MOTION
IIT-JEE|AIEEE
CBSE|SAT|NTSE
OLYMPIADS
Nurturing potential through education

PHYSICS
TARGET IIT JEE

GRAVITATION

THEORY AND EXERCISE BOOKLET

CONTENTS

Syllabus :

Law of gravitation; Gravitational potential and field; Acceleration due to gravity; Motion of planets and satellites in circular orbits.

INTRODUCTION

Newton observed that an object, an apple, when released near the earth surface is accelerated towards the earth. As acceleration is caused by an unbalanced force, there must be a force pulling objects towards the earth. If someone throws a projectile with some initial velocity, then instead of that object moving off into space in a straight line, it is continosly acted on by a force pulling it back to earth. If we throw the projectile with greater velocity then the path of projectile would be different as well and its range is also increased with initial velocity. If the projection velocity is further increased until at some initial velocity, the body would not hit the earth at all but would go right around it in an orbit. But at any point along its path the projectile would still have a force acting on it pulling it toward the surface of earth.

Newton was led to the conclusion that the same force that causes the apple to fall to the earth also causes the moon to be pulled to the earth. Thus the moon moves in its orbit about the earth because it is pulled toward the earth. But if there is a force between the moon and the earth, why not a force between the sun and the earth or why not a force between the sun and the other planets ? Newton proposed that the same force, named gravitational force which acts on objects near the earth surface also acts on all the heavenly bodies. He proposed that there was a force of gravitation between each and every mass in the universe.

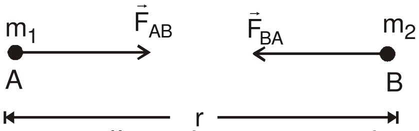
1.1 Newtons's Law of Universal Gravitation

All physical bodies are subjected to the action of the forces of mutual gravitational attraction. The basic law describing the gravitational forces was stated by Sir Issac Newton and it is called Newton's Law. of Universal gravitation.

The law is stated as : "Between any two particles of masses m_I and m_2 at separation r from each other there exist attractive forces $\vec{F}_{AB}$ and $\vec{F}_{BA}$ directed from one body to the other and equal in magnitude which is directly proportional to the product of the masses of the bodies and inversely proportional to the square of the distance between the two". Thus we can write

$$F_{AB} = F_{BA} = G\frac{m_1 m_2}{r^2} \qquad ...(1)$$

Where G is called universal gravitational constant. Its value is equal to 6.67×10^{-11} Nm^2/kg. The law of gravitation can be applied to the bodies whose dimensions are small as compared to the separation between the two or when bodies can be treated as point particles.



If the bodies are not very small sized, we can not directly apply the expression in equation-(1) to find their natural gravitational attraction. In this case we use the following procedure to find the same. The bodies are initially split into small parts or a large number of point masses. Now using equation-(1) the force of attraction exerted on a particle of one body by a particle of another body can be obtained. Now we add all forces vectorially which are exerted by all independent particles of second body on the particle of first body. Finally the resultants of these forces is summed over all particles of the first body to obtain the net force experinced by the bodies. In general we use integration or basic summation of these forces.

⇒ Gravitational force is a conservative force.
⇒ Gravitational force is a central force.
⇒ Gravitational force is equal in magnitude & opposite in direction
⇒ Gravitational forces are action - reaction pair.
⇒ Gravitational force acts along the line joining the two masses.
⇒ Gravitational force doesn't depend upon the medium
⇒ Gravitational force is an attractive force.

$$\vec{F} = \frac{-Gm_1 m_2 \vec{r}}{|\vec{r}|^3}$$

[Head of $\vec{r}$ is placed at that position where we have to evaluate force]

2. GRAVITATIONAL FIELD

We can state by Newton's universal law of gravitation that every mass M produces, in the region around it, a physical situation in which, whenever any other mass is placed, force acts on it, is called gravitational field. This field is recognized by the force that the mass M exerts another mass, such as *m*, brought into the region.

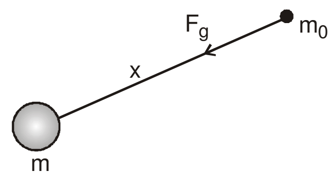
2.1 Strength of Gravitational Field

We define gravitational field strength at any point in space to be the gravitational force per unit mass on a test mass (mass brought into the field for experimental observation). Thus for a point in space if a test mass m_0, experiences a force $\vec{F}$, then at that point in space, gravitational field strength which is denoted by $\vec{g}$, is given as $\vec{g} = \frac{\vec{F}}{m_0}$

Gravitational field strength $\vec{g}$ is a vector quantity and has same direction as that of the force on the test mass in field.

Generally magnitude of test mass is very small such that its gravitational field does not modify the field that is being measured. It should be also noted that gravitational field strength is just the acceleration that a unit mass would experience at that point in space.

2.2 Gravitational Field Strength of Point Mass



As per our previous discussion we can state that every point mass also produces a gravitational field in its surrounding. To find the gravitational field strength due to a point mass, we put a test mass m_0 at a point P at distance x from a point mass m then force on m_0 is given as

$$F_g = \frac{Gmm_0}{x^2}$$

Now if at point P, gravitational field strength due to m is g_p then it is given as

$$g_p = \frac{F_g}{m_0} = \frac{Gm}{x^2}$$

The expression written in above equation gives the gravitational field strength at a point due to a point mass.

It should be noted that the expression in equation written above is only applicable for gravitational field strength due to point masses. It should not be used for extended bodies.

However, the expression for the gravitational field strength produced by extended masses has already been derived in electrostatics section.

[Just replace k by –G & Q by M in those expression]

So we will just revise the expression of gravitational field strength at points due to various extended masses. Gravitational field strength :

1. At a point on the axis of Ring = $\frac{-GMx}{(x^2+R^2)^{3/2}}$

2. At a point on the axis of disc = $\frac{2GM}{R^2}\left[1-\frac{x}{\sqrt{R^2+x^2}}\right]$

3. At an axial point of a rod = $\frac{-GM}{L}\left[\frac{1}{x}-\frac{1}{x+L}\right]$

4. Due to a circular Arc = $\frac{-2GM\sin(\phi/2)}{\phi R^2}$

5. Due to a long infinite thread = $\frac{-2G\lambda}{x}$

6. Due to long solid cylinder

(a) at an outer point = $\frac{-2G\rho\pi R^2}{x}$ (where ρ is mass density per volume)

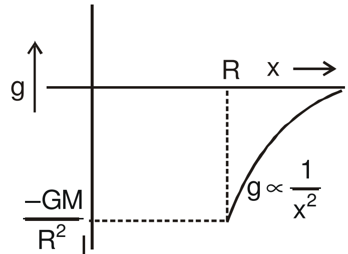
(b) at an inner point = $-2G\rho\pi x$

7. Due to hollow sphere :

(a) for outer points = $\frac{-GM}{x^2}$ (Behaving as a point mass)

(b) for points on surface = $\frac{-GM}{R^2}$ (Behaving as a point mass)

(c) for inner points = 0 (As no mass is enclosed within it)

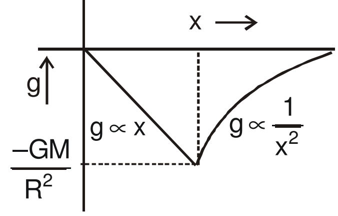


8. Due to solid sphere

(a) For outer points = $\frac{-GM}{x^2}$ (Behaving as a point mass)

(b) For points on surface = $\frac{-GM}{R^2}$ (Behaving as a point mass)

(c) For inner points = $\frac{-GMx}{R^3}$

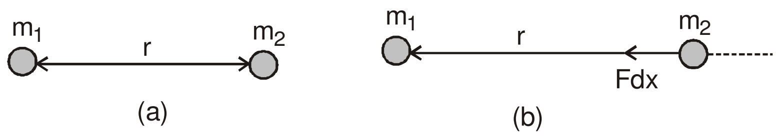


3. INTERACTION ENERGY

This energy exists in a system of particles due to the interaction forces between the particles of system. Analytically this term is defined as the work done against the interaction of system forces in assembling the given configuration of particles. To understand this we take a simple example of interaction energy of two points masses.

Figure (a) shows a system of two point masses m_1 and m_2 placed at a distance r apart in space. here if we wish to find the interaction potential energy of the two masses, this must be the work done in bringing the two masses from infinity (zero interaction state) to this configuration. For this we first fix m_1 at its position and bring m_2 slowly from infinity to its location. If in the process m_2 is at a distance x from m_1 then force on it is

$$F = -\frac{Gm_1m_2}{x^2}\hat{i}$$



This force is applied by the gravitational field of m_1 to m_2. If it is further displaced by a distance dx towards m_1 then work done by the field is

$$dW = \vec{F}.\vec{dx} = \frac{Gm_1m_2}{x^2}dx$$

Now in bringing m_2 from infinity to a position at a distance r from $m_{1,}$ the total work done by the field is

$$W = \int dW - \int_{\infty}^{r} \frac{Gm_1m_2}{x^2}dx = -G\, m_1\, m_2 \left[-\frac{1}{x}\right]_{\infty}^{r}$$

$$W = +\frac{Gm_1m_2}{r}$$

Thus during the process field of system has done $\frac{Gm_1m_2}{r}$ amount of work. The work is positive because the displacment of body is in the direction of force.

Initially when the separation between m_1 and m_2 was very large (at infinity) there was no interaction between them. We conversely say that as a reference when there is no interaction the interaction energy of the system is zero and during the process system forces (gravitational forces) are doing work so system energy will decrease and becomes negative (as initial energy was zero). As a consequence we can state that in general if system forces are attractive, in assembling a system of particles work will be done by the system and it will spend energy in assembling itself. Thus finally the interaction energy of system will be negative. On the other hand if in a given system of particles, the system forces are repulsive, then in assembling a system some external forces have to be work against the system forces and in this case some work must be done by external forces on the system hence finally the interaction energy of the system of particles must be positive.

In above example as work is done by the gravitaional forces of the system of two masses, the interaction energy of system must be negative and it can be given as

$$U_{12} = -\frac{Gm_1m_2}{r} \qquad ...(1)$$

As gravitational forces are always attractive, the gravitational potential energy is always taken negative.

3.1 Interaction Energy of a System of Particles

If in a system there are more than two particles then we can find the interaction energy of particle in pairs using equation (1) and finally sum up all the results to get the total energy of the system. For example in a system of N particles with masses m_1, m_2........m_n separated from each other by a distance r_{12}........ where r_{12} is the separation between m_1 and m_2 and so on.

In the above case the total interaction energy of system is given as

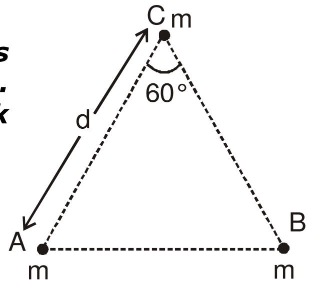
$$U = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_im_j}{r_{ij}}$$

In this expression the factor $\frac{1}{2}$ is taken because the interaction energy for each possible pair of system is taken twice during summation as for mass m_1 and m_3

$$U = -\frac{Gm_1m_3}{r_{13}} = -\frac{Gm_3m_1}{r_{31}}$$

Now to understand the applications of interaction energy we take few examples.

Ex.1 ***Three particles each of the mass m are placed at the corners of an equilateral triangle of side d and shown in figure. Calculate (a) the potential energy of the system, (b) work done on this system if the side of the triangle is changed from d to 2d.***



Sol. (a) As in case of two-particle system potential energy is given by $(-Gm_1m_2/r)$, so

$$U = U_{12} + U_{23} + U_{31} \quad \text{or} \quad U_i = -3\frac{Gmm}{d} = -\frac{3Gm^2}{d}$$

(b) When d is changed to 2d,

$$U_f = -\frac{3Gm^2}{2d}$$

Thus work done in changing in potential energy is given as

$$W = U_f - U_i = \frac{3Gm^2}{2d}$$

Ex.2 ***Two particles m_1 and m_2 are initially at rest at infinite distance. Find their relative velocity of approach due to gravitational atraction when their separation is d.***

Sol. Initiallly when the separation was large there was no interaction energy and when they get closer the system gravitational energy decreases and the kinetic energy increases.

When separation between the two particles is d, then according to energy conservation we have

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{d} = 0$$

As no other force is present we have according to momentum conservation

$$m_1v_1 = m_2v_2$$

From equations written above

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}\frac{m_1^2}{m_2}v_1^2 = \frac{Gm_1m_2}{d} \quad \text{or} \quad v_1 = \sqrt{\frac{2Gm_2^2}{d(m_1+m_2)}} = \sqrt{\frac{2G}{d(m_1+m_2)}}m_2$$

And on further solving we get

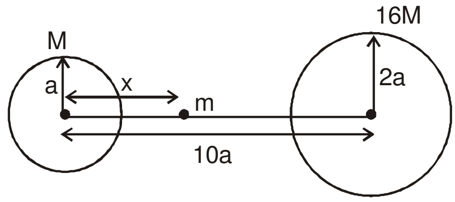
$$v_2 = \sqrt{\frac{2G}{d(m_1 + m_2)}}\, m_1$$

Thus approach velocity is given as

$$v_{ap} = v_1 + v_2 = \sqrt{\frac{2G(m_1 + m_2)}{d}}$$

Ex.3 ***If a particle of mass 'm' is projected from a surface of bigger sphere of mass '16M' and radius '2a' then find out the minimum velocity of the paticle such that the particle reaches the surface of the smaller sphere of mass M and radius 'a'. Given that the distance between the centres of two spheres is 10 a.***

Sol. When the particle is at the surface of bigger sphere it is attracted more by the bigger sphere and less by the smaller sphere. As it is projected the force of attraction from bigger sphere decreases and that from smaller sphere increass and thus the particle reaches the state of equilibrium at distance x from the centre smaller sphere



$$\frac{GMm}{x^2} = \frac{G(16M)m}{(10a - x)^2}$$

$(10a - x)^2 = 16x^2$

$10a - x = 4x \Rightarrow \quad x = 2a$

After this point the attraction on the particle from the smaller sphere becomes more than that from the bigger sphere and the particle will automatically move towards the smaller sphere. Hence the minimum velocity to reach the smaller sphere is the veloicty required to reach the equilibrium state according to energy conservation, we have,

$$-\frac{G(16M)m}{2a} - \frac{GMm}{8a} + \frac{1}{2}mv^2 = \frac{-G(16M)m}{8a} - \frac{GMm}{2a}$$

$$v^2 = \frac{45GM}{4a} \quad \Rightarrow \quad v = \sqrt{\frac{45GM}{4a}}$$

4 GRAVITATIONAL POTENTIAL

The gravitational potential at a point in gravitational field is the gravitional potenial energy per unit mass placed at that point in gravitational field. Thus at a certain point in gravitational field, a mass m_0 has a potential energy U then the gravitational potential at that point is given as

$$V = \frac{U}{m_0}$$

or if at a point in gravitational field gravitational potential V is known then the interaction potential energy of a point mass m_0 at that point in the field is given as

$$U = m_0 v$$

Interaction energy of a point mass m_0 in a field is defined as work done in bringing that mass from infinity to that point. In the same fashion we can define gravitational potential at a point in field, alternatively as "*Work done in bringing a unit mass from infinity to that point against gravitational forces.*"

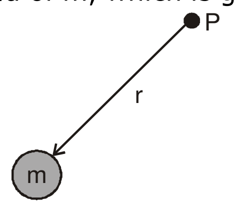
When a unit mass is brought to a point in a gravitational field, force on the unit mass is $\vec{g}$ at a point in the field. Thus the work done in bringing this unit mass from infinity to a point P in gravitational field or gravitational potential at point P is given as

$$V_P = -\int_{\infty}^{P} \vec{g} \cdot \vec{dx}$$

Here negative sign shown that V_P is the negative of work done by gravitation field or it is the external required work for the purpose against gravitational forces.

4.1 Gravitational Potential due to a Point Mass

We know that in the surrounding of a point mass it produces its gravitational field. If we wish to find the gravitational potential at a point P situated at a distance r from it as shown in figure, we place a test mass m_0 at P and we find the interaction energy of m_0 with the field of m, which is given as



$$U = -\frac{Gmm_0}{r}$$

Now the gravitational potential at P due to m can be written as

$$V = \frac{U}{m_0} = -\frac{Gm}{r}$$

The expression of gravitational potential in equation is a standard result due to a point mass which can be used as an elemental form to find other complex results, we'll see later.

The same thing can also be obtained by using equation

$$V_P = \int_{\infty}^{P} \vec{g}\,.\,\vec{dx} \quad \text{or} \quad V_P = \int_{\infty}^{r} \frac{Gm}{x^2}dx \quad \text{or} \quad V_P = -\frac{Gm}{r}$$

4.2 Gravitational potential

1. Due to a rod at an axial point = $-G\lambda \ln\left(\frac{a+l}{a}\right)$

2. Due to ring at an axial point = $\frac{-GM}{\sqrt{R^2+x^2}}$

3. Due to ring at the centre = $\frac{-GM}{R}$

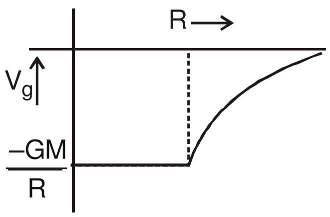
4. Due to Disc = $-G\sigma 2\pi[\sqrt{R^2+x^2} - x]$ (where σ is mass density per unit area)

5. Due to hollow sphere

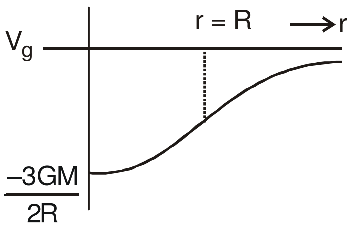
for outer points = $\frac{-GM}{r}$

For surface points = $\frac{-GM}{R}$

For inner points = $-\frac{GM}{R}$



6. Due to solid sphere



For outer points = $\frac{-GM}{r}$

For surface points = $\frac{-GM}{R}$

For inner points = $\frac{-GM}{2R^3}(3R^2 - r^2)$

Potential energy of hollow sphere = $\frac{-GM^2}{2R}$

Potential energy of solid sphere = $\frac{-3GM^2}{5R}$

5. GRAVITATIONAL LINES OF FORCES

Gravitational field can also be represented by lines of force. A line of force is drawn in such a way that at each point the direction of field is tangent to line that passes through the point. Thus tangent to any point on a line of force gives the direction of gravitational field at that point. By convention lines of force are drawn in such a way that their density is proportional to the strength of field. Figure shown shows the field of a point mass in its surrounding. We can see that the lines of force are radially inward giving direction of field and as we go closer to the mass the density of lines is more which shows that field strength is increasing.

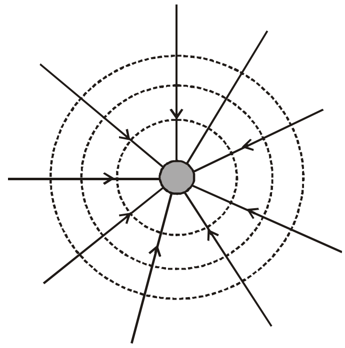
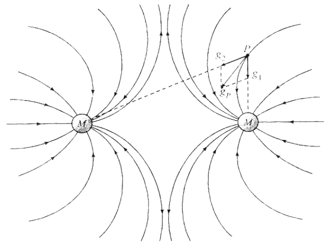


Figure shown shows the configuration of field lines for a system of two equal masses separated by a given distance.



Here we can see that there is no point where any two lines of force intersects or meet. The reason is obvious that at one point in space there can never be two direction of gravitational fields. It should be noted that a line of force gives the direction of net gravitational field in the region. Like electric field gravitational field never exists in closed loops.

- **Gravitational Flux :** $\phi = \int \vec{g}.\vec{ds}$
- **Gravitational Gauss law :** $\oint \vec{g}.\vec{ds} = -4\pi G M_{in}$

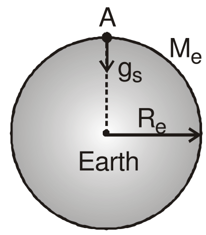
Here $\vec{g}$ is the gravitational field due to all the masses. M_{in} is the mass inside the assumed gaussian surface.

5.1 Gravitational Field Strength of Earth:

We can consider earth to be a very large sphere of mass M_e and radius R_e. Gravitational field strength due to earth is also regarded as acceleration due to gravity or gravitational acceleration. Now we find values of g at different points due to earth.

- Earth behaves as a non conducting solid sphere

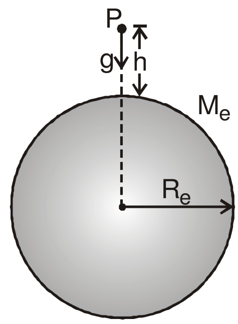
5.2 Value of g on Earth's Surface :



If g_s be the gravitational field strength at a point A on the surface of earth, then it can be easily obtained by using the result of a solid sphere. Thus for earth, value of g_s can be given as

$$g_s = \frac{GM_e}{R_e^2} \qquad ...(1)$$

5.3 Value of g at a Height *h* Above the Earth's Surface:



If we wish to find the value of g at a point P as shown in figure shown at a height h above the Eath's surface. Then the value can be obtained as

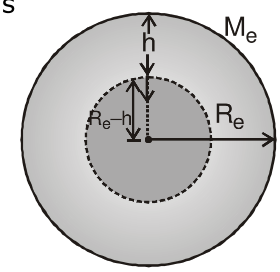
$$g_s = \frac{GM_e}{(R_e + h)^2} \quad \text{or} \quad g_s = \frac{GM_e}{R_e^2\left(1+\frac{h}{R_e}\right)^2} = \frac{g_s}{\left(1+\frac{h}{R_e}\right)^2}$$

If point P is very close to Earth's surface then for h << R_e we can rewrite the expression in given equation as

$$g_h = g_s\left(1+\frac{h}{R_e}\right)^{-2} \simeq g_s\left(1-\frac{2h}{R_e}\right) \quad \text{[Using binomial approximation]} \qquad ...(2)$$

5.4 Value of g at a Depth *h* Below the Earth's Surface

If we find the value of g inside the volume of earth at a depth *h* below the earth's surface at point P as shown in figure, then we can use the result of *g* inside a solid sphere as



$$g_{in} = \frac{GM_e x}{R_e^3}$$

Here x, the distance of point from centre of earth is given as

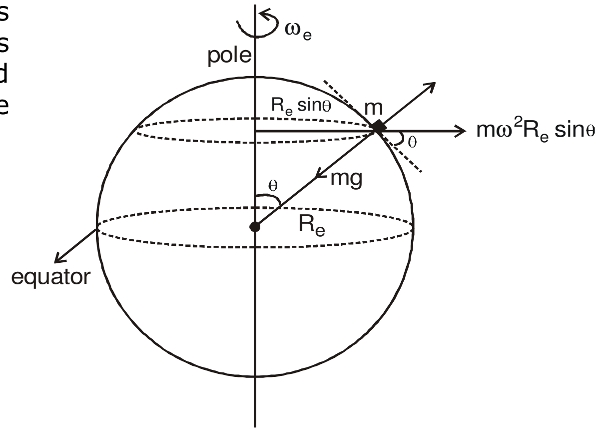
$$x = R_e - h$$

Thus we have $g_h = \frac{GM_e(R_e - h)}{R_e^3} = g_s\left(1 - \frac{h}{R_e}\right)$ ...(3)

From equation (1), (2) and (3) we can say that the value of *g* at eath's surface is maximum and as we move above the earth's surface or we go below the surface of earth, the value of *g* decrease.

5.5 Effect of Earth's Rotation on Value of *g*

Let us consider a body of mass *m* placed on Earth's surface at a latitude θ as shown in figure. This mass experiences a force mg_s towards the centre of earth and a centrifugal force $mw_e^2 R_e \sin\theta$ relative to Earth's surface as shown in figure.



If we consider g_{eff} as the effective value of *g* on earth surface at a latitude θ then we can write

$$g_{eff} = \frac{\vec{F}_{net}}{m} = g_{eff} = \sqrt{(\omega_e^2 R_e \sin\theta)^2 + g^2 + 2\omega_e^2 R_e \sin\theta . g\cos(90+\theta)}$$

ω_e^4 is very very small

So we can write $g_{eff} = \sqrt{g^2 - 2\omega_e^2 R_e \sin^2\theta g}$

$$g_{eff} = g\left(1 - \frac{2\omega_e^2 R_e \sin^2\theta}{g}\right)^{1/2} \approx g - \omega_e^2 R \sin^2\theta \qquad ...(i)$$

From equation (1) we can find the value of effective gravity at poles and equatorial points on Earth as

At poles $\theta = 0 \Rightarrow$ $g_{poles} = g_s = 9.83 \text{ m/s}^2$

At equator $\theta = \frac{\pi}{2} \Rightarrow g_{equator} = g_s - \omega^2 R_e = 9.78 \text{ m/s}^2$

Thus we can see that the body if placed at poles of Earth, it will only have a spin, not circular motion so there is no reduction in value of *g* at poles due to rotation of earth. Thus at poles value of g on Earth surface is maximum and at equator it is minimum. But an average we take 9.8 m/s^2, the value of g everywhere on earth's surface.

5.6 Effect of Shape of Earth on Value of g

Till now we considered that earth is spherical in its shape but this is not actually true. Due to some geological and astromonical reasons, the shape of earth is not exact spherical. It is ellipsoidal.

As we've discussed that the value of *g* at a point on earth surface depends on radius of Earth.It is observed that the approximate difference in earth's radius at different points on equator and poles is $r_e - r_p \simeq$ 21 to 34 km. Due to this the difference in value of g at poles and equatorial points is approximately $g_p - g_e \simeq$ 0.02 to 0.04 m/s^2, which is very small. So for numerical calculations, generally, we ignore this factor while taking the value of *g* and we assume Earth is spherical in shape.

Ex.4 ***Calculate the mass and density of the earth. Given that Gravitational constant G = 6.67 × 10^{-11} Nm^2/kg^2, the radius of the earth = 6.37 × 10^6 m and g = 9.8 m/s^2.***

Sol. The acceleration due to gravity on earth surface is given as

$$g_e = \frac{GM_e}{R_e^2} \quad \text{or} \quad M_e = \frac{g_s R_e^2}{G} = \frac{9.8 \times (6.37 \times 10^6)^2}{6.67 \times 10^{-11}} = 6 \times 10^{24} \text{ kg}$$

If ρ be the density of earth, then

$$M = \frac{4}{3}\pi R^3 \times \rho \quad \text{or} \quad \rho = \frac{3M}{4\pi R^3} = \frac{3 \times (6 \times 10^{24})}{4 \times 3.14 \times (6.37 \times 10^6)^3} = 5.5 \times 10^3 \text{ kg/m}^3$$

Ex.5 ***If the radius of the earth were to shrink by one percent, its mass remaining the same, what would happen to the acceleration due to gravity on the earth's surface?***

Sol. Consider the case of body of mass *m* placed on the earth's surface (mass of the earth M and radius R). If *g* is acceleration due to gravity, then we known that

$$g_s = \frac{GM_e}{R_e^2} \quad ...(1)$$

Now, when the radius is reduced by 1%, i.e. radius becomes 0.99 R, let acceleration due to gravity be g', then

$$g' = \frac{GM}{(0.99R)^2} \quad ...(2)$$

From equation (1) and (2), we get

$$\frac{g'}{g} = \frac{R^2}{(0.99R)} = \frac{1}{(0.99)^2} \quad \text{or} \quad g' = g \times \left(\frac{1}{0.99}\right)^2 \quad \text{or} \quad g' = 1.02\text{ g}$$

Thus, the value of *g* is increased by 2%.

Ex.6 ***At what rate should the earth rotate so that the apparent g at the equator becomes zero? What will be the length of the day in this situation ?***

Sol. At earth's equator effective value of gravity is

$$g_{eq} = g_s - \omega^2 R_e$$

If g_{eff} at equator is zero, we have

$$g_s = \omega^2 R_e \quad \text{or} \quad \omega = \sqrt{\frac{g_s}{R_e}}$$

Thus length of the day will be $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_e}{g_s}} = 2 \times 3.14 \sqrt{\frac{6.4 \times 10^6}{9.8}} = 5074.77\text{ s} \simeq 84.57$ min.

Ex.7 ***Calculate the acceleration due to gravity at the surface of Mars if its diameter is 6760 km and mass is one-tenth that of earth. The diameter of earth is 12742 km and acceleration due to gravity on earth is 9.8 m/s^2.***

We know that $g = \left(\frac{GM}{R^2}\right)$

So $$\frac{g_M}{g_E} = \left(\frac{M_M}{M_E}\right)\left(\frac{R_E}{R_M}\right)^2 = \left(\frac{1}{10}\right)\left(\frac{12742}{6760}\right)^2 \Rightarrow \frac{g_M}{g_E} = 0.35 \quad \text{or } g_M = 9.8 \times 0.35 = 3.48 \text{ m/s}^2$$

Ex.8 ***Calculate the apparent weight of a body of mass m at a latitude λ when it is moving with speed v on the surface of the earth from west to east at the same latitude.***

Sol. If W be the apparent weight of body at a latitude λ then from figure shown, we have

$$W = mg - m\omega^2 R\cos^2\lambda \quad ...(1)$$

When body moves at speed v from west to east relative to earth, its net angular speed ω can be given as

$\omega = \omega_e + \frac{v}{R\cos\lambda}$ [$\omega_e \to$ earth's angular velocity]

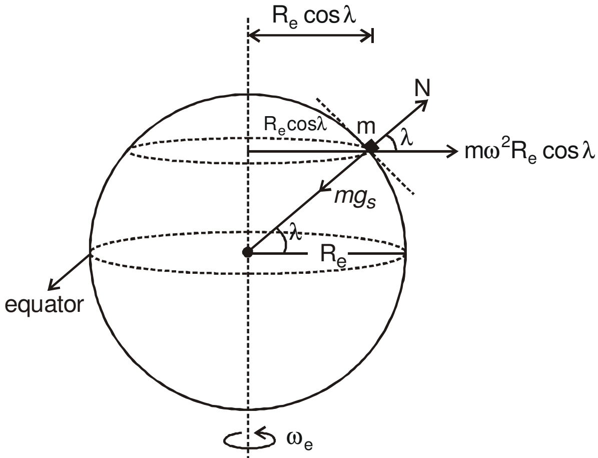
Now from equation (1) we have

$$W = mg - m\left[\omega_e + \frac{v}{R\cos\lambda}\right]^2 R\cos^2\lambda$$

or $$W = mg - m\left[\omega_e^2 + \frac{v^2}{R^2\cos^2\lambda} + \frac{2\omega_e v}{R\cos\lambda}\right] R\cos^2\lambda$$

$$= mg - m\omega_e^2 R\cos^2\lambda - \frac{mv^2}{R} - 2m\omega_e v\cos\lambda$$

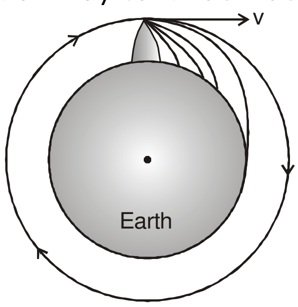
$$= mg\left(1 - \frac{w_e^2 R\cos^2\lambda}{g} - \frac{2\omega_e v\cos\lambda}{g}\right)$$ [Neglecting $\frac{mv^2}{R}$ as being very small]



6. SATELLITE AND PLANETARY MOTION

6.1 Motion of a Satellite in a Circular Orbit

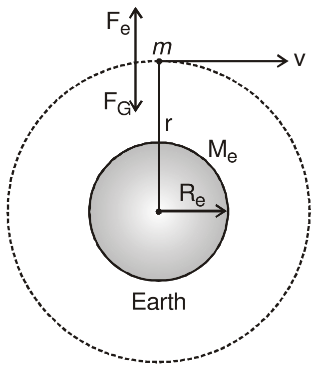
To understand how a satellite continously moves in its orbit, we consider the projection of a body horizontally from the top of a high mountain on earth as shown in figure. Here till our discussion ends we neglect air friction. The distance the projectile travels before hitting the ground depends on the launching speed. The greater the speed, the greater the distance. The distance the projectile travels before hitting the ground is also affected by the curvature of earth as shown in figure shown. This figure was given by newton in his explanantion of laws of gravitation. it shows different trajectories for diferent launching speeds. As the launching speed is made greater, a speed is reached where by the projectile's path follow the curvature of the earth. This is the launching speed which places the projectile in a circular orbit. Thus an object in circular orbit may be regarded as falling, but as it falls its path is concentric with the earth's spherical surface and the object maintains a fixed distance from the earth's centre. Since the motion may continue indefinitely, we may say that the orbit is stable.



Let's find the speed of a satellite of mass *m* in a circular orbit around the earth. Consider a satellite revolving around the earth in a circular orbit of radius r as shown in figure.

If its orbit is stable during its motion, the net gravitational force on it must be balanced by the centrifugal force on it relative to the rotating frame as

$$\frac{GM_e m}{r^2} = \frac{mv^2}{r} \quad \text{or} \quad v = \sqrt{\frac{GM_e}{r}}$$



Expression in above equation gives the speed of a statellite in a stable circular orbit of radius r.

6.2 Energies of a Satellite in a Circular Orbit

When there is a satellite revolving in a stable circular orbit of radius *r* around the earth, its speed is given by above equation. During its motion the kinetic energy of the satellite can be given as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GM_e m}{r} \quad ...(1)$$

As gravitational force on satellite due to earth is the only force it experiences during motion, it has gravitational interaction energy in the field of earth, which is given as

$$U = -\frac{GM_e m}{r} \qquad ...(2)$$

Thus the total energy of a satellite in an orbit of radius r can be given as

Total energy E = Kinetic energy K + Potential Energy U

$$= \frac{1}{2}\frac{GM_e m}{r} - \frac{GM_e m}{r} \quad \text{or} \quad E = -\frac{1}{2}\frac{GM_e m}{r} \qquad ...(3)$$

From equation (1), (2) and (3) we can see that $|k| = \left|\frac{U}{2}\right| = |E|$

The above relation in magnitude of total, kinetic and potential energies of a satelline is very useful in numerical problem so it is advised to keep this relation in mind while handing satellite problems related to energy.

Now to understand satellite and planetary motion in detail, we take few example.

Ex.9 ***Estimate the mass of the sun, assuming the orbit of the earth round the sun to be a circle. The distance between the sun and earth is 1.49 × 10¹¹ m and G = 6.66 × 10⁻¹¹ Nm²/kg².***

Sol. Here the revolving speed of earth can be given as

$$v = \sqrt{\frac{GM}{r}} \qquad \text{[Orbital speed]}$$

Where M is the mass of sun and r is the orbit radius of earth.

We known time period of earth around sun is T = 365 days, thus we have

$$T = \frac{2\pi r}{v} \quad \text{or } T = 2\pi r\sqrt{\frac{r}{GM}} \quad \text{or } M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\times(3.14)^2\times(1.49\times10^{11})^3}{(365\times24\times3600)^2\times(6.66\times10^{-11})} = 1.972 \times 10^{22}\text{ kg}$$

Ex.10 ***If the earth be one-half of its present distance from the sun, how many days will be in one year ?***

Sol. If orbit of earth's radius is R, in previous example we've discussed that time period is given as

$$T = 2\pi r\sqrt{\frac{r}{Gm}} = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

If radius changes or $r' = \frac{r}{2}$, new time period becomes

$$T' = \frac{2\pi}{\sqrt{GM}} r'^{3/2}$$

From above equations, we have

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2} \quad \text{or} \quad T' = T\left(\frac{r'}{r}\right)^{3/2} = 365\left(\frac{1}{2}\right)^{3/2} = \frac{365}{2\sqrt{2}}\text{ days}$$

Ex.11 ***A satellite revolving in a circular equatorial orbit of radius r = 2.00 × 10⁴ km from west to east appear over a certain point at the equator every t = 11.6 hours. Using this data, calculate the mass of the earth. The gravitational constant is supposed to be known.***

Sol. Here the absolute angular velocity of satellite is given by

$$\omega = \omega_s + \omega_E$$

Where ω_E is the angular velocity of earth, which is from west to east.

or $$\omega = \frac{2\pi}{t} + \frac{2\pi}{T} \qquad \text{[Where t = 11.6 hr. and T = 24 hr.]}$$

From Kepler's III law, we have $$\omega = \frac{\sqrt{GM}}{r^{3/2}}$$

Thus we have $\frac{\sqrt{GM}}{r^{3/2}} = \frac{2\pi}{t} + \frac{2\pi}{T}$

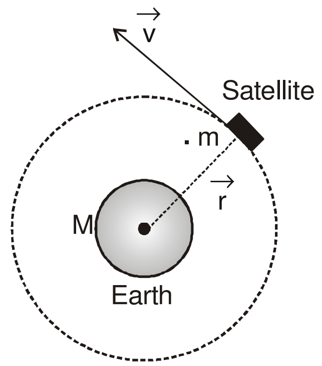
or $M = \frac{4\pi^2 r^3}{G}\left[\frac{1}{t} + \frac{1}{T}\right]^2 = \frac{4\pi^2 (2\times10^7)^3}{(6.67\times10^{-11})}\left[\frac{1}{11.6\times3600} + \frac{1}{24\times3600}\right]^2$ = 6.0 × 10^{24} kg

Ex.12 ***A satellite of mass m is moving in a circular orbit of radius r. Calculate its angular momentum with respect to the centre of the orbit in terms of the mass of the earth.***

Sol. The situation is shown in figure

The angular momentum of the satellite with respect to the centre of orbit is given by

$$\vec{L} = \vec{r} \times m\,\vec{v}$$



Where $\vec{r}$ is the position vector of satellite with respect to the centre of orbit and $\vec{v}$ is its velocity vector of satellite.

In case of circular orbit, the angle between $\vec{r}$ and $\vec{v}$ is 90°. Hence

$$L = m\,v\,r \sin 90° = m\,v\,r \quad ...(1)$$

The direction is perpendicular to the plane of the orbit.

We know orbital speed of satellite is

$$v = \sqrt{\frac{GM}{r}} \quad ...(2)$$

From equation (1) and (2), we get

$$L = m\sqrt{\frac{GM}{r}} \Rightarrow L = (GMm^2 r)^{1/2}$$

Now we will understand the concept of ***double star system*** through an example.

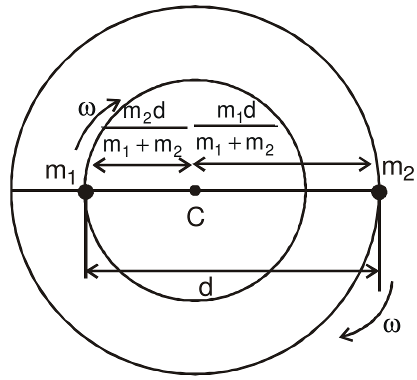
Ex.13 ***In a double star, two stars of masses m_1 and m_2. distance d apart revolve about their common centre of mass under the influence of their mutual gravitational attraction. Find an expression for the period T in terms of masses m_1, m_2 and d. Find the ratio of their angular momenta about centre of mass and also the ratio of their kinetic energies.***

Sol. The centre of mass of double star from mass m_1 is given by

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 \times 0 + m_2 d}{m_1 + m_2} = \frac{m_2 d}{m_1 + m_2}$$

∴ Distance of centre of mass from m_2 is

$$r'_{cm} = d - r_{cm} = d - \frac{m_2 d}{m_1 + m_2} = \frac{m_1 d}{m_1 + m_2}$$



Both the stars rotate around centre of mass in their own circular orbits with the same angular speed ω. the gravitational force acting on each star provides the necessary centripetal force. if we consider the rotation of mass m_1, then

$$m_1 (r_{cm})\omega^2 = \frac{Gm_1 m_2}{d^2} \quad \text{or} \quad m_1\left(\frac{m_2 d}{m_1 + m_2}\right)\omega^2 = \frac{Gm_1 m_2}{d^2}$$

This gives $\omega = \frac{2\pi}{T} = \sqrt{\left(\frac{G(m_1 + m_2)}{d^3}\right)}$

or Period of revolution

$$T = 2\pi\sqrt{\left(\frac{d^3}{G(m_1 + m_2)}\right)}$$

Ratio of Angular Momenta is

$$\frac{J_1}{J_2} = \frac{I_1\omega}{I_2\omega} = \frac{I_1}{I_2} = \frac{m_1\left(\frac{m_2 d}{m_1 + m_2}\right)^2}{m_2\left(\frac{m_1 d}{m_1 + m_2}\right)^2} = \frac{m_2}{m_1}$$

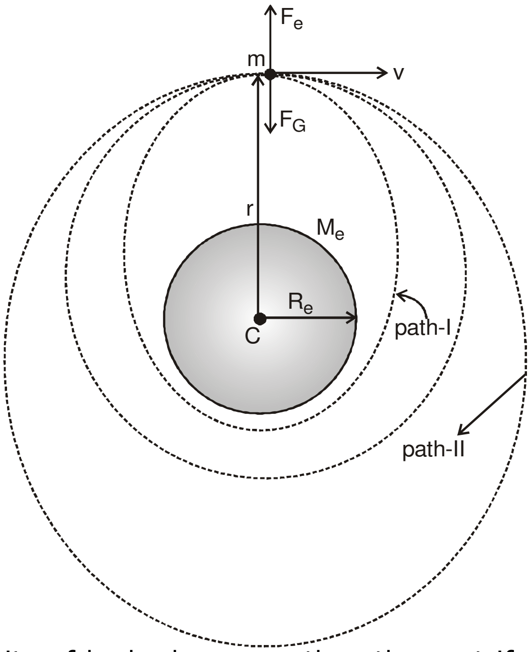
Ratio of kinetic energies is

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}I_1\omega^2}{\frac{1}{2}I_2\omega^2} = \frac{I_1}{I_2} = \frac{m_2}{m_1}$$

7. MOTION OF A SATELLITE IN ELLIPTICAL PATH

Whenever a satellite is in a circular or elliptical path, these orbits are called bounded orbits as satellite is moving in an orbit bounded to earth. The bound nature of orbit means that the kinetic energy of satellite is not enough at any point in the orbit to take the satellite to infinity. In equation shown negative total energy of a revolving satellite shows its boundness to earth. Even when a body is in elliptical path around the earth, its total energy must be negative. Lets first discuss how a satellite or a body can be in elliptical path.

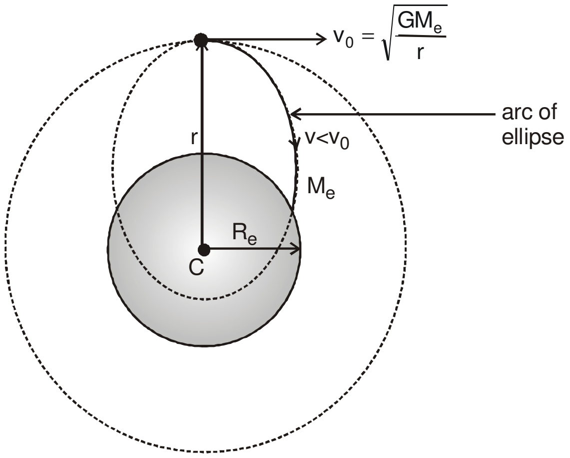
Consider a body (satellite) of mass *m* in a circular path of radius r around the earth as shown in figure. we've discussed that in circular path the net gravitational frame on body is exactly balancing the centrifugal force on it in radial direction relative to a rotating frame with the body.



If suddenly the velocity of body decreases then the centrifugal force on it becomes less then the gravitational force acting on it and due to this it can not continue in the circular orbit and will come inward from the circular orbit due to unbalanced force. Mathematical analysis shown that this path-I along which the body is now moving is an ellipse. The analytical calculations of the laws for this path is beyond the scope of this book. But it should be kept in mind that if velocity of a body at a distance r from earth's centre tangential to the circular orbit is less than $\sqrt{\frac{GM_e}{r}}$ then its path will be elliptical with earth centre at one of the foci of the ellipse.

Similarly if the speed of body exceeds $\sqrt{\frac{GM_e}{r}}$ then it must move out of the circular path due to unbalancing of forces again but this time $F_e > F_g$. Due to this if speed of body is not increased by such a value that its kinetic energy can take the particle to infinity then it will follow in a bigger elliptical orbit as shown in figure in path-II, with earth's at one of the foci of the orbit.

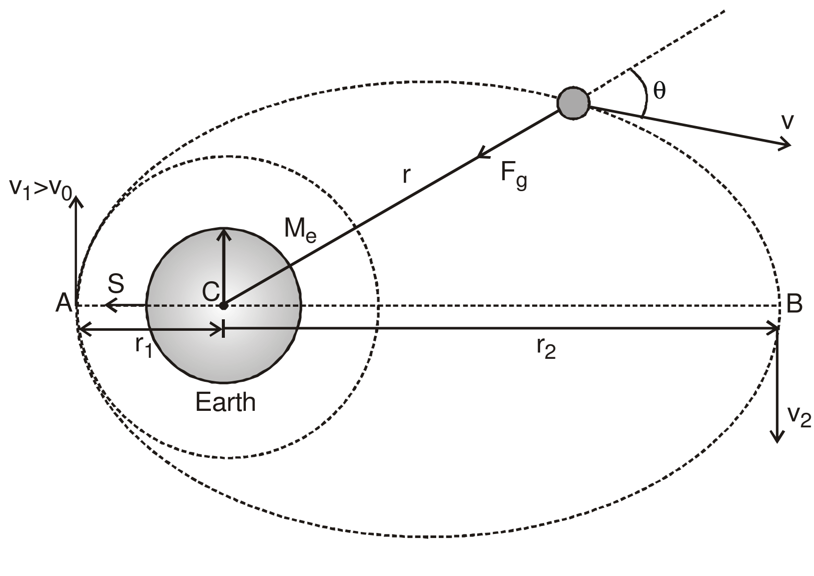
In above case when speed of body was decreased and its value is lesser than $\sqrt{\frac{GM_e}{r}}$ and the speed is decreased to such a value that the elliptical orbit will intersect the earth's surface as shown in figure, then body will follow an arc of ellipse and will fall back to earth.



7.2 Satellite Motion and Angular Momentum Conservation

We've discussed that when a body is in bounded orbit around a planet it can be in circular or elliptical path depending on its kinetic energy at the time of launching. Lets consider a case when a satellite is launched in an orbit around the earth.

A satellite S is first fired away from earth source in vertical direction to penetrate the earth's atmosphere. When it reaches point A, it is imparted a velocity in tangential direction to start its revolution around the earth in its orbit.



This velocity is termed as insertion velocity, if the velocity imparted to satellite is $v_0 = \sqrt{\frac{GM_e}{r_1}}$ then it starts following the circular path shown in figure. If velocity imparted is $v_1 > v_0$ then it will trace the elliptical path shown. During this motion the only force acting on satellite is the gravitational force due to earth which is acting along the line joining satellite and centre of earth.

As the force on satellite always passes through centre of earth during motion, we can say that on satellite there is no torque acting about centre of earth thus total angular momentum of satellite during orbital motion remains constant about earth's centre.

As no external force is involved for earth-satellite system, no external work is being done here so we can also state that total mechanical energy of system also remains conserved.

In the elliptical path of satellite shown in figure if r_1 and r_2 are the shortest distance (perigee) and farthest distance (appogee) of satellite from earth and at the points, velocities of satellite are v_1 and v_2 then we have according to conservation of angular momentum, the angular momentum of satellite at a general point is given as

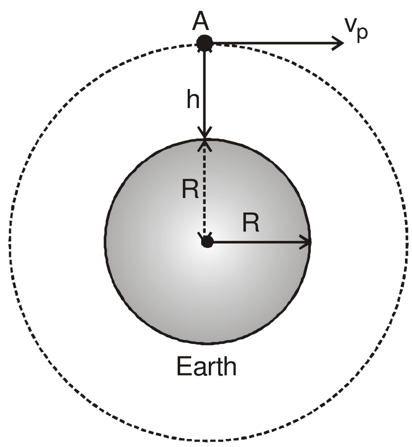
$$L = mv_1r_1 = mv_2r_2 = mvr \sin\theta$$

During motion the total mechanical energy of satellite (kinetic + potential) also remains conserved. Thus the total energy of satellite can be given as

$$E = \frac{1}{2}mv_1^2 - \frac{GM_em}{r_1} = \frac{1}{2}mv_2^2 - \frac{GM_em}{r_2} = \frac{1}{2}mv^2 - \frac{GM_em}{r}$$

Using the above relations in equation written above we can find velocities v_1 and v_2 of satellite at nearest and farthest locations in terms of r_1 and r_2.

7.3 Projection of Satellites and Spaceships From Earth



To project a body into space, first it should be taken to a height where no atmopshere is present then it is projected with some initial speed. The path followed by the body also depends on the projection speed. Lets discuss the cases step by step.

Consider the situation shown in figure. A body of mass *m* is taken to a height h above the surface of earth to a point A and then projected with an insertion velocity v_p as shown in figure.

If we wish to launch the body as an earth's satellite in circular path the velocity of projection must be

$$v_p = \sqrt{\frac{GM_e}{R_e + h}} \qquad ...(1)$$

If *h* is small compared to radius of earth, we have

$$v_1 = v_p = \sqrt{\frac{GM_e}{R_e}} = \sqrt{g_sR_e} = 7.93 \text{ km/s}.$$

This velocity v_1 = 7.93 km/s with which, when a body is thrown from earth's surface tangentially so that after projection it becomes a satellite of earth in a circular orbit around it, is called orbital speed or first cosmic velocity.

We've already discussed that if projection speed is lesser the orbital speed, body will start following the inner ellipse and if velocity of projection is increased the body will follow the outer ellipse. If projection speed of the satellite is further increased, the outer ellipse will also become bigger and at a particular higher projection speed, it may also be possible that body will go to infinity and will never come back to earth again.

We have discussed that negative total energy of body shows its boundness. If we write the total energy of a body projected from point A as shown in figure is

$$E = \frac{1}{2}mv_p^2 - \frac{GM_e m}{R_e + h}$$

If after projection body becomes a satellite of earth then it implies it is bounded to earth and its total energy is negative. If at point A, that much of kinetic energy is imparted to the body so that total energy of body becomes zero then it implies that the body will reach to infinity and escape from gravitational field of earth. If v_{II} is such a velocity then we have

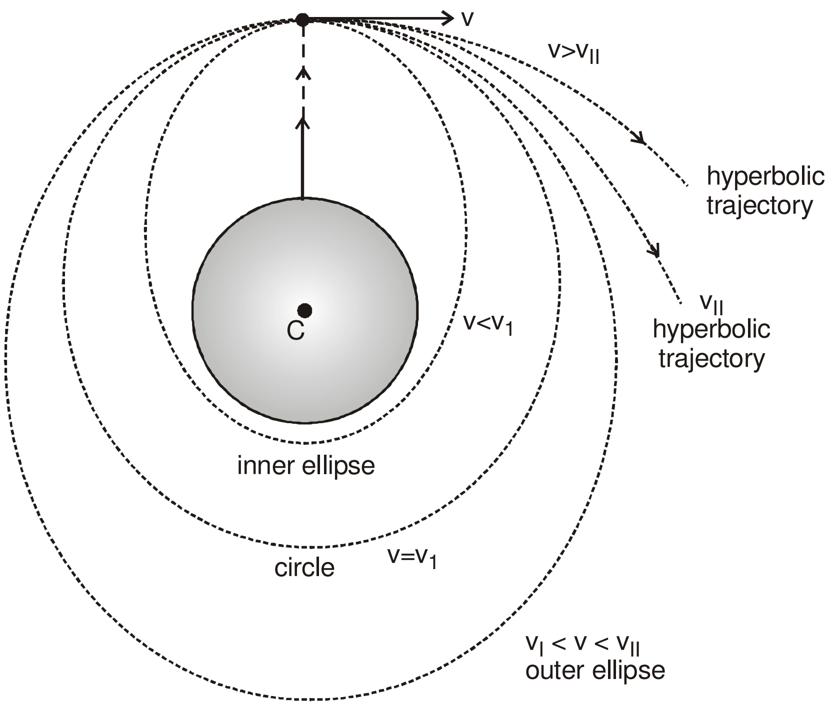
$$\frac{1}{2}mv_{II}^2 - \frac{GM_e m}{R_e + h} = 0$$

or $$v_{II} = \sqrt{\frac{2GM_e}{R_e + h}} = \sqrt{2}v_1 \quad ....(2)$$

For h << R_e, we have $$v_{II} = \sqrt{\frac{2GM_e}{R_e}} = \sqrt{2g_sR_e} = 11.2\,km/s \quad ....(3)$$

Thus from earth's surface a body is thrown at a speed of 11.2 km/s, it will escape from earth's gravitation. If the projection speed of body is less than this value total energy of body is negative and it wil orbit the earth in elliptical orbit. This velocity is referred as the second cosmic velocity or escape velocity. When a body is thrown with this speed, it follows a parabolic trajectory and will become free from earth's gravitational attraction.

When body is thrown with speed more then v_{II} then it moves along a hyperbolic trajectory and also leaves the region where the earth's gravitational attraction acts. Also when it reaches infinity some kinetic energy will be left in it and it becomes a satellite of sun, that is small artificial planet.



All the calculations we've performed till now do not take into account the influence of the sun and of the planets on the motion of the projected body. In other words we have assumed that the reference frame connected with the earth is an inertial frame and the body moves relative to it. But in reality the whole system body and the earth is in a non inertial frame which is permanently accelerated relative to sun.

Lets take some examples to understand some basic concepts related to gravitational energy and projection.

Ex.14 ***A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity has now to be imparted to the spaceship in the orbit of overcome the gravitational pull. (Radius of the earth = 6400 km and g = 9.8 m/sec.)***

Sol. In an orbit close to earth's surface velocity of space ship is $v = \sqrt{\frac{GM}{R}} = \sqrt{gR}$

We know escape velocity is $v_{II} = \sqrt{2gR}$

Hence additional velocity required to be imparted is $\Delta v = v_{II} - v = (\sqrt{2} - 1)\sqrt{gR}$

$$= (\sqrt{2} - 1)\sqrt{9.8 \times 6400 \times 10^3} = 3.28 \times 10^3 \text{ m/s}$$

Ex.15 ***A particle is fired vertically upward with a speed of 9.8 km/s. Find the maximum height attained by the particle. Radius of the earth = 6400 km and g at the surface = 9.8 m/s². Consider only earth's gravitation.***

Sol. Initial energy of particle on earth's surface is

$$E_r = \frac{1}{2}mu^2 - \frac{GMm}{R}$$

If the particle reaches upto a height h above the surface of earth then its final energy will only be the gravitational potential energy.

$$E_f = -\frac{GMm}{R+h}$$

According to energy conservation, we have

$$E_t = E_f$$

or $\frac{1}{2}mu^2 - \frac{GMm}{R} = -\frac{GMm}{R+h}$ or $\frac{1}{2}u^2 - gR = -\frac{gR^2}{R+h}$

or $h = \frac{2gR^2}{2gR - u^2} - R = \frac{2 \times 9.8 \times (6400 \times 10^3)^2}{2 \times 9.8 \times 6400 \times 10^3 - (9.8)^2} - 6400 \times 10^3 = (27300 - 6400) \times 10^3 = 20900 \text{ km}$

Ex.16 ***A satellite of mass m is orbiting the earth in a circular orbit of radius r. It starts losing energy slowly at a constant rate C due to friction. If M_e and R_e denote the mass and radius of the earth respectively, show the the satellite falls on the earth in a limit t given by***

$$t = \frac{GmM_e}{2C}\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

Sol. Let velocity of satellite in its orbit of radius r be v then we have

$$v = \sqrt{\frac{GM_e}{r}}$$

When satellite approaches earth's surface, if its velocity becomes v', then it is given as

$$v' = \sqrt{\frac{GM_e}{R_e}}$$

The total initial energy of satellite at a distance r is

$$E_{T_f} = K_f + U_r = \frac{1}{2}mv^2 - \frac{GM_em}{R_e} = -\frac{1}{2}\frac{GM_em}{r}$$

The total final energy of satellite at a distance R_e is

$$E_{T_f} = K_f + U_r = \frac{1}{2}mv'^2 - \frac{GM_em}{R_e} = -\frac{1}{2}\frac{GM_em}{R_e}$$

As satellite is loosing energy at rate C, if it takes a time t in reaching earth, we have

$$Ct = E_{T_i} - E_{T_f} = \frac{1}{2}GM_em\left[\frac{1}{R_e} - \frac{1}{r}\right] \Rightarrow t = \frac{GM_em}{2C}\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

Ex.17 ***An artifical satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.***
(i) Determine the height of the satellite above earth's surface.
(ii) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth.

Sol. (i) Let M and R be the mass and radius of the earth respectively. If *m* be the mass of satellite, then escape velocity from earth $v_c = \sqrt{2gR_e}$

velocity of satellite $= \sqrt{\frac{gR_e}{2}}$

Further we know orbital speed of satallite at a height *h* is

$$v_s = \sqrt{\left(\frac{GM_e}{r}\right)} = \sqrt{\left(\frac{R_e^2 g}{R_e + h}\right)} \quad \text{or} \quad v_s^2 = \frac{R^2 g}{R + h}$$

From equation written above, we get

$h = R = 6400$ km

(ii) Now total energy at height h = total energy at earth's surface (principle of conservation of energy)

or $$0 - GM_e \frac{m}{R+h} = \frac{1}{2}mv^2 - GM_e \frac{m}{R_e}$$

or $$\frac{1}{2}mv^2 = \frac{GM_e m}{R_e} - \frac{GM_e m}{2R_e} \qquad [\text{As } h = R]$$

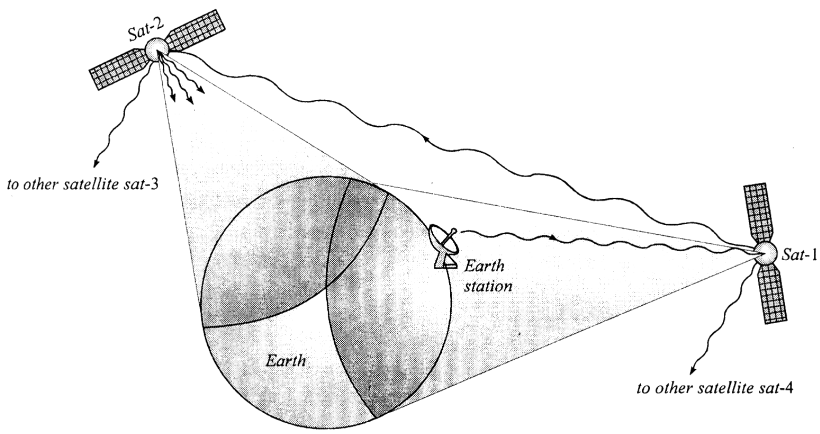
Solving we get $v = \sqrt{gR_e}$

or $\sqrt{9.8 \times 6400 \times 10^3} = 7.919$ km/s

8. COMMUNICATION SATELLITES

Communication satellite around the earth are used by Information Technology for spreading information through out the globe.

Figure shows as to how using satellites an information from an earth station, located at a point on earth's surface ca be sent throughout the world.



First the information is sent to the nearest satellite in the range of earth station by means of electromanetic waves then that satellite broadcasts the signal to the region of earth exposed to this satellite and also send the same signal to other satellite for broadcasting in other parts of the globe.

8.1 Geostationary Satellite and Parking Orbit

There are so many types of communication satellites revolving around the arth in different orbits at different heights depending on their utility. Some of which are Geostationary satellites, which appears at rest relative to earth or which have same angular velocity as that of earth's rotation i.e., with a time peiod of 24 hr. such satellite must be orbiting in an orbit of specific radius. This orbit is called parking orbit. If a Geostationary satellite is at a height h above the earth's surface then its orbiting speed is given as

$$v_{gs} = \sqrt{\frac{GM_e}{(R_e + h)}}$$

The time period of its revolution can be given Kepler's third law as

$$T^2 = \frac{4\pi^2}{GM_e}(R_e + h)^3$$

or $$T^2 = \frac{4\pi^2}{g_s R_e^2}(R_e + h)^3$$

or $$h = \left(\frac{g_s R_e^2}{4\pi^2}T^2\right)^{1/3} - R_e$$

or $$h = \left[\frac{9.8 \times [6.4 \times 10^6] \times [86400]^2}{4 \times (3.14)^2}\right]^{1/3} - 6.4 \times 10^6 = 35954.6 \text{ km} \simeq 36000 \text{ km}$$

Thus when a satellite is launched in an orbit at a height of about 36000 km above the quator then it will appear to be at rest with respect to a point on Earth's surface. A Geostationary satellite must have in orbit in equatorial plane due to the geographic limitation because of irregular geometry of earth (ellipsoidal shape.)

In short

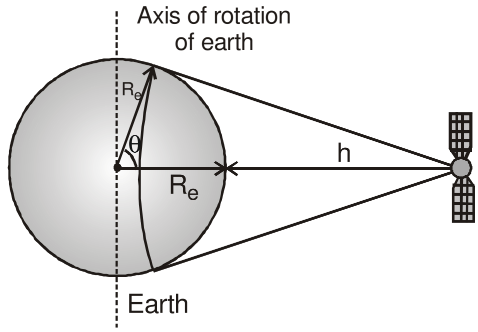
- Plane of the satellite should pass through centre of the planet
- For geostationary satallites plane should be equatorial plane
- Time peirod should be 24 hrs & direction should be west to east
- For any point on the earth, geostationary satellite is stationary.

8.2 Broadcasting Region of a Satellite

Now as we known the height of a geostationary satellite we can easily find the area of earth exposed to the satellite or area of the region in which the comunication can be mode using this satellite.

Figure shown earth and its exposed area to a geostationary satellite. Here the angle θ can be given as

$$\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$$



Now we can find the solid angle Ω which the exposed area subtend on earth's centre as

$$\Omega = 2\pi(1 - \cos\theta)$$

$$= 2\pi\left(1 - \frac{R_e}{R_e + h}\right) = \frac{2\pi h}{R_e + h}$$

Thus the area of earth's surface to geostationary satellite is

$$S = \Omega R_e^2 = \frac{2\pi h R_e^2}{R_e + h}$$

Lets take some examples to understand the concept in detail.

Ex.18 ***A satellite is revolving around the earth in an orbit of radius double that of the parking orbit and revolving in same sense. Find the periodic time duration between two instants when this satallite is closest to a geostationary satellite.***

Sol. We know that the time period of revolution of a satellite is given as

$$T^2 = \frac{4\pi^2}{GM_e} r^3 \qquad \text{[Kepler's III law]}$$

For satellite given in problem and for a geostationary satellite we have

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^3 \quad \text{or} \quad T_1 = \left(\frac{r_1}{r_2}\right)^3 \times T_2 = (2)^3 \times 24 = 192 \text{ hr}$$

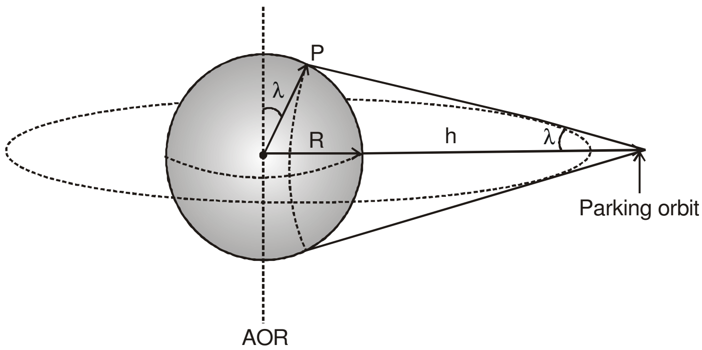
If Δt be the time between two sucessive instants when the satellite are closed then we must have

$$\Delta t = \frac{\theta}{\omega_1} = \frac{2\pi + \theta}{\omega_2} = \frac{2\pi}{\omega_2 - \omega_1}$$

Where ω_1 and ω_2 are the angular speeds of the two planets

Ex.19 ***Find the minimum colatitude which can directly receive a signal from a geostationary satellite.***

Sol. The farthest point on earth, which can receive signals from the parking orbit is the point where a length is drawn on earth surface from satellite as shown in figure.



The colatitude λ of point P can be obtained from figure as

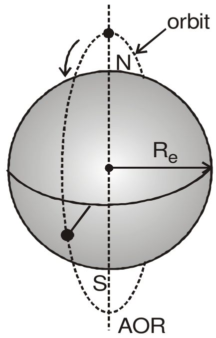
$$\sin\lambda = \frac{R_e}{R_e + h} \simeq \frac{1}{7}$$

We known for a parking orbit $h \simeq 6R_e$

Thus we have $\qquad \lambda = \sin^{-1}\left(\frac{1}{7}\right)$

Ex.20 ***If a satellite is revolving around the earth in a circular orbit in a plane containing earth's axis of rotation. if the angular speed of satellite is equal to that of earth, find the time it takes to move from a point above north pole of a point above the equator.***

Sol. A satellite which rotates with angular speed equal to earth's rotation has an orbit radius 7 R_e and the angular speed of revolution is



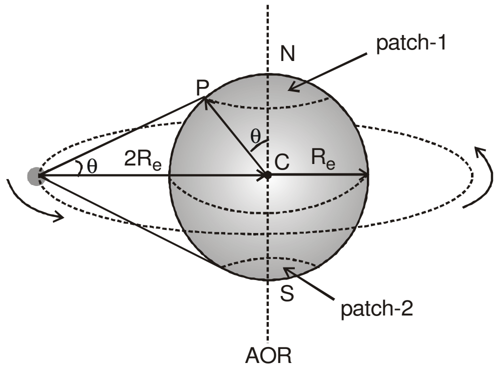
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86400} = 7.27 \times 10^{-5} \text{rad/s}$$

When satellite moves from a point above north pole to a point above equator, it traverses an angle $\pi/2$, this time taken is

$$t = \frac{\pi/2}{\omega} = 21600 \text{ s} = 6 \text{ hrs.}$$

Ex.21 ***A satellite is orbiting around the earth in an orbit in equatorial plane of radius 2R$_e$ where R$_e$ is the radius of earth. Find the area on earth, this satellite covers for communication purpose in its complete revolution.***

Sol. As shown in figure when statelite S revolves, it covers a complete circular belt on earth's surface for communication. If the colatitude of the farthest point on surface upto which singals can be received (point P) is θ then we have



$$\sin\theta = \frac{R_e}{2R_e} = \frac{1}{2} \qquad \text{or } \theta = \frac{\pi}{6}$$

During revolution satellite leaves two spherical patches 1 and 2 on earth surface at north and south poles where no signals can be transmitted due to curvature. The areas of these patches can be obtained by solid angles.

The solid angle subtended by a patch on earth's centre is

$$\Omega = 2\pi\,(1 - \cos\theta) = \pi\,(2 - \sqrt{3})\ \text{st.}$$

Area of patch 1 and 2 is

$$A_P = \Omega R_e^2 = \pi(2-\sqrt{3})R_e^2$$

Thus total area on earth's surface to which communication can be made is

$$A_C = 4\pi R_e^2 - 2A_P \;= 4\pi R_e^2 - 2\pi(2-\sqrt{3})R_e^2$$

$$= 2\pi R_e^2(2-2+\sqrt{3}) = 2\sqrt{3}\,\pi R_e^2$$

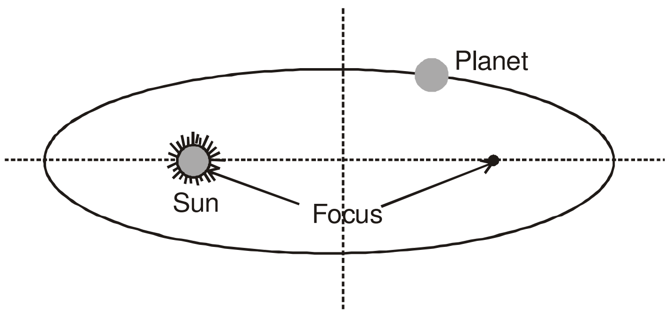
9. KEPLER'S LAWS OF PLANETARY MOTION

The motions of planet in universe have always been a puzzle. In 17th century Johannes Kepler, after a life time of study worded out some empirical laws based on the analysis of astronomical measurements of Tycho Brahe. Kepler formulated his laws, which are kinematical description of planetary motion. Now we discuss these laws step by step.

9.1 Kepler's First Law [The Law of Orbits]

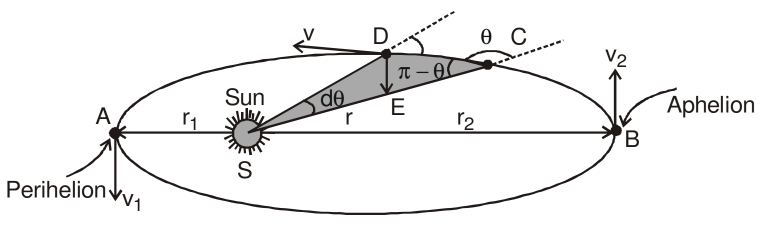
Kepler's first law is illustrated in the image shown in figure. It states that "All the planets move around the sun in ellipitcal orbits with sun at one of the focus not at centre of orbit."

It is observed that the orbits of planets around sun are very less ecentric or approximately circular

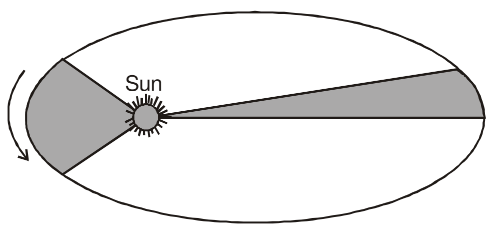


9.2 Kepler's Second Law [The Law of Areas]

Kepler's second Law is basically an alternative statement of law of conservation of momentum. It is illustrated in the image shown in figure(a). We know from angular momentum conservation, in elliptical orbit plane will move faster when it is nearer to the sun. Thus when a planet executes elliptical orbit its angular speed changes continuously as it moves in the orbit. The point of nearest approach of the planet to the sun is termed perihelion. The point of greatest seperation is termed aphelion. Hence by angular momentum conservation we can state that the planet moves with maximum speed when it is near perihelion and moves with slowest speed when it is near aphelion.



(a)



(b)

Kepler's second law states that "The line joining the sun and planet sweeps out equal areas in equal time or the rate of sweeping area by the position vector of the planet with respect to sun remains constant. "This is shown in figure (b).

The above statement of Kepler's second law can be verified by the law of conservation of angular momentum. To verify this consider the moving planet around the sun at a general point C in the orbit at speed v. Let at this instant the distance of planet from sun is *r*. If θ be the angle between position vector $\vec{r}$ of planet and its velocity vector then the angular momentum of planet at this instant is

$$L = m v r \sin\theta \qquad ...(1)$$

In an elemental time the planet will cover a small distance CD = dl and will travel to another adjacent point D as shown in figure (a), thus the distance CD = vdt. In this duration dt, the position vector $\vec{r}$ sweeps out an area equal to that of triangle SCD, which is calculated as

Area of triangle SCD is $dA = \frac{1}{2} \times r \times vdt \sin(\pi - \theta)$

$$= \frac{1}{2} r v \sin\theta .\ dt$$

Thus the rate of sweeping area by the position vector $\vec{r}$ is

$$\frac{dA}{dt} = \frac{1}{2} rv\sin\theta$$

Now from equation (1)

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant} \qquad ...(2)$$

The expression in equation (2) verifies the statement of Kepler' II law of planetary motion.

9.3 Kepler's Third law [The Law of Periods]

Kepler's Third Law is concerned with the time period of revolution of planets. It states that "The time period of revolution of a planet in its orbit around the sun is directly proportional to the cube of semi-major axis of the elliptical path around the sun"

If 'T' is the period of revolution and 'a' be the semi-major axis of the path of planet then according to Kepler's III law, we have

$$T^2 \propto a^3$$

For circular orbits, it is a special case of ellipse when its major and minor axis are equal. If a planet is in a circular orbit of radius r around the sun then its revolution speed must be given as

$$v = \sqrt{\frac{GM_s}{r}}$$

Where M_s is the mass of sun. Here you can recall that this speed is independent from the mass of planet. Here the time period of revolution can be given as

$$T = \frac{2\pi r}{v} \qquad \text{or} \qquad T = \frac{2\pi r}{\sqrt{\frac{GM_s}{r}}}$$

Squaring equation written above, we get

$$T^2 = \frac{4\pi^2}{GM_s} r^3 \qquad ...(1)$$

Equation (1) verifies the statement of Kepler's third law for circular orbits. Similarly we can also verify it for elliptical orbits. For this we start from the relation we've derived earlier for rate of sweeping area by the position vector of planet with respect to sun which is given as

$$\frac{dA}{dt} = \frac{L}{2m}$$

Ex.22 ***The moon revolves around the earth 13 times per year. If the ratio of the distance of the earth from the sun to the distance of the moon from the earth is 392, find the ratio of mass of the sun to the mass of the earth.***

Sol. The time period T_e of earth around sun of mass M_s is given by

$$T_e^2 = \frac{4\pi^2}{GM_s} \times r_e^3 \quad ...(1)$$

Where r_e is the radius of the earth.

Similarly, time period T_m of moon around earth is given by

$$T_m^2 = \frac{4\pi^2}{GM_e} \times r_m^3 \quad ...(2)$$

Dividing equation(1) by equation (2), we get

$$\left(\frac{T_e}{T_m}\right)^2 = \left(\frac{M_e}{M_s}\right)\left(\frac{r_e}{r_m}\right)^3$$

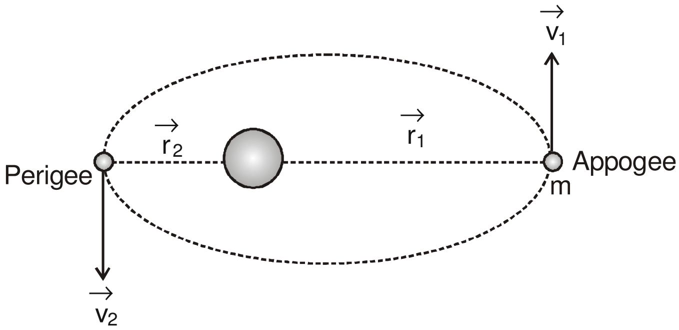
or $$\left(\frac{M_s}{M_e}\right) = \left(\frac{T_m}{T_e}\right)^2 \times \left(\frac{r_e}{r_m}\right)^3 \quad ...(3)$$

Substituting the given values, we get

$$\left(\frac{M_s}{M_e}\right) = \left\{\frac{(13)}{1}\right\}^2 \times (392)^3 = 3.56 \times 10^5$$

Ex.23 ***A satellite revolves around a planet in an elliptical orbit. Its maximum and minimum distances from the planet are 1.5×10^7 m and 0.5×10^7 m respectively. If the speed of the satellite at the farthest point be 5×10^3 m/s, calculate the speed at the nearest point.***

Sol.



In case of elliptical orbit, the speed of satellite varies constantly as shown in figure. Thus according to the law of conservation of angular momentum, the satellite must move faster at a point of closest approach (Perigee) than at a farthest point (Appogee).

We know that $\vec{L} = \vec{r} \times m\vec{v}$

Hence, at the two points, $L = m\,v_1\,r_1 = m\,v_2\,r_2$

or $$\frac{v_1}{v_2} = \frac{r_2}{r_1}$$

Substituting the given values, we get

$$\frac{5\times10^3}{v_2} = \frac{0.5\times10^7}{1.5\times10^7} \Rightarrow v_2 = 1.5 \times 10^4 \text{ m/s}$$

Ex.24 ***Imagine a light planet revolving around a very massive star in a circular orbit of radius r with a period of revolution T. On what power of r, will the square of time period depend if the gravitational force of attraction between the planet and the star is proportional to $r^{-5/2}$.***

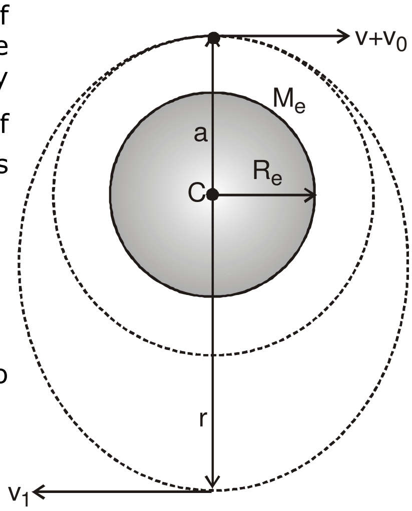
Sol. As gravitation provides centripetal force

$$\frac{mv^2}{r} = \frac{K}{r^{5/2}}, \quad \text{i.e.,} \quad v^2 = \frac{K}{mr^{3/2}}$$

So that $$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{mr^{3/2}}{K}} \quad \text{or} \quad T^2 = \frac{4\pi^2 m}{K} r^{7/2}; \quad \text{so } T^2 \propto r^{7/2}$$

Ex.25 A satellite is revolving round the earth in a circular orbit of radius a with velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v = (\sqrt{5/4} - 1)v_0$. Calculate, during subsequent motion of the particle its minimum and maximum distances from earth's centre.

The corresponding situation is shown in figure.



Initial velocity of satellite $v_0 = \sqrt{\left(\frac{GM}{a}\right)}$

When particle is thrown with the velocity v relative to satellite, the resultant velocity of particle will become

$$v_R = v_0 + v$$

$$= \sqrt{\left(\frac{5}{4}\right)}v_0 = \sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}$$

As the particle velocity is greater than the velocity required for circular orbit, hence the particle path deviates from circular path to elliptical path. At position of minimum and maximum distance velocity vectors are perpendicular to instantaneous radius vector. In this elliptical path the minimum distance of particle from earth's centre is a and maximum speed in the path is v_R and let the maximum distance and minimum speed in the path is r and v_1 respectively.

Now as angular momentum and total energy remain conserved. Applying the law of conservation of angular momentum, we have

$$m v_1 r = m(v_0 + v)\,a \qquad [m = \text{mass of particle}]$$

or $$v_1 = \frac{(v_0 + v)a}{r} = \frac{a}{r}\left[\sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}\right] = \frac{1}{r}\left[\sqrt{\left(\frac{5}{4} \times GMa\right)}\right]$$

Applying the law of conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r} = \frac{1}{2}m(v_0 + v)^2 - \frac{GMm}{a}$$

or $$\frac{1}{2}m\left(\frac{5}{4}\frac{GMa}{r^2}\right) - \frac{GMm}{r} = \frac{1}{2}m\left(\frac{5}{4}\frac{GM}{a}\right) - \frac{GMm}{a}$$

$$\frac{5}{8} \times \frac{a}{r^2} - \frac{1}{r} = \frac{5}{8} \times \frac{1}{a} - \frac{1}{a} = -\frac{3}{8a}$$

or $3r^2 - 8\,ar + 5\,a^2 = 0$ or $r = a$ or $\frac{5a}{3}$

Thus minimum distance of the particle = a

And maximum distance of the particle = $\frac{5a}{3}$

Ex.26 ***A sky lab of mass 2×10^3 kg is first launched from the surface of earth in a circular orbit of radius 2 R (from the centre of earth) and then it is shifted from this circular orbit to another circular orbit of radius 3 R. Calculate the minimum energy required (a) to place the lab in the first orbit (b) to shift the lab from first orbit to the second orbit. Given, R = 6400 km and g = 10 m/s².***

Sol. (a) The energy of the sky lab on the surface of earth

$$E_S = KE + PE = 0 + \left(-\frac{GMm}{R}\right) = -\frac{GMm}{R}$$

And the total energy of the sky lab in an orbit of radius 2 R is

$$E_1 = -\frac{GMm}{4R}$$

So the energy required to placed the lab from the surface of earth to the orbit of radius 2R is given as

$$E_1 - E_s = -\frac{GMm}{4R} - \left[-\frac{GMm}{R}\right] = \frac{3}{4}\frac{GMm}{R}$$

or $$\Delta E = \frac{3}{4}\frac{m}{R} \times gR^2 = \frac{3}{4}mgR \qquad \left[\text{As } g = \frac{GM}{R^2}\right]$$

or $$\Delta E = \frac{3}{4}(2 \times 10^3 \times 10 \times 6.4 \times 10^6) = \frac{3}{4}(12.8 \times 10^{10}) = 9.6 \times 10^{10} \text{ J}$$

(b) As for II orbit of radius 3R the total energy of sky lab is

$$E_2 = -\frac{GMm}{2(3R)} = -\frac{GMm}{6R}$$

or $$E_2 - E_1 = -\frac{GMm}{6R} - \left(-\frac{GMm}{4R}\right) = \frac{1}{12}\frac{GMm}{R}$$

or $$\Delta E = \frac{1}{12}mgR = \frac{1}{12}(12.8 \times 10^{10}) = 1.1 \times 10^{10} \text{ J}$$

Ex.27 ***A satellite is revolving around a planet of mass M in an elliptic orbit of semimajor axis a. Show that the orbital speed of the satellite when it is at a distance r from the focus will be given by :***

$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

Sol. As in case of elliptic orbit with semi major axes a, of a satellite total mechanical energy remains constant, at any position of satellite in the orbit, given as

$$E = -\frac{GMm}{2a}$$

or $$KE + PE = -\frac{GMm}{2a} \qquad ...(1)$$

Now, if at position r, v is the orbital speed of satellite, we have

$$KE = \frac{1}{2}mv^2 \text{ and } PE = -\frac{GMm}{r} \qquad ...(2)$$

So from equation (1) and (2), we have

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}, \text{ i.e., } \qquad v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$