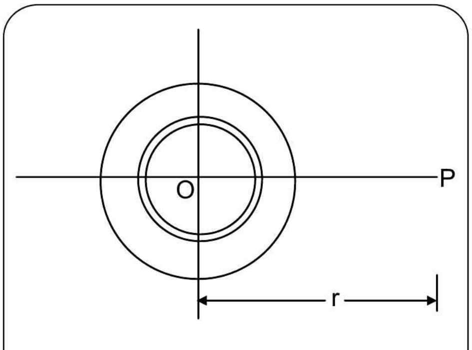


Fig. 3.9

The solid sphere can be divided into concentric uniform thin shells, each of mass dm

The gravitational field at P due to thin shell is $dE = \frac{Gdm}{r^2}$

$\therefore$ Total gravitational field at P due to solid sphere

$$E = \int dE = \int \frac{Gdm}{r^2} = \frac{GM}{r^2}$$

$$\therefore \qquad E = \frac{GM}{r^2} \text{ for outside points}$$

Hence, *a uniform solid sphere can be treated as a point mass(particle) kept at its geometric centre for evaluation of gravitational field at all outside points of the solid sphere.*

(ii) P is inside the sphere (r < a)

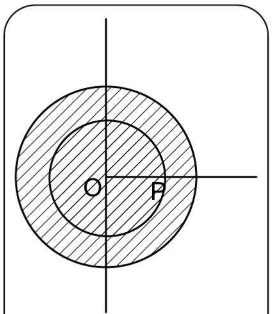


Fig. 3.10

In this case, we can treat the solid sphere as made of two parts, namely, (1) solid sphere of radius 'r' and (2) uniform spherical shell of inside radius r and outside radius a. The gravitational field at P is due to the superposition of the gravitational fields due to these two portions. We know that gravitational field at P due to the shell is zero (as P is in the inside of the shell). Hence gravitational field intensity at P is due to a solid sphere of radius r (instead of a).

Mass of reduced sphere, $M' = \dfrac{M}{\left(\dfrac{4}{3}\pi a^3\right)} \times \dfrac{4}{3}\pi r^3 = \dfrac{M}{a^3} r^3$

Gravitational field at P is $E = \dfrac{GM'}{r^2}$

$$= \frac{G}{r^2}\frac{M}{a^3}r^3 = \frac{GMr}{a^3}$$

$\therefore \quad E = \left(\dfrac{GM}{a^3}\right) r$ for inside points of solid sphere i.e., $E \propto r$ for inside points

Hence *the gravitational field intensity at inside points of a uniform solid sphere, is directly proportional to the distance of that point from the geometric centre of the solid sphere.*

Since r = 0 at the centre of the solid sphere, the gravitational field intensity at the centre of the solid sphere is zero.

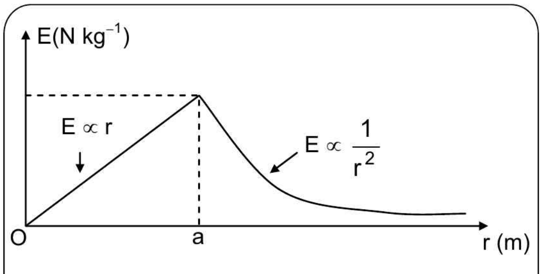


Fig. 3.11

Variation of gravitational field intensity due to a uniform solid sphere of radius 'a' is shown in Fig.3.11.

GRAVITATIONAL POTENTIAL (V)

The gravitational potential at a point is equal to the work done by an external force on a particle of unit mass in bringing it from infinity to its position in the gravitational field. Gravitational potential is a scalar quantity. Its SI unit is joule per kilogramme(J kg^{-1}) and its dimensional formula is $M^0L^2T^{-2}$. While bringing the unit mass from infinity to its position in the gravitational field, at every point in the path, the applied external force is equal and opposite to the gravitational force on the particle at those points i.e., the particle is brought slowly from infinity to its position so that its kinetic energy is zero at all positions. *Hence gravitational potential (V) can also be defined as the negative of the work done by the gravitational force as a particle of unit mass is brought from infinity to its position in the gravitational field.* If 'W' is the work done by an external force, in bringing a particle of mass 'm' from infinity to its position in the gravitational field, then the gravitational potential at that point (V) is given by

$$V = \frac{W}{m}$$

Also $W = -W_G$, where W_G = work done by gravitational force in bringing the particle from infinity to its position and $W_G = \int_\infty^r \overline{F}_G.\overline{dr}$, where $\overline{F}_G$ = gravitational force on particle and $\overline{dr}$ = displacement of particle

$$\Rightarrow \quad W = -\int_\infty^r \overline{F}_G.\overline{dr} \qquad \therefore V = \frac{W}{m} = -\frac{W_G}{m} = \frac{-\int_\infty^r \overline{F}_G.\overline{dr}}{m}$$

$$= -\int_\infty^r \frac{\overline{F}_G}{m}.\overline{dr}$$

$$\therefore \qquad V = -\int_\infty^r \overline{E}.\overline{dr}$$

$$\left(\because \frac{\overline{F}_G}{m} = \overline{E}\right)$$

Conventionally the potential of a particle at infinity is taken as zero.

Gravitational potential (V) at a distance r from a point mass (M)

Consider a particle of mass M placed at point O. We want to determine the gravitational potential at point P, distant r from O.

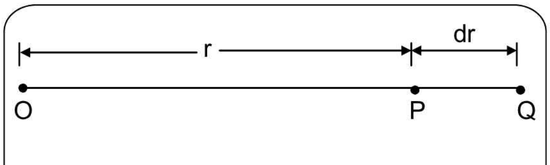


Fig. 3.12

The gravitational force acting on a particle of unit mass at P is the gravitational field intensity at P due to the mass M at O.

$$\therefore \quad F_G = E = \frac{GM}{r^2}, \text{ along PO.}$$

If the unit mass is displaced from P through a small distance $\overline{dr}$ to Q, the small amount of work done by gravitational force

$$dW_G = \overline{F}_G.\overline{dr} = \overline{E}.\overline{dr}$$

$$= E\, dr \cos 180°$$

$$(\because \overline{E} \text{ and } \overline{dr} \text{ are in opposite directions})$$

$$= -E\, dr = -\frac{GM}{r^2}dr$$

Work done by the gravitational force in transferring the unit mass from infinity to P is given by

$$W_G = \int_\infty^r dW_G = \int_\infty^r -\frac{GM}{r^2}dr$$

Gravitational potential at P is given by

$$V = -W_G = -\int_\infty^r -\frac{GM}{r^2}dr = -GM\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= -\frac{GM}{r} + \frac{GM}{\infty} = -\frac{GM}{r}$$

$$V = -\frac{GM}{r}$$

Since gravitational potential at infinite distance is considered to be zero $\left(\text{i.e., } \frac{GM}{\infty} = 0\right)$, the gravitational potential comes out to be always negative

Relation between gravitational field intensity ($\overline{E}$) and gravitational potential (V)

Since V is obtained by integration of $\overline{E}$, the converse, namely differentiation of V gives $\overline{E}$. Since V is in general $V \equiv V(x, y, z)$,

$$\overline{E} = -\overline{\nabla}V,$$

where $\overline{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, i.e.,

$$\overline{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

is the relation between gravitational field intensity $\overline{E}$ and gravitational potential V

When V depends on x alone,

$$E_x = -\frac{dV}{dx}, \text{ where } E = |\overline{E}|$$

Similarly, $\quad E_y = \frac{-dV}{dy}$ or $E_z = -\frac{dV}{dz}$

we can also write $E = -\frac{dV}{dr}$, when V is spherically symmetric

Gravitational potential due to a thin uniform ring along the axis of the ring

Consider a thin, uniform ring of mass m and radius a with centre at O.

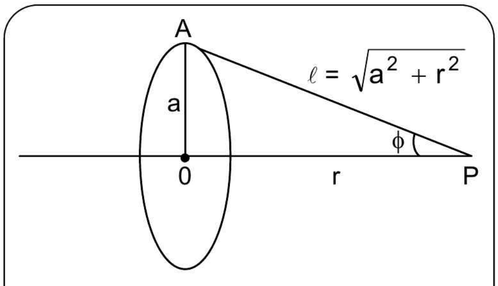


Fig. 3.13

An element of length $d\ell$ of the ring at A has a mass dm $= \frac{Md\ell}{2\pi a}$

The gravitational potential at P on the axis of the ring, distant r from the centre O, due to dm is given by

$$dV = \frac{-G\, dm}{\ell}$$

$\therefore$ Total gravitational potential at P due to the ring

$$V = \int dV = \int_0^M -\frac{Gdm}{\ell} = -\frac{GM}{\ell}$$

$$\therefore \quad V = \frac{-GM}{(a^2 + r^2)^{1/2}}$$

At the centre of the ring, r = 0

$$\therefore \quad V = -\frac{GM}{a}.$$

If r >> a, $V = -\frac{GM}{r}$ *i.e., for distant points along the axis, the thin uniform ring behaves like a particle of mass m at its centre.*

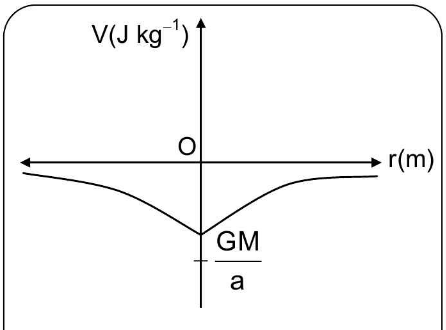


Fig. 3.14

Variation of gravitational potential along the axis of a thin, uniform ring of radius 'a' is given in Fig.3.14.

Gravitational potential due to a thin uniform spherical shell

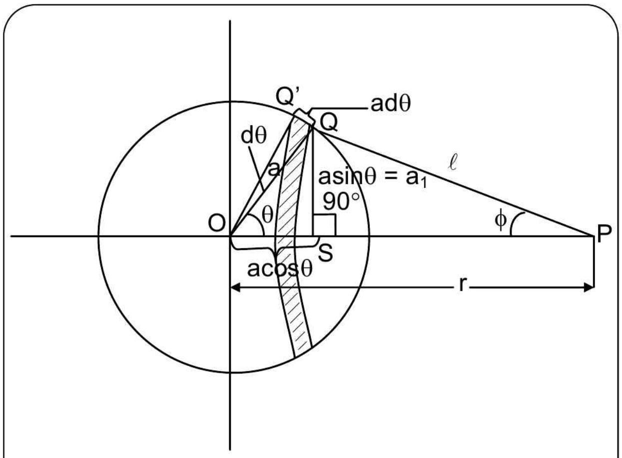


Fig. 3.15

Consider a thin, uniform shell of mass M and radius a with centre at point O. The mass of the element ring(shaded area) is $dm = \frac{M}{2}\sin\theta d\theta$ (already derived in the section on gravitational field)

Also $\ell^2 = a^2 + r^2 - 2ar\cos\theta$ (from Δ POQ in Fig. 3.15)

$$\therefore \quad 2\ell d\ell = 2ar\sin\theta d\theta$$

$$\Rightarrow \quad \sin\theta d\theta = \frac{\ell d\ell}{ar}$$

$\therefore \; dm = \frac{M\ell d\ell}{2ar}$ (This is also derived in the section on gravitational field)

The gravitational potential at P due to dm is given by

$$dV = -\frac{G\,dm}{\ell} \quad \text{(for ring)}$$

$$= -\frac{GM\ell d\ell}{2ar\ell}$$

i.e., $$dV = -\frac{GMd\ell}{2ar}$$

As we vary θ from zero to π, the rings formed on the shell cover up the whole shell. The potential due to the shell is obtained by integrating dV within the limits θ = 0 to θ = π

(i) P outside the shell (r > a)

$\ell^2 = a^2 + r^2 - 2ar\cos\theta$

$\Rightarrow$ when θ = 0, ℓ = r – a and when θ = π, ℓ = r + a

$$\therefore \quad V = \int dV = \int_{\ell=(r-a)}^{\ell=(r+a)} -\frac{GM}{2ar}d\ell$$

$$= -\frac{GM}{2ar}\left|\ell\right|_{r-a}^{r+a}$$

$$= -\frac{GM}{2ar}\left[(r+a)-(r-a)\right] = -\frac{GM}{r}$$

$$\therefore \quad V = -\frac{GM}{r} \text{ for all external points}$$

Hence, *the thin uniform shell can be treated as a point mass, of same mass as the shell, placed at the centre of the shell for calculation of gravitational potential at all external points.*

(ii) P inside the shell (r < a)

In this case, when θ = 0, ℓ = a – r and when θ = π, ℓ = a + r

$$V = \int dV = \int_{a-r}^{a+r} -\frac{GM}{2ar}d\ell = -\frac{GM}{a}$$

$$\therefore \quad V = -\frac{GM}{a} \text{ (inside the shell, V is independent of 'r')}$$

Hence, *the gravitational potential due to a thin, uniform spherical shell is the same at all points inside it and also on its surface. i.e., the gravitational field is uniform.* Hence the interior of a thin, uniform spherical shell is an equipotential volume.

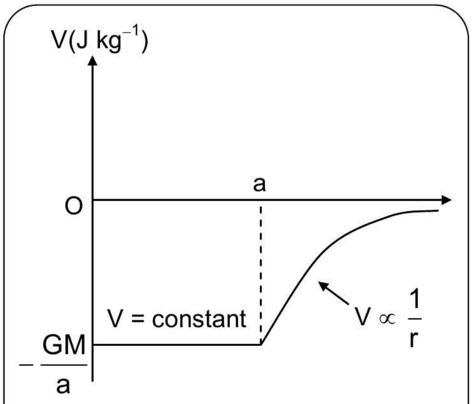


Fig. 3.16

Variation of gravitational potential due to a thin, uniform spherical shell of radius 'a' is shown in Fig. 3.16.

We know that E = 0, inside a spherical shell due to its mass alone. Also $E = -\frac{dV}{dr} \Rightarrow \frac{dV}{dr} = 0 \Rightarrow V =$ constant inside a thin, uniform spherical shell. *Thus, it is not necessary that if the gravitational field is zero at a point, the gravitational potential is zero at that point.*

Gravitational potential due to a uniform solid sphere

(i) P outside the sphere (r > a)

Consider a uniform solid sphere of mass M and radius a, with centre at O.

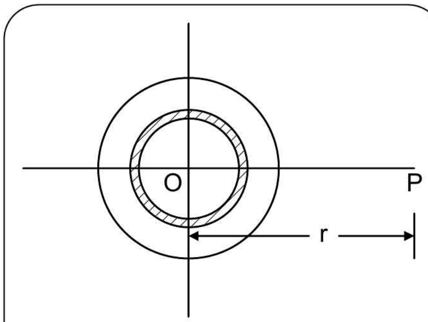


Fig. 3.17

The solid sphere can be divided into a large number of concentric uniform spherical shells, each of mass dm. The gravitational potential at P due to the shell of mass dm is

$$dV = -\frac{G\,dm}{r}$$

Total gravitational potential at P due to the entire sphere,

$$V = \int dV = \int_0^M -\frac{Gdm}{r} = -\frac{GM}{r}$$

$$\therefore \quad V = -\frac{GM}{r} \text{ for all external points}$$

Hence *a solid uniform sphere can be treated as a particle at its centre, having the same mass as the sphere, for calculation of gravitational potential at all external points.*

(ii) P inside the sphere (r < a)

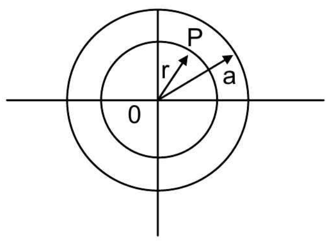


Fig. 3.18

The gravitational potential V at P is due to

(a) a uniform solid sphere of radius r and mass M_S and

(b) a hollow sphere of outside radius a and inside radius r

$\therefore V = V_S + V_H$, where V_S = potential due to solid sphere of mass M_S and

V_H = potential due to hollow sphere

$$M_S = \frac{M}{\left(\frac{4}{3}\pi a^3\right)} \times \frac{4}{3}\pi r^3 = \frac{Mr^3}{a^3}$$

$$\therefore \quad V_S = -\frac{GM_S}{r} = -\frac{GMr^3}{a^3 r} = \frac{-GMr^2}{a^3} \qquad \text{— (i)}$$

For calculation of V_H, we take an elemental shell of radius x and thickness $dx(x_{min} = r, x_{max} = a)$

$$dM_H = \frac{M}{\left(\frac{4}{3}\pi a^3\right)} . 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}$$

$$dV_H = -\frac{GdM_H}{x} = \frac{-3GMxdx}{a^3}$$

$$\therefore \quad V_H = \int dV_H = \int_{x=r}^{x=a} \frac{-3GMxdx}{a^3} = -\frac{3GM}{a^3}\left|\frac{x^2}{2}\right|_r^a$$

$$= -\frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

$$\therefore \quad V = V_S + V_H = -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

$$= -\frac{GM}{2a^3}[2r^2 + 3a^2 - 3r^2] = -\frac{GM}{2a^3}\left[3a^2 - r^2\right]$$

$$\therefore \quad V = -\frac{GM}{2a^3}\left(3a^2 - r^2\right)$$

for interior points. At the centre of the sphere, r = 0

$$\Rightarrow \quad V = -\frac{3}{2}\frac{GM}{a}$$

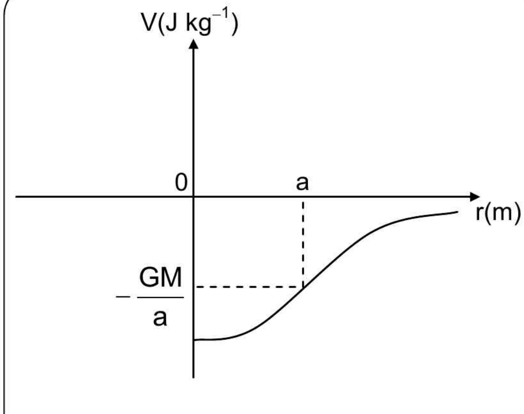


Fig. 3.19

Variation of gravitational potential due to a uniform solid sphere of radius 'a' is shown in Fig.3.19.

GRAVITATIONAL POTENTIAL ENERGY

Since gravitational force is a conservative force, we can define gravitational potential energy associated with a system of particles, interacting through gravitational force.

We had earlier defined the gravitational potential $V = -\frac{W_G}{m}$, where W_G = work done by gravitational force on a particle of mass m, in bringing it slowly from infinity to its position in the gravitational field.

Since $-W_G$ is the negative of the work done by a conservative force, it is equal to the change in potential energy between the final and initial positions($\because \Delta U = -W_G$, where ΔU equals the change in potential energy)

$$\therefore \quad V = \frac{\Delta U}{m} = \frac{U(r) - U(\infty)}{m}$$, where

U(r) = gravitational potential energy of the particle at position r

U(∞) = gravitational potential energy at infinity (conventionally taken as zero)

$$V = \frac{U(r)}{m}$$

That is *gravitational potential at a point in a gravitational field is the gravitational potential energy of a unit mass placed at that point.*

$\therefore$ U = mV *is the gravitational potential energy of a particle of mass m, placed at a point in a gravitational field, where the gravitational potential is V.* Gravitational potential energy of a particle is zero at infinite distance or it is always negative.

CONCEPT STRAND

Concept Strand 6

What is the gravitational potential energy of a particle of mass 5 kg, when placed at a point where the gravitational potential is -10^{-6} J kg^{-1}?

Solution

$$U = mV$$
$$= 5 \times (-10^{-6}) \text{ J}$$
$$= -5 \times 10^{-6} \text{ J}$$

Gravitational potential difference

The gravitational potential difference between two points in a gravitational field is the change in potential energy of a particle of unit mass, when it is moved from one point to the other, against the gravitational force.

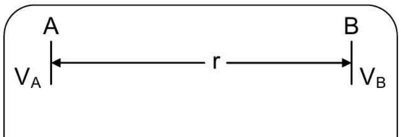


Fig. 3.20

Let A and B be two points, at gravitational potentials V_A and V_B respectively. If a particle of mass m is placed at A, its potential energy at A is $U(A) = mV_A$ — (i)

If this particle is moved very slowly from A to B, at B, its potential energy is $U(B) = mV_B$ — (ii)

$$\therefore \quad \Delta U = \text{change in potential energy} = U(B) - U(A) = mV_B - mV_A = m(V_B - V_A)$$

Potential difference between A and B is given by

$$\Delta V = \frac{\Delta U}{m} = \frac{m(V_B - V_A)}{m} = V_B - V_A$$

If A is at infinity, $V_A = 0 \Rightarrow \Delta V = V_B$

i.e., Gravitational potential at any point in a gravitational field is the change in potential energy of a particle of unit mass brought from infinity to that point in the gravitational field.

Gravitational potential energy of a system of two particles

The gravitational potential energy of a system of two particles is the negative of the work done by the gravitational force in assembling the system by bringing the particles from infinity to the desired configuration.

Consider two particles of masses m_1 and m_2, placed at A and B respectively, separated by a distance 'r'.

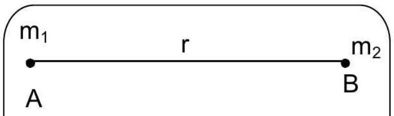


Fig. 3.21

If we consider that these particles were initially at infinite distance, initial potential energy of A = $m_1V_\infty = 0$ and initial potential energy of B = $m_2V_\infty = 0$ ($\because V_\infty = 0$)

$\therefore U_i$ = Initial potential energy of the system = $m_1V_\infty + m_2V_\infty = 0$

If particle m_1 was brought from infinity to A, no work is done as there is no gravitational field(or force). But m_1 sets up a gravitational field all around A so that gravitational potential at B is $V_B = -\frac{Gm_1}{r}$

Potential energy of m_2 when it is brought to B is = $m_2V_B = -\frac{Gm_1m_2}{r}$

$\therefore$ Total potential energy of system of A and B

$$= -\frac{Gm_1m_2}{r} + 0 = \frac{-Gm_1m_2}{r}$$

$$\therefore \quad U = -\frac{Gm_1m_2}{r}$$

This is the potential energy of the two particle system. If we consider m_2 was brought to B first from infinity, it will produce a gravitational potential at A which is $V_A = -\frac{Gm_2}{r}$

The change in potential energy of m_1 when it is brought to A is $U = m_1V_A = -\frac{Gm_1m_2}{r}$

Hence it is immaterial how the particles were brought to their configuration. The final energy of the configuration is the same and is $U = \frac{-Gm_1m_2}{r}$

Gravitational potential energy of three particle system

Let us consider that all the three particles were at infinite distances apart initially so that they do not exert any gravitational force on each other.

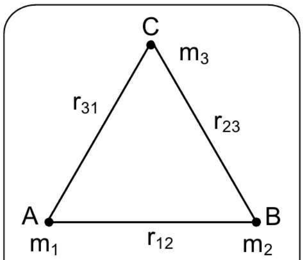


Fig. 3.22

If m_1 was brought first to A, no work is done as there is no gravitational force.

$\therefore$ $U_A = 0$ (single particle system where V = 0)

Now m_1 establishes a gravitational field all around it and the potential at B due to A is $V_{BA} = -\frac{Gm_1}{r_{12}}$

When particle m_2 is brought to B, the potential energy of the two particle system of m_1 and m_2 is $U_{AB} = m_2 V_{BA} = -\frac{Gm_1 m_2}{r_{12}}$

The gravitational potential at C due to A is $-\frac{Gm_1}{r_{31}}$ and at C due to B is $-\frac{Gm_2}{r_{23}}$

Hence the potential energy of 2 particle system B and C is $U_{BC} = -\frac{Gm_2 m_3}{r_{23}}$

and the potential energy of 2 particle system C and A is $U_{CA} = -\frac{Gm_3 m_1}{r_{31}}$

Hence the total gravitational potential energy of the 3 particles system is

$$U = U_{AB} + U_{BC} + U_{CA}$$

$$= -\frac{Gm_1 m_2}{r_{12}} - \frac{Gm_2 m_3}{r_{23}} - \frac{Gm_3 m_1}{r_{31}}$$

i.e., $U = -G\left[\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}}\right]$, is the gravitational potential energy of a three particles system.

Note:

If there are n particles in a system, they form $\frac{n(n-1)}{2}$ pairs of particles. The total gravitational potential energy of the system is the sum of the gravitational potential energy of all such pairs.

'Self energy' of bodies

The energy possessed by a body due to interaction of the particles inside the body, is called the self energy of the body.

For a single particle, there is no self energy. For other bodies, self energy of the body is the negative of the work done by the gravitational forces in assembling the body from infinity to their corresponding configuration to make the desired body.

Self energy of a thin, uniform hollow sphere

Consider a thin, uniform hollow sphere of mass m and radius R. Its initial mass is zero and as particles of mass dm get added to it, its mass increases (becomes more positive) while potential energy decreases(becomes more negative).

When the mass of the shell is 'm', if a particle of mass dm is added to the shell, the potential energy of the system is

$$dU = -Vdm = \frac{Gm}{R}dm \text{, where V = potential at surface}$$

$$= \frac{-Gm}{R}$$

$\therefore$ Total potential energy of the system(self energy)

$$U = \int_0^U dU = \int_{m=0}^{m=M} -\frac{Gm}{R}dm = -\frac{G}{R}\left|\frac{m^2}{2}\right|_0^m = \frac{-GM^2}{2R}$$

$$U = -\frac{GM^2}{2R}$$

is the self energy of a thin, uniform spherical shell of mass M and radius R.

Self energy of a uniform solid sphere

Consider a uniform solid sphere of mass M and radius R. Initially, its mass is zero as no particle is in the system. As particles are brought from infinity, both mass and radius keep on increasing.

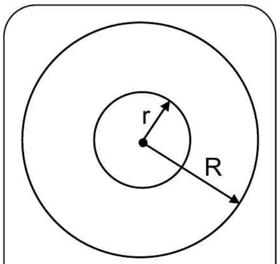


Fig. 3. 23

At an instant, when its radius is r, its mass is m, so that the potential at its surface is $dV = -\frac{Gm}{r}$

When a small particle of mass dm is added to the system, potential energy of the system is dV dm.

i.e., $$dU = -\frac{Gm}{r}dm \quad \text{— (i)}$$

But m = volume × density

$$= \frac{4}{3}\pi r^3 \times \frac{M}{\frac{4}{3}\pi R^3} = \frac{Mr^3}{R^3}$$

$$\therefore \quad dm = \frac{3Mr^2}{R^3}dr \qquad \text{— (i)}$$

$$\therefore\ dU = -\frac{Gm}{r}dm = -\frac{GMr^3}{rR^3}.\frac{3Mr^2}{R^3}dr = -\frac{3GM^2}{R^6}.r^4dr$$

$$\therefore \quad U = \int dU = \int_{r=0}^{r=R} \frac{-3GM^2}{R^6}r^4dr$$

$$= \frac{-3GM^2}{R^6}\left|\frac{r^5}{5}\right|_0^R = \frac{-3GM^2}{5R^6} \times R^5$$

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

is the self energy of the uniform solid sphere of mass M and radius R.

CONCEPT STRAND

Concept Strand 7

Considering Earth as a uniform solid sphere of radius 6400 km and mass 5.98×10^{24} kg, calculate the minimum energy required to separate all particles of Earth to infinite distance of separation.

Solution

$R = 6400$ km $= 6.4 \times 10^6$ m; $M = 5.98 \times 10^{24}$ kg

$$U_i = \text{The self energy of Earth} = \frac{-3GM^2}{5R}$$

When all particles are at infinite distance, $U_f = 0$

$$\therefore \quad \Delta U = U_f - U_i = 0 + \frac{3GM^2}{5R}$$

$$= \frac{3 \times 6.67 \times 10^{-11} \times (5.98 \times 10^{24})^2}{5 \times (6.4 \times 10^6)}$$

$$= 22.36 \times 10^{31}\ J$$

$$= 2.36 \times 10^{32}\ J$$

Hence it is not easy to disintegrate Earth.

ACCELERATION DUE TO GRAVITY

The force of attraction between two bodies produces acceleration of the bodies towards each other.

Consider the attraction between the Earth (mass M) and a body (mass m). The force of attraction F produces acceleration of both bodies given by

$$F = \frac{GMm}{R^2} \qquad \text{— (i)}$$

$$F = mg = Ma \qquad \text{— (ii)}$$

But, M being very large, a is unnoticeable. However, g, the acceleration due to gravity of the body of mass m is measurable. Equations (i) and (ii) yield

$$g = \frac{GM}{R^2}$$

The acceleration due to gravity of Earth on a particle is independent of its mass. The value of g at most places on Earth is about 9.8 m s^{-2} and the value has been standardized as 9.8066 m $s^{-2} \approx 9.8$ m s^{-2}.

Note that, in problems, unless otherwise mentioned, we take $g = 9.8$ m s^{-2}.

CONCEPT STRANDS

Concept Strand 8

What is the acceleration of Earth towards a spherical body of radius 1 m and density same as that of the Earth and which is on the surface of the Earth?

Solution

$$\frac{GMm}{R^2} = Ma \Rightarrow a = \frac{Gm}{R^2} = \frac{m}{M}g$$

$$= \frac{1}{(6.4\times10^6)^3}g \approx 10^{-20}\ \text{m s}^{-2} \text{ (negligibly small)}$$

Concept Strand 9

If the acceleration due to gravity on a planet is $\frac{1}{4}$ of that on Earth and the radius of Earth is 20 times that of the planet, what is the density ratio of the planet to that of the Earth?

Solution

$$\frac{g_p}{g_e} = \frac{M_p R_e^2}{M_e R_p^2} \Rightarrow \frac{1}{4} = \frac{1}{20}\frac{\rho_p}{\rho_e} \Rightarrow \frac{\rho_p}{\rho_e} = 5$$

$$\left(\because M_p = \frac{4}{3}\pi R_p^3 \rho_p \text{ and } M_e = \frac{4}{3}\pi R_e^3 \rho_e\right)$$

Concept Strand 10

What is the distance above Earth where a body will experience equal values of acceleration due to gravity due to the Earth and due to a planet with $\frac{1}{4}$ of the mass of the Earth, and at a distance D from centre of Earth?

Solution

Let D be the distance between the Earth and the planet. Then,

$$\frac{GM_e m}{(R+h)^2} = \frac{GM_p m}{[D-(R+h)]^2}$$

$$\Rightarrow \frac{[D-(R+h)]^2}{(R+h)^2} = \frac{M_p}{M_e} = \frac{1}{4}$$

$$\Rightarrow \frac{D-(R+h)}{R+h} = \frac{1}{2}$$

$$\Rightarrow \frac{D}{R+h} = \frac{3}{2}$$

$$\Rightarrow h = \frac{2}{3}D - R$$

Concept Strand 11

If an outer shell of thickness 100 km disintegrates from the Earth, what will be the percentage change in the acceleration due to gravity on the surface of the Earth?

Solution

$$g' = G\frac{M'}{R'^2} = g\frac{R^2}{M}\frac{M'}{R'^2} = g\frac{R^2R'^3}{R^3R'^2} = g\frac{R'}{R}$$

$$\therefore \frac{g-g'}{g}\times100 = 1.5\%$$

Variation of acceleration due to gravity

If acceleration due to gravity is considered to be the net effect of Earth on a body on or near it, it can vary due to the following:

(i) Effect of altitude

The force of attraction experienced by a body at a height 'h' above the surface of the Earth is

$$F = \frac{GMm}{(R+h)^2} \qquad \text{— (4)}$$

where R is the radius of the Earth. In the above formula we have assumed that all the mass of the Earth is concentrated at the centre. This assumption is valid so long as the force is measured outside the bodies in question. The acceleration due to gravity can be calculated by writing

$$mg' = \frac{GMm}{(R+h)^2}$$

$$\text{where } g' = \frac{GM}{R^2}\frac{1}{\left(1+\frac{h}{R}\right)^2} = \frac{g}{\left(1+\frac{h}{R}\right)^2} = g\left(\frac{R}{R+h}\right)^2$$

If h << R, as is usual for bodies near the surface of the Earth,

$$g' = g\left(1-\frac{2h}{R}\right)$$

If we treat Earth as a uniform solid sphere of mass M and radius R, the gravitational field of Earth at height 'h' above Earth,

$$E = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}\cdot\left(\frac{R}{R+h}\right)^2 = g\left(\frac{R}{R+h}\right)^2$$

which is the same expression as the effective acceleration due to gravity at height 'h'.

It should be understood that gravitational field intensity and acceleration due to gravity are two different physical quantities but their magnitudes and directions are same.

(ii) Effect of depth below the surface of the Earth

The average density of Earth can be calculated from the knowledge of g and R

$$g = \frac{GM}{R^2} = \frac{GM}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi R$$

$$g = \rho_{av}\frac{4\pi GR}{3}$$

$$\rho_{av} = \frac{3g}{4\pi GR} = 5.483\ \text{g cm}^{-3}$$ (substituting standard values of g, G, and R)

The Earth consists of an outer crust (~40 km) of density ~3 g cm^{-3}, a mantle (~3000 km) of density varying from 3.5 to 5.5 g cm^{-3} and an inner solid core, of density ~13 g cm^{-3}. However, in gravitational problems we assume that the density of Earth is a constant, and equal to ρ_{av}(~5.5 g cm^{-3})

A body at a depth d below the surface of the Earth is subject to an attractive force due to the volume of Earth below it, that is, due to the mass $\frac{4}{3}\pi(R-d)^3\rho$, where ρ is the average density of the Earth. Hence,

$$mg' = G\frac{\frac{4}{3}\pi(R-d)^3\rho.m}{(R-d)^2}$$

leading to

$$g' = \frac{G}{R^3}\left(\frac{4}{3}\pi R^3\rho\right)(R-d) = \frac{GM}{R^2}\frac{(R-d)}{R}$$

$$g' = g\left(1-\frac{d}{R}\right)$$

At the centre of Earth, d = R ⇒ g' = 0

It may be noted that g' = 0 at the centre of the Earth.

Treating Earth as a uniform solid sphere, gravitational field at depth 'd' below surface of Earth = gravitational field at radius r = (R – d)

i.e.,
$$E = \frac{GMr}{R^3} = \frac{GM(R-d)}{R^2.R}$$

$$= g\left[1-\frac{d}{R}\right]$$

which is the same expression as g' at depth d

(iii) Apparent change in acceleration due to gravity due to Earth's rotation

Earth rotates about the N–S axis at an angular frequency $\omega = 7.3\times10^{-5}$ rads^{-1}Any point particle P on the surface is subject to a pseudo force.(because of rotation of Earth, a frame of reference attached to Earth is a non-inertial frame and hence the pseudo force)

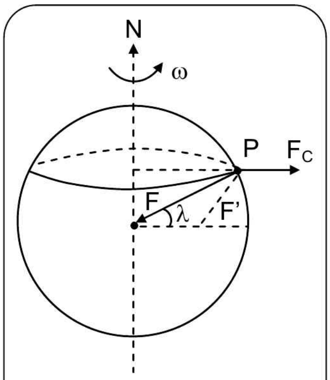


Fig. 3.24

$$F_c = m\omega^2 r = m\omega^2 R\cos\lambda$$

where, λ is the latitude at the point P. Therefore, the total force on the particle is the resultant F' of F_c and the gravitational attraction F

$$F' = \sqrt{F^2 + F_c^2 + 2FF_c\cos(180^\circ - \lambda)}$$

$$= \sqrt{\left(\frac{GMm}{R^2}\right)^2 + (m\omega^2R\cos\lambda)^2 - 2\frac{GMm}{R^2}.m\omega^2R\cos^2\lambda}$$

$$= \frac{GMm}{R^2}\left[1+\left(\frac{\omega^2R^3\cos\lambda}{GM}\right)^2 - 2.\frac{\omega^2R^3\cos^2\lambda}{GM}\right]^{1/2}$$

The second term in the bracket << the first term and the last term. Expanding binomially and neglecting higher order terms

$$\therefore \quad F' = \frac{GMm}{R^2}\left[1-\frac{\omega^2R^3\cos^2\lambda}{GM}\right]$$

$$\therefore \quad mg' = \frac{GMm}{R^2}\left[1 - \frac{\omega^2 R^3 \cos^2\lambda}{GM}\right]$$

$$g' = g\left(1 - \frac{\omega^2 R \cos^2\lambda}{g}\right)$$

Note that, at the poles, $\lambda = 90°$ and $g' = g$ while at the equator, g' has the lowest value $g' = g(1 - \omega^2 R)$

(iv) Effect of the shape of the Earth

In all the cases above, we have assumed that the Earth is perfectly spherical, but the shape is ellipsoidal, bulging at the equator and flattened at the poles. Thus

Equatorial radius > polar radius.

As $g \propto \frac{1}{R^2}$, it increases from the equator to the poles, being minimum at equator and maximum at the poles.

CONCEPT STRAND

Concept Strand 12

How much faster will the Earth have to rotate for a particle on the equator to fly off?

Solution

Weight at the equator $W_e = m\,g_e = m(g - \omega^2 R)$

For $W_e = 0$, $g = \omega^2 R \Rightarrow \omega$

$= 1.237 \times 10^{-3}$ rad s^{-1}

The increase is $\frac{1.237 \times 10^{-3}}{7.275 \times 10^{-5}} \approx 17$

The Earth will have to rotate 17 times faster.

ESCAPE VELOCITY

Escape velocity is the minimum velocity required for a particle to escape from the gravitational field of a celestial body such as a planet. The magnitude of escape velocity depends on the mass of the planet, and the distance from it. Consider a particle of mass m, at a distance r from a body of mass M. Then, the potential energy of the configuration is given by

$$PE = \frac{-GMm}{r}$$

This can be taken as the PE of the particle in the field of the massive body M.

For the particle to escape, its total energy should be zero.

i.e., $\quad KE + PE = 0$

$$\Rightarrow \quad KE = -PE = \frac{GMm}{r}$$

$$KE = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v_e = \sqrt{\frac{2GM}{r}}$$

For a particle on the surface of Earth, r = R, the radius of Earth.

$$\Rightarrow \quad v_e = \sqrt{\frac{2GM}{R}}$$

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}$$

With $g = 9.8$ m s^{-2} and $R = 6.4 \times 10^6$ m, the escape velocity on the surface of the earth is $v_e = 11.2 \times 10^3$ ms^{-1}

Escape velocity does not depend on (i) the mass of the particle (ii) the angle of projection.

Note:

If a body is launched from the surface of Earth with speed greater than v_e (i.e., escape velocity), at infinite distance from Earth, only its PE becomes zero. Its kinetic energy will not be zero.

CONCEPT STRAND

Concept Strand 13

What is the energy necessary for a particle of mass 1kg to escape from the gravity of the Earth?

Solution

$$W = \int dW = GMm\int_R^{\infty}\frac{dr}{r^2}$$

$$= GMm\left[\frac{-1}{r}\right]_R^{\infty} = \frac{GMm}{R}$$

$$= \frac{6.67\times10^{-11}\times5.98\times10^{24}\times1}{6.4\times10^{6}}$$

$$=6.23\times10^{7}\,J$$

SATELLITES IN CIRCULAR ORBITS

Satellites are usually launched into circular orbits. Consider a satellite in an orbit of radius r = R + h where R is the radius of the Earth and h is the height above the Earth at which the satellite is orbiting. The centripetal force necessary to keep the satellite in its orbit is provided by the gravitational attraction of the Earth directed towards the centre of the Earth. Thus we may write

$$\frac{mv_0^2}{r} = \frac{GMm}{r^2}$$

where v_0 is the orbital velocity.

$\therefore v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{gR^2}{r}} = \sqrt{\frac{gR^2}{R+h}}$. Hence v_0 is independent of mass of satellite.

$$v_0 \text{ near surface of Earth } = \sqrt{\frac{Gm}{R}} = \frac{v_e}{\sqrt{2}}\left(\because v_e = \sqrt{\frac{2Gm}{R}}\right)$$

$$v_0 = \frac{v_e}{\sqrt{2}}$$

Angular velocity ω is given by

$$\omega = \frac{v_0}{r} = \sqrt{\frac{gR^2}{(R+h)^3}} \qquad \omega = \frac{\sqrt{gR^2}}{r^{3/2}}$$

The time period T is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gR^2}}r^{3/2}$ $\quad T^2 \propto r^3$

Geostationary satellites

Geostationary satellites have time period same as that of the Earth so that their position in the orbit is stationary with respect to the Earth. The radius of the geostationary orbit can be calculated as

$$r = R + h = \left[gR^2\left(\frac{T}{2\pi}\right)^2\right]^{1/3}$$

Substituting the values of g and R we get h ≈ 36000 km.

A geostationary satellite moves in the same direction as rotation of Earth (West to East), with a time period of 24 hour and its orbital plane passes through the equatorial plane of Earth.

CONCEPT STRAND

Concept Strand 14

Satellite A is launched such that its time period is 3 times that of a geostationary satellite B. What is the height of A above the Earth?

Solution

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$\therefore \frac{T_1^2}{T_2^2} = 9 = \frac{r_1^3}{r_G^3}$$

But $r_G = 6.625$ R for a geostationary satellite, where R is the radius of Earth.

$$\therefore 9 = \frac{r_1^3}{(6.625R)^3} \Rightarrow h = 88196 \text{ km}$$ where h is the height of satellite A above the surface of the Earth.

First and second cosmic velocities

For satellites in orbits close to the Earth, equation for orbital speed v_0 can be written as

$v_0 = \sqrt{gR} = 7.92\ \text{km s}^{-1}$ where v_0 is called the first cosmic velocity.

The escape velocity from the surface of Earth is also known as the second cosmic velocity.

$$v_e = \sqrt{2gR} = \sqrt{2}v_0 = 11.2\ \text{km s}^{-1}$$

CONCEPT STRAND

Concept Strand 15

A planet revolving around the star in a circular orbit of radius R with a time period T is subject to a gravitational force proportional to R^{-3}. Calculate T as a function of R.

Solution

$$\frac{mv^2}{R} = kR^{-3} \quad \Rightarrow v = \sqrt{\frac{kR^{-2}}{m}}$$

$$\frac{v}{R} = \omega = \frac{2\pi}{T}$$

$$\Rightarrow T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{mR^2}{kR^{-2}}}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}}R^2$$

Energy of a satellite

Consider a satellite in a circular orbit of radius r around a planet of mass M.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\text{Kinetic energy} = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$\text{Potential energy} = \frac{-GMm}{r}$$

$$\text{TE} = \text{Total energy} = \text{PE} + \text{KE} = \frac{-GMm}{2r}$$

$$\Rightarrow \quad E = -KE = \frac{PE}{2}$$

For satellites,

$$KE : PE : TE = 1 : -2 : -1$$

Kepler's laws

First law (The law of orbits)

"The path of a planet is an elliptical orbit around the sun with the sun at one of its foci".

Second law (The law of areas)

"The radius vector, drawn from the sun to the planet sweeps out equal areas in equal intervals of time". Alternatively, "The areal velocity of a planet in its orbit is a constant".

Consider a small elemental area dA swept out by the planet;

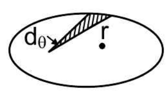


Fig. 3.25

$$dA = \frac{1}{2}r^2\,d\theta$$

$$\therefore \quad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m}$$

where L is the angular momentum, a constant.

Third law (The law of periods)

"The square of the time period of a planet around the sun is proportional to the cube of the semi major axis of its orbit". This has been proved for satellites in circular orbits in.

Angular momentum of a satellite

If m is the mass of satellite, v_0 is its orbiting speed and r is the radius of the orbit, then

$$\bar{v}_0 \perp \bar{r}$$

Angular momentum of the satellite in its orbit is $\bar{L} = \bar{r} \times \bar{p}$

$$= \bar{r} \times m\bar{v}_0$$

$$= m\left(\bar{r} \times \bar{v}_0\right)$$

$$\therefore \quad L = mrv_0 \ (\because \sin\theta = \sin 90° = 1)$$

$$L = mr\sqrt{\frac{Gm}{r}} = \sqrt{m^2Gmr}$$

$$\therefore \quad L = mrv_0 = \sqrt{m^3Gr}$$

is the angular momentum of the satellite. Here, we have considered that there is no spin for the satellite. Since no external torque acts on the satellite during its orbital motion, its angular momentum is conserved. i.e. L = constant

CONCEPT STRAND

Concept Strand 16

A planet in its elliptical orbit has the farthest distance from the Sun(r_1) equal to three times its nearest distance from the Sun(r_2). Will the orbital speed of the planet be different at those points? Explain

Solution

Let m be the mass of the planet. If v_1 and v_2 are the orbital speeds of the planet with respect to the Sun, at positions r_1 and r_2 respectively, then

Angular momentum of planet at position $\bar{r}_1$ is $\bar{L}_1 = m\left(\bar{r}_1 \times \bar{v}_1\right) \Rightarrow L_1 = mr_1v_1 \ \left(\because \bar{r}_1 \perp \bar{v}_1\right)$

Angular momentum of planet at position $\bar{r}_2$ is $\bar{L}_2 = m\left(\bar{r}_2 \times \bar{v}_2\right) \Rightarrow L_2 = mr_2v_2 \ \left(\because \bar{r}_2 \perp \bar{v}_2\right)$. Since angular momentum of the planet is conserved ($\because$ no torque acts on the planet), $L_1 = L_2 \Rightarrow mr_1v_1 = mr_2v_2$

$$\therefore \quad \frac{v_1}{v_2} = \frac{r_2}{r_1} = \frac{r_2}{3r_2} = \frac{1}{3}$$

$$\therefore \quad v_2 = 3v_1$$

Hence the orbital speed of planet in the nearest position to Sun is three times the orbital speed of planet in the farthest position.

Nature of trajectory

Let a satellite be projected from certain height from Earth's surface with a velocity v along the x-direction as shown in Fig. 3.24.

(i) If v = 0, it will fall vertically down towards Earth.

(ii) If $0 < v < v_0$, where $v_0 = \sqrt{gR}$, then the projectile will fall back to Earth in a spiral path.

(iii) If $v = v_0$, where $v_0 = \sqrt{gR}$, the satellite will move in a circular orbit of radius R.

(iv) If $v_0 < v < v_e$, where $v_e = \sqrt{2gR}$, the satellite will move in an elliptical orbit.

(v) If $v = v_e$, the projectile will escape from Earth's gravity in a parabolic path.

(vi) If $v > v_e$, the projectile will escape from Earth's gravity in a hyperbolic path.

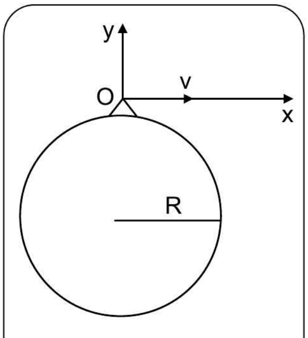


Fig. 3.26

1.	P.E of body on surface of Earth	$\frac{-GMm}{R}$	$-mgR$	$-WR$
2.	P.E at altitude h = R	$\frac{-GMm}{2R}$	$-\frac{mgR}{2}$	$\frac{-WR}{2}$
3.	P.E at finite altitude 'h'	$\frac{-GMm}{(R+h)}$	$-\frac{mgR}{\left(1+\frac{h}{R}\right)}$	$\frac{-WR}{\left(1+\frac{h}{R}\right)}$
4.	P.E at infinite altitude	Zero	Zero	Zero
5.	Minimum work done to transfer a body from Earth's surface to an altitude h = R	$\frac{GMm}{2R}$	$\frac{mgR}{2}$	$\frac{WR}{2}$

Potential Energy at different positions

M = mass of Earth, m = mass of body,

G = universal gravitational constant

R = radius of Earth, W = weight of body (mg)

g = acceleration due to gravity (on surface of Earth)

SUMMARY

$F=\frac{Gm_1m_2}{r^2}$	F→ Gravitational force between two particles of masses m_1 and m_2. G → Gravitational Constant r → Distance between the two masses
$g=\frac{GM_e}{R^2_e}=\frac{4}{3}\pi GR_e\rho_e$	g → Acceleration due to gravity on the surface of Earth R_e → Radius of Earth M_e → Mass of Earth ρ_e → Density of Earth
$g=\frac{GM}{R^2}$	g → Acceleration due to gravity on the surface of a planet or satellite M → Mass of the planet or satellite R → radius of the planet or satellite
$g_\lambda=g-\omega^2R\cos^2\lambda$	$g\lambda$ → Acceleration due to gravity at a given latitude λ ω → Angular velocity of rotation of Earth
$g_e=g-\omega^2R_e$ $g_p=g$	g_e → Acceleration due to gravity at the equator g_p → Acceleration due to gravity at the poles
$g_h=\frac{GM}{(R_e+h)^2}=g\left(\frac{R_e}{R_e+h}\right)^2$	g_h → Acceleration due to gravity at a height h above the surface of Earth
If $\frac{g_h}{g}=n$, $h=R\left(\sqrt{n}-1\right)$	g = acceleration due to gravity on the surface of Earth. R = radius of Earth
$g_h=g\left(1-\frac{2h}{R_e}\right)$	g_h → Acceleration due to gravity at a height h above the surface of Earth (h << R_e)

$g_d = \frac{4}{3}\pi G\rho(R_e - d)$	$g_d \rightarrow$ Acceleration due to gravity at a depth "d" from surface of earth. $\rho \rightarrow$ Average density of Earth
$g_d = g\left(1 - \frac{d}{R_e}\right)$	$\rho \rightarrow$ density of Earth. $R_e \rightarrow$ radius of Earth d = depth below surface of Earth
If $\frac{g_d}{g} = n$, then $d = R_0\left[\frac{n-1}{n}\right]$ $g_{\text{centre of Earth}} = 0$	
$I = \frac{GM}{r^2}$	$I \rightarrow$ Gravitational intensity at a distance 'r' from the centre of a particle of mass M
$V = -\frac{GM}{r}$	$V \rightarrow$ Gravitational Potential at a distance 'r' from a particle of mass M
$P.E = -\frac{Gm_1m_2}{r}$	$PE \rightarrow$ Potential Energy of a system of two particles of masses m_1 and m_2 separated by a distance 'r'
$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}$	$v_e \rightarrow$ Escape velocity from the surface of Earth
$v_o = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{r}}$	$v_o \rightarrow$ Orbital velocity of a satellite which is orbiting in a circular path of radius 'r'(at a height h from the surface)
$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$	$T \rightarrow$ Time period of revolution of a satellite which is revolving at a height 'h'
Energy of a satellite $KE = \frac{1}{2}mv_0^2 = \frac{1}{2}\frac{GMm}{(R+h)}$	$KE \rightarrow$ Kinetic Energy of a satellite $PE \rightarrow$ Potential Energy of a satellite $TE \rightarrow$ Total Energy of a satellite
$PE = -\frac{GMm}{(R+h)} = -2K.E$	
$TE = -\frac{1}{2}\frac{GMm}{(R+h)} = -K.E$	
K.E:P.E:T.E =1:-2:-1	
$T = 2\pi\sqrt{\frac{R}{g}}$	$T \rightarrow$ Time Period of a planet around the Sun $g \rightarrow$ Acceleration due to gravity of Sun at the orbit

CONCEPT CONNECTORS

Connector 1: Two particles of masses 1 kg and 2 kg are placed at a separation of 20 cm. If they start moving towards each other due to mutual gravitational force, find the velocity of the 1 kg mass after 10 s.

Solution: $F = \dfrac{Gm_1m_2}{r^2} = 3.3\times10^{-9}\,N$

$\therefore$ Acceleration $= a = \dfrac{F}{m_1} = \dfrac{3.3\times10^{-9}}{1} = 3.3\times10^{-9}\,m\,s^{-2}$.

Since this is very low, the distance does not change appreciably during motion; so acceleration can be taken as uniform.

$\therefore$ Velocity after 10 s, $v = u + at = 0 + 3.3\times10^{-9}\times10$

$= 3.3\times10^{-8}\,m\,s^{-1}$.

Connector 2: The weight of a body of mass $m_1 = 1$ kg on the surface of the Moon is $\dfrac{1}{6}$ of its weight on the surface of the Earth. If radius of the Moon is 1.738×10^6 m, find the mass of the Moon.

Solution: On the Moon m_1 weighs $\dfrac{1}{6}m_1(9.8)\,N = \dfrac{9.8}{6}\,N$ ($\because m_1 = 1$ kg)

$\therefore \quad W_1 = \dfrac{Gm_1m_2}{r^2}$

$$\frac{1}{6}\times(9.8) = 6.67\times10^{-11}\times\frac{1\times m_2}{\left(1.7368\times10^6\right)^2}$$

$$\Rightarrow m_2 = \frac{9.8\times\left(1.738\times10^6\right)^2}{6\times\left(6.67\times10^{-11}\right)} = 7.4\times10^{22}\text{ kg}$$

Connector 3: If the percentage error in the measurement of radius of Earth is known to be 0.2%, that of G is known to be 0.1% and that of is g is known to be 0.2%, what is the accuracy in the measurement of the mass of Earth?

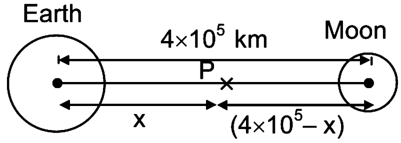
Solution: $g = \dfrac{GM}{R^2}$

$\Rightarrow \quad M = \dfrac{gR^2}{G}$

$$\frac{dM}{M} = \frac{dg}{g} + \frac{2dR}{R} + \frac{dG}{G} = 0.2 + 2\times0.2 + 0.1 = 0.7\%$$

Connector 4: Mass of Earth is 6×10^{24} kg and that of Moon is 7.4×10^{22} kg. The separation between the Earth and the Moon is 4×10^5 km. Find the distance of the point from the centre of the Earth where the resultant gravitational field is zero.

Solution: The point P where the resultant gravitational field is zero should be on the line joining the two centres. Let it be at a distance x from the centre of the Earth. Let E_e be the Earth's gravitational field and E_m that of moon



For resultant field to be zero, $E_e = E_m$

$$\Rightarrow \frac{GM_e}{x^2} = \frac{GM_m}{\left(4\times10^5 - x\right)^2} \qquad \Rightarrow M_e(4\times10^5 - x)^2 = M_m x^2 \Rightarrow x = 3.6\times10^5\text{ km}$$

Connector 5: A point mass M is kept at the centre of a uniform spherical shell of the same mass. Find the gravitational potential at a point P = $\frac{r}{4}$ from the centre (where r is the radius of the shell).

Solution: The potential at P due to the particle at the centre is

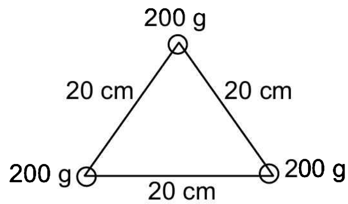
$$V_1 = -\frac{G \cdot M}{r/4} = -\frac{4GM}{r}$$

The potential at the point P due to the shell

$$V_2 = -\frac{GM}{r} \quad (\because \text{ potential due to a shell at a point inside} = \frac{-GM}{r})$$

$$\therefore \text{ The net potential} = V_1 + V_2 = \frac{-4GM}{r} + \frac{-GM}{r} = -\frac{5GM}{r}$$

Connector 6: From infinite separation, three bodies of mass 200 g each are brought to the vertices of an equilateral triangle of side 20 cm. Find the work required to be done by the external force for this process.



Solution: Here the work done will be equal to the gravitational potential energy of the system

$$\text{Gravitational P.E.} = U = 3\left[-\frac{Gm_1m_2}{r}\right]$$

$$= -3\left[\frac{6.67\times10^{-11}\text{N m}^2/\text{kg}^2\times(0.2\text{ kg})(0.2\text{ kg})}{0.2\text{ m}}\right] = -4 \times 10^{-11}\text{ J}$$

Connector 7: A simple pendulum has a time period 2 s at the North Pole. If it is taken to the equator, find the change in the time period (due to Earth's rotation). ($g = 9.8$ m s^{-2} and $R_e = 6400$ km)

Solution: At pole $T = 2\pi\sqrt{\frac{\ell}{g}}$

At equator $T' = 2\pi\sqrt{\frac{\ell}{g'}}$ where $g' = (g - \omega^2 R)$

where $\omega = \frac{2\pi}{24\times60\times60}$ rad s^{-1}

$$\therefore \quad \frac{T'}{T} = \sqrt{\frac{g}{g-\omega^2 R}} = \left[1-\frac{\omega^2 R}{g}\right]^{-1/2}$$

Expanding and considering the first term,

$$\therefore \quad T' = T\left[1+\frac{\omega^2 R}{2g}\right] \text{ where } \omega = \frac{2\pi}{24\times60\times60} \text{ rad s}^{-1}$$

Substituting the values $\Rightarrow$ $T' = 2.004$ second

Connector 8: A particle is projected vertically upwards with a velocity v. Find the height reached by the particle in terms of the radius of the Earth R and mass of Earth M.

Solution: Initial energy = $\frac{1}{2}mv^2 - \frac{GMm}{R}$

Final Energy $= -\dfrac{GMm}{R+h}$ $\Rightarrow \dfrac{1}{2}mv^2 = GMm\left(\dfrac{1}{R}-\dfrac{1}{R+h}\right)$

$\dfrac{v^2}{2GM} = \dfrac{1}{R} - \dfrac{1}{R+h}$ $\Rightarrow \dfrac{1}{R+h} = \dfrac{1}{R} - \dfrac{v^2}{2GM}$

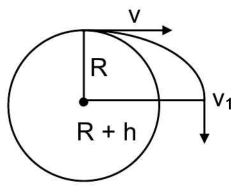
$\Rightarrow \dfrac{1}{R+h} = \dfrac{2GM - v^2R}{2GMR}$ $\Rightarrow \dfrac{R+h}{R} = \dfrac{2GM}{2GM - v^2R}$

$\Rightarrow \dfrac{h}{R} = \dfrac{v^2R}{2GM - v^2R}$ $\Rightarrow h = \dfrac{v^2R^2}{2GM - v^2R}$

Connector 9: A particle is projected horizontally with a speed v. Find the maximum height reached. (R = radius of Earth, M = mass of Earth).

Solution: $v.\,R = v_1\,(R + h)$ (conservation of angular momentum) — (1)

$\dfrac{1}{2}mv^2 - \dfrac{GMm}{R} = \dfrac{1}{2}mv_1^{\,2} - \dfrac{GMm}{R+h}$ (conservation of energy) — (2)



$\therefore \quad \dfrac{1}{2}m\left(v^2 - v_1^2\right) = GMm\left(\dfrac{1}{R} - \dfrac{1}{R+h}\right)$

$\Rightarrow \quad (v^2 - v_1^{\,2}) = 2GM\left[\dfrac{1}{R} - \dfrac{1}{R+h}\right]$

$v^2\left(1 - \left(\dfrac{v_1}{v}\right)^2\right) = 2GM\left(\dfrac{1}{R} - \dfrac{1}{R+h}\right);$

substituting for $\dfrac{v_1}{v}$ from (1), $v^2\left[1 - \left(\dfrac{R}{R+h}\right)^2\right] = \dfrac{2GMh}{R(R+h)}$

$\Rightarrow v^2\dfrac{(2R+h)}{(R+h)} = 2v_0^{\,2}$ where $\upsilon_0^{\,2} = \dfrac{GH}{R}$

$\left(\dfrac{v}{v_0}\right)^2 = 2\left(\dfrac{R+h}{2R+h}\right) = \eta^2,$ where $\eta = \dfrac{v}{v_0}$ $\Rightarrow (2R+h)\eta^2 = 2(R+h)$

$(2 - \eta^2)h = 2R\,(\eta^2 - 1) \Rightarrow h = \dfrac{2R(\eta^2 - 1)}{(2 - \eta^2)}$

Connector 10: Find the escape velocity from the Moon if the mass of the Moon is 7.4×10^{22} kg and radius of the Moon is 1740 km.

Solution: Escape velocity $v = \sqrt{\dfrac{2GM}{R}}$

$= \sqrt{\dfrac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1740 \times 10^3}} = 2.4\ \text{km s}^{-1}$

Connector 11: From the surface of a rocky sphere of density 3.0 g cm^{-3} a body is thrown with a velocity 40 m s^{-1}, which is the minimum velocity in any direction so that it never comes back to the sphere. Find the radius of the sphere.

Solution: Escape velocity v_e is given by

$$\frac{1}{2}mv_e^2 = \frac{GmM}{R} = \frac{Gm}{R}.\frac{4}{3}\pi R^3\rho,$$

where $M = \frac{4}{3}\pi R^3\rho$ $\quad \therefore R = v_e\sqrt{\frac{3}{8\pi G\rho}} \Rightarrow R = 31$ km

Connector 12: A binary system consists of two masses m each. If the radius of the orbit is R, find the escape velocity.

Solution: $PE = \frac{-Gm^2}{2R}$

For escape, $KE = PE \Rightarrow \frac{1}{2}mv^2 = \frac{Gm^2}{2R}$

$$v^2 = \frac{Gm}{R} \Rightarrow v = \sqrt{\frac{Gm}{R}}$$

Connector 13: A triple system consists of three stars of mass m each at the corners of an equilateral triangle of side a. Find the escape velocity of one of the stars.

Solution: $PE = \frac{-Gm^2}{a} \times 2;\quad KE = \frac{1}{2}mv^2$

$\Rightarrow v^2 = 2 \times 2\,\frac{Gm}{a} \quad v = 2\sqrt{\frac{Gm}{a}}$

Connector 14: Show that the average kinetic energy of a mass m revolving around a large mass M is $\frac{-1}{2}$ times its average potential energy.

Solution: $\overline{F} = -\frac{GMm}{r^2}\hat{r}$ where $\hat{r}$ points from M to m $\Rightarrow \overline{F} = -\frac{GMm\overline{r}}{r^3}$

Let $Q = m\overline{v}.\overline{r}$, $\dot{Q} = m\overline{a}.\overline{r} + m\overline{v}.\overline{v} = \overline{F}.\overline{r} + mv^2$

But $\overline{F}.\overline{r} = -\frac{GMm}{r}$

Taking the time average over a period, $\dot{Q}_{av} = 0$ (for closed orbits)

$$\therefore \quad 0 = -\frac{GMm}{r}\bigg|_{av} + mv^2\Big|_{av}$$

But $mv^2 = 2KE$ $\quad \Rightarrow KE_{av} = \frac{GMm}{2r}\bigg|_{av} = -\frac{PE_{av}}{2}$

Connector 15: A satellite is revolving in a circular orbit around the Earth at a height of 600 km above the surface of Earth.

Find

(i) the orbital speed of the satellite,

(ii) Its time period.

(Radius of Earth R_e = 6400 km and Mass of Earth $M_e = 6 \times 10^{24}$ kg)

Solution: (i) $v = \sqrt{\dfrac{GM}{r}}$ (where $r = R_e + h$); $r = 6400 + 600 = 7000$ km $= 7 \times 10^6$ m

$$= \sqrt{\frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{7000 \times 10^3}} = 7.56 \times 10^3 \text{ m s}^{-1} = 7.6 \text{ km s}^{-1}$$

(ii) Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 7000 \times 10^3}{7.6 \times 10^3} = 5.8 \times 10^3$ s

Connector 16: The Moon orbits the Earth in an approximately circular path of radius 3.8×10^8 m. It takes 27days to complete 1 revolution. Find the mass of the Earth.

Solution: $\dfrac{mv^2}{r} = \dfrac{GMm}{r^2}$

$\therefore\ M = \dfrac{v^2 r}{G} = \dfrac{\omega^2 r^3}{G}.$

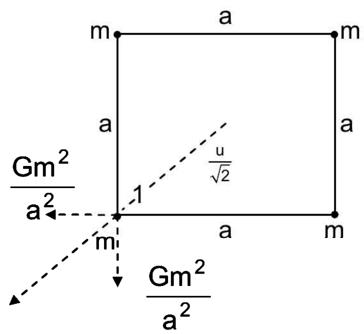
Here $\omega = 1$ rev/27 days $= 2.7 \times 10^{-6}$ rad s^{-1}

$r = 3.8 \times 10^8$ m, and $G = 6.7 \times 10^{-11}$ N m^2 kg^{-2}

$\therefore\ M = 6 \times 10^{24}$ kg

Connector 17: Four particles of mass m are at the vertices of a square of side a. Find the time period of rotation for which the relative distances remain unchanged.

Solution: Force on particle 1 are $\dfrac{Gm^2}{a^2}$ at right angles and $\dfrac{Gm^2}{\left(\sqrt{2}\,a\right)^2}$ along the diagonal.



$$F = \frac{Gm^2}{a^2}.\frac{1}{\sqrt{2}} \times 2 + \frac{Gm^2}{(\sqrt{2}\,a)^2} = \frac{Gm^2}{a^2}\left(\sqrt{2} + \frac{1}{2}\right) = \frac{mv^2\sqrt{2}}{a}$$

$$\Rightarrow v^2 = \frac{Gm}{a}\left(1 + \frac{1}{2\sqrt{2}}\right)$$

$$T = \frac{2\pi.\dfrac{a}{\sqrt{2}}}{v},\ v = \sqrt{\frac{Gm}{a}\left(1 + \frac{1}{2\sqrt{2}}\right)}$$

$$\Rightarrow T = \frac{\sqrt{2}\pi a}{\sqrt{1 + \dfrac{1}{2\sqrt{2}}}}.\sqrt{\frac{a}{Gm}}$$

$$= \sqrt{\frac{2}{1 + \dfrac{1}{2\sqrt{2}}}}.\frac{a^{\frac{3}{2}}\pi}{\sqrt{Gm}} = \left[\frac{2^{5/2}\pi^2 a^3}{(2^{3/2} + 1)Gm}\right]^{1/2}$$

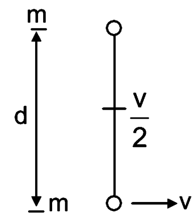
Connector 18: A spherical region of space contains dust of mass M and radius R. If the dust has uniform density and offers no frictional drag, find the time period of motion of a body at a distance r from the centre of the region.

Solution: $\dfrac{G\left(\dfrac{r}{R}\right)^3.M}{r^2} = \dfrac{v^2}{r}$

$$\Rightarrow \frac{GM}{R^3}.r^2 = v^2 \Rightarrow \left(\frac{r}{v}\right)^2 = \frac{R^3}{GM} \qquad \text{But } T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{R^3}{GM}}$$

Connector 19: Two equal masses m each are placed at a distance d apart. Find the velocity that should be imparted to one of the masses perpendicular to the line joining the two masses so that their relative separation is unchanged.

Solution: Let the velocity be v. The centre of mass moves with a velocity $\frac{v}{2}$.



In the centre of mass frame,

$$\frac{m\left(\frac{v}{2}\right)^2}{\frac{d}{2}} = \frac{Gm^2}{d^2}, \quad \frac{v^2}{2d} = \frac{Gm}{d^2} \quad \Rightarrow v = \sqrt{2\frac{Gm}{d}}$$

Connector 20: A sphere of mass m and radius r rolls on top of another fixed sphere of mass M and radius R along its circumference with a time period T. Find the contact force between them.

Solution:

$$\frac{mv^2}{(R+r)} = \frac{GMm}{(R+r)^2} - N \Rightarrow N = -m\omega^2 (R+r) + \frac{GMm}{(R+r)^2}$$

$$\omega = \frac{2\pi}{T} \Rightarrow N = m\left[\frac{-4\pi^2 (R+r)}{T^2} + \frac{GM}{(R+r)^2}\right]$$

TOPIC GRIP

Subjective Questions

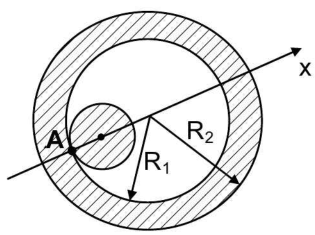
1. A pendulum clock keeps correct time at each of the following three locations.
   (i) in a laboratory at the equator.
   (ii) at an altitude h' above Earth's surface at latitude 30°.
   (iii) in a mine at depth h at latitude 45°.

   Determine h and h' in terms of g_0, ω, R where g_0 = acceleration due to gravity of Earth at the pole, ω = angular velocity of Earth's rotation, R = Radius of Earth

2. A certain planet's day is one-fourth of Earth day. Apparent acceleration due to gravity of Earth at its pole is 0.2 times that at Earth's pole (g_0). Apparent acceleration due to gravity at its 60° latitude is 0.1 times that at Earth at 60° latitude. Determine the planet's radius.

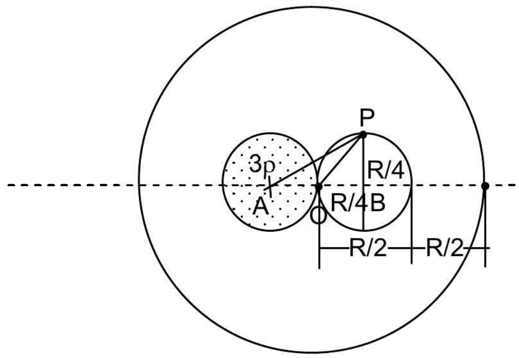
   (Use g_0, acceleration due to gravity at Earth's pole = π^2, R_e = Radius of Earth = 6400 km, Earth day = 24 h = 86400 s).

3. A solid sphere of mass m and radius r is placed inside a hollow sphere of mass M and inner radius R_1 and outer radius R_2, touching the inner surface at point A as shown. Determine the gravitational field at a distance x from point A along the line joining the centres for the following cases;



   Case
   (i) $0 < x < r$
   (ii) $x = r$
   (iii) $r < x < 2r$
   (iv) $x = 2r$
   (v) $2r < x < 2R_1$
   (vi) $2R_1 < x < R_2 + R_1$
   (vii) $x = R_2 + R_1$
   (viii) $x > R_2 + R_1$

4. A sphere of radius R has a hollow spherical cavity of a radius $\frac{R}{4}$ and a spherical portion of radius $\frac{R}{4}$ of material of density 3ρ. The other part of the sphere has density ρ (See figure). Determine the gravitational field strength and direction at a point P (shown) on the circumference of the cavity.



5. A wire of length ℓ and mass m is bent into an arc of a circle, subtending an angle of $\frac{\pi}{3}$ radian at the centre. Determine the gravitational field and gravitational potential at the centre of the circle, of which the arc forms a part.

6. Four particles, each of mass m, are placed at the four corners of a square of side a. What is the work to be done to increase the side of the square to 1.5a?

7. A projectile is fired vertically up from the surface of Earth with an initial velocity of 2 km s^{-1}. How far above the surface of Earth does it go? (Radius of Earth = 6400 km, g = 10 m s^{-2}, Neglect air drag).

8. A planet's radius is 4 times that of Earth, its average density is $\frac{1}{4}$ of Earth's and its day is half of Earth day. If the speed of a geostationary satellite is 3000 m s^{-1}, determine the escape velocity of a planet–stationary satellite. $\left(2^{\frac{1}{6}} = \frac{9}{8}\right)$

9. A satellite of mass m in circular orbit around Earth is acted upon by constant frictional force f due to cosmic dust and gradually spirals towards Earth. Assuming the decrease in radius very small so that at each instant the orbit is very nearly circular, determine change in radius per revolution as a function of 'r'. Also determine the time taken for the radius r to become $\frac{r}{2}$.

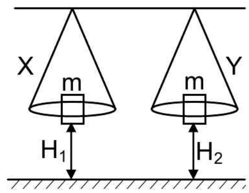
(Mass of Earth = M, Universal gravitational constant = G)

10. Determine the time period of rotation of a binary star system about its centre of mass if the individual masses of binary members are m_1 and m_2 and their distance of separation is d.

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

11. In a physics laboratory a block X is balanced by a block Y of mass 200 kg, each are suspended from a balance of equal lever arms. The scale pan on X is at a height. H_1 = 2cm, where as scale pan on Y is at height 1cm above the floor. The error in weighing is(in gram)



(a) $\frac{1}{650}$ (b) $\frac{1}{320}$

(c) $\frac{1}{1600}$ (d) $\frac{1}{540}$

12. If gravitation field varies as $E = \frac{-k^2}{h}$, and the reference point is h_i whose potential is V_i, then potential at any point at h is given by:

(a) $V = k^2 \ln\frac{1}{V_i}$ (b) $V = k^2 \ln\frac{h}{h_i} + V_i$

(c) $V = \ln\frac{h}{h_i} + k^2 V_i$ (d) $V = \ln\frac{h}{h_i} + \frac{V_i}{k}$

13. The potential at the center of a planet of mass M and radius R whose density is proportional to the distance from the center is

(a) $\frac{-GM}{R}$ (b) $\frac{-3}{2}\frac{GM}{R}$ (c) $\frac{-4}{3}\frac{GM}{R}$ (d) $\frac{-2GM}{R}$

14. A projectile is fired vertically up from the Earth's surface, with a velocity equal to 3/4th of the escape velocity v_e from the surface of the Earth. The maximum distance of the projectile from centre of Earth is (radius of Earth = R)

(a) 2R (b) $\frac{4}{3}R$ (c) $\frac{16}{7}R$ (d) $\frac{16}{9}R$

15. Two satellites A and B of the same mass m, are at distances $2R_e$ and $6R_e$, above the Earth's surface where R_e = radius of Earth. The ratio of the potential energy of A to that of B is

(a) 3 (b) 7/3 (c) 7/4 (d) 3/7

Assertion–Reason Type Questions

Directions: **Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

16. Statement 1

The fractional change in weight of a body at a particular depth below surface of Earth as compared to its weight at the surface of Earth, is more at the pole than at the equator.

and

Statement 2

Acceleration due to gravity is more at the pole than at the equator.

17. Statement 1

If the density of a solid sphere is non-uniform, it is quite possible that magnitude of gravitational potential, $|V|$ at a point inside the solid sphere is less than that at its surface.

and

Statement 2

g at points inside is less than that at surface for a solid sphere of uniform density.

18. Statement 1

If a body at rest on the surface of Earth is given a velocity $v \geq v_{escape,}$ the body will escape Earth's gravity, irrespective of the angle $\vec{v}$ makes with the radius vector $\vec{R}$.

and

Statement 2

Energy is a scalar.

19. Statement 1

If two satellites, one in circular orbit and another in elliptical orbit, are of equal time period and equal angular momentum, then the one in circular orbit has lower mass.

and

Statement 2

Areal velocity of a satellite can be estimated using the relation $\frac{LT}{2m}$ where L is its angular momentum, T period and m its mass.

20. Statement 1

For a planet in elliptical orbit, its linear momentum and angular momentum are conserved.

and

Statement 2

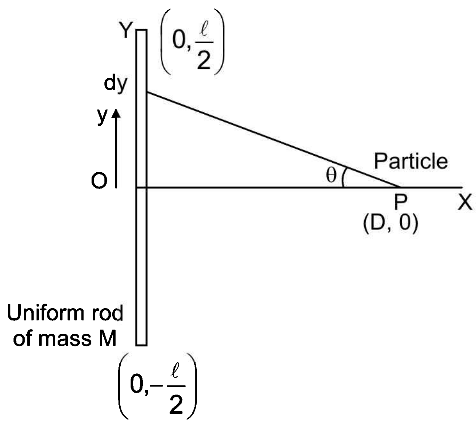
The torque due to a radial force is zero.

Linked Comprehension Type Questions

Directions: **This section contains 2 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Sir Isaac Newton demonstrated that the gravitational force exerted by a finite sized spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass were concentrated at its centre. Initially it was an intelligent guess which was later proved to be true. Can we do a similar guess work regarding the force of gravitation exerted by a uniform rod of mass M and length ℓ on a particle p (mass m) placed on the perpendicular bisector. Refer the adjoining figure.



21. Let the rod be split into small length elements. One such element dy, which can be treated as a particle, is shown in the figure at position (0, y). According to the universal law of gravitation, the force of attraction by the element dy on the particle at P is given by dF =

(a) $\dfrac{GmM(dy)}{\ell D^2}$ (b) $\dfrac{GmM(dy)}{\ell(y^2+D^2)}$

(c) $\dfrac{GmM(dy)}{\ell D^2 \sec^2\theta}$ (d) both (b) and (c) are correct

22. The direction of the gravitational force dF is given by the unit vector

(a) $\cos\theta\hat{i}+\sin\theta\hat{j}$ (b) $\cos\theta\hat{i}-\sin\theta\hat{j}$ (c) $-\cos\theta\hat{i}+\sin\theta\hat{j}$ (d) $-\cos\theta\hat{i}-\sin\theta\hat{j}$

23. The force of gravitation of the rod on the particle P is

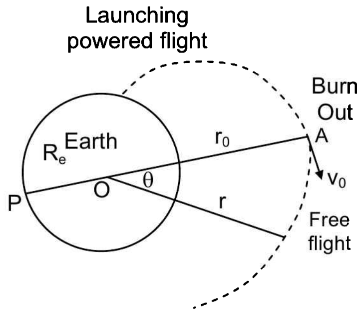
(a) $\dfrac{GmM}{D^2\sqrt{1+\dfrac{\ell}{2D}}}$ (b) $\dfrac{GmM}{D^2\sqrt{1+\left(\dfrac{\ell}{2D}\right)^2}}$ (c) $\dfrac{GmM}{D^2\sqrt{1+\left(\dfrac{2D}{\ell}\right)^2}}$ (d) $\dfrac{GmM}{D^2\sqrt{1+\dfrac{2D}{\ell}}}$

Passage II

The trajectory of a space vehicle is determined from the position and the velocity of the space vehicle at the beginning of its free flight. The figure shows the steps of launching the space vehicle from the Earth's surface. It is assumed that the powered phase of the flight is so programmed that the velocity of the vehicle is parallel to the Earth's surface, when the last stage of the launching rocket burns out at A. Let r_0 and v_0 denote the radius and velocity of the vehicle at the start of its free flight. We define the following parameters related to the launching of the vehicle.

$$\mu = v_0 r_0$$

$$\varepsilon = \frac{v_0^2 r_0}{Gm_e} - 1$$

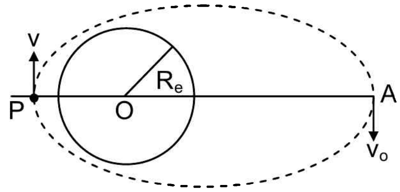


During the free flight the vehicle (mass m) is acted upon by the gravitational attraction of the Earth alone. The position of the vehicle is given by $r = \frac{\mu^2}{Gm_e(1+\varepsilon\cos\theta)}$. Atmospheric friction is neglected.

24. If the path of the vehicle is to be a circle with the centre O, the value of ε is

(a) 1 (b) ≤ 1 (c) ≥ 1 (d) 0

25. After launching the space vehicle, at A on its free flight path, it is desired that the vehicle becomes an Earth satellite by just clearing the point P diametrically opposite to A such that $OP \simeq R_e$. The velocity v_0 required is



(a) $\sqrt{\frac{Gm_e}{R_e}}$ (b) $\sqrt{\frac{Gm_e}{r_0}}$

(c) $\sqrt{\frac{Gm_e}{R_e}}\left(\sqrt{\frac{2}{1+\frac{r_0}{R_e}}}\right)$ (d) $\sqrt{\frac{Gm_e}{R_e}}\left(\sqrt{\frac{2}{\frac{r_0}{R_e}\left(1+\frac{r_0}{R_e}\right)}}\right)$

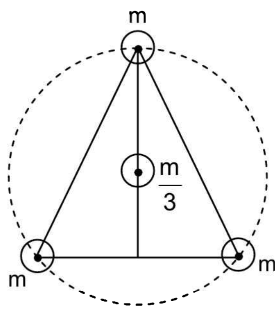
26. Consider the case $r_0 = 4R_e$. In order to become an Earth satellite, the minimum total mechanical energy of the space vehicle is

(a) $-\frac{Gmm_e}{4R_e}$ (b) $-\frac{Gmm_e}{5R_e}$ (c) $-\frac{Gmm_e}{8R_e}$ (d) $-\frac{Gmm_e}{2R_e}$

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

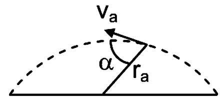
27. The masses m each are placed at the corners of an equilateral triangle of side a and a mass $\frac{m}{3}$ is placed at the centre of the triangle. The 3 masses are in uniform circular motion with velocity v on the circumcircle of the triangle, the centripetal force being provided by their mutual attraction. Then



(a) $v = \sqrt{\frac{GM}{a}\frac{(\sqrt{3}+1)}{\sqrt{3}}}$ (b) $v = \sqrt{\frac{GM}{a}\frac{(\sqrt{3}-1)}{\sqrt{3}}}$

(c) The mass $\frac{m}{3}$ experiences no force (d) If $\frac{m}{3}$ is absent, $v = \sqrt{\frac{GM}{a}}$

28. An artificial satellite of Earth, moving in an elliptical orbit of eccentricity e, has a maximum velocity v_m and the velocity at a radial distance r_a equal to v_a and the angle between radius and the velocity at this position is α. Then if r_1 is the minimum distance and v_{min} is the minimum velocity (Mass of earth = M)



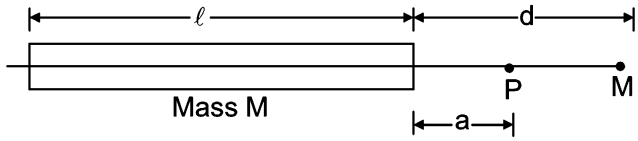
(a) $v_{min} = v_m \dfrac{1-e}{1+e}$

(b) $r_1 = \dfrac{v_a \cdot r_a \sin\alpha}{v_m}$

(c) $r_1 = \dfrac{v_a r_a \cos\alpha}{v_m}$

(d) $v_m > \sqrt{\dfrac{GM}{r_1}}$

29.



In free space, a rod of mass M and length ℓ has another point mass M on its axial line at a distance d from one end as shown in figure. At a point P in between the rod and the mass, at a distance 'a' from one end of the rod, the gravitational field is zero. Then

(a) $d = a + \sqrt{a(\ell + a)}$

(b) $d = 2a - a\sqrt{\ell + a}$

(c) If the point mass is shifted to P, its P.E = 0

(d) If the point mass is shifted to P, the total P.E of the system is $-\dfrac{GM^2}{\ell}\ln\left(1 + \ell/a\right)$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

30. Match the items in column I to those in column II with regard to a body projected from the surface of earth

Column I	**Column II**
Total energy E (U is potential energy)	Path of the body
(a) $E = 0$	(p) circular orbit
(b) $E < 0$	(q) parabolic path
(c) $E > 0$	(r) hyperbolic path
(d) $E = \dfrac{U}{2}$	(s) closed orbit

IIT ASSIGNMENT EXERCISE

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

31. The distance between two particles is x and the gravitational force between them is F. If x is quadrupled, the gravitational force between them will be

(a) 2F (b) 4F (c) 16F (d) $\frac{F}{16}$

32. Which is not true about the gravitational force?

(a) Obeys inverse square law (b) is a conservative force
(c) both attractive and repulsive (d) is weakest of all fundamental forces

33. The gravitational force between two particles (Point out the wrong statement)

(a) Depends on the masses of the particles (b) Depends on the separation between the particles
(c) Depends on the presence of other bodies (d) Is independent of the presence of other bodies

34. 1 newton of force is approximately

(a) The force needed to hold 1 kg of a body near the surface of Earth.
(b) The force needed to hold 102 g of body near the surface of Earth.
(c) The force required to give an acceleration of 1 m s^{-2} to a body of mass 1 kg.
(d) Both (b) and (c).

35. The ratio of the inertial mass to gravitational mass is equal to

(a) 0.5 (b) 1 (c) 2 (d) no fixed number

36. If F is the gravitational force acting on a mass 'M' kept at a place where the acceleration due to gravity is g, the gravitational field at that point is

(a) $\frac{Mg}{F}$ (b) $\frac{F}{Mg}$ (c) $\frac{F}{g}$ (d) $\frac{F}{M}$

37. If a body's weight at a place is 100 N where the acceleration due to gravity is 10 m s^{-2}, its gravitational mass is equal to

(a) 100 kg (b) 1000 kg (c) 10 kg (d) $\frac{1}{10}$ kg

38. If the radius of Earth is 6400 km and mean value of 'g' on the surface is 9.8 m s^{-2}, the value of acceleration due to the gravity at a height 100 km above the surface of Earth is

(a) 7.5 m s^{-2} (b) 8.7 m s^{-2} (c) 9.5 m s^{-2} (d) 10.0 m s^{-2}

39. The value of acceleration due to gravity on the surface of Earth is 'g'. It becomes g/2 at the depth 'h' equal to

(a) R (b) 2R (c) $\frac{R}{2}$ (d) $\frac{R}{4}$

40. g_e and g_m are the acceleration due to gravity on the surface of Earth and Moon such that $g_m = 0.2g_e$. If R_e and R_m are the maximum ranges of a projectile on Earth and moon for the same initial velocity and angle of projection, then

(a) $R_m = 0.2R_e$ (b) $R_m = 2R_e$ (c) $R_m = 0.5R_e$ (d) $R_m = 5R_e$

41. If the Earth suddenly stops rotating, the apparent weight of a body on the surface of Earth other than poles,

(a) increases (b) decreases
(c) does not change (d) becomes half the original value

42. If a second's pendulum is taken from Earth to another planet whose mass and radius are twice that of Earth, its period on the planet will be

(a) 2 s (b) $\sqrt{2}$ s (c) $\frac{1}{\sqrt{2}}$ s (d) $2\sqrt{2}$ s

43. A body of mass m is taken to the bottom of a deep mine then

(a) Its mass increases (b) Its weight increases (c) Its mass decreases (d) Its weight decreases

44. If the Earth shrinks to half of its radius with mass remaining the same, the weight of an object on the Earth will

(a) decrease 2 times (b) increase 4 times (d) decrease 4 times (d) increase 2 times

45. If $g = 9.8$ m s^{-2} and the radius of Earth is 6400 km, the first orbital velocity is equal to

(a) 5.7 km s^{-1} (b) 9.72 m s^{-1} (c) 7.92 km s^{-1} (d) 7.92 m s^{-1}

46. If the mass of Earth is eighty times mass of a planet and diameter of the planet is one fourth that of Earth, then the acceleration due to gravity on the planet would be

(a) 3.12 m s^{-2} (b) 9.8 m s^{-2} (c) 4.9 m s^{-2} (d) 1.96 m s^{-2}

47. If the radius of the Earth shrinks by one percent, its mass remaining the same, the acceleration due to gravity on the Earth's surface will

(a) decrease (b) remains the same (c) increase (d) cannot be predicted

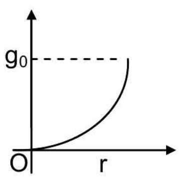
48. Consider the Earth to be a homogenous sphere. Scientist A goes deep down in a mine and scientist B goes high up in a balloon. The gravitational field measured by

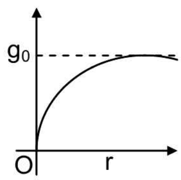
(a) A goes on decreasing and that by B goes on increasing
(b) B goes on decreasing and that by A goes on increasing
(c) each remain unchanged
(d) each goes on decreasing

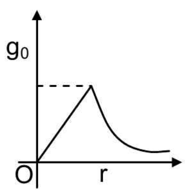
49. Choose the correct statement.

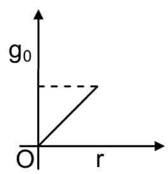
(a) Weight of a body is greater at the poles than at equator.
(b) Weight of the body is greater in planes than on hill tops.
(c) Weight of a body on the Moon is less than that on the Earth and is more on Sun.
(d) All the above are true.

50. If g_0 is the value of acceleration due to gravity on the surface of Earth, the variation of acceleration due to gravity from the centre of Earth to the surface of Earth is best represented by :

(a) 

(b) 

(c) 

(d) 

51. If the point P is at a distance 'r' from the centre of the Earth, the work done by an external agency in taking a body of mass m from an infinite distance to P against the gravitational force of Earth is

(a) $\frac{GMm}{r^2}$ (b) $-\frac{GM}{r^2}$ (c) $-\frac{GMm}{r}$ (d) $\frac{GMm}{r}$

52. Consider a point in the gravitational field of the Moon. The gravitational potential at that point is

(a) The work done in moving a body from infinity to that point.
(b) The work done in moving a body from that point to infinity.
(c) Work done in moving a unit mass from infinity to that point.
(d) Work done in moving a unit mass from that point to infinity.

53. If the potential at the center of a uniform solid sphere is V, the potential at the face center of a solid hemisphere of the same radius is

(a) V (b) $\frac{V}{2}$ (c) $\frac{3V}{4}$ (d) $\frac{V}{4}$

54. The dimensional formula for gravitational potential is

(a) ML^2T^{-2} (b) $M^2L^2T^{-2}$ (c) $M^0L^2T^{-2}$ (d) ML^0T^{-3}

55. If a satellite of Earth is stopped suddenly in its orbit at a height equal to radius of Earth and is allowed to fall into Earth's surface, with what speed will it hit the surface of Earth?

(a) 7.92 km s^{-1} (b) 11.2 km s^{-1} (c) 8.92 km s^{-1} (d) 5.6 km s^{-1}

56. A body of mass m rises to a height h = 1/5th of radius of Earth above the surface of Earth. The increase in its P.E is

(a) mgh (b) $\frac{5}{6}$ mgh (c) $\frac{4}{5}$mgh (d) $\frac{6}{5}$mgh

57. A body starts from rest from a point outside the Earth at a distance R_0 from the centre of the Earth. The velocity acquired by the body when it reaches the surface of the Earth will be (R represents radius of the Earth)

(a) $2GM\left(\frac{1}{R}-\frac{1}{R_0}\right)$ (b) $\sqrt{2GM\left(\frac{1}{R}-\frac{1}{R_0}\right)}$ (c) $GM\left(\frac{1}{R}-\frac{1}{R_0}\right)$ (d) $2GM\sqrt{\left(\frac{1}{R}-\frac{1}{R_0}\right)}$

58. If the first cosmic velocity of a satellite is v, its escape velocity is equal to

(a) $\frac{v}{\sqrt{2}}$ (b) $\sqrt{\frac{v}{2}}$ (c) $\sqrt{2}$ v (d) $\frac{\sqrt{2}}{v}$

59. The escape velocity of a body from Earth's surface is 11 km/s. What will be its value from a planet whose radius is double but density same as that of Earth?

(a) 11 km/s (b) 22 km/s (c) 33 km/s (d) 44 km/s

60. The escape velocity from a planet of mass M and radius R is given by

(a) $\sqrt{\frac{2GM}{R}}$ (b) $2\sqrt{\frac{GM}{R}}$ (c) $\sqrt{\frac{2MR}{G}}$ (d) $2\sqrt{\frac{R}{GM}}$

61. A missile is launched with a velocity less than the escape velocity. The sum of its K.E and P.E is

(a) positive.
(b) negative.
(c) zero.
(d) may be positive or negative depending upon its initial velocity.

62. The escape velocity of a projectile from the surface of Earth is approximately

(a) 7 km/s (b) 112 km/s (c) 11.2 km/s (d) 1.1 km/s

63. The weight of a body at Earth's surface is W. At a depth half way to the centre of Earth its weight will be

(a) W (b) $\frac{W}{2}$ (c) $\frac{W}{4}$ (d) $\frac{W}{8}$

64. A boy can jump 10 m vertically on the surface of Moon, where the value of acceleration due to gravity is 1/5 of the acceleration due to gravity on the surface of Earth. How high will he be able to jump on Earth?

(a) 9.8 m (b) 6 m (c) 5 m (d) 2 m

65. To what height should a body be raised so that the force of gravity on it is 16% of the corresponding value on the surface of Earth? (Radius of Earth = 6400 km)

(a) 6400 km (b) 1024 km (c) 9600 km (d) 2048 km

66. If the P.E. of a satellite close to the surface of Earth is equal to $-\frac{GMm}{R}$, its K.E is given by

(a) $-\frac{GMm}{R}$ (b) $\frac{GMm}{R}$ (c) $-\frac{GMm}{2R}$ (d) $\frac{GMm}{2R}$

67. If an artificial satellite revolves around Earth at a height 'h' above the surface of Earth then its orbital velocity is (g = acceleration due to gravity on the surface of Earth)

(a) $\frac{gR^2}{R+h}$ (b) $R^2\sqrt{\frac{g}{R+h}}$ (c) $\sqrt{\frac{gR^2}{R+h}}$ (d) $\sqrt{\frac{gM}{R+h}}$

68. If a satellite, very close to Earth, is in the first cosmic orbit, the corresponding period is

(a) 400 π s (b) 800 π s (c) 1200 π s (d) 1600 π s

69. The period of a geo-synchronous satellite is

(a) 1 month (b) 1 week (c) 1 year (d) 1 day

70. Orbital speed of an artificial satellite very close to Earth's surface is v_0. Its orbital speed at a height equal to three times the radius of Earth is

(a) $4v_0$ (b) $2v_0$ (c) $0.5v_0$ (d) v_0

71. Two satellites P and Q are in the same circular orbit around the Earth. The mass of P is greater than that of Q, it follows that

(a) P and Q have the same speed
(b) Speed of P is greater than that of Q
(c) Speed of P is less than that of Q
(d) KE of P is equal to that of Q

72. Geo-stationary satellite is one which

(a) remains stationary at a fixed height from the Earth's surface
(b) revolves like other satellites but in the opposite direction of Earth's rotation
(c) revolves round the Earth at a suitable height with same time period and in the same direction as Earth does about its own axis
(d) Both (a) and (c) are correct

73. The time period of a communication satellite in the parking orbit is :

(a) 24 hour (b) 365 day (c) 60 s (d) 12 hour

74. Which of the following cannot be used to measure time in a space ship orbiting the Earth?

(a) Atomic watch
(b) A watch using elastic vibrations
(c) Pendulum clock
(d) A watch using electric oscillations

75. A geostationary satellite is orbiting the Earth at a height of 6R above the surface R being radius of Earth. The time period of another satellite at a height 2.5 R from the surface is

(a) $6\sqrt{2}$ hour (b) 6 hour (c) 12 hour (d) 10 hour

76. A satellite is orbiting close to Earth. In order to make it escape the gravitational field of Earth, its velocity is to be increased by

(a) 20% (b) 41.4% (c) 0.414% (d) 33%

77. A satellite moving in a circular orbit of radius R_0 round the Earth of mass M and radius R has angular velocity

(a) $\sqrt{\frac{GM}{R_0^3}}$ (b) $\sqrt{GMR_0^3}$ (c) $\frac{R_0^3}{\sqrt{GM}}$ (d) $\sqrt{GMR_0^2}$

78. If the period of a satellite in a circular orbit of radius R_0 is T, the period of another satellite in a circular orbit of radius $4R_0$ is

(a) T/4 (b) T/8 (c) 4T (d) 8T

79. Time period of a satellite revolving around the Earth at a height h from the Earth's surface is given by

(a) $2\pi\sqrt{\frac{GM}{R^3}}$ (b) $2\pi\sqrt{\frac{(R+h)^3}{GM}}$ (c) $2\pi\sqrt{\frac{(R+h)^3}{2GM}}$ (d) $\frac{2\pi}{GMR^3}$

80. If v_e and v_0 represent the escape velocity and orbital velocity of a satellite corresponding to a circular orbit of radius R, then,

(a) $v_e = v_0$ (b) $v_e = \sqrt{2}.v_0$ (c) $v_e = \frac{1}{\sqrt{2}}v_0$ (d) v_e and v_0 are not related

81. The magnitude of the gravitational fields at distances d_1 and d_2 from the centre of a uniform solid sphere of radius r and mass m are F_1 and F_2. Then

(a) $\frac{F_1}{F_2} = \frac{d_1}{d_2}$ if $d_1 < r$ and $d_2 < r$ and $\frac{F_1}{F_2} = \frac{d_2^2}{d_1^2}$ if $d_1, d_2 > r$. (b) $\frac{F_1}{F_2} = \frac{d_1}{d_2}$ if $d_1, d_2 < r$; $\frac{F_1}{F_2} = \left(\frac{d_1}{d_2}\right)^{3/2}$ if $d_1, d_2 > r$.

(c) $\frac{F_1}{F_2} = \frac{d_1^2}{d_2^2}$ if $d_1, d_2 < r$. (d) $\frac{F_1}{F_2} = \frac{d_1^3}{d_2^3}$ if $d_1, d_2 < r$. and $\frac{F_1}{F_2} = \left(\frac{d_1}{d_2}\right)^{1/3}$ if $d_1, d_2 > r$

82. Two bodies of mass M each are separated by 2a. A small mass m placed midway, and is displaced slightly.

(a) axial displacement results in force which tend to restore it to original position.
(b) perpendicular displacement results in force which tend to restore it to original position.
(c) both axial and perpendicular displacement results in force which tend to restore it to original position.
(d) both are unstable, body moves away from the original position.

83. The height above the surface of Earth where 'g' decreases to 9% of its maximum value on the Earth's surface is (Radius of Earth = 6400 km)

(a) 14980 km (b) 14933 km (c) 14960 km (d) 14970 km

84. If different planets have the same density, but different radii, then g on a planet's surface is related to its diameter D as

(a) $\propto \frac{1}{D^2}$ (b) $\propto \frac{1}{D}$ (c) $\propto D$ (d) $\propto D^2$

85. The acceleration due to gravity of Earth at a depth d below the surface is equal to $\frac{g}{n}$, where g = acceleration due to gravity on the surface of Earth. If R is the radius of Earth, d is equal to

(a) $\left(\frac{n-1}{n}\right)R$ (b) $\frac{n}{R}$ (c) $\frac{R}{(n-1)}$ (d) $\frac{nR}{(n+1)}$

86. If the acceleration due to gravity on the surface of Earth is g and the radius of Earth, is R, the density of Earth is

(a) $\frac{g}{RG}$ (b) $\frac{\pi g}{RG}$ (c) $\frac{3g}{4\pi RG}$ (d) $\frac{gR}{G}$

87. A simple pendulum has a time period T_1 when at a distance R from Earth's surface and T_2 when taken to a height 2R above the first point. If R is radius of Earth, $\frac{T_1}{T_2}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{1}{16}$ (d) $\frac{1}{4}$

88. Distance between a planet and its natural satellite is 4×10^5 km. If the mass of the planet is 64 times the mass of the satellite, the distance from the centre of the planet where gravitational field of planet–satellite system vanishes is

(a) 3.57×10^8 m (b) 3.56×10^8 m (c) 3.6×10^8 m (d) 3.58×10^8 m

89. If the Earth was a solid sphere of iron of radius 6.5 million metre and density 7.8 g/cc and if Universal gravitational constant is taken as 6.5×10^{-8} in C.G.S units, 'g' at the surface of Earth would be

(a) $12.7\ m\ s^{-2}$ (b) $13.8\ m\ s^{-2}$ (c) $14.8\ m\ s^{-2}$ (d) $15.2\ m\ s^{-2}$

90. A uniform circular wire of mass M and radius 'a' has a unit mass placed at point A on the axis at a distance b from the ring's centre. The gravitational field at A is

(a) $\frac{GMa}{\sqrt{b^2+a^2}}$ (b) $\frac{GMb}{\sqrt{(b^2+a^2)^3}}$ (c) $\frac{GMb}{\sqrt{(b^2+a^2)}}$ (d) $\frac{GMb}{(b^2+a^2)}$

91. The gravitational potential of 2 homogenous spherical shells A and B of same material and of same thickness, at their respective centres are in the ratio 2 : 3. If these 2 shells combine to form a single shell C with the same original thickness the potentials at the centres of the shells are in the ratio ($V_A : V_B : V_C$)

(a) 4 : 9 :16 (b) 4 : 15 : 9 (c) 2:3 : $\sqrt{13}$ (d) 2 : 3 : 5

92. A point Q lies on the axis of a ring of mass M and radius a, at a distance a from the centre of the ring P. A small particle starts from Q and reaches P under gravitational attraction, its speed at P is

(a) $\sqrt{\frac{2GM}{a}}$ (b) $\sqrt{\frac{2GM}{a}\times 0.30}$ (c) $\sqrt{\frac{GM}{2a}\times 0.414}$ (d) $\sqrt{\frac{GM}{a}\times 0.6}$

93. A particle of mass m is transferred from the centre of a uniform solid hemisphere of mass M and radius D/2 to infinity. The work performed in the process by the gravitational force exerted on the particle by the hemisphere is

(a) $\frac{3GMm}{D}$ (b) $\frac{-3GMm}{D}$ (c) $\frac{-3}{2}\frac{GMm}{D}$ (d) $\frac{3}{4}\frac{GMm}{D}$

94. The energy necessary to move a satellite of mass m from a circular orbit of radius 2R to an orbit of radius 4R is (mass of Earth = M)

(a) $\frac{GMm}{4R}$ (b) $\frac{GMm}{2R}$ (c) $\frac{GMm}{8R}$ (d) $\frac{GMm}{R}$

95. A large number of particles of the same mass M are kept horizontally at distances of 2R, 8R, 32R etc from the origin. The total gravitational potential at the origin is

(a) $\frac{-GM}{R}$ (b) $\frac{-GM}{4R}$ (c) $\frac{-2}{3}\frac{GM}{R}$ (d) $\frac{-3}{2}\frac{GM}{R}$

96. Gravitational potential at the centre of a semi circle formed from a thin wire of mass 100 gram and radius 20 cm is

(a) $-3.5 \times 10^{-12}\ J\ kg^{-1}$ (b) $-3.3 \times 10^{-11}\ J\ kg^{-1}$ (c) $-3.6 \times 10^{-11}\ J\ kg^{-1}$ (d) $-3.8 \times 10^{-12}\ J\ kg^{-1}$

97. A body is raised first by a small distance z from the surface of the Earth and then by a distance R from the surface of Earth. The work done in each case is

(a) mgz, mgR (b) mgz, 2mgR (c) mgz, $\frac{mgR}{2}$ (d) mgz, $\frac{mgR}{4}$

98. Two bodies of masses 400 g and 900 g are placed at 10 cm apart. The gravitational potential where the gravitational field due to them is zero is given by (in $nJ\ kg^{-1}$)

(a) −1.6 (b) −1.67 (c) −1.65 (d) −1.7

99. A hole is dug along a diameter of the Earth, (densityρ). If a body is dropped into the tunnel, the acceleration at a distance x from the centre is

(a) $\frac{4}{3}\pi G\rho x^2$ (b) $\frac{4}{3}\frac{\pi G\rho}{x}$ (c) $\frac{4}{3}\pi G\rho x$ (d) $\frac{4}{3}\pi G\rho x^3$

100. The escape velocity from the surface of a solid spherical planet is v. A tunnel is dug along a diameter of the planet and a small body is dropped into it at the surface. When the body reaches the centre of the planet, its speed will be

(a) v (b) $\frac{v}{\sqrt{2}}$ (c) $\frac{v}{2}$ (d) $\sqrt{2}v$

101. A body is projected vertically upwards from a planet with an initial speed, which is half the escape velocity from the surface of the planet. The maximum height reached by the particle is (R = radius of the planet)

(a) 3R (b) 4R (c) $\frac{R}{3}$ (d) $\frac{R}{4}$

102. A particle is projected vertically upwards from the surface of Earth of radius R with a kinetic energy equal to 75% of the minimum value needed for it to escape. The height to which it rises above in the Earth is

(a) h = Re (b) h = 2Re (c) h = 3Re (d) h = −3Re

103. The escape velocity from the surface of a planet of mass M and radius R which is in the form of hollow shell is

(a) $\sqrt{\frac{GM}{R}}$ (b) $\sqrt{\frac{2GM}{R}}$ (c) $\sqrt{\frac{3}{2}\frac{GM}{R}}$ (d) $\sqrt{\frac{9}{5}\frac{GM}{R}}$

104. If the Earth is at one fourth of its present distance from the Sun, the duration of the year will be

(a) Half the present year (b) One-fourth of the present year
(c) One-eight of the present year (d) One-sixth of the present year

105. Gravitational field E outside a planet is given by $E(r) = \frac{-k}{r^n}$, k is a constant, and r is the distance from the centre of the planet. For two satellites of different circular orbits of radii r_1 and r_2, $\frac{r_1}{r_2} = \frac{1}{2}$ and $\frac{v_1}{v_2} = 2$. Then the value of n is

(a) 2 (b) 3 (c) 4 (d) 5

106. A satellite orbits the Earth in a circular orbit of diameter 16000 km, with period T. The time period of a monitoring satellite in the neighbourhood above the Earth's surface is

(a) $(0.85)^{3/2}$ T (b) $(0.81)^{3/2}$ T (c) $(0.9)^{3/2}$ T (d) $(0.8)^{3/2}$ T

107. A satellite is launched into a circular orbit of radius R around the Earth. A second satellite is launched into an orbit of radius 1.21R. The period of second satellite is larger than the first one by

(a) 21% (b) 33% (c) 11% (d) 42%

108. The force on a planet of mass m in a circular orbit of radius R is $\frac{\pi^2 m}{\sqrt{R}}$. The period is

(a) $2R^{3/4}$ (b) $\frac{R^{1/2}}{2}$ (c) $R^{3/2}$ (d) $R^{5/2}$

109. The diameters of the orbits of two satellites of the Earth are D and 4D. Their frequencies of rotation are in the ratio

(a) 8 (b) 16 (c) 4 (d) 12

110. Two small satellites move in circular orbits around the Earth at distances a and a + Δa from the centre of the Earth. If f is the frequency of the former

(a) $\Delta f = -\frac{3}{2} f \frac{\Delta a}{a}$ (b) $\Delta f = -\frac{2}{3} f \frac{a}{\Delta a}$

(c) $\Delta f = \frac{\Delta a}{af}$ (d) $\Delta f = f\left(\frac{2\Delta a}{a}\right)^3$

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

111. Statement 1

Escape velocity from the surface of Moon is much less than that from the surface of Earth.

and

Statement 2

Density of Moon is much less than that of Earth.

112. Statement 1

The apparent weight of a body in a satellite does not depend on the radius of orbit.

and

Statement 2

In the reference frame of the satellite force acting on the body is zero, irrespective of radius of orbit.

113. Statement 1

If a bolt on the outer surface of an orbiting satellite gets detached, it will never fall to Earth.

and

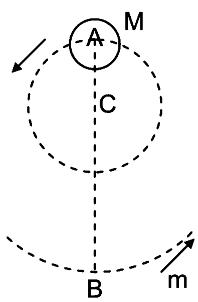
Statement 2

Angular momentum of a body in orbit is conserved

Linked Comprehension Type Questions

***Directions:* This section contains a paragraph. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Two masses, moving under the influence of mutual gravitational force, are in collision mode if they move in a straight line. Two body systems, such as a binary star, consists of two masses rotating about a common centre under mutual gravity. The adjacent figure shows the two masses, m and M, moving in circular orbits about the centre of mass C. The distance between the two AB = d, is constant.



114. The orbital radii of the masses m and M are, respectively

(a) $\dfrac{md}{m+M}, \dfrac{Md}{m+M}$ (b) $\left(\dfrac{m+M}{m}\right)d, \left(\dfrac{m+M}{M}\right)d$ (c) $\dfrac{Md}{m+M}, \dfrac{md}{m+M}$ (d) $\dfrac{dM}{m}, \dfrac{dm}{M}$

115. The total mechanical energy of the system is

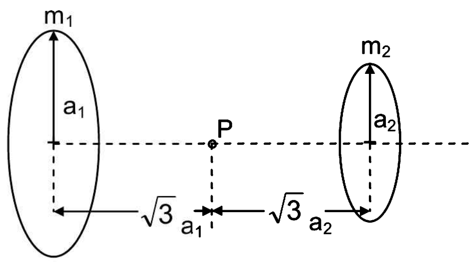
(a) $\dfrac{-GmM}{d}$ (b) $\dfrac{-GmM}{2d}$ (c) $\dfrac{GmM}{d}$ (d) $\dfrac{-2GmM}{d}$

116. The following quantities are conserved for the system.
(a) Momentum and total mechanical energy only
(b) Angular momentum about C and total mechanical energy only
(c) Momentum and angular momentum about C only
(d) Momentum, total mechanical energy and angular momentum about C

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

117. Two thin uniform rings of mass m_1 and m_2 and radii a_1 and a_2 respectively, kept coaxially as shown, create a zero field at P at a distance from the two centres as shown. Then



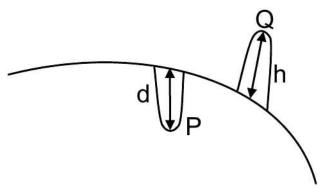
(a) $\dfrac{m_1}{m_2} = \left(\dfrac{a_1}{a_2}\right)^2$

(b) $\dfrac{m_1}{m_2} = \sqrt{\dfrac{a_1}{a_2}}$

(c) Potential at P = 0, if potential at infinity is $\dfrac{-G}{2}\left(\dfrac{m_1}{a_1} + \dfrac{m_2}{a_2}\right)$

(d) Modulus of potential at $P = \dfrac{G}{2}\left(\dfrac{m_1}{a_1} + \dfrac{m_2}{a_2}\right)$, if potential at infinity is zero.

118. P is a point at a depth $d = 0.02\ R_E$ ($R_E \rightarrow$ radius of Earth) and Q is a point at a height h, from the surface of Earth, where the g' value is same. Then



(a) $h = 0.01\ R_E$.
(b) Escape velocity at Q is 0.5 % less than escape velocity on surface of Earth.
(c) Escape velocity at P is 1% less than escape velocity on surface of Earth.
(d) Escape velocity at P is 2% less than escape velocity on surface of Earth.

119. A satellite revolving around the Earth in a circular orbit of radius r, is brought to a smaller circular orbit of radius $(r - \Delta r)$. If T_0, v_0 and E_0 are in the original period, velocity and modulus value of total energy, then after the change in radius : $(\Delta r << r)$

(a) period decreases by $\Delta T = T_0 \dfrac{3}{2}\dfrac{\Delta r}{r}$ (b) period increases by $\Delta T = T_0 \dfrac{3}{2}\dfrac{\Delta r}{r}$

(c) velocity increases by $\Delta v = v_0 \dfrac{1}{2}\dfrac{\Delta r}{r}$ (d) Total energy decreases by $\Delta E = E_0 \dfrac{1}{2}\dfrac{\Delta r}{r}$

Matrix-Match Type Question

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

120. Two concentric spherical shells of radii r_1 and r_2 ($r_2 > r_1$), each having mass m, have a point mass m at a distance r from the centre. Then match the items in column I to those in column II.

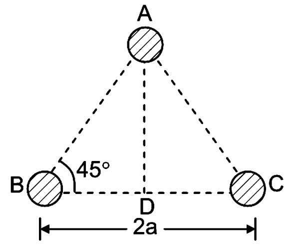
Column I	Column II
(a) Potential energy of point mass at $r < r_1$	(p) $-\dfrac{Gm^2}{r_2}+\dfrac{Gm^2}{r_1}$
(b) Potential energy of point mass at $r = r_1$	(q) $-\dfrac{Gm^2}{r}-\dfrac{Gm^2}{r_2}$
(c) Work done in taking the point mass m from $r = r_1$ to $r = r_2$	(r) $-Gm^2\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}\right)$
(d) Potential energy at r ($r_1 < r < r_2$)	(s) Work done in taking mass m from $r < r_1$ to $r = r_2$

ADDITIONAL PRACTICE EXERCISE

Subjective Questions

121. A double star is a system moving around the centre of inertia of the system due to gravitation. Calculate the distance between the components of the double star, if their total mass is 10 times that of Earth and the time period of revolution is $\sqrt{5}\times10^6$ second. ($g = 10$ m s^{-2}, radius of Earth $= 6.4 \times 10^6$ m, $\pi^2 = 10$).

122. Point masses m, 2m and an unknown mass are located at A, B, C respectively which form an isosceles triangle with $BC = 2a$ and $\angle B = 45°$. Median AD is a tangent to an equi potential curve at a point $\frac{a}{2}$ from D. Determine the gravitational field at that point in terms of G, m, a.



123. Two spherical, equal density stars, one of mass m, radius R and another mass 8m are approaching each other for a head-on collision. Initially, when their centres are at distance d, their speeds are negligible. When they collide, determine

(i) the velocity of the smaller star

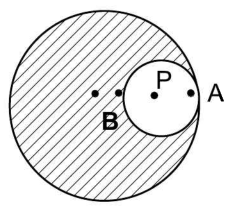
(ii) distance travelled by the smaller star from initial position.

124. The density ρ, at a distance r from the centre of a solid sphere of radius R, is given by the expression $\rho = \rho_0\sqrt{\frac{R}{r}}$, where ρ_0 is the density near the surface of the sphere (i.e., at radius R). Calculate the gravitational field and potential at.

(i) The surface of the sphere.

(ii) A point P, distant r from the centre of the sphere such that $r < R$.

125. Uniform sphere, radius R has a spherical cavity, of radius $\frac{R}{3}$, as shown. Determine the ratio of potentials $V_A : V_P : V_B$, treating potential at infinity as zero. (P is the centre of the cavity).



126. Determine the binding energy of a spherical planet whose density varies linearly as its distance from centre, with a density of 10^4 kg m^{-3} at the centre and density of 10 kg m^{-3} at its surface. Neglect gravitational effects of all other heavenly bodies. R = radius of planet. Also determine the escape velocity from the planet's surface.

127. Satellite A in a circular orbit around Earth, collides head-on with another satellite B which has been launched on the same orbit in opposite sense by mistake. The collision reduces the speed of A to half of its orbital speed. A hits the surface of Earth with same speed as what its orbital speed was. Radius of Earth = 6400 km. Determine the orbital radius.

128. A projectile is launched from Earth's surface at an angle of 37° with the horizontal, at an initial speed of $\sqrt{\dfrac{5GM}{4R}}$, where M = mass of Earth, R = radius of Earth. Determine the maximum height to which it will rise.

129. Two satellites are in circular orbits around the Earth, moving in the same sense of rotation, one at 2600 km above Earth's surface, other at 2700 km above Earth's surface. At t = 0, they are farthest from each other. Determine when they will be closest to each other next. Give the answer in nearest integral number of days.
Radius of Earth = 6400 km. Take $g = \pi^2$. Use Binomial approximation when applicable.

130. A satellite of mass m is in circular orbit around a planet with a total energy of $-E_{0.}$
(i) What is its escape velocity from the orbit?
(ii) How much work is to be performed to increase its time period n fold?

Straight Objective Type Questions

Directions: **This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

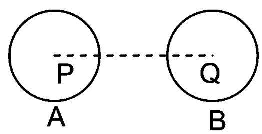
131. Acceleration due to gravity at a planet's surface in more than that at Earth's surface. But it is less at a small depth h below planet's surface than at the same h above Earth's surface. Then the ratio of radius of the planet to that of Earth is
(a) less than1/2 (b) between1/2and 1 (c) between; 1and 2 (d) more than2

132. If the radius of Earth shrinks, mass unchanged, then acceleration due to gravity at the surface will
(a) decrease (b) increase
(c) remain same (d) cannot be concluded

133. If the radius of the Earth shrinks, mass unchanged, then acceleration due to gravity at the some height above surface as earlier will be
(a) less (b) more
(c) same (d) cannot be concluded

134. Two pendulums are taken one upto a height $\frac{R}{2}$ and other upto a depth $\frac{R}{2}$ from the surface of Earth respectively. The respective change in their time periods is (R = radius of Earth)
(a) 50%, 40% (b) +50%, –40% (c) +50%, –100% (d) +50%, +100%

135. A and B are of identical masses and radii. A is a spherical shell, B a solid sphere. P and Q are two points on the line joining centres and located within A and B respectively and at equal distances from the respective centres. Comparing the magnitudes of gravitational fields at P and Q.



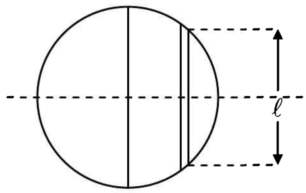
(a) they are equal (b) more at P
(c) more at Q (d) any of above is possible

136. Refer to the same data as per above question
Comparing the magnitudes of gravitational potentials at P and Q.
(a) they are equal (b) more at P
(c) more at Q (d) any of above is possible.

137. If the radius of Earth shrinks, mass unchanged, then acceleration due to gravity at the same depth below surface as earlier will be

(a) less
(b) more
(c) same
(d) cannot be concluded

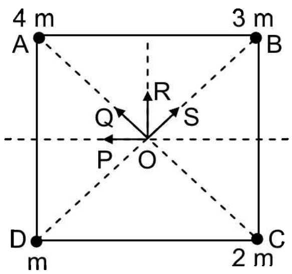
138.



A tunnel of length ℓ is dug parallel to a diameter of Earth and opening at two ends to the surface of Earth as shown. If a mass m is dropped at one end of the tunnel, its initial acceleration is proportional to ℓ^n where n is equal to

(a) 2
(b) 1
(c) $\frac{1}{2}$
(d) 0

139. 4 masses having values as shown are kept at the 4 corners of a square of side a. The resultant gravitational field at the center of the square is along the direction:

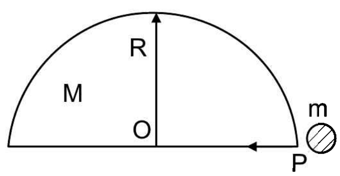


(a) $\overline{OP}$
(b) $\overline{OQ}$
(c) $\overline{OR}$
(d) $\overline{OS}$

140. Two large spheres made of material of densities ρ_1 and ρ_2 and having radii R_1 and R_2 and each having a tunnel made along the diameter (loss of mass due to tunnel is negligible) are taken to outer space and a unit mass is dropped into the tunnel. Their initial accelerations are in the ratio

(a) $\frac{R_1\rho_2}{R_2\rho_1}$
(b) $\sqrt{\frac{R_1\rho_2}{R_2\rho_1}}$
(c) $\frac{R_2\rho_2^2}{R_1\rho_1^2}$
(d) $\frac{R_1\rho_1}{R_2\rho_2}$

141. A point mass m is kept as shown, at the periphery of a hemisphere of mass M and radius R, in free space. If m is constrained to move along PO, its initial acceleration is



(a) $\frac{GM}{R^2}$
(b) $\frac{GM}{4R^2}$
(c) $\frac{2}{5}\frac{GM}{R^2}$
(d) $\frac{GM}{2R^2}$

142. Due to two concentric spherical shells, if P and Q are points in space with maximum magnitudes of gravitational potential and field respectively and if maximum and minimum possible distance PQ are in ratio 4:1, the ratio of the radii of the shells is .

(a) 5 : 3
(b) 4 : 1
(c) 5 : 2
(d) 3 : 1

143. A mass is taken vertically upwards from point P on surface of Earth to point Q, then to point R and then finally to infinity. Work done in each of these three steps is the same. Then $\frac{QR}{PQ}$ is

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 3

144. If the gravitational potential at the surface of the Earth is V (taking potential at infinity as zero), the escape velocity from the surface of Earth is

(a) $\sqrt{\frac{V}{2}}$ (b) $\sqrt{2V}$ (c) $\sqrt{-\frac{V}{2}}$ (d) $\sqrt{-2V}$

145. Potential at surface of a solid sphere is V. A concentric spherical cavity is made so that the potential the center is now V. The fraction of material taken out is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{3\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

146. If a spherical shell is expanded to a larger spherical shell, the gravitational potential at the centre will

(a) remain zero as before (b) be unchanged from its non-zero value
(c) decrease (d) increase

147. A hole is drilled along the diameter of Earth. From what height above the hole should a body be released so that it has a speed of 11.2 km s^{-1} at the center? (R = radius of Earth)

(a) ∞ (b) $\frac{R}{2}$ (c) R (d) 2R

148. A body falls from rest on to the Earth vertically with a speed of $\frac{2}{\sqrt{5}}$ v_{esc}. It must have travelled a distance of (R = radius of Earth)

(a) 2R (b) 3R (c) 4 R (d) 5 R

149. A uniform sphere of mass m and radius 2a is at the origin. Then, among the following equipotential is

(a) The curve $x^2 + y^2 = a^2$ in the plane $z = 0$ (b) The curve $x^2 + y^2 = a^2$ in the plane $z = 3a$
(c) The surface $x^2 + y^2 + z^2 = 3a^2$ (d) All of the above

150. A planet has density, escape velocity and radius, of magnitudes $\frac{1}{2}$, 2 and x times respectively as those of Earth. x is

(a) $2\sqrt{2}$ (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2\sqrt{2}}$

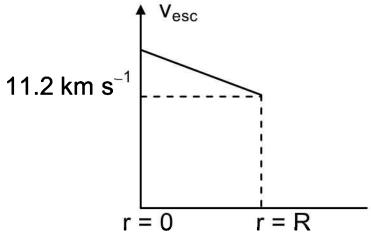
151. If a body on the surface of Earth is provided a velocity more than escape velocity at an angle to the vertical, the body will escape

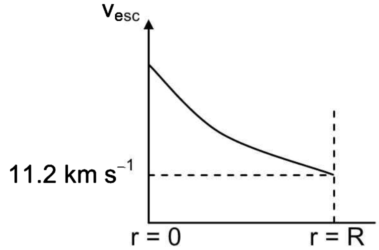
(a) only if the vertical component of velocity is more than escape velocity
(b) only if the angle of projection to horizontal is more than 45°
(c) always
(d) never

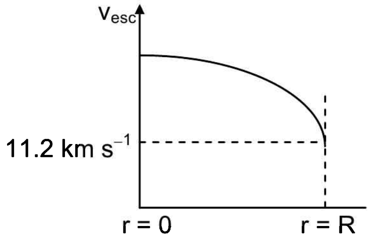
152. Escape velocity of a body at altitude $\frac{R}{2}$ is how many times that at the surface?

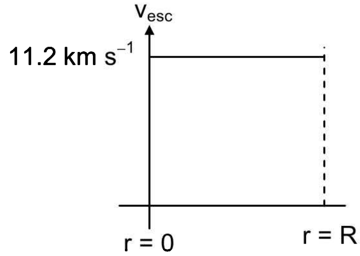
(a) $\sqrt{\frac{3}{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

153. A hole is drilled along the diameter of the Earth. The plot of escape velocity vs distance from the centre of the Earth is given by.

(a) 

(b) 

(c) 

(d) 

154. When a satellite in circular orbit around Earth, with period T, is suddenly stopped and released it will fall towards earth with an initial acceleration = KT^n where n is

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{-2}{3}$ (d) $\frac{-4}{3}$

155. In the above question, the proportionality constant K is

(a) $(2\pi)^{1/3}(GM)^{2/3}$ (b) $(2\pi)^{2/3}(GM)^{1/3}$ (c) $(2\pi)^{4/3}(GM)^{1/3}$ (d) $(2\pi)^{4/3}(GM)^{2/3}$

156. If a satellite is in a circular orbit around the Earth so that its orbital velocity is equal to half the velocity of escape (from Earth's surface), its altitude is (if R is radius of Earth)

(a) $\frac{R}{2}$ (b) R (c) $\frac{3}{2}R$ (d) 2R

157. A satellite of mass m is in a circular orbit of time period T and with angular momentum L. The radius of orbit is

(a) $\sqrt{\frac{TL}{2\pi m}}$ (b) $\frac{T}{2\pi}\sqrt{\frac{L}{m,}}$ (c) $\frac{L}{m}\sqrt{\frac{T}{2\pi}}$ (d) $\frac{L^2}{2m}\sqrt{\frac{T}{2\pi}}$

158. If $\overline{p}$ denotes linear momentum and $\overline{L}$ 'the angular momentum, for a satellite in elliptical orbit, the quantities which vary with time are

(a) $|\overline{p}|$ and $\overline{p}$ only (b) $|\overline{p}|$, $\overline{p}$ and $\overline{L}$ only (c) $|\overline{p}|$, $\overline{p}$, $|\overline{L}|$ and $\overline{L}$ (d) $\overline{p}$ and $\overline{L}$ only.

159. A satellite is in circular orbit of radius r with angular velocity ω around a planet whose mass is

(a) $\frac{\omega^2 r^2}{G}$ (b) $\frac{\omega^2 r^3}{G}$ (c) $\frac{\omega^2 r^4}{G}$ (d) $\frac{\omega^2 r^5}{G}$

160. A planet A has its day, radius and synchronous orbit radius, each of these twice as that of another planet B. The ratio of the densities of A and B is.

(a) 1 : 8 (b) 1 : 4 (c) 4 : 1 (d) 8 : 1

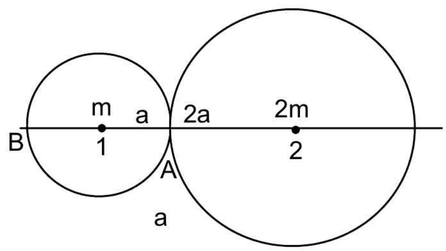
161. A satellite is in circular orbit around Earth. At a particular instant, its velocity is doubled by firing a rocket. Then
(a) it will go into a higher circular orbit.
(b) it will go into elliptical orbit with initial point as nearest to Earth.
(c) it will go into elliptical orbit with initial point as farthest to Earth.
(d) None of the above.

162. Two planets in elliptical orbits have same angular momentum and same time period. Their areas of orbit will be in
(a) direct proportion of their masses.
(b) inverse proportion of their masses.
(c) direct proportion of square of their masses.
(d) inverse proportion of square of their masses.

163. Two satellites are in elliptical orbits, area of the two orbits being equal. Then time period will be
(a) more for the satellite with longer minor axis.
(b) more for the one with shorter minor axis.
(c) same.
(d) more for the one with larger product of major and minor axes.

164. How many of the following quantities remain constant during elliptical motion of a planet around the sun?
Angular speed, angular momentum, kinetic energy
(a) nil (b) one (c) two (d) three

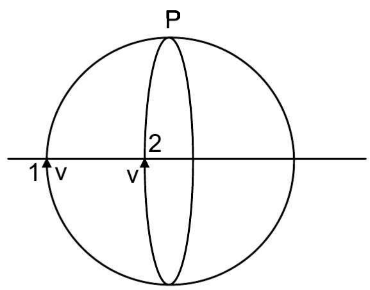
165. A satellite is revolving around planet 1 of mass m in a circular orbit of radius 'a'. Another satellite is revolving around planet 2 of mass 2m in a circular orbit of radius 2 a. If the two orbits touch each other at A, the ratio of the gravitational potentials at A and a diametrically opposite point B is



(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) 4

166. The shape of the Orbit of a comet is
(a) Parabolic (b) Hyperbolic (c) Elliptical (d) All of the above

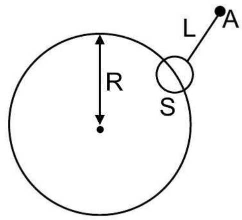
167. Satellites 1 and 2 are moving in circular orbits such that their orbital planes are circular (each of radius R) but mutually perpendicular to each other. At the instant shown, they are equidistant from P and move with the same speed v, in the directions shown. The magnitudes of their minimum and maximum relative velocities are



(a) $\sqrt{2}$ v, $\sqrt{2}$ v
(b) 0, 2 v
(c) 0, $\sqrt{2}$ v
(d) $\sqrt{2}$ v, 2 v

168. The speed of rotation of Earth increases such that the height of geostationary satellites is R. An object on the equator, originally weighing W, will now weigh (R = radius of Earth)
(a) W (b) $\frac{W}{2}$ (c) $\frac{7}{8}$W (d) $\frac{3}{4}$W

169. A cord of length 2m is used to connect an astronaut to a massive spaceship. If the radius of the orbit is R and the spaceship orbits the centre of a uniform spherical gas cloud, and the weight of the astronaut (if stationary at the point) is W, the tension in the rope is



(a) $W.\frac{L}{R}$ (b) W

(c) $\frac{3WL}{R}$ (d) 0

170. A satellite orbits the Earth at a height of 900 km above the surface. (Radius of Earth is 6400 km.) Its period of revolution is (in second). Take g on Earth's surface = 10 m s^{-2}

(a) 1800 π (b) 1949 π (c) 1600 π (d) 2100 π

Assertion–Reason Type Questions

***Directions:* Each question contains Statement-1 and Statement-2 and has the following choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True

171. Statement 1

If the radius of a solid sphere is decreased to half without change in density, the slope of gravitational field vs distance from centre for points inside the sphere will not change.

and

Statement 2

Gravitational field at any point on the surface of an imaginary sphere is not dependant on mass lying outside the imaginary sphere.

172. Statement 1

The acceleration due to gravity at a point below and at a point above the surface of Earth can be same if they are equidistant from the surface.

and

Statement 2

The decrease in g is linear in both cases but slopes are different.

173. Statement 1

Consider the rotation of Earth, the angle between the radius vector $\vec{R}$ and the acceleration due to gravity vector $\vec{g}$ will continuously increase from pole to equator.

and

Statement 2

g is maximum at pole and minimum equator.

174. Statement 1

If variation of g is taken into consideration, maximum height of a projectile is more than $\frac{u^2 \sin^2\theta}{2g}$.

and

Statement 2

As g decreases with height work done by gravity in upward motion is less negative than the case with constant g.

175. Statement 1

If variation of g is taken into consideration, range of a projectile will be more than $\frac{u^2 \sin 2\theta}{g}$.

and

Statement 2

Time of flight increases in this case.

176. Statement 1

Escape velocity is independent of angle of projection.

and

Statement 2

Energy is a scalar

177. Statement 1

The path of a projectile is an ellipse if variation of g is taken into consideration.

and

Statement 2

If perigee is less than radius of earth, it falls to ground.

178. Statement 1

Work done by gravity on a satellite during any time interval is zero whether it is circular or elliptical orbit.

and

Statement 2

Gravitational force is a central force.

179. Statement 1

If two satellites nearly miss colliding when each of them are at their apogees, then their time periods are same.

and

Statement 2

Square of period is proportional to cube of semi major axis.

180. Statement 1

When a body is projected with a velocity less than the escape velocity, the body will reach maximum altitude if its initial angular momentum about Earth's centre is zero.

and

Statement 2

Both initial angular momentum about centre of Earth and total energy are conserved.

Linked Comprehension Type Questions

***Directions:* This section contains 3 paragraphs. Based upon the paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.**

Passage I

Consider Earth as a uniform solid sphere.

Consider Earth's rotation about its own axis.

Speed of a point on equator is v_0.

Escape velocity at pole is v_e.

Point P is above pole. Accelerations due to gravity at P and on the surface of Earth at latitude 45° are equal. A stone dropped from P will hit Earth with speed.

Case (i)

v' if we assume g to be constant and equal to the value at the surface of Earth.

and

Case (ii)

v", otherwise.

181. v' =

(a) $\frac{v_0}{2}$ (b) $\frac{v_0}{\sqrt{2}}$ (c) $\sqrt{2}\,v_0$ (d) $2\,v_0$

182. (a) v' = v" (b) v' > v" (c) v' < v" (d) cannot be concluded

183. v" will differ from v' by a fraction

(a) zero (b) $\frac{v_0}{v_e}$ (c) $\left(\frac{v_0}{2v_e}\right)^2$ (d) $\frac{v_0}{2v_e}$

Passage II

It is proposed to put a satellite of mass m in an elliptical orbit around Earth (Radius of Earth = R, acceleration due to gravity at Earth's surface = g). Such that its mechanical energy is E and angular momentum about centre of earth is L. (Gravitational potential energy at infinity is zero). The mission is successful if it does not hit Earth during its motion.

184. The necessary and sufficient condition for success of mission is

(a) E should be negative (b) $m^3g^2R^4 + 2EL^2 > 0$ (c) $E + mgR < \frac{L^2}{2mR^2}$ (d) $E > \frac{L^2}{2mR^2} - mgR$

185. If $E = -\frac{mgR}{12}$ and $L = m\sqrt{2gR^3}$, then the closest distance from surface is

(a) $\left(3-\sqrt{3}\right)R$ (b) $\left(6-2\sqrt{6}\right)R$ (c) $\left(3-\sqrt{2}\right)R$ (d) $\left(2-\sqrt{3}\right)R$

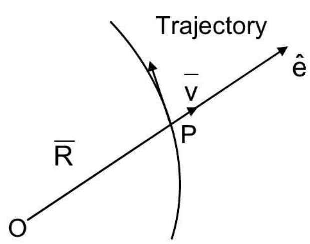
186. With data as per above question, time period of the satellite is

(a) $6\sqrt{3}\pi\sqrt{\frac{R}{g}}$ (b) $3\sqrt{3}\pi\sqrt{\frac{R}{g}}$ (c) $12\sqrt{6}\pi\sqrt{\frac{R}{g}}$ (d) $4\sqrt{2}\pi\sqrt{\frac{R}{g}}$

Passage III

The motion of an object under the action of a force, which is always directed towards or away from a fixed point, is called central force motion. Gravitational force is one such example. Consider the motion of an object P of mass m with respect to a fixed point O, called inertial reference.

Let $\overline{R}$ be the position vector, $\overline{v}$ the instantaneous velocity vector and $\hat{e}$, the radial direction.



187. The force $\overline{F}$ acting on the object can be expressed as ($\left|\overline{R}\right| = r, \phi(r)$ = function of r)

(a) $\overline{F} = \varphi(r)\frac{\overline{v}}{\left|\overline{v}\right|}$ (b) $\overline{F} = \varphi(r)\hat{e}$

(c) $\overline{F} = -\frac{GmM}{r^2}\hat{e}$ (d) Both (b) and (c) are acceptable

188. The torque of the force $\overline{F}$ about O is zero as the line of action of $\overline{F}$ passes through O. That is $\overline{R} \times \overline{F} = 0$. It can be concluded that

(a) $\overline{R}.m\overline{V}$ = constant (b) $m\overline{V}$ = constant (c) $\overline{R} \times m\overline{V} = 0$ (d) $\overline{R} \times m\overline{V}$ = constant

189. Under central force motion, the trajectory of the object

(a) may be a straight line (b) must be a circle
(c) may be any curve that lies in a fixed plane (d) Both (a) and (c)

Multiple Correct Objective Type Questions

Directions: **Each question in this section has four suggested answers out of which ONE OR MORE answers will be correct.**

190. P is a point on the axis of a ring centered at O. The gravitational field at P does not change its magnitude but changes direction by 90° when a spherical shell of same mass and same radius as the ring is placed in the region with centre at O'. Then,

(a) $O'P \geq R$ (b) $O'P < R$
(c) O'PO is an acute angled triangle (d) O'PO is obtuse

191. A satellite revolves with time period T in a circular orbit of radius R_2 around a planet of radius R_1. Then

(a) Energy of the satellite is negative
(b) The velocity of the satellite is definitely less than half of the escape velocity from surface of planet
(c) The velocity of the satellite can be more than $\frac{1}{\sqrt{2}}$ times the escape velocity from surface of planet
(d) The acceleration due to gravity at surface of planet is $\frac{4\pi^2}{T^2}.\frac{R_2^{\,2}}{R_1^{\,2}}$

192. A body is projected vertically from the surface of Earth with kinetic energy K_1 and when it comes to momentary rest, it is given a horizontal velocity so that its kinetic energy is K_2 and it goes into circular orbit. Then

(a) For escape from surface kinetic energy required is $K_1 + 2K_2$
(b) The altitude of the orbit is $\left(\frac{K_1}{2K_2}\right)$ times radius of earth
(c) If initially, the body had been projected horizontally with a kinetic energy of $\frac{K_1}{2} + K_2$, it would have gone into circular orbit
(d) $K_1 > K_2$

193. Due to a solid sphere, the gravitational

(a) fields at some point inside and some point outside can be same
(b) potentials at some point inside and some point outside can be same
(c) field at centre is zero
(d) potential at centre is zero

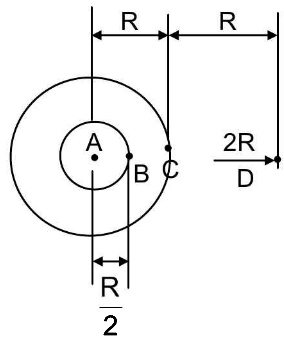
194. Due to a spherical shell, the gravitational

(a) fields at some point inside and some point outside can be same
(b) potentials at some point inside and some point outside can be same
(c) field at centre is zero
(d) potential at centre is zero

195. A satellite of earth
(a) must be above a pole at some time
(b) must be above equator at some time
(c) cannot have time period less than $2\pi\sqrt{\frac{R}{g}}$
(d) cannot have a time period more than that of a geostationary satellite

196. A geostationary satellite
(a) need not be above equator at all times
(b) has an orbit radius of about 36,000 km
(c) cannot have an elliptical orbit
(d) has a period of one day

197. A solid sphere of mass M and radius $\frac{R}{2}$ is kept concentric with a shell of equal mass and radius R.



(a) The fields at A, B, C, D are in arithmetic progression.
(b) The fields at A, B, C, D are in geometric progression.
(c) The potentials at A, B, C, D are in arithmetic progression.
(d) The potentials at A, B, C, D are in geometric progression.

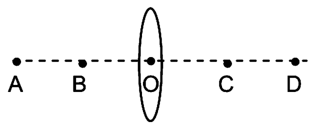
Matrix-Match Type Questions

Directions: **Match the elements of Column I to elements of Column II. There can be single or multiple matches.**

198. Match the columns.

Column I	Column II
(a) Satellite in circular orbit around Earth	(p) Potential energy varies
(b) Satellite in elliptical orbit around Earth	(q) Kinetic energy varies
(c) Body projected vertically upwards from the surface of Earth	(r) Mechanical energy is constant
(d) A payload with multistage rocket in motion in space	(s) Angular momentum about centre of Earth is constant

199.



Ring of radius R, centre O, ABCD is the axis. AB = BO = OC = CD = $\frac{R}{\sqrt{2}}$
Gravitational potential due to the ring is taken as zero at infinity.
Match the columns.

Column I	Column II
(a) Moving from A to B	(p) Magnitude of gravitational field increases
(b) Moving from B to O	(q) Magnitude of gravitational field decreases
(c) Moving from O to C	(r) Gravitational potential increases
(d) Moving from C to D	(s) Gravitational potential decreases

200. Consider the following parameters related to the motion of a satellite: height of the satellite above the surface of Earth = h; mass of satellite = m; acceleration due to gravity at height h = g(h); acceleration due to gravity on the surface of Earth = g(0); radius of Earth = R; mass of Earth = M; angular frequency of satellite = ω; then, match the following:

Column I	**Column II**
(a) $\left\lvert\dfrac{g(0)}{g(h)}\right\rvert^{1/2}$	(p) $\left\lvert\dfrac{\text{escape velocity from surface of earth}}{\text{velocity of satellite in orbit}}\right\rvert^{2}$
(b) $\dfrac{2(R+h)}{R}$	(q) $\dfrac{R+h}{R}$
(c) $\dfrac{\text{Energy of satellite on the launch pad}}{\text{Energy of a satellite in orbit}}$	(r) $\dfrac{\text{kinetic energy to be imparted to a body to pull it out of earth's gravity}}{\text{additional kinetic energy to be imparted to pull a satellite out of its orbit beyond earth's gravity}}$
(d) $\dfrac{\text{Potential due to earth on its surface}}{\text{Potential due to earth at a height of }(R+h)}$	(s) $\sqrt{\dfrac{g(0)}{\omega^2(R+h)}}$

ANSWERS KEYS

Topic Grip

1. (i) $g_0 - \omega^2 R$

(ii) $\dfrac{g_0}{\left(1+\dfrac{h}{R}\right)^2}$

(iii) $\dfrac{g_0}{\left(1+\dfrac{h}{R}\right)^2}$,

$$h = \left[\sqrt{\frac{g_0}{g_0 - \omega^2 R}} - 1\right] R$$

$$h' = \frac{\omega^2 R}{\left[\dfrac{2g_0}{R} - \omega^2\right]}$$

2. 46, 656 km

3. (i) $E = \dfrac{Gm\,(r-x)}{r^3}$

(ii) $E = 0$

(iii) $E = \dfrac{Gm(x-r)}{r^3}$

(iv) $E = \dfrac{Gm}{r^2}$

(v) $E = \dfrac{Gm}{(x-r)^2}$

(vi) $E = \dfrac{Gm}{(x-r)^2} + \dfrac{Gm\left[(x-R_1)^2 - R_1^2\right]}{\left[R_2^1 - R_1^2\right]}$ unit

(vii) $E = \dfrac{Gm}{(R_2 + R_1 - r)^2} + \dfrac{GM}{R_2^2}$

(viii) $E = \dfrac{Gm}{(x-r)^2} + \dfrac{GM}{(x-R_1)^2}$

4. $\overline{g} = G\dfrac{4}{3}\pi\rho R$

$$\left[\left(\frac{1}{4} + \frac{1}{5\sqrt{5}}\right)\check{i} + \frac{1}{10\sqrt{5}}\check{j}\right] \text{unit}$$

5. $E = \dfrac{\pi}{3} \cdot \dfrac{Gm}{\ell^2}\hat{j}$

$$v = -\frac{\pi}{3}\frac{Gm}{\ell}$$

6. $\left(\dfrac{4+\sqrt{2}}{3}\right)\dfrac{Gm^2}{a}$ unit

7. 206 km

8. 13500 m s^{-1}

9. $(\Delta)r = -\dfrac{4\pi f r^3}{GMm}$,

$$t = \sqrt{GM}\,\frac{m}{f}\frac{1}{\sqrt{r}}\left(\sqrt{2}-1\right)$$

10. $T = \dfrac{2\pi d^{3/2}}{\sqrt{G\,(m_1 + m_2)}}$

11. (c)	**12.** (b)	**13.** (c)
14. (c)	**15.** (b)	**16.** (b)
17. (d)	**18.** (b)	**19.** (a)
20. (d)	**21.** (d)	**22.** (c)
23. (b)	**24.** (c)	**25.** (d)

26. (b)
27. (a), (c), (d)
28. (a), (b), (d)
29. (a), (d)
30. (a) – (q)
(b) – (p), (s)
(c) – (r)
(d) – (p), (s)

IIT Assignment Exercise

31. (d)	**32.** (c)	**33.** (c)
34. (d)	**35.** (b)	**36.** (d)
37. (c)	**38.** (c)	**39.** (c)
40. (d)	**41.** (a)	**42.** (d)
43. (d)	**44.** (b)	**45.** (c)
46. (d)	**47.** (c)	**48.** (d)
49. (d)	**50.** (d)	**51.** (c)
52. (c)	**53.** (b)	**54.** (c)
55. (a)	**56.** (b)	**57.** (b)
58. (c)	**59.** (b)	**60.** (a)
61. (b)	**62.** (c)	**63.** (b)
64. (d)	**65.** (c)	**66.** (d)
67. (c)	**68.** (d)	**69.** (d)
70. (c)	**71.** (a)	**72.** (c)
73. (a)	**74.** (c)	**75.** (a)
76. (b)	**77.** (a)	**78.** (d)
79. (b)	**80.** (b)	**81.** (a)
82. (b)	**83.** (b)	**84.** (c)
85. (a)	**86.** (c)	**87.** (b)
88. (b)	**89.** (b)	**90.** (b)
91. (c)	**92.** (d)	**93.** (b)
94. (c)	**95.** (c)	**96.** (b)
97. (c)	**98.** (b)	**99.** (c)
100. (b)	**101.** (c)	**102.** (c)
103. (b)	**104.** (c)	**105.** (b)
106. (d)	**107.** (b)	**108.** (a)
109. (a)	**110.** (a)	**111.** (b)
112. (a)	**113.** (b)	**114.** (c)
115. (b)	**116.** (d)	

117. (a), (d)
118. (a), (b)
119. (a), (c), (d)
120. (a) – (r)
(b) – (q), (r)
(c) – (p), (s)
(d) – (q)

Additional Practice Exercise

121. 8×10^8 m

122. $\dfrac{8\left(5\sqrt{5}-4\right)}{5\sqrt{5}}\dfrac{Gm}{a^2}\check{i}, \hat{i}$ along DC

123.

(i) $\dfrac{8}{3}\sqrt{\dfrac{2}{3}}\sqrt{\dfrac{Gm\,(d-3R)}{Rd}}$

(ii) $\dfrac{8}{9}(d-3R)$

124. (i) $E_R = \dfrac{8}{5}\pi G\rho_0 R$,

$$V_R = -\frac{8}{5}\pi G\rho_0 R^2$$

(ii) $E_r = \dfrac{8}{5}\pi G\rho_0\sqrt{R}\,r^{1/2}$,

$$v_r = -\frac{8}{3}\pi G\rho_0 r^{\pm} + \frac{16}{15}\pi Gp\sqrt{R}\,v^{\pm}$$

125. 4 : 5 : 6

126. $\frac{152}{7}.10^6\,\pi^2\,GR^5;\ 40R\sqrt{10\pi G}$

127. 8800 km

128. $\left(\frac{1+4\sqrt{0.4}}{3}\right)R$

129. 3 days

130. (i) $2\sqrt{\frac{E_0}{m}}$

(ii) $E_0\left[1-\frac{1}{n^{2/3}}\right]$

131. (a) **132.** (b) **133.** (b)
134. (a) **135.** (d) **136.** (c)
137. (d) **138.** (b) **139.** (c)
140. (d) **141.** (a) **142.** (a)
143. (d) **144.** (d) **145.** (b)
146. (d) **147.** (c) **148.** (c)
149. (d) **150.** (a) **151.** (c)
152. (d) **153.** (c) **154.** (d)
155. (c) **156.** (b) **157.** (a)
158. (a) **159.** (b) **160.** (b)
161. (d) **162.** (b) **163.** (b)
164. (b) **165.** (b) **166.** (d)
167. (c) **168.** (c) **169.** (d)
170. (b) **171.** (a) **172.** (c)
173. (d) **174.** (c) **175.** (a)
176. (a) **177.** (b) **178.** (d)
179. (d) **180.** (a) **181.** (b)
182. (b) **183.** (c) **184.** (c)
185. (b) **186.** (c) **187.** (d)
188. (d) **189.** (d) **190.** (a), (d)
191. (a), (d)
192. (a), (b), (c)
193. (a), (c)
194. (c)
195. (b), (c)
196. (c), (d)
197. (c)

198. (a) → (r), (s)
(b) → (p), (q), (r), (s)
(c) → (p), (q), (r), (s)
(d) → (p), (q)

199. (a) → (p), (s)
(b) → (q), (s)
(c) → (p), (r)
(d) → (q), (r)

200. (a) → (q), (s)
(b) → (p), (r)
(c) → (p), (r)
(d) → (q), (s)

HINTS AND EXPLANATIONS

Topic Grip

1. $T = 2\pi\sqrt{\frac{\ell}{g}}$

g at equator $= g_0 - \omega^2 R$

g at altitude h, $= \dfrac{g_0}{\left(1+\frac{h}{R}\right)^2}$

g at depth h at 45° latitude

$$= g_0\left(1-\frac{h'}{R}\right) - \omega^2\left(R-h'\right)\cos^2 45°$$

$$= g_0\left(1-\frac{h'}{R}\right) - \frac{\omega^2(R-h')}{2}$$

$$= g_0 - g_0.\frac{h'}{R} - \frac{\omega^2 R}{2} + \frac{\omega^2 h'}{2}$$

$\therefore$ Equations are

$$g_0 - \omega^2 R = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

$$\Rightarrow \left(1+\frac{h}{R}\right)^2 = \left(\frac{g_0}{g_0-\omega^2 R}\right)$$

$$\Rightarrow h = \left[\sqrt{\frac{g_0}{g_0-\omega^2 R}} - 1\right]R$$

$$g_0 - \omega^2 R = g_0 - g_0\frac{h'}{R} - \frac{\omega^2 R}{2} + \frac{\omega^2 h'}{2}$$

$$\Rightarrow \frac{\omega^2 R}{2} = \frac{g_0}{R}h' - \frac{\omega^2 h'}{2} = \frac{h'}{2}\left(\frac{2g_0}{R} - \omega^2\right)$$

$$\Rightarrow h' = \frac{\omega^2 R}{\left[\frac{2g_0}{R} - \omega^2\right]}$$

2. $T_p = \dfrac{T_e}{4} \Rightarrow \omega_p = 4.\omega_e$ — .(1)

$g_{0,p}$ (at pole of planet) $= 0.2\, g_0$ — (2)

$g_{0,p} - \omega_p^2 . R_p . \cos^2 60^0$

$= 0.1\,(g_0 - \omega_e^2 R_e \cos^2 60^0)$

$$\Rightarrow g_{0,p} - \frac{\omega_p^2 .Rp}{4} = 0.1 g_0 - \frac{0.1\,\omega_e^2 .R_e}{4}$$

$\Rightarrow$ Using (1) and (2)

$$0.2\,g_0 - 0.1\,g_0 = 4\,\omega_e^2 R_p - \frac{0.1\,\omega_e^2 .R_e}{4}$$

$$\Rightarrow R_p = \frac{1}{4}\left[\frac{0.1 g_0}{\omega_e^2} + 0.025\,R_e\right]$$

Using $g_0 = \pi^2$, $R_e = 6.4 \times 10^6$m

And $\omega_e = \dfrac{2\pi}{T}$ where T = 86400 s

$$\Rightarrow R_p = \frac{1}{4}\left[\frac{\pi^2}{10}\frac{(86400)^2}{4\pi^2} + 0.025 \times 6.4 \times 10^6\right]$$

We get $R_p = 46{,}696 \times 10^3$m = 46,696 km

3.

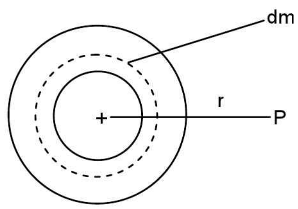
(a) Some results useful for solving the problem :

Field inside a hollow sphere = 0

(b) Field inside a solid sphere at x from centre is $= \dfrac{GMx}{R^3}$ where M = mass of sphere,

R = radius of sphere.

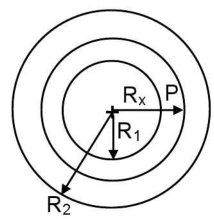
(c)



Field due to a thick spherical shell, at P at a distance r from centre. Consider a thin shell of mass dm, within. The field due to this is $\dfrac{Gdm}{r^2}$.

$$\text{Total field at P} = \int\frac{Gdm}{r^2} = \frac{G}{r^2}\int dm = \frac{GM}{r^2}$$

(d) Field at intermediate point P on the thickness, of the hollow sphere : at a distance R_x.



Field E_p =

$$\frac{GM'}{R_x^2} \Rightarrow M' \text{ is the mass up to radius } R_x$$

$$M' = \frac{4}{3}\pi R_x^3\rho - \frac{4}{3}\pi R_1^3\rho;\ M = \frac{4}{3}\pi R_2^3\rho - \frac{4}{3}\pi R_1^3\rho$$

$$\frac{M'}{M} = \frac{R_x^3 - R_1^3}{R_2^3 - R_1^3} \Rightarrow M' = M\frac{R_x^3 - R_1^3}{R_2^3 - R_1^3}$$. At P, the field due to the outer layer is zero. Hence field at P

$$E_p = \frac{GM'}{R_x^2} = \frac{GM}{R_x^2}\frac{R_x^3 - R_1^3}{R_2^3 - R_1^3}$$

(e) Gravitational filed due to several masses is vector sum of the field due to individual masses.

Hence

Case (i)

$$E = \frac{Gm(r-x)}{r^3}$$

Case (ii)

$E = 0$

Case (iii)

$$E = \frac{Gm(x-r)}{r^3}$$

Case (iv)

$$E = \frac{Gm}{r^2}$$

Case (v)

$$E = \frac{Gm}{(x-r)^2}$$

Case (vi)

$$E = \frac{Gm}{(x-r)^2} + \frac{GM\left[(x-R_1)^3 - R_1^3\right]}{\left[R_2^3 - R_1^3\right](x-R_1)^2}$$

Case (vii)

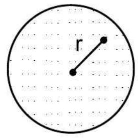
$$E = \frac{Gm}{(R_2 + R_1 - r)^2} + \frac{GM}{R_2^2}$$

Case (viii)

$$E = \frac{Gm}{(x-r)^2} + \frac{GM}{(x-R_1)^2}$$

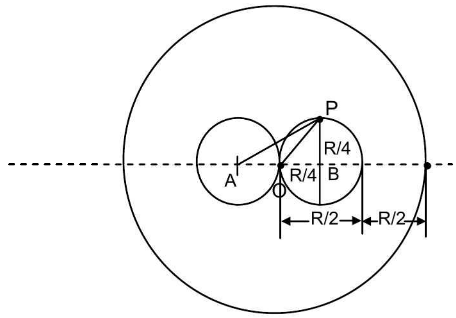
4. The gravitational field at a point inside a solid sphere, distant r from centre of a solid sphere, is given by

$$g = G.\frac{4}{3}\frac{\pi r^3\rho}{r^2}$$



(where ρ is density) and is directed towards centre.

$$\Rightarrow \bar{g} = -G.\frac{4}{3}\pi\rho.\bar{r}$$



Using this, $\bar{g}$ at P = $\bar{g_1} + \bar{g_2} + \bar{g_3}$ where $\bar{g_1}$ due to solid sphere ρ, R, directed along PO, $\bar{g_2}$, due to solid sphere (3ρ – ρ), R/4, directed along PA, $\bar{g_3}$, due to sphere (–ρ), R/4 directed along PB[The hollow part can be considered as a negative mass of density –ρ superimposed on the existing mass, so that the density at the hollow region is zero]

$$|\bar{g_1}| = G.\frac{4}{3}\pi\rho(PO) = G.\frac{4}{3}\pi\rho.\frac{R}{2\sqrt{2}}$$

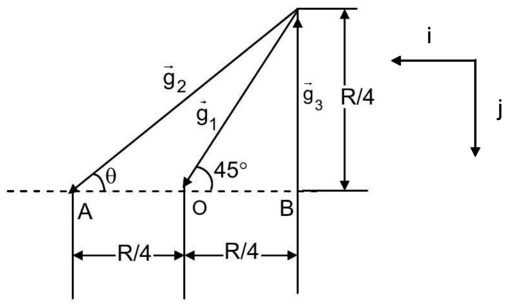
$$|\bar{g_2}| = \frac{G\frac{4}{3}\pi\left(\frac{R^3}{64}\right)\times 2\rho}{5R^2/16} = G.\frac{4}{3}\pi\rho.\frac{R}{10}$$

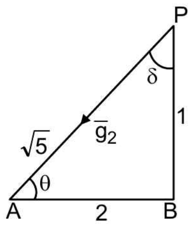
$$\left(\because AP^2 = \frac{R^2}{16} + \frac{R^2}{4} = \frac{5R^2}{16}\right)$$

$$|\bar{g_3}| = G.\frac{4}{3}\pi\rho(PB) = G.\frac{4}{3}\pi\rho.\frac{R}{4}$$

$\bar{g} = \bar{g_1} + \bar{g_2} + \bar{g_3}$ is vector addition as shown below.

(Note the positive directions of $\hat{i}$ and $\hat{j}$)





$$\bar{g}_1 = G.\frac{4}{3}\pi\rho R.\frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{i}\right)$$

$$= G\frac{4}{3}\pi R\rho\left(\frac{1}{4}\hat{j} + \frac{1}{4}\hat{i}\right)$$

$$\bar{g}_2 = G.\frac{4}{3}\pi\rho\frac{R}{10}\left(\frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{i}\right)$$

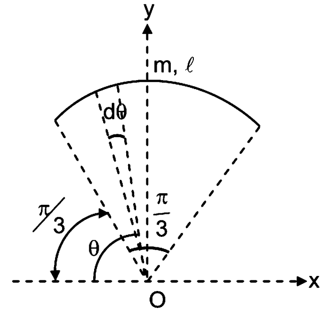
$$= G\frac{4}{3}\pi R\rho\left(\frac{\hat{j}}{10\sqrt{5}} + \frac{\hat{i}}{5\sqrt{5}}\right)$$

$$\bar{g}_3 = G.\frac{4}{3}\pi\rho R.\frac{1}{4}(-\hat{j}) = G\frac{4}{3}\pi R\rho\left(-\frac{1}{4}\right)\hat{j}$$

∴ g_3 is due to -ve mass, hence acts in opposite direction.

Adding, $\bar{g} = G.\frac{4}{3}\pi\rho R.\left[\left(\frac{1}{4} + \frac{1}{5\sqrt{5}}\right)\hat{i} + \frac{1}{10\sqrt{5}}\hat{j}\right]$ unit

5.



Take any element of angle dθ at θ with horizontal. Let R be radius of circle, and μ be linear density.

Then $\overrightarrow{dE}_{atO} = \frac{G.dm}{R^2}\left[\cos\theta(-\hat{i}) + \sin\theta\hat{j}\right]$

Integrating between limits $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{2\pi}{3}$ and using $dm = Rd\theta.\mu$

$$\vec{E} = \frac{G\mu}{R}\left[-\sin\theta\Big|_{\pi/3}^{2\pi/3}\hat{i} - \cos\theta\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\hat{j}\right]$$

$$\vec{E} = \frac{G\mu}{R}\hat{j} = \frac{G}{R}\cdot\frac{m}{\ell}\hat{j}$$

$$= \frac{G}{\ell/\left(\frac{\pi}{3}\right)}\cdot\frac{m}{\ell}\hat{j} = \frac{\pi}{3}\frac{Gm}{\ell^2}\hat{j}$$

$$\left[\because \ell = \theta R = \frac{\pi}{3}R \Rightarrow R = \frac{3\ell}{\pi}\right]$$

Potential at O $= \int_0^m \frac{Gdm}{R} = -\frac{Gm}{R}$

$$= \frac{-Gm}{\frac{\ell}{\left(\pi/3\right)}} = \frac{-\pi}{3}\cdot\frac{Gm}{\ell}$$

6. Potential energy of initial system

$$= -4\frac{Gm^2}{a} - \frac{2Gm^2}{\sqrt{2}a} = \frac{-Gm^2}{a}\left(4 + \sqrt{2}\right)$$

Potential energy of final system

$$= -4\frac{Gm^2}{\frac{3}{2}a} - \frac{2Gm^2}{\sqrt{2}.\frac{3}{2}a} = \frac{-2}{3}\frac{Gm^2}{a}\left(4 + \sqrt{2}\right)$$

Work to be done = Final P.E – Initial PE

$$= \frac{-2}{3}\frac{Gm^2}{a}\left(4 + \sqrt{2}\right) + \frac{Gm^2}{a}\left(4 + \sqrt{2}\right)$$

$$= \left(\frac{4 + \sqrt{2}}{3}\right)\frac{Gm^2}{a} \text{ unit}$$

7. Energy conservation equation:

$$\frac{-GMm}{R} + \frac{1}{2}mu^2 = \frac{-GMm}{R + h} + 0$$

⇒ (Using $GM = gR^2$)

$$\frac{1}{2}u^2 = gR^2\left(\frac{1}{R} - \frac{1}{R + h}\right) = \frac{gRh}{R + h}$$

$$\Rightarrow \quad (R + h)\, u^2 = 2gRh \Rightarrow h(2gR - u^2) = Ru^2$$

$$\Rightarrow \quad h = \frac{Ru^2}{\left(2gR - u^2\right)}$$

Putting values,

$$h = \frac{6.4 \times 10^6 \times 4 \times 10^6}{20 \times 6.4 \times 10^6 - 4 \times 10^6} = 206 \text{ km}$$

8. For a satellite in synchronous orbit around a planet,

Let T = time period, R = orbit radius,

v = orbital velocity, M = mass of planet,

ρ = density of planet, R = radius of planet.

Then, $v = \sqrt{\frac{GM}{r}}$

Using $T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$

$$\Rightarrow \quad r = \left(\sqrt{GM}.\frac{T}{2\pi}\right)^{2/3}, v = \sqrt{\frac{GM(2\pi)^{2/3}}{(GM)^{1/3}T^{2/3}}}$$

$$\Rightarrow \quad v = G^{1/3}M^{1/3}\frac{2^{1/3}\pi^{1/3}}{T^{1/3}}$$

Using $M = \frac{4}{3}\pi R^3\rho$

$$v = G^{1/3}\frac{2}{3^{1/3}}.\pi^{2/3}.R.\frac{\rho^{1/3}}{T^{1/3}} \Rightarrow v \propto \frac{R\rho^{1/3}}{T^{\frac{1}{3}}}$$

Let suffix 2 represent the planet, suffix 1 for Earth

$$\frac{v_2}{v_1} = \frac{R_2}{R_1}\left(\frac{\rho_2}{\rho_1}\right)^{1/3}\left(\frac{T_1}{T2}\right)^{1/3}$$

$$\text{Then } \frac{v_2}{v_1} = 4.\left(\frac{1}{4}\right)^{1/3}.2^{1/3} = 2^{5/3} \Rightarrow v_2 = 2^{5/3}.v_1$$

v_2 is the planet's stationary orbit velocity.

$$\therefore \quad \text{Escape velocity} = \sqrt{2}v_2 = 2^{13/6}.v_1$$

$$= 2^2.2^{1/6}\, v_1$$

$$= 4 \times (1.125).v_1 = 4.5 \times 3000 = 13500 \text{ m s}^{-1}$$

9. $$v = \sqrt{\frac{GM}{r}} \Rightarrow \frac{dv}{dr} = \frac{-1}{2}\frac{\sqrt{GM}}{r\sqrt{r}} \qquad \text{— (1)}$$

$$\frac{dv}{dt} = \frac{f}{m} \Rightarrow \frac{dv}{dr}.\frac{dr}{dt} = \frac{f}{m}$$

$$\Rightarrow \quad (\text{using (1)})\ dt = -\frac{1}{2}\sqrt{GM}.\frac{m}{f}.\frac{dr.}{r\sqrt{r}} \qquad \text{— (2)}$$

If Δt is time per revolution, $\Delta t = \frac{2\pi r}{v} = \frac{2\pi r\sqrt{r}}{\sqrt{GM}}$

$$\left[\because v = \sqrt{\frac{GM}{r}}\right]$$

Putting in (2),

$$\frac{2\pi r\sqrt{r}}{\sqrt{GM}} = -\frac{1}{2}\sqrt{GM}.\frac{m}{f}.\frac{\Delta r.}{r\sqrt{r}} \Rightarrow \Delta r = -\frac{4\pi f r^3}{GMm}$$

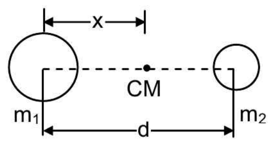
From (2)

$$t = \int dt = -\frac{1}{2}\sqrt{GM}.\frac{m}{f}\int_r^{\frac{r}{2}}\frac{dr}{r\sqrt{r}}$$

$$= -\frac{1}{2}\sqrt{GM}\frac{m}{f}\frac{r^{\left(-\frac{3}{2}+1\right)}}{\left(-\frac{3}{2}+1\right)} = \sqrt{GM}\frac{m}{f}\frac{1}{\sqrt{r}}\Bigg]_r^{\frac{r}{2}}$$

$$= \sqrt{GM}\frac{m}{f}\frac{1}{\sqrt{r}}\left(\sqrt{2}-1\right)$$

10.



Centripetal force for rotation is given by mutual attraction

$$m_1\omega^2 x = \frac{Gm_1m_2}{d^2} \Rightarrow x = \frac{Gm_2}{\omega^2 d^2}$$

$$m_2\omega^2(d - x) = \frac{Gm_1m_2}{d^2}$$

substituting for x

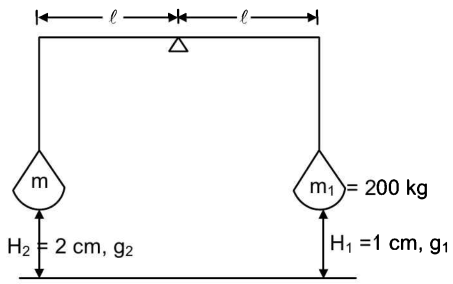
$$\Rightarrow \quad \omega^2\left(d - \frac{Gm_2}{\omega^2 d^2}\right) = \frac{Gm_1}{d^2}$$

$$\Rightarrow \quad \omega^2 d - \frac{Gm_2}{d^2} = \frac{Gm_1}{d^2}$$

$$\Rightarrow \quad \omega^2 = \frac{G}{d^3}(m_1 + m_2)$$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = \frac{2\pi.d^{3/2}}{\sqrt{G(m_1 + m_2)}}$$

11.



We have $mg_2\ell = m_1g_1\ell$

$$\rightarrow m = m_1\left(\frac{g_1}{g_2}\right) \qquad -(1)$$

$$g_h = g\left[1-\frac{2h}{R}\right] \Rightarrow g_1 = g\left[1-\frac{2H_1}{R}\right]$$

$$g_2 = g\left[1-\frac{2H_2}{R}\right]$$

$$\therefore \quad (1) \Rightarrow m = m_1\frac{\left[1-\frac{2H_1}{R}\right]}{\left[1-\frac{2H_2}{R}\right]}$$

$$= m_1\frac{(R-2H_1)}{(R-2H_2)}$$

$$\therefore \Delta m = m_1 - m$$

$$= m_1\left[1-\frac{(R-2H_1)}{(R-2H_2)}\right]$$

$$= \frac{m_1 \cdot 2(H_2-H_1)}{(R-2H_2)}$$

$R - 2H_2 \approx R$

$$\therefore \quad \Delta m = \frac{m_1 2(H_2-H_1)}{R}$$

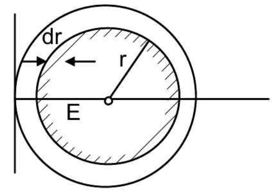
$$= \frac{200\times 2\times(2-1)\times 10^{-2}}{6400\times 10^3}\text{ kg}$$

$$= \frac{4}{6400\times 10^3}\text{ kg} = \frac{1}{1600}\text{ gram}$$

12. $E = -\frac{dV}{dh} = \left(\frac{-k^2}{h}\right) \qquad \int_{V_i}^{V} dV = k^2\int_{h_i}^{h}\frac{dh}{h}$

$$V = V_i + k^2 \ln\left(\frac{h}{h_i}\right)$$

13.



$$\int_0^R 4\pi r^2.kr\,dr = M \Rightarrow k = \frac{M}{\pi R^4}$$

$\Rightarrow$ mass 'm' up to radius $r \Rightarrow$

$$m = \int_0^r 4\pi kr^3 dr = 4\pi k\frac{r^4}{4}\Big]_0^r = k\pi r^4$$

$\therefore$ workdone in moving from an intermediate level of 'm' inward by dr:

$dw = Edr = \frac{Gm}{r^2}dr \Rightarrow$ change in potential energy is equal to the –ve of work done by internal forces.

Consider the potential at center as V_0 and on the surface as $V = -\frac{GM}{R}$. The work done in moving a unit mass from the centre to the surface is

$$1\times(V-V_0)\int_0^R F.dr = \int_0^R .\frac{G\pi kr^4}{r^2}dr \Rightarrow$$

$$\therefore \qquad V_0 = -\frac{GM}{R} - \int_0^R \frac{G\pi kr^4}{r^2}dr$$

$$= \frac{-GM}{R} - \frac{1}{3}\frac{GM}{R} = \frac{-4}{3}\frac{GM}{R}$$

14. $v_e = \sqrt{\frac{2GM}{R}} \Rightarrow$ at maximum distance R'

$$\frac{GMm}{R} - \frac{GMm}{R'} = \frac{1}{2}m\left(\frac{3}{4}\right)^2\frac{2GM}{R}$$

$$\Rightarrow \qquad R' = \frac{16}{7}R$$

15. From centre of the earth, the satellite distances are, $r_1 = 3R_e$ and $r_2 = 7R_e$.

$$U_A = \frac{-GMm}{3R_e}$$

$$U_B = \frac{-GMm}{7R_e} \qquad \frac{U_A}{U_B} = \frac{7}{3}$$

16. $g' = \frac{gr}{R} = \frac{g}{R}\left[R - d\right]$

$= g\left[1 - \frac{d}{R}\right]$

$\left|\frac{\Delta g}{g}\right| = \frac{d}{R}$

R at pole is < R at equator

(due to bulging at equator)

17. Gravitational field is always directed inward. Hence when a body is taken from outer surface inward, work done by the field is positive and hence the external force is negative. Hence potential reduces. Since V outside is negative, the magnitude of V inside should increase.

18. At escape velocity initially total energy on surface of Earth is zero. Hence final potential energy U also is zero. i.e U at zero potential. (infinite distance).

19. $T^2 \propto a^3 \Rightarrow$ for same T, a of ellipse = R of circle

$\Rightarrow$ Area of ellipse πab < Area of circle πR^2

$\frac{dA}{dt} = \frac{L}{2m} \Rightarrow A = \frac{L}{2m}T$

same L and T $\Rightarrow$

Am = constant

20. Linear momentum is not conserved

mvr = constant; r changes

21. From symmetric position of P and the rod being uniform, the direction will be negative X direction. The particles on the rod near the end are at a larger distance than D from P.

Mass of the element $= \frac{M}{\ell}(dy)$

Distance of the element from

$P = \sqrt{y^2 + D^2} = D\sec\theta$

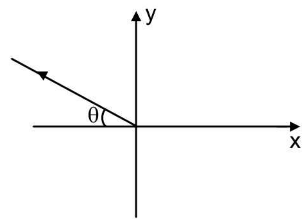
$dF = \frac{GmMdy}{\ell.(D\sec\theta)^2} = \frac{GmMdy}{\ell.D^2\sec^2\theta}$

$= \frac{GmMdy}{\ell.(D^2 + y^2)}$

22. Angle with x-axis = $\pi - \theta$

$\Rightarrow$ Unit vector = $\cos(\pi - \theta)\hat{i} + \sin(\pi - \theta)\hat{j}$

$= -\cos\theta\hat{i} + \sin\theta\hat{j}$



23. $d\bar{F} = \frac{GMm\,dy}{\ell D^2\sec^2\theta}(-\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

we have $y = D\tan\theta \Rightarrow dy = D\sec^2\theta\,d\theta$

$\therefore\quad d\bar{F} = \frac{GMm}{\ell D^2\sec^2\theta}.D\sec^2 d\theta\,(-\cos\theta\hat{i} + \sin\theta\hat{j})$

$= \frac{GMm}{\ell D}\left[-\cos\theta\,d\theta\,\hat{i} + \sin\theta\,d\theta\,\hat{j}\right]$

$\therefore \bar{F} = \int_0^F d\bar{F} = \frac{GMm}{\ell D}\int_{-\theta}^{+\theta}(-\cos\theta\,d\theta\hat{i} + \sin\theta\,d\theta\,\hat{j})$

$= \frac{GMm}{\ell D}\left|-\sin\theta\hat{i} - \cos\theta\,\hat{j}\right|_{-\theta}^{+\theta}$

$= \frac{GMm}{\ell D}\left[(-\sin\theta - \sin\theta)\hat{i} - 0\right]$

$= -\frac{GMm}{\ell D}.2\sin\theta\hat{i}$

$\therefore\quad F = \frac{2GMm}{\ell D}\sin\theta\left[\because \sin\theta = \frac{\left(\ell/2\right)}{\sqrt{\left(\frac{\ell^2}{2}\right)^2 + D^2}}\right]$

$= \frac{2GMm}{\ell D}.\frac{\ell}{2.\sqrt{\left(\frac{\ell}{2}\right)^2 + D^2}}$

$= \frac{GMm}{D^2\sqrt{\left(\frac{\ell}{2D}\right)^2 + 1}}$

24. Since the path is circular with radius r_0 we have $v_0 = \sqrt{\frac{GM_e}{r_0}}$

$\in = \frac{v_0{}^2 r_0}{Gm_e} - 1 = \frac{v_0^2}{\left(\frac{Gm_e}{r_0}\right)} - 1 = 0$

25. $mvR_e = mv_0r_0 \Rightarrow v = \dfrac{v_0r_0}{R_e}$

$$\frac{1}{2}mv^2 - \frac{Gmm_e}{R_e} = \frac{1}{2}mv_0^2 - \frac{Gmm_e}{r_0}$$

$$\Rightarrow \frac{1}{2}\frac{mv_0^2r_0^2}{R_e^2} - \frac{Gmm_e}{R_e} = \frac{1}{2}mv_0^2 - \frac{Gmm_e}{r_0}$$

$$\frac{v_0^2}{2}\left(\frac{r_0^2}{R_e^2} - 1\right) = \frac{Gm_e}{r_0}\left[\frac{r_0}{R_e} - 1\right]$$

$$v_0^2 = Gm_e \frac{2}{r_0\left(\frac{r_0}{R_e} + 1\right)} = \frac{Gm_e}{R_e}\frac{2}{\frac{r_0}{R_e}\left(1 + \frac{r_0}{R_e}\right)}$$

$$v_0 = \sqrt{\frac{Gm_e}{R_e}}\sqrt{\frac{2}{\frac{r_0}{R_e}\left(1 + \frac{r_0}{R_e}\right)}}$$

26. Substituting $r_0 = 4\,R_e$, in the solution v_o of Qn. 25

$$v_0 = \sqrt{\frac{Gm_e}{R_e}} \times \sqrt{\frac{2}{4\frac{R_e}{R_e}\left(1 + \frac{4R_e}{Re}\right)}}$$

$$v_0^2 = \frac{Gm_e}{10R_e}$$

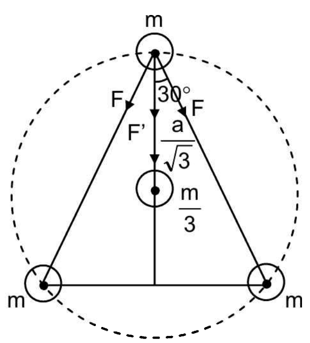
Total mechanical energy

$$= \frac{1}{2}mv_0^2 - \frac{Gmm_e}{r_0} = -\frac{Gmm_e}{5R_e}$$

Note : $-\dfrac{Gmm_e}{2(4R_e)}$ would be wrong.

This is the total energy if a circular orbit of radius 4 R_e is derived.

27.



The resultant of all forces on each m acts in the direction towards centre

$$F_r = F' + 2\,F\cos 30° = F' + \sqrt{3}\,F$$

$$= \frac{Gm\frac{m}{3}}{a^2/3} + \frac{\sqrt{3}\,Gmm}{a^2} = \frac{Gm^2}{a^2}\left[1 + \sqrt{3}\right]$$

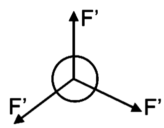
Since this provides the centripetal force:

$$\frac{Gm^2}{a^2}\left(1 + \sqrt{3}\right) = \frac{mv^2}{\left(\frac{a}{\sqrt{3}}\right)}$$

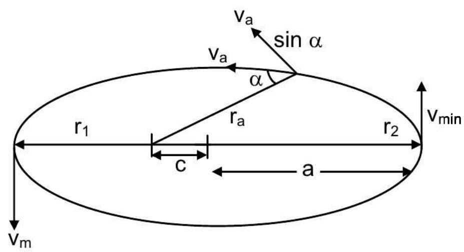
$\therefore\quad v = \sqrt{\dfrac{Gm}{a}\left(\dfrac{1 + \sqrt{3}}{\sqrt{3}}\right)} \Rightarrow$ If $\dfrac{m}{3}$ is absent, then

$$F_r = \sqrt{3}F \Rightarrow \therefore \sqrt{3}\frac{Gmm}{a^2} = \frac{mv^2}{\frac{a}{\sqrt{3}}} \Rightarrow v = \sqrt{\frac{Gm}{a}}$$

Resultant force on $\dfrac{m}{3} \Rightarrow = 0$



28.



Obviously the maximum velocity occurs at the minimum radial distance

$$v_{min} = v_m\frac{r_1}{r_2} = v_m \cdot \frac{a - c}{a + c} = v_m\frac{\left(1 - \frac{c}{a}\right)}{1 + \frac{c}{a}} = v_m\frac{1 - e}{1 + e};$$

$mv_mr_1 = mv_a\,r_a\sin\alpha$ (conservation of angular momentum)

$$\therefore\quad r_1 = \frac{v_a\,r_a\sin\alpha}{v_m}$$

At the left extreme radius of curvature of the orbit is $r_1 + c$.

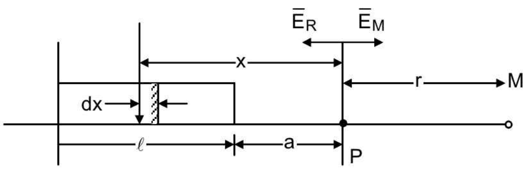
Consider centripetal force at the left extreme. (m = mass of satellite)

$$\frac{mv_m^2}{r_1 + c} = \frac{GMm}{r_1^2} \Rightarrow$$

$$v_m^2 = \frac{r_1 + c}{r_1} . \frac{GM}{r_1} \Rightarrow v_m^2 = \left(1 + \frac{c}{r_1}\right)\frac{GM}{r_1}$$

$$\Rightarrow v_m > \sqrt{\frac{GM}{r_1}}$$

29.



Considering a small element dx at x from P.

$$E_R = \int_a^{(a+\ell)} -G\frac{M}{\ell}.dx . \frac{1}{x^2} = -\frac{GM}{\ell}\left[-\frac{1}{x}\right]_a^{\ell+a}$$

$$= -\frac{GM}{\mathbf{n}}\left(\frac{1}{a} - \frac{1}{(\mathbf{n} + a)}\right)$$

$$= -\frac{GM}{\ell}\frac{\ell + a - a}{a(\ell + a)} = -\frac{GM}{a(\ell + a)} = E_M = -\frac{GM}{r^2}$$

$$\Rightarrow \quad r = \sqrt{a(\ell + a)}$$

$$\therefore \quad d = a + \sqrt{a(\ell + a)}$$

At P, potential is $\int_a^{a+\ell} -\frac{GM}{\ell}\frac{dx}{x} = -\frac{GM}{\ell}\ell nx\Big]_a^{\ell+a}$

$$= -\frac{GM}{\ell}\ell n\left(\frac{\ell + a}{a}\right)$$

Hence the work done in bringing M from infinity to P is $W = -\frac{GM^2}{\ell}\ell n\left(1 + \frac{\ell}{a}\right)$ which is the P.E of the system. Obviously P.E of the point mass M is not zero.

30. For a circular orbit $\frac{mv^2}{r} = \frac{GMm}{r^2}$

$\therefore$ K.E $= \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$

P.E $= -\frac{GMm}{r}$

$\therefore$ Total energy = K.E + P.E $= -\frac{1}{2}\frac{GMm}{r}$

$\therefore$ (b) → (p), (s)

(d) → (p), (s)

If $E = \frac{U}{2}$; for gravitation, $u < 0 \Rightarrow E < 0$

$\Rightarrow$ same as (b)

If E = 0, K.E is too large to hold the body in a closed circular orbit. The body goes through a parabolic path. (a) → (q)

If E > 0 the path is a hyperbola

(c) → (r)

IIT Assignment Exercise

31. $F = \frac{k}{x^2} \Rightarrow F' = \frac{k}{(4x)^2} \Rightarrow \frac{F'}{F} = \frac{1}{16}$

32. $F = \frac{Gm_1m_2}{r^2}$, and m_1, m_2 both are always positive, F is always attractive.

33. The force between two bodies depends only on their masses and the distance between them.

34. $g = 9.8 \text{ m s}^{-2}$

$m = \frac{F}{g} = \frac{1}{9.8} = 0.102 \text{ kg} = 102 \text{ g}$

$F = ma = 1 \times 1 = 1 \text{ N}$

35. Inertial mass = gravitational mass, from experiments.

36. Definition of field = Force acting on a unit mass

= Force per unit mass $= \frac{F}{M}$

37. $F = m . g \Rightarrow m = \frac{F}{g} = \frac{100}{10} = 10 \text{ kg}$

38. $g' = \frac{GM}{(R + h)^2}$, $g = \frac{GM}{R^2}$

$$\frac{g'}{g} = \left(\frac{R}{R + h}\right)^2 = \frac{1}{\left(1 + \frac{h}{R}\right)^2}$$

$\Rightarrow$ If h << R, then

$\frac{g'}{g} = 1 - \frac{2h}{R} \Rightarrow g'$

$$= 9.8\left(1 - \frac{2 \times 100}{6400}\right) \simeq 9.5 \text{ m s}^{-2}$$

39. Inside earth,

$g' = K.\, r$ where r is the distance from the centre of the earth, $g = KR$

$$\Rightarrow \quad g' = Kr \Rightarrow \frac{g}{2} = Kr \Rightarrow r = \frac{R}{2}$$

$$\Rightarrow \quad \text{depth} = R - \frac{R}{2} = \frac{R}{2}$$

40. $R_m = \dfrac{u^2}{g_m}, R_e = \dfrac{u^2}{g_e}$

$$\Rightarrow \; R_m = \left(\frac{g_e}{g_m}\right) R_e = 5\, R_e$$

41. $g_a = g - \omega^2 r \cos\phi$

If $\omega \rightarrow 0$, g_a increases.

42. Time period of second's pendulum = 2 s

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

$$g' = \frac{G(2M)}{(2R)^2} = \frac{GM}{2R^2} = \frac{g}{2}$$

$$T' = 2\pi\sqrt{\frac{\ell}{g'}} = 2\pi\sqrt{\frac{2\ell}{g}} = \sqrt{2}\, T$$

$$T' = \sqrt{2}\,.\,2 = 2\sqrt{2} \text{ s}$$

43. The acceleration due to gravity decreases with depth, so weight decreases.

44. $g = \dfrac{GM}{R^2}, g' = \dfrac{GM}{\left(\dfrac{R}{2}\right)^2} = 4g$

Weight = mg' = m 4g = 4mg

45. First cosmic velocity corresponds to velocity in an orbit close to earth

$v = \sqrt{gR} = \sqrt{9.8 \times 6400 \times 10^3} \Rightarrow v = 7.92$ km/s

46. $g = \dfrac{GM}{R^2}$

$$g' = g\left(\frac{M_1}{M}\right)\left(\frac{R}{R_1}\right)^2 = 9.8 \times \frac{16}{80} = 1.96 \text{ m s}^{-2}$$

47. $g = \dfrac{GM}{R^2}$

$$g' = \frac{GM}{(0.99R)^2} \approx 1.02g$$

Aliter:

$g = KR^{-2} \Rightarrow \Delta g = K - 2R^{-3}\,\Delta R$

$$\frac{\Delta g}{g} = -2\frac{\Delta R}{R} = -2(-1\%) = 2\%$$

48. g is maximum at the surface and decreases as we go below or above the surface of the earth.

49. A body is closer to the centre of Earth at the poles, compared to the equator, and closer on planes compared to hill tops.

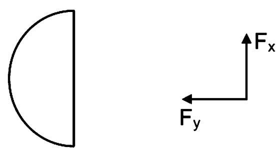
50. $g = \dfrac{GM'}{r'^2} = G\,\dfrac{4}{3}\,\dfrac{\pi r'^3}{r'^2}\rho \qquad g = \dfrac{4}{3}G\pi\, r'\; \rho \propto r'$

51. The work done = mass × gravitational potential at P

$$= -m\frac{GM}{r} = -\frac{GMm}{r}$$

52. By definition of gravitational potential.

53.



$$V' = \int (F_x\hat{i} + F_y\hat{j}).dx\,\hat{i}$$

$$= \int F_x dx = \int \frac{F}{2} dx = \frac{V}{2}$$

Aliter:

Potential is a scalar and can be added arithmetically. Potential at the centre of full sphere = Sum of potentials at the centre of two half spheres

54. Gravitational potential $= \dfrac{-GM}{r} = \dfrac{\text{work done}}{\text{mass}}$

$$= \frac{MLT^{-2}.L}{M} = M^0 L^2 T^{-2}$$

55. $\dfrac{\Delta PE}{m} = \dfrac{GM}{R} - \dfrac{GM}{2R} = \dfrac{GM}{2R} = \dfrac{\Delta KE}{m}$

$$= \frac{1}{2}\frac{mv^2}{m} \Rightarrow v^2 = \frac{GM}{R}$$

But $v_e = \sqrt{\dfrac{2GM}{R}} = 11.2$

$$v^2 = \frac{GM}{R} = \frac{v_e^2}{2} \Rightarrow v = \frac{11.2}{\sqrt{2}} = 7.92 \text{ km s}^{-1}.$$

56. $\Delta PE = \left(\frac{GM}{R} - \frac{GM}{\frac{6R}{5}}\right) m = \frac{GMm}{6R} = m\left(\frac{GM}{R^2}\right)\frac{R}{6}$

$= mg\frac{R}{6} = mg.\frac{R}{5}.\frac{5}{6} = \frac{5}{6}\ mgh$

57. $\Delta KE = \Delta PE$

$\frac{1}{2}mv^2 = GMm\left(\frac{1}{R} - \frac{1}{R_0}\right) \Rightarrow$

$v = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{R_0}\right)}$

58. First cosmic velocity is the velocity of a satellite close to earth's surface

$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow v^2 = \frac{GM}{r}$

$\frac{1}{2}mv_e^2 = \frac{GMm}{r}$

$\Rightarrow\ v_e^2 = 2\frac{GM}{r} = 2v^2 \Rightarrow v_e = \sqrt{2}\,v$

59. $v_e^2 = \frac{2GM}{R} = 2\,G.\,\frac{4}{3}\pi R^2\rho$

$\left(\frac{v_e'}{v_e}\right)^2 = \left(\frac{R'}{R}\right)^2$

$= 4 \Rightarrow v_e' = 2\,v_e = 2 \times 11 = 22\ km\ s^{-1}.$

60. $\frac{1}{2}mv_e^2 = \frac{GM}{R}m \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$

61. We take the potential at infinity as zero. For a body to escape, $E_{total} \geq 0$. Thus if it does not escape,

$E_{total} < 0$.

62. $v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = \sqrt{2 \times 10 \times 6400 \times 10^3}$

$= \sqrt{128} \times 10^3\ m\ s^{-1} \approx 11.2\ km\ s^{-1}$

63. g varies linearly with distance from the centre of the Earth inside it .

$\Rightarrow\ g' = \frac{g}{2} \Rightarrow W' = mg' = \frac{mg}{2} = \frac{W}{2}$

64. $H_e = \frac{u^2}{2g_e}, H_m = \frac{u^2}{2g_m}$

$H_e = H_m.\frac{g_m}{g_e} = 10 \times \frac{1}{5} = 2\ m$

65. $\frac{GM}{(R+h)^2} = \left(\frac{16}{100}\right).\frac{GM}{R^2} \Rightarrow \frac{R+h}{R} = \frac{10}{4} = 2.5$

$\frac{h}{R} = 1.5 \Rightarrow h = 1.5 \times 6400 = 9600\ km.$

Aliter:

$g' = 0.16\,g = \frac{16}{100}g = \frac{g}{n}\left(n = \frac{100}{16}\right)$

$h = R\left[\sqrt{n} - 1\right] = R\left[\sqrt{\frac{100}{16}} - 1\right]$

$= R\left[\frac{10}{4} - 1\right] = 1.5\,R\ \ = 9600\ km$

66. $\frac{v^2}{R} = \frac{GM}{R^2} \Rightarrow v^2 = \frac{GM}{R}$

$KE = \frac{1}{2}mv^2 = \frac{GMm}{2R}$

67. For a satellite at altitude R + h centrifugal force

$\frac{mv^2}{(R+h)} = \frac{GMm}{(R+h)^2}$

$\frac{v^2}{R+h} = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}.\frac{R^2}{(R+h)^2}$

$= \frac{gR^2}{(R+h)^2} \Rightarrow v = \sqrt{\frac{gR^2}{R+h}}$

Aliter:

$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R+h)}} = \sqrt{\frac{GM}{R^2}.\frac{R^2}{(R+h)}}$

$= \sqrt{\frac{gR^2}{R+h}}$

68. $\frac{v^2}{R} = \frac{GM}{R^2} = g \Rightarrow v = \sqrt{gR}$

$T = \frac{2\pi R}{v} = 2\pi.\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6400 \times 1000}{10}}$

$= 800 \times 2\pi = 1600\,\pi\ s$

69. By definition, a geo-synchronous satellite revolves at the same period as Earth.

70. $v_0^2 = \dfrac{GM}{R}$

$$v^2 = \frac{GM}{(4R)} = \frac{v_0^2}{4} \Rightarrow v = \frac{v_0}{2}$$

71. The orbital speed is independent of mass of satellite.

72. A geostationary satellite remains above the same point on Earth by orbiting at the same frequency as earth.

73. The communication satellite in the parking orbit has to be above the location it covers at all times $\Rightarrow$ geostationary

$\Rightarrow$ Time period = 24 hour.

74. The spaceship is in free fall and all the bodies will have apparent weight = 0

$\Rightarrow$ $g_{apparent} = 0$.

So a pendulum clock cannot be used.

75. $T^2 \propto a^3$

$$\left(\frac{24}{T}\right)^2 = \left(\frac{7R}{(1+2.5)R}\right)^3$$

$$\frac{24}{T} = \sqrt{8} = 2\sqrt{2} \Rightarrow T = 6\sqrt{2}\text{ h.}$$

76. $\dfrac{v^2}{R} = \dfrac{GM}{R^2} \Rightarrow v^2 = \dfrac{GM}{R}$

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \Rightarrow v_e^2 = \frac{2GM}{R} = 2v^2$$

$$\therefore \text{ Percentage increase} = \frac{(\sqrt{2}-1)v}{v} \times 100$$

$$= 41.4\%.$$

77. $\dfrac{v_0^2}{R_0} = \dfrac{GM}{R_0^2}$

$$\omega^2 = \frac{v_0^2}{R_0^2} = \frac{GM}{R_0^3} \Rightarrow \omega = \sqrt{\frac{GM}{R_0^3}}$$

78. $T^2 \propto a^3$

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{4R_0}{R_0}\right)^3 = 64 \Rightarrow T' = 8T$$

79. $\Rightarrow$ $v = \sqrt{\dfrac{GM}{R+h}}$

$$T = \frac{2\pi(R+h)}{v}$$

$$= \frac{2\pi(R+h)}{\sqrt{GM}\Big/\sqrt{R+h}} = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

80. $\dfrac{v_0^2}{r} = \dfrac{GM}{r^2} \Rightarrow v_0^2 = \dfrac{GM}{r}$

$$\frac{1}{2}mv_e^2 = \frac{GMm}{r^2}$$

$$v_e^2 = \frac{2GM}{r} = 2v_0^2 \Rightarrow v_e = \sqrt{2}\,v_0$$

81. When point is outside the sphere, i.e. $d > r$

$$\text{gravitational field} = E = \frac{GM}{d^2}$$

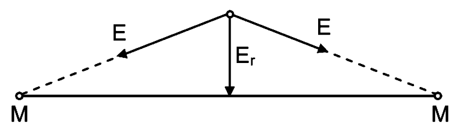
$$\therefore \quad \frac{F_1}{F_2} = \frac{GM}{d_1^2}\Big/\frac{GM}{d_2^2} = \frac{d_2^2}{d_1^2}$$

When point is inside the sphere, $(d < r)$

$$\text{Gravitational field} = \frac{G\left(\frac{4}{3}\pi d^3\rho\right)}{d^2} = \frac{4}{3}\pi G d\rho$$

$$\Rightarrow \quad \frac{F_1}{F_2} = \frac{d_1}{d_2}$$

82.



When displaced axially it will not return to the original position. But when displaced perpendicular, it will tend to restore to original position.

83. Take R = 6400 km = 6400×10^3 m

$$g \propto \frac{1}{R^2} \Rightarrow \frac{g_h}{g} = \frac{R^2}{(R+h)^2}$$

$$\Rightarrow \frac{9}{100} = \left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{3}{10} = \frac{R}{R+h}$$

$\therefore \quad 10R = 3R + 3h \Rightarrow 3h = 7R \;\therefore\; h = \dfrac{7R}{3}$

$\Rightarrow \quad h = \dfrac{7}{3} \times 6400 \text{ km} = 14933 \text{ km}$

Aliter:

$$g_h = \frac{g}{n} \Rightarrow n = \frac{100}{9}$$

$$h = R\left[\sqrt{n} - 1\right] = R\left[\sqrt{\frac{100}{9}} - 1\right]$$

$$= R\left[\frac{10}{3} - 1\right] = \frac{7}{3}R = 14933 \text{ km}$$

84. $GM = R^2 g \Rightarrow g = \dfrac{GM}{R^2} \text{ or } \dfrac{4GM}{D^2}$

$\Rightarrow \quad GM = \dfrac{D^2 g}{4}$

Now $G\left(\dfrac{4}{3}\pi\dfrac{D^3}{8}\rho\right) = \dfrac{D^2}{4}g \Rightarrow g = \dfrac{2}{3}\pi\rho D$

$\therefore \quad g \propto D$

85. $g_d = \dfrac{g.r}{R}$, where r = distance from centre of Earth

$$= \frac{g(R-d)}{R}$$

$$\therefore \quad \frac{g}{n} = \frac{g(R-d)}{R} \Rightarrow d = \frac{(n-1)}{n}R$$

86. $g = \dfrac{GM}{R^2} = \dfrac{G\frac{4}{3}\pi R^3\rho}{R^2}$

$$\rho = \frac{3g}{4\pi RG}$$

87. $g_1 = \dfrac{GM}{(2R)^2};\; g_2 = \dfrac{GM}{(2R+2R)^2}$

$$\Rightarrow \frac{g_1}{g_2} = 4 \Rightarrow \frac{T_1}{T_2} = \sqrt{\frac{g_2}{g_1}} = \frac{1}{2}\left(T = 2\pi\sqrt{\frac{R}{g}}\right)$$

88. Let x be the distance from planet where field vanishes when

$$\frac{GM_p}{x^2} = \frac{GM_s}{(4\times10^8 - x)^2}$$

$$\Rightarrow \frac{G\times 64M_s}{x^2} = \frac{GM_s}{(4\times10^8 - x)^2}$$

$$\Rightarrow \frac{64}{x^2} = \frac{1}{(4\times10^8 - x)^2} \Rightarrow \frac{8}{x} = \frac{1}{(4\times10^8 - x)}$$

$\Rightarrow \quad x = 32 \times 10^8 - 8x$

$\Rightarrow \quad 9x = 32 \times 10^8$ m

$\therefore \quad x = 3.56 \times 10^8$ m

89. Let M be the mass of the earth $g' = \dfrac{GM}{R^2}$

But $M = \dfrac{4}{3}\pi R^3\rho$

$$g' = \frac{4}{3}\frac{\pi R^3\rho G}{R^2} = \frac{4}{3}\pi R\rho G$$

$R = 6.5 \times 10^6$ m

$\rho = 7.8$ g/cc $= 7800$ kg m^{-3}

$G = 6.5 \times 10^{-8}$ CGS unit $= 6.5 \times 10^{-11}$ N m^2 kg^{-2}

$$g' = \frac{4}{3}\times\frac{22}{7}\times 6.5\times10^6 \times 7800 \times 6.5\times10^{-11}$$

$= 13.8$ m s^{-2}

Aliter:

Use all CGS units only conversion is

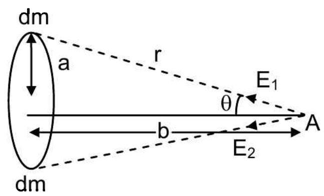
$R = 6.5 \times 10^6\text{m} = 6.5 \times 10^8$ cm

$$g' = \frac{4}{3}\pi \times 6.5\times10^8 \times 7.8 \times 6.5 \times 10^{-8}$$

$= 1380$ cm s^{-2}

$= 13.8$ ms^{-2}

90.



If we consider fields E_1 and E_2 due to small, diametrically opposite elements dm each, $|E_1| = |E_2|$ and $E_1 \cos\theta$ components add up while $E_1 \sin\theta$ components cancel.

$$\therefore \quad \text{Total field } E = \int\frac{Gdm}{r^2}\cos\theta$$

$$E = \int\frac{Gdm}{(a^2+b^2)}\cdot\frac{b}{\sqrt{a^2+b^2}} = \frac{Gb}{(a^2+b^2)^{3/2}}\int dm$$

$$\Rightarrow \quad E = \frac{GMb}{(b^2+a^2)^{3/2}} = \frac{GMb}{\sqrt{(b^2+a^2)^3}}$$

91. Mass of single shell = sum of masses of 2 shells i.e., $4\pi r^2\sigma\,\Delta t = 4\pi r_1^{\,2}\sigma\Delta t + 4\pi r_2^{\,2}\,\sigma\,\Delta t$

$\Rightarrow$ $r_1^{\,2} + r_2^{\,2} = r^2$, where σ is surface density, r is radius of shell after combining, r_1 and r_2 are radii of individual shells.

If V_1 & V_2 be potentials at the centres of 2 shells,

$\Rightarrow$ $M = 4\pi r^2\sigma\Delta t$ ($\Delta t \to$ thickness of the shell)

$$V_1 = -\frac{GM_1}{r_1} = -4\pi G r_1\sigma\Delta t \text{ and}$$

$$V_2 = -\frac{GM}{r_2} = -4\pi G r_2\sigma\Delta t$$

$V \propto r$

$$\therefore \frac{V_1}{V_2} = \frac{r_1}{r_2} = \frac{2}{3}$$

For combined shell $r = \sqrt{4k^2 + 9k^2} = \sqrt{13}\,k$

$V_1 : V_2 : V \equiv 2:3:\sqrt{13}$

92. $V_Q = \dfrac{-GM}{(a^2 + a^2)^{1/2}}, V_P = -\dfrac{GM}{a}$

$$\therefore \frac{1}{2}mv^2 = \frac{GMm}{a}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$v = \sqrt{\frac{GM}{a} \times 0.6}$$

93. Potential at the center of a solid sphere is $-\dfrac{3}{2}\dfrac{GM'}{R}$ and potential at infinity is zero. M' = 2M where M is the mass of half of the sphere. Since the potential is additive, potential at the centre due to half the sphere (i.e., hemisphere) is :

$$V = \frac{1}{2}\left(-\frac{3}{2}\frac{GM'}{R}\right) = -\frac{3}{2}\frac{GM}{R}$$

$$\therefore \text{ Work done} = \Delta V.\, m = \frac{-3GMm}{D}$$

Note:

The work done by gravitational force is negative since the body moves against the gravitational force.

94. $E_i = -\dfrac{GMm}{4R}$

$E_f = -\dfrac{GMm}{8R}$

$$\Delta E = E_f - E_i = \frac{GMm}{4R}\left[1 - \frac{1}{2}\right] = \frac{GMm}{8R}$$

95. We know $V = -\dfrac{GM}{r}$

$$\therefore \quad V_{net} = \left(-\frac{GM}{2} - \frac{GM}{8} - \frac{GM}{32}\ldots\right)\frac{1}{R}$$

and this is a geometric progression with $r = \dfrac{1}{4}$

$$V_{net} = \frac{-GM}{R}\frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{-2GM}{3R}$$

96. Gravitational potential

$$= -\frac{Gm}{r} = \frac{-6.6 \times 10^{-11} \times 0.1}{0.2} = -3.3 \times 10^{-11}\text{J kg}^{-1}$$

97. In the first case

$$U_i = \frac{-GMm}{R};\ U_{f_1} = -\frac{GMm}{(R + z)}$$

$$\Delta U_1 = GMm\left[\frac{1}{R} - \frac{1}{(R+z)}\right]$$

$$= \frac{GMmz}{R(R+z)} = \frac{GMmz}{R^2}$$

$(\therefore R + z \approx R)$

$$= mgz \left(\because \frac{GM}{R^2} = g\right)$$

In the second case

$$U_{f_2} = -\frac{GMm}{2R}$$

$$\Delta U_2 = GMm\left[\frac{1}{R} - \frac{1}{2R}\right] = \frac{GMm}{R}\left[1 - \frac{1}{2}\right]$$

$$= \frac{GMm}{2R}$$

$$= \frac{GM}{R^2}\frac{mR}{2} = \frac{mgR}{2}$$

$\therefore$ Work done are mgz and $\dfrac{mgR}{2}$ respectively.

98. Let the gravitational force be 0 at a point lying at distance x from M (= 900 gram)

$$\frac{GM}{x^2} = \frac{Gm}{(d - x)^2} \Rightarrow \frac{d}{x} - 1 = \sqrt{\frac{m}{M}} \Rightarrow x = \frac{3}{5}d$$

$$d - x = \frac{2}{5}d$$

Given $V_p = -\left(\frac{Gm}{d-x} + \frac{GM}{x}\right)$

We get $V_p = -\frac{Gm}{d}\left(\frac{1}{2/5} + \frac{9/4}{3/5}\right)$

$$= -\frac{6.67 \times 10^{-11} \times 0.4}{0.1}\left[\frac{25}{4}\right]$$

$$= -1.67 \times 10^{-9}\ \text{J kg}^{-1}$$

$$= -1.67\ \text{nJ kg}^{-1}$$

99. Acceleration due to gravity$= \frac{GM}{x^2} = \frac{G\frac{4}{3}\pi x^3 \rho}{x^2}$

$$= \frac{4}{3}\pi G x \rho$$

100. On the surface of the planet, gravitational field

$$E = \frac{GM}{R^2} = G\frac{4}{3}\frac{\pi R^3 \rho}{R^2} = G\frac{4}{3}\pi\rho R$$

Field at any radial distance r

$$E_r = -G\frac{4}{3}\pi\rho r$$

Change in PE = –ve of the work done by internal forces

$$\Delta V = -G\frac{4}{3}\pi\rho\int_0^R r.\,dr$$

$$|\Delta V| = G\frac{4}{3}\pi\rho\frac{R^2}{2} = \frac{GM}{2R} = \text{KE at the centre.}$$

$$KE_{esc} = \frac{GM}{R} \Rightarrow \left(\frac{v'}{v}\right)^2 = \frac{GM/2R}{GM/R}$$

$$v' = \frac{v}{\sqrt{2}}$$

Aliter:

Total energy at surface = PE + KE

$$= -\frac{GMm}{R} + 0 = -\frac{GMm}{R}$$

Total energy at centre = PE + KE

$$= -\frac{3}{2}\frac{GMm}{R} + \frac{1}{2}mv_c^2$$

($\therefore$ potential at centre $= -\frac{3GM}{2R}$)

v_C = velocity at the centre

$\Rightarrow$ Total energy is conserved

$$-\frac{3}{2}\frac{GMm}{R} + \frac{1}{2}mv_c^2 = -\frac{GMm}{R}$$

$$\Rightarrow \frac{1}{2}mv_c^2 = \frac{GMm}{2R}$$

$$v_c^2 = \frac{GM}{R} = \frac{1}{2}.\frac{2GM}{R} = \frac{1}{2}.v^2$$

$$\therefore\ v_c = \frac{v}{\sqrt{2}}$$

101. Let the maximum height reached be βR from centre of the planet

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{\beta R}$$

$$\frac{1}{2}mv_{esc}^2 = \frac{GMm}{R};\ \text{when } v = \frac{v_{esc}}{2},$$

$$\frac{1}{2}mv^2 = \frac{1}{2}m\frac{v_{esc}^2}{4} = \frac{GMm}{4R}$$

$$\therefore\ \frac{GMm}{4R} - \frac{GMm}{R} = -\frac{GMm}{\beta R}$$

$$\frac{1}{4} = \left(1 - \frac{1}{\beta}\right)$$

$$\frac{1}{\beta} = \frac{3}{4},\ \beta = \frac{4}{3}$$

$$H = \frac{4}{3}R - R = \frac{R}{3}$$

102. $(K.E)_{escape} = \frac{GMm}{R_e}$

(K.E) body initially $= \frac{3}{4}G\frac{Mm}{R_e}$

By law of conservation of energy

(KE + PE) initial = (KE + PE) at height h

$\frac{3}{4}\frac{GMm}{R_e} - \frac{GMm}{R_e} = 0 - \frac{GMm}{(R_e + h)}$ (as velocity at maximum height is 0)

$$-\frac{1}{4}\frac{GMm}{R_e} = -\frac{GMm}{R_e + h}$$

$$\therefore\ 4R_e = R_e + h \Rightarrow 3R_e = h \Rightarrow h = 3R_e$$

103. $\frac{1}{2}mv^2 = \frac{GMm}{R}$, $v = \sqrt{\frac{2GM}{R}}$

104. $T^2 \propto a^3$

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

$$\Rightarrow T' = \frac{T}{8}$$

105. $\frac{mv^2}{r} = \frac{km}{r^n} \Rightarrow \frac{1}{v} \propto r^{\frac{n-1}{2}}$

(Force = field × m = $\frac{km}{r^n}$)

$$\left(\frac{r_1}{r_2}\right)^{\frac{n-1}{2}} = \left(\frac{v_2}{v_1}\right) \Rightarrow \left(\frac{1}{2}\right)^{\frac{n-1}{2}} = \frac{1}{2}$$

$$\frac{n-1}{2} = 1 \Rightarrow n = 3$$

106. From Kepler's law, $T^2 \propto R^3$

$$\therefore T' = \left(\frac{6400}{8000}\right)^{3/2} T = (0.8)^{3/2}\, T$$

107. $\frac{T_1}{T} = \left(\frac{R_1}{R}\right)^{\frac{3}{2}} = (1.21)^{\frac{3}{2}} = 1.331 \Rightarrow 33\%$

108. $\frac{\pi^2 m}{R^{1/2}} = \frac{mv^2}{R}$

$$v^2 = \pi^2 R^{1/2} \Rightarrow v = \pi R^{1/4}$$

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{\pi R^{1/4}} = 2R^{3/4}$$

Aliter:

$$\frac{mv^2}{R} = mR\omega^2 = mR\frac{4\pi^2}{T^2} \qquad \text{— (i)}$$

$$\left(\omega = \frac{2\pi}{T}\right)$$

Also $\frac{mv^2}{R} = \frac{\pi^2 m}{\sqrt{R}}$ — (ii) (data)

From (i) & (ii) $\frac{mR4\pi^2}{T^2} = \frac{\pi^2 m}{\sqrt{R}}$

$$\Rightarrow T = 2R^{3/4}$$

109. Here by Kepler's law

$$T^2 \propto r^3 \propto \left(\frac{D}{2}\right)^3 \propto D^3$$

$$D_1 = D; D_2 = 4D$$

$$\therefore \left(\frac{T_1}{T_2}\right)^2 = \left(\frac{D_1}{D_2}\right)^3 = \left(\frac{D}{4D}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

$$\therefore \frac{T_1}{T_2} = \frac{1}{8}$$

$$\Rightarrow \frac{f_1}{f_2} = \frac{T_2}{T_1} = 8$$

110. $\frac{1}{f} \propto a^{\frac{3}{2}} \Rightarrow f \propto a^{-3/2}$

$$\frac{\Delta f}{f} = \frac{-3}{2}\frac{\Delta a}{a}$$

111. $v_{esc} = \sqrt{\frac{2GM}{R}}$

$$\propto \sqrt{\frac{M}{R}} = \sqrt{\frac{4}{3}\pi R^2 \rho}$$

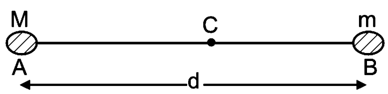
$$\Rightarrow v_{esc} \propto \sqrt{R^2\rho}.$$

Density of moon is lesser than that of the earth. But that alone is not the total reason.

112. Gravitational force = centrifugal force, so that the net force in the accelerated frame is zero.

113. The bolt will continue in the same orbit because it experiences apparent weightlessness.

114.



As C is the centre of mass M (AC) = m (BC)

AC + BC = d ⇒ AC = (d – BC)

M(d – BC) = m BC ⇒ Md = BC(m +M)

$$\Rightarrow BC = \frac{Md}{M+m} \Rightarrow AC = d - \frac{Md}{M+m} = \frac{md}{M+m}$$

115. $\frac{GmM}{d^2} = \frac{mv_1^2}{r_1} = \frac{mv_2^2}{r_2}$ = centripetal force

Total energy = KE + PE

$$KE = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{GmM}{2d^2}(r_1 + r_2)$$

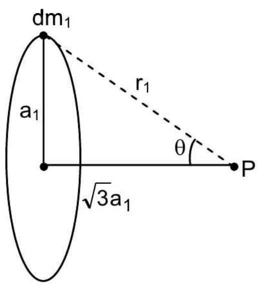
$$= \frac{GmM(d)}{2d^2} = \frac{GmM}{2d}$$

$$PE = -\frac{GmM}{d}$$

$$TE = -\frac{GmM}{2d}$$

116. Since external force is zero, momentum is conserved. Angular momentum is conserved (central force). Total mechanical energy is conserved (conservative force)

117.



Considering a small element dm_1 in m_1, $dE_1 = \frac{Gdm_1}{r_1^2}$.

Since sin θ components of $\overline{dE}$ from radially opposite elements cancel out, only cos θ components add up. Total field E is

$$E_1 = \int dE_1 = \int \frac{Gdm_1}{r_1^2} \cos\theta \, d\theta$$

But $\cos\theta = \frac{\sqrt{3}}{2}$ and

$$E_1 = \frac{Gm_1}{r_1^2} \cdot \frac{\sqrt{3}}{2} \text{ (where } r_1 = \sqrt{a_1^2 + \left(\sqrt{3}a_1\right)^2} = 2a_1)$$

$$= \frac{Gm_1}{4a_1^2} \cdot \frac{\sqrt{3}}{2}$$

$\Rightarrow$ For P to have zero field

$$\frac{Gm_1\sqrt{3}}{8a_1^2} = \frac{Gm_2\sqrt{3}}{8a_2^2} \Rightarrow \frac{m_1}{m_2} = \frac{a_1^2}{a_2^2}$$

$$\text{Potential at P} : V_P = -\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)$$

$$= -\left(\frac{Gm_1}{2a_1} + \frac{Gm_2}{2a_2}\right) = \frac{-G}{2}\left(\frac{m_1}{a_1} + \frac{m_2}{a_2}\right)$$

PE is maximum at infinity, hence (c) is wrong.

118. $g\left(1 - \frac{2h}{R_E}\right) = g\left(1 - \frac{d}{R_E}\right) \Rightarrow 2h = d$

$$h = \frac{d}{2} = \frac{0.02R_E}{2} = 0.01R_E$$

Escape velocity at Q $\Rightarrow \frac{1}{2}mv_Q^2$

$$= \frac{GMm}{\left(R_E + 0.01R_E\right)}$$

$$v_Q = \sqrt{\frac{2GM}{R_E}}\left(1 + 0.01\right)^{-\frac{1}{2}}$$

$$= v_{esc}(1 - 0.005) = 0.995\, v_{esc}$$

$\Rightarrow$ 0.5 % less than surface value

$$\text{Potential at P} : V_P = -\frac{GM}{2R_E^3}\left(3R_E^2 - 0.98^2 R_E^2\right)$$

$$= -\frac{GM}{R_E} \cdot \frac{1}{2}\left(3 - \left(1 - 0.02\right)^2\right)$$

$$\simeq -\frac{GM}{R_E} \cdot \frac{1}{2}\left(2 + 0.04\right) = -\frac{GM}{R_E}\left(1 + 0.02\right)$$

$$\therefore \quad \frac{1}{2}m\,v_P^2 = \frac{GM}{R_E}m\left(1 + 0.02\right)$$

$$\Rightarrow v_P = \sqrt{\frac{2GM}{R_E}}\left(1 + 0.02\right)^{1/2}$$

$$v_{esc}(1 + 0.01) = 1.01\, v_{esc}$$

$\therefore$ $v_P \to$ 1% more than v_{esc}

Clearly, the escape velocity from P will be more than that from surface as some energy is required to first bring it to the surface and then make it escape.

119. $T = \frac{2\pi}{\sqrt{GM}} r^{3/2}; \; v = \sqrt{\frac{GM}{r}}; \; E = -\frac{GMm}{r}$

$$\Rightarrow \quad T \propto r^{3/2}; \; v \propto r^{-\frac{1}{2}}; \; E \propto r^{-1}$$

[use the result $A \propto x^n \Rightarrow \frac{\delta A}{A} = n\frac{\delta x}{x}$]

$$\frac{\delta T}{T} = \left(\frac{3}{2}\right)\frac{\delta r}{r}; \; \frac{\delta v}{v} = \left(-\frac{1}{2}\right)\frac{\delta r}{r}; \; \frac{\delta E}{E} = (-1)\frac{\delta r}{r}$$

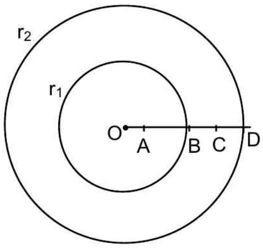
Put $T = T_0$; $v = v_0$; $E = -E_0$ (Since E is negative and E_0 stands for its modulus). Also put $\delta r = -\Delta r$ to get

$$\delta T = -T_0 \frac{3}{2}\frac{\Delta r}{r};\ \delta v = v_0\left(\frac{1}{2}\right)\frac{\Delta r}{r};\ \delta E = -E_0\frac{\Delta r}{r}$$

$\Rightarrow$ T decreases; v increases; E decreases.

Correct options a, c, d

120.



Potential at internal points and on the surface of spherical shell is $-\frac{GM}{R}$ (where, R is radius) and at external points is $-\frac{GM}{r}$ (where, r is the distance of the point from the centre). Using this result

At A (where $r < r_1$) $V_A = -\frac{Gm}{r_1} - \frac{Gm}{r_2}$

At B (where $r = r_1$) V_B

$$= -\frac{Gm}{r_1} - \frac{Gm}{r_2} = -\frac{Gm}{r} - \frac{Gm}{r_2}$$

At C (where $r_1 < r < r_2$) $V_C = -\frac{Gm}{r} - \frac{Gm}{r_2}$

At D (where $r = r_2$) V_D

$$= -\frac{Gm}{r_2} - \frac{Gm}{r_2} = -\frac{Gm}{r_2} - \frac{Gm}{r}\ (\text{since } r = r_2)$$

$$V_D - V_B = -\frac{Gm}{r_2} + \frac{Gm}{r_1}$$

$$V_D - V_A = -\frac{Gm}{r_2} + \frac{Gm}{r_1}$$

(a) $mV_A = -\frac{Gm^2}{r_1} - \frac{Gm^2}{r_2}$ (p) $-\frac{Gm^2}{r_2} + \frac{Gm^2}{r_1}$

(b) mV_B (q) $-\frac{Gm^2}{r} - \frac{Gm^2}{r_2}$

$$= -\frac{Gm^2}{r} - \frac{Gm^2}{r_2}$$

(c) $m(V_D - V_B)$ (r) $-Gm^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$

$$= -\frac{Gm^2}{r_2} + \frac{Gm^2}{r_1}$$

(d) mv_D (s) $m(V_D - V_A)$

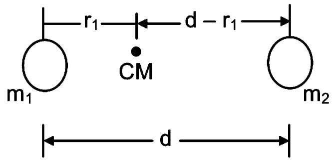
$$= -\frac{Gm^2}{r_2} - \frac{Gm^2}{r_2}$$

$$= -\frac{Gm^2}{r_2} - \frac{Gm^2}{r}$$

a $\rightarrow$ r; b $\rightarrow$ q, r; c $\rightarrow$ p, s; d $\rightarrow$ q

Additional Practice Exercise

121.



$m_1 r_1 = m_2 (d - r_1)$

$\Rightarrow$ $(m_1 + m_2) r_1 = m_2 d$ —(1)

centripetal force:

$$\frac{Gm_1 m_2}{d^2} = m_1\omega^2 r_1 = m_1\frac{4\pi^2}{T^2}r_1$$

$$\Rightarrow \frac{Gm_2}{d^2} = \frac{4\pi^2}{T^2}\cdot\frac{m_2 d}{m_1 + m_2}\ (\text{using (1)})$$

$$\Rightarrow d = \left[\frac{G\cdot 10M\cdot T^2}{4\pi^2}\right]^{1/3}\ [\text{where M = mass of earth}]$$

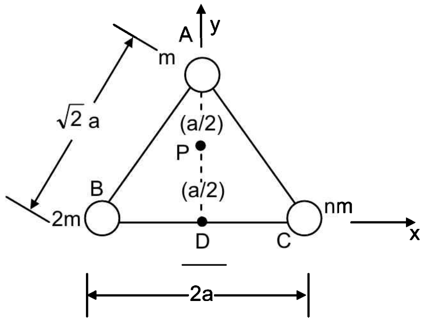
$GM = gR^2$

Using data,

$$d = \left[\frac{10\cdot 10\cdot(6.4\times10^6)^2\times 5\times 10^{12}}{4\times 10}\right]^{1/3}$$

$= 8 \times 10^8$ m.

122. Let D be the origin



Equipotential curve has AD as tangent at P

Field is perpendicular to AD at P

$\Rightarrow$ $\overline{E}_P$ is parallel to x-axis

$\Rightarrow$ $y_{component}$ = zero

Considering P as origin and using vector form

$$\overline{E} = \frac{Gm\overline{r}}{r^3},$$

where $\overline{r}$ is radius vector to the point mass from origin.

Due to mas m:

$$\overline{E}_{P,m} = \frac{Gm}{\left(a/2\right)^3}\left[0\check{i} + \frac{a}{2}\check{j}\right]$$

$$\overline{E}_{P,2m} = \frac{G.2m}{\left[a^2 + \left(\frac{a}{2}\right)^2\right]^{3/2}}\left[-a\hat{i} - \frac{a}{2}\hat{j}\right]$$

$$\overline{E}_{P,nm} = \frac{G.nm}{\left[a^2 + \left(\frac{a}{2}\right)^2\right]^{3/2}}\left[a\hat{i} - \frac{a}{2}\hat{j}\right]$$, where n is a positive number.

Equating $\hat{j}$ component to zero

$$\frac{Gm}{\left(\frac{a}{2}\right)^3}\frac{a}{2} - \frac{G2m}{\left(\frac{5a^2}{4}\right)^{3/2}}\frac{a}{2} - \frac{Gnm}{\left(\frac{5a^2}{4}\right)^{3/2}}\frac{a}{2} = 0$$

$\Rightarrow$ $n = 5\sqrt{5} - 2$

$\therefore$ $\hat{i}$ component is:

$$E = \frac{G}{\left(\frac{5a^2}{4}\right)^{3/2}}\left(2m(-a) + \left(5\sqrt{5} - 2\right)ma\right)$$

$$\overline{E} = \frac{8\,(5\sqrt{5} - 4)}{5\sqrt{5}} \cdot \frac{Gm}{a^2}\hat{i}.$$

123. (i) Initial energy of system $= \dfrac{-Gm\,8m}{d}$

Let final velocity of smaller star be v.

Then final velocity of bigger star $= \dfrac{v}{8}$

($\because$ momentum conservation)

$\therefore$ Final KE of system

$$= \frac{1}{2}mv^2 + \frac{1}{2}8m\frac{v^2}{64} = \frac{9}{16}mv^2.$$

$\because$ Radius of bigger star is 2R,

density same, mass 8 times

Final distance between centres = 3R

$\therefore$ Final PE $= \dfrac{-G.m.8m}{3R}$

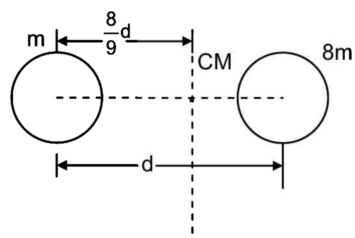
Equating energies, $\Rightarrow \dfrac{9}{16}mv^2 - \dfrac{8}{3}\dfrac{Gm^2}{R}$

$$= \frac{-8Gm^2}{d}$$

$$v = \frac{8\sqrt{2}}{3\sqrt{3}}\sqrt{\frac{Gm\left(d - 3R\right)}{Rd}}$$

(ii) Initial position of CM: mx = 8 m (d – x)

$$x = \frac{8}{9}d$$

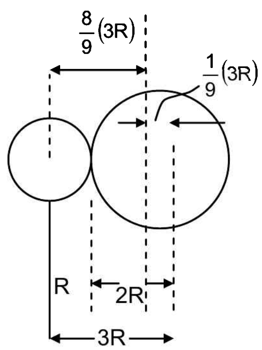


No displacement for centre of mass.

Final CM:

Final d when they touch each other = 3 R

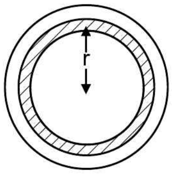
Position from centre of small star $= \dfrac{8}{9}$ (3 R)



$\therefore$ Distance travelled by smaller star

$$= \frac{8}{9}\left(d - 3R\right).$$

124.



Calculation of mass of sphere of radius r

Consider an elemental shell of radius r and thickness dr.

Its mass $dm = 4\pi r^2.dr.\rho$

$$= 4\pi r^2.dr.\rho_0.\frac{\sqrt{R}}{\sqrt{r}}$$

$$= 4\pi\rho_0\sqrt{R}r^{3/2}.dr$$

$$\therefore m_r = \int_0^r dm = 4\pi\rho_0\sqrt{R}.\int_0^r r^{3/2}dr$$

$$= 4\pi\rho_0\sqrt{R}\left.\frac{r^{5/2}}{5/2}\right|_0^r$$

$$m_r = \frac{8}{5}\pi\rho_0\sqrt{R}.r^{5/2} \qquad \text{— (1)}$$

Total mass : $M = m_R$ (Put $r = R$)

$$= \frac{8}{5}\pi\rho_0.\sqrt{R}.R^{5/2} = \frac{8}{5}\pi\rho_0 R^3$$

$$E_R = \frac{Gm_R}{R^2} = \frac{8}{5}\pi\rho_0 R$$

$$E_r = \frac{Gm_r}{r^2} = \frac{8}{5}\pi G\rho_0\sqrt{R}\, r^{1/2}$$

$$V_R = \frac{-Gm_R}{R} = -\frac{8}{5}\pi G\rho_0 R^2$$

$$V_r = V_R - \int_R^r \overline{E}_r.\overline{dr}$$

$$(\overline{E}_r.\overline{dr} = E_r\, dr \cos\pi = -E_r dr) = V_R + \int_R^r E_r dr$$

$$= V_R + \int_R^r \frac{8}{5}G\pi\rho_0\sqrt{R}.r^{1/2}dr$$

$$= \frac{-8}{5}\pi G\rho_0 R^2 + \frac{8G}{5}\pi\rho_0\sqrt{R}\frac{\left(r^{3/2} - R^{3/2}\right)}{\frac{3}{2}}$$

$$= \frac{-8}{5}\pi G\rho_0 R^2 - \frac{8}{5}\pi G\rho_0 R^2.\frac{2}{3} + \frac{8}{5}.\frac{2}{3}G\pi\rho_0\sqrt{R}r^{3/2}$$

$$= -\frac{8}{3}\pi G\rho_0 R^2 + \frac{16}{15}\pi G\rho_0\sqrt{R}.r^{3/2}$$

125. Potential inside a uniform solid sphere at a radial distance r from centre of sphere

$$= -\frac{GM}{2R^3}\left(3R^2 - r^2\right)$$

(R – radius of the sphere, M – mass of original sphere)

Potential at the centre $V_0 = -\frac{GM}{R}.\frac{3}{2}$

Treat cavity as the result of super imposition of –ve mass of same density, over the existing mass. Negative mass produces positive potential, Mass |m'| of cavity is

$$|m'| = \frac{4}{3}\pi\left(\frac{R}{3}\right)^3\rho = \frac{M}{27} \quad \text{and its radius is } \frac{R}{3}.$$

At A, the point is on the surface of both the spheres

$$V_A = -\frac{GM}{R} + G\left(\frac{M}{27}\right).\frac{1}{R/3} = -\frac{GM}{R}.\frac{8}{9}$$

At P, the point is at the centre of –ve mass sphere, and at $r = \frac{2R}{3}$ for the original sphere.

$$V_P = -\frac{GM}{2R^3}\left(3R^2 - \frac{4R^2}{9}\right) + G\left(\frac{M}{27}\right)\frac{3}{2.\left(\frac{R}{3}\right)}$$

$$= -\frac{GM}{2R}\left[\frac{27-4}{9} - \frac{9}{27}\right]$$

$$= -\frac{GM}{R}\frac{10}{9}$$

At B, the point is on the surface of the –ve mass and at $r = \frac{R}{3}$ for the original sphere.

$$V_B = -\frac{GM}{2R^3}\left(3R^2 - \frac{R^2}{9}\right) + G\left(\frac{M}{27}\right).\frac{1}{R/3}$$

$$= -\frac{GM}{R}\left[\frac{1}{2}.\frac{26}{9} - \frac{1}{9}\right]$$

$$= -\frac{GM}{R}\times\frac{12}{9}$$

Ratio $V_A : V_P : V_B = 8 : 10 : 12 = 4 : 5 : 6$

126. Density at any r is

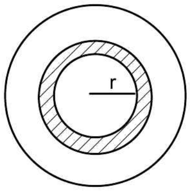
$$\rho_r = 10^4 - \frac{(10^4 - 10^3)}{R}r$$

$$= 10^4 - 10^3\,(10-1)\,\frac{r}{R} = 10^3\left[10 - \frac{9r}{R}\right] \text{kg m}^{-3}$$

m at r:

dm of a thin strip thickness dr at r is:

$dm = 4\pi r^2 dr.\rho(r)$



$$\therefore \quad m_r = \int_0^r 4\pi r^2 dr\,\rho(r)$$

$$= \int_0^r 4\pi r^2 dr\left[10^3\left(10 - \frac{9r}{R}\right)\right]$$

$$= 4\pi 10^3 \int_0^r 10r^2 dr - \frac{9r^3}{R}.dr$$

$$= 4\pi\ 10^3.\left(\frac{10\ r^3}{3} - \frac{9r^4}{4R}\right)$$

(i) Mass of the planet upto a radius r

$$\int_0^r 4\pi r^2 \rho dr = \int_0^r 4\pi r^2.10^3\left[10 - \frac{9r}{R}\right]dr$$

$$= 4\pi.10^3\left[\frac{10}{3}r^3 - \frac{9}{4R}r^4\right]$$

Potential at its surface when there is no matter outside

$$= -\frac{Gm}{r}$$

$$= -\frac{G}{r}.4\pi.10^3\left[\frac{10}{3}r^3 - \frac{9}{4R}r^4\right]$$

Work done to add an elemental shell of mass dm and thickness dr = V dm

$$= -\frac{G}{r}.4\pi.10^3\left[\frac{10}{3}r^3 - \frac{9}{4R}r^4\right].$$

$$4\pi r^2.10^3\left[10 - \frac{9r}{R}\right]dr$$

$$= -G(4\pi)^2.10^6.\ \frac{r^2}{r}\left[\frac{10}{3}r^3 - \frac{9}{4R}r^4\right]\left[10 - \frac{9r}{R}\right]dr$$

Total work done to form the planet of radius R

$$= \text{self energy of the planet} = \int V\,dm$$

$$= -G(4\pi)^2.10^6 \int_0^R r\left[\frac{10}{3}r^3 - \frac{9}{4R}r^4\right]\left[10 - \frac{9r}{R}\right]dr$$

$$= -G(4\pi)^2.10^6 \times$$

$$\left[\int_0^R \frac{100}{3}r^4 dr - \int_0^R \frac{30}{R}r^5 dr - \int_0^R \frac{45}{2R}r^5 dr + \int_0^R \frac{81}{4R^2}r^6 dr\right]$$

$$= -G(4\pi)^2.10^6.\left(\frac{17}{21}\right)R^5$$

Binding energy = −(self energy)

$$= G.(4\pi)^2.10^6.\left(\frac{17}{21}\right)R^5$$

$$= \frac{272}{21}.10^6\pi^2 GR^5$$

$$\text{(ii)}\quad \therefore\ M_1 = 4\pi.10^3\left(\frac{10}{3}R^3 - \frac{9R^4}{4R}\right)$$

$$= 4\pi.10^3.R^3\left(\frac{10}{3} - \frac{9}{4}\right)$$

$$= 4\pi.10^3.R^3.\frac{13}{12}$$

$$\text{V on surface}\quad -\frac{GM}{R} = -G4\pi.10^3.\frac{13}{12}R^2$$

$\therefore$ v_{esc} is:

$$\frac{1}{2}mv_{esc}^2 = |\,mV\,| = G.\pi.10^3\frac{13}{3}R^2.m$$

$$v_{esc} = \sqrt{\frac{26}{3}G\pi 10^3}\ R$$

$$= R.\sqrt{\frac{26}{3\times 10}G\pi 10^3 \times 10}$$

$$= 100\ R\sqrt{\frac{26}{30}G\pi}$$

$$= 100\ R\sqrt{\frac{13}{15}G\pi}$$

127. Let masses of A and B be m, fm.

Let orbital radius be R + h where

R = radius of Earth. M = mass of Earth.

Then orbital velocity $v = \sqrt{\dfrac{GM}{R+h}}$

Energy of A after collision:

$$(PE)_A = \frac{-GMm}{R+h},\ \left(K.E\right)_A = \frac{1}{2}\frac{mv^2}{4} = \frac{GMm}{8(R+h)}$$

$$E_A = \frac{-7}{8}\frac{GMm}{R+h}$$

Energy of A on reaching earth:

$$(PE)'_A = \frac{-GMm}{R},\ \left(KE\right)'_A = \frac{1}{2}mv^2 = \frac{GMm}{2(R+h)}$$

$$\text{[Data given] } E'_A = -\frac{GMm}{R} + \frac{GMm}{2(R+h)}$$

Equating energies,

$$\Rightarrow \frac{-GMm}{R},\ \left(K.E\right)'_A = \frac{1}{2}mv^2 = \frac{GMm}{2(R+h)}$$

$$\frac{1}{R} = \frac{11}{8}\frac{1}{(R+h)} \Rightarrow h = \frac{3}{8}R = 2400 \text{ km.}$$

$\therefore$ Orbit radius = R + h = 6400 + 2400 = 8800 km

128. Let v be final velocity (Perpendicular to radius vector, r).

For conservation of angular momentum about the centre of Earth ($\therefore$ g acts along centre of Earth)

$$mvr = m\sqrt{\frac{5GM}{4R}} \cdot R\cos 37^\circ$$

$$\Rightarrow \quad v = \frac{0.4\sqrt{5GMR}}{r} \qquad \text{— (1)}$$

Energy equation:

$$\frac{-GMm}{R} + \frac{1}{2}m\cdot\frac{5GM}{4R} = \frac{-GMm}{r} + \frac{1}{2}m(0.8)\frac{GMR}{r^2}$$

$$\Rightarrow -1 + \frac{5}{8} = -\frac{R}{r} + 0.4\frac{R^2}{r^2}$$

$$\Rightarrow \text{Let } \frac{R}{r} = f \Rightarrow 0.4f^2 - f + \frac{3}{8} = 0$$

$$f = \frac{1 \pm \sqrt{1-0.6}}{0.8} = \frac{1 \pm \sqrt{0.4}}{0.8}$$

since r > R, f < 1

$$\therefore \quad f = \frac{1-\sqrt{0.4}}{0.8} \Rightarrow \frac{R}{r} = \frac{1-\sqrt{0.4}}{0.8}$$

$$\Rightarrow r = \frac{0.8}{1-\sqrt{0.4}}R$$

height to which rocket rises is

$$r - R = R\left(\frac{0.8}{1-\sqrt{0.4}} - 1\right)$$

$$= R\left(\frac{0.8 - 1 + \sqrt{0.4}}{\left(1-\sqrt{0.4}\right)}\right)$$

$$= \frac{\left(\sqrt{0.4} - 0.2\right)}{\left(1-\sqrt{0.4}\right)} \times \frac{\left(1+\sqrt{0.4}\right)}{\left(1+\sqrt{0.4}\right)} \cdot R$$

$$= \left(\frac{1 + 4\sqrt{0.4}}{3}\right)R.$$

129. Orbital radius $r_1 = 9 \times 10^6$ m

$r_2 = r_1 + \Delta r = 9.1 \times 10^6$ m

$$\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{2\pi r_1/v_1}$$

$$= \frac{1}{r_1}\sqrt{\frac{GM}{r_1}} = \frac{\sqrt{GM}}{r_1^{3/2}};\left[\because v = \sqrt{\frac{GM}{r}}\right]$$

$$\omega_2 = \frac{\sqrt{GM}}{(r_1 + \Delta r)^{3/2}}$$

$$= \frac{\sqrt{GM}\left(1 - \frac{3}{2}\frac{\Delta r}{r_1}\right)}{r_1^{3/2}}$$

$$\omega_1 - \omega_2 = \frac{\sqrt{GM}}{r_1^{3/2}}\cdot\frac{3}{2}\cdot\frac{\Delta r}{r_1} = \sqrt{GM}\cdot\frac{3}{2}\cdot\frac{\Delta r}{r_1^{5/2}}$$

$$= R\sqrt{g}\cdot\frac{3}{2}\cdot\frac{\Delta r}{r_1^{5/2}} = \frac{3}{2}\pi R\frac{\Delta r}{r_1^{5/2}} \quad [\because g \simeq \pi^2]$$

$$\text{Required } t = \frac{\pi}{\omega_1 - \omega_2} = \frac{2r_1^{5/2}}{3R\Delta r}$$

Substituting [r_1 = 6400 km + 2600 km

= 9×10^6 m, $\Delta r = 0.1 \times 10^6$ m]

$$\therefore \quad t = \frac{2\times\left(9\times10^6\right)^{5/2}}{3\times6.4\times10^6\times10^5}$$

$$= \frac{81\times10^4}{3.2}\ s = \frac{81\times10^4}{3.2\times8.64\times10^4}\ \text{day}$$

= 3 day.

130. In orbit P.E = $-2E_0$, KE = $+E_0$

In order to escape, KE should be $+2E_0$

$$\Rightarrow \frac{1}{2}mv_{esc}^2 = 2E_0 \Rightarrow v_{esc} = 2\sqrt{\frac{E_0}{m}}$$

$$T_2 = nT_1 \Rightarrow r_2^{3/2} = nr_1^{3/2} \Rightarrow r_2 = n^{2/3} \cdot r_1$$

$$\text{Initial total energy} = -E_0 = \frac{-GMm}{2r_1}$$

$$\text{Final total energy required} = \frac{-GMm}{2r_2}$$

$$= \frac{-GMm}{2n^{2/3} \cdot r_1} = \frac{-E_0}{n^{2/3}}$$

$$\text{Work to be done} = \frac{-E_0}{n^{2/3}} + E_0 = E_0\left(1 - \frac{1}{n^{2/3}}\right)$$

131. Suffixes : 1 : Earth, 2 : planet

$$\text{Given} : g_2 > g_1 \Rightarrow \frac{g_2}{g_1} > 1 \qquad \text{— (1)}$$

$$\text{and } g_2\left(1 - \frac{h}{R_2}\right) < g_1\left(1 - \frac{2h}{R_1}\right)$$

$$\Rightarrow \frac{g_2}{g1} < \frac{\left[1 - \dfrac{h}{R_1/2}\right]}{1 - \dfrac{h}{R_2}} \qquad \text{— (2)}$$

$$\text{(1) and (2)} \quad \frac{\left(1 - \dfrac{h}{R_1/2}\right)}{1 - \dfrac{h}{R_2}} > 1$$

$$\Rightarrow \frac{R_1}{2} > R_2 \Rightarrow \frac{R_2}{R_1} < \frac{1}{2}$$

132. $g = \dfrac{GM}{R^2}$

133. $g = \dfrac{GM}{(R+h)^2}$

h same, R less $\Rightarrow$ g more

134. key (a)

$$T = 2\pi\sqrt{\frac{\ell}{g}},\ T_{depth} = 2\pi\sqrt{\frac{\ell}{\frac{g}{2}}} = \sqrt{2}T = 1.4\ T$$

$$\frac{T_h}{T} = \sqrt{\frac{g}{g_h}} = \frac{3}{2} \Rightarrow T_h = \frac{3}{2}T$$

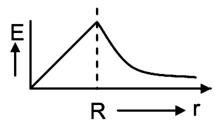
$$T_{depth} = \sqrt{2}\ T \Rightarrow \text{increase} = +40\%$$

$$T_{height} = \frac{3}{2}T \Rightarrow \text{increase} = +50\%$$

135. At P : due to A : Zero

due to B : + E(say)

total :+E



At Q :due to A : - E

due to B : +E' (not zero)

total : -(E-E').

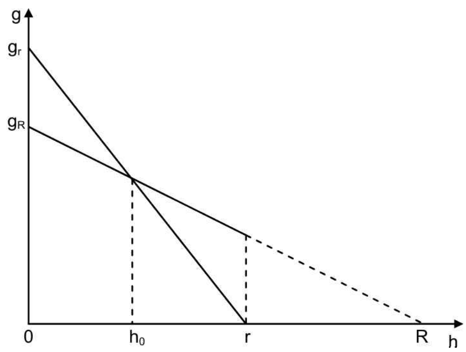
E' can be >, = or < E

136. Since the distance d from the centre of A to Q and centre of B to P are same the potential at P due to B and at Q due to A are same, say $-V'$. At P, the potential due to A is same as its surface say $-V$. Hence $V_P = -(V' + V)$.

At Q the potential due to B is (modulus value) larger than that at its surface, say $-(V + \Delta V)$

Hence $V_Q = -(V' + V + \Delta V) \Rightarrow |V_Q| > |V_P|$.

137.



Let R and r be the initial and final radii. The value of g at a depth h is:

$$g_R = \frac{GM}{R^3}(R - h) \text{(for initial R)}.$$

$$g_r = \frac{GM}{r^3}(r - h) \text{(for final r)}.$$

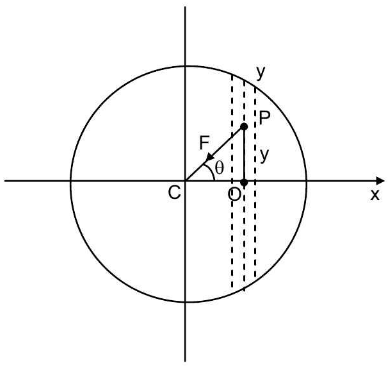
The variation of g with h when plotted for the two cases will be as shown where

$$g_R = \frac{GM}{R^3}.R = \frac{GM}{R^2} \text{ at h = 0 (surface)}$$

$$\text{and } g_r = \frac{GM}{r^3}.r = \frac{GM}{r^2} \text{ at h = 0}$$

Obviously h < r. Hence from the plot we can conclude that at a particular $h = h_0$ where the plots meet the value of $g'_R = g_r$. For $h < h_0$ $g''_r > g''_R$ and for $h > h_0$ $g'''_R > g'''_r$.

138.



Consider the particles at a position P at a distance PO = y as shown in figure. The force F on the particles is only due to Gravitational field exerted by mass m, of the sphere of radius r = CP

$$F = \frac{Gm'}{r^2}. m[m \to \text{mass of the particles}]$$

$$m' = (\frac{4}{3}\pi r^3 \rho) \Rightarrow F = G\frac{\frac{4}{3}\pi r^3 \rho}{r^2} m$$

$$= G\frac{4}{3}\pi r \rho.m$$

Force downwards towards O is F' = F sin θ

$$= G\frac{4}{3}\pi \, r \, \rho \, m. \frac{y}{r}$$

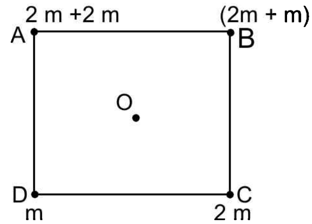
$$F' = (G\frac{4}{3}\pi\rho)\ y.\ m. \Rightarrow$$

$F \propto y$

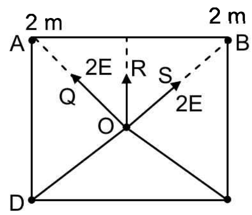
$$\propto \frac{\ell}{2} \text{(initially)}$$

$\propto \ell \Rightarrow n = 1$

139. The equivalent is



$\Rightarrow$ The 2 m mass at corner A and C will produce fields at O neutralising each other and so also masses m at B and D. Hence the field at O is only due to the left over masses as shown. If $|E|$ is the modulus value of field at O due to mass at any corner, then $|2E|$ acts along OA and OB. The resultant of vector addition is obviously along OR. $\bar{E}_R = 2(2E\cos 45°)$



140. (d)

Force F on unit mass at a distance x from the centre is

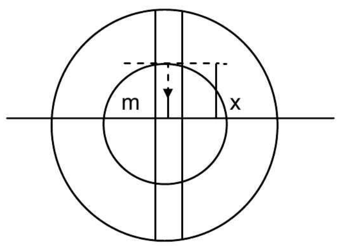
$$F = \frac{Gm}{x^2} \times 1 = \frac{G\frac{4}{3}\pi x^3 \rho}{x^2}$$

$$= G\frac{4}{3}\pi\rho.\ x.$$

$\Rightarrow$ $a \propto x\rho$

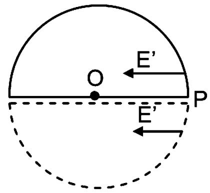
At top position

$$\frac{a_1}{a_2} = \frac{R_1\rho_1}{R_2\rho_2}$$



141. Let M' = 2 M be the mass of the total sphere.

E at P = 2E'



$$\frac{GM'}{R^2} = 2E'$$

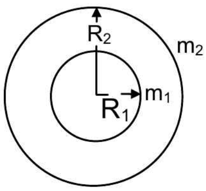
$$E' = \frac{GM'}{2R^2} = \frac{G(2M)}{2R^2}$$

$$\therefore \quad F = m\,E' = \frac{GMm}{R^2}$$

Note: only component of E' along PO is considered.

Hence acceleration = $\frac{F}{m} . = \frac{GM}{R^2}$

142.



Potential at any point Q on the surface of outer shell is (symbols m_1, R_1 for inner shell and m_2, R_2 for outer shell).

$$V_Q = -\frac{Gm_1}{R_2} - \frac{Gm_2}{R_2} = -\frac{G}{R_2}(m_1 + m_2)$$

Potential at any point P on the surface of inner shell is

$$V_P = -\frac{Gm_1}{R_1} - \frac{Gm_2}{R_2} = -\frac{G}{R_2}\left[\frac{R_2}{R_1}m_1 + m_2\right]$$

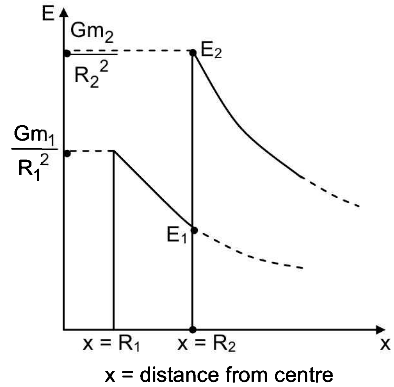
$\Rightarrow R_2 > R_1$

$\therefore$ $|V_P| > |V_Q|$ and $+|V_P|$ is the maximum value of the system and is constant within the inner shell.

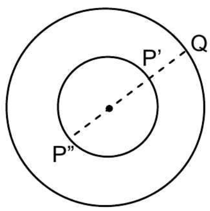
The field distribution of the system will be as shown. (Given $\frac{Gm_2}{R_2^2} > \frac{Gm_1}{R_1^2}$)

Obviously on the point on surface of outer shell field E is maximum.

If $\frac{Gm_1}{R_1^2} > \frac{Gm_2}{R_2^2}$, the maximum field E will be on the surface of inner sphere, then field and potential points coincide, not acceptable as per given condition.



Hence Q is point of maximum field and P' and P" are diametrically opposite points for maximum modulus value of potential and to get the minimum and maximum distance they lie on the same diameter as shown



Minimum PQ = P'Q = $R_2 - R_1$

Maximum PQ = P"Q = $R_2 + R_1$

Given $R_2 + R_1 = 4R_2 - 4R_1$

$R_2 = \frac{5}{3}R_1$

143. Let W be work done in each step.

$$\text{Then } 3\,W = \frac{GMm}{R} \Rightarrow W = \frac{GMm}{3R}$$

$$\therefore \quad U_P = -\frac{GMm}{R},$$

$$U_Q = U_P + W = -\frac{2}{3}\frac{GMm}{R}$$

$$= \frac{-GMm}{\frac{3}{2}R} \Rightarrow R + PQ = \frac{3}{2}R \Rightarrow PQ = \frac{R}{2} \text{—(1)}$$

$$U_R = U_Q + W = \frac{-3}{2}\frac{GMm}{R} + \frac{GMm}{3R}$$

$$= -\frac{GMm}{3R} \Rightarrow PR = 2R$$

$$\Rightarrow \quad QR = \frac{3}{2}R \qquad \text{— (2)}$$

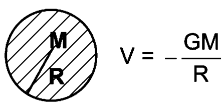
$$\therefore \quad \frac{QR}{PQ} = 3$$

144. $V = -\dfrac{GM}{R}$, $v_{esc} = \sqrt{\dfrac{2GM}{R}} \Rightarrow v_{esc} = \sqrt{-2V}$

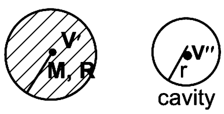
145. Well known formula for solid sphere:

V at centre = $\dfrac{3}{2}$ times V at surface

(i)



(ii)



$V = V' - V''$

For cavity: Radius r

Mass $\dfrac{M.r^3}{R^3}$

Potential at the centre of full sphere

$$V' = -\frac{3}{2}\frac{GM}{R},$$

Potential at centre of a sphere identical to cavity:

$$V'' = -\frac{3}{2}\frac{GM\frac{r^3}{R^3}}{r}$$

$$= -\frac{3}{2}\frac{GM}{R}.\frac{r^2}{R^2}$$

To get the potential at the centre of a sphere with cavity, delete the potential at the centre of the equivalent mass absent (V") from the potential of original sphere at centre which implies V = V' – V".

$$\therefore \quad V = -\frac{3}{2}\frac{GM}{R}\left(1 - \frac{r^2}{R^2}\right)$$

$$\text{Equating } -1 = -\frac{3}{2}\left(1 - \frac{r^2}{R^2}\right)$$

$$\Rightarrow \quad \frac{r^2}{R^2} = \frac{1}{3} \Rightarrow \frac{r^3}{R^3} = \frac{1}{3\sqrt{3}}$$

Since masses are proportional to $(\text{radius})^3$

$\Rightarrow$ Fraction taken out = $\dfrac{1}{3\sqrt{3}}$

146. V at centre = V at surface = $-\dfrac{GM}{R}$

R increases (M constant) $\Rightarrow$ V becomes less negative $\Rightarrow$ increases.

147. (c)

Let the particle be dropped form a height H above the surface of earth. Energy conservation requries

$$PE_h = PE_c + KE_c$$

$$-\frac{GM}{(R+h)} = -\frac{3}{2}\frac{GM}{R} + \frac{1}{2}mV_e^2$$

$$\frac{GM}{R} = \frac{1}{2}mV_e^2 \text{ (escape velocity formula)}$$

$$\Rightarrow \quad \frac{-GM}{(R+h)} = -\frac{3}{2}\frac{GM}{R} + \frac{GM}{R} = -\frac{GM}{2R}$$

$$\Rightarrow \quad 2R = R + h \Rightarrow h = R$$

148. (c)

Assume the body falls from height h. Energy conservation requires.

$$PE_h = PE_s + \frac{1}{2}mV^2$$

$$\frac{GM}{R} = \frac{1}{2}mV_e^2 - \frac{Gm}{(R+h)} = -\frac{Gm}{R} + \frac{1}{2}m.\frac{4}{5}V_e^2$$

$$= \frac{-Gm}{R} + \frac{4}{5}.\frac{Gm}{R} = \frac{-Gm}{5R}$$

$R + h = 5R$

$h = 4R$

149. (d)

In each of the curves (a), (b), (c) the points are at the same distance from the centre, and the potential is a function of distance from the center only.

150. $v_{esc}^2 = \frac{2GM}{R} \Rightarrow v_{esc}^2 \propto \rho R^2$

$\Rightarrow 2^2 = \left(\frac{1}{2}\right).x^2$

$\Rightarrow x = 2\sqrt{2}$

151. Energy is the criterion, angle, immaterial.

152. At surface: (at R); $V_e = \sqrt{\frac{2GM}{R}}$

at $\frac{3R}{2}$, $\frac{1}{2}mv'^2_e = \frac{GMm}{r} = \frac{GMm}{\left(\frac{3}{2}R\right)}$

$$v_e' = \sqrt{2 \times \frac{2}{3}\frac{GM}{R}}$$

Ratio required $= \sqrt{\frac{2}{3}}$

153. (c)

$$V_{inside} = -\frac{GM}{2R^3}(3R^2 - r^2)$$

$$= -\frac{GM}{2R^3}(2R^2 + R^2 - r^2)$$

$$= -\frac{GM}{R} - \frac{GM}{2R}\left(\frac{R^2 - r^2}{R^2}\right)$$

$$V' = V + \frac{V}{2}\frac{(R^2 - r^2)}{R^2} \left(V = -\frac{GM}{R}\right)$$

$$V' = V + \frac{(R^2 - r^2)}{R^2}.\frac{V}{2}$$

$v_{esc} \propto \sqrt{V'}$

$\propto \sqrt{a_1 - a_2 r^2}$

Apply condition f"(x).y < 0 for concave curve with respect to x. Hence curve (c).

154. Initial acceleration, $a = \frac{GM}{r^2}$

From kepler's third law $r^3 \propto T^2$

$\Rightarrow$ initial acceleration $a \propto r^{-2} \propto T^{-4/3}$

155. We know $T = \frac{2\pi}{\sqrt{GM}} r^{3/2}$

$\Rightarrow T^{-4/3} \cdot \frac{\left(\sqrt{GM}\right)^{-4/3}}{(2\pi)^{-4/3}} = r^{-2}$

$\therefore (GM)r^{-2} = T^{-4/3}.(GM)^{1/3}.(2\pi)^{4/3}$

156. Orbital velocity $= \sqrt{\frac{GM}{r}}$

Escape velocity $= \sqrt{\frac{2GM}{R}}$

Given $\sqrt{\frac{GM}{r}} = \frac{1}{2}\sqrt{\frac{2GM}{R}}$

$\Rightarrow r = 2R \Rightarrow h = R$

157. $L = I\omega = mr^2 . \frac{2\pi}{T}$

$\Rightarrow r = \sqrt{\frac{TL}{2\pi m}}$

158. Since r varies PE varies, since PE + KE is constant KE varies $\Rightarrow$ v varies

Since central force, $\bar{r} \times \bar{F} = 0$

$\Rightarrow \bar{L}$ is constant

159. $m\omega^2 r = \frac{GMm}{r^2} \Rightarrow M = \frac{\omega^2 r^3}{G}$

160. $m\omega^2 r = \frac{GMm}{r^2} \Rightarrow G = \frac{\omega^2 r^3}{M}$

$$= \frac{\omega^2 r^3}{\frac{4}{3}\pi R^3 \rho} \Rightarrow \frac{\omega^2 r^3}{R^3 \rho} = \text{constant}$$

Ratios of ω is 1 : 2 ($\because$ day ratio is 2 : 1) of r is 2 : 1 and of R is 2 : 1

$\therefore$ Ratio of density $\rho = \frac{\left(\frac{1}{2}\right)^2.2^3}{2^3} = 1:4$

161. It will escape

If v is orbital velocity, more than $\sqrt{2}v$ will make it escape.

Initial : E = –(K.E.)

Final E' = E + 3(K.E.) ($\because$ v doubles $\Rightarrow$ KE four times)

= 2(K.E.) > 0 $\Rightarrow$ it will escape.

Aliter:

Initially KE = $\frac{1}{2}mv^2 = \frac{GMm}{2r}$

Final v' = 2 v

$\therefore$ KE' = $\frac{1}{2}m\,4v^2 = 4.\frac{GMm}{2r} = \frac{2GMm}{r}$

$$TE = \frac{2GMm}{r} - \frac{GMm}{r} = \frac{GMm}{r} > 0$$

It will escape.

162. $\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}r^2 d\theta\right) = \frac{1}{2}r^2\frac{d\theta}{dt}$

$$= \frac{1}{2}r^2\omega = \frac{1}{2}mr^2\omega.\frac{1}{m}$$

$$= \frac{1}{2}\frac{I\omega}{m} = \frac{1}{2}\frac{L}{m}$$

Integrating, $A = \frac{L}{2m}T \Rightarrow mA = \frac{LT}{2}$

Here $\frac{LT}{2}$ is constant.

Am = constant

163. $T^2 \propto a^3 = \frac{(\pi ab)^3}{\pi^3 b^3} = \frac{A^3}{\pi^2 b^3} \propto \frac{1}{b^3}$

164. Only angular momentum.

165. (b)

$$V_A = -G\frac{2m}{2a} + \frac{-Gm}{a} = -2\frac{Gm}{a}$$

$$V_B = -\frac{G2m}{4a} - \frac{Gm}{a} = -\frac{3}{2}\frac{Gm}{a}$$

$$\frac{V_A}{V_B} = \frac{4}{3}$$

166. (d)

Comets which return have elliptical orbits. Others have parabolic ($\varepsilon = 0$) or hyperbolic ($\varepsilon > 0$) orbits.

167. (c)

$v_{vel(min)} = 0$ (at the initial position and similar position where v is parallel)

$v_{rel(max)} = \sqrt{2}\,v$ (at P and similar position where v is normal to each other).

168. (c)

$$\omega_0 = \frac{v}{r} = \sqrt{\frac{GM}{r}}.\frac{1}{r} \Rightarrow \omega_0^2 = \frac{GM}{r^3} \Rightarrow r = \left(\frac{GM}{\omega_0^2}\right)^{1/3}$$

[r = R + h]

$$h = \left[\frac{Gm}{\omega_0^2}\right]^{1/3} - R$$

Let ω be the new angular velocity, when h = R

$\therefore$ $R = \left[\frac{Gm}{\omega^2}\right]^{1/3} - R \Rightarrow (2R)^3 = \frac{Gm}{\omega^2}$

$\therefore$ $\omega^2 R = \frac{Gm}{8R^2} = \frac{g}{8}\left(\because \frac{Gm}{R^2} = g\right)$

$$g' = g - \omega^2 R\cos^2\lambda = g - \frac{g}{8}\cos^2 0 = \frac{7g}{8}$$

$$W' = mg' = \frac{7mg}{8} = \frac{7}{8}W$$

169. (d)

The mass of the cloud can be considered as that of a planet and an orbit within it will have $g' = g\frac{r}{R}$, where g is the value at the outermost radius R of the cloud.

$$\frac{mv^2}{r} = mg' \quad \omega^2 r = g\frac{r}{R}$$

where g is a constant

$\Rightarrow$ $\omega^2 = \frac{g}{R} \quad \Rightarrow T = 0$

170. $v = \sqrt{\frac{GM}{r}}$; $GM = gR^2, r = R + x$

$$v = \sqrt{\frac{gR^2}{R + x}} = \sqrt{\frac{10 \times (6.4 \times 10^6)^2}{(7.3 \times 10^6)}} = 7490 \text{ m s}^{-1}$$

$$T = \frac{2\pi(R + x)}{v} = \frac{2\pi \times 7.3 \times 10^6}{v} = 1949\,\pi \text{ s}$$

Aliter:

$v = \sqrt{\frac{Gm}{R}}$ for satellite close to Earth's surface

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{\frac{Gm}{R}}} = \frac{2\pi R}{\sqrt{gR}} \quad \left(\because \sqrt{\frac{Gm}{R^2}} = g\right)$$

$$= 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{10}} = 1600\,\pi \text{ second}$$

$T^2 \propto r^3$

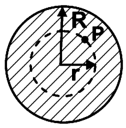
$T_1^2 = k.(6400 + 900)^3 = k(7300)^3$

$T^2 = k.(6400)^3$

$\Rightarrow$ $T_1 = T\left(\frac{7300}{6400}\right)^{3/2} = 1600\,\pi \times 1.218 = 1949\,\pi$ s

Note:
In such problems, time period of satellite in first cosmic orbit($r = R$) can be taken 5077 second.

171. (a)



$$E_p = \frac{G.\frac{4}{3}\pi r^3.\rho}{r^2}$$

$$= \left(G.\frac{4}{3}\pi\rho.\right)r$$

$$\frac{E_P}{r} = \text{constant}$$

172. Below surface: $g' = g\left(1 - \frac{d}{R}\right)$

Above surface: $g' = g\left(1 - \frac{2h}{R}\right)$

is only approximate formula.

Correct formula is $g' = g\left(1 + \frac{h}{R}\right)^{-2}$

173. The angle is 0 at pole and 0 at equator. It increases and then decreases.

174. (c)
Work done = ΔKE is same in both cases.

But loss of kinetic energy = gain in potential energy over a greater height than with constant g. In other words, with decreasing value of g with height, larger distance has to be moved to achieve same loss in KE than in the case of constant g.

175. (a)

$T = \frac{2u\sin\theta}{g}$; $g_{average}$ decreases, T increases

$\Rightarrow R = u\cos\theta\ T \rightarrow$ increases.

176. (a)

177. (b)
A projectile motion is the reverse of a satellite motion with inadequate velocity.

178. (d)
Not valid for elliptical orbits. Kinetic energy varies due to work done.

179. (d)
Perigees need not be same.

Major axis = sum of perigee and apogee distances

180. $-\frac{GM}{R_e} + KE = -\frac{GE}{R_e + h} + KE_f$

$\Rightarrow$ total values on both side are negative, say $-k$

$$KE_f = -k + \frac{GM}{R_e + h}$$

$$\Rightarrow KE_f \downarrow, \frac{GM}{R_e + h} \downarrow, h \uparrow$$

If initial angular momentum $L_0 \neq 0$, final angular momentum

$L_f \neq 0$, hence $v \neq 0$, $KE_f \neq 0$. Hence h is not maximum possible

181. Variation with altitude is given by $g' = \frac{g_0 R^2}{(R + h)^2}$ or by $g_0\left(1 - \frac{2h}{R}\right)$ if h is small.

Variation with latitude λ is given by

$g_0 - \omega^2 R\cos^2\lambda$. We know that the term $\omega^2 R\cos^2\lambda$ is small.

From given fact of g being equal at P and at latitude, it is clear that

$$g_0 \frac{2h}{R} = \omega^2 R\cos^2 45°, \text{ where}$$

h = altitude of P

$$\Rightarrow 2g_0 h = \frac{(\omega^2 R^2)}{2} = \frac{v_0^2}{2}$$

LHS equals v'^2

$$\therefore v' = \frac{v_0}{\sqrt{2}}$$

182. By above, v' > v"

$\therefore$ $KE = FS = gH = g_{avg}.H$

$g_{avg} < g$ if we take variation of g.

183. By work – KE theorem,

$$\frac{1}{2}mv''^2 - 0 = \int_0^H mg_0\left(1 - \frac{2h}{R}\right).dh$$

$$= mg_0.\left[H - \frac{2H^2}{2R}\right]$$

$$= m\,g_0 H\left(1 - \frac{H}{R}\right)$$

$$\Rightarrow \quad v''^2 = 2g_0H\left(1 - \frac{H}{R}\right)$$

$$2g_0H = \frac{v_0^{\ 2}}{2} = v'^2 \text{ and } 2g_0R = v_e^{\ 2}$$

$$\Rightarrow \quad \frac{H}{R} = \frac{v_0^{\ 2}}{2v_e^{\ 2}}$$

Substituting,

$$v''^2 = \frac{v_0^{\ 2}}{2}\left(1 - \frac{v_0^{\ 2}}{2v_e^{\ 2}}\right) = v'^2\left(1 - \frac{v_0^{\ 2}}{2v_e^{\ 2}}\right)$$

$$\Rightarrow \quad v'' = v'.\left(1 - \frac{v_0^{\ 2}}{2v_e^{\ 2}}\right)^{1/2} \approx v'\left(1 - \frac{v_0^{\ 2}}{4v_e^{\ 2}}\right)$$

$$\Rightarrow \quad \frac{v''}{v'} = 1 - \frac{v_0^{\ 2}}{4v_e^{\ 2}} \Rightarrow \frac{v' - v''}{v'} = \frac{v_0^{\ 2}}{4v_e^{\ 2}} = \left(\frac{v_0}{2v_e}\right)^2$$

184. Consider position apogee or perigee (where velocity is perpendicular to radius vector)

$$mvr = L \Rightarrow v = \frac{L}{mr} \qquad \text{— (1)}$$

$$-\frac{GMm}{r} + \frac{1}{2}mv^2 = E$$

$$\Rightarrow \quad -\frac{gR^2m}{r} + \frac{L^2}{2mr^2} = E$$

$$\Rightarrow \quad Er^2 + mgR^2r - \frac{L^2}{2m} = 0 \qquad \text{— (2)}$$

(2) is a quadratic, gives two values of r (apogee and perigee).

Required condition is smaller value > R.

$$r = \frac{-mgR^2 \pm \sqrt{m^2g^2R^4 + \frac{4EL^2}{2m}}}{2E} > R$$

Noting $E < 0$,

The smaller value

$$= \frac{-mgR^2 + \sqrt{m^2g^2R^4 + \frac{4EL^2}{2m}}}{2E} > R$$

$$\Rightarrow \quad 2ER + mgR^2 < \sqrt{m^2g^2R^4 + \frac{4EL^2}{2m}}$$

$$\Rightarrow \quad 4E^2R^2 + m^2g^2R^4 + 4EmgR^3 < m^2g^2R^4 + \frac{2EL^2}{m}$$

$$\Rightarrow \quad E + mgR < \frac{L^2}{2mR^2}$$

185. (2) $\Rightarrow r^2 + \frac{mgR^2}{E}r - \frac{L^2}{2mE} = 0$

Using data,

$\Rightarrow \quad r^2 - 12Rr + 12R^2 = 0$

$\Rightarrow \quad$ smaller $r = \left(6 - 2\sqrt{6}\right)R$

186. Semi major axis $= \frac{r_1 + r_2}{2} = 6R$

$$T = \frac{2\pi}{\sqrt{GM}}.a^{3/2} = \frac{2\pi}{R\sqrt{g}}6\sqrt{6}R^{3/2} = 12\sqrt{6}\pi\sqrt{\frac{R}{g}}$$

187. Line of action of F passes through O. $\overline{F}$ may be directed towards or away from O.

188. $\overline{\tau} = \overline{R} \times \overline{F} = \frac{d}{dt}\overline{L}_0$ where $\overline{L}_0$ = Angular momentum about O = $\overline{R} \times m\overline{v}$ as $\overline{\tau} = 0$, $\overline{R} \times m\overline{v}$ = constant

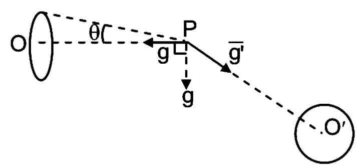
189. As $\overline{R} \times m\overline{v}$ = constant vector. The vectors $\overline{R}$ and $\overline{v}$ are perpendicular to a constant vector. Hence the trajectory must be on a fixed plane.

190. (a), (d)

Field at axial point is axial and towards centre O.

Let $\overline{g}$ due to ring and $\overline{g}'$ due to shell.

Resultant = g at 90° to $\overline{g}$ can only be if angle between $\overline{g}'$ and $\overline{g}$ is obtuse as below:



$\Rightarrow \quad g' = \sqrt{2}g$

If P lies within the shell, g' will be zero $\Rightarrow$ Resultant will be same as $\overline{g}$. Hence P lies on or outside shell.

$\Rightarrow \quad O'P \geq R$

191. (a), (d)

$$E = -K.E = \frac{P.E}{2} = -\frac{GMm}{2R_2}$$

$$K.E = \frac{GMm}{2R_2}$$

Kinetic energy to from surface $= \frac{GMm}{R_1}$

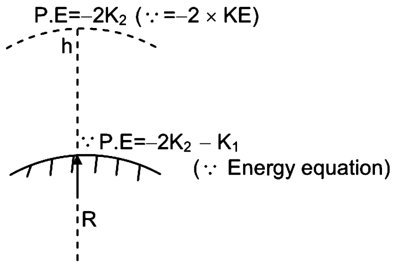
$$\text{Ratio} = \frac{R_1}{2R_2} \le \frac{1}{2} \qquad (\because R_2 \ge R_1)$$

$\therefore$ Velocity ratio $\le \frac{1}{\sqrt{2}}$

$$T = \frac{2\pi}{\sqrt{GM}}.R_2^{3/2} \Rightarrow GM = gR_1^2 = \frac{4\pi^2}{T^2}.R_2^3$$

$$g = \frac{4\pi^2}{T^2}.\frac{R_2^3}{R_1^2}$$

192. (a), (b), (c)



$\therefore$ For escape K.E = –P.E = $K_1 + 2K_2$

$$-\frac{GMm}{R+h} = -2K_2$$

$$-\frac{GMm}{R} = -2K_2 - K_1$$

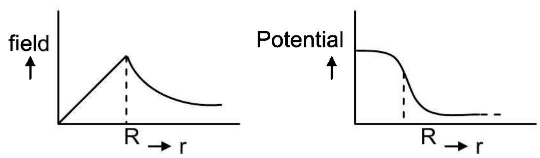
Divide, solve, $h = \frac{K_1}{2K_2}.R$

For circular orbit at R,

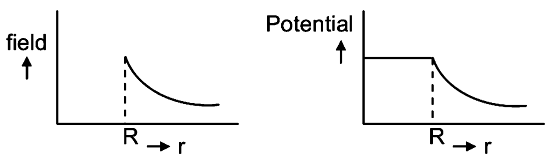
$$\text{K.E required} = -\frac{\text{P.E}}{2} = \frac{2K_2 + K_1}{2}$$

K_1 and K_2 are independent, cannot be compared.

193. (a), (c)



194. (c)



195. (b), (c)

Radius vector will pass through centre. $\therefore$ plane of orbit will contain centre $\Rightarrow$ equator will be crossed.

$$T = \frac{2\pi}{\sqrt{GM}}r^{3/2} = \frac{2\pi}{\sqrt{g}}\frac{r^{3/2}}{R}$$

$$= 2\pi\sqrt{\frac{R}{g}}.\left(\frac{r}{g}\right)^{3/2} \qquad \ge 2\pi\sqrt{\frac{R}{g}}$$

Since $r \ge R$

196. (c), (d)

197. Field due to shell, within the shell, is 0.

Hence field : at A $\to$ 0, B $\to$

$$-\frac{GM}{\left(\frac{R}{2}\right)^2} = \frac{-4GM}{R^2};$$

$$C \to -\frac{GM}{R^2} - \frac{GM}{R^2} = \frac{-2GM}{R^2}$$

$$D \to -\frac{GM}{4R^2} - \frac{GM}{4R^2} = -\frac{GM}{2R^2}$$

$\Rightarrow$ value as 0, 4k, 2k, $\frac{k}{2}$, neither AP or GP

Potential due to shell, within the shell is constant $-\frac{GM}{R}$

$$\therefore \quad V_A = -\frac{3}{2}\frac{GM}{\frac{R}{2}} - \frac{GM}{R} = -4\frac{GM}{R}$$

$$V_B = -\frac{GM}{\frac{R}{2}} - \frac{GM}{R} = -\frac{3GM}{R}$$

$$V_C = -\frac{GM}{R} - \frac{GM}{R} = -2\frac{GM}{R}$$

$$V_D = -\frac{GM}{2R} - \frac{GM}{2R} = -\frac{GM}{R}$$

$\therefore$ Value are 4k, 3k, 2k, k in arithmetic progression

198. (a) $\to$ r, s

(b) $\to$ p, q, r, s

(c) $\to$ p, q, r, s

(d) $\to$ p, q

199. (i) Field is maximum at $\frac{R}{\sqrt{2}}$

(ii) Remember potential is negative.

(a) $\rightarrow$ p, s

(b) $\rightarrow$ q, s

(c) $\rightarrow$ p, r

(d) $\rightarrow$ q, r

200.

(a) $\left[\frac{GM}{R^2} \div \frac{GM}{(R+h)^2}\right]^{\frac{1}{2}} = \frac{R+h}{R}$

(b) $\frac{2(R+h)}{R}$

(c) $-\frac{GMm}{R} \div \frac{-GMm}{2(R+h)} = \frac{2(R+h)}{R}$

(d) $-\frac{GM}{R} \div \frac{-GM}{R+h} = \frac{R+h}{R}$

(p) $\left[\sqrt{\frac{2GM}{R}} \div \sqrt{\frac{GM}{R+h}}\right]^2 = \frac{2(R+h)}{R}$

(q) $\frac{R+h}{R}$

(r) $\frac{GMm}{R} \div \frac{GMm}{2(R+h)} = \frac{2(R+h)}{R}$

(s) $\left[\frac{GM}{R^2} \div \left(\frac{V_0}{R+h}\right)^2 (R+h)\right]^{\frac{1}{2}} = \left[\frac{GM(R+h)}{R^2 V_0^2}\right]^{\frac{1}{2}}$

$$= \left[GM(R+h) \div R^2 \frac{GM}{R+h}\right]^{\frac{1}{2}} = \frac{R+h}{R}$$

(a) $\rightarrow$ q, s

(b) $\rightarrow$ p, r

(c) $\rightarrow$ p, r

(d) $\rightarrow$ q, s