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**PHYSICS**

**TARGET IIT JEE**

# WAVES

*THEORY AND EXERCISE BOOKLET*

## CONTENTS


| S.NO. | TOPIC | PAGE NO. |
|---|---|---|
| 1. | Waves | 3 – 8 |
| 2. | Particle Velocity and Acceleration | 8 |
| 3. | Different forms of Wave function | 8 – 9 |
| 4. | Linear Wave equation | 10 – 13 |
| 5. | Speed of wave on a string | 13 – 15 |
| 6. | Energy calculation in waves | 15 – 18 |
| 7. | Superposition | 19 – 24 |
| 8. | Reflection and transmission | 24 – 27 |
| 9. | Standing waves | 27 – 32 |
| 10. | Stationary waves in string | 32 – 34 |
| 11. | Exercise - I | 35 – 43 |
| 12. | Exercise - II | 44 – 49 |
| 13. | Exercise - III | 50 - 58 |
| 14. | Exercise - IV | 59 |
| 15. | Exercise - V | 60 – 66 |
| 16. | Answer key | 67 – 68 |

**Syllabus ::**

Wave motion (plane waves only), longitudinal and transverse waves, Superposition of waves; progressive and stationary waves.

## 1. WAVES :

Waves is distributed energy or distributed "disturbance (force)"

- **Following points regarding waves :**

**1.** The disturbance (force) is transmitted from one point to another.

**2.** The energy is transmitted from one point to another.

**3.** The energy or distrubance passes in the form of wave without any net displacement of medium.

**4.** The oscillatory motion of preceding particle is imparted to the adjacent particle following it.

**5.** We need to keep creating disturbance in order to propagate wave (energy or disturbance) continuously.

### (a) Waves classification

The waves are classified under two high level headings :

**1.** **Mechanical waves :** The motion of the particle constituting the medium follows mechanical laws i.e. Newton's laws of motion. Mechanical waves originate from a distrubance in the medium (such as a stone dropping in a pond) and the disturbance propagates through the medium. The force between the atoms in the medium are responsible for the propagation of mechanical waves. Each atom exerts a force on the atoms near it, and through this force the motion of the atom is transmitted to the others. The atoms in the medium do not experience any net displacement.

Mechanical waves is further classified in two categories such that

1. Transverse waves (waves on a string)

2. Longitudnal waves (sound waves)

**2.** **Non Mechanical waves :** These are electro magnetic waves. The electromagnetic waves do not require a medium for propagation. Its speed in vacuum is a universal constant. The motion of the electromagnetic waves in a medium depends on the electromagnetic properties of the medium.

### 2.1 Transverse waves

If the disturbance travels in the x direction but the particles move in a direction, perpendicular to the x axis as the wave passes it is called a transverse waves.

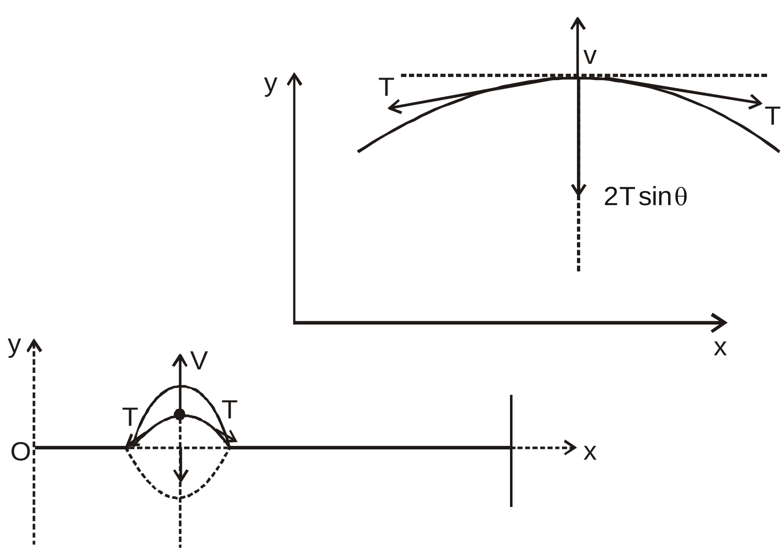



figure - I

Consider a sinusoidal harmonic wave travelling through a string and the motion of a particle as shown in the figure I$^{st}$ (only one unit of wave shown for illustration purpose). Since the particle is displaced from its natural (mean) position, the tension in the string arising from the deformation tends to restore the position of the particle. On the other hand, velocity of the particle (kinetic energy) move the particle farther is zero. Therefore, the particle is pulled down due to tension towards mean position. In the process, it acquires kinetic energy (greater speed) and overshoots the mean position in the downward direction. The cycle of restoration of position continues as vibration (oscillation) of particle takes place.

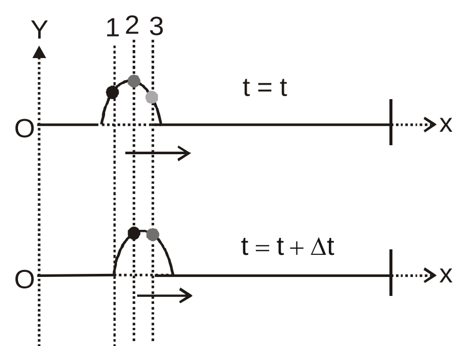
### 2.2 Longitudinal waves

Longitudinal waves are characterized by the direction of vibration (disturbance) and wave motion. They are along the same direction. It is clear that vibration in the same direction needs to be associated with a "restoring" mechanism in the longitudinal direction.

### (b) Mathematical description of waves

We shall attempt here to evolve a mathematical model of a travelling transverse wave. For this, we choose a specific set up of string and associated transverse wave travelling through it. The string is tied to a fixed end, while disturbance is imparted at the free end by up and down motion. For our purpose, we consider that pulse is small in dimension; the string is light, elastic and homogeneous. The assumptions are required as we visualize a small travelling pulse which remains undiminished when it moves through the strings. We also assume that the string is long enough so that our observation is not subjected to pulse reflected at the fixed end.

For understanding purpose, we first consider a single pulse as shown in the figure (irrespective of whether we can realize such pulse in practice or not). Our objective here is to determine the nature of a mathematical description which will enable us to determine displacement (disturbance) of string as pulse passes through it. We visualize two snap shots of the travelling pulse at two close time instants "t" and "t + $\Delta$t". The single pulse is moving towards right in the positive x-direction.




The vibration and wave motion are at right angle to each other.

Three position along x-axis named "1", "2" and "3" are marked with three vertical dotted lines. At either of two instants as shown, the positions of string particles have different displacements from the undisturbed position on horizontal x-axis. We can conclude from this observation that displacement in y-direction is a function of positions of particle in x-direction. As such, the displacement of a particle constituting the string is a function of "x".

Let us now observe the positions of a given particle, say "1". It has certain positive displacement at time t = t, At the next snapshot at t = t + $\Delta$t, the displacement has reduced to zero. The particle at "2" has maximum displacement at t = t, but the same has reduced at t = t + $\Delta$t. The third particle at "3' has certain positive displacement at t = t, At t = t + $\Delta$t, it acquires additional positive displacement and reaches the position of maximum displacement. From these observation, we conclude that displacement of a particle at any position along the string is a function of "t".

Combining two observations, we conclude that displacment of a particle is a function of both position of the particle along the string and time.

$$y = f(x, t)$$

We can further specify the nature of the mathematical function by association the speed of the wave in our consideration. Let "v" be the constant speed with which wave travels from the left end to the right end. We notice that wave function at a given position of the string is a function of time only as we are considering displacement at a particular value of "x". Let us consider left hand end of the string as the origin of reference (x = 0 and t = 0). The displacement in y-direction (disturbance) at x = 0 is a function of time, "t" only :

$$y = f(t) = A \sin \omega t$$

The disturbance travels to the right at constant speed "v'. Let it reaches a point specified as x = x after time "t". If we visualize to describe the origin of this disturbance at x = 0, then time elapsed for the distrubance to move from the origin (x = 0) to the point (x = x) is "x/v". Therefore, if we want to use the function of displacement at x = 0 as given above, then we need to subtract the time elapsed and set the equation is :

$$y = f\left(t - \frac{x}{v}\right) = A \sin \omega \left(t - \frac{x}{v}\right)$$

This can also be expressed as

$$\Rightarrow \quad f\left(\frac{vt - x}{v}\right) \quad \Rightarrow \quad -f\left(\frac{x - vt}{v}\right)$$

$$y(x, t) = g(x - vt)$$

using any fixed value of t (i.e. at any instant), this shows shape of the string.

If the wave is travelling in –x direction, the wave equation is written as

$$y\ (x, t) = f(t + \frac{x}{v})$$

The quantity x – vt is called phase of the wave function. As phase of the pulse has fixed value

$$x - vt = \text{const.}$$

Taking the derivative w.r.t. time $\frac{dx}{dt} = v$

where v is the phase velocity although often called wave velocity. It is the velocity at which a particular phase of the distrubance travels through space.

In order for the function to represent a wave travelling at speed v, the quantities x, v and t must appear in the combination (x + vt) or (x – vt). Thus $(x - vt)^2$ is acceptable but $x^2 - v^2 t^2$ is not.

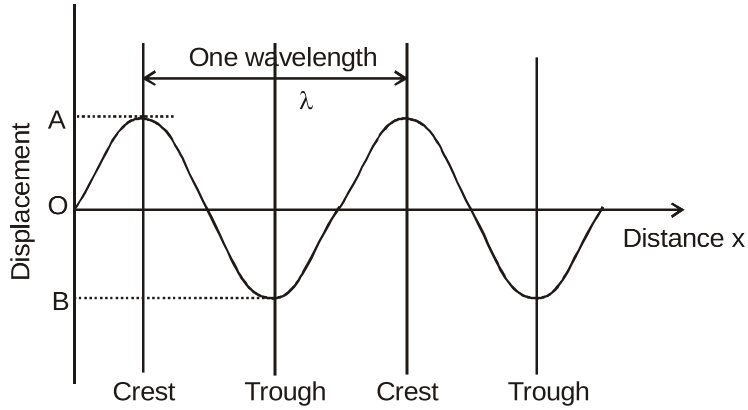
**(c) Describing Waves :**

Two kinds of graph may be drawn displacement - distance and displacement-time.

A displacement-distance graph for a transverse mechanical waves shows the displacement y of the vibrating particles of the transmitting medium at different distance x from the source at a certain instant i.e. it is like a photograph showing shape of the wave at that particular instant.

The maximum displacement of each particle from its undisturbed position is the amplutude of the wave.

In the figure 1, it OA or OB.




The wavelength $\lambda$ of a wave is generally taken as the distance between two successive crests or two successive trough. To be more specific, it is the distance between two consecutive points on the wave which have same phase.

A displacement-time graph may also be drawn for a wave motion, showing how the displacement of one particle at a particular distance from the source varies with time. If this is simple harmonic variation then the graph is a sine curve.

- **Wave Length, Frequency, Speed**

If the source of a wave makes f vibrations per second, so they will the particles of the transmitting medium. That is, the frequency of the waves equals frequency of the source.

When the source makes one complete vibration, one wave is generated and the disturbance spreads out a distance $\lambda$ from the source. If the source continues to vibrate with constant frequency f, then f waves will be produced per second and the wave advances a distance f $\lambda$ in one second. If v is the wave speed then

$$v = f\lambda$$

This relationship holds for all wave motions.

☛ Frequency depends on source (not on medium), v depends on medium (not on source frequency), but wavelength depend on both medium and source.

**(d) Initial Phase :**

At x = 0 and t = 0, the sine function evaluates to zero and as such y-displacement is zero. However, a wave form can be such that y-displacement is not zero at x =0 and t = 0. In such case, we need to account for the displacement by introducting an angle like :

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where "$\phi$" is initial phase. At x = 0 and t = 0.

$$y(0, 0) = A\sin(\phi)$$

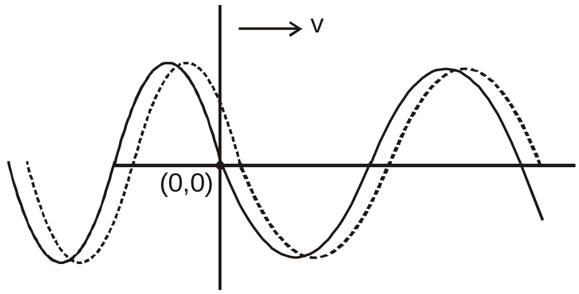
The measurement of angle determines following two aspects of wave form at x = 0, t = 0 : (i) whether the displacement is positive or negative and (ii) whether wave form has positive or negative slope.

For a harmonic wave represented by sine function, there are two values of initial phase angle for which displacement at reference origin (x = 0, t = 0) is positive and has equal magnitude. We know that the sine values of angles in first and second quadrants are positive. A pair of initial phase angles, say $\phi = \pi/3$ and $2\pi/3$, correspond to equal positive sine values are :

$$\sin\theta = \sin(\pi - \theta)$$

$$\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

To choose the initial phase in between the two values $\pi/3$ & $\frac{2\pi}{3}$. We can look at a wave motion in yet another way. A wave form at an instant is displaced by a distance $\Delta x$ in very small time interval $\Delta t$ then then speed to the particle at t = 0 & x = 0 is in upward +ve direction in further time $\Delta t$




***Ex.1 Find out the expression of wave equation which is moving is +ve x direction and at x = 0, t = 0 y = $\frac{A}{\sqrt{2}}$***

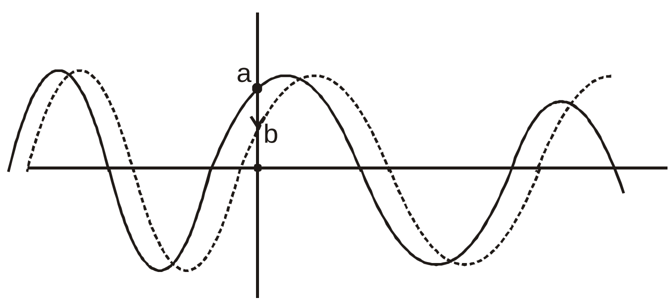
**Sol.** Let $y = A\sin(\omega t - kx + \phi)$

at $t = 0$ and $x = 0$

$$\frac{A}{\sqrt{2}} = A\sin\phi \Rightarrow \sin\phi = \frac{1}{\sqrt{2}}$$

$$\phi = \frac{\pi}{4}, \ \frac{3\pi}{4}$$

To choose the correct phase angle $\phi$ we displaced to wave. Slightly in +ve x direction such that




In above figure Paticle at a is move downward towards point b i.e. particle at x = 0 & y = $\frac{A}{\sqrt{2}}$ have negative velocity which gives

$$\frac{\partial y}{\partial t} = \omega A\cos(\omega - kx + \phi) \text{ at}$$

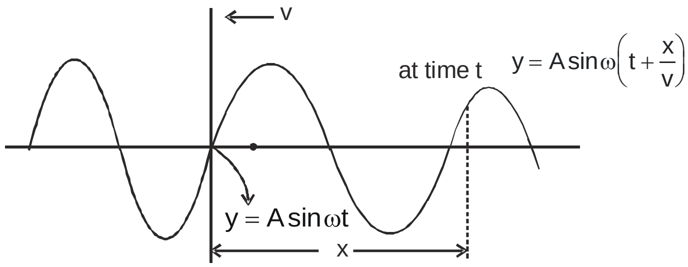
$t = 0, \ x = 0$

is $\cos\phi = -$ ve (from figure) ...(2)

from above discussion $3\pi/4$ gives $\sin\phi$ + ve and $\cos\phi$ negative i.e.

$$\phi = \frac{3\pi}{4}$$

**Note :** Equation of wave which is moving –ve x direction.




$y = A \sin (\omega t + kx + \phi)$

***Ex.2 If ($\omega$t) & (kx) terms have same sign then the wave move toward –ve x direction and vice versa and with diffierent initial phase.***

$\left.\begin{array}{l} y = A \sin (\omega t - kx) \\ y = A \sin (-kx + \omega t) \end{array}\right\}$ Wave move toward +ve x direction

$\left.\begin{array}{l} y = A \sin (-kx - \omega t) \\ \quad = A \sin (kx + \omega t + \pi) \\ y = A \sin (kx + \omega t) \end{array}\right\}$ Wave move toward –ve x direction.

## 2. PARTICLE VELOCITY AND ACCELERATION :

Particle velocity at a given position x = x is obtained by differentiating wave function with respect to time "t". We need to differentiate equation by treating "x" as constant. The partial differentiation yields particle velocity as :

$$v_p = \frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t} A\sin(kx - \omega t) = -\omega A \cos (kx - \omega t)$$

We can use the property of cosine function to find the maximum velocity. We obtain maximum speed when cosine function evaluates to "–1" :

$$\Rightarrow v_{pmax} = \omega A$$

The acceleration of the particle is obtained by differentiating expression of velocity partially with respect to time :

$$\Rightarrow a_p = \frac{\partial}{\partial t} v_p = \frac{\partial}{\partial t}\{-\omega A\cos(kx - \omega t)\} = -\omega^2 A \sin (kx - \omega t) = -\omega^2 y$$

Again the maximum value of the acceleration can be obtained using property of sine function :

$$\Rightarrow a_{pmax} = \omega^2 A$$

## 3. DIFFERENT FORMS OF WAVE FUNCTION :

Different forms give rise to bit of confusion about the form of wave function. The forms used for describing wave are :

$$y (x, t) = A \sin (kx - \omega t)$$
$$y(x, t) = A \sin (\omega t - kx + \pi)$$

Which of the two forms is correct ? In fact, both are correct so long we are in a position to accurately interpret the equation. Starting with the first equation and using trigonometric identity :

We have,

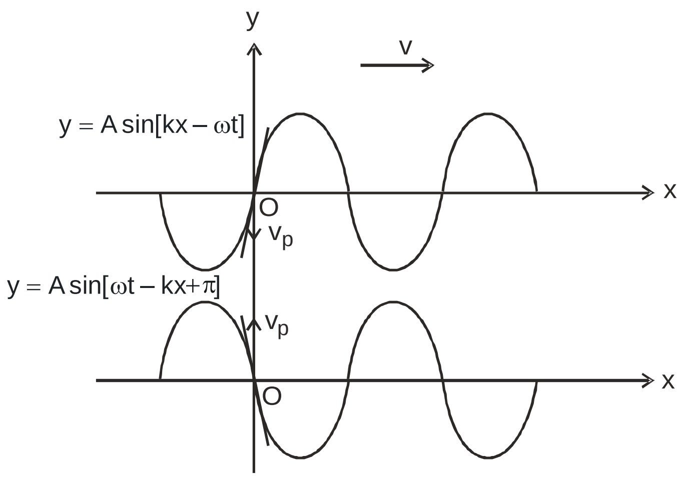
$$\Rightarrow \ A \sin (kx - \omega t) = A \sin (\pi - kx + \omega t) = A \sin (\omega t - kx + \pi)$$

Thus we see that two forms represent waves along at the same speed $\left(v = \frac{\omega}{k}\right)$. They differ, however, in phase. There is phase difference of "$\pi$". This has implication on the waveform and the manner particle oscillates at any given time instant and position. Let us consider two waveforms at x = 0, t = 0. The slopes of the waveforms are :

$$\frac{\partial}{\partial x} y(x,t) = kA\cos(kx - \omega t) = kA = \text{a positive number}$$

and $$\frac{\partial}{\partial x} y(x,t) = -kA\cos(\omega t - kx) = -kA = \text{a negative number}$$

**Forms of wave functions**




Exchange of terms in the argument of sine function results in a phase difference of $\pi$.

In the first case, the slope is positive and hence particle velocity is negative. It means particle is moving from reference origin or mean position to negative extreme position. In the second case, the slope is negative and hence particle velocity is positive. It means particle is moving from positive extreme position to reference origin or mean position. Thus two forms represent waves which differ in direction in which particle is moving at a given position.

Once we select the appropriate wave form, we can write wave equation in other forms as given here :

$$y(x, t) = A \sin (kx - \omega t) = A \sin k\left(x - \frac{\omega t}{k}\right) = A \sin \frac{2\pi}{\lambda}(x - vt)$$

Further, substituting for "k" and "$\omega$" in wave equation, we have :

$$y(x, t) = A \sin \left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

If we want to represent waveform moving in negative "x" direction, then we need to replace "t" by "–t".

## 4. THE LINEAR WAVE EQUATION :

By using wave function $y = A \sin(\omega t - kx + \phi)$, we can describe the motion of any point on the string. Any point on the string moves only vertically, and so its x coordinate remains constant. The transverse velocity $v_y$ of the point and its transverse acceleration $a_y$ are therefore.

$$v_y = \left[\frac{dy}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial y}{\partial t} = \omega A \cos(\omega t - kx + \phi) \quad ...(1)$$

$$a_y = \left[\frac{dv_y}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(\omega t - kx + \phi) ..(2)$$

and hence

$v_{y.\ max} = \omega A$

$a_{y.max} = \omega^2 A$

The transverse velocity and transverse acceleration of any point on the string do not reach their maximum value simultaneously. Infact, the transverse velocity reaches its maximum value ($\omega A$) when the displacement y = 0, whereas the transverse acceleration reaches its maximum magnitudes ($\omega^2 A$) when $y = \pm A$

further

$$\left[\frac{dy}{dx}\right]_{t=\text{constant}} \Rightarrow \frac{\partial y}{\partial x} = -kA\cos(wt - kx + \phi) \quad ...(3)$$

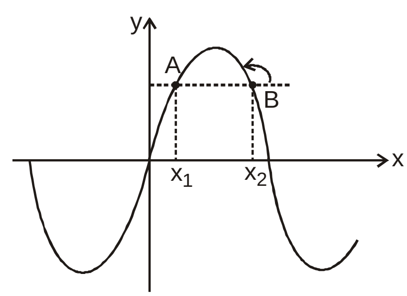
$$= \frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(\omega t - kx + \phi) \quad ...(4)$$

From (1) and (3) $\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x}$

$\Rightarrow$ $v_p = -v_w \times$ slope

i.e. if the slope at any point is negative, particle velocity and vice-versa, for a wave moving along positive x axis i.e. $v_w$ is positive.

For example, consider two points A and B on the y-curve for a wave, as shown. The wave is moving along positive x-axis.




Slope at A is positive therefore at the given moment, its velocity is negative. That means it is coming downward. Reverse is the situation for particle at point B.

Now using equation (2) and (4)

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This is known as the linear wave equation or diffential equation representation of the travelling wave model. We have developed the linear wave equation from a sinusoidal mechanical wave travelling through a medium. But it is much more general. The linear wave equation successfully describes waves on strings, sound waves and also electromagnetic waves.

Thus, the above equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad ...(i)$$

The general solution of this equation is of the form

$$y(x, t) = f(ax \pm bt) \qquad \text{...(ii)}$$

Thus, any function of x and t which satisfies Eq. (i) or which can be written as Eq. (ii) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v = \frac{\text{coefficient of t}}{\text{coefficient of x}} = \frac{b}{a}$$

Thus plus (+) sign between ax and bt implies that the wave is travelling along negative x-direction and minus (–) sign shows that it is travelling along positive x-direction.

***Ex.3 Verify that wave function***

$$\mathbf{y = \frac{2}{(x-3t)^2+1}}$$

***is a solution to the linear wave equation x and y are in cm.***

**Sol.** By taking partial derivatives of this function w.r.t x and to t.

$$\frac{\partial^2 y}{\partial x^2} = \frac{12(x-3t)^2-4}{[(x-3t)^2+1]^3}, \text{ and}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{108(x-3t)^2-36}{[(x-3t)^2+1]^3}$$

or $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{9}\frac{\partial^2 x}{\partial t^2}$$

Comparing with linear wave equation, we see that the wave function is a solution to the linear wave equation if the speed at which the pulse moves is 3 cm/s. It is apparent from wave function therefore it is a solution to the linear wave equation.

***Ex 4. A wave pulse is travelling on a string at 2 m/s. displacement y of the particle at x = 0 at any time t is given by***

$$\mathbf{y = \frac{2}{t^2+1}}$$

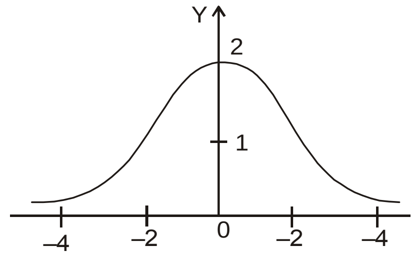
***Find***
***(i) Expression of the function y = (x, t) i.e., displacement of a particle position x and time t.***
***(ii) Shape of the pulse at t = 0 and t = 1s.***

**Sol.** (i) By replacing t by $\left(t - \frac{x}{v}\right)$, we can get the desired wave function i.e.,

$$y = \frac{2}{\left(t-\frac{x}{2}\right)^2+1}$$

(ii) We can use wave function at a particular instant, say t = 0, to find shape of the wave pulse using different values of x.

at t = 0 $y = \dfrac{2}{\dfrac{x^2}{4}+1}$



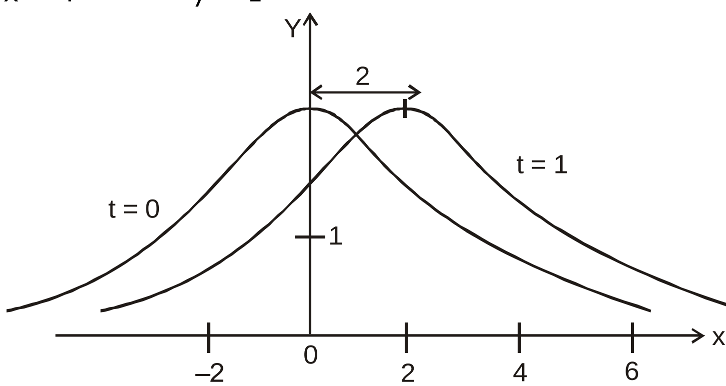

at x = 0 y = 2
x = 2 y = 1
x = – 2 y = 1
x = 4 y = 0.4
x = −4 y = 0.4

Using these value, shape is drawn.

Similarly for t = 1s, shape can drawn. What do you conclude about direction of motion of the wave from the graphs? Also check how much the pulse has move in 1s time interval. This is equal to wave speed. Here is the procedure.

$$y = \frac{2}{\left(1-\frac{x}{2}\right)^2+1} \qquad \text{at } t = 1s$$

at x = 2 y = 2(maximum value)
at x = 0 y = 1
at x = 4 y = 1



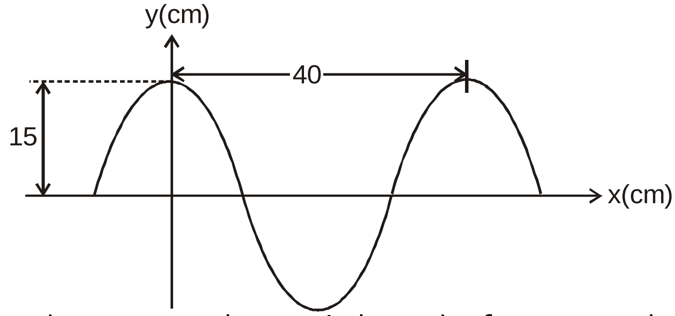

The pulse has moved to the right by 2 units in 1 s interval.

Also as $t - \dfrac{x}{2}$ = constt.

Differentiating w.r.t time

$1 - \dfrac{1}{2} \cdot \dfrac{dx}{dt} = 0 \qquad \Rightarrow \quad \dfrac{dx}{dt} = 2$

**Ex.5** ***A sinusoidal wave travelling in the positive x direction has an amplitude of 15 cm, wavelength 40 cm and frequency 8 Hz. The vertical displacement of the medium at t =0 and x = 0 is also 15 cm, as shown***




(a) Find the angular wave number, period angular frquency and speed of the wave.
(b) Determine the phase constant $\phi$, and write a general expression for the wave function.

**Sol.** (a) $k = \frac{2\pi}{\lambda} = \frac{2\pi rad}{40cm} = \frac{\pi}{20} rad/cm$

$T = \frac{1}{f} = \frac{1}{8} s$ $\omega = 2\pi f = 16\ s^{-1}$

$v = f\lambda = 320$ cm/s

(b) It is given that A = 15 cm

and also $y = 15$ cm at $x = 0$ and $t = 0$

then using $y = A \sin(\omega t - kx + \phi)$

$15 = 15 \sin\phi \Rightarrow \sin\phi = 1$

Therefore, the wave function is

$y = A \sin(\omega t - kx + \frac{\pi}{2})$ $= (15\,cm) \sin\left[(16\pi s^{-})t - \left(\frac{\pi}{20}\frac{rad}{cm}\right).x + \frac{\pi}{2}\right]$

## 5. SPEED OF A TRANSVERSE WAVE ON A STRING

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse in stationary and the string moves with a speed v to the left. Figure shows a small segment of the string of length $\Delta l$. This segment forms part of a circular arc of radius R. Instantaneosly the segment is moving with speed v in a circular path, so it has centripetal acceleration $v^2/R$. The forces acting on the segment are the tension T at each end. The horizontal component of these forces are equal and opposite and thus cancel. The vertical component of these forces point radially inward towards the centre of the circular. arc. These radial forces provide centripetal acceleration. Let the angle substended by the segment at centre be $2\theta$. The net radial force acting on the segment is

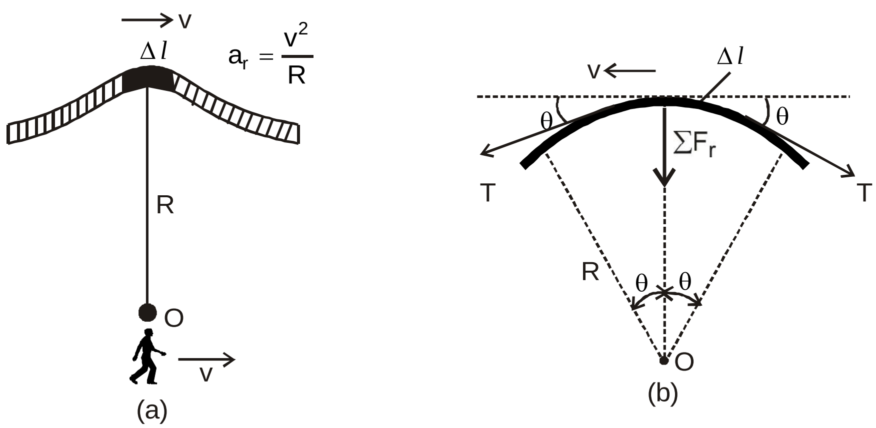



Fig. (a) To obtain the speed v of a wave on a stretched string. It is convenient to describe the motion of a small segment of the string in a moving frame of reference.

Fig. (b) In the moving frame of reference, the small segment of length $\Delta l$ moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension force cancel.

$$\sum F_r = 2T\sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta \approx \theta$ for small $\theta$.

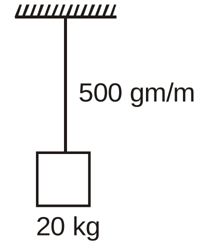
If $\mu$ is the mass per unit length of the string, the mass of the segment of length $\Delta l$ is

$m = \mu\Delta l = 2\mu R\theta$ (as $\Delta l = 2R\theta$)

From Newton's second law $\Sigma F_r = ma = \frac{mv^2}{R}$

or $2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right)$ $\therefore$ $v = \sqrt{\frac{T}{\mu}}$

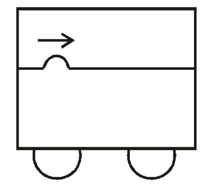
**Ex.6** ***Find speed of the wave generated in the string as in the situation shown. Assume that the tension in not affected by the mass of the cord.***




**Sol.** T = 20 × 10 = 200 N

$$v = \sqrt{\frac{200}{0.5}} = 20 m/s$$

**Ex.7** ***A taut string having tension 100 N and linear mass density 0.25 kg/m is used inside a cart to generate a wave pulse starting at the left end, as shown. What should be the velocity of the cart so that pulse remains stationary w.r.t ground.***



**Sol.** Velocity of pulse = $\sqrt{\frac{T}{\mu}} = 20 m/s$

Now $\vec{v}_{PG} = \vec{v}_{PC} + \vec{v}_{CG}$

$0 = 20\ \hat{i} + \vec{v}_{CG}$

$\vec{V}_{CG} = -20\hat{i} m/s$

**Ex.8** ***One end of 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to a fixed support. A cord attached to the other and passes over a pulley and supports an object with a mass of 5.0 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end (g = 9.8 m/s²)***

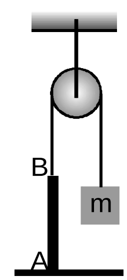
**Sol.** Tension in the rubber tube AB, T = mg

T = (5.0) (9.8) = 49 N

or

Mass per unit length of rubber tube,

$$\mu = \frac{0.9}{12} = 0.075 \text{ kg/m}$$




∴ Speed of wave on the tube,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{49}{0.075}} = 25.56 m/s$$

∴ The required time is,

$$t = \frac{AB}{v} = \frac{12}{25.56} = 0.47 s$$

**Ex.9** ***A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling***

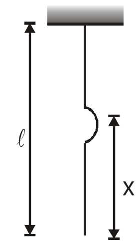
***(a) Find the speed of transverse wave in the rope at a point 0.5 m distant from the lower end.***

***(b) Calculate the time taken by a transverse wave to travel the full length of the rope.***

**Sol.** (a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So, if m is the mass of string of length $l$, the mass of length x of the string will be, $\left(\frac{m}{l}\right)x$.

$\therefore$ $$T = \left(\frac{m}{l}\right)xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$$

$\therefore$ $$\frac{T}{\mu} = xg$$




or $$v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \qquad \text{...(i)}$$

At x = 0.5 m, $v = \sqrt{0.5 \times 9.8}$ = 2.21 m/s

(b) From Eq. (i) we see that velocity of the wave is different at different points. So, if at point x the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$\therefore$ $$\int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

or $$t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}} \qquad = \quad 1.0 \text{ s } \textbf{Ans.}$$

## 6. ENERGY CALCULATION IN WAVES :

### (a) Kinetic energy per unit length

The velocity of string element in transverse direction is greatest at mean position and zero at the extreme positions of waveform. We can find expression of transverse velocity by differentiating displacement with respect to time. Now, the y-displacement is given by :

$$y = A \sin (kx - \omega t)$$

Differentiating partially with respect to time, the expression of particle velocity is :

$$v_p = \frac{\partial y}{\partial t} = -\,\omega A \cos (kx - \omega t)$$

In order to calculate kinetic energy, we consider a small string element of length "dx" having mass per unit length "$\mu$". The kinetic energy of the element is given by :

$$dK = \frac{1}{2}dmv_p^2 = \frac{1}{2}\mu dx\omega^2 A^2 \cos^2(kx - \omega t)$$

This is the kinetic energy associated with the element in motion. Since it involves squared of cosine function, its value is greatest for a phase of zero (mean position) and zero for a phase of $\frac{\pi}{2}$ (maximum displacement).

Now, we get kinetic energy per unit length, "$K_L$", by dividing this expression with the length of small string considered :

$$K_L = \frac{dK}{dx} = \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx - \omega t)$$

- **Rate of transmission of kinetic energy**

The rate, at which kinetic energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, "dt" :

$$\frac{dK}{dt} = \frac{1}{2}\mu\frac{dx}{dt}\omega^2A^2\cos^2(kx-\omega t)$$

But, wave or phase speed, v, is time rate of position i.e. $\frac{dx}{dt}$. Hence,

$$\frac{dK}{dt} = \frac{1}{2}\mu v\omega^2A^2\cos^2(kx-\omega t)$$

Here kinetic energy is a periodic function. We can obtain average rate of transmission of kinetic energy by integrating the expression for integral wavelengths. Since only $\cos^2(kx - \omega t)$ is the varying entity, we need to find average of this quantity only. Its integration over intergal wavelengths give a value of "$\frac{1}{2}$". Hence, average rate of transmission of kinetic energy is :

$$\frac{dK}{dt}\Big|_{avg} = \frac{1}{2}\times\frac{1}{2}\mu v\omega^2A^2 = \frac{1}{4}\mu v\omega^2A^2$$

**(b) Elastic potential energy**

The elastic potential energy of the string element results as string element is stretched during its oscillation. The extension or stretching is maximum at mean position. We can see in the figure that the length of string element of equal x-length "dx" is greater at mean position than at the extreme. As a matter of fact, the elongation depends on the slope of the curve. Greater the slope, greater is the elongation. The string has the least length when slope is zero. For illustration purpose, the curve is purposely drawn in such a manner that the elongation of string element at mean position is highlighted.

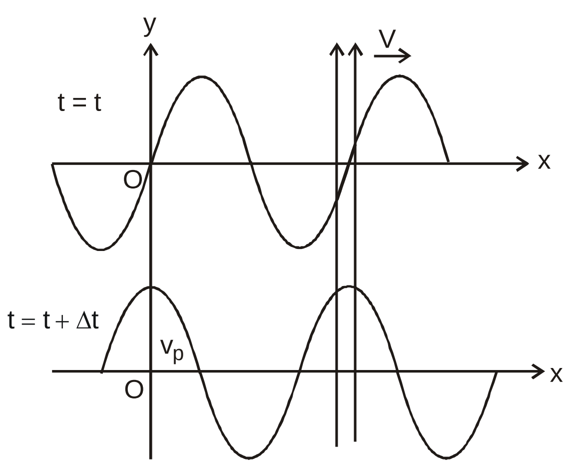
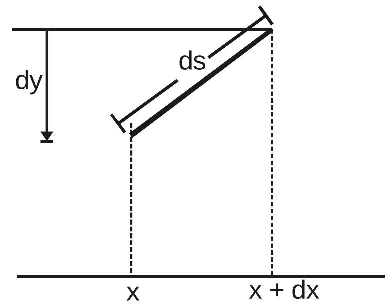



**fig :** The string element stretched most at equilibrium position

Greater extension of string element corresponds to greater elastic energy. As such, it is greatest at mean position and zero at extreme position. This deduction in contrary to the case of SHM in which potential energy is greatest at extreme position and zero at mean position.

- **Potential energy per unit length**

When the string segment is stretched from the length dx to the length ds an amount of work = T (ds – dx) is done. This is equal to the potential energy stored in the stretched string segment. So the potential energy in this case is :




$$U = T\,(ds - dx)$$

Now $$ds = \sqrt{(dx^2 + dy^2)}$$

$$= dx\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

from the binomial expansion

so $$ds \approx dx + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$U = T\,(ds - dx) \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

or the potential energy density

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \qquad \text{...(i)}$$

$$\frac{dy}{dx} = kA\cos\,(kx - \omega t)$$

and $T = v^2\,\mu$

Put above value in equation (i) then we get

$$\frac{dU}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t)$$

- **Rate of transmission of elastic potential energy**

The rate, at which elastic potential energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, "dt". This expression is same as that for kinetic enegy.

$$\frac{dU}{dt} = \frac{1}{2}\mu v\omega^2A^2\cos^2(kx-\omega t)$$

and average rate of transmission of elastic potential energy is :

$$\frac{dU}{dt}\Big|_{avg} = \frac{1}{2}\times\frac{1}{2}\mu v\omega^2A^2 = \frac{1}{4}\mu v\omega^2A^2$$

**(c) Mechanical energy per unit length**

Since the expression elastic potential energy is same as that of kinetic energy, we get mechanical energy expression by multiplying expression of kinetic energy by "2". The mechanical energy associated with small string element, "dx", is :

$$dE = 2xdK = 2x\frac{1}{2}dmv_p^2 = \mu dx\omega^2A^2\cos^2\,(kx - \omega t)$$

Similarly, the mechanical energy per unit length is :

$$E_L = \frac{dE}{dx} = 2x\frac{1}{2}\mu\omega^2A^2\cos^2(kx-\omega t) = \mu\omega^2\,A^2\cos^2\,(kx - \omega t)$$

**(d) Average power transmitted**

The average power transmitted by wave is equal to time rate of transmission of mechanical energy over integral wavelengths. It is equal to :

$$P_{avg} = \frac{dE}{dt}\Big|_{avg} = 2 \times \frac{1}{4}\mu v \omega^2 A^2 = \frac{1}{2}\mu v \omega^2 A^2$$

If mass of the string is given in terms of mass per unit volume, "$\rho$", then we make appropriate change in the derivation. We exchange "$\mu$" by "$\rho s$" where "s" is the cross section of the string :

$$P_{avg} = \frac{1}{2}\rho s v \omega^2 A^2$$

**(e) Energy density**

Since there is no loss of energy involved, it is expected that energy per unit length is uniform throughout the string. As much energy enters that much energy goes out for a given length of string. This average value along unit length of the string length is equal to the average rate at which energy is being transferred.

The average mechanical energy per unit length is equal to integration of expression over integral wavelength

$$E_L|_{avg} = 2x\frac{1}{4}\ \mu v \omega^2\ A^2 = \frac{1}{2}\mu v \omega^2 A^2$$

We have derived this expression for harmonic wave along a string. The concept, however, can be extended to two or three dimensional transverse waves. In the case of three dimensional transverse waves, we consider small volumetric element. We, then, use density, $\rho$, in place of mass per unit length, $\mu$. The corresponding average energy per unit volume is referred as energy density (u) :

$$u = \frac{1}{2}\rho v w^2 A^2$$

**(f) Intensity**

Intensity of wave (I) is defined as power transmitted per unit cross section area of the medium :

$$I = \rho s v \omega^2 \frac{A^2}{2s} = \frac{1}{2}\rho v w^2 A^2$$

Intensity of wave (I) is a very useful concept for three dimensional waves radiating in all direction from the source. This quantity is usually referred in the context of light waves, which is transverse harmonic wave in three dimensions. Intensity is defined as the power transmitted per unit cross sectional area. Since light spreads uniformly all around, intensity is equal to power transmitted, divided by spherical surface drawn at that point with source at its center.

☛ **Phase difference between two particles in the same wave :**

The general expression for a sinusoidal wave travelling in the positive x direction is

$y(x, t) = A \sin(\omega t - kx)$

Eq$^n$ of Particle at $x_1$ is given by $y_1 = A \sin(\omega t - kx_1)$

Eq$^n$ of particle which is at $x_2$ from the origin

$y_2 = A\sin(\omega t - kx_2)$

Phase difference between particles is $k(x_2 - x_1) = \Delta\phi$

$$K\Delta x = \Delta\phi \Rightarrow \quad \Delta x \Rightarrow \frac{\Delta\phi}{k}$$

## 7. PRINCIPLE OF SUPERPOSITION :

This principle defines the displacement of a medium particle when it is oscillating under the influence of two or more than two waves. The principle of superposition is stated as :

"When two or more waves superpose on a medium particle than the resultant displacement of that medium particle is given by the vector sum of the individual displacements produced by the component waves at that medium particle independently."

Let $\vec{y}_1, \vec{y}_2, \ldots\ldots \vec{y}_N$ are the displacements produced by N independent waves at a medium particle in absence of others then the displacemnt of that medium, when all the waves are superposed at that point, is given as

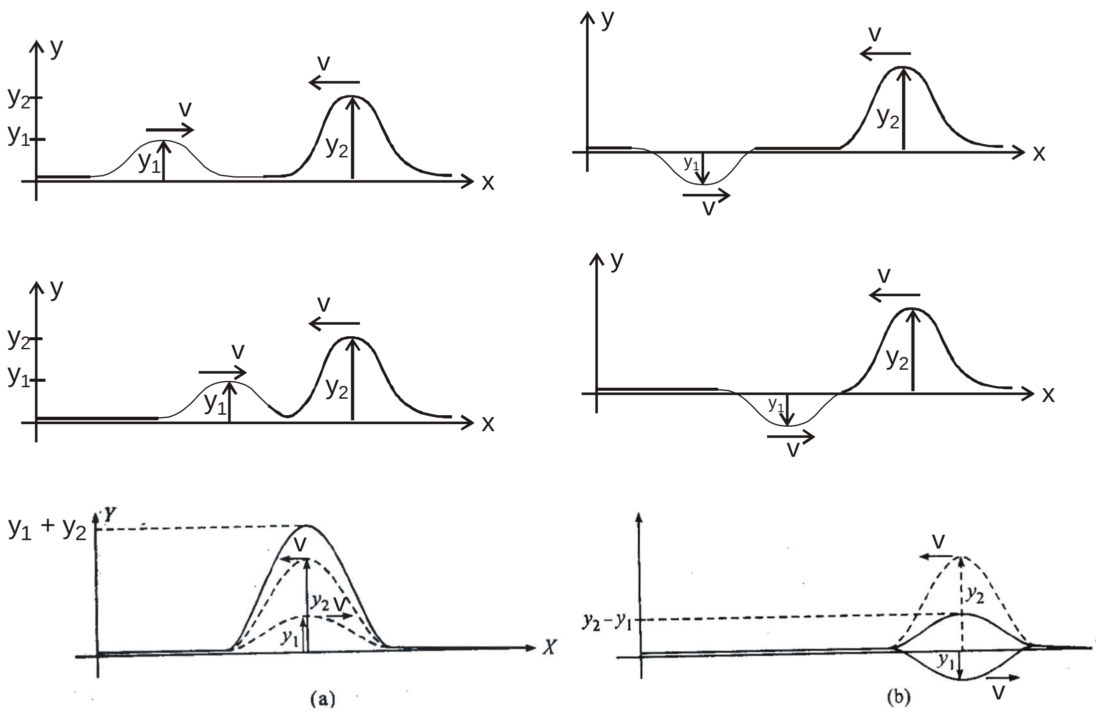
$$\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \ldots\ldots + \vec{y}_N$$

If all the waves are producing oscillations at that point are collinear then the displacement of the medium particle where superposition is taking place can be simply given by the algebric sum of the individual displacement. Thus we have

$$y = y_1 + y_2 + \ldots\ldots\ldots\ldots + y_N$$

The above equation is valid only if all individual displacements $y_1$, $y_2$ ........... $y_N$ are along same straight line.

A simple example of superposition can be understood by figure shown. Suppose two wave pulses are travelling simultaneously in opposite directions as shown. When they overlap each other the displacement of particle on string is the algebric sum of the two displacement as the displacements of the two pulses are in same direction. Figure shown (b) also shows the similar situation when the wave pulses are in opposite side.




### (a) Applications of Principle of Superposition of Waves

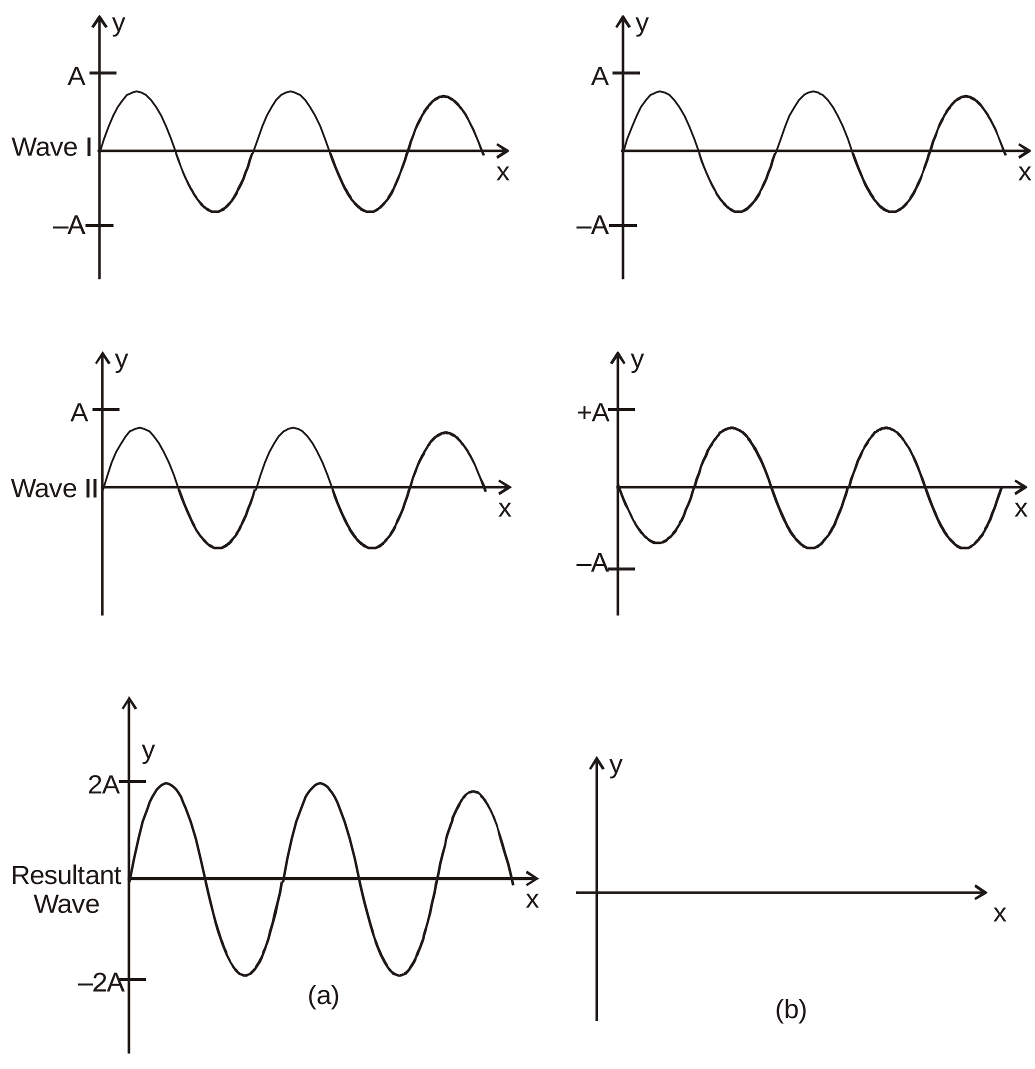
There are several different phenomenon which takes place during superposition of two or more wave depending on the wave characteristics which are being superposed. We'll discuss some standard phenomenons, and these are :

(1) Interference of Wave
(2) Stationary Waves
(3) Beats
(4) Lissajou's Figures (Not discussed here in detail.)

Lets discuss these in detail.

**(b) Interference of Waves**

Suppose two sinusoidal waves of same wavelength and amplitude travel in same direction along the same straight line (may be on a stretched string) then superposition principle can be used to define the resultant displacement of every medium particle. The resultant wave in the medium depends on the extent to which the waves are in phase with respect to each other, that is, how much one wave form is shifted from the other waveform. If the two waves are exactly in same phase, that is the shape of one wave exactly fits on to the other wave then they combine to double the displacement of every medium particle as shown in figure (a). This phenomenon we call as constructive interference. If the superposing waves are exactly out of phase or in opposite phase then they combine to cancel all the displacements at every medium particle and medium remains in the form of a straight line as shown in figure (b)



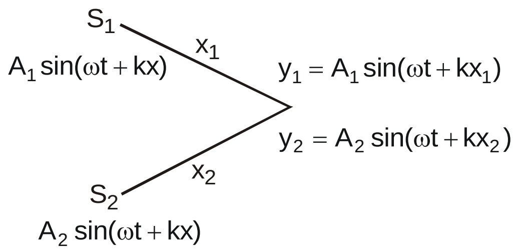
(a)
(b)

This phenomenon we call destructive interference. Thus we can state that when waves meet, they interfere constructively if they meet in same phase and destructively if they meet in opposite phase. In either case the wave patterns do not shift relative to each other as they propagates. Such superposing waves which have same form and wavelength and have a fixed phase relation to each other, are called coherent waves. Sources of coherent waves are called coherent source. Two indepedent sources can never be coherent in nature due to practical limitations of manufacturing process. Generally all coherent sources are made either by spliting of the wave forms of a single source or the different sources are fed by a single main energy source.

In simple words interference is the phenomenon of superposition of two coherent waves travelling in same direction.

We've discussed that the resultant displacement of a medium particle when two coherent waves interfere at that point, as sum or difference of the individual displacements by the two waves if they are in same phase (phase difference = 0, $2\pi$, .....) or opposite phase (phase difference = $\pi$, $3\pi$,.....) respectively. But the two waves can also meet at a medium particle with phase difference other then 0 or $2\pi$, say if phase difference $\phi$ is such that $0 < \phi < 2\pi$, then how is the displacement of the point of superposition given ? Now we discuss the interference of waves in details analytically.

## (c) Analytical Treatment of Interference of Waves




Interference implies super position of waves. Whenever two or more than two waves superimpose each other they give sum of their individual diplacement.

Let the two waves coming from sources $S_1$ & $S_2$ be

$y_1 = A_1 \sin(\omega t + kx_1)$

$y_2 = A_2 \sin(\omega t + kx_2)$ respectively.

Due to superposition

$y_{net} = y_1 + y_2$

$y_{net} = A_1 \sin(\omega t + kx_1) + A_2 \sin(\omega t + kx_2)$

Phase difference between $y_1$ & $y_2 = k(x_2 - x_1)$

i.e., $\Delta\phi = k(x_2 - x_1)$

As $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$ (where $\Delta x$ = path difference & $\Delta\phi$ = phase difference)

$$A_{net} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

$$\Rightarrow \quad A_{net}{}^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi$$

$\therefore \quad I_{net} = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi$ (as $I \propto A^2$)

When the two displacements are in phase, then the resultant amplitude will be sum of the two amplitude & $I_{net}$ will be maximum, this is known of constructive interference.

For $I_{net}$ to be maximum

$\cos\phi = 1 \Rightarrow \phi = 2n\pi$ where n = {0,1,2,3,4,5..........}

$\frac{2\pi}{\lambda}\Delta x = 2n\pi \Rightarrow \mathbf{\Delta x = n\lambda}$

For constructive interference

$I_{net} = (\sqrt{I_1} + \sqrt{I_2})^2$

When $I_1 = I_2 = I$

$I_{net} = 4I$

$A_{net} = A_1 + A_2$

When superposing waves are in opposite phase, the resultant amplitude is the difference of two amplitudes & $I_{net}$ is minimum; this is known as destructive interference.

For $I_{net}$ to be minimum,

$\cos \Delta\phi = -1$

$\Delta\phi = (2n + 1)\pi$ where n = {0,1,2,3,4,5..........}

$\frac{2\pi}{\lambda}\Delta x = (2n+1)\pi \Rightarrow$ $\mathbf{\Delta x = (2n+1)\frac{\lambda}{2}}$

For destructive interfence

$I_{net} = (\sqrt{I_1} - \sqrt{I_2})^2$

If $I_1 = I_2$

$I_{net} = 0$

$A_{net} = A_1 - A_2$

Ratio of $I_{max}$ & $I_{min}$ = $\frac{(\sqrt{I_1}+\sqrt{I_2})^2}{(\sqrt{I_1}-\sqrt{I_2})^2}$

Generally,

$I_{net} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$

If $I_1 = I_2 = I$

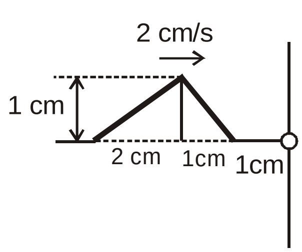
$I_{net} = 2I + 2I\cos\phi$

$I_{net} = 2I(1 + \cos\phi) = 4I\cos^2\frac{\Delta\phi}{2}$

**Ex.10** ***Wave from two source, each of same frequency and travelling in same direction, but with intensity in the ratio 4 : 1 interfere. Find ratio of maximum to minimum intensity.***

**Sol.** $\frac{I_{max}}{I_{min}} = \left(\frac{\sqrt{I_1}+\sqrt{I_2}}{\sqrt{I_1}-\sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}}+1}{\sqrt{\frac{I_1}{I_2}}-1}\right)^2 = \left(\frac{2+1}{2-1}\right)^2 = 9 : 1$

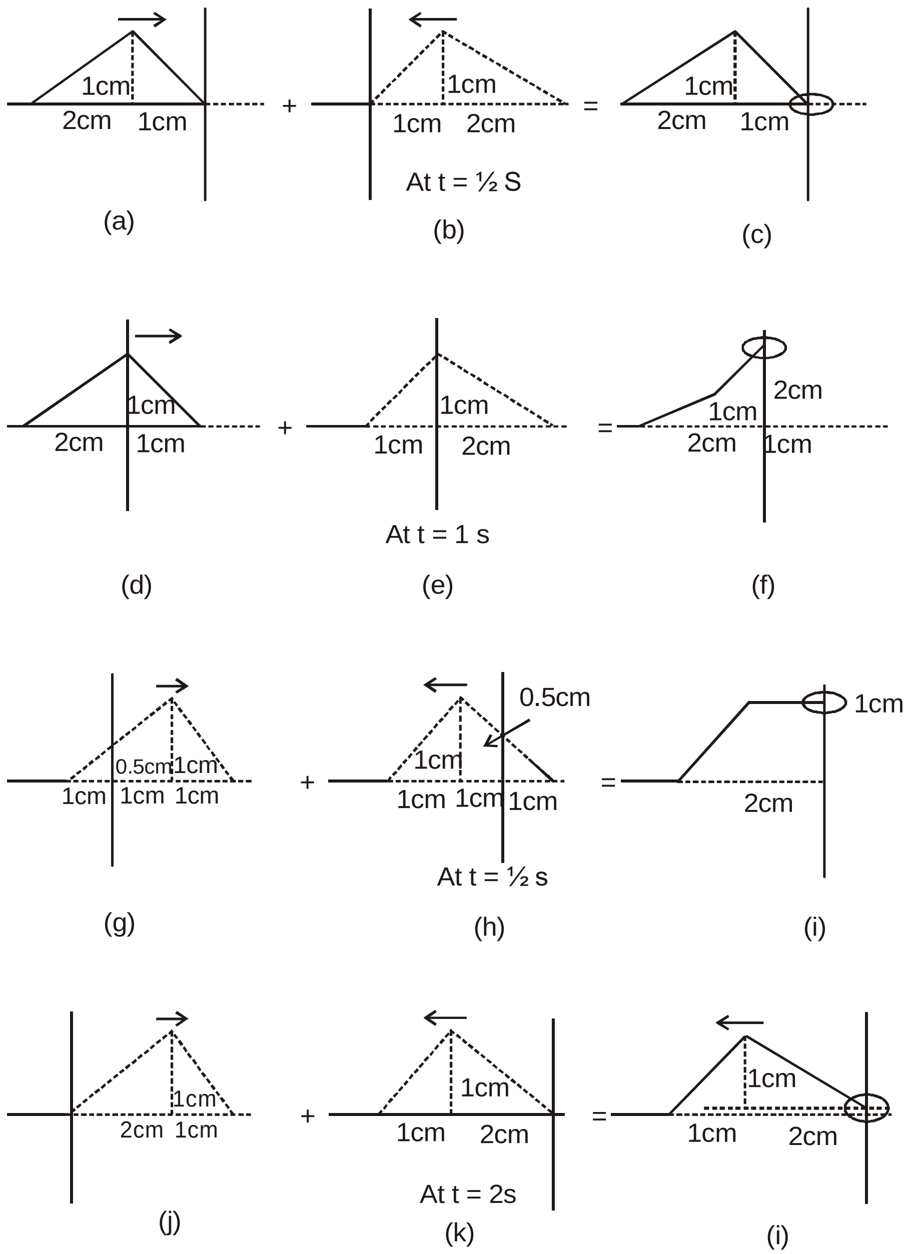
***Ex.11 A triangular pulse moving at 2 cm/s on a rope approaches an end at which it is free to slide on a vertical pole.***




(a) Draw the pulse at $\frac{1}{2}$ s interval until it is completely reflected.

(b) What is the particle speed on the trailing edge at the instant depicted ?

**Sol.** (a) Reflection of a pulse from a free boundary is really the superposition of two identical waves travelling in opposite direction. This can be shown as under.



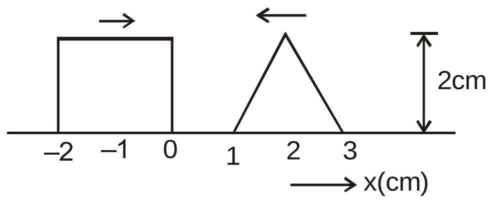

In every $\frac{1}{2}$s, each pulse (one real moving towards right and one imaginary moving towards left travels a distance of 1 cm, as the wave speed is 2 cm/s.)

(b) Particle speed, $v_p = |-v \text{ (slope)}|$

Here, v = wave speed = 2 cm/s and slope = $\frac{1}{2}$

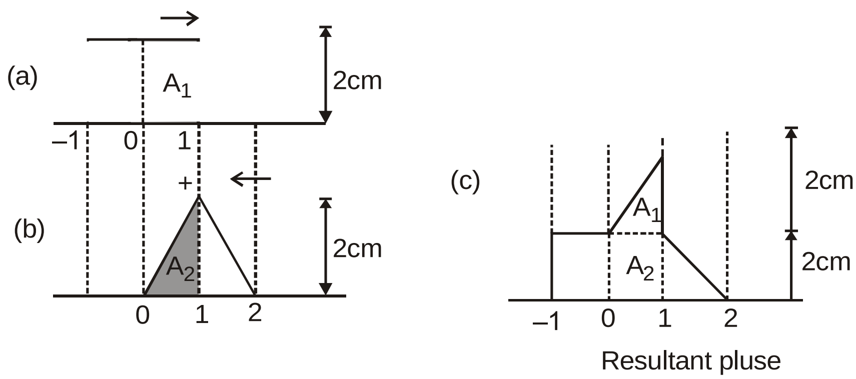
$\therefore$ Particle speed = 1 cm/s **Ans.**

***Ex.12 Figure shows a rectanglar pulse and triangular pulse approaching each other. The pulse speed is 0.5 cm/s. Sketch the resultant pulse at t = 2 s***




**Sol.** In 2 s each pulse will travel a distance of 1 cm.

The two pulses overlap between 0 and 1 cm as shown in figure. So, $A_1$ and $A_2$ can be added as shown in figure (c).

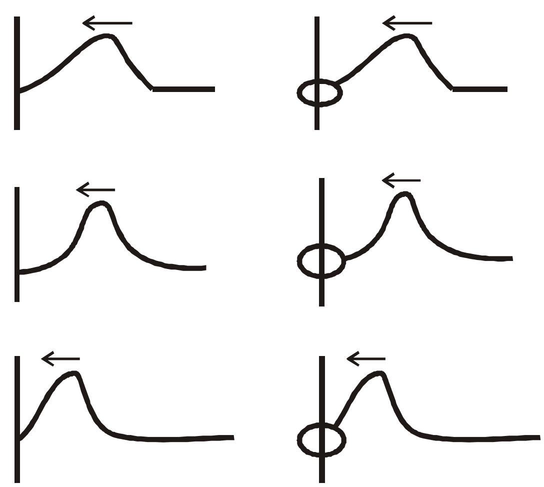


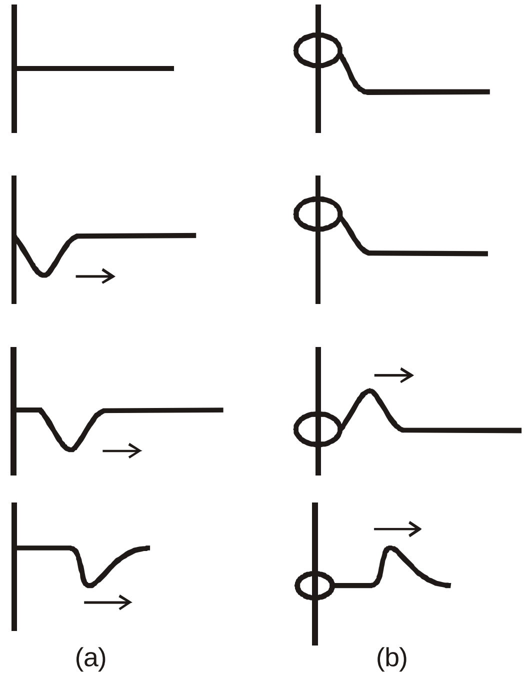
Resultant pluse at t = 2s

**8. REFLECTION AND TRANSMISSION IN WAVES :**

**1.** When a pulse travelling along a string reaches the end, it is reflected. If the end is fixed as shown in figure (a), the pulse returns inverted. This is bacause as the leading edge reaches the wall, the string pulls up the wall. According to Newton's third law, the wall will exert an equal and opposite force on the string as all instants. This force is therefore, directed first down and then up. It produces a pulse that is inverted but otherwise identical to the original.

The motion of free end can be studied by letting a ring at the end of string sliding smoothly on the rod. The ring and rod maintain the tension but exert no transverse force.

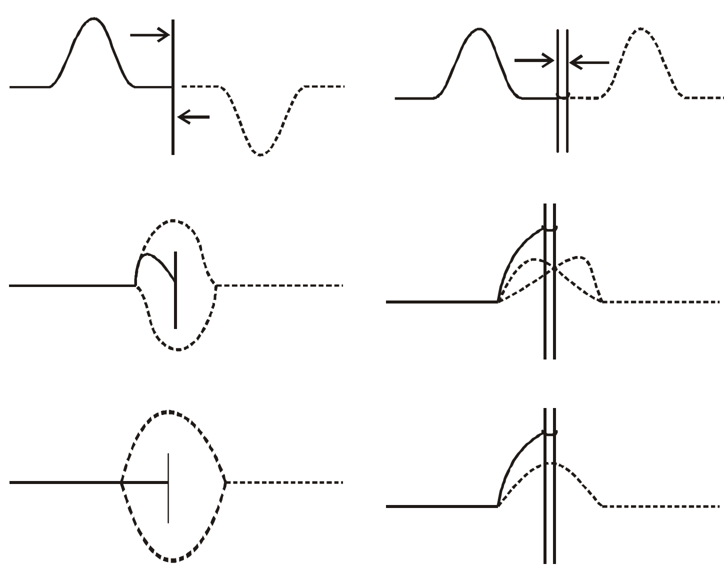





Reflection of wave pulse (a) at a fixed end of a string and (b) at a free end. Time increases from top to bottom in each figure.

When a wave arrives at this free end, the ring slides the rod. The ring reaches a maximum displacement. At this position the ring and string come momentarily to rest as in the fourth drawing from the top in figure (b). But the string is stretched in this position, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced, but now the direction of the displacement is the same as for the initial pulse.

**2.** The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite directions. The net displacement at any point is given by the principle of superposition.



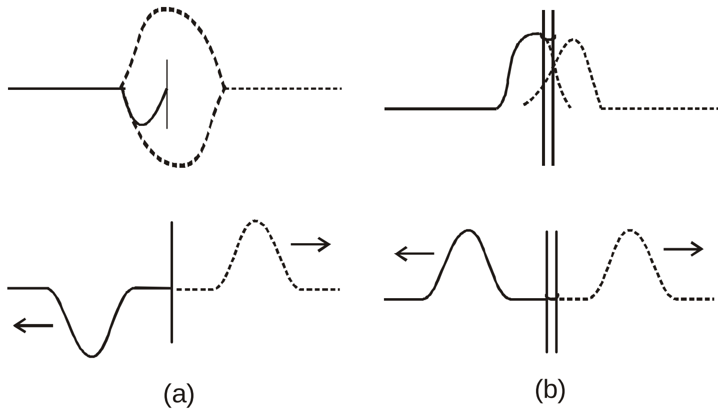



**Fig (a) :** shows two pulses with the same shape, one inverted with respect to the other, travelling in opposite directions. Because these two pulses have the same shape the net displacement of the point where the string is attached to the wall is zero at all times.

**Fig (b) :** shows two pulses with the same shape, travelling in oppoiste directions but not inverted relative to each other. Note that at one instant, the displacement of the free end is double the pulse height.

## 9. REFLECTION AND TRANSMISSION BETWEEN TWO STRING :

Here we are dealing with the case where the end point is neither completely fixed nor completely free to move As we consider an example where a light string is attached to a heavy string as shown is figure a.

If a wave pulse is produced on a light string moving towards the friction a part of the wave is reflected and a part is transmitted on the heavier string the reflected wave is inverted with respect to the original one.

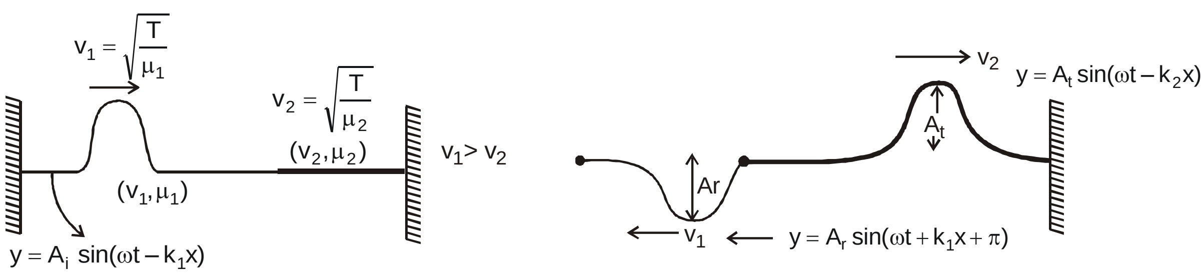
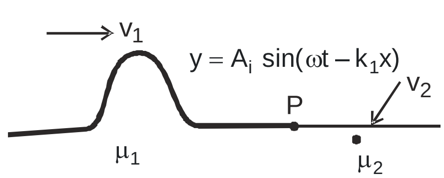



**figure (a)**

On the other hand if the wave is produced on the heavier string which moves toward the junction a part will the reflected and a part transmitted, no inversion in waves shape will take place.

The wave velocity is smaller for the heavier string lighter string



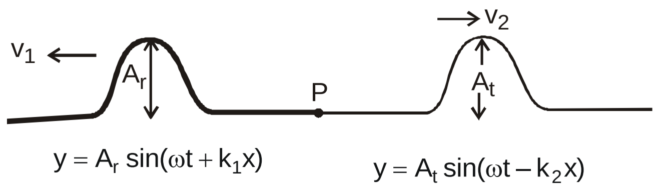



figure : (b)

Now to find the relation between $A_i$, $A_r$, $A_t$ we consider the figure (b)

Incident Power = Reflected Power + Transmitted Power

$$P_i = P_r + P_t$$

$$2\pi^2 f^2 A_i^{\,2}\mu_1 v_1 = 2\pi^2 f^2 A_r^{\,2}\mu_1 v_1 + 2\pi^2 f^2 A_t^{\,2}\mu_2 v_2 \qquad ...(i)$$

Put $\mu_1 = \dfrac{T}{v_1^{\,2}}$ and $\mu_2 = \dfrac{T}{v_2^{\,2}}$

in equation (i) their

$$\frac{A_i^{\,2}}{v_1} = \frac{A_r^{\,2}}{v_1} + \frac{A_t^{\,2}}{v_2}$$

$$A_i^{\,2} - A_r^{\,2} = \frac{v_1}{v_2}A_t^2 \qquad .......(ii)$$

Maximum displacement of joint particle P (as shown in figure) due to left string

$$= A_i + A_r$$

Maximum displacement of joint particle due to right string = $A_t$

At the boundary (at point P) the wave must be continuous, that is there are no kinks in it. Then we must have $A_i + A_r = A_t$ ...(iii)

from equation (ii) & (iii)

$$A_i - A_r = \frac{v_1}{v_2}A_t \qquad ...(iv)$$

from eq. (iii) & (iv)

$$A_t = \left[\frac{2v_2}{v_1 + v_2}\right]A_i$$

$$A_r = \left[\frac{v_2 - v_1}{v_1 + v_2}\right]A_i$$

## 10. STANDING WAVES :

In previous section we've discussed that when two coherent waves superpose on a medium particle, phenomenon of interference takes place. Similarly when two coherent waves travelling in opposite direction superpose then simultaneous interference if all the medium particles takes place. These waves interfere to produce a pattern of all the medium particles what we call, a stationary wave. If the two interfering waves which travel in opposite direction carry equal energies then no net flow of energy takes place in the region of superposition. Within this region redistribution of energy takes place between medium particles. There are some medium particles where constructive interference takes place and hence energy increases and on the other hand there are some medium particles where destructive interference takes place and energy decreases. Now we'll discuss the stationary waves analytically.

Let two waves of equal amplitude are travelling in opposite direction along x-axis. The wave equation of the two waves can be given as

$y_1$ = A sin ($\omega$t – kx) [Wave travelling in +x direction] ...(1)

and $y_2$ = A sin ($\omega$t + kx) [Wave travelling in –x direction] ...(2)

When the two waves superpose on medium particles, the resultant displacement of the medium particles can be given as

$y = y_1 + y_2$

or $y = A \sin(\omega t - kx) + A \sin(\omega t + kx)$

or $y = A [\sin\omega t \cos kx - \cos \omega t \sin kx + \sin \omega t \cos kx + \cos \omega t \sin kx]$

or $y = 2A \cos kx \sin \omega t$ ...(3)

Equation (3) can be rewritten as

$y = R \sin \omega t$ ...(4)

Where $R = 2 A \cos kx$ ...(5)

Here equation (4) is an equation of SHM. It implies that after superposition of the two waves the medium particles executes SHM with same frequency $\omega$ and amplitude R which is given by equation (5) Here we can see that the oscillation amplitude of medium particles depends on x i.e. the position of medium particles. Thus on superposition of two coherent waves travelling in opposite direction the resulting interference pattern, we call stationary waves, the oscillation amplitude of the medium particle at different positions is different.

At some point of medium the resultant amplitude is maximum which are given as

R is maximum when $\cos kx = \pm 1$

or $\frac{2\pi}{\lambda}x = N\pi$ [$N \in I$]

or $x = \frac{N\lambda}{2}$

or $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ .....

and the maximum value of R is given as

$R_{max} = \pm 2A$ ...(6)

Thus in the medium at position $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, .......... the waves interfere constructively and the amplitude of oscillations becomes 2A. Similarly at some points of the medium, the waves interfere destructively, the oscillation amplitude become minimum i.e. zero in this case. These are the points where R is minimum, when

$\cos kx = 0$

or $\frac{2\pi x}{\lambda} = (2N+1)\frac{\pi}{2}$

or $x = (2N + 1)\frac{\lambda}{4}$ [$N \in I$]

or $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$..........

and the minimum value of R is given as

$R_{min} = 0$ [7]

Thus in the medium at position $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ ......... the waves interfere destructively and the amplitude of oscillation becomes zero. These points always remain at rest. Figure (a) shows the oscillation amplitude of different medium particles in a stationary waves.

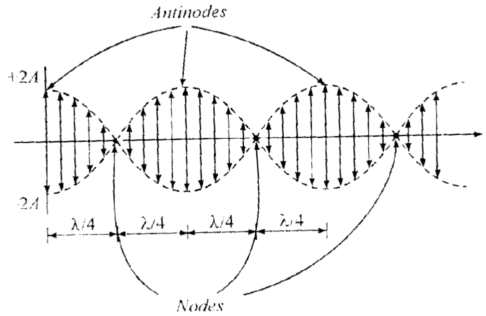



**figure (a)**

In figure (a) we can see that the medium particles at which constructive interference takes place are called antinodes of stationary wave and the points of destructive interference are called nodes of stationary waves which always remain at rest.

Figure (b) explain the movement of medium particles with time in the region where stationary waves are formed. Let us assume that at an instant t = 0 all the medium particles are at their extreme positions as shown in figure - (b - 1). Here points ABCD are the nodes of stationary waves where medium particles remains at rest. All other starts moving towards their mean positions and t = $T/4$ all particles cross their mean position as shown in figure (b – 3), you can see in the figure that the particles at nodes are not moving. Now the medium crosses their mean position and starts moving on other side of mean position toward the other extreme position. At time t = T/2, all the particles reach their other extreme position as shown in figure (b - 5) and at time t = 3T/4 again all these particles cross their mean position in opposite direction as shown in figure (b - 7).

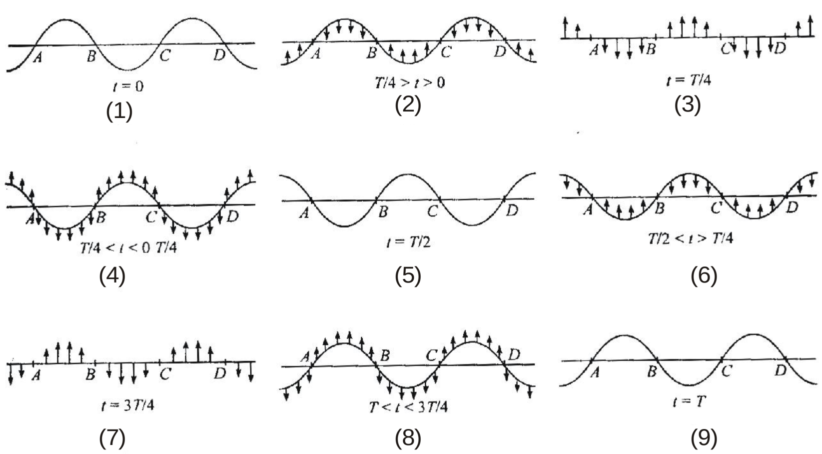



**figure (b)**

Based on the above analysis of one complete oscillations of the medium particles, we can make some interference for a stationary waves. These are :

(i) In oscillations of stationary wave in a region, some points are always at rest (nodes) and some oscillates with maximum amplitudes (antinodes). All other medium particles oscillate with amplitudes less then those of antinodes.

(ii) All medium particles between two successive nodes oscillate in same phase and all medium particles on one side of a node oscillate in opposite phase with those on the other side of the same node.

(iii) In the region of a stationary wave during one complete oscillation all the medium particles come in the form of a straight line twice.

(iv) If the component wave amplitudes are equal, then in the region where stationary wave is formed, no net flow of energy takes place, only redistribution of energy takes place in the medium.

**(a) Different Equation for a Stationary Wave**

Consider two equal amplitude waves travelling in opposite direction as

$y_1 = A \sin (\omega t - kx)$ ...(11)

and $y_2 = A \sin (\omega t + kx)$ ...(12)

The result of superposition of these two waves is

$y = 2A \cos kx \sin \omega t$ ...(13)

Which is the equation of stationary wave where 2A cos kx represents the amplitude of medium particle situated at position x and sin $\omega$t is the time sinusoidal factor. This equation (13) can be written in several ways depending on initial phase differences in the component waves given by equation (11)) can (12). If the superposing waves are having an initial phase difference $\pi$, then the component waves can be expressed as

$y_1 = A \sin (\omega t - kx)$ ...(14)

$y_2 = - A \sin (\omega t - kx)$ ...(15)

Superposition of the above two waves will result

$y = 2A \sin kx \cos \omega t$ ...(16)

Equation (16) is also an equation of stationary wave but here amplitude of different medium particles in the region of interference is given by

$R = 2A \sin kx$ ...(17)

Similarly the possible equations of a stationary wave can be written as

$y = A_0 \sin kx \cos (\omega t + \phi)$ ....(18)

$y = A_0 \cos kx \sin (\omega t + \phi)$ ...(19)

$y = A_0 \sin kx \sin (\omega t + \phi)$ ...(20)

$y = A_0 \cos kx \cos (\omega t + \phi)$ ...(21)

Here $A_0$ is the amplitude of antinodes. In a pure stationary wave it is given as

$A_0 = 2A$

Where A is the amplitude of component waves. If we care fully look at equation (18) to (21), we can see that in equation (18) and (20), the particle amplitude is given by

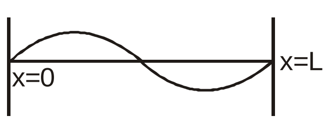
$R = A_0 \sin kx$ ...(22)

Here at x = 0, there is nodes as R = 0 and in equation (19) and (21) the particle amplitude is given as

$R = A_0 \cos kx$ ...(23)

Here at x = 0, there is an antinode as R = $A_0$. Thus we can state that in a given system of co-ordinates when origin of system is at a node we use either equation (18) or (20) for analytical representation of a stationary wave and we use equation (19) or (21) for the same when an antinode is located at the origin of system.

**Ex.13** ***Find out the equation of the standing waves for the following standing wave pattern.***




(A) $A\sin\frac{2\pi}{L}x\cos\omega t$ (B) $A\sin\frac{\pi x}{L}\cos\omega t$ (C) $A\cos\frac{\pi x}{2L}\cos\omega t$ (D) $A\cos\frac{\pi x}{L}\cos\omega t$

**Sol.** General Equation of standing wave

$y = A' \cos \omega t$

where

$A' = A \sin (kx + \theta)$

here $\lambda = L$

$\Rightarrow \quad k = \dfrac{2\pi}{L}$

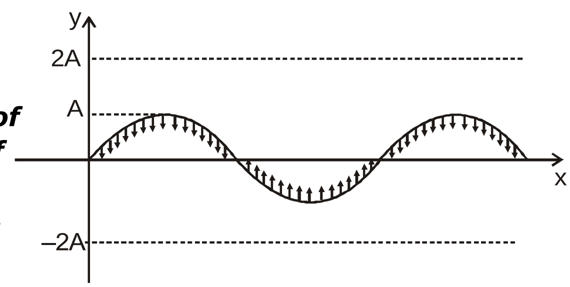
$A' = A \sin (kx + \theta) = A \sin\left(\dfrac{2\pi}{L}x + \theta\right)$

at x = 0 node

$\Rightarrow$ $A' = 0$ at $x = 0$

$\Rightarrow$ $\theta = 0$

eq. of standing wave $= A \sin \dfrac{2\pi}{L} x \cos \omega t$




***Ex.14 Figure shows the standing waves pattern in a string at t = 0. Find out the equation of the standing wave where the amplitude of antinode is 2A.***

**Sol.** Let we assume the equation of standing waves is $= A' \sin (\omega t + \phi)$

where $A' = 2A \sin (kx + \theta)$

$\because$ x = 0 is node $\Rightarrow A' = 0$, at x = 0

$2A \sin \theta = 0 \quad \Rightarrow \quad \theta = 0$

at t = 0 Particle at is at y = A and going towards mean position.

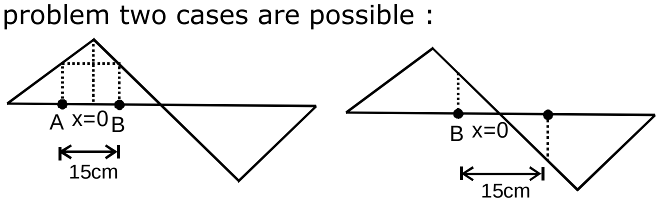
$\Rightarrow \quad \phi = \dfrac{\pi}{2} + \dfrac{\pi}{3} = \dfrac{5\pi}{6}$

so eq. of standing waves is

$y = 2A\sin kx \sin \left(\omega t + \dfrac{5\pi}{6}\right)$

***Ex.15 A string 120 cm in length sustains standing wave with the points of the string at which the displacement amplitude is equal to 3.5 mm being separated by 15.0 cm. The maximum displacement amplitude is X. 95 mm then find out the value of X.***

**Sol.** In this problem two cases are possible :




**Case - I** is that A and B have the same displacement amplitude and **case - 2** is that C and D have the same amplitude viz 3.5 mm. In case 1, if x = 0 is taken at antinode then

$A = a \cos kx$

In case -2, if x = 0 is taken at node, then

$A = a \sin kx$

But since nothing is given in the question.

Hence from both the cases, result should be same. This is possible only when

$a \cos kx = a \sin kx$

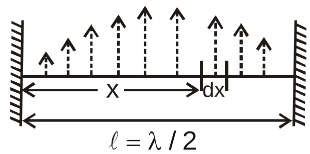
or $kx = \dfrac{\pi}{4}$ or $a = \dfrac{A}{\cos kx} = \dfrac{3.5}{\cos \pi/4} = 4.95\text{mm}$

**(b) Energy of standing wave in one loop**

When all the particles of one loop are at extreme position then total energy in the loop is in the form of potential energy only when the particles reaches its mean position then total potential energy converts into kinetic energy of the particles so we can say total energy of the loop remains constant

Total kinetic energy at mean position is equal to total energy of the loop because potential energy at mean position is zero.

Small kinetic energy of the particle which is in element dx is




$$d\,(KE) = \frac{1}{2} dmv^2$$

$$dm = \mu\, dx$$

Velocity of particle at mean position

$$= 2A \sin kx\, \omega$$

then $d\,(KE) = \frac{1}{2}\mu dx \cdot 4A^2 \omega^2 \sin^2 kx \Rightarrow d\,(KE) = 2A^2\omega^2 \mu \cdot \sin^2 kx\, dx$

$$\int d(K.E) = 2A^2\omega^2\mu \int_0^{\lambda/2} \sin^2 kx dx$$

Total K.E = $A^2\omega^2\mu \int_0^{\lambda/2} (1 - \cos 2kx)dx = A^2\omega^2\mu \left[ x - \frac{\sin 2kx}{2k} \right]_0^{\lambda/2} = \frac{1}{2}\lambda A^2 \omega^2 \mu$

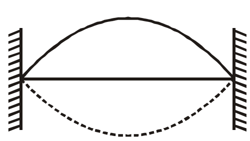
**11. STATIONARY WAVES IN STRINGS :**

**(a) When both end of string is fixed :**

A string of length L is stretched between two points. When the string is set into vibrations, a transverse progressive wave begins to travel along the string. It is reflected at the other fixed end. The incident and the reflected waves interfere to produce a stationary transverse wave in which the **ends are always nodes, if both ends of string are fixed.**

**Fundamental Mode**

(a) In the simplest form, the string vibrates in one loop in which the ends are the nodes and the centre is the antinode. This mode of vibration is known as the fundamental mode and frequency of vibration is known as the fundamental frequency or first harmonic.



Since the distance between consecutive nodes is $\frac{\lambda}{2}$

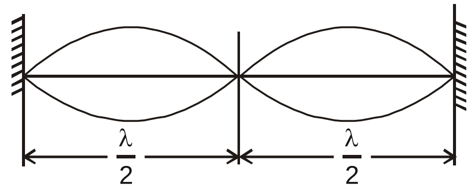
$\therefore \quad L = \frac{\lambda_1}{2} \qquad \therefore \quad \lambda_1 = 2L$

If $f_1$ is the fundamental frequency of vibration, then the velocity of transverse waves is given as,

$$v = \lambda_1 f_1 \qquad \text{or} \qquad f_1 = \frac{v}{2L} \qquad \text{...(i)}$$

**First Overtone**

(b) The same string under the same conditions may also vibrate in two loops, such that the centre is also the node




$\therefore \quad L = \frac{2\lambda_2}{2} \qquad \therefore \ \lambda_2 = L$

If $f_2$ is frequency of vibrations

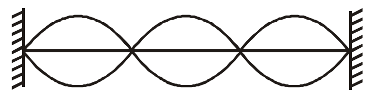
$$\therefore \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L}$$

$$\therefore \quad f_2 = \frac{v}{L} \qquad \text{...(ii)}$$

The frequency $f_2$ is known as second harmonic or first overtone.

**Second Overtone**

(c) The same string under the same conditions may also vibrate in three segments.



$$\therefore \quad L = \frac{3\lambda_3}{2}$$

$$\therefore \quad \lambda_3 = \frac{2}{3}L$$

If $f_3$ is the frequency in this mode of vibration, then,

$$f_3 = \frac{3v}{2L} \qquad \text{...(iii)}$$

The frequency $f_3$ is known as third harmonic or second overtone.
Thus a stretched string vibrates with frequencies, which are integral multiples of the fundamental frequencies. These frequencies are known as harmonics.

The velocity of transverse wave in stretched string is given as $v = \sqrt{\frac{T}{\mu}}$. Where T = tension in the string.
$\mu$ = linear density or mass per unit length of string. If the string fixed at two ends, vibrates in its fundamental mode, then

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad \text{....(17)}$$

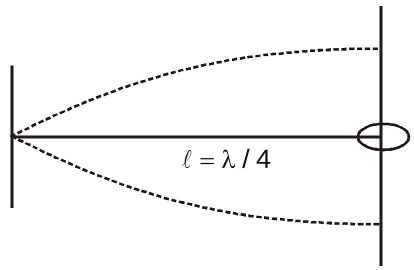
In general f = $\frac{n}{2\ell}\sqrt{\frac{T}{\mu}}$ $\qquad$ $n^{th}$ harmonic
$(n - 1)^{th}$ overtone

☛ In general, any integral multiple of the fundamental frequency is an allowed frequency. These higher frequenceis are called overtones. Thus, $v_1 = 2v_0$ is the first overtone, $v_2 = 3v_0$ is the second overtone etc. An integral multiple of a frequency is called its harmonic. Thus, for a string fixed at both the ends, all the overtones are harmonics of the fundamental frequency and all the harmonics of the fundamental frequency are overtones.

**(b) When one end of the string is fixed and other is free :**

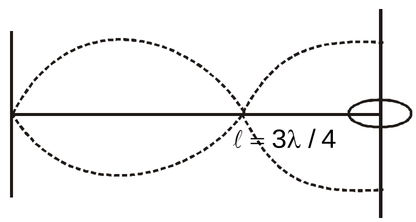
**free end acts as antinode**

**1.**




$f = \frac{1}{4\ell}\sqrt{\frac{T}{\mu}}$ fundamental or I$^{st}$ harmonic

**2.**




$f = \frac{3}{4\ell}\sqrt{\frac{T}{\mu}}$ III$^{rd}$ harmonic or I$^{st}$ overtone

In general : $f = \frac{(2n+1)}{4\ell}\sqrt{\frac{T}{\mu}}$ } ((2n + 1)$^{th}$ harmonic, n$^{th}$ overtone)