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PHYSICS
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VECTOR & CALCULUS

THEORY AND EXERCISE BOOKLET

CONTENTS

394 - Rajeev Gandhi Nagar Kota, Ph. No. 0744-2209671, 93141-87482, 93527-21564
www. motioniitjee.com , email-hr.motioniitjee@gmail.com

VECTOR

1. SCALAR :

In physics we deal with two type of physical quantity one is scalar and other is vector. Each scalar quantity has a magnitude and a unit.

For example mass = 4kg

Magnitude of mass = 4

and unit of mass = kg

Example of scalar quantities : mass, speed, distance etc.

Scalar quantities can be added, subtracted and multiplied by simple laws of algebra.

2. VECTOR :

Vector are the physical quantites having magnitude as well as specified direction.

For example :

Speed = 4 m/s (is a scalar)

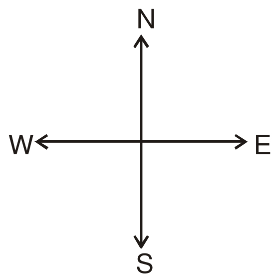
Velocity = 4 m/s toward north (is a vector)

If someone wants to reach some location then it is not sufficient to provide information about the distance of that location it is also essential to tell him about the proper direction from the initial location to the destination.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.

Example of vector quantity : Displacement, velocity, acceleration, force etc.

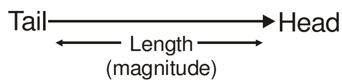
Knowledge of direction

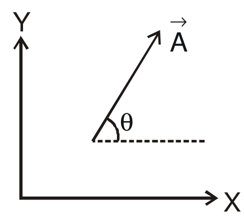


3. GENERAL POINTS REGARDING VECTORS :

3.1 Representation of vector :

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as





Mathematically, vector is represented by $\vec{A}$.

Sometimes it is represented by bold letter **A.**

Thus, the arrow in abow figure represents a vector $\vec{A}$ in xy-plane making an angle θ with x-axis.

A representation of vector will be complete if it gives us direction and magnitude.

Symbolic form : $\vec{v}, \vec{a}, \vec{F}, \vec{s}$ used to separate a vector quantity from scalar quantities (u, i, m)

Graphical form : A vector is represented by a directed straight line, having the magnitude and direction of the quantity represented by it.

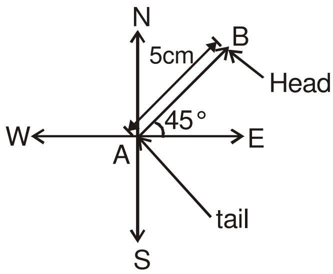
e.g. if we want to represent a force of 5 N acting 45° N of E

(i) We choose direction co-ordinates.

(ii) We choose a convenient scale like 1 cm ≡ 1 N

(iii) We draw a line of length equal in magnitude and in the direction of vector to the chosen quantity.

(iv) We put arrow in the direction of vector.



1cm ≡ 1N

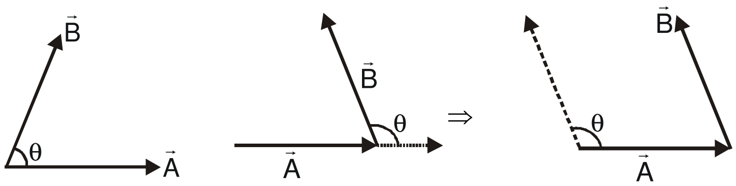
$\overrightarrow{AB}$

Magnitude of vector :

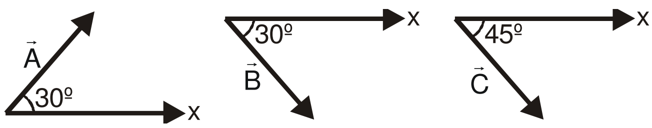
$|\overrightarrow{AB}| = 5N$

3.2 Angle between two Vectors (θ)

Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e $0 \le \theta \le \pi$).



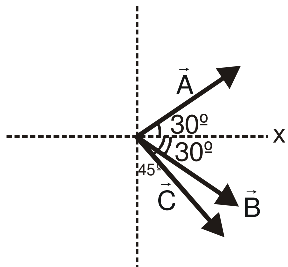
Ex.1 ***Three vectors $\vec{A}, \vec{B}, \vec{C}$ are shown in the figure. Find angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.***



Sol. To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ & $\vec{C}$ such that

tails of $\vec{A}, \vec{B}$ and $\vec{C}$ are connected as shown in figure.

Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60º, $\vec{B}$ and $\vec{C}$ is 15º and between $\vec{A}$ and $\vec{C}$ is 75º.

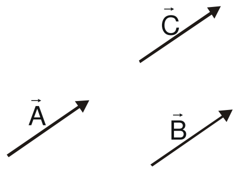


3.3 Negative of Vector

It implies vector of same magnitude but opposite in direction.

3.4 Equality of Vectors.

Vectors having equal magnitude and same direction are called equal vectors



if $|\vec{A}| = |\vec{B}| = |\vec{C}|$

and $\hat{A} = \hat{B} = \hat{C}$

then $\vec{A} = \vec{B} = \vec{C}$

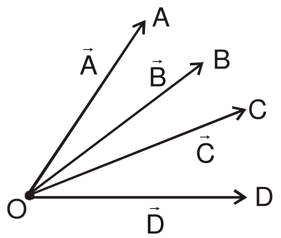
3.5 Collinear vectors :

Any two vectors are co-linear then one can be express in the term of other.

$\vec{a} = \lambda\vec{b}$ (where λ is a constant)

3.6 Co-initial vector : If two or more vector start from same point then they called co-initial vector.

e.g.



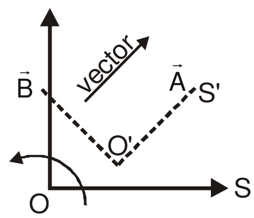
here A, B, C, D are co-initial.

3.7 Coplanar vectors :

Three (or more) vectors are called coplanar vectors if they lie in the same plane or are parallel to the same plane. Two (free) vectors are always coplanar.

Important points

☛ If the frame of reference is translated or rotated the vector does not change (though its components may change).



Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.

3.8 Multiplication and division of a vector by a scalar

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $(\vec{B} = \lambda\,\vec{A})$ whose magnitude become λ times but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times $|\vec{A}|$.

The division of vector $\vec{A}$ by a non-zero scalar m is defined as multiplication of $\vec{A}$ by $\frac{1}{m}$.

At here $\vec{A}$ and $\vec{B}$ are co-linear vector

Ex.2 ***A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if***

(i) $\vec{a}$ = 3m/s² East wards

(ii) $\vec{a}$ = –4 m/s² North wards

Sol. (i) $\vec{F} = m\vec{a} = 3 \times 3 ms^{-2}$ East wards

= 9 N East wards

(ii) $\vec{F} = m\vec{a} = 3 \times (-4)N$ North wards

= –12 N North wards

= 12 N South wards

4. LAWS OF ADDITION AND SUBTRACTION OF VECTORS :

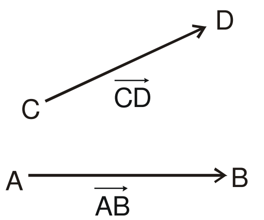
4.1 Triangle rule of addition :
Steps for additing two vector representing same physical quantity by triangle law.

(i) Keep vectors s.t. tail of one vector coincides with head of other.

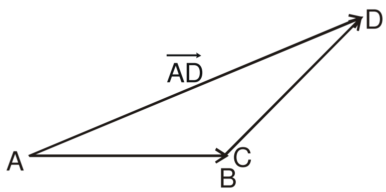
(ii) Join tail of first to head of the other by a line with arrow at head of the second.

(iii) This new vector is the sum of two vectors. (also called reultant)

(i)



(ii)



(iii) $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$

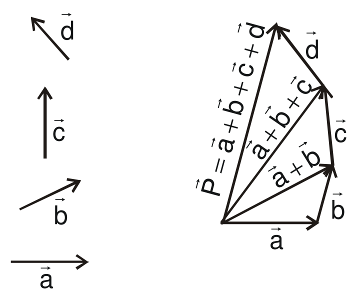
Take example here.

Q. A boy moves 4 m south and then 5 m in direction 37° E of N. Find resultant displacement.

4.2 Polygon Law of addition :

This law is used for adding more than two vectors. This is extension of triangle law of addition. We keep on arranging vectors s.t. tail of next vector lies on head of former.

When we connect the tail of first vector to head of last we get resultant of all the vectors.



Note : $\vec{P} = \left((\vec{a} + \vec{b}) + \vec{c}\right) + \vec{d} = \left((\vec{c} + \vec{a} + \vec{d})\right) + \vec{d}$ [Associative Law]

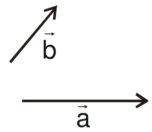
4.3 Parallelogram law of addition :

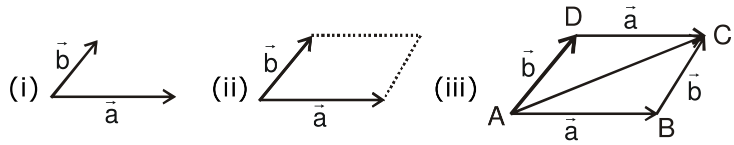
Steps :

(i) Keep two vectors such that there tails coincide.

(ii) Draw parallel vectors to both of them considering both of them as sides of a parallelogram.

(iii) Then the diagonal drawn from the point where tails coincide represents the sum of two vectors, with its tail at point of coincidence of the two vectors.



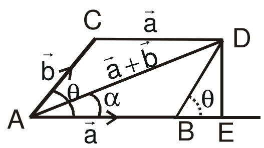


$\overrightarrow{AC} = \vec{a} + \vec{b}$

Note : $\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{AC} = \vec{b} + \vec{a}$ thus $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ [Cummutative Law]

Note : Angle between 2 vectors is the angle between their positive directions.

Suppose angle between these two vectors is θ, and $|\vec{a}| = a, |\vec{b}| = b$



$(AD)^2 = (AE)^2 + (DE)^2$

$= (AB + BE)^2 + (DE)^2$

$= (a + b\cos\theta)^2 + (b\sin\theta)^2$

$= a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta$

$= a^2 + b^2 + 2ab\cos\theta$

Thus, $AD = \sqrt{a^2 + b^2 + 2ab\cos\theta}$

or $|\vec{a} + \vec{b}| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$

angle α with vector a is

$$\tan\alpha = \frac{DE}{AE} = \frac{b\sin\theta}{(a + b\cos\theta)}$$

Important points :

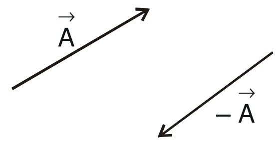
- To a vector, only a vector of same type can be added that represents the same physical quantity and the resultant
  is also a vector of the same type.
- As $R = [A^2 + B^2 + 2AB\cos\theta]^{1/2}$ so R will be maximum when, $\cos\theta = \max = 1$,
  i.e., $\theta = 0^o$, i.e. vectors are like or parallel and $R_{max} = A + B$.
- $|\vec{A}| = |\vec{B}|$ and angle between them θ then $R = 2A\cos\theta/2$
- $|\vec{A}| = |\vec{B}|$ and angle between them $\pi - \theta$ then $R = 2A\sin\theta/2$
- The resultant will be minimum if, $\cos\theta = \min = -1$, i.e., $\theta = 180^o$, i.e. vectors are antiparallel and $R_{min} = A - B$.
- If the vectors A and B are orthogonal, i.e., $\theta = 90^o$, $R = \sqrt{A^2 + B^2}$

- As previously mentioned that the resultant of two vectors can have any value from (A – B) to (A + B) depending on the angle between them and the magnitude of resultant decreases as θ increases 0º to 180º.
- Minimum number of unequal coplanar vectors whose sum can be zero is three.
- The resultant of three non-coplanar vectors can never be zero, or minimum number of non coplanar vectors whose sum can be zero is four.

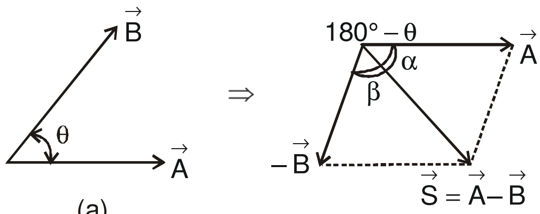
5. SUBTRACTION OF VECTOR :

Negative of a vector say $-\vec{A}$ is a vector of the same magnitude as vector $\vec{A}$ but pointing in a direction opposite to that of $\vec{A}$.

Thus, $\vec{A}-\vec{B}$ can be written as $\vec{A}+(-\vec{B})$ or $\vec{A}-\vec{B}$ is really the vector addition of $\vec{A}$ and $-\vec{B}$.



Suppose angle between two vectors $\vec{A}$ and $\vec{B}$ is θ. Then angle between $\vec{A}$ and $-\vec{B}$ will be 180° – θ as shown in figure.



(a) (b)

Magnitude of $\vec{S} = \vec{A} - \vec{B}$ will be thus given by

$$S = |\vec{A}-\vec{B}| = \sqrt{A^2+B^2+2AB\cos(180°-\theta)}$$

or $$S = \sqrt{A^2+B^2-2AB\cos\theta} \qquad \text{...(i)}$$

For direction of $\vec{S}$ we will either calculate angle α or β, where,

$$\tan\alpha = \frac{B\sin(180°-\theta)}{A+B\cos(180°-\theta)} = \frac{B\sin\theta}{A-B\cos\theta} \qquad \text{...(ii)}$$

or $$\tan\beta = \frac{A\sin(180°-\theta)}{B+A\cos(180°-\theta)} = \frac{A\sin\theta}{B-A\cos\theta} \qquad \text{...(iii)}$$

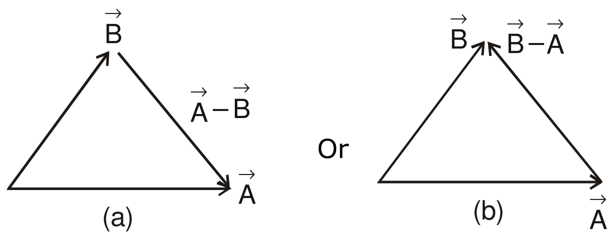
Ex.3 ***Two vectors of 10 units & 5 units make an angle of 120° with each other. Find the magnitude & angle of resultant with vector of 10 unit magnitude.***

Sol. $|\vec{a}+\vec{b}| = \sqrt{a^2+b^2+2ab\cos\theta} = \sqrt{100+25+2\times10\times5(-1/2)} = 5\sqrt{3}$

$$\tan\alpha = \frac{5\sin120°}{10+5\cos120°} = \frac{5\sqrt{3}}{20-5} = \frac{5\sqrt{3}}{5\times3} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30°$$

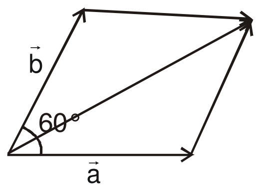
[Here shows what is angle between both vectors = 120° and not 60°]

Note : $\vec{A}-\vec{B}$ **or** $\vec{B}-\vec{A}$ can also be found by making triangles as shown in figure. (a) and (b)

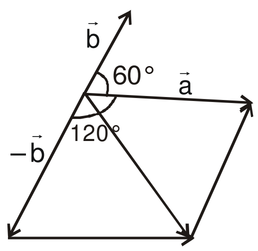


Ex.4 ***Two vectors of equal magnitude 2 are at an angle of 60° to each other find magnitude of their sum & difference.***

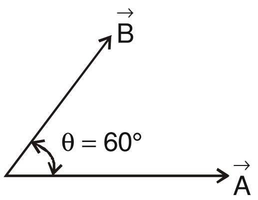
Sol. $|\vec{a}+\vec{b}| = \sqrt{2^2+2^2+2\times2\times2\cos60°} = \sqrt{4+4+4} = 2\sqrt{3}$



$|\vec{a}-\vec{b}| = \sqrt{2^2+2^2+2\times2\times2\cos120°} = \sqrt{4+4-4} = 2$



Ex.5 ***Find*** $\vec{A}+\vec{B}$ ***and*** $\vec{A}-\vec{B}$ ***in the diagram shown in figure. Given A = 4 units and B = 3 units.***



Sol. **Addition :**

$$R = \sqrt{A^2+B^2+2AB\cos\theta}$$

$$= \sqrt{16+9+2\times4\times3\cos60°} = \sqrt{37} \text{ units}$$

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \frac{3\sin60°}{4+3\cos60°} = 0.472$$

$\therefore$ $\alpha = \tan^{-1}(0.472) = 25.3°$

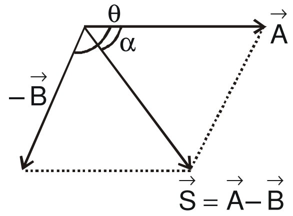
Thus, resultant of $\vec{A}$ and $\vec{B}$ is $\sqrt{37}$ units at angle 25.3° from $\vec{A}$ in the direction shown in figure.

Subtraction : $S = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

$= \sqrt{16+9-2\times4\times3\cos60^\circ} = \sqrt{13}$ units

and $\tan\theta = \dfrac{B\sin\theta}{A - B\cos\theta}$

$= \dfrac{3\sin60^\circ}{4-3\cos60^\circ} = 1.04$



$\therefore \quad \alpha = \tan^{-1}(1.04) = 46.1^\circ$

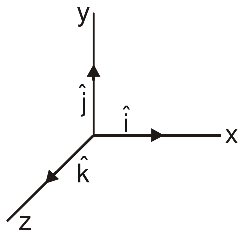
Thus, $\vec{A} - \vec{B}$ is $\sqrt{13}$ units at 46.1° from $\vec{A}$ in the direction shown in figure.

6. UNIT VECTOR AND ZERO VECTOR

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector $(\vec{A})$ can be written as the product of unit vector $(\hat{A})$ in that direction and magnitude of the given vector.

$$\vec{A} = A\,\hat{A} \quad \text{or} \quad \hat{A} = \frac{\vec{A}}{A}$$

A unit vector has no dimensions and unit. Unit vectors along the positive x-, y-and z-axes of a rectangular coordinate system are denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$.



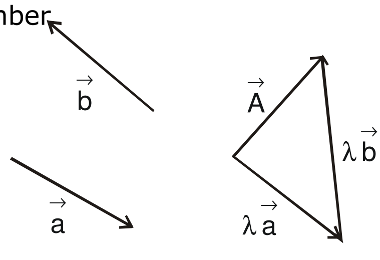
A vector of zero magnitude is called a **zero or a null vector**. Its direction is arbitrary.

Ex.6 ***A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts west wards. Represent the force in terms of $\hat{i}$.***

Sol. $\vec{F} = -10^5\,\hat{i}$ dynes

7. RESOLUTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two non-zero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors-one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.



$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real numbers)

We say that $\vec{A}$ has been resolved into two component vectors namely

$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real number)

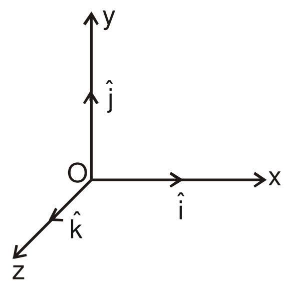
We say that $\vec{A}$ has been resolved into two component vectors namely

$\lambda\vec{a}$ and $\mu\vec{b}$

$\lambda\vec{a}$ and $\mu\vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane.

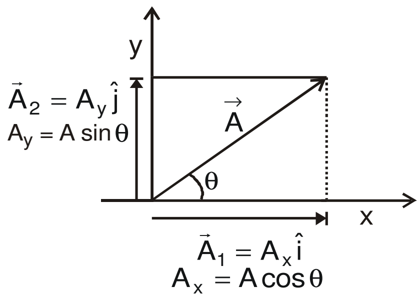
7.1 Resolution along rectangular component :

It is convenient to resolve a general vector along axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit along x, y and z-axis as shown in figure below :



7.2 Resolution in two Dimension

Consider a vector $\vec{A}$ that lies in xy plane as shown in figure,



$$\vec{A} = \vec{A}_1 + \vec{A}_2$$

$$\vec{A}_1 = A_x\hat{i},\ \vec{A}_2 = A_y\hat{j} \qquad \Rightarrow \qquad \vec{A} = A_x\hat{i} + A_y\hat{j}$$

The quantities A_x and A_y are called x-and y-components of the vector $\vec{A}$.

A_x is itself not a vector but $A_x\hat{i}$ is a vector and so it $A_y\hat{j}$.

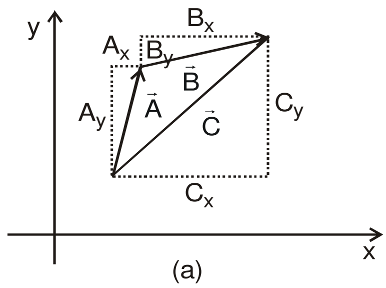
$$A_x = A\cos\theta \text{ and } A_y = A\sin\theta$$

It's clear from above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways :

(a) its magnitude A and the direction θ it makes with the x-axis; or

(b) its components A_x and A_y $\quad A = \sqrt{A_x^2 + A_y^2}, \qquad \theta = \tan^{-1}\dfrac{A_y}{A_x}$

Note : If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e., components of a vector perpendicular to itself is always zero. The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in figure. We saw that



(a)

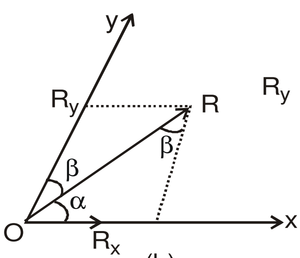
$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both

$C_x = A_x + B_x$

and $C_y = A_y + B_y$

Refer figure (b)

Vector $\vec{R}$ has been resolved in two axes x and y not perpendicular to each other. Applying sine law in the triangle shown, we have



(b)

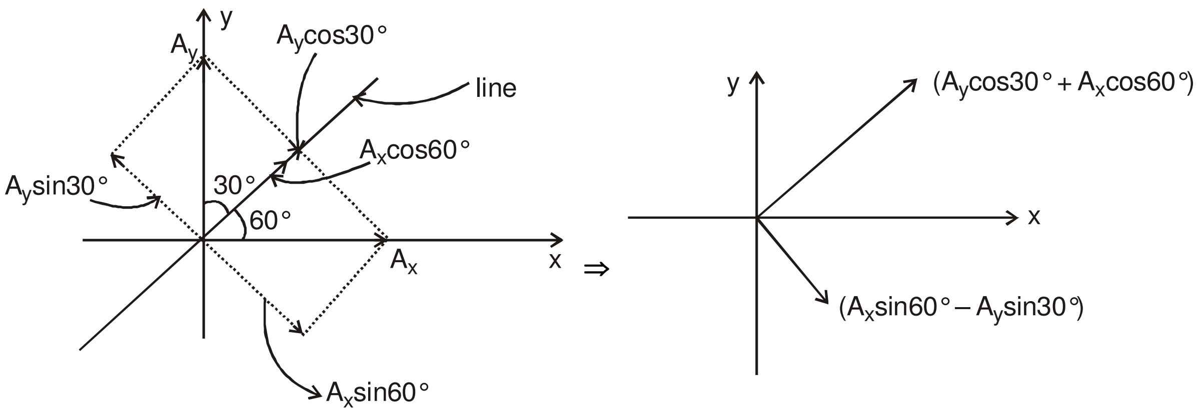
$$\frac{R}{\sin[180° - (\alpha + \beta)]} = \frac{R_x}{\sin\beta} = \frac{R_y}{\sin\alpha}$$

or $\quad R_x = \dfrac{R\sin\beta}{\sin(\alpha+\beta)} \quad$ and $\quad R_y = \dfrac{R\sin\alpha}{\sin(\alpha+\beta)}$

If $\alpha + \beta = 90°$, $R_x = R\sin\beta$ and $R_y = R\sin\alpha$

Ex.7 ***Resolve the vector $A = A_x\hat{i} + A_y\hat{j}$ along an perpendicular to the line which make angle 60° with x-axis.***

Sol.



so the component along line = $|A_y\cos30° + A_x\cos60°|$

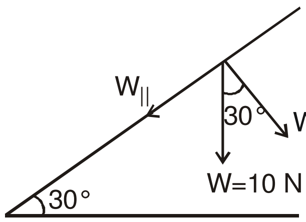
and perpendicular to line = $|A_x\sin60° - A_y\sin30°|$

Ex.8 ***Resolve a weight of 10 N in two directions which are parallel and perpendicular to a slope inclined at 30° to the horizontal***

Sol. Component perpendicular to the plane

$$W_\perp = W\cos30°$$

$$= (10)\frac{\sqrt{3}}{2} = 5\sqrt{3}\text{ N} \quad \textbf{Ans.}$$

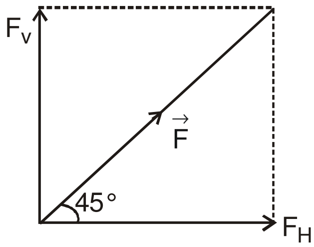


and component parallel to the plane

$$W_{||} = W\sin30° = (10)\left(\frac{1}{2}\right) = 5\text{ N}$$

Ex.9 ***Resolve horizontally and vertically a force F = 8 N which makes an angle of 45° with the horizontal.***

Sol. Horizontal component of $\vec{F}$ is



$$F_H = F\cos45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{N}$$

and vertical component of $\vec{F}$ is

$$F_v = F\sin45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{N} \quad \textbf{Ans.}$$

8. PROCEDURE TO SOLVE THE VECTOR EQUATION

$$\vec{A} = \vec{B} + \vec{C} \quad ...(1)$$

(a) There are 6 variables in this equation which are following :

(1) Magnitude of $\vec{A}$ and its direction

(2) Magnitude of $\vec{B}$ and its direction

(3) Magnitude of $\vec{C}$ and its direction.

(b) We can solve this equation if we know the value of 4 variables [Note : two of them must be directions]

(c) If we know the two direction of any two vectors then we will put them on the same side and other on the different side.

For example

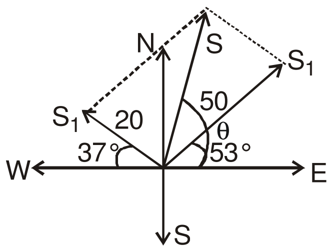
If we know the directions of $\vec{A}$ and $\vec{B}$ and $\vec{C}$'s direction is unknown then we make equation as follows:-

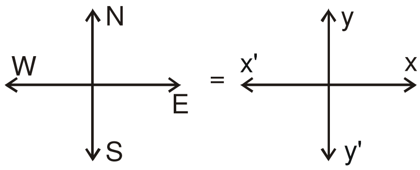
$$\vec{C} = \vec{A} - \vec{B}$$

(d) Then we make vector diagram according to the equation and resolve the vectors to know the unknown values.

Ex.10 Find the net displacement of a particle from its starting point if it undergoes two sucessive displacement given by $\vec{S}_1 = 20\text{m}$, 37° North of West, $\vec{S}_2 = 50\text{m}$, 53° North of East

Sol.





$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$S_x = S_{1x} + S_{2x}$

$= -20 \cos 37° + 50 \cos 53°$

$= 14$

$S_y = S_{1y} + S_{2y}$

$= 20 \sin 37° + 50 \sin 53°$

$= 52$

$$S = \sqrt{S_x^2 + S_y^2} = \sqrt{(14)^2 + (52)^2} = 53.85$$

Angle from west - east axis (x- axis)

$$\tan\theta = \frac{S_y}{S_x} = \frac{52}{14} = \frac{26}{7}$$

$$\theta = \tan^{-1}\left(\frac{26}{7}\right)$$

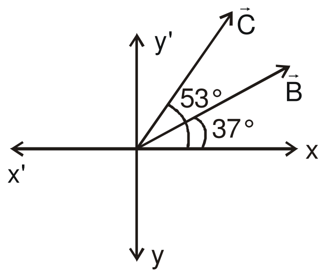
Ex.11 ***Find magnitude of $\vec{B}$ and direction of $\vec{A}$. If $\vec{B}$ makes angle 37° and $\vec{C}$ makes 53° with x axis and $\vec{A}$ has magnitude equal to 10 and $\vec{C}$ has 5. (given $\vec{A} + \vec{B} + \vec{C} = 0$)***

Sol.

$$-\vec{A} = \vec{C} + \vec{B}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

$$\Rightarrow -\vec{A} = -A_x\hat{i} + -A_y\hat{j}$$



$A_x = -(B\cos 37° + C\cos 53°)$

$A_y = -(B \sin 37° + C \cos 53°)$

$$|\vec{A}|^2 = A_x{}^2 + A_y{}^2$$

$$A^2 = \left(B \times \frac{4}{5} + C \times \frac{3}{5}\right)^2 + \left(B \times \frac{3}{5} + C \times \frac{4}{5}\right)^2$$

$10^2 = \left(\frac{4B}{5}+3\right)^2 + \left(\frac{3B}{5}+4\right)^2$

$\Rightarrow\ 100 = \frac{16}{25}B^2 + \frac{9}{25}B^2 + 25 + 2\left(\frac{3\times4}{5}+\frac{4\times3}{5}\right)B$

$\Rightarrow\ B^2 + \frac{48}{5}B - 75 = 0$

B = 5 (magnitude can not be negative)

& Angle made by A

$\Rightarrow\ A_x = -(\frac{20}{5} + 3) = -12$

$A_y = -(\frac{15}{5} + 4) = -7$

$\tan\theta = \frac{A_y}{A_x} = \frac{-7}{-12}$

$\theta = 180° + 25° = 205°$

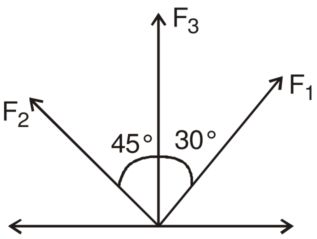
Ex.12 ***Find the magnitude of F_1 and F_2. If F_1, F_2 make angle 30° and 45° with F_3 and magnitude of F_3 is 10 N. (given $\vec{F}_1 + \vec{F}_2 = \vec{F}_3$)***

Sol. $|\vec{F}_3| = F_1\cos30° + F_2\cos45°$

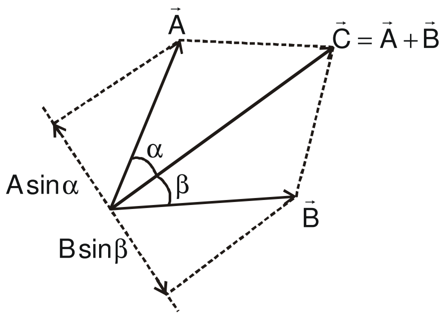
& $F_2 \sin 45° = F_1 \sin 30°$

$\Rightarrow 10 = \frac{\sqrt{3}F_1}{2} + \frac{F_2}{\sqrt{2}},\ \frac{F_2}{\sqrt{2}} = \frac{F_1}{2}$

$\Rightarrow\ F_1 = \frac{20}{\sqrt{3}+1}$ & $F_2 = \frac{20\sqrt{2}}{\sqrt{3}+1}$



9. SHORT - METHOD

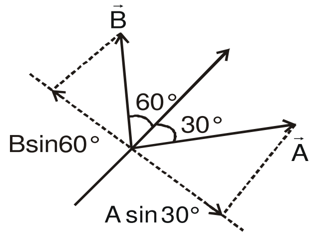


If their are two vectors $\vec{A}$ and $\vec{B}$ and their resultent make an anlge α with $\vec{A}$ and β with $\vec{B}$.

then $A \sin\alpha = \beta \sin\beta$

Means component of $\vec{A}$ perpendicular to resultant is equal in magnitude to the component of $\vec{B}$ perpendicular to resultant.

Ex.13 ***If two vectors $\vec{A}$ and $\vec{B}$ make angle 30° and 45° with their resultant and $\vec{B}$ has magnitude equal to 10, then find magnitude of $\vec{A}$.***



So $B \sin 60° = A \sin 30°$

$\Rightarrow$ $10 \sin 60° = A \sin 30°$

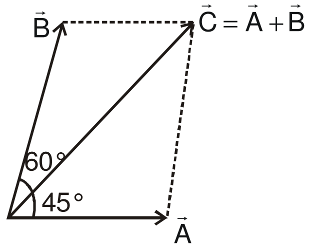
$\Rightarrow$ $A = 10\sqrt{3}$

Ex.14 ***If $\vec{A}$ and $\vec{B}$ have angle between them equals to 60° and their resultant make, angle 45° with $\vec{A}$ and $\vec{A}$ have magnitude equal to 10. Then Find magnitude of $\vec{B}$.***

Sol. here $\alpha = 45°$ and $\beta = 60° - 45° = 15°$

so $A \sin \alpha = B \sin \beta$

$10 \sin 45° = B \sin 45°$



So $B = \frac{10}{\sqrt{2}} \sin 15°$

$$= \frac{10}{\sqrt{2}}\sqrt{\frac{1-\cos(2\times 15)}{2}} = \frac{5}{\sqrt{2}}\sqrt{2-\sqrt{3}}$$

10. ADDITION AND SUBTRACTION IN COMPONENT FORM :

Suppose there are two vectors in component form. Then the addition and subtraction between these two are

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\mathbf{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}$$

Also if we are having a third vector present in component form and this vector is added or subtracted from the addition or subtraction of above two vectors then

$$\mathbf{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

$$\mathbf{A} \pm \mathbf{B} \pm \mathbf{C} = (A_x \pm B_x \pm C_x)\hat{i} + (A_y \pm B_y \pm C_y)\hat{j} + (A_z \pm B_z \pm C_z)\hat{k}$$

Note : Modulus of vector A is given by

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Ex.15 ***Obtain the magnitude of $2\vec{A} - 3\vec{B}$ if***

$$\vec{A} = \hat{i} + \hat{j} - 2\hat{k} \quad \textit{and} \quad \vec{B} = 2\hat{i} - \hat{j} + \hat{k}$$

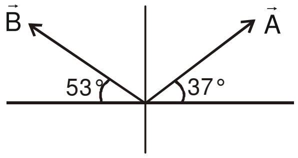
Sol. $2\vec{A} - 3\vec{B} = 2(\hat{i} + \hat{j} - 2\hat{k}) - 3(2\hat{i} - \hat{j} + \hat{k})$

$\therefore$ Magnitude of $2\vec{A} - 3\vec{B} = \sqrt{(-4)^2 + (5)^2 + (-7)^2}$

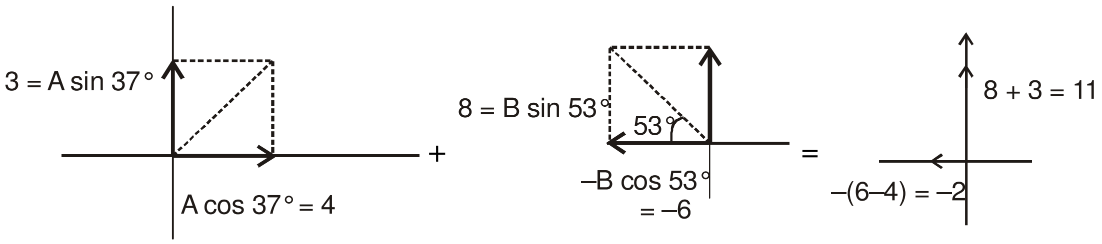
$= \sqrt{16 + 25 + 49} = \sqrt{90}$ **Ans.**

Ex.16 ***Find $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ if $\vec{A}$ make angle 37° with positive x-axis and $\vec{B}$ make angle 53° with negative x-axis as shown and magnitude of $\vec{A}$ is 5 and of B is 10.***

Sol.



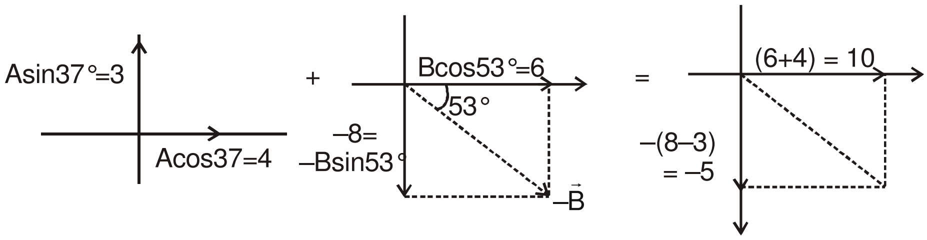
for $\vec{A}+\vec{B}$



so the magnitude of resultant will be = $\sqrt{11^2+(-2)^2}$ = $5\sqrt{5}$

and have angle θ = $\tan^{-1}\left(\frac{11}{2}\right)$ from negative x - axis towards up

for $\vec{A}-\vec{B}$



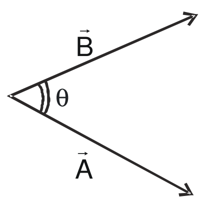
So the magnitude of resultant will be

$= \sqrt{10^2+(-5)^2} = 5\sqrt{5}$

and have angle $\theta = \tan^{-1}\left(\frac{5}{10}\right)$ from positive x-axis towards down.

11. MULTIPLICATION OF VECTORS (The Scalar and vector products) :

11.1 Scalar Product



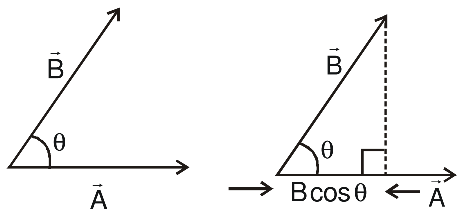
The scalar product or dot product of any two vector $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}\cdot\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus,

$\vec{A}.\vec{B} = AB\cos\theta$ (here θ is the angle between the vectos)

Properties :

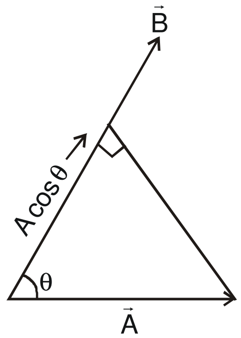
- It is always a scalar which is positive if angle between the vectors is acute (i.e.< 90°) and negative if angle between them is obtuse (i.e., 90° < q ≤ 180°)
- It is commutative i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
- It is distributive, i.e. $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$
- As by definition $\vec{A}.\vec{B}$ = AB cos θ. The angle between the vectors θ = $\cos^{-1}\left[\dfrac{\vec{A}.\vec{B}}{AB}\right]$
- $\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta)$

  Geometrically, B cos θ is the projection of $\vec{B}$ onto $\vec{A}$ and vice versa



Component of $\vec{B}$ along $\vec{A}$ = B cos θ = $\dfrac{\vec{A}.\vec{B}}{A}$ = $\hat{A}.\vec{B}$ (Projection of $\vec{B}$ on $\vec{A}$)

Component of $\vec{A}$ along $\vec{B}$ = A cos θ = $\dfrac{\vec{A}.\vec{B}}{B}$ = $\vec{A}.\hat{B}$ (Projection of $\vec{A}$ on $\vec{B}$)



- Scalar product of two vectors will be maximum when cos θ = max = 1, i.e., θ = 0°,

  i.e., vectors are parallel $\Rightarrow (\vec{A}.\vec{B})_{max} = AB$
- If the scalar product of two non-zero vectors vanishes then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by

  $(\vec{A})^2 = \vec{A}.\vec{A}$ = AA cos θ = $A^2 \Rightarrow$ $A = \sqrt{\vec{A}.\vec{A}}$
- In case of unit vector $\hat{n}$,

  $\hat{n}.\hat{n}$ = 1 × 1 × cos 0° = 1 $\Rightarrow$ $\hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

  In case of orthogonal unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

  $\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

Ex.17 ***If the vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?***

Sol. If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$\Rightarrow \vec{P}.\vec{Q} = 0$ $\Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0$

$\Rightarrow a^2 - 2a - 3 = 0$ $\Rightarrow a^2 - 3a + a - 3 = 0$

$\Rightarrow a(a-3) + 1(a-3)$ $\Rightarrow a = -1, 3$

Ex.18 Find the component of $3\hat{i}+4\hat{j}$ along $\hat{i}+\hat{j}$?

Sol. Component of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}.\vec{B}}{B}$ hence required component

$$= \frac{(3\hat{i}+4\hat{j}).(\hat{i}+\hat{j})}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

Ex.19 Find angle between $\vec{A}=3\hat{i}+4\hat{j}$ and $\vec{B}=12\hat{i}+5\hat{j}$?

Sol. We have $\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{(3\hat{i}+4\hat{j}).(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$$\cos\theta = \frac{36+20}{5\times13} = \frac{56}{65} \qquad \theta = \cos^{-1}\left(\frac{56}{65}\right)$$

Ex.20 ***(i) For what value of m the vector $\vec{A}=2\hat{i}+3\hat{j}-6\hat{k}$ is perpendicular to $\vec{B}=3\hat{i}-m\hat{j}+6\hat{k}$***

(ii) Find the component of vector $\vec{A}=2\hat{i}+3\hat{j}$ along the direction of $\hat{i}+\hat{j}$?

Sol. (i) m = – 10 (ii) $\frac{5}{\sqrt{2}}$

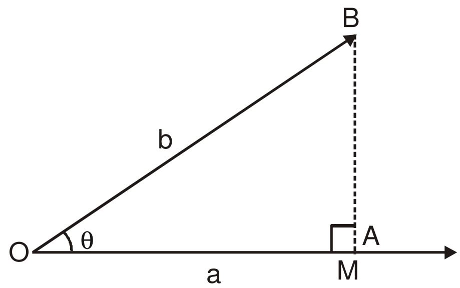
Important Note :

Components of **b** along and perpendicular to **a**.

Let $\overrightarrow{OA}$. $\overrightarrow{OB}$ represent two (non-zero) given vectors **a**, **b** respectively. Draw BM perpendicular to $\overrightarrow{OA}$

From ΔOMB, $\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$

$\Rightarrow$ **b** = $\overrightarrow{OM} + \overrightarrow{MB}$



Thus $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are components of **b** along **a** and perpendicular to **a**.

Now $\overrightarrow{OM} = (\overrightarrow{OM})\hat{a}$ = (OB cos θ) $\hat{a}$

= |**b**| cos$\theta\hat{a}$ = |b| . **a** . **b** / |**a**| |**b**| . $\hat{a}$

= **a** . **b** / |**a**| . **a**/|**a**| = (**a** . **b**) **a** / |**a**|^2

= (**a** . **b**) **a** / a^2

$\overrightarrow{MB}$ = **b** – $\overrightarrow{OM}$ = **b** – (**a** . **b** / |**a**|^2) . **a**

Hence, components of **b** along a perpendicular to **a** are.

(**a** . **b**/ |**a**|^2) **a** and **b** – (**a** . **b** / |**a**|^2) **a** respectively.

Ex.21 The velocity of a particle is given by $\vec{v}=3\hat{i}+2\hat{j}+3\hat{k}$. Find the vector component of its velocity parallel to the line $\vec{l}=\hat{i}-\hat{j}+\hat{k}$.

Sol. Component of $\vec{v}$ along $\vec{l}$

$$= v\cos\theta\hat{l} = v\frac{\vec{v}.\vec{l}}{vl}\hat{l} = \frac{\vec{v}.\vec{l}}{l^2}\vec{l}$$

$$= \frac{(3\hat{i}+2\hat{j}+3\hat{k})(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|^2} = \frac{4}{3}(\hat{i}-\hat{j}+\hat{k})$$

11.2 Vector product

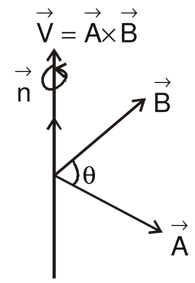
The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as

$\vec{A}\times\vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as :

$$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$$

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right - hand - thumb rule.

Right - Hand - Thumb Rule :



To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$. The direction of the thumb gives the direction of $\hat{n}$.

Properties :

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.
- Vector product of two vectors is not commutative i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

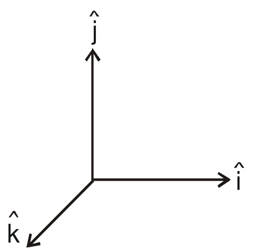
  But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$
- The vector product is distributive when the order of the vectors is strictly maintained i.e.

  $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
- The magnitude of vector product of two vectors will be maximum when sin θ = max = 1. i.e. θ = 90°

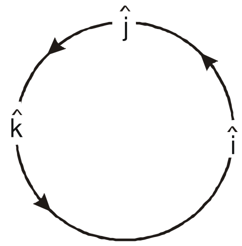
  $|\vec{A}\times\vec{B}|_{max} = AB$
- The magnitude of vector product of two non-zero vectors will be minimum when $|\sin\theta|$ = minimum = 0, i.e., θ = 0° or 180° and $|\vec{A}\times\vec{B}|_{min} = 0$ i.e., if the vector product of two non-zero vectors vanishes, the vectors are collinear.
- The self cross product i.e. product of a vector by itself vanishes i.e. is a null vector.

  $\vec{A}\times\vec{A} = AA\sin 0°\hat{n} = \vec{0}$
- In case of unit vector $\hat{n}$, $\quad \hat{n}\times\hat{n} = \vec{0} \quad \Rightarrow \quad \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- In case of orthogonal unit vectors $\hat{i},\hat{j}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,

$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{i} = \hat{j}$



(A)



(B)

- In terms of components, $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

Ex.22 ***$\vec{A}$ is East wards and $\vec{B}$ is downwards. Find the direction of $\vec{A} \times \vec{B}$?***

Sol. Applying right hand thumb rule we find that $\vec{A}\times\vec{B}$ is along North.

Ex.23 ***If $\vec{A}.\vec{B} = |\vec{A}\times\vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$***

Sol. $\vec{A}.\vec{B} = |\vec{A}\times\vec{B}|$ AB cos θ = AB sin θ tan θ = 1 ⇒ θ = 45°

Ex.24 $\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$ ⇒ $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ ***here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$***

Ex.25 **Find $\vec{A}\times\vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$**

Sol. $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}(-4-(-4)) - \hat{j}(2-12) + \hat{k}(-1-(-6)) = 10\hat{j} + 5\hat{k}$

Ex.26 ***(i) $\vec{A}$ is North-East and $\vec{B}$ is down wards, find the direction of $\vec{A}\times\vec{B}$***

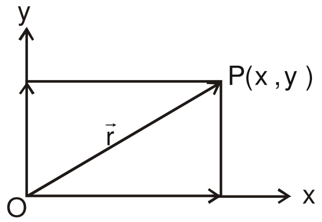
(ii) Find $\vec{B} \times \vec{A}$ if $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$

Ans. (i) North - West. (ii) $-4\hat{i} - 3\hat{j} + \hat{k}$

12. POSITION VECTOR :

Positin vector for a point is vector for which tail is origin & head is the given point itself.

Position vector of a point defines the position of the point w.r.t. the origin.

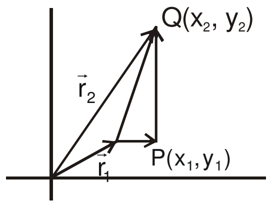


$\overrightarrow{OP} = \vec{r}$

$\vec{r} = x\hat{i} + y\hat{j}$

13. DISPLACEMENT VECTOR :

Change in position vector of particle is known as displacement vector.



$\overrightarrow{OP} = \vec{r}_1 = x_1\hat{i} + y_1\hat{j}$

$\overrightarrow{OQ} = \vec{r}_2 = x_2\hat{i} + y_2\hat{j}$

$\overrightarrow{PQ} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

Thus we can represent a vector in space starting from (x_1, y_1) & ending at (x_2, y_2) as $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

CALCULUS

14. CONSTANTS : They are fixed real number which value does not change

Ex. 3, e, a, – 1, etc.

15. VARIABLE :

Somthing that is likly to vary, somthing that is subject to variation.

or

A quantity that can assume any of a set of value.

Types of variables.

(i) Independent variables : Indepedent variables is typically the variable being manipulated or change

(ii) dependent variables : The dependent variables is the object result of the independent variable being manipulated.

Ex. $y = x^2$

here y is dependent variable and x is independent variable

16. FUNCTION :

Function is a rule of relationship between two variables in which one is assumed to be dependent and the other independent variable.

The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent & temperature is the dependent variable.

The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each case, the value of one variable quantity (dependent variable), which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as y = f(x).

all possible values of independent variables (x) are called domain of function.

all possible values of dependent variable (y) are called Range of fucntion.

Think of function f as a kind machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usualy call the area A and the radius r. Since area depends on radius, we say that A is a function of r, A = f(r). The eauation $A = \pi r^2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(x) = \pi r^2$. (Here the rule of relationship which describes the function may be described as square & multiply by π)

if $r = 1$ $A = \pi$

if $r = 2$ $A = 4\pi$

if $r = 3$ $A = 9\pi$

The set of all possible input values for the radius is called the domain of the function. The set of all output values of the area is the range of the function.

We usually denote functions in one of the two ways :

1. By giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the fucntion.
2. By giving a formula such as $f(x) = x^2$ that defines a functions symbols f to name the function. Strictly speaking, we should call the function f and not f(x). y = sinx. Here the function is y since, x is the independent variable.

Ex.27 The volume V of ball (solid sphere) of radius r is given by the function V(r) = $\frac{4}{3}\pi(r)^3$

The volume of a ball of radius 3m is ?

Sol. $V(3) = \frac{4}{3}\pi(3)^3 = 36\pi m^3$.

Ex.28 Suppose that the function F is defined for all real numbers r by the formula.

F(r) = 2 (r – 1) + 3.

Evaluate F at the input values 0, 2 x + 2, and F(2).

Sol. In each case we substitute the given input value for r into the formula for F:

$F(0) = 2(0 - 1) + 3 = -2 + 3 = 1$

$F(2) = 2(2 - 1) + 3 = 2 + 3 = 5$

$F(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$

$F(F(2)) = F(5) = 2(5 - 1) + 3 = 11$

Ex.29 function f(x) is defined as

f(x) = x² + 3, Find

f(0), f(1), f(x²), f(x + 1) and f(f(1))

Sol.

$f(0) = 0^2 + 3 = 3$

$f(1) = 1^2 + 3 = 4$

$f(x^2) = (x^2)^2 + 3 = x^4 + 4$

$f(x + 1) = (x + 1)^2 + 3 = x^2 + 2x + 4$

$f(f(1)) = f(4) = 4^2 + 3 = 19$

17. DIFFERENTIATION

Finite difference :

The finite difference between two values of a physical is represented by Δ notation.

For example :

Difference in two values of y is written as Δy as given in the table below.

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

Infinitey small difference :

The infinitely small difference means very-very small difference. And this difference is represented by 'd' notation insted of 'Δ'.

For example infinitely small difference in the values of y is written as 'dy'

if $y_2 = 100$ and $y_1 = 99.9999999999999.....$

then dy = 0.0000000000000..........00001

Definition of differentiation

Another name of differentiation is derivative. Suppose y is a function of x or y = f(x)

Differentiation of y with respect to x is denoted by sumbols $f'(x)$

where $f'(x) = \frac{dy}{dx}$; dx is very small change in x and dy is corresponding very small change in y.

Notation : There are many ways to denote the derivative of function y = f(x), the most common notations are these :

y'	"y prime"	Nice and brief and does not name the independent variable
$\frac{dy}{dx}$	" dy by dx"	Names the variables and uses d for derivative
$\frac{df}{dx}$	" df by dx"	Emphasizes the function's name
$\frac{d}{dx}f(x)$	" d by dx of f "	Emphasizes the idea that differentiation is an operation performed on f.
$D_x f$	" dx of f "	A common operator notation
$\dot{y}$	" y dot"	One of Newton's notations, now common for time derivative i.e. dy/dt

Average rates of change :

Given an arbitrary function y = f(x) we calculate the average rate of change of y with respect to x over the interval (x, x+ Δx) by dividing the change in value of y, i.e., Δy = f(x+ Δx) – f(x), by length of interval Δx over which the change occurred.

The average rate of change of y with respect to x over the interval [x, x + Δx]

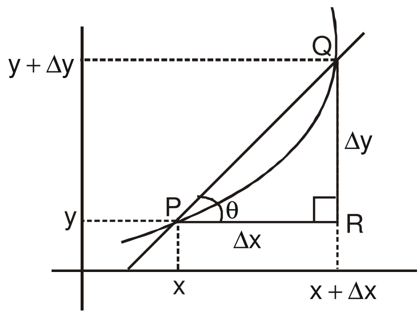
$$= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Geometrically

$$\frac{\Delta y}{\Delta x} = \frac{QR}{PR} = \tan\theta = \text{Slope of the line PQ}$$

In triangle QPR $\tan\theta = \frac{\Delta y}{\Delta x}$

therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P & Q.



The derivative of a fucntion

We know that Average rate of change of y w.r.t x is -

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If the limit of this ratio exists as $\Delta x \to 0$, then it is called the derivative of given function f(x) and is denoted as

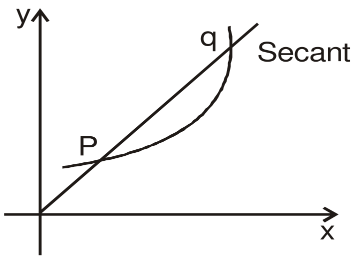
$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

18. GEOMETRICAL MEANING OF DIFFERENTIATION :

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve.

Secant and Tangent to a Curve

Secant : - A secant to a curve is a straight line, which intersects the curve at any two points.



Tangent :

A tangent is straight line, which touches the curve a particular point. Tangent is limiting case of secant which intersects the curve at two overlapping point.

In the figure - 1 shown, if value of Δx is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (Figure-2) value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P.

Therefore

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = \tan\theta$$

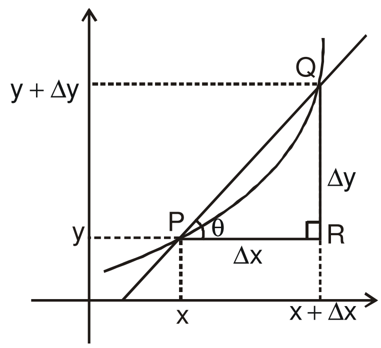


Figure-1

we can say that differentiation of y with respect to x, i.e. $\left(\frac{dy}{dx}\right)$ is equal to slope of the tangent at point P (x,y)

or $\tan\theta = \frac{dy}{dx}$

(From fig-1 the average rate change of y from x to x + Δx is identical with the slope of secant PQ)

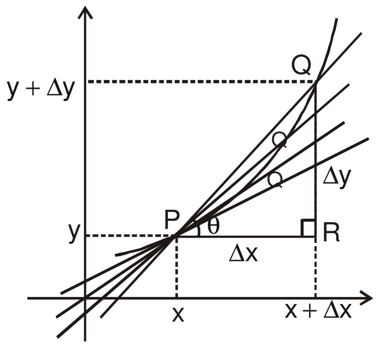


Figure-2

Rule No. 1 Derivative Of A Constant

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\frac{d}{dx}c = 0$

Ex.30 $\frac{d}{dx}(8) = 0$, $\frac{d}{dx}\left(-\frac{1}{2}\right) = 0$, $\frac{d}{dx}\left(\sqrt{3}\right) = 0$

Rule No.2 Power Rule

If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$

To apply the power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

Ex.31

f	x	x^2	x^3	x^4	…..
f'	1	2x	$3x^2$	$4x^3$	…..

Ex.32 (i) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$ (ii) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$

Ex.33 (a) $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Function defined for $x \geq 0$ derivative defined only for $x > 0$

(b) $\frac{d}{dx}(x^{1/5}) = \frac{1}{5}x^{-4/5}$

Function defined for $x \geq 0$ derivative not defined at $x = 0$

Rule No.3 The Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$

In particular, if n is a positive integer, then $\frac{d}{dx}(cx^n) = cn\ x^{n-1}$

Ex.34 ***The derivative formula***

$$\frac{d}{dx}(3x^2) = 3(2x) = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3.

Ex.35 A useful special case

The derivative of the negative of a differentiable function is the negative of the function’s derivative. Rule 3 with c = – 1 gives.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1.u) = -1.\frac{d}{dx}(u) = -\frac{d}{dx}(u)$$

Rule No.4 The Sum Rule

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point where u and v are both differentiable functions in their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The sum Rule also extends to sums of more than two functions, as long as there are only finite functions in the sum. If u_1, u_2, ... u_n are differentiable at x, then so if $u_1 + u_2 + ... + u_n$, then

$$\frac{d}{dx}(u_1 + u_2 + ... + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + ... + \frac{du_n}{dx}$$

Ex.36 (a) $y = x^4 + 12x$ (b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$= 4x^3 + 12$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$= 3x^2 + \frac{4}{3}.2x - 5 + 0$

$= 3x^2 + \frac{8}{3}x - 5$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in above example.

Rule No. 5 The Product Rule

If u and v are differentiable at x, then if their product uv is considered, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In prime notation

$(uv)' = uv' + vu'$.

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$\frac{d}{dx}(x.x) = \frac{d}{dx}(x^2) = 2x,$ while $\frac{d}{dx}(x).\frac{d}{dx}(x) = 1.1 = 1$, which is wrong

Ex.37 ***Find the derivatives of y = $(x^2 + 1)(x^3 + 3)$***

Sol. Using the product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}[(x^2+1)(x^3+3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x$$

Example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check :

$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

This is in agreement with our first calculation.

There are times, however, when the product Rule must be used. In the following examples. We have only numerical values to work with.

Ex.38 Let y = uv be the product of the functions u and v. Find y′(2) if u(2) = 3, u′(2) = – 4, v(2) = 1, and v′(2) = 2.

Sol. From the Product Rule, in the form

$y' = (uv)' = uv' + vu'$,

we have $y'(2) = u(2)\,v'(2) + v(2)\,u'(2)$

$= (3)(2) + (1)(-4) = 6 - 4 = 2$

Rule No.6 The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

Ex.39 Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$

Sol. We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$

$$\frac{dy}{dt} = \frac{(t^2+1)2t - (t^2-1).2t}{(t^2+1)^2} \qquad \left[\text{As } \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\right]$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

Rule No. 7 Derivative Of Sine Function

$$\frac{d}{dx}(\sin x) = \cos x$$

Ex.40 (a) $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$ Difference Rule

(b) $y = x^2 \sin x$: $\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x$ Product Rule

$= x^2\cos x + 2x \sin x$

(c) $y = \frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2}$ Quotient Rule

$= \frac{x\cos x - \sin x}{x^2}$

Rules No.8 Derivative Of Cosine Function

$$\frac{d}{dx}(\cos x) = -\sin x$$

Ex.41 (a) $y = 5x + \cos x$ Sum Rule

$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x$

(b) $y = \sin x \cos x$

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$ Product Rule

$= \sin x(-\sin x) + \cos x(\cos x)$

$= \cos^2 x - \sin^2 x = \cos 2x$

Rule No. 9 Derivatives Of Other Trigonometric Functions

Because sin x and cos x are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \;; \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \;; \qquad \text{cosec } x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. There derivatives, Calculated from the Quotient Rule, are given by the following formulas.

$$\frac{d}{dx}(\tan x) = \sec^2 x \;; \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x \;; \qquad \frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$$

Ex.42 Find dy / dx if y = tan x.

Sol. $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Ex.43 (a) $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \text{cosec}^2 x$

(b) $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\text{cosec } x) = 2\frac{d}{dx}(\text{cosec } x)$

$= 2(-\text{cosec } x \cot x) = -2 \text{ cosec } x \cot x$

Rule No. 10 Derivative Of Logrithm And Exponential Functions

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \;, \qquad \frac{d}{dx}(e^x) = e^x$$

Ex.44 $y = e^x . \log_e (x)$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x).\log(x) + \frac{d}{dx}[\log_e(x)]e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x.\log_e(x) + \frac{e^x}{x}$$

Rule No. 11 Chain Rule Or 'Outside Inside' Rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

It sometime helps to think about the Chain Rule the following way. If y = f(g(x)),

$$\frac{dy}{dx} = f'[g(x)] . g'(x)$$

In words : To find dy/dx, differentiate the "outside" function f and leave the "inside" g(x) alone; then multiply by the derivative of the inside.

We now know how to differntiate sin x and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$?

The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

Ex.45 ***The function y = 6x – 10 = 2(3x – 5) is the composite of the functions y = 2u and u = 3x – 5. How are the derivatives of these three functions related ?***

Sol. We have $\frac{dy}{dx} = 6$, $\frac{dy}{du} = 2$, $\frac{du}{dx} = 3$

Since 6 = 2 × 3 $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

Is it an accident that $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$?

If we think of the derivative as a rate of change, our intution allows us to see that this relationship is reasonable. For y = f(u) and u = g(x), if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

Ex.46 ***Let us try this again on another function.***

$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$

is the composite $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives. We see that

$\frac{dy}{du} . \frac{du}{dx} = 2u.6x$ $= 2(3x^2 + 1). 6x = 36x^3 + 12x$

and $\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1)$ $= 36x^3 + 12x$

Once again, $\frac{dy}{du} . \frac{du}{dx} = \frac{dy}{dx}$

The derivative of the composite function f(g(x)) at x is the derivative of f at g(x) times the derivative of g at x.

Ex.47 ***Find the derivation of*** $y = \sqrt{x^2 + 1}$

Sol. Here y = f(g(x)), where f(u) = $\sqrt{u}$ and u = g(x) = $x^2 + 1$. Since the derivatives of f and g are

$f'(u) = \frac{1}{2\sqrt{u}}$ and $g'(x) = 2x$,

the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} f(g(x)) = f'(g(x)).g'(x) = \frac{1}{2\sqrt{g(x)}}.g'(x) = \frac{1}{2\sqrt{x^2+1}}.(2x) = \frac{x}{\sqrt{x^2+1}}$$

Ex.48

$$\frac{d}{dx}\overset{\text{outside}}{\sin}(\underbrace{x^2 + x}_{\text{Inside}}) = \overset{\text{derivative of the outside}}{\cos}(\underbrace{x^2 + x}_{\text{Inside left along}}).\underbrace{(2x + 1)}_{\text{derivative of the inside}}$$

Ex.49 (a) $\frac{d}{dx}(1 - x^2)^{1/4} = \frac{1}{4}(1 - x^2)^{-3/4}(-2x)$ $u = 1 - x^2$ and $n = 1/4$

(Function defined) on [–1, 1]

$= \frac{-x}{2(1 - x^2)^{3/4}}$ (derivative defined only on (–1, 1))

(b) $\frac{d}{dx}\sin 2x = \cos 2x \frac{d}{dx} 2x = \cos 2x .2 = 2 \cos 2x$

(c) $\frac{d}{dt}(A\sin(\omega t+\phi)) = A \cos(\omega t + \phi) \frac{d}{dt}(\omega t + \phi) = A \cos(\omega t + \phi).\ \omega = A\,\omega \cos(\omega t + \phi)$

Rull No. 12 Power Chain Rule

* If $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

Ex.50 $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = -1\,(3x-2)^{-2}\frac{d}{dx}(3x-2)$

$= -1\,(3x-2)^{-2}\,(3) = -\frac{3}{(3x-2)^2}$

In part (d) we could also have found the derivation with the Quotient Rule.

Ex.51 (a) $\frac{d}{dx}(Ax+B)^n$

Sol. Here u = Ax + B, $\frac{du}{dx} = A$

$\therefore \frac{d}{dx}(Ax+B)^n = n(Ax+B)^{n-1}.A$

(b) $\frac{d}{dx}\sin(Ax+B) = \cos(Ax+B).A$ (c) $\frac{d}{dx}\log(Ax+B) = \frac{1}{Ax+B}.A$

(d) $\frac{d}{dx}\tan(Ax+B) = \sec^2(Ax+B).A$ (e) $\frac{d}{dx}e^{(Ax+B)} = e^{(Ax+B)}.A$

Note : These results are important

19. DOUBLE DIFFERENTIATION

If *f* is differentiable function, then its derivative f' is also a function, so *f'* may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of y = f(x) as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Another notation is $f''(x) = D_2 f(x)$.

Ex.52 If f(x) = x cos x, find f'' (x)

Sol. Using the Product Rule, we have $f'(x) = x\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x) = -x \sin x + \cos x$

To find f'' (x) we differentiate f'(x) :

$f''(x) = \frac{d}{dx}(-x\sin x+\cos x) = -x\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(-x) + \frac{d}{dx}(\cos x)$

$= -x\cos x - \sin x - \sin x = -x \cos x - 2\sin x$

20. APPLICATION OF DERIVATIVE DIFFERENTIATION AS A RATE OF CHANGE

$\frac{dy}{dx}$ is rate of change of 'y' with respect to 'x' :

For examples :

(i) $v = \frac{dx}{dt}$ this means velocity 'v' is rate of change of displacement 'x' with respect to time 't'

(ii) $a = \frac{dv}{dt}$ this means acceleration 'a' is rate of change of velocity 'v' with respect to time 't'.

(iii) $F = \frac{dp}{dt}$ this means force 'F' is rate of change of monentum 'p' with respect to time 't'.

(iv) $\tau = \frac{dL}{dt}$ this means torque 'τ' is rate of change of angular momentum 'L' with respect to time 't'

(v) Power $= \frac{dW}{dt}$ this means power 'P' is rate of change of work 'W' with respect to time 't'

Ex.53 The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$.

How fast is the area changing with respect to the diameter when the diameter is 10 m ?

Sol. The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4}2D = \frac{\pi D}{2}$$

When D =10m, the area is changing at rate (π/2) = 5π m^2/m. This mean that a small change ΔD m in the diameter would result in a changed of about 5p ΔD m^2 in the area of the circle.

Physical Example :

Ex.54 Boyle's Law state that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant : PV = C. Find the rate of change of volume with respect to pressure.

Sol. $\frac{dV}{dP} = -\frac{C}{P^2}$

Ex.55 (a) Find the average rate of change of the area of a circle with respect to its radius r as r changed from

(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1

(b) Find the instantaneous rate of change when r = 2.

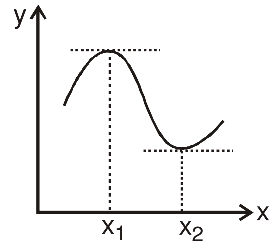
(c) Show that thre rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically when this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small ?

Sol. (a) (i) 5π (ii) $4.5\,\pi$ (iii) $4.1\,\pi$

(b) $4\,\pi$

(c) $\Delta A \approx 2\,\pi\, r\, \Delta r$

21. MAXIMA & MINIMA

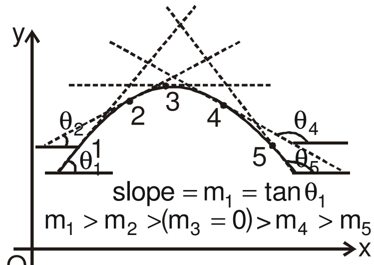


Suppose a quantity y depends on another quantity x in a manner shown in figure. It becomes maximum at x_1 and minimum at x_2. At these points the tangent to the curve is parallel to the x-axis and hence its slope is $\tan\theta = 0$. Thus, at a maxima or a minima slope

$$\Rightarrow \frac{dy}{dx} = 0$$

Maxima

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decrease at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at maximum. The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$. Conditions for maxima are : (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} < 0$



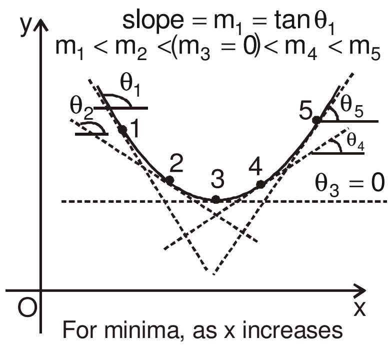
For maxima, as x increases the slope decreases

Minima

Similarly, at a minimum the slope changes from negative to positive,

Hence with the increases of x. The slope is increasing that means the rate of change of slope with respect to x is positive.

Hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$



For minima, as x increases slope increases

Conditions for minima are :

(a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} > 0$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

Ex.56 ***Find maximum or minimum values of the functions :***
(A) $y = 25x^2 + 5 - 10x$ (B) $y = 9 - (x - 3)^2$

Sol. (A) For maximum and minimum value, we can put $\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = 50x - 10 = 0 \quad \therefore \quad x = \frac{1}{5}$

Further, $\frac{d^2y}{dx^2} = 50$

or $\frac{d^2y}{dx^2}$ has positive value at x = $\frac{1}{5}$. Therefore, y has minimum value at x = $\frac{1}{5}$. Therefore, y has minimum value at x = $\frac{1}{5}$. Substituting x = $\frac{1}{5}$ in given equation, we get

$$y_{min} = 25\left(\frac{1}{5}\right)^2 + 5 - 10\left(\frac{1}{5}\right) = 4$$

(B) $y = 9 - (x - 3)^2 = 9 - x^2 - 9 + 6x$
or $y = 6x - x^2$

$\therefore$ $\frac{dy}{dx} = 6 - 2x$

For minimum or maximum value of y we will substitute $\frac{dy}{dx} = 0$
or $6 - 2x = 0$
$x = 3$

To check whether value of y is maximum or minimum at x = 3 we will have to check whether $\frac{d^2y}{dx^2}$ is positive or negative.

$$\frac{d^2y}{dx^2} = -2$$

or $\frac{d^2y}{dx^2}$ is negative at x = 3. Hence, value of y is maximum. This maximum value of y is,

$$y_{max} = 9 - (3 - 3)^2 = 9$$

22. INTEGRATION

Definitions :

A function F(x) is a antiderivative of a function f(x) if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of integration.

For example $f(x) = x^3$ then $f'(x) = 3x^2$

So the integral of $3x^2$ is x^3

Similarly if $f(x) = x^3 + 4$

there for general integral of $3x^2$ is $x^3 + c$ where c is a constant

One antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way :

$$\int f(x)\,dx = F(x) + C \qquad .....(i)$$

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, "The indefinite integral of f with respect to x is F(x) + C." When we find F(x) + C, we say that we have integrated f and evaluated the integral.

Ex.57 Evaluate $\int 2x\,dx$

Sol. $\int 2x\,dx = \underbrace{x^2}_{\text{an antiderivative of } 2x} + \underbrace{C}_{\text{the arbitrary constant}}$

The formula x^2 + C generatres all the antiderivatives of the function 2x. The function $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function 2x, as you can check by differentiation.

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

Integral Formulas

	Indefinite Integral	**Reversed derivated formula**
1.	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, n ≠ – 1, n rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
	$\int dx = \int 1dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2.	$\int \sin kx\,dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3.	$\int \cos kx dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4.	$\int \sec^2 x dx = \tan x + C$	$\frac{d}{dx}\tan x = \sec^2 x$
5.	$\int \mathrm{cosec}^2 x dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6.	$\int \sec x \tan x dx = \sec x + C$	$\frac{d}{dx}\sec x$ = sec x tan x
7.	$\int \mathrm{cosec}\,x \cot x\,dx$ **=** – cosec x + C	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

Ex.58 *Examples based on above formulas :*

(a) $\int dx = x + c$

(b) $\int x^5 dx = \frac{x^6}{6} + C$ — Formula 1 with n = 5

(c) $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = 2x^{1/2} + C = 2\sqrt{x} + C$ — Formula 1 with n = $-\frac{1}{2}$

(d) $\int \sin 2x\,dx = \frac{-\cos 2x}{2} + C$ — Formula 2 with k = 2

(e) $\int \cos\frac{x}{2} dx = \int \cos\frac{1}{2} x dx = \frac{\sin(1/2)x}{1/2} + C = \int 2\sin\frac{x}{2} + C$ Formula 3 with k = $\frac{1}{2}$

Ex.59 Right : $\int x\cos x\,dx$ = x sin x + cos x + C

Reason : The derivative of the right-hand side is the integrand :

Check : $\frac{d}{dx}(x\sin x + \cos x + C)$ = x cos x + sin x – sin x + 0 = x cos x.

Wrong : $\int x\cos x\,dx$ = x sin x + C

Reason : The derivative of the right-hand side is not the integrand :

Check : $\frac{d}{dx}(x\sin x + C)$ = x cos x + sin x + 0 ≠ x cos x

Rule No. 1 Constant Multiple Rule

- A function is an antiderivative of a constant multiple k of a function f if and only if it is k times an antiderivative of f.

$$\int kf(x)dx = k\int f(x)\,dx$$

Ex.60 $\int 5x^n dx$ **=** $5\int x^n dx$ **=** $\frac{5(x)^{n+1}}{n+1} + c$

Rule No.2 Sum And Difference Rule

A function is an antiderivative of a sum or difference f ± g if and only if it is the sum or difference of an antiderivative of f an antiderivative of g.

$$\int [f(x) \pm g(x)dx] = \int f(x)dx \pm \int g(x)dx$$

Ex.61 Term-by-term integration

Evaluate : $\int (\mathbf{x^2 - 2x + 5})\,\mathbf{dx}$

Sol. If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

arbitrary constant

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + C$$

If we do not recognize the antiderivative right away, we can generate it term by term with the sum and difference Rule :

$$\int (x^2 - 2x + 5)\,dx = \int x^2 dx - \int 2x dx + \int 5dx$$

$$= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3$$

This formula is more complicated than it needs to be. If we combine C_1, C_2 and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x dx + \int 5 dx = \frac{x^3}{3} - x^2 + 5x + C$$

Find the simplest antiderivative you can for each part add the constant at the end.

Ex.62 ***We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals. The inetgral formulas for sin² x and cos² x arise frequently in applications.***

(a) $\int \sin^2 x\, dx \quad = \int \frac{1-\cos 2x}{2} dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$

$$= \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$\frac{x}{2} + \left(-\frac{1}{2}\right)\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $\int \cos^2 x\, dx \quad = \int \frac{1+\cos 2x}{2} dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{x}{2} + \frac{\sin 2x}{4} + C$ As in part (a), but with a sign change

23. SOME INDEFINITE INTEGRALS (AN ARBITRARY CONSTANT SHOULD BE ADDED TO EACH OF THESE INTEGRALS.

(a) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$ (provided n ≠ –1) + C

(b) $\int \frac{1}{x} dn = \ln x + C$

(c) $\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

(e) $\int \sin(ax+b) = \frac{-1}{a}\cos(ax+b) + C$

(f) $\int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + C$

Ex.63 (a) $\int (3x+2)^3 dx = \frac{(3x+2)^4}{4\times 3} + C = \frac{(3x+2)^4}{12} + C$

(b) $\int \frac{2dx}{x} = 2\ln x + C$

(c) $\int \frac{dx}{5+2x} = \frac{1}{2}\ln(5+2x) + C$

(d) $\int \frac{dx}{3-5x} = -\frac{1}{5}\ln(3-5x) + C$

(e) $\int e^{3x} dx = \frac{1}{3}e^{3x} + C$

(f) $\int e^{-x/2} dx = -2e^{-x/2} + C$

(g) $\int \sin(3x+5)dx = -\frac{1}{3}\cos(3x+5) + C$

(h) $\int \cos(2x-5)dx = \frac{1}{2}\sin(2x-5) + C$

24. DEFINITE INTEGRATION OR INTEGRTION WITH LIMITS

Upper limit of integration

The function is the integrand

x is the variable of integration

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of f from a to b}}$$

Lower limit of integration

$$\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$$

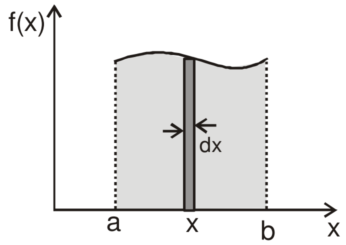
Ex.64 $\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3[x]_{-1}^{4}$ = 3 [4 – (–1)] = (3) (5) = 15

$\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right)$ + cos (0) = – 0 + 1 = 1

Ex.65 (1) $\int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$ (2) $\int_3^5 x dx = \left[\frac{x^2}{2}\right]_3^5 = \frac{5^2 - 3^2}{2} = 8$ (3) $\int_0^b x^{3/2} dx = \left[\frac{x^{5/2}}{5/2}\right]_0^b = \frac{2}{5} b^{5/2}$

25. APPLICATION OF DEFINITE INTERGRAL

Calculation Of Area Of A Curve.



From graph shown in figure if we divide whole area in infinitely small strips of dx width.

We take a strip at x position of dx width.

Small area of this strip dA = f(x) dx

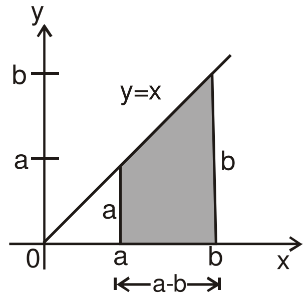
So, the total area between the curve and x-axis = sum of area of all strips = $\int_a^b f(x)dx$

Let f(x) ≥ 0 be continuous on [a,b]. The area of the region between the graph of f and the x-axis is

$$A = \int_a^b f(x)dx$$

Ex.66 ***Using an area to evaluate a definite integral***

Evaluate $\int_a^b xdx$ ***$0 < a < b$.***



The region in Example

Sol. We sketch the region under the curve y = x, $a \le x \le b$ (figure) and see that it is a trapezoid with height (b – a) and bases a and b. The value of the integral is the area of this trapezoid :

Thus $\int_a^b xdx = (b-a)\bullet\frac{a+b}{2} = \frac{b^2}{2}-\frac{a^2}{2}$

$$\int_1^{\sqrt{5}} xdx = \frac{(\sqrt{5})^2}{2}-\frac{(1)^2}{2} = 2$$

and so on.

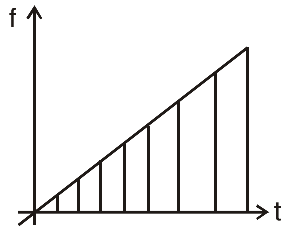
Notice that $x^2/2$ is an antiderivative of x, further evidence of a connection between antiderivatives and summation.

(i) **To find impulse**

$$dF = \frac{dp}{dt} \text{ so imples } = \int F.dt$$

Ex.67 If F = kt then find impulse at t = 3 sec.

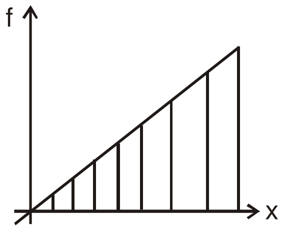
so impulse will be area under f - t curve



$$I = \int_0^3 kt\,dt = K\left[\frac{t^2}{2}\right]_0^3$$

$$\Rightarrow \quad I = \frac{9k}{2}$$

2. **To calculate work done by force :**



$$w = \int \vec{f}.d\vec{x}$$

So area under f - x curve will give the value of work done.