1. WORK :

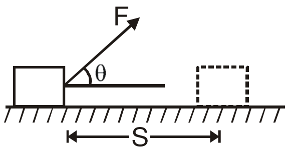
Work is said to be done by a force when the force produces a displacement in the body on which it acts in any direction except perpendicular to the direction of the force.

1.1 Work done by constant force

Consider an object undergoes a displacement S along a straight line while acted on a force F that makes an angle θ with S as shown.

The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement.

i.e., $W = FS \cos \theta$ ...(1)



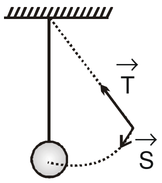
Work done is a scalar quantity and its S.I. unit is N-m or joule (J). We can also write work done as a scalar product of force and displacement.

$W = \vec{F}.\vec{S}$ ...(2)

where S is the displacement of the point of application of the force

From this definition, we conclude the following points

(A) work done by a force is zero if displacement is perpendicular to the force (θ = 90°)



Example.

The tension in the string of a simple pendulum is always perpendicular to displacement. (Figure)

So, work done by the tension is zero.

(B) if the angle between force and displacement is acute (θ < 90°), we say that the work done by the force is positive.

Example :

When a load is lifted, the lifting force and the displacement act in the same direction. So, work done by the lifting force is positive.

Example :

When a spring is stretched, both the stretching force and the displacement act in the same direction. So work done by the stetching force is positive.

(C) If the angle between force and displacement is obtuse (θ > 90°), we say that the work done by the force is negative.

Example :

When a body is lifted, the work done by the gravitational force is negative. This is because the gravitational force acts vertically downwards while the displacement is in the vertically upwards direction.

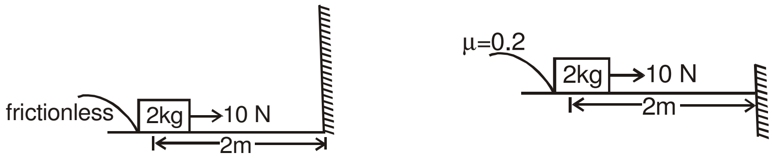
Important points about work :

1. Work is said to be done by a force when its point of application moves by some distance.Force does no work if point of application of force does not move (S = 0)

Example :

A person carrying a load on his head and standing at a given place does no work.

2. Work is defined for an interval or displacement. There is no term like instantaneous work similar to instantaneous velocity.



Work done by 10 N force in both the cases are same = 20 N

3. For a particular displacement, work done by a force is independent of type of motion i.e. whether it moves with constant velocity, constant acceleration or retardation etc.
4. If a body is in dynamic equilibrium under the action of certain forces, then total work done on the body is zero but work done by individual forces may not be zero.
5. When several forces act, work done by a force for a particular displacement is independent of other forces.
6. A force is independent of reference frame. Its displacement depends on frame so work done by a force is frame dependent therefore work done by a force can be different in different reference frame.

---

2. UNITS OF WORK :

In cgs system, the unit of work is erg.

One erg of work is said to be done when a force of one dyne displaces a body through one centimetre in its own direction.

$\therefore$ 1 erg = 1 dyne × 1 cm = 1 g cm s^{-2} × 1 cm = 1 g cm^2 s^{-2}

Note : Another name for joule is newton metre.

<u>Relation between joule and erg</u>

1 joule = 1 newton × 1 metre

1 joule = 10^5 dyne × 10^2 cm = 10^7 dyne cm

1 joule = 10^7 erg

1 erg = 10^{-7} joule

Dimensions of Work :

[Work] = [Force] [Distance] = $[MLT^{-2}]$ [L] = $[ML^2T^{-2}]$

Work has one dimension in mass, two dimensions in length and '−2' dimensions in time, On the basis of dimensional formula, the unit of work is kg m^2 s^{-2}.

Note that 1 kg m^2 s^{-2} = (1 kg m s^{-2}) m = 1 N m = 1 J.

3. WORK DONE BY MULTIPLE FORCES :

If several forces act on a particle, then we can replace $\vec{F}$ in equation W = $\vec{F}\,.\,\vec{S}$ by the net force $\sum \vec{F}$ where

$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$

$\therefore$ W = $\left[\sum \vec{F}\right].\ \vec{S}$ ...(i)

This gives the work done by the net force during a displacement $\vec{S}$ of the particle.

We can rewrite equation (i) as :

W = $\vec{F}_1.\vec{S} + \vec{F}_2.\vec{S} + \vec{F}_3.\vec{S} + ...$

or W = $W_1 + W_2 + W_3 + ...$

So, the work done on the particle is the sum of the individual work done by all the forces acting on the particle.

Ex.1 A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with horizontal. Coefficient of friction between block and surface is μ. If the block travels with uniform velocity, find the work done by this applied force during a displacement d of the block.

Sol. The forces acting on the block are shown in Figure. As the block moves with uniform velocity the resultant force on it is zero.

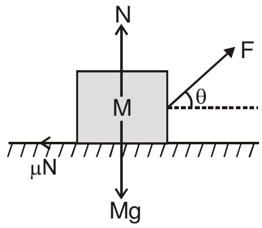
$\therefore$ $F\cos\theta = \mu N$ ...(i)

$F\sin\theta + N = Mg$ ...(ii)

Eliminating N from equations (i) and (ii),

$F\cos\theta = \mu(Mg - F\sin\theta)$

$$F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta}$$



Work done by this force during a displacement d

$$W = F\,.\,d\cos\theta = \frac{\mu Mgd\cos\theta}{\cos\theta + \mu\sin\theta}$$

Ex.2 A particle moving in the xy plane undergoes a displacement $\vec{S} = (2.0\hat{i} + 3.0\hat{j})$m ***while a constant force*** $\vec{F} = (5.0\hat{i} + 2.0\hat{j})$N ***acts on the particle.***

(a) Calculate the magnitude of the displacement and that of the force.

(b) Calculate the work done by the force.

Sol. (a) $\vec{s} = (2.0\hat{i} + 3.0\hat{j})$ $\quad \vec{F} = (5.0\hat{i} + 2.0\hat{j})$

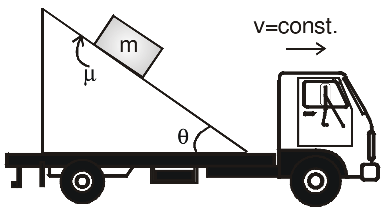
$|\vec{s}| = \sqrt{x^2 + y^2} = \sqrt{(2.0)^2 + (3.0)^2} = \sqrt{13}$ m

$|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = 5.4$N

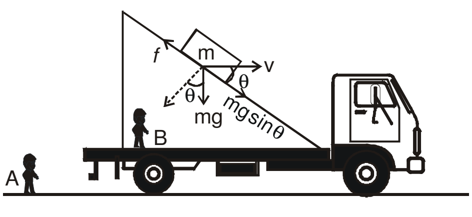
(b) Work done by force, $W = \vec{F}.\vec{s}$

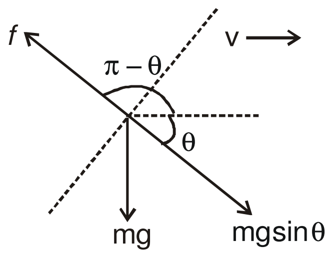
$= (5.0\hat{i} + 2.0\hat{j}) \cdot (2.0\hat{i} + 3.0\hat{j})$N.m $\quad$ = 10 + 0 + 0 + 6 = 16 N.m = 16 J

Ex.3 A block of mass m is placed on an inclined plane which is moving with constant velocity v in horizontal direction as shown in figure. Then find out work done by the friction in time t if the block is at rest with respect to the incline plane.



Sol. F.B.D of block with respect to ground.





Block is at rest with respect to wedge
$\Rightarrow$ f = mg sin θ
In time t the displacement of block with respect to ground d = vt
Work done by friction for man A
W_f = (component of friction force along displacement) × displacement
$W_f = mg\sin\theta . vt\cos(180° - \theta)$
$W_f = -\, mg\, vt \cos\theta \sin\theta$
W_f for man B = 0 (displacement is zero with respect to man B)

4. WORK DONE BY A VARIABLE FORCE :

(A) When F as a function of x, y, z

When the magnitude and direction of a force vary in three dimensions, it can be expressed as a function of the position. For a variable force work is calculated for infinitely small displacement and for this displacement force is assumed to be constant.

$$dW = \vec{F}.d\vec{s}$$

The total work done will be sum of infinitely small work

$$W_{A \to B} = \int_A^B \vec{F}.d\vec{s} = \int_A^B (\vec{F}\cos\theta)d\vec{s}$$

It terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\hat{k}$$

$$W_{A \to B} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Ex.4 A force F = (4.0 x $\hat{i}$ + 3.0 y $\hat{j}$) N acts on a particle which moves in the x-direction from the origin to x = 5.0 m. Find the work done on the object by the force.

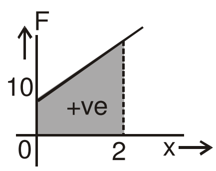
Sol. Here the work done is only due to x component of force because displacement is along x-axis.

$$\text{i.e., } W = \int_{x_1}^{x_2} F_x dx = \int_0^5 4x\,dx = \left[2x^2\right]_0^5 = 50 \text{ J}$$

Ex.5 A force F = 0.5x + 10 acts on a particle. Here F is in newton and x is in metre. Calculate the work done by the force during the displacement of the particle from x = 0 to x = 2 metre.

Sol. Small amount of work done dW in giving a small displacement $\overrightarrow{dx}$ is given by

$dW = \overrightarrow{F}.\overrightarrow{dx}$
or $dW = Fdx \cos 0°$
or $dW = Fdx$ $[\therefore \cos 0° = 1]$



$$\text{Total work done, } W = \int_{x=0}^{x=2} Fdx = \int_{x=0}^{x=2} (0.5x + 10)dx$$

$$= \int_{x=0}^{x=2} 0.5x dx + \int_{x=0}^{x=2} 10 dx = 0.5\left|\frac{x^2}{2}\right|_{x=0}^{x=2} + 10|x|_{x=0}^{x=2}$$

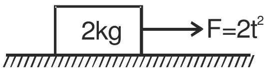
$= \frac{0.5}{2}[2^2 - 0^2] + 10[2 - 0] = (1 + 20) = 21$ J

(B) When F is given as a function of Time(t) :

Ex.6 The force F = $2t^2$ is applied on the 2 kg block. Then find out the work done by this force in 2sec. Initially at time t = 0, block is at rest.

at t = 0, v = 0

Sol.



F = ma

$\Rightarrow \quad 2t^2 = 2a \quad \Rightarrow \quad a = t^2$

$\Rightarrow \quad \frac{dv}{dt} = t^2 \quad \Rightarrow \quad \int_0^v dv = \int_0^t t^2 dt$ (At t = 0 it is at rest)

$\Rightarrow \quad v = \frac{t^3}{3}$

Let the displacement of the block be dx from t = t to t = t +dt then, work done by the force F in this time interval dt is.

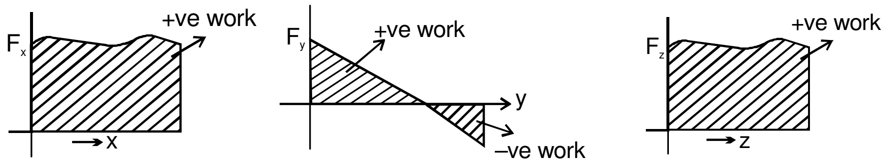
$dw = F.dx = 2t^2.dx$

$dw = 2t^2.\frac{dx}{dt}.dt \quad \Rightarrow \quad dw = 2t^2(v)dt$

$\int_0^w dw = \int_0^2 2t^2.\frac{t^3}{3}dt \quad \Rightarrow \quad W = \frac{2}{3}\int_0^2 t^5 dt \quad \Rightarrow \quad W = \frac{2}{3}\left[\frac{t^6}{6}\right]_0^2 = \frac{64}{9}$ Joule

5. AREA UNDER FORCE DISPLACEMENT CURVE :

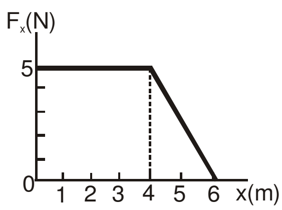
Graphically area under the force-displacement is the work done



The work done can be positive or negative as per the area above the x-axis or below the x-axis respectively.

Ex.7 Force acting on a particle varies with x as shown in figure. Calculate the work done by the force as the particle moves from x = 0 to x = 6.0 m.

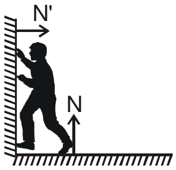
Sol. The work done by the force is equal to the area under the curve from x = 0 to x = 6.0 m. This area is equal to the area of the rectangular section from x = 0 to x = 4.0 m plus the area of the triangular section from x = 4.0 m to x = 6.0 m. The area of the rectangle is (4.0) (5.0) N.m = 20 J, and the area of the triangle is $\frac{1}{2}$(2.0), (5.0) N.m = 5.0 J. Therefore, the total work done is 25 J.



6. INTERNAL WORK :

Suppose that a man sets himself in motion backward by pushing against a wall. The forces acting on the man are his weight 'W' the upward force N exerted by the ground and the horizontal force

N′ exerted by the wall. The works of 'W' and of N are zero because they are perpendicular to the motion. The force N' is the unbalanced horizontal force that imparts to the system a horizontal acceleration. The work of N', however, is zero because there is no motion of its point of application. We are therefore confronted with a curious situation in which a force is responsible for acceleration, but its work, being zero, is not equal to the increase in kinetic energy of the system.



The new feature in this situation is that the man is a composite system with several parts that can move in relation to each other and thus can do work on each other, even in the absence of any interaction with externally applied forces. Such work is called internal work. Although internal forces play no role in acceleration of the composite system, their points of application can move so that work is done; thus the man's kinetic energy can change even though the external forces do no work.

"Basic concept of work lies in following lines

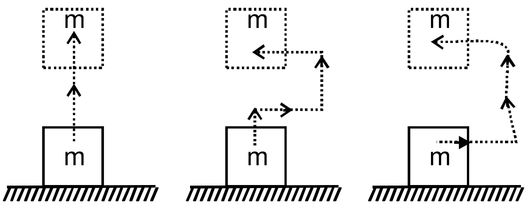
Draw the force at proper point where it acts that give proper importance of the point of application of force.

Think independently for displacement of point of application of force, Instead of relation the displacement of applicant point with force relate it with the observer or reference frame in which work is calculated.

$$W = (\text{Force vector}) \times \begin{pmatrix} \text{displacement vector of point of} \\ \text{application of force as seen by} \\ \text{observer} \end{pmatrix}$$

7. CONSERVATIVE FORCE :

A force is said to be conservative if work done by or against the force in moving a body depends only on the initial and final positions of the body and does not depend on the nature of path followed between the initial and final positions.



(a) (b) (c)

Consider a body of mass m being raised to a height h vertically upwards as shown in above figure. The work done is mgh. Suppose we take the body along the path as in (b). The work done during horizontal motion is zero. Adding up the works done in the two vertical parts of the paths, we get the result mgh once again. Any arbitrary path like the one shown in (c) can be broken into elementary horizontal and vertical portions. Work done along the horizontal path is zero. The work done along the vertical parts add up to mgh. Thus we conclude that the work done in raising a body against gravity is independent of the path taken. It only depends upon the intial and final positions of the body. We conclude from this discussion that the force of gravity is a conservative force.

Examples of Conservative forces.

(i) Gravitational force, not only due to Earth due in its general form as given by the universal law of gravitation, is a conservative force.

(ii) Elastic force in a stretched or compressed spring is a conservative force.

(iii) Electrostatic force between two electric charges is a conservative force.

(iv) Magnetic force between two magnetic poles is a conservative force.

Forces acting along the line joining the centres of two bodies are called central forces. Gravitational force and Electrosatic forces are two important examples of central forces. Central forces are conservative forces.

Properties of Conservative forces

- **Work done by or against a conservative force depends only on the initial and final position of the body.**
- Work done by or against a conservative force does not depend upon the nature of the path between initial and final position of the body.
- Work done by or against a conservative force in a round trip is zero.
  If a body moves under the action of a force that does no total work during any round trip, then the force is conservative; otherwise it is non-conservative.
  The concept of potential energy exists only in the case of conservative forces.
- The work done by a conservative force is completely recoverable.
  Complete recoverability is an important aspect of the work done by a conservative force.

Work done by conservative forces

Ist format : (When constant force is given)

Ex.8 Calculate the work done to displace the particle from (1, 2) to (4, 5). if $\vec{F} = 4\hat{i} + 3\hat{j}$

Sol. dw = $\vec{F}.d\vec{r}$ ($d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$)

dw = $(4\hat{i} + 3\hat{j}).(dx\hat{i} + dy\hat{j})$ $\Rightarrow$ dw = 4dx + 3dy

$$\int_0^w dw = \int_1^4 4dx + \int_2^5 3dy \Rightarrow w = [4x]_1^4 + [3y]_2^5$$

w = (16 – 4) + (15 – 6) $\Rightarrow$ w = 12 + 9 = 21 Joule

II format : (When F is given as a function of x, y, z)

If $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

then

dw = $(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$ $\Rightarrow$ dw = $F_x dx + F_y dy + F_z dz$

Ex.9 An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})m$ ***to*** $\vec{r}_2 = (4\hat{i} + 6\hat{j})m$ ***under a force*** $\vec{F} = (3x^2\hat{i} + 2y\hat{j})N$***. Find the work done by this force.***

Sol. $W = \int_{r_i}^{\vec{r}_f} \vec{F}.d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \bullet (dx\hat{i} + dy\hat{j} + dz\hat{k}) = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2dx + 2ydy) = [x^3 + y^2]_{(2,3)}^{(4,6)} = 83\,J$ **Ans.**

IIIrd format (perfect differential format)

Ex.10 ***If $\vec{F} = y\hat{i} + x\hat{j}$ then find out the work done in moving the particle from position (2, 3) to (5, 6)***

Sol. dw = $\vec{F}.d\vec{s}$

dw = $(y\hat{i} + x\hat{j}).(dx\hat{i} + dy\hat{j})$

dw = ydx + xdy

Now ydx + xdy = d(xy) (perfect differential equation)

$\Rightarrow$ dw = d(xy)

for total work done we integrate both side

$$\int dw = \int d(xy)$$

Put xy = k

then at (2, 3) $k_i = 2 \times 3 = 6$

at (5, 6) $k_f = 5 \times 6 = 30$

then w = $\int_6^{30} dk = [k]_6^{30}$ $\Rightarrow$ w = (30 – 6) = 24 Joule

8. NON-CONSERVATIVE FORCES :

A force is said to be non-conservative if work done by or against the force in moving a body depends upon the path between the initial and final positions.

The frictional forces are non-conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does not depend only on the initial and final positions. Note that the work done by fricitional force in a round trip is not zero.

The velocity-dependent forces such as air resistance, viscous force, magnetic force etc., are non conservative forces.

Ex.11 ***Calculate the work done by the force $\vec{F} = y\hat{i}$ to move the particle from (0, 0) to (1, 1) in the following condition***

(a) y = x (b) y = x²

Sol. We know that

dw = $\vec{F}.d\vec{s}$ $\Rightarrow$ dw = $(y\hat{i}).(dx\,\hat{i})$

dw = ydx ...(1)

In equation (1) we can calculate work done only when we know the path taken by the particle.

either

y = x or y = x^2 so now

(a) when y = x

$$\int dw = \int_0^1 x dx \Rightarrow w = \frac{1}{2} \text{Joule}$$

(b) when y = x^2

$$\int dw = \int_0^1 x^2 dx \Rightarrow w = \frac{1}{3} \text{Joule}$$

Difference between conservative and Non-conservative forces

S. No.	Conservative forces	Non-Conservative forces
1	Work done does not depend upon path	Work done depends on path.
2	Work done in a round trip is zero.	Work done in a round trip is not zero.
3	Central in nature.	Forces are velocity-dependent and retarding in nature.
4	When only a conservative force acts within a system, the kinetic energy and potential energy can change. However their sum, the mechanical energy of the system, does not change.	Work done against a non-conservative force may be disipiated as heat energy.
5	Work done is completely recoverable.	Work done in not completely recoverable.

9. ENERGY

A body is said to possess energy if it has the capacity to do work. When a body possessing energy does some work, part of its energy is used up. Conversely if some work is done upon an object, the object will be given some energy. Energy and work are mutually convertiable.

There are various forms of energy. Heat, electricity, light, sound and chemical energy are all familiar forms. In studying mechanics, we are however concerned chiefly with mechanical energy. This type of energy is a property of movement or position.

9.1 Kinetic Energy

Kinetic energy (K.E.), is the capacity of a body to do work by virtue of its motion.
If a body of mass m has velocity v its kinetic energy is equivalent to the work, which an external force would have to do to bring the body from rest up to its velocity v.
The numerical value of the kinetic energy can be calculated from the formula

$$K.E. = \frac{1}{2}mv^2 \qquad ...(8)$$

- Since both m and v^2 are always positive, K.E. is always positive and does not depend upon the direction of motion of the body.

9.2 Potential Energy

Potential energy is energy of the body by virtue of its position. A body is capable to do work by virtue of its position, configuration or state of strain.
Now relation between Potential energy and work done is
W.D = $-\Delta U$
where ΔU is change in potential energy
There are two common forms of potential energy, gravitational and elastic.

Important points related to Potential energy :

1. Potential energy is a straight function (defined only for position)
2. Potential energy of a point depends on a reference point
3. Potential energy difference between two position doesn't depend on the frame of reference.
4. Potential energy is defined only for conservative force because work done by conservative force is path independent.
5. If we define Potential energy for non conservative force then we have to define P.E. of a single point through different path which gives different value of P.E. at single point that doesn't make any sense.

9.2.1 (a) Gravitational Potential Energy :

It is possessed by virtue of height.

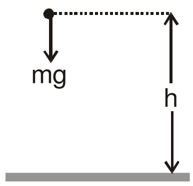
When an object is allowed to fall from one level to a lower level it gains speed due to gravitational pull, i.e., it gains kinetic energy. Therefore, in possessing height, a body has the ability to convert its gravitational potential energy into kinetic energy.

The gravitational potential energy is equivalent to the negative of the amount of work done by the weight of the body in causing the descent.

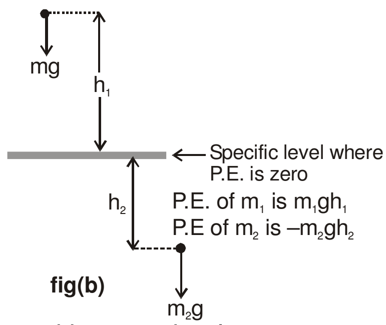
If a mass m is at a height h above a lower level the P.E. possessed by the mass is (mg) (h).

Since h is the height of an object above a specified level, an object below the specified level has negative potential energy.

Therefore GPE = ± mgh ...(9)



fig(a)



fig(b)

- The chosen level from which height is measured has no absolute position. It is important therefore to indicate clearly the zero P.E. level in any problem in which P.E. is to be calculated.
- GPE = ± mgh is applicable only when h is very small in comparison to the radius of the earth. We have discussed GPE in detail in 'GRAVITATION'.

9.2.2 (b) Elastic Potential Energy : It is a property of stretched or compressed springs.

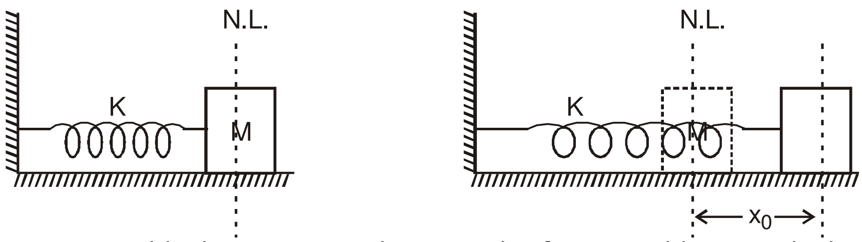
The end of a stretched elastic spring will begin to move if it is released. The spring. therefore possesses potential energy due to its elasticity. (i.e., due to change in its configuration)

The amount of elastic potential energy stored in a spring of natural length a and spring constant k when it is extended by a length x **(from the natural length)** is equivalent to the amount of work necessary to produce the extension.

Elastic Potential Energy = $\frac{1}{2}kx^2$ ...(10)

It is never negative whether the spring is extended or compressed.

Proof :



Consider a spring block system as shown in the figure and let us calculate work done by spring when the block is displaceed by x_0 from the natural length.

At any moment if the elongation in spring is x, then the force on the block by the spring is kx towards left. Therefore, the work done by the spring when block further displaces by dx

dW = – kx dx

$$\therefore \text{ Total work done by the spring, } W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2$$

Similarly, work done by the spring when it is given a compression x_0 is $-\frac{1}{2}kx_0^2$.

: We assume zero potential energy at natural length of the spring :

10. CONSERVATIVE FORCE AND POTENTIAL ENERGY :

$$F_s = -\frac{\partial U}{\partial s}$$

i.e. the projection of the force field , the vector F, at a given point in the direction of the displacement r equals the derivative of the potential energy U with respect to a given direction, taken with the opposite sign. The designation of a partial derivative $\partial/\partial s$ emphasizes the fact of deriving with respect to a definite direction.

So, having reversed the sign of the partial derivatives of the function U with respect to x, y, z, we obtain the projection F_x, F_y and F_z of the vector F on the unit vectors i, j and k. Hence, one can readily find the vector itself :

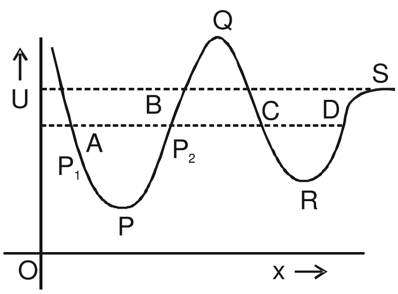
$$F = F_x i + F_y j + F_z k, \text{ or } F = -\left(\frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k\right)$$

The quantity in parentheses is referred to as the scalar gradient of the function U and is denoted by grad U or ∇U. We shall use the second, more convenient, designation where ∇ ("nabla") signifies the symbolic vector or operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Potential Energy curve :

- A graph plotted between the PE of a particle and its displacement from the centre of force field is called PE curve.
- Using graph, we can predict the rate of motion of a particle at various positions.
- Force on the particle is $F_{(x)} = -\frac{dU}{dx}$



Case : I On increasing x, if U increases, force is in (–) ve x direction i.e. attraction force.

Case : II On increasing x, if U decreases, force is in (+) ve x-direction i.e. repulsion force.

Different positions of a particle :

Position of equilibrium

If net force acting on a body is zero, it is said to be in equilibrium. For equilibrium $\frac{dU}{dx} = 0$. Points P, Q, R and S are the states of equilbrium positions.

Types of equilirbium :

- **Stable equilibrium :**

When a particle is displaced slightly from a position and a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position.

Necessary conditions:$-\frac{dU}{dx} = 0$,and $\frac{d^2U}{dx^2} = +$ ve

In figure P and R point shows stable equilibrium point.

- **Unstable Equilibrium :**

  When a particle is displaced slightly from a position and force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium.

  Condition : $\frac{dU}{dx} = 0$ potential energy is maximum i.e. = $\frac{d^2U}{dx^2}$ = – ve

  Q point in figure shows unstable equilibrium point

- **Neutral equilibrium** :

  In the neutral equilibrium potential energy is constant. When a particle is displaced from its position it does not experience any force acting on it and continues to be in equilibrium in the displaced position. This is said to be neutral equilibrium.

  In figure S is the neutral point

  Condition : $\frac{dU}{dx} = 0$, $\frac{d^2U}{dx^2}$ = 0

Ex.12 The potential energy between two atoms in a molecule is given by, $U_{(x)} = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants and x is the distance between the atoms. The system is in stable equilibrium when -

(A) x = 0 ***(B) $x = \frac{a}{2b}$*** ***(C) $x = \left(\frac{2a}{b}\right)^{1/6}$*** ***(D) $x = \left(\frac{11a}{5b}\right)$***

Sol. **(C)**

Given that, $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$

We, know $F = -\frac{du}{dx}$

$= (-12)\ a\ x^{-13} - (-6\ b)\ x^{-7} = 0$

or $\frac{6b}{x^7} = \frac{12a}{4x^{13}}$ or $x^6 = 12a/6b = 2a/b$ or $x = \left(\frac{2a}{b}\right)^{1/6}$

Ex.13 The potential energy of a conservative system is given by U = ax² – bx where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

Sol. In a conservative field $F = -\frac{dU}{dx}$ $\therefore F = -\frac{d}{dx}(ax^2 - bx) = b - 2ax$

For equilibrium F = 0 or b – 2ax = 0 $\therefore x = \frac{b}{2a}$

From the given equation we can see that $\frac{d^2U}{dx^2} = 2a$ (positive), i.e., U is minimum.

Therefore, $x = \frac{b}{2a}$ is the stable equilibrium positon.

11. WORK ENERGY THEOREM :

If the resultant or net force acting on a body is F_{net} then Newton's second law states that

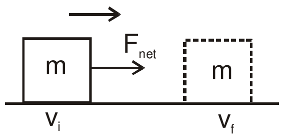
$F_{net} = ma$ ...(1)

If the resultant force varies with x, the acceleration and speed also depend on x.

then $a = v\dfrac{dv}{dx}$ ...(2)

from eq. (1)

$F_{net} = mv\dfrac{dv}{dx} \Rightarrow F_{net}.dx = m\,v\,dv$



$\int F_{net}.dx = \int_{v_i}^{v_f} mvdv$

$W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

$W_{net} = k_f - k_i$

$W_{net} = \Delta K$ ...(3)

Work done by net force F_{net} in displacing a particle equals to the change in kinetic energy of the particle i.e.

we can write eq. (3) in following way

$(W.D)_c + (W.D)_{N.C} + (W.D)_{ext.} + (W.D)_{pseudo}$

$= \Delta K$ ...(4)

where $(W.D)_c$ = work done by conservative force

$(W.D)_{N.C}$ = work done by non conservative force.

$(W.D)_{ext}$ = work done by external force

$(W.D)_{pseudo}$ = work done by pseudo force.

we know that

$(W.D)_c = -\Delta U$

$\Rightarrow -\Delta U + (W.D)_{N.C} + (W.D)_{ext} + (W.D)_{pseudo} = \Delta K$

$\Rightarrow (W.D)_{N.C} + (W.D)_{ext.} + (W.D)_{pseudo} = (k_f + u_f) - (k_i + u_i)$

$\because$ k + u = Mechanical energy.

$\Rightarrow$ work done by forces (except conservative forces)

= change is mechanical energy.

If $(W.D)_{N.C} = (W.D)_{ext} = (W.D)_{pseudo} = 0$

$K_f + U_f = K_i + U_i$

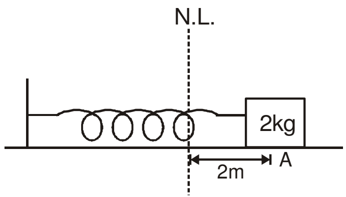
Initial mechanical energy = final mechanical energy

This is called mechanical energy conservation law.

Questions Based on work Energy Theorem :

(A) When only one conservative force is acting

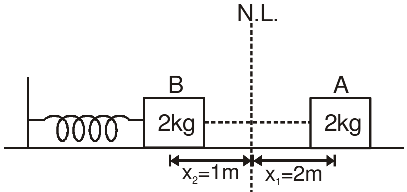
Ex.14 The block shown in figure is released from rest. Find out the speed of the block when the spring is compressed by 1 m.



Sol. In the above problem only one conservative force (spring force) is working on the block so from mechanical energy conservation

$k_f + u_f = k_i + u_i$ ...(i)

at A block is at rest so $k_i = 0$



$$u_i = \frac{1}{2}kx_1^2 = \frac{1}{2}k(2)^2 = 2k \text{ Joule}$$

At position B if speed of the block is v then

$$k_f = \frac{1}{2}mv^2 = \frac{1}{2}\times 2\times v^2 = v^2$$

$$u_f = \frac{1}{2}kx_2^2 = \frac{1}{2}\times k\times 1 = \frac{k}{2}$$

Putting the above values in equation (i), we get

$$\Rightarrow \quad v^2 + \frac{k}{2} = 2k \quad \Rightarrow \quad v^2 = \frac{3k}{2} \quad \Rightarrow \quad v = \sqrt{\frac{3k}{2}} \text{ m/sec}$$

Ex.15 ***A block of mass m is dropped from height h above the ground. Find out the speed of the block when it reaches the ground.***

Sol.

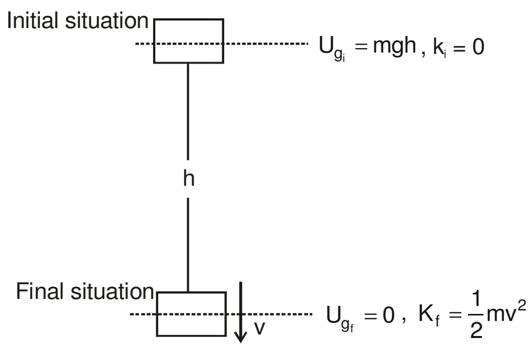


Figure shows the complete description of the problem only one conservative force is working on the block. So from mechanical energy conservation

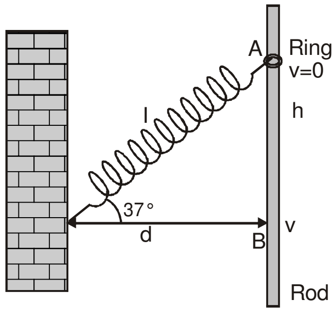
$$k_f + u_f = k_i + u_i \quad \Rightarrow \quad \frac{1}{2}mv^2 + 0 = 0 + mgh$$

$$v = \sqrt{2gh} \text{ m/sec}$$

(B) When two conservative force are acting in problem.

Ex.16 ***One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially the spring makes an angle of 37º with the horizontal as shown in fig. When the system is released from rest, find the speed of the ring when the spring becomes horizontal.***

[sin 37º = 3/5]



Sol. If l is the stretched length of the spring, then from figure

$$\frac{d}{l} = \cos 37^\circ = \frac{4}{5}, \text{ i.e., } \quad l = \frac{5}{4}d$$

So, the stretch $\quad y = l - d = \frac{5}{4}d - d = \frac{d}{4}$

and $\quad h = l \sin 37^\circ = \frac{5}{4}d \times \frac{3}{5} = \frac{3}{4}d$

Now, taking point B as reference level and applying law of conservation of mechanical energy between A and B,

$$E_A = E_B$$

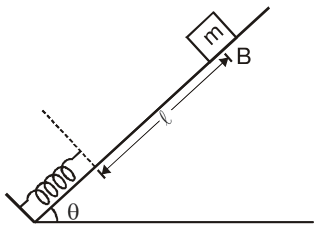
or $\quad mgh + \frac{1}{2}ky^2 = \frac{1}{2}mv^2$

[as for, B, h = 0 and y = 0]

or $\quad \frac{3}{4}mgd + \frac{1}{2}k\left(\frac{d}{4}\right)^2 = \frac{1}{2}mv^2 \qquad$ [as for A, $h = \frac{3}{4}d$ and $y = \frac{1}{4}d$]

or $\quad v = d\sqrt{\frac{3g}{2d} + \frac{k}{16m}} \qquad$ **Ans.**

Ex.*17* ***The block shown in figure is released from rest and initially the spring is at its natural length. Write down the energy conservation equation. When the spring is compressed by l_1?***



Sol. Here two conservative forces are included in the problem.

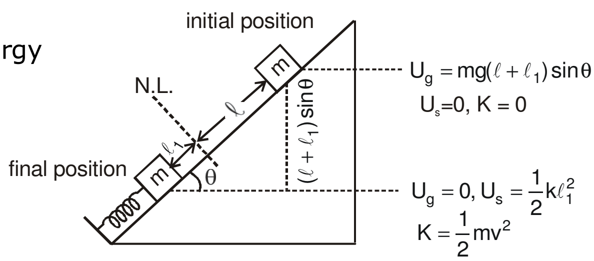
(i) Gravitational force (ii) spring force

We assume zero gravitational potential energy at A as shown in figure.

from mechanical energy conservation

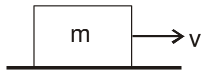
$k_f + u_f = k_i + u_i \qquad$ ...(i)

$$\frac{1}{2}mv^2 + \frac{1}{2}k\ell_1^2 = mg(\ell_1 + \ell)\sin\theta$$

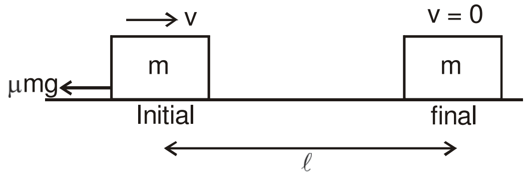


(C) When only one non conservative force is included in problem.

Ex.18 Find out the distance travelled by the block as shown in figure. If the initial speed of the block is v and μ is the friction coefficient between the surface of block and ground.



Sol. Applying work energy theorem, we get



$\Rightarrow \quad (-\mu\, mg\, \ell) = (0 + 0) - \left(\frac{1}{2}mv^2 + 0\right)$

$\Rightarrow \quad \mu\, g\, \ell = \frac{1}{2}v^2 \quad \Rightarrow \quad \ell = \frac{v^2}{2\mu g}$

(D) When both conservative and non-conservative force in the problem

Ex.19 A particle slides along a track with elevated ends and a flat central part as shown in figure. The flat portion BC has a length l = 3.0 m. The curved portions of the track are frictionless. For the flat part the coefficient of kinetic friction is μ_k = 0.20, the particle is released at point A which is at height h = 1.5 m above the flat part of the track. Where does the particle finally comes to rest?

Sol. As initial mechanical energy of the particle is mgh and final is zero, so loss in mechanical energy = *mgh*. This mechanical energy is lost in doing work against friction in the flat part,

So, loss in mechanical energy = work done against friction

or $mgh = \mu mgs$ i.e., $\quad s = \frac{h}{\mu} = \frac{1.5}{0.2} = 7.5$ m

After starting from B the particle will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend and at C will have the same value as it had when ascending, but now it will move from C to B. The same will be repeated and finally the particle will come to rest at E such that

BC + CB + BE = 7.5

or 3 + 3 + BE = 7.5

i.e., BE = 1.5

So, the particle comes to rest at the centre of the flat part.

Ex.20 A 0.5 kg block slides from the point A on a horizontal track with an initial speed 3 m/s towards a weightless horizontal spring of length 1 m and force constant 2 N/m. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as 0.22 and 0.20 respectively. If the distance AB and BD are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. [g = 10 m/s²]

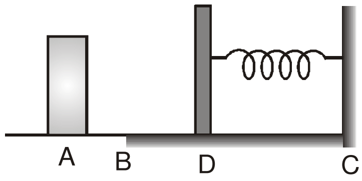
Sol. As the track AB is frictionless, the block moves this distance without loss in its initial

KE = $\frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 3^2 = 2.25$ J. In the path BD as friction is present, so work done against friction

= $\mu_k\, mgs = 0.2 \times 0.5 \times 10 \times 2.14 = 2.14$ J

So, at D the KE of the block is = 2.25 – 2.14 = 0.11 J.

Now, if the spring is compressed by x



$0.11 = \frac{1}{2} \times k \times x^2 + \mu_k mgx$

i.e., $0.11 = \frac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x$

or $x^2 + x - 0.11 = 0$

which on solving gives positive value of x = 0.1 m

After moving the distance x = 0.1 m the block comes to rest. Now the compressed spring exerts a force :

$F = kx = 2 \times 0.1 = 0.2$ N

on the block while limiting frictional force between block and track is $f_L = \mu_s mg = 0.22 \times 0.5 \times 10 = 1.1$ N.

Since, $F < f_L$. The block will not move back. So, the total distance moved by block

= AB + BD + 0.1

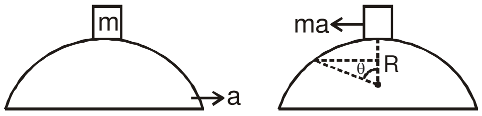
= 2 + 2.14 + 0.1

= 4.24 m

(E) Important Examples :

Ex.21 A smooth sphere of radius R is made to translate in a straight line with a constant acceleration a. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the paritcle with respect to the sphere as a function of the angle θ it slides.

Sol. We solve the above problem with respect to the sphere. So apply a pseudo force on the particle



Now from work energy theorem.

work done by ma = change in mechanical energy

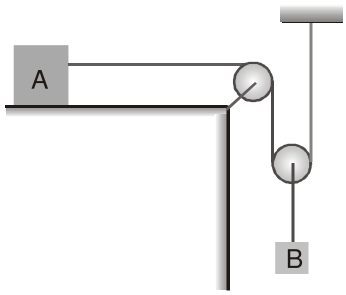
$\Rightarrow$ ma R sin $\theta = (k_f + u_f) - (k_i + u_i)$

$maR \sin\theta = \frac{1}{2}mv^2 - mgR(1-\cos\theta) \Rightarrow \frac{1}{2}mv^2 = maR \sin\theta + mgR(1 - \cos\theta)$

$\Rightarrow v^2 = 2R(a \sin\theta + g - g\cos\theta) \Rightarrow v = [2R(a \sin\theta + g - g\cos\theta)]^{1/2}$ m/sec

Ex.22 In the arrangement shown in figure $m_A = 4.0$ kg and $m_B = 4.0$ kg. The system is released from rest and block B is found to have a speed 0.3 m/s after it has descended through a distance of 1m. Find the coefficient of friction between the block and the table. Neglect friction elsewhere.

(Take g = 10 m/s²)



Sol. From constraint relations, we can see that

$v_A = 2v_B$

Therefore, $v_A = 2(0.3) = 0.6$ m/s

as $v_B = 0.3$ m/s (given)

Applying $W_{nc} = \Delta U + \Delta K$

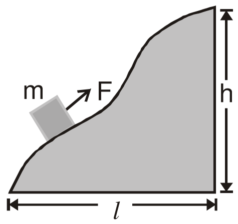
we get $-\mu m_A g S_A = -m_B g S_B + \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$

Here, $S_A = 2S_B = 2m$ as $S_B = 1$ m (given)

$\therefore \quad -\mu(4.0)(10)(2) = -(1)(10)(1) + \frac{1}{2}(4)(0.6)^2 + \frac{1}{2}(1)(0.3)^2$

or $-80\mu = -10 + 0.72 + 0.045$ or $80\mu = 9.235$ or $\mu = 0.115$ **Ans.**

Ex.23 A body of mass 'm' was slowly hauled up the hill as shown in the figure by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.



Sol. Four forces are acting on the body :

1. weight (mg) 2. normal reaction (N)
3. friction (f) and 4. the applied force (F)

Using work-energy theorem

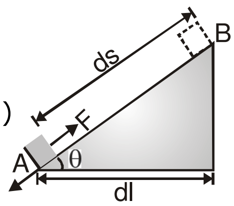
$$W_{net} = \Delta KE$$

or $W_{mg} + W_N + W_f + W_F = 0$

Here, $\Delta KE = 0$, because $K_i = 0 = K_f$

$W_{mg} = -mgh \Rightarrow W_N = 0$

(as normal reaction is perpendicular to displacement at all points)



W_f can be calculated as under :

$f = \mu mg \cos\theta$

$\therefore \quad (dW_{AB})_f = -f\,ds$

$= -(\mu mg\cos\theta)\,ds = -\mu mg\,(dl)$ (as $ds\cos\theta = dl$)

$\therefore \quad f = -\mu mg \sum dl = -\mu mgl$

Substituting these values in Eq. (i), we get

$$W_F = mgh + \mu mgl$$

: Here again, if we want to solve this problem without using work-energy theorem we will first find magnitude of applied force $\vec{F}$ at different locations and then integrate dW ($= \vec{F}.d\vec{r}$) with proper limits.

12. POWER

Power is defined as the time rate of doing work.

When the time taken to complete a given amount of work is important, we measure the power of the agent doing work.

The average power ($\bar{P}$ or P_{av}) delivered by an agent is given by

$$\bar{P} \text{ or } P_{av} = \frac{\Delta W}{\Delta t} = \frac{\text{Total work done}}{\text{Total time}}$$

where ΔW is the amount of work done in time Δt.

Power is the ratio of two scalars-work and time. So, power is a scalar quantity. If time taken to complete a given amount of work is more, then power is less.

- The instantaneous power is, $P = \frac{dW}{dt}$ where dW is the work done by a force $\vec{F}$ in a small time dt.
- $P = \frac{dW}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{v}$ where $\vec{v}$ is the velocity of the body.

By definition of dot product,

$P = Fv\cos\theta$

where θ is the smaller angle between $\vec{F}$ and $\vec{v}$

This P is called as instantaneous power if dt is very small.

12.1 Unit of Power :

A unit power is the power of an agent which does unit work in unit time.

The power of an agent is said to be one watt if it does one joule of work in one second.

1 watt = 1 joule/second = 10^7 erg/second

Also, 1 watt = $\frac{1\text{ newton} \times 1\text{ metre}}{1\text{ second}}$ = 1 N ms^{-1}.

Dimensional formula of power

$$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]} = \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$$

Ex.24 A one kilowatt motor pumps out water from a well 10 metre deep. Calculate the quantity of water pumped out per second.

Sol. Power, P = 1 kilowatt = 10^3 watt

S = 10 m ; Time, t = 1 second ; Mass of water, m = ? Power = $\frac{mg \times S}{t}$

$\therefore \quad 10^3 = \frac{m \times 9.8 \times 10}{1}$

or $\quad m = \frac{10^3}{9.8 \times 10}\text{kg} = 10.204$ kg

Ex.25 The blades of a windmill sweep out a circle of area A. (a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through in time t? (b) What is the kinetic energy or the air? (c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that A = 30m², v = 36 km h⁻¹ and the density of air is 1.2 kg m⁻³. What is the electrical power produced?

Sol. (a) Volume of wind flowing per second = Av

Mass of wind flowing per second = $Av\rho$

Mass of air passing in t second = $Av\rho t$

(b) Kinetic energy of air $= \frac{1}{2}mv^2 = \frac{1}{2}(Av\rho t)v^2 = \frac{1}{2}Av^3\rho t$

(c) Electrical energy produced = $\frac{25}{100} \times \frac{1}{2}Av^3\rho t = \frac{Av^3\rho t}{8}$

Electrical power $= \frac{Av^3\rho t}{8t} = \frac{Av^3\rho}{8}$

Now, A = 30 m^2, v = 36 km h^{-1} = 36 × $\frac{5}{18}$ m s^{-1} = 10 m s^{-1}, ρ = 1.2 kg ms^{-1}

$\therefore$ Electrical power $= \dfrac{30 \times 10 \times 10 \times 1.2}{8}$ W = 4500 W = 4.5 kW

Ex.26 One coolie takes one minute to raise a box through a height of 2 metre. Another one takes 30 second for the same job and does the same amount of work. Which one of the two has greater power and which one uses greater energy?

Sol. Power of first coolie = $\dfrac{\text{Work}}{\text{Time}} = \dfrac{M \times g \times S}{t} = \dfrac{M \times 9.8 \times 2}{60} Js^{-1}$

Power of second coolie = $\dfrac{M \times 9.8 \times 2}{30} Js^{-1} = 2\left(\dfrac{M \times 9.8 \times 2}{60}\right)$ J s^{-1} = 2 × Power of first coolie

So, the power of the second coolie is double that of the first. Both the coolies spend the same amount of energy.

We know that W = Pt

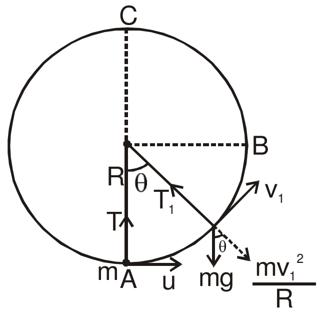
For the same work,

$W = p_1t_1 = P_2t_2$

or $\dfrac{P_2}{P_1} = \dfrac{t_1}{t_2} = \dfrac{1\,\text{minute}}{30\,\text{s}} = 2$ or $P_2 = 2P_1$

13. VERTICAL CIRCULAR MOTION

To understand this consider the motion of a small body (say stone) tied to a string and whirled in a vertical circle.



Now we study the circular motion of the body in two parts.

A) Motion of a body from A to B.

B) Motion of a body from B to C.

A. Motion of a body from A to B.

$T_1 = mg\cos\theta + \dfrac{mv_1^2}{R}$ ...(1)

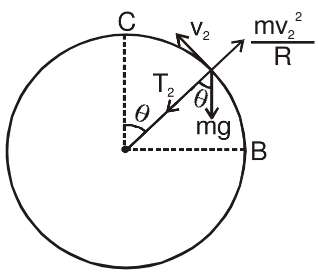
During the motion of the body from A to B.

θ will increase so cos θ will decrease.

Due to which mg cos θ will decrease. From A to B speed of the body also decreases due to which $\dfrac{mv^2}{R}$ decreases. Therefore tenstion in the string decreases from A to B.

But due to mg cos θ tension can never be zero.

B. Motion of a body from B to C.



$T_2 = \dfrac{mv_2^2}{R} - mg\cos\theta$ ...(2)

From B $\rightarrow$ C

speed decreases due to which $\dfrac{mv_2^2}{R}$ decreases.

θ decreases due to which mg cos θ increases.

Therefore from B $\rightarrow$C. Tension in the string decreases.

String slacks at a point where $\dfrac{mv_2^2}{R}$ = mg cos θ i.e., T = 0

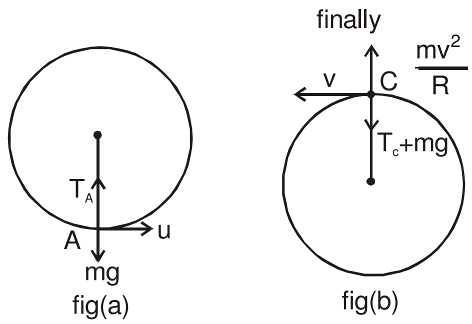
13.1 Minimum velocity at point A for which body can complete the vertical circle

The condition for the body to complete the vertical circle is that the string should be taut all the time i.e. the tension is greater than zero.

So the body can complete the vertical circle if the tension is not zero in between the region B to C.

Initially.

from figure (b) $\frac{mv^2}{R} = T_C + mg$ ...(1)



Apply energy conservation from A to C then

$K_f + U_f = K_i + U_i$

$\frac{1}{2}mv^2 + 2mgR = \frac{1}{2}mu^2 + 0$ ...(2)

body can complete vertical circle, when

$T_C \geq 0$

$\frac{mv^2}{R} - mg \geq 0$

$\Rightarrow v^2 \geq gR$ ...(3)

Put the value from (3) to (2) and

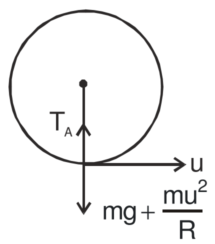
$u = u_{min}$

$\Rightarrow \frac{1}{2}m(Rg) + 2mgR = \frac{1}{2}mu_{min}^2$

$\Rightarrow u_{min}^2 = 5gR \Rightarrow u_{min} = \sqrt{5gR}$

It the velocity is greater than equal to $\sqrt{5gR}$ then the body will complete the vertical circle.

Tension at A

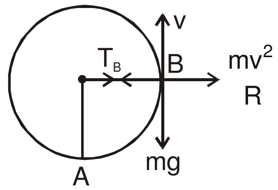


$T_A = mg + \frac{mu^2}{R}$

If $u = u_{min} = \sqrt{5gR}$

then $T_A = mg + \frac{5mgR}{R} \Rightarrow T_A = 6mg$

Tension at B



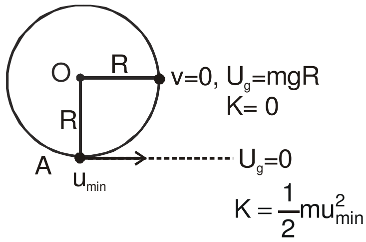
$T_B = \frac{mv^2}{R}$

energy conservation from A to B

$\frac{1}{2}mu_{min}^2 = mgR + \frac{1}{2}mv^2$

$\Rightarrow v^2 = 3gR \Rightarrow T_B = 3mg$

13.2 Condition for the body to reach B :



Let us calculate the u_{min} such that the body just reaches B.

Work done by tension = 0

Only gravitational force is working on the body which is a conservative force. Therefore Applying conservation of energy, we get

$$mgR = \frac{1}{2}mu_{min}^2 \Rightarrow u_{min} = \sqrt{2gR}$$

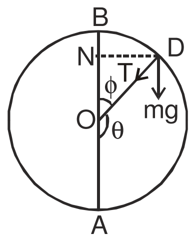
$\therefore$ if $u \leq \sqrt{2gR}$ then the body will oscillate about A.

13.3 When $\sqrt{2gR} < u < \sqrt{5gR}$

If the velocity of projection is greater than $\sqrt{2gR}$ but less than $\sqrt{5gR}$, the particle rises above the horizontal diameter and the tension vanishes before reaching the highest point.
We have seen that the tension in the string at the highest point is lower than the tension at the lowest point.
At the point D, the string OD makes an angle ϕ with the vertical. The radial component of the weight is mg cos ϕ **towards** the centre O.

$$T + mg\cos\phi = \frac{mv^2}{R} \quad \Rightarrow \quad T = m\left(\frac{v^2}{R} - g\cos\phi\right) \qquad \text{...(i)}$$

Kinetic energy at D = $\frac{1}{2}mv^2$



Potential energy at D = mg(AN)
= mg (AO + ON) $\Rightarrow$ mg(R + R cosϕ) = mgR(1 + cosϕ)
From conservation of energy

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgR(1 + \cos\phi)$$

$v^2 = u^2 - 2gR(1 + \cos\phi)$
Substituting in equation (i),

$$T = m\left[\frac{u^2}{R} - 2g(1+\cos\phi) - g\cos\phi\right]$$

$$T = m\left[\frac{u^2}{R} - 3g\left(\cos\phi + \frac{2}{3}\right)\right]$$

This equation shows that the tension becomes zero. if

$$\frac{u^2}{R} = 3g\left(\cos\phi + \frac{2}{3}\right) \qquad \text{...(ii)}$$

If the tension is not to become zero.

$$u^2 > 3Rg\left(\cos\phi + \frac{2}{3}\right)$$

Equation (ii) gives the values of ϕ at which the string becomes slack.

$$\cos\phi + \frac{2}{3} = \frac{u^2}{3Rg}$$

$$\cos\phi = \frac{u^2}{3Rg} - \frac{2}{3}$$

$$\cos\phi = \frac{u^2 - 2gR}{3gR}$$

It is the angle from the vertical at which tension in the string vanishes to zero. And after that its motion is projectile.

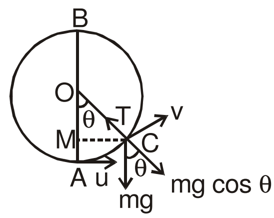
13.4 Tension in the string versus θ

We may find an expression for the tension in the string when it makes an angle θ with the vertical. At C, the weight of the body acts vertically downwards, and the tension in the string is towards the centre O.

The weight mg is resolved radially and tangentially.
The radial component is mg cos θ and the tangential component is mg sin θ.

$T - mg\cos\theta = \frac{mv^2}{R}$, where v is the velocity at C.

i.e., $T = m\left(\frac{v^2}{R} + g\cos\theta\right)$ ...(i)



The velocity v can be expressed in terms of velocity u at A.

The total energy at A = $\frac{1}{2}mu^2$

The kinetic energy at C = $\frac{1}{2}mv^2$

The potential energy at C = mg (AM)
= mg (AO – MO) = mg (R – R cosθ)
= mgR (1 – cos θ)

The total energy at C = $\frac{1}{2}mv^2 + mgR(1-\cos\theta)$

$\therefore$ From conservation of energy

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgR(1-\cos\theta)$$

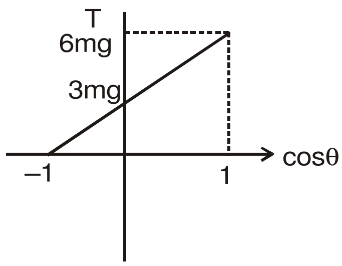
$u^2 = v^2 + 2gR(1-\cos\theta)$ or $v^2 = u^2 - 2gR(1-\cos\theta)$

Substituting in equation (v),

$$T = m\left[g\cos\theta + \frac{u^2}{R} - 2g(1-\cos\theta)\right] = \frac{mu^2}{R} + 3mg\left(\cos\theta - \frac{2}{3}\right) \quad ...(ii)$$

This expression gives the value of the tension in the string in terms of the velocity at the lowest point and the angle θ.

Equation (i) shows that tension in the string decreases as θ increases, since the term 'g cos θ' decreases as θ increases.



when $u = \sqrt{5gR}$

$\Rightarrow T = 3mg(1 + \cos\theta)$

Now $\theta = 0 \Rightarrow \cos\theta = 1 \quad \Rightarrow \quad T_A = 6$ mg

if, $\theta = 90° \quad \Rightarrow \quad T_B = 3$ mg

$\theta = 180°$, $\cos\theta = -1$

$T_c = 0$

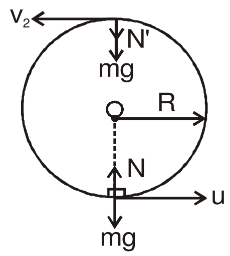
13.5 Different situations :

(A) A BODY MOVING INSIDE A HOLLOW TUBE OR SPHERE

The previous discussion holds good for this case, but instead of tension in the string we have the normal reaction of the

surface. If N is the normal reaction at the lowest point, then

the condition $u \geq \sqrt{5Rg}$ for the body to complete the circle holds for this case also. All other equations (can be) similarly obtained by replacing tension T by normal reaction N.



(B) WHEN BODY IS ATTACHED TO A ROD OF LENGTH R

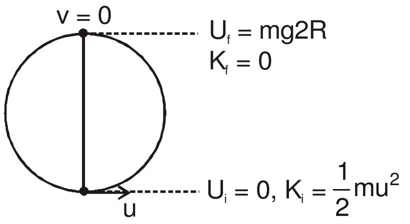
In this case since the body is attached to a rigid rod. The body can not leave the circular path.

Therefore, if the speed of the body becomes zero before the highest point C. It's motion will be oscillatory about the centre of the rod.

Condition for completing the circle :

If the body just reaches the highest point then it will completes the vertical circle

Applying energy conservation between the lowest and highest point of circle, we get

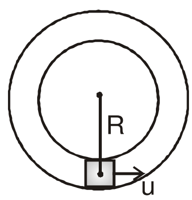


$$2mgR = \frac{1}{2}mu^2 \Rightarrow u = \sqrt{4gR}$$

So, If the velocity at point A is greater than equal to $\sqrt{4gR}$ then

body will compete the vertical circle.

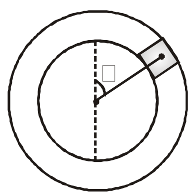
(C) VERTICAL MOTION IN A DUAL RING



This system will behave as the preivious system. So u_{min} to

complete vertical circle $u_{min} = \sqrt{4gR}$

Angle at which the normal reaction on the body will change its direction from inward to outward the ring is given by

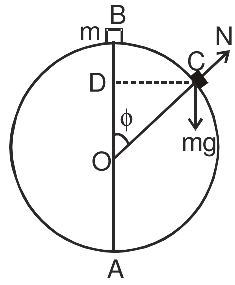


$$\cos\phi = \frac{u^2 - 2gR}{3gR}$$

(D) BODY MOVING ON A SPHERICAL SURFACE

The small body of mass m is placed on the top of a smooth sphere of radius R and the body slides down the surface.

At any instant, i.e., at point C the forces are the normal reaction N and the weight mg. The radial component of the weight is mgcos ϕ acting towards the centre. The centripetal force is



$$mg\cos\phi - N = \frac{mv^2}{R},$$

where v is the velocity of the body at O.

$$N = m\left(g\cos\phi - \frac{v^2}{R}\right) \qquad ...(i)$$

The body flies off the surface at the point where N becomes zero.

i.e., $g\cos\phi = \frac{v^2}{R}$; $\cos\phi = \frac{v^2}{Rg}$ ...(ii)

To find v, we use conservation of energy

i.e., $\frac{1}{2}mv^2 = mg\,(BD)$

$= mg\,(OB - OD) = mgR\,(1 - \cos\phi)$

$v^2 = 2Rg\,(1 - \cos\phi)$

$2(1 - \cos\phi) = \frac{v^2}{Rg}$ ...(iii)

From equation (ii) and (iii) we get

$\cos\phi = 2 - 2\cos\phi$; $3\cos\phi = 2$

$\cos\phi = \frac{2}{3}$; $\phi = \cos^{-1}\left(\frac{2}{3}\right)$ ...(iv)

This gives the angle at which the body goes of the surface. The height from the ground of that point

$= AD = R(1 + \cos\phi)$

$= R\left(1 + \frac{2}{3}\right) = \frac{5}{3}R$