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PHYSICS

TARGET IIT JEE

SIMPLE HARMONIC MOTION

THEORY AND EXERCISE BOOKLET

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SIMPLE HARMONIC MOTION

Many motions in this universe are periodic i.e., they repeat themselves at regular intervals. Simple harmonic motion is one of the simplest periodic motion in which an object oscillates between two spatial positions for an indefinite period of time with no loss of mechanical energy.

In this lesson we will learn about simple harmonic motion in detail. We will calculate time period of simple harmonic motions in various cases. We will also discuss combination of simple harmonic motions.

IIT - JEE Syllabus :

Linear and angular simple harmonic motions.

1. PERIODIC MOTION :

When a body or a moving particle repeats its motion along a definite path after regular intervals of time its motion is said to be Periodic Motion and interval of time is called time period (T). The path of periodic motion may be linear, circular, elliptical or any other curve. For example rotation of earth around the sun.

2. OSCILLATORY MOTION :

To and fro type of motion is called Oscillatory Motion. A particle has oscillatory motion when it moves about stable equilibrium position. It need not be periodic and need not have fixed extreme positions.

The oscillatory motions in which energy is conserved are also periodic.For example motion of pendulum of a wall clock.

The force / torque (directed towards equillibrium point) acting in oscillatory motion is called restoring force/torque **Damped Oscillations** are those in which energy consumed due to some resistive forces and hence total mechanical energy decreases and after some time oscillation will stop.

Oscillatory Equation : Consider a particle free to move on x-axis is being acted upon by a force given by

$$F = -kx^n$$

Above equation is called oscillatory equation. Here k is a positive constant and x is the displacement from mean position

Now following cases are possible depending on the value of n.

(i) If n is an even interger (0, 2, 4......... etc)$_n$ force is always along negative x-axis whether x is positive or negative Hence, the motion of the particle is not oscillatory. If the particle is released from any position on the x-axis (except x = 0) a force in – ve direction of x-axis acts on it and it moves rectilinearly along – ve x axis.

(ii) If n is an odd integer (1, 3, 5 .... etc), force is along – ve x-axis for x > 0 and along +ve x-axis for x < 0 and zero for x = 0. Thus the particle will oscillate about stable equillibrium position x = 0. The force in this case is called the restoring force.

If n = 1 i.e., F = – kx the motion is said to be SHM (Simple Harmonic Motion)

If the restoring force / torque acting on the body in oscillatory motion is directly proportional to the displacement of body / particle w.r.t. mean position and is always directed towards equillibrium position then the motion is called Simple Harmonic motion. It is the simplest form of oscillatory motion.

3. TYPES OF SHM :

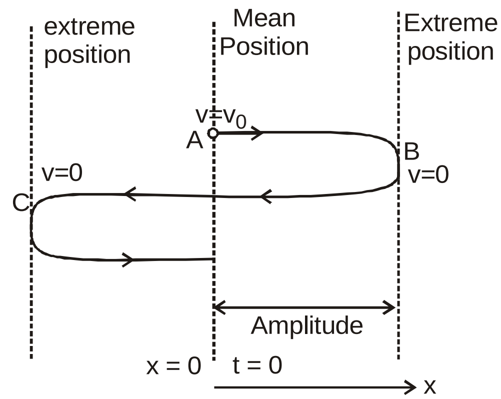
(a) Linear SHM : When a particle moves to and fro about an equilibrium point, along a straight line here A and B are extreme positions and M is mean position so AM = MB = Amplitude.

(b) Angular SHM : When body/particle is free to rotate about a given axis and executing angular oscillations.

4. ANALYSIS OF MOTION IN LINEAR SHM :

When the particle is moved away from the mean position or equillibrium position and released, a force (–kx) comes into play to pull it back towards mean position. By the time it gets at mean position it has picked up some kinetic energy and so it overshoots, stopping some where on the other side and it is again pulled back towards the mean position.

It is necessary to study the change in speed and acceleration of particle during SHM. Let us consider a particle whose position is x = 0 at t = 0 and $v = v_0$. Then we divide the motion of particle in one time period in four parts.



(A) from A to B (B) from B to A (C) from A to C (D) from C to A

NOTE : In the figure shown, path of the particle is a straight line.

(1) Motion of a particle from A to B :

Initially the particle is at A (mean position) and is moving towards +ve x direction with speed v_0. As the particle is moving towards B, force acting on it towards A is increasing. Consequently its acceleration towards A is increasing in magnitude while its speed decreases and finally it comes to rest momentarily at B.

(2) Motion of a particle from B to A :

Now the particle starts moving towards A with initial speed v = 0. As the particle is moving towards A, force is acting on it towards A and decreasing as it approaches A. Consequently its acceleration towards A is decreasing in magnitude while its speed increases and finally it comes to A with same speed $v = v_0$.

(3) Motion of a particle from A to C :

The motion of a particle from A to C is qualitatively same as motion of a particle from A to B.

(4) Motion of a particle from C to A :

It is qualitatively same as motion of a particle from B to A.

Summary :

Motion from	**Velocity (Direction/Magnitude)**	**Acceleration (Direction/ Magnitude)**
$A \rightarrow B$	$V \rightarrow \downarrow$	$a \leftarrow \uparrow$
$B \rightarrow A$	$V \leftarrow \uparrow$	$a \leftarrow \downarrow$
$A \rightarrow C$	$V \leftarrow \downarrow$	$a \rightarrow \uparrow$
$C \rightarrow A$	$V \rightarrow \uparrow$	$a \rightarrow \downarrow$

5. CHARACTERISTICS OF SHM :

(1) Mean Position : It is the position where net force on the particle is zero.

(2) Extreme Point : Point where speed of the particle is zero.

(3) Displacement : It is defined as the distance of the particle from the mean position at that instant.

(4) Amplitude : It is the maximum value of displacement of the particle from its mean position.

Extreme position – Mean position = Amplitude.

It depends upon the energy of the system.

(5) Frequency : The frequency of SHM is equal to the number of complete oscillations per unit time.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \sec^{-1} \text{ or Hz.}$$

(6) Time Period : Smallest time interval after which the oscillatory motion gets repeated is called time period.

$$T = \frac{2\pi}{\omega}$$

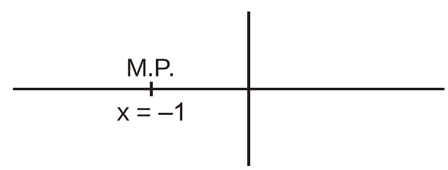
Ex.1 Describe the motion of a particle acted upon by a force.

(A) F = 3x + 3 (B) F = – 3x – 3 (C) F = – 3x + 3 (D) F = 3x – 3

Sol. (a) Given F = 3x + 3 ...(i)

We find the mean position at which net force on the particle is zero.

$\Rightarrow$ 3x + 3 = 0 $\Rightarrow$ x = – 1



If we put x = 0 in eq. (i) then

F = 3N (away from M.P.) ...(a)

Now put x = –2 in eq. (i)

F = – 3 N (away from M.P.) ...(b)

From (a) and (b) we conclude that particle doesn't perform S.H.M.

(b) Given F = – 3x – 3 ...(i)

at M.P. F = 0

$\Rightarrow$ x = –1

Now put x = 0 in eq. (i)

$\Rightarrow$ F = – 3N (towards M.P.)

If x = – 2 ; F = 3N (towards M.P.)

We conclude from the above calculation that in every case (whether the particle is left from M.P. or right from M.P.) force acts towards M.P. so the particle performs S.H.M.

(c) Given F = – 3x + 3

when F = 0

x = 1 (M.P.)

Now put x = 0

Then F = 3N (towards M.P.)

If x = 2 F = – 3 (towards M.P.)

i.e. particle performs S.H.M.

(D) Given F = 3x – 3

Mean position at x = 1.

When x = 0 ; F = – 3N (away from M.P.)

x = 2 ; F = 3N (away from M.P.)

Particle doesn't perform S.H.M.

6. EQUATION OF SIMPLE HARMONIC MOTION :

The necessary and sufficient condition for SHM is

$F = -kx$

we can write above equation in the following way:

$ma = -kx$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\mathbf{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \quad \textbf{...(1)}$$

Equation (1) is Double Differential Equation of SHM.

Now $\frac{d^2x}{dt^2} + \omega^2 x = 0$

It's solution is $x = A\sin(\omega t + \phi)$

where ω = angular frequency = $\sqrt{\frac{k}{m}}$

x = displacement from mean position

k = SHM constant.

The equality ($\omega t + \phi$) is called the phase angle or simply the phase of the SHM and ϕ is the initial phase i.e., the phase at t = 0 and depends on initial position and direction of velocity at t = 0.

To understand the role of ϕ in SHM, we take two particles performing SHM in the following condition:

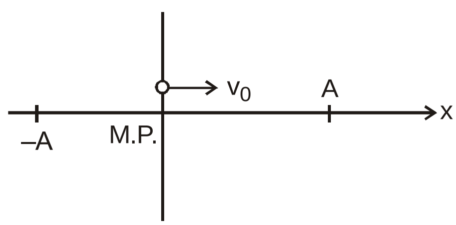


figure I

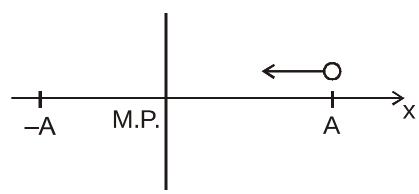


figure II

Suppose we choose t = 0 at an instant when the particle is passing through its mean position towards right (i.e.positive direction) as shown in figure I[st] then

In figure I at t = 0 x = 0

i.e., $x = A\sin\omega t$

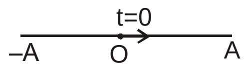
$\therefore$ The particle is at its mean position.

In figure II at t = 0 x = A ; and the particle is moving towards the mean position.

i.e., $x = A\sin(\omega t + \pi/2)$

Here $\pi/2$ is the only phase possible.

Ex.2 A particle starts from mean position and moves towards positive extreme as shown below. Find the equation of the SHM. Amplitude of SHM is A.



Sol. General equation of SHM can be written as $x = A \sin (\omega t + \phi)$

At $t = 0$, $x = 0$

$\therefore$ $0 = A \sin \phi$

$\therefore$ $\phi = 0, \pi$ $\phi \in [0, 2\pi)$

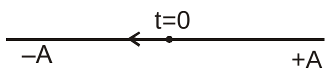
Also; at $t = 0$, $v = +$ ve

$\therefore$ $A \omega \cos \phi = +$ ve or, $\phi = 0$

Hence, if the particle is at mean position at t = 0 and is moving towards +ve extreme, then the equation of SHM is given by $x = A \sin \omega t$.

Similarly

for particle moving towards –ve extreme then

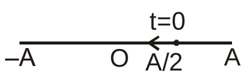


$\phi = \pi$

$\therefore$ equation of SHM is $x = A \sin (\omega t + \pi)$

or, $x = - A \sin \omega t$

Ex.3 Write the equation of SHM for the situation shown below :



Sol. General equation of SHM can be written as

$x = A \sin (\omega t + \phi)$

At $t = 0$, $x = A/2$

$\Rightarrow$ $\frac{A}{2} = A \sin \phi$

$\Rightarrow$ $\phi = 30°, 150°$

Also at $t = 0$, $v = -$ ve

$A\omega \cos \phi = -$ ve $\Rightarrow$ $\phi = 150°$

7. VELOCITY :

It is the rate of change of particle displacement with respect to time at that instant.

Let the displacement from mean position is given

by $x = A \sin (\omega t + \phi)$

velocity $v = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$

$v = A\omega \cos (\omega t + \phi)$

$v = \omega\sqrt{A^2 - x^2}$

At mean position (x = 0), velocity is maximum.

$V_{max} = \omega A$

At extreme position (x = A), velocity is minimum.

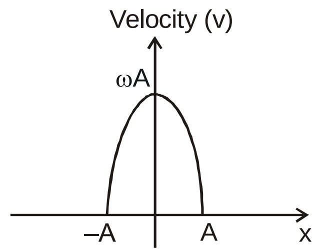
v_{min} = zero.

7.1 Graph of Velocity (v) V/S Displacement (x) :

$v = \omega\sqrt{A^2 - x^2}$ $\qquad v^2 = \omega^2(A^2 - x^2)$

$v^2 + \omega^2 x^2 = \omega^2 A^2$ $\qquad \dfrac{v^2}{\omega^2 A^2} + \dfrac{x^2}{A^2} = 1$

Graph would be a half ellipse.



8. ACCELERATION :

It is the rate of change of particle's velocity w.r.t. time at that instant.

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d}{dt}[A\omega\cos(\omega t + \phi)]$

$a = -\omega^2 A \sin(\omega t + \phi)$

$a = -\omega^2 x$

Note : Negative sign shows that acceleration is always directed towards the mean position. At mean position (x =0), acceleration is minimum.

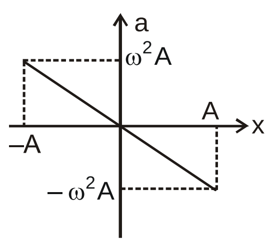
a_{min} = zero

At extreme position (x = A), acceleration is maximum.

$|a_{max}| = \omega^2 A$

8.1 Graph of Acceleration (A) v/s Displacement (x):

$a = -\omega^2 x$



9. GRAPHICAL REPRESENTATION OF DISPLACEMENT, VELOCITY & ACCELERATION IN SHM:

Displacement, $x = A \sin \omega t$

Velocity, $v = A\omega\cos\omega t = A\omega\sin(\omega t + \frac{\pi}{2})$

or $v = \omega\sqrt{A^2 - x^2}$

Acceleration, $a = -\omega^2 A \sin \omega t = \omega^2 A \sin(\omega t + \pi)$

or $a = -\omega^2 x$

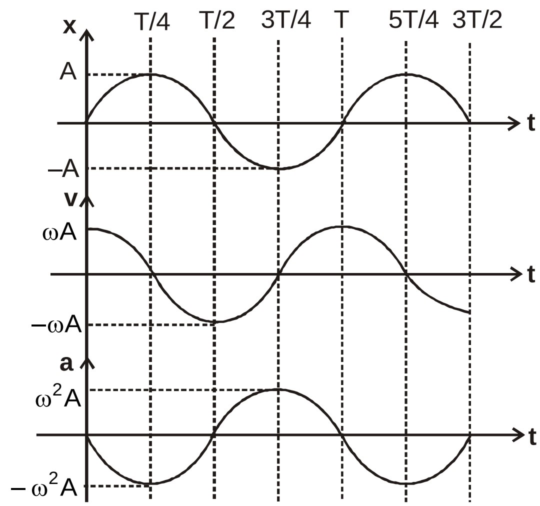
Note :

- $v = \omega\sqrt{A^2 - x^2}$

  $a = -\omega^2 x$

These relations are true for any equation of x.

[redacted]



1. All the three quantities displacement, velocity and acceleration vary harmonically with time, having same period.

2. The maximum velocity is ω times the amplitude ($V_{max} = \omega A$).

3. The acceleration is ω^2 times the displacement amplitude ($a_{max} = \omega^2 A$).

4. In SHM, the velocity is ahead of displacement by a phase angle of $\frac{\pi}{2}$.

5. In SHM, the acceleration is ahead of velocity by a phase angle of $\frac{\pi}{2}$.

Ex.4 The equation of particle executing simple harmonic motion is $x = (5m)\sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$.

Write down the amplitude, time period and maximum speed. Also find the velocity at t = 1 s.

Sol. Comparing with equation $x = A \sin(\omega t + \phi)$, we see that the amplitude = 5m,

and time period = $\frac{2\pi}{\omega}$ = $\frac{2\pi}{\pi s^{-1}} = 2s$

The maximum speed = $A\omega = 5\text{ m} \times \pi\text{ s}^{-1} = 5\pi\text{ m/s}$

The veloity at time $t \Rightarrow \frac{dx}{dt}$ = $A\omega\cos(\omega t + \delta)$

At t = 1 s,

$$v = (5\,m)\,(\pi\,s^{-1})\,\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}m/s$$

Ex.5 A particle executing simple harmonic motion has angular frequency 6.28 s^{-1} and amplitude 10 cm. Find (a) the time period, (b) the maximum speed, (c) the maximum acceleration, (d) the speed when the displacement is 6 cm from the mean position, (e) the speed at t = 1/6 s assuming that the motion starts from rest at t = 0.

Sol. (a) Time period = $\frac{2\pi}{\omega} = \frac{2\pi}{6.28}$ s = 1 s.

(b) Maximum speed = $A\omega$ = (0.1m) (6.28 s^{-1})

(c) Maximum acceleration = $A\omega^2$

= (0.1m) (6.28 $s^{-1})^2$

= 4 m/s^2

(d) $v = \omega\sqrt{A^2 - x^2}$ = (6.28 s^{-1}) $\sqrt{(10cm)^2 - (6cm)^2}$ = 50.2 cm/s.

(e) At t = 0, the velocity is zero i.e., the particle is at an extreme. The equation for displacement may be written as

$x = A \cos \omega t$.

The velocity is $v = -A\omega \sin \omega t$.

At $t = \frac{1}{6}s$, $v = -(0.1\text{ m})(6.28\text{ s}^{-1}) \sin\left(\frac{6.28}{6}\right)$

$= (-0.628\text{ m/s}) \sin\frac{\pi}{3} = -54.4$ cm/s. (towards mean position)

Note : If mean position is not at the origin, then we can replace x by $x - x_0$ and the eqn. becomes

$x - x_0 = -A \sin \omega t$, where x_0 is the position co-ordinate of the mean position.

Ex.6 A particle of mass 2 kg is moving on a straight line under the action force F = (8 – 2x) N. It is released at rest from x = 6m.

(A) Is the particle moving simple harmonically?

(B) Find the equilibrium position of the particle.

(C) Write the equation of motion of the particle.

(D) Find the time period of SHM.

Sol. F = 8 – 2x

or F = −2(x − 4)

for equilibrium position F = 0

⇒ x = 4m is equilibrium position.

Hence the motion of particle is SHM with force constant 2 and equilibrium position x =4.

(a) Yes, motion is SHM.

(b) Equilibrium position is x = 4m.

(c) At x = 6 m, particle at rest i.e. it is one of the extreme position. Hence amplitude is A = 2 m and initially particle at the extreme position.

∴ Equation of SHM can be written as

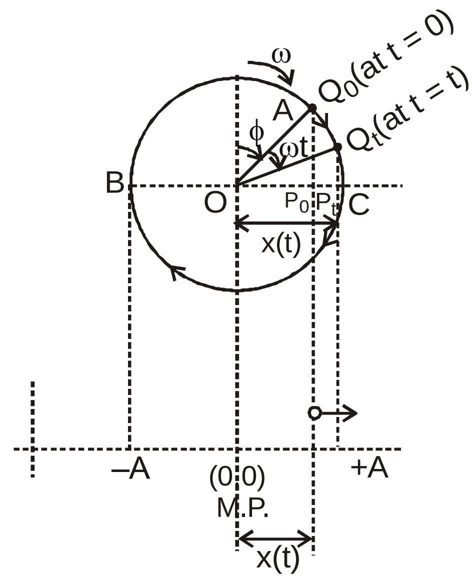
$x - 4 = 2 \cos \omega t$, where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$ $(sec)^{-1}$

i.e. $x = 4 + 2 \cos t$

(d) Time period, $T = \frac{2\pi}{\omega} = 2\pi \sec$.

10. SHM AS A PROJECTION OF UNIFORM CIRCULAR MOTION.

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P excecutes a simple harmonic motion on the x-axis between B and C. The angle that the radius OQ makes with the +ve vertical in clockwise direction in at t = 0 is equal to phase constant (ϕ).



Let the radius OQ_0 makes an angle ωt with the OQ_t at time t. Then

$$x(t) = A \sin(\omega t + \phi)$$

In the above discussion the foot of projection is x-axis so it is called horizontal phasor. Similarly the foot of perpendicular on y axis will also executes SHM of amplitude A and angular frequency ω [$y(t) = A\cos\omega t$]. This is called vertical phasor. The phaser of the two SHM differ by $\pi/2$.

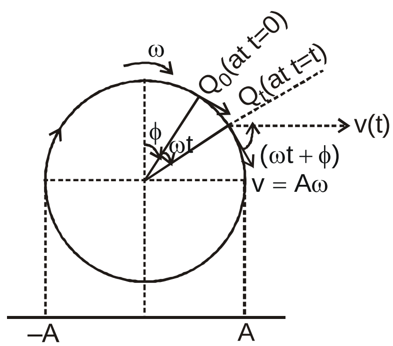
Problem solving strategy in horizontal phasor:

(1) First assume circle of radius equal to amplitude of S.H.M.

(2) Assume a particle rotating in a circular path moving with constant ω same as that of S.H.M in clockwise direction.

(3) Angle made by the particle at t = 0 with the upper vertical is equal to phase constant.

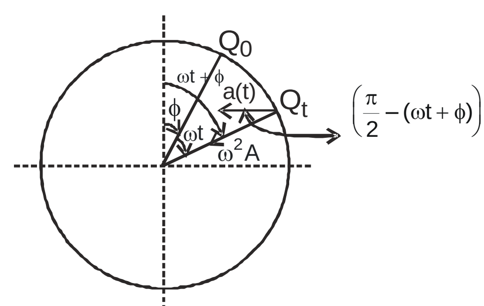
(4) Horizontal component of velocity of particle gives you the velocity of particle performing S.H.M. for example



from figure

$$v(t) = A\,\omega \cos(\omega t + \phi)$$

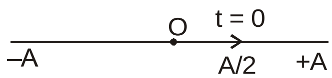
(5) Component of acceleration of particle in horizontal direction is equal to the acceleration of particle performing S.H.M. The acceleration of a particle in uniform circular motion is only centripetal and has a magnitude $a = \omega^2 A$.

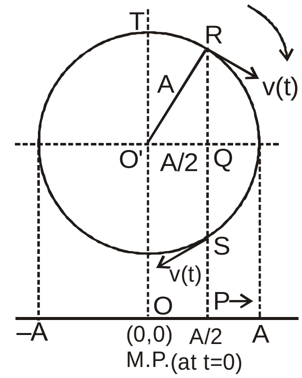


From figure

$$a(t) = -\,\omega^2 A \sin(\omega t + \phi)$$

Ex.7 A particle starts from A/2 and moves towards positive extreme as shown below. Find the equation of the SHM. Given amplitude of SHM is A.



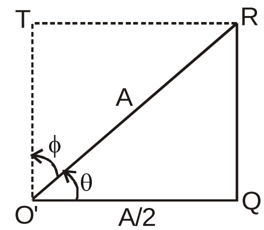


Sol. We will solve the above problem with the help of horizontal phasor.

Step 1. Draw a perpendicular line in upward direction from point P on the circle which cuts it at point R & S

Step 2. Horizontal compoment of v(t) at R gives the direction P to A while at S gives P to O. So at t = 0 particle is at R

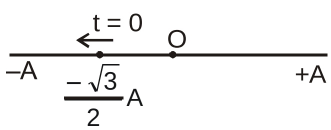
Step 3. In Δ O'RQ



$\cos\theta = \frac{A/2}{A} = 60° \quad \Rightarrow \quad \phi = 30°$

So equation of the SHM is x = A sin (ωt + 30°)

Ex.8 A particle starts from point $x = \frac{-\sqrt{3}}{2}A$ **and move towards negative extreme as shown**

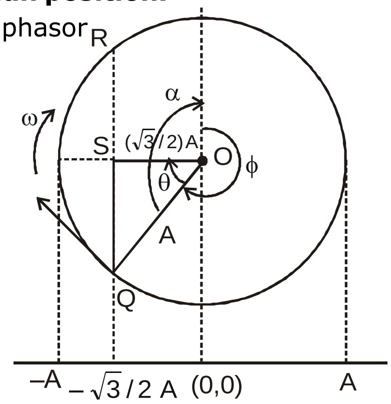


(a) Find the equation of the SHM.

(b) Find the time taken by the particle to go directly from its initial position to negative extreme.

(c) Find the time taken by the particle to reach at mean position.

Sol. Figure shows the solution of the problem with the help of phasor Horizontal component of velcoity at Q gives the required direction of velocity at t = 0.



In Δ OSQ $\quad \cos\theta = \frac{\sqrt{3}/2A}{A} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$

Now $\quad \phi = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

So equation of SHM is

$$x = A\sin\left(\omega t + \frac{4\pi}{3}\right)$$

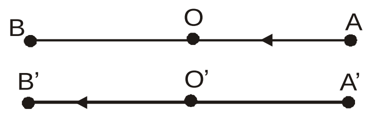
(b) Now to reach the particle at left extreme point it will travel angle θ along the circle. So time taken.

$$t = \frac{\theta}{\omega} = \frac{\pi}{6\omega} \quad \Rightarrow \quad t = \frac{T}{12} \text{ sec}$$

(c) To reach the particle at mean position it will travel an angle $\alpha = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

So, time taken = $\frac{\alpha}{\omega} = \frac{T}{3}$sec

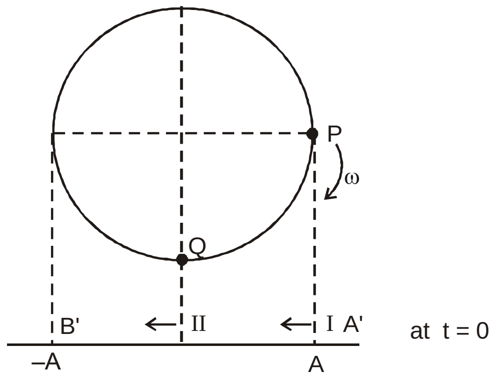
Ex.9 Two particles undergoes SHM along parallel lines with the same time period (T) and equal amplitudes. At a particular instant, one particle is at its extreme position while the other is at its mean position. They move in the same direction. They will cross each other after a further time.



(A) T/8 **(B) 3T/8** **(C) T/6** **(D) 4T/3**

Sol. This problem is easy to solve with the help of phasor diagram.

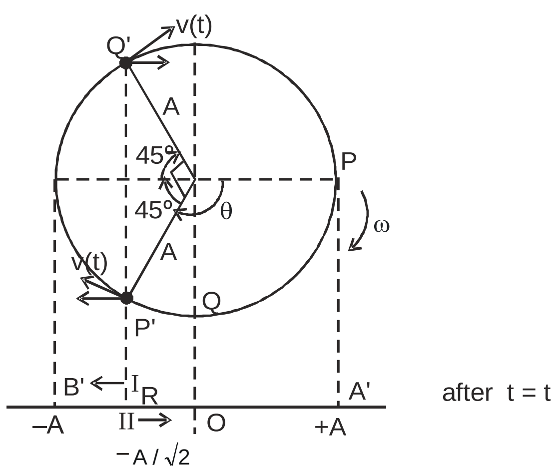
First we draw the initial position of both the particle on the phasor as shown in figure.



From above figure phase difference between both the particles is $\pi/2$.

They will cross each other when their projection from the circle on the horizontal diameter meet at one point.

Let after time t both will reach at P'Q' point having phase difference $\pi/2$ as shown in figure.



Both will meet at $-A/\sqrt{2}$

When they meet angular displacement of P is

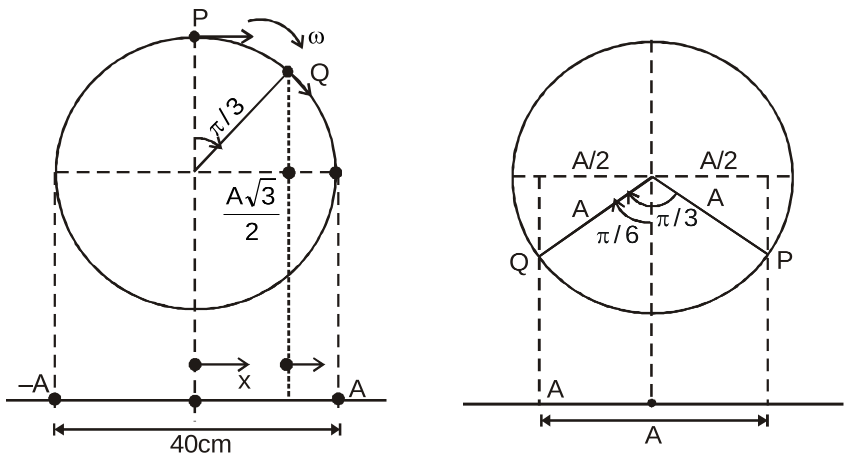
$$\theta = \pi/2 + \pi/4 = 3\pi/4$$

So they will meet after time $t = \dfrac{3\pi}{4 \times \omega}$

$$t = \frac{3\pi}{4 \times 2\pi} \times T = \frac{3T}{8} \text{ sec}$$

Ex.10 Two particles execute SHM of same amplitude of 20 cm with same period along the same line about the same equilibrium position. If phase difference is $\pi/3$ then find out the maximum distance between these two.

Sol. Let us assume that one particle starts from mean position and another starts at a distance x having $\phi = \pi/3$. This condition is shown in figure.



Fig(i) Fig ii

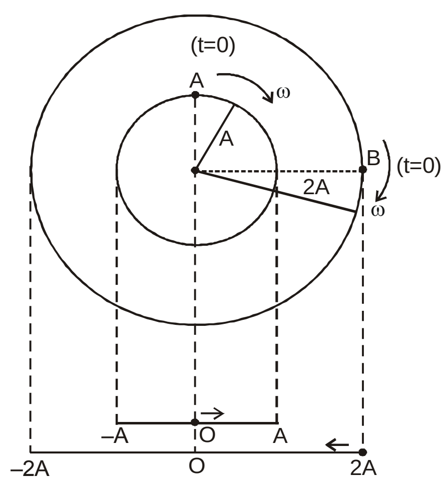
Above figure shows the situation of maximum distance between them.

So maximum distance = A = 10 cm. (as 2A = 20 cm)

Ex.11 Two particles execute SHM of same time period but different amplitudes along the same line. One starts from mean position having amplitude A and other starts from extreme position having amplitude 2A. Find out the time when they both will meet?

Sol. We solve the above problem with the help of phasor diagram.

First we draw the initial position of both the particle on the phasor.



From figure phase difference between both the particle is $\pi/2$.

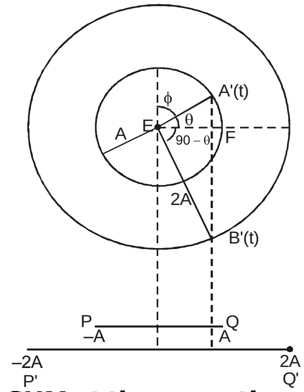
They will meet each other when their projection from the circle on the horizontal diameter meet at one point.

Now from figure:

$$EF = A\cos\theta = 2A\sin\theta$$

$$\tan\theta = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$



So time taken by the particle to cross each other

$$t = \frac{\text{angle travelled by A}}{\omega} \Rightarrow t = \frac{\pi/2-\theta}{\omega}$$

Ex.12 Two particles have time periods T and 5T/4. They start SHM at the same time from the mean postion. After how many oscillations of the particle having smaller time period, they will be again in the same phase ?

Sol. They will be again at m.p. and moving in same direction when the particle having smaller time period makes n_1 oscillations and the other one makes n_2 oscillations.

$$\Rightarrow \quad n_1T = \frac{5T}{4}\times n_2$$

$$\frac{n_1}{n_2} = \frac{5}{4} \quad \Rightarrow \quad n_1 = 5,\ n_2 = 4$$

11. ENERGY OF SHM :

11.1 Kinetic Energy (KE):

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}m\,\omega^2(A^2 - x^2) \qquad \therefore\ \omega^2 = \frac{k}{m}$$

$$\Rightarrow \quad K.E = \frac{1}{2}K(A^2 - x^2)$$

$$K.E_{max} = \frac{1}{2}KA^2 \text{ (at } x = 0)$$

$$K.E_{min} = 0 \text{ (at } x = A) ; \qquad \langle KE\rangle_{0-T} = \frac{1}{4}kA^2 ; \qquad \langle KE\rangle_{0-A} = \frac{1}{3}kA^2$$

Frequency of KE = 2 × (frequency of SHM)

11.2 Potential Energy (PE):

Simple harmonic motion is defined by the equation

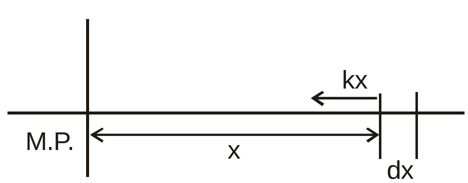
$$F = -kx$$

The work done by the force F during a displacement from x to x + dx is

$$dW = Fdx = -kx\,dx$$

The work done in a displacement from x = 0 to x is

$$W = \int_0^x(-kx)\,dx = -\frac{1}{2}kx^2$$



Let U(x) be the potential energy of the system when the displacement is x. As the change in potential energy corresponding to a conservative force is the negative of the work done by that force.

$$U(x) - U_{M.P.} = -W = \frac{1}{2}kx^2$$

Let us choose the potential energy to be zero when the particle is at the mean position oscillation x = 0.

Then $U_{M.P.} = 0$ and $U(x) = \frac{1}{2}kx^2$

$\because$ $k = m\omega^2$

$\therefore$ $U(x) = \frac{1}{2}m\omega^2 x^2$

$$U = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

But $x = A \sin(\omega t + \phi)$

Kinetic energy of the particle at any instant is

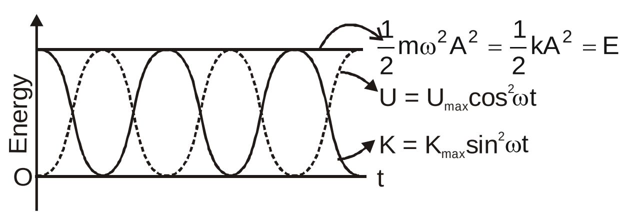
$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi) = \frac{1}{2}m\omega^2(A^2 - x^2)$$

So the total mechanical energy at time 't' is

$$E = U + K \quad \Rightarrow \quad E = \frac{1}{2}m\omega^2 A^2$$

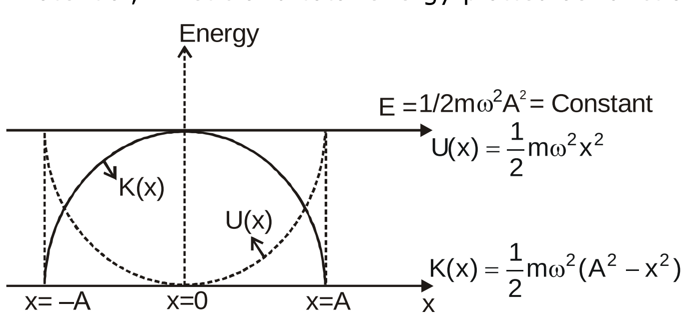
Note : $U_{min} = U_{M.P.}$ (which is not always = 0)

(i)



Potential, Kinetic and total energy plotted as function of time

(ii)



Potential, Kinetic and total energy are plotted as a function of displacement from the mean position.

Ex.13 A particle of mass 0.50 kg executes a simple harmonic motion under a force F = – (50 N/m)x. If it crosses the centre of oscillation with a speed of 10 m/s, find the amplitude of the motion.

Sol. The kinetic energy of the particle when it is at the centre of oscillation is

$$E = \frac{1}{2}mv^2 \quad = \quad \frac{1}{2}(0.50\text{kg})(10\,\text{m/s})^2 \quad = \quad 2.5\text{ J}.$$

The potential energy is zero here. At the maximum displacement x = A, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2}kA^2$. As there is no loss of energy,

$$\frac{1}{2}kA^2 = 2.5\text{J}$$

The force on the particle is given by

$$F = -(50\text{ N/m})\,x.$$

Thus the spring constant is k = 50 N/m.

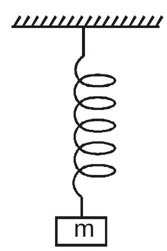
Equation (i) gives

$$\frac{1}{2}(50\text{N/m})A^2 = 2.5\text{J} \qquad \text{or,} \qquad A = \frac{1}{\sqrt{10}}\text{ m}.$$

12. METHOD TO DETERMINE TIME PERIOD AND ANGULAR FREQUENCY IN SIMPLE HARMONIC MOTION :

To understand the steps which are usually followed to find out the time period we will take one example.

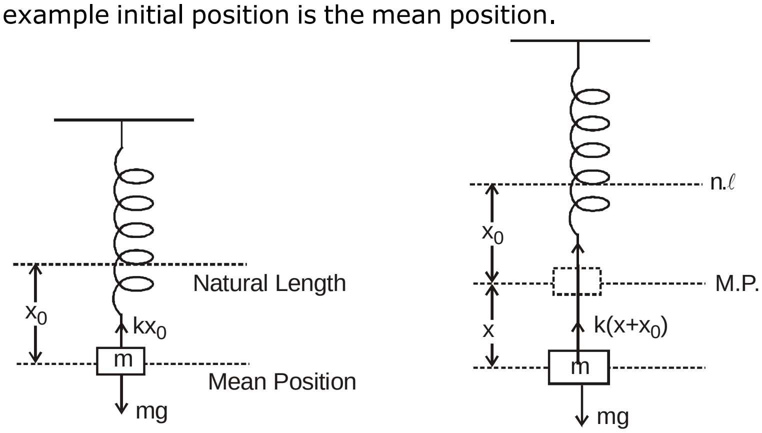
Ex.14 A mass m is attached to the free end of a massless spring of spring constant k with its other end fixed to a rigid support as shown in figure. Find out the time period of the mass, if it is displaced slightly by an amount x downward.



Sol. The following steps are usually followed in this method:

Step 1. Find the stable equillibrium position which is usually known as the mean position. Net force or torque on the particle at this position is zero. Potential energy is minimum.

In our example initial position is the mean position.



Step 2. Write down the mean position force relation. In above figure at mean position

$kx_0 = mg$ ...(1)

Step 3. Now displace the particle from its mean position by a small displacement x (in linear SHM) or angle θ (in case of an angular SHM) as shown in figure.

Step 4. Write down the net force on the particle in the displaced position.

From the above figure.

$F_{net} = mg - k(x + x_0)$ ...(2)

Step 5. Now try to reduce this net force equation in the form of $F = -kx$ (in linear S.H.M.) or $\tau = -k\theta$ (in angular SHM) using mean position force relation in step 2 or binomial theorem.

from eq. (2) $F_{net} = mg - kx - kx_0$

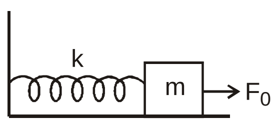
Using eq (i) in above equation

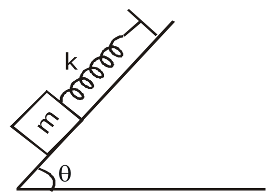
$F_{net} = -kx$ ...(3)

Equation (3) shows that the net force acting towards mean position and is proportional to x, but in this S.H.M. constant $K_{S.H.M.}$ is replaced by spring constant k. So

$$T = 2\pi\sqrt{\frac{m}{K_{S.H.M.}}} = 2\pi\sqrt{\frac{m}{k}}$$

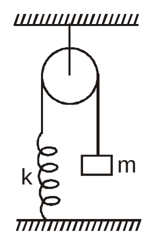
Note : If we apply constant force on the string then time period T is always same $T = 2\pi\sqrt{\frac{m}{K_{S.H.M.}}}$





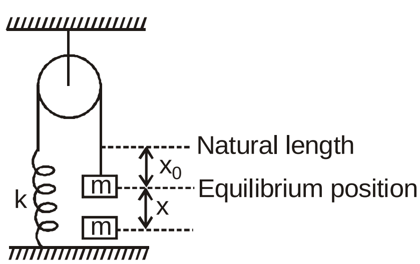
In above both cases $T = \left(2\pi\sqrt{\frac{m}{k}}\right)$

Ex.15 The string, the spring and the pulley shown in figure are light. Find the time period of the mass m.



Sol. Let in equilibrium position of the block, extension in spring is x_0.

$\therefore$ $kx_0 = mg$ ...(1)



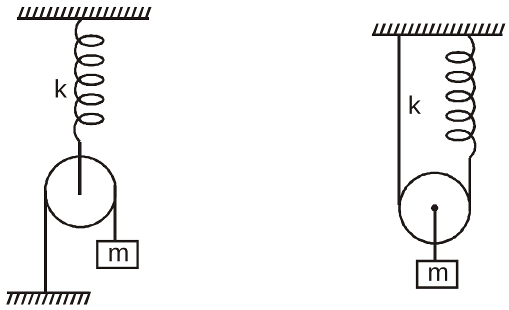
Now if we displace the block by x in the downward direction, net force on the block towards mean position is

$F = k(x + x_0) - mg = kx$ using (1)

Hence the net force is acting towards mean position and is also proportional to x. So, the particle will perform S.H.M. and its time period would be

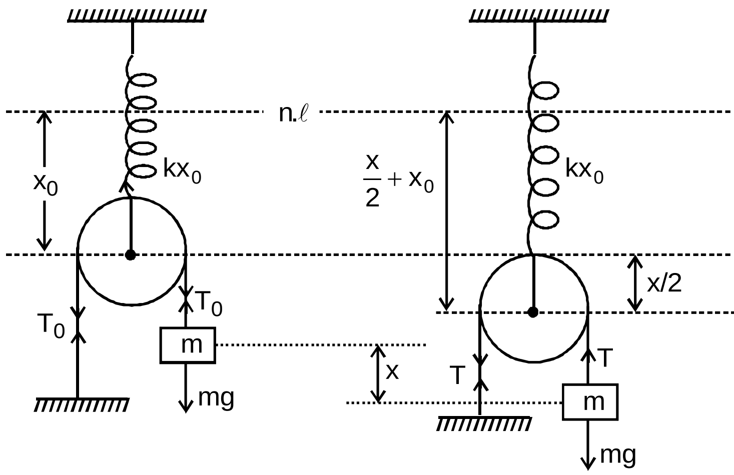
$$T = 2\pi\sqrt{\frac{m}{k}}$$

Ex.16 Figure shows a system consisting of a massless pulley, a spring of force constant k and ablock of mass m. If the block is slightly displaced vertically down from its equillibrium position and then released, find the period of its vertical oscillation in cases (a) & (b).



Sol. Let us assume that in equillibrium condition spring is x_0 elongate from its natural length

Case (a)



When equillibrium   When displaced block by 'x'

In equilibrium $T_0 = mg$

and $kx_0 = 2T_0$

$\Rightarrow$ $kx_0 = 2mg$ ...(1)

If the mass m moves down a distance x from its equilibrium position then pulley will move down by $\frac{x}{2}$. So the extra force in spring will be $\frac{kx}{2}$. From figure

$$F_{net} = mg - T = mg - \frac{k}{2}\left(x_0 + \frac{x}{2}\right)$$

$$F_{net} = mg - \frac{kx_0}{2} - \frac{kx}{4}$$

from eq. (1)

$$F_{net} = \frac{-kx}{4} \quad ...(3)$$

Now compare eq. (3) with $F = -K_{S.H.M}\,x$

then $K_{S.H.M} = \frac{K}{4}$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{m}{K_{S.H.M}}} = 2\pi\sqrt{\frac{4m}{K}}$$

Case (b) :

In this situation if the mass m moves down distance x from its equilibrium position, then pulley will also move by x and so the spring will stretch by 2x.

At equilibrium $kx_0 = \frac{T_0}{2} = \frac{mg}{2}$

When block is displaced

$F_{net} = mg - T$

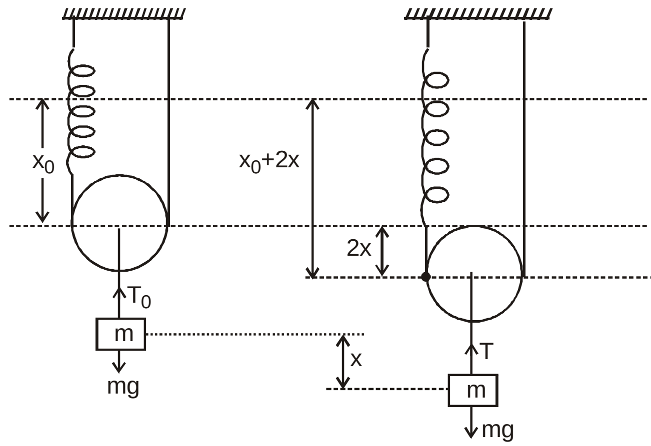
$= mg - 2k(x_0 + 2x)$

$= -4kx$

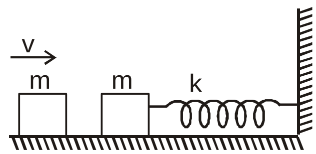
Now $F = -K_{SHM}\, x$ then

$K_{SHM} = 4K$

So time period $T = 2\pi\sqrt{\frac{m}{4k}}$



Ex.17 The left block in figure collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.



Sol. The collision is for a small interval only, we can apply the principal of conservation of momentum. The common velocity after the collision is $\frac{v}{2}$. The kinetic energy $= \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is A, the total energy can also be written as $\frac{1}{2}kA^2$. Thus,

$$\frac{1}{2}kA^2 = \frac{1}{4}mv^2, \text{ giving } A = \sqrt{\frac{m}{2k}}v$$

Ex.18 The system is in equilibrium and at rest. Now mass m_1 is removed from m_2. Find the time period and amplitude of resultant motion. (Given : spring constant is K.)

Sol. Initial extension in the spring

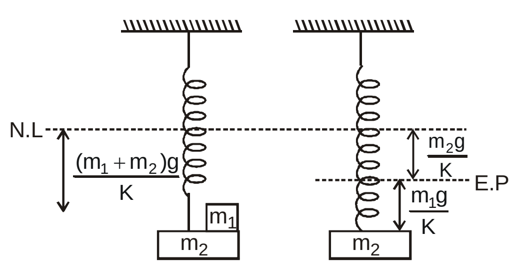
$$x = \frac{(m_1 + m_2)g}{k}$$

Now, if we remove m_1. equillibrium position (E.P.) of m_2 will be $\frac{m_2 g}{K}$ below natural length of spring.

At the initial position, since velocity is zero i.e. it is the extreme position.

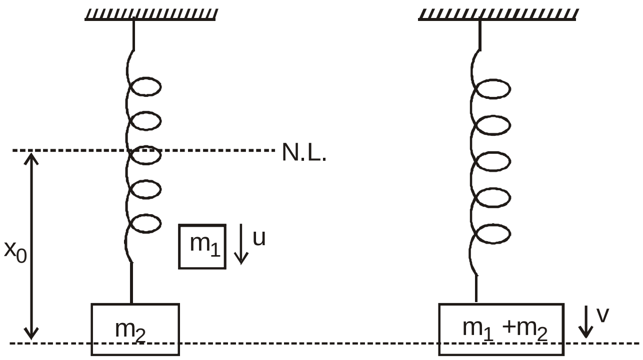
Hence Amplitude $= \frac{m_1 g}{K}$

Time period $= 2\pi\sqrt{\frac{m_2}{K}}$



Ex.19 Block of mass m_2 is in equilibrium and at rest. The mass m_1 moving with velocity u vertically downwards collides with m_2 and sticks to it. Find the energy of oscillation.

Sol.



At equilibrium position $m_2g = kx_0$

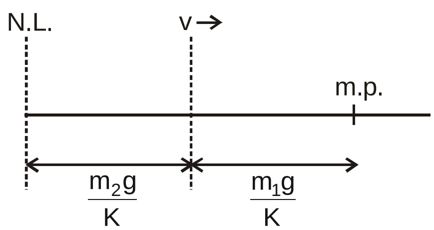
$$\Rightarrow \quad x_0 = \frac{m_2 g}{K}$$

After collision m_2 sticks to m_1. $\therefore$ By momentum conservation.

$$m_1u = (m_1 + m_2)\, v$$

$$v = \frac{m_1 u}{m_1 + m_2}$$

Now both the blocks are executing S.H.M. which can be interpreted as follows:



Now, we know that $v^2 = \omega^2(A^2 - x^2)$ ...(1)

$$\omega^2 = \frac{k}{m_1 + m_2}$$

$$\Rightarrow \quad x = \frac{m_1 g}{k}$$

Put the values of v, ω^2 & x in eq. (1)

$$\left(\frac{m_1 u}{m_1 + m_2}\right)^2 = \left(\frac{k}{m_1 + m_2}\right)\left[A^2 - \left(\frac{m_1 g}{k}\right)^2\right]$$

$$\Rightarrow \quad kA^2 = \left[\left(\frac{m_1^2 u^2}{m_1 + m_2}\right) + \left(\frac{m_1 g}{k}\right)^2\right]$$

$$\Rightarrow \text{ Energy of oscillation } = \frac{1}{2}kA^2 = \frac{1}{2}\left[\left(\frac{m_1^2 u^2}{m_1 + m_2}\right) + \left(\frac{m_1{}^2 g^2}{k}\right)\right]$$

Ex.20 A body of mass m falls from a height h on to the pan of a spring balance. The masses of the pan and spring are negligible. The spring constant of the spring is k. Having stuck to the pan the body starts performing harmonic oscillations in the vertical direction. Find the amplitude and energy of oscillation.

Sol. Suppose by falling down through a height h, the mass m compresses the spring balance by a length x.

$x = \frac{mg}{k}$, $\omega = \sqrt{\frac{k}{m}}$

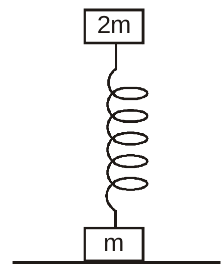
velocity at Q $v = \sqrt{2gh}$

$\because \; v = \omega\sqrt{A^2 - x^2}$

$$\sqrt{2gh} = \sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\frac{mg}{k}\right)^2} \Rightarrow A = \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}}$$

Energy of oscillation $= \frac{1}{2}kA^2 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2\left(1 + \frac{2kh}{mg}\right) = mgh + \frac{(mg)^2}{2k}$

Ex.21 A body of mass 2m is connected to another body of mass m as shown in figure. The mass 2m performs vertical S.H.M. Then find out the maximum amplitude of 2m such that mass m doesn't lift up from the ground.

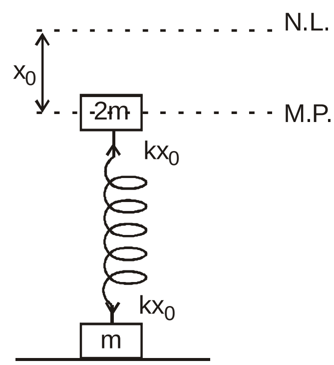


Sol. In the given situation 2m mass is in equilibrium condition.

Let assume spring is compressed x_0 distance from its natural length.

$\Rightarrow \quad kx_0 = 2mg$

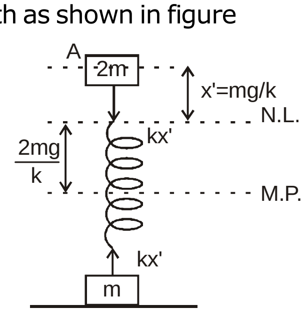
$\Rightarrow \quad x_0 = \frac{2mg}{k}$



The lower block will be lift up, only in the case when the spring force on it will be greater than equal to mg and in upward direction

$\Rightarrow \quad kx' = mg \quad \Rightarrow \quad x' = \frac{mg}{k}$

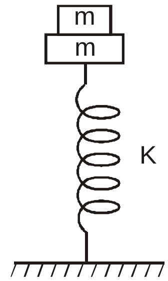
Above situation arises when 2m block moves upward mg/k from natural length as shown in figure



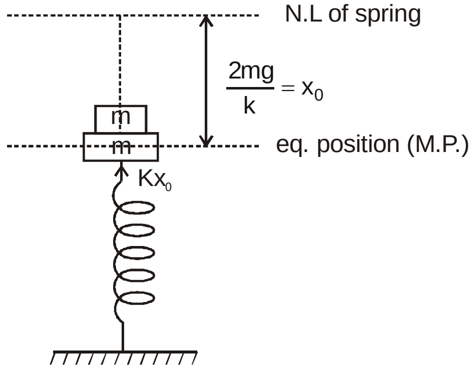
Block m doesn't lift up if the maximum amplitude of the 2m block is

$$= \frac{2mg}{k} + \frac{mg}{k} = \frac{3mg}{k}$$

Ex.22 A block of mass m is at rest on the another block of same mass as shown in figure. Lower block is attached to the spring then determine the maximum amplitude of motion so that both the block will remain in contact.



Sol.



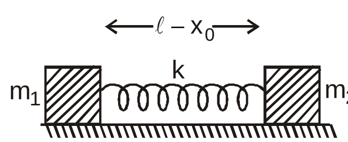
The blocks will remain in contact till the blocks do not go above the natural length of the spring, because after this condition the deceleration of lower block becomes more then upper block due to spring force. So they will get seprated.

So maximum possible amplitude = $x_0 = \dfrac{2mg}{k}$

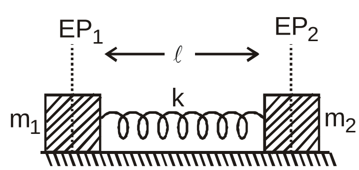
12.1 Two Block Systems:

Ex.23 Two blocks of mass m_1 and m_2 are connected with a spring of natural length ℓ and spring constant k. The system is lying on a smooth horizontal surface. Initially spring is compressed by x_0 as shown in figure.

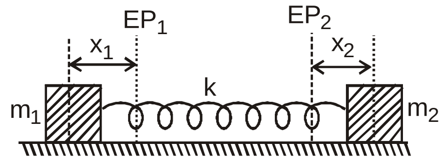
Show that the two blocks will perform SHM about their equilibrium position. Also (a) find the time period, (b) find amplitude of each block and (c) length of spring as a function of time.



Sol. (a) Here both the blocks will be in equilibrium at the same time when spring is in its natural length. Let EP_1 and EP_2 be equilibrium positions of block A and B as shown in figure.



Let at any time during oscillations, blocks are at a distance of x_1 and x_2 from their equilibrium positions.



As no external force is acting on the spring block system

$\therefore \quad (m_1 + m_2)\Delta x_{cm} = m_1x_1 - m_2x_2 = 0 \quad$ or $\quad m_1x_1 = m_2x_2$

For 1st particle, force equation can be written as

$$k(x_1 + x_2) = -m_1\frac{d^2x_1}{dt^2} \qquad \text{or,} \qquad k\left(x_1 + \frac{m_1}{m_2}x_1\right) = -m_1a_1$$

$$\text{or,} \quad a_1 = -\frac{k(m_1+m_2)}{m_1m_2}x_1 \qquad \therefore \quad \omega^2 = \frac{k(m_1+m_2)}{m_1m_2}$$

Hence, $T = 2\pi\sqrt{\dfrac{m_1m_2}{k(m_1+m_2)}} = 2\pi\sqrt{\dfrac{\mu}{K}}$ where $\mu = \dfrac{m_1m_2}{(m_1+m_2)}$ which is known as reduced mass

(b) Let the amplitude of blocks be A_1 and A_2.

$m_1A_1 = m_2A_2$

By energy conservation ;

$$\frac{1}{2}k(A_1+A_2)^2 = \frac{1}{2}kx_0^2 \qquad \text{or,} \qquad A_1 + A_2 = x_0$$

$$\text{or,} \quad A_1 + A_2 = x_0 \qquad \text{or,} \qquad A_1 + \frac{m_1}{m_2}A_1 = x_0$$

$$\text{or,} \quad A_1 = \frac{m_2x_0}{m_1+m_2} \qquad \text{Similarly,} \qquad A_2 = \frac{m_1x_0}{m_1+m_2}$$

(c) Let equilibrium position of 1st particle be origin, i.e., x = 0.

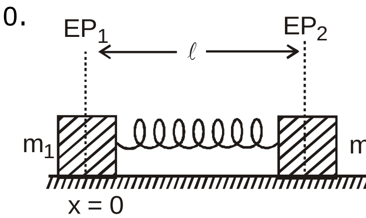
x co-ordinate of particles can be written as

$x_1 = A_1 \cos\omega t$ and $x_2 = \ell - A_2\cos\omega t$

Hence, length of spring can be written as :

length = $x_2 - x_1$

$= \ell - (A_1 + A_2)\cos\omega t$



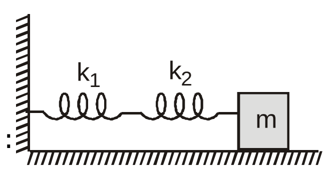
13. COMBINATION OF SPRINGS :

13.1 Series Combination :

Total displacement $x = x_1 + x_2$

Tension in both springs $= k_1x_1 = k_2x_2$

$\therefore$ Equivalent constant in series combination K_{eq} is given by :



$$1/k_{eq} = 1/k_1 + 1/k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

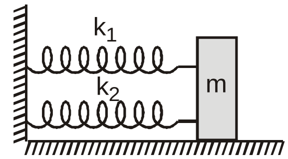
In series combination, tension is same in all the springs & extension will be different. (If k is same then deformation is also same)

13.2 Parallel combination :

Extension is same for both springs but force acting will be different.

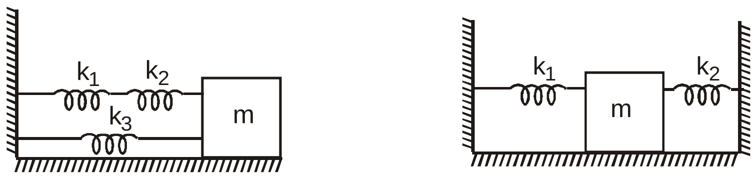
Force acting on the system = F

$\therefore$ $F = -(k_1x + k_2x) \Rightarrow F = -(k_1 + k_2)\,x \Rightarrow F = -k_{eq}x$



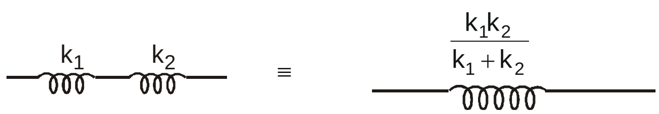
$\therefore$ $k_{eq} = k_1 + k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\dfrac{m}{k_{eq}}}$

Ex.24 Find the time period of the oscilltion of mass m in figure a and b. What is the equivalent spring constant of the spring in each case. ?

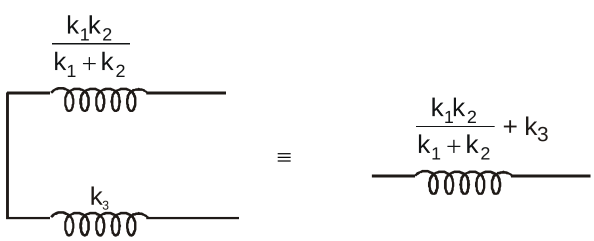


(a) (b)

Sol. In figure (a)



Which gives



$$k_{eq} = \frac{k_1k_2}{k_1+k_2} + k_3 = \frac{k_1k_2 + k_2k_3 + k_1k_3}{k_1+k_2}$$

Now $T = 2\pi\sqrt{\dfrac{m}{k_{eq}}} = 2\pi\sqrt{\dfrac{m(k_1+k_2)}{k_1k_2 + k_2k_3 + k_1k_3}}$

Case (b)

If the block is displaced slightly by an amount x then both the spring are displaced by x from their natural length so it is parallel combination of springs.

which gives

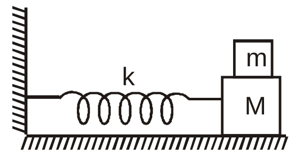
$$k_{eq} = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_{eq}}} = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$

Note :

- In series combination, extension of springs will be reciprocal of its spring constant.
  $\therefore$ $k \propto 1/\ell$
  $\therefore k_1\ell_1 = k_2\ell_2 = k_3\ell_3$
- If a spring is cut in 'n' equal pieces then spring constant of one piece will be nk.

Ex.25 The friction coefficient between the two blocks shown in figure is μ and the horizontal plane is smooth. (a) If the system is slightly displaced and released, find the time period. (b) Find the magnitude of the frictional force between the blocks when the displacement from the mean position is x. (c) What can be the maximum amplitude if the upper block does not slip relative to the lower block?



Sol. (a) For small amplitude, the two blocks oscillate together.
The angular frequency is

$\omega = \sqrt{\dfrac{k}{M+m}}$ and so the time period $T = 2\pi\sqrt{\dfrac{M+m}{k}}$

(b) The acceleration of the blocks at displacement x from the mean position is

$$a = -\omega^2 x = \left(\frac{-kx}{M+m}\right)$$

The resultant force on the upper block is, therefore, $ma = \left(\dfrac{-mkx}{M+m}\right)$

This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\left(\dfrac{mk\,|x|}{M+m}\right)$

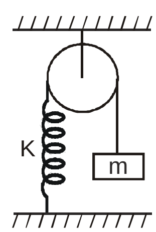
(c) Maximum force of friction required for simple harmonic motion of the upper block is $\dfrac{mkA}{M+m}$ at the extreme positions. But the maximum frictional force can only be μ mg. Hence

$$\frac{mkA}{M+m} = \mu mg \quad \text{or,} \quad A = \frac{\mu(M+m)g}{k}$$

14. ENERGY METHOD :

Another method of finding time period of SHM is energy method. To understand this method we will consider the following example.

Ex.26 Figure shows a system consisting of pulley having radius R, a spring of force constant k and a block of mass m. Find the period of its vertical oscillation.

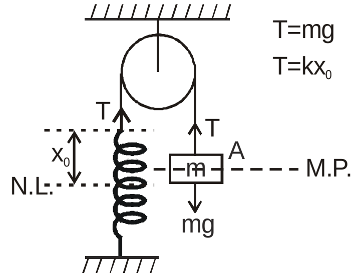


Sol. The following steps are usually followed in this method:

Step 1. Find the mean position. In following figure point A shows mean position.

Step 2. Write down the mean position force relation from figure.

$$mg = kx_0$$



Step 3. Assume that particle is performing SHM with amplitude A. Then displace the particle from its mean position.

Step 4. Find the total mechanical energy (E) in the displaced position since, mechanical energy in SHM remains constant $\frac{dE}{dt} = 0$

* $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x + x_0)^2 - mgx$

$E = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} + \frac{1}{2}k(x + x_0)^2 - mgx$

$$\frac{dE}{dt} = \frac{2mv}{2}\frac{dv}{dt} + \frac{2Iv}{2R^2}\frac{dv}{dt} + \frac{2k(x + x_0)}{2}\frac{dx}{dt} - mg\frac{dx}{dt} \quad ...(1)$$

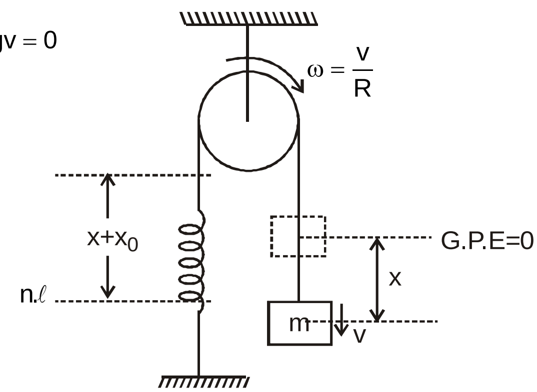
Put $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

in eq. (1) put

$$\frac{dE}{dt} = 0 \Rightarrow mv\frac{d^2x}{dt^2} + \frac{Iv}{R^2}\frac{d^2x}{dt^2} + kxv + kx_0v - mgv = 0$$

which gives $\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + kx = 0$

$$\frac{d^2x}{dt^2} + \frac{k}{\left(m + \frac{I}{R^2}\right)}x = 0 \quad ...(2)$$



compare eq. (2) with S.H.M eq. the

$$\omega^2 = \frac{k}{\left(m + \frac{I}{R^2}\right)} \Rightarrow T = 2\pi\sqrt{\frac{(m + I/R^2)}{k}}$$

15. ANGULAR S.H.M. :

If the restoring torque acting on the body in oscillatory motion is directly proportional to the angular displacement of body from its equillibrium position i.e.,

$\tau = -k\theta$

k = S.H.M. constant

θ = angular displacement from M.P.

S.H.M. equation is given by

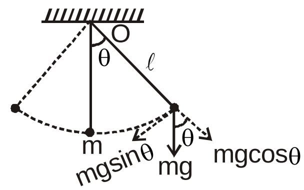
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad \text{Here} \qquad \omega = \sqrt{\frac{K}{I}}$$

Here I is moment of inertia of the body/particle about a given axis.

16. SIMPLE PENDULUM :

If a heavy point-mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple penduluml.

Time period of a simple pendulum $T = 2\pi\sqrt{\frac{\ell}{g}}$.



(some times we can take g = π^2 for making calculation simple)

Proof :

Now taking moment of forces acting on the bob about point O.

$$\tau = \tau_T + \tau_{mg}$$

$$\tau_T = 0$$

⇒ $\tau = -(mg \sin\theta)\ell$

if θ is very small then $\sin\theta \simeq \theta$

⇒ $\tau = -\, mg\, \theta\, \ell$ ...(1)

Now compare eq. (1) with

$$\tau_{net} = -\, K_{S.H.M}\, \theta$$

which gives $K_{S.H.M} = mg\,\ell$

⇒ $T = 2\pi\sqrt{\frac{I}{K_{S.H.M}}} = 2\pi\sqrt{\frac{m\ell^2}{mg\ell}} = 2\pi\sqrt{\frac{\ell}{g}}$

Note :

- Time period of second pendulum is 2 seconds.
- Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear S.H.M.
- If time period of clock based upon simple pendulum increases then clock will become slow but if time period decreases then clock will become fast.

17. TIME PERIOD OF SIMPLE PENDULUM IN ACCELERATING REFERENCE FRAME :

$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$ where

g_{eff} = Effective acceleration due to gravity in reference system = $|\vec{g} - \vec{a}|$

$\vec{a}$ = acceleration of the point of suspension w.r.t. ground.

Condition for applying this formula : $|\vec{g} - \vec{a}|$ **=** constant

If the acceleration $\vec{a}$ is upwards, then $|\vec{g}_{eff}| = g + a$ and $T = 2\pi\sqrt{\frac{\ell}{g+a}}$

Time lost or gained in time t is given by

$$\Delta T' = \frac{\Delta T}{T}.t$$

Ex.27 If T = 2 sec T_{new} = 3 sec. then ΔT = 1 sec.

Since time lost by clock in 3 sec is = 1 sec

then time lost by clock in 1 sec = $\frac{1}{3}$sec

$\therefore$ Time lost by the clock in an hour = $\frac{1}{3}\times 3600$ = 1200 sec.

Ex.28 A simple pendulum is suspended from the ceiling of a car which is accelerating uniformly on a horizontal road. The acceleration of car is a_0 and the length of the pendulum is ℓ_1. Then find the time period of small oscillations of pendulum about the mean position.

Sol. We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a psuedo force ma_0 on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If θ_0 be the angle made by the string with the vertical, the tension, weight and the peusdo force will add to zero in this position.

Hence, resultant of mg and ma_0 (say F = $m\sqrt{g^2+a_0^2}$) has to be along the string.

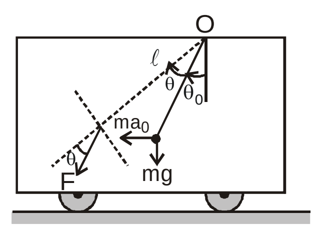
$\therefore \quad \tan\theta_0 = \frac{ma_0}{mg} = \frac{a_0}{g}$

Now, suppose the string is further deflected by an angle θ as shown in figure.

Now, restoring torque about point O can be given by $\tau = I\alpha$

$(F \sin\theta)\,\ell = -\,m\,\ell^2\,\alpha$

Substituting F and using $\sin\theta = \theta$, for small θ.



$(m\sqrt{g^2+a_0^2})\,\ell\,\theta = -\,m\ell^2\,\alpha$

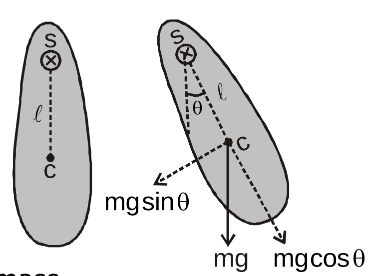
or, $\alpha = -\frac{\sqrt{g^2+a_0^2}}{\ell}\theta$ so ; $\omega^2 = \frac{\sqrt{g^2+a_0^2}}{\ell}$

This is an equation of simple harmonic motion with time period.

$$T = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{\ell}}{(g^2+a_0^2)^{1/4}}$$

18. COMPOUND PENDULUM / PHYSICAL PENDULUM :

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.



C = Position of centre of mass

S = Point of suspension

ℓ = Distance between point of suspension and centre of mass (it remains constant during motion for small angular displacement "θ" from mean position

The restoring torque is given by

$\tau = -mg\,\ell\sin\theta$

$\tau = -mg\,\ell\,\theta$ $\quad\therefore$ for small θ, $\sin\theta = \theta$

or, $I\alpha = -mg\,\ell\,\theta$ $\quad$ where, I = Moment of inertia about point of suspension.

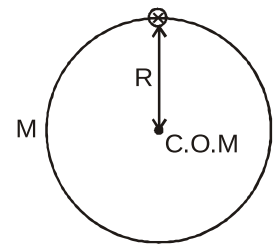
or, $a = -\dfrac{mg\ell}{I}\theta$ $\quad$ or, $\omega^2 = \dfrac{mg\ell}{I}$

Time period, $T = 2\pi\sqrt{\dfrac{I}{mg\ell}}$

Ex. 29 A ring is suspended at a point on its rim and it behaves as a second's pendulum when it oscillates such that its centre move in its own plane. The radius of the ring would be ($g = \pi^2$)

Sol. Time period of second pendulum T = 2 cm.

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$



Moment of inertia with respect to axis O

$I = MR^2 + MR^2 = 2MR^2$

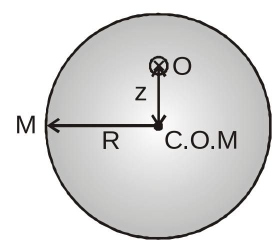
the distance between centre of mass and the axis O

d = R

$$2 = 2\pi\sqrt{\frac{2MR^2}{MgR}} \Rightarrow R = 0.5\text{ m}$$

Ex.30 A circular disc has a tiny hole in it, at a distance z from its center. Its mass is M and radius R(R > 2). Horizontal shaft is passed through the hole and held fixed so that the disc can freely swing in the vertical plane. For small distrubance, the disc performs SHM whose time period is minimum for z . Find the value of z.

Sol. The time period w.r.t the axis $T = 2\pi\sqrt{\dfrac{I}{Mgd}}$



where I = moment of inertia w.r.t the axis O

d = distance between C.O.M and O

$\Rightarrow \quad I = \dfrac{MR^2}{2} + Mz^2$

d = z

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{\frac{MR^2}{2} + Mz^2}{Mgz}} = 2\pi\sqrt{\frac{R^2}{2gz} + \frac{z}{g}}$$

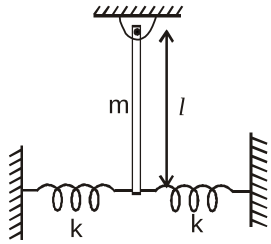
the time period will be minimum when $\dfrac{R^2}{2z} + z$ = minimum

Let say $f = \dfrac{R^2}{2z} + z$

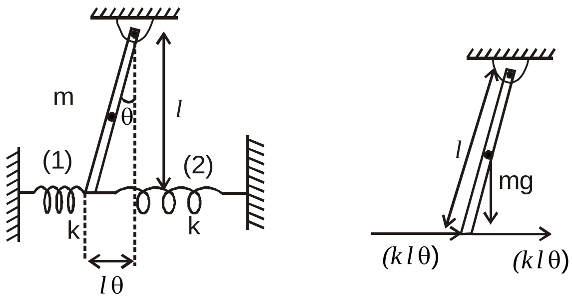
f will be minimum when $\dfrac{df}{dz} = 0$

$\Rightarrow \quad -\frac{R^2}{2z^2}+1=0 \Rightarrow \quad z=\frac{R}{\sqrt{2}}$

Ex.31 Find out the angular frequency of small oscillation about axis O



Sol.



The compression in spring (1) = $l\,\theta$

and the extension in spring (2) = $l\,\theta$

Net torque opposite to the mean position = – $(2kl\theta)\,l - mg\frac{l}{2}\sin\theta = \tau_{net}$

θ is small $\Rightarrow \sin\theta \simeq \theta$

$\tau_{net} = -I\omega^2\theta = -(2kl\theta)\,l - mg\frac{l}{2}\sin\theta = \tau_{net}$

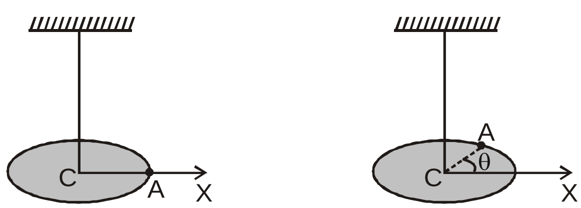
$I = \frac{ml^2}{3}$

$\Rightarrow \quad \omega = \sqrt{\frac{3}{2}\frac{(4kl+mg)}{ml}}$

19. TORSIONAL PENDULUM :

In torsional pendulum, an extended object is suspended at the centre by a light torsion wire. A torsion wire is essentially inextensible, but is free to twist about its axis. When the lower end of the wire is rotated by a slight amount, the wire applies a restoring torque causing the body to oscillate rotationally when released.

The restoring torque produced is given by



$\tau = -C\theta$ where, C = Torsional constant

or, $I\alpha = -C\theta$ where, I = Moment of inertia about the vertical axis.

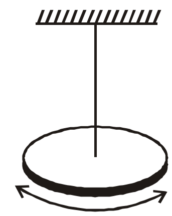
or, $\alpha = -\frac{C}{I}\theta$ $\quad\therefore$ Time Period, $T = 2\pi\sqrt{\frac{I}{C}}$

: The above concept of torsional pendulum is used in inertia table to calculate the moment of inertia of unknown body.

Ex.32 A uniform disc of radius 5.0 cm and mass 200 g is fixed at its centre to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

Sol. The situation is shown in figure. The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0.200\text{kg})(5.0\times10^{-2}\text{m})^2}{2} = 2.5 \times 10^{-4}\ \text{kg-m}^2.$$



The time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad \text{or,} \qquad C = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4\pi^2(2.5\times10^{-4}\text{kg}-\text{m}^2)}{(0.20\,\text{s})^2} \qquad = 0.25\frac{\text{kg}-\text{m}^2}{\text{s}^2}$$

20. VECTOR METHOD OF COMBINING TWO OR MORE SIMPLE HARMONIC MOTIONS:

A simple harmonic motion is produced when a force (called restoring force) proportional to the displacement acts on a particle. If a particle is acted upon by two such forces the resultant motion of the particle is a combination of two simple harmonic motions.

20.1 In Same direction :

(a) Having same Frequencies:

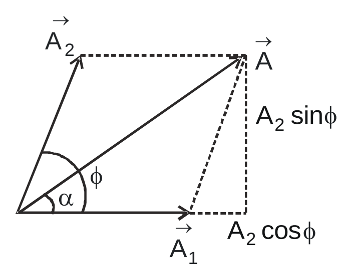
Suppose the two individual motions are represented by,

$$x_1 = A_1 \sin \omega t \ \text{ and } \quad x_2 = A_2 \sin(\omega t + \phi)$$

Both the simple harmonic motions have same angular frequency ω.

$$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin(\omega t + \phi)$$

$$= A \sin(\omega t + \alpha)$$



Here, $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$

and $\tan\alpha = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi}$

Thus, we can see that this is similar to the vector addition. The same method of vector addition can be applied to the combination of more than two simple harmonic motions.

Important points to remember before solving the questions:

1. Convert all the trignometric ratios into **sine** form and ensure that ωt term is with +ve sign.
2. Make the sign between two term +ve.
3. A_1 is the amplitude of that S.H.M whose phase is small.
4. Then resultant $x = A_{net} \sin(\text{phase of } A_1 + \alpha)$

Where A_{net} is the vector sum of A_1 & A_2 with angle between them is the phase difference between two S.H.M.

Ex.33 $x_1 = 3 \sin \omega t$; $x_2 = 4 \cos \omega t$

Find (i) amplitude of resultant SHM. (ii) equation of the resultant SHM.

Sol. First right all SHM's in terms of sine functions with positive amplitude. Keep "ωt" with positive sign.

$\therefore$ $x_1 = 3 \sin \omega t$

$x_2 = 4 \sin (\omega t + \pi/2)$

$$A = \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \cos\frac{\pi}{2}} = \sqrt{9+16} = \sqrt{25} = 5$$

$$\tan\phi = \frac{4\sin\frac{\pi}{2}}{3 + 4\cos\frac{\pi}{2}} = \frac{4}{3} \qquad \phi = 53°$$

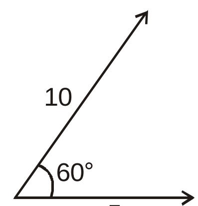
equation $x = 5 \sin (\omega t + 53°)$

Ex.34 $x_1 = 5 \sin (\omega t + 30°)$; $x_2 = 10 \cos (\omega t)$

Find amplitude of resultant SHM.

Sol. $x_1 = 5 \sin (\omega t + 30°)$

$x_2 = 10 \sin (\omega t + \frac{\pi}{2})$



Phasor Diagram

$$A = \sqrt{5^2 + 10^2 + 2 \times 5 \times 10 \cos 60°} = \sqrt{25 + 100 + 50} = \sqrt{175} = 5\sqrt{7}$$

Ex.35 A particle is subjected to two simple harmonic motions

$x_1 = A_1 \sin \omega t$

and $x_2 = A_2 \sin (\omega t + \pi/3)$

Find (a) the displacment at t = 0, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle.

Sol. (a) At t = 0, $x_1 = A_1 \sin \omega t = 0$

and $x_2 = A_2 \sin (\omega t + \pi/3) = A_2 \sin (\pi/3) = \frac{A_2\sqrt{3}}{2}$

Thus, the resultant displacement at t = 0 is

$$x = x_1 + x_2 = A_2 \frac{\sqrt{3}}{2}$$

(b) The resultant of the two motion is a simple harmonic motion of the same angular frequency ω. The amplitude of the resultant motion is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\pi/3)} = \sqrt{A_1^2 + A_2^2 + A_1A_2}$$

The maximum speed is

$$u_{max} = A\,\omega = \omega\sqrt{A_1^2 + A_2^2 + A_1A_2}$$

(c) The maximum acceleration is

$$a_{max} = A\,\omega^2 = \omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}$$

(b) Having different frequencies

$x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$ This resultant motion is not SHM.

20.2 In two perpendicular directions

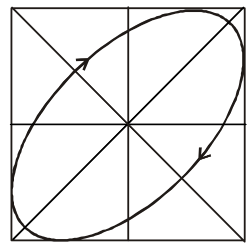
$x = A_1 \sin \omega t$ ...(1)

$y = A_2 \sin (\omega t + \phi)$ ...(2)

The Amplitudes A_1 and A_2 may be different and Phase difference ϕ and ω is same.

So equation of the path may be obtained by eliminating t from (1) & (2)

$\sin \omega t = \frac{x}{A_1}$ ...(3)



$\cos \omega t = \sqrt{1 - \frac{x^2}{A_1^2}}$ ...(4)

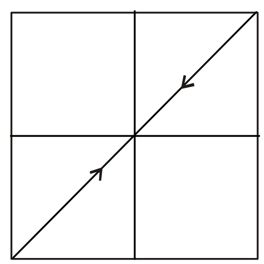
On rearranging we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy\cos\phi}{A_1 A_2} = \sin^2 \phi \quad ...(5)$$

(general eq. of ellipse)

special case :

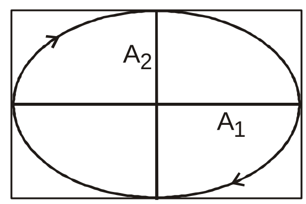
(1) If $\phi = 0$



$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

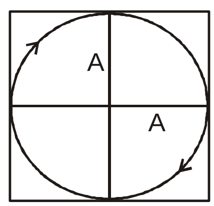
$\therefore y = \frac{A_2}{A_1}.x$ (eq. of straight line)

(2) If $\phi = 90°$



$\Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$ (Eq. of ellipse)

(3) If $\phi = 90°$ & $A_1 = A_2 = A$



then $x^2 + y^2 = A^2$ (Eq. of circle.)

The above figures are called Lissajous figures.

MIND MAP

SHM

1. Equation of S.H.M

(i) **Linear** : $a = -\omega^2 x$

(ii) **Angular** : $\alpha = -\omega^2\theta$

2. Linear SHM

(i) Displacement of particle :

$x = A\sin(\omega t + \phi)$

(ii) **Velocity**

$\frac{dx}{dt} = A\omega\cos(\omega t + \phi)$

$= \omega\sqrt{A^2 - x^2}$

(iii) **Acceleration :**

$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi)$

$= -\omega^2 x$

(iv) **Phase :** $\omega t + \phi$

(v) **Phase constant :** ϕ

3. Angular S.H.M

(i) **Displacement :**

$\theta = \theta_0\sin(\omega t + \phi)$

(ii) **Angular velocity**

$\frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi)$

(iii) **Acceleration**

$\frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi) = -\omega^2\theta$

(iv) **Phase :** $\omega t + \phi$

(v) **Phase constant :** ϕ

4. Energy in S.H.M

(i) $K = \frac{1}{2}m\omega^2(A^2 - x^2)$

(ii) $U = \frac{1}{2}m\omega^2x^2$

(iii) $E = K + U = \frac{1}{2}m\omega^2A^2$

= constant

5. Time Period :

Pendulums :

(a) **Simple pendulum :**

$T = 2\pi\sqrt{\frac{l}{g}}$

(b) **Physical pendulum :**

$T = 2\pi\sqrt{\frac{I}{mgl}}$

(c) **Torsional pendulum :**

$T = 2\pi\sqrt{\frac{I}{C}}$

6. Mass-spring system

(a) $T = 2\pi\sqrt{\frac{m}{k}}$

(b) **Two bodies system :**

$T = 2\pi\sqrt{\frac{\mu}{k}}$;

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Combination of springs :

(a) series : $\frac{1}{K_{eff}} = \frac{1}{K_1} + \frac{1}{K_2}$

(b) parallel : $K_{eff} = K_1 + K_2$

(c) Spring cut into two parts m : n

$K_1 = \frac{(m+n)K}{m}, K_2 = \frac{(m+n)K}{n}$

7. Composition of 2 SHMs :

$x_1 = A_1\sin\omega t$

$x_2 = A_2\sin(\omega t + \phi)$

$x = x_1 + x_2$

$x = A\sin(\omega t + \delta)$

where, $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$

and $\tan\delta = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$