Solutions to I.E. Irodov's Problems in General Physics

Volume I

Mechanics • Heat • Electrodynamics

SECOND EDITION

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CBS

CBS PUBLISHERS & DISTRIBUTORS
4596/1A, 11 DARYAGANJ, NEW DELHI - 110 002 (INDIA)

Dedicated to
my Teacher
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ISBN : 81-239-0399-5

First Edition : 1995
Reprint : 1997
Second Edition : 1998
Reprint : 2000
Reprint : 2001
Reprint : 2002
Reprint : 2003
Reprint : 2004
Reprint : 2005

Published by S.K. Jain for CBS Publishers & Distributors,
4596/1A, 11 Darya Ganj, New Delhi - 110 002 (India)

Printed at :
India Binding House, Delhi - 110 032

FOREWORD

Science, in general, and physics, in particular, have evolved out of man's quest to know beyond unknowns. Matter, radiation and their mutual interactions are basically studied in physics. Essentially, this is an experimental science. By observing appropriate phenomena in nature one arrives at a set of rules which goes to establish some basic fundamental concepts. Entire physics rests on them. Mere knowledge of them is however not enough. Ability to apply them to real day-to-day problems is required. Prof. Irodov's book contains one such set of numerical exercises spread over a wide spectrum of physical disciplines. Some of the problems of the book long appeared to be notorious to pose serious challenges to students as well as to their teachers. This book by Prof. Singh on the solutions of problems of Irodov's book, at the outset, seems to remove the sense of awe which at one time prevailed. Traditionally a difficult exercise to solve continues to draw the attention of concerned persons over a sufficiently long time. Once a logical solution for it becomes available, the difficulties associated with its solutions are forgotten very soon. This statement is not only valid for the solutions of simple physical problems but also to various physical phenomena.

Nevertheless, Prof. Singh's attempt to write a book of this magnitude deserves an all out praise. His ways of solving problems are elegant, straight forward, simple and direct. By writing this book he has definitely contributed to the cause of physics education. A word of advice to its users is however necessary. The solution to a particular problem as given in this book is never to be consulted unless an all out effort in solving it independently has been already made. Only by such judicious uses of this book one would be able to reap better benefits out of it.

As a teacher who has taught physics and who has been in touch with physics curricula at I.I.T., Delhi for over thirty years, I earnestly feel that this book will certainly be of benefit to younger students in their formative years.

Dr. Dilip Kumar Roy
Professor of Physics
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FOREWORD

A proper understanding of the physical laws and principles that govern nature require solutions of related problems which exemplify the principle in question and leads to a better grasp of the principles involved. It is only through experiments or through solutions of multifarious problem-oriented questions can a student master the intricacies and fall outs of a physical law. According to Ira M. Freeman, professor of physics of the state university of new Jersy at Rutgers and author of "physic--principles and Insights" -- "In certain situations mathematical formulation actually promotes intuitive understanding....... Sometimes a mathematical formulation is not feasible, so that ordinary language must take the place of mathematics in both roles. However, Mathematics is far more rigorous and its concepts more precise than those of language. Any science that is able to make extensive use of mathematical symbolism and procedures is justly called an exact science". I.E. Irodov's problems in General Physics fulfills such a need. This book originally published in Russia contains about 1900 problems on mechanics, thermodynamics, molecular physics, electrodynamics, waves and oscillations, optics, atomic and nuclear physics. The book has survived the test of class room for many years as is evident from its number of reprint editions, which have appeared since the first English edition of 1981, including an Indian Edition at affordable price for Indian students.

Abhay Kumar Singh's present book containing solutions to Dr. I.E. Irodov's Problems in General Physics is a welcome attempt to develop a student's problem solving skills. The book should be very useful for the students studying a general course in physics and also in developing their skills to answer questions normally encountered in national level entrance examinations conducted each year by various bodies for admissions to professional colleges in science and technology.

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Professor of Physics
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PREFACE TO THE SECOND EDITION

Nothing succeeds like success, they say. Now, consequent upon the warm welcome on the part of students and the teaching fraternity this revised and enlarged edition of this volume is before you. In order to make it more up-to-date and viable, a large number of problems have been streamlined with special focus on the complicated and ticklish ones, to cater to the needs of the aspiring students.

I extend my deep sense of gratitude to all those who have directly or indirectly engineered the cause of its existing status in the book world.

Patna

June 1997 **Abhay Kumar Singh**

PREFACE TO THE FIRST EDITION

When you invisage to write a book of solutions to problems, one pertinent question crops up in the mind that—why solution! Is this to prove one's erudition? My only defence against this is that the solution is a challenge to save the scientific man hours by channelizing thoughts in a right direction.

The book entitled "Problems in General Physics" authored by I.E. Irodov (a noted Russian physicist and mathematician) contains 1877 intriguing problems divided into six chapters.

After the acceptance of my first book "Problems in Physics", published by Wiley Eastern Limited, I have got the courage to acknowledge the fact that good and honest ultimately win in the market place. This stimulation provided me insight to come up with mv second attempt—"Solutions to I.E. Irodov's Problems in General Physics."

This first volume encompasses solutions of first three chapters containing 1052 problems. Although a large number of problems can be solved by different methods, I have adopted standard methods and in many of the problems with helping hints for other methods.

In the solutions of chapter three, the emf of a cell is represented by ξ (xi) in contrast to the notation used in figures and in the problem book, due to some printing difficulty.

I am thankful to my students Mr. Omprakash, Miss Neera and Miss Punam for their valuable co-operation even in my hard days while authoring the present book. I am also thankful to my younger sister Prof. Ranju Singh, my younger brother Mr. Ratan Kumar Singh, my junior friend Miss Anupama Bharti, other well wishers and friends for their emotional support. At last and above all I am grateful to my Ma and Pappaji for their blessings and encouragement.

ABHAY KUMAR SINGH

CONTENTS

PART ONE

PHYSICAL FUNDAMENTALS OF MECHANICS

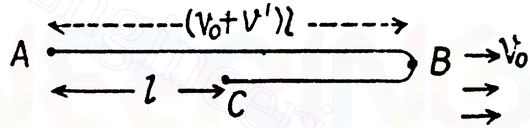
1.1 KINEMATICS

1.1 Let v_0 be the stream velocity and v' the velocity of motorboat with respect to water. The motorboat reached point B while going downstream with velocity $(v_0 + v')$ and then returned with velocity $(v' - v_0)$ and passed the raft at point C. Let t be the time for the raft (which flows with stream with velocity v_0) to move from point A to C, during which the motorboat moves from A to B and then from B to C.

Therefore

$$\frac{l}{v_0} = \tau + \frac{(v_0 + v')\tau - l}{(v' - v_0)}$$

On solving we get $v_0 = \dfrac{l}{2\tau}$



1.2 Let s be the total distance traversed by the point and t_1 the time taken to cover half the distance. Further let $2t$ be the time to cover the rest half of the distance.

Therefore $\qquad \dfrac{s}{2} = v_0 t_1 \quad$ or $\quad t_1 = \dfrac{s}{2 v_0} \qquad (1)$

and $\qquad \dfrac{s}{2} = (v_1 + v_2)t \quad$ or $\quad 2t = \dfrac{s}{v_1 + v_2} \qquad (2)$

Hence the sought average velocity

$$\langle v \rangle = \frac{s}{t_1 + 2t} = \frac{s}{[s/2 v_0] + [s/(v_1 + v_2)]} = \frac{2 v_0 (v_1 + v_2)}{v_1 + v_2 + 2 v_0}$$

1.3 As the car starts from rest and finally comes to a stop, and the rate of acceleration and deceleration are equal, the distances as well as the times taken are same in these phases of motion.

Let Δt be the time for which the car moves uniformly. Then the acceleration / deceleration time is $\dfrac{\tau - \Delta t}{2}$ each. So,

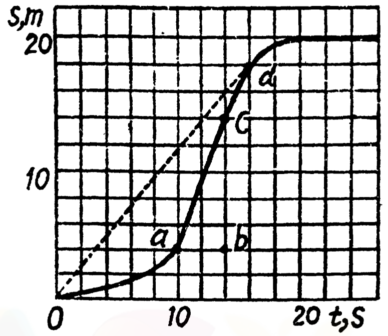
$$<v>\tau = 2\left\{\frac{1}{2}w\frac{(\tau-\Delta t)^2}{4}\right\} + w\frac{(\tau-\Delta t)}{2}\Delta t$$

or $$\Delta t^2 = \tau^2 - \frac{4<v>\tau}{w}$$

Hence $$\Delta t = \tau\sqrt{1 - \frac{4<v>}{w\tau}} = 15 \text{ s.}$$

1.4 (a) Sought average velocity

$$<v> = \frac{s}{t} = \frac{200 \text{ cm}}{20\text{s}} = 10 \text{ cm/s}$$



(b) For the maximum velocity, $\frac{ds}{dt}$ should be maximum. From the figure $\frac{ds}{dt}$ is maximum for all points on the line ac, thus the sought maximum velocity becomes average velocity for the line ac and is equal to :

$$\frac{bc}{ab} = \frac{100 \text{ cm}}{4\text{s}} = 25 \text{ cm/s}$$

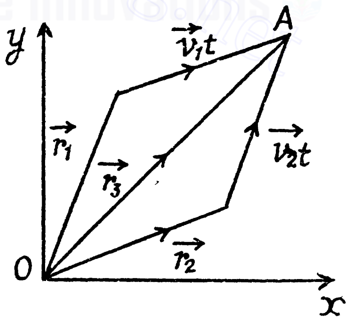
(c) Time t_0 should be such that corresponding to it the slope $\frac{ds}{dt}$ should pass through the point O (origin), to satisfy the relationship $\frac{ds}{dt} = \frac{s}{t_0}$. From figure the tangent at point d passes through the origin and thus corresponding time $t = t_0 = 16$ s.

1.5 Let the particles collide at the point A (Fig.), whose position vector is $\vec{r}_3$ (say). If t be the time taken by each particle to reach at point A, from triangle law of vector addition :

$$\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$

so, $$\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1)t \quad (1)$$

therefore, $$t = \frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|} \quad (2)$$



From Eqs. (1) and (2)

$$\vec{r}_1 = \vec{r}_2 - (\vec{v}_2 - \vec{v}_1)\frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$$

or, $$\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|},$$ which is the sought relationship.

1.6 We have

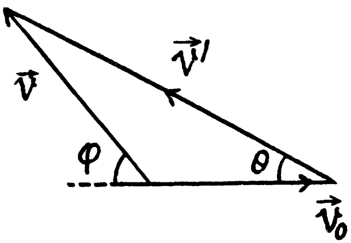
$$\vec{v}' = \vec{v} - \vec{v}_0 \quad (1)$$

From the vector diagram [of Eq. (1)] and using properties of triangle

$$v' = \sqrt{v_0^2 + v^2 + 2\,v_0\,v\cos\varphi} = 39.7\ \text{km/hr} \qquad (2)$$

and $$\frac{v'}{\sin(\pi - \varphi)} = \frac{v}{\sin\theta} \quad \text{or,} \quad \sin\theta = \frac{v\sin\varphi}{v'}$$

or $$\theta = \sin^{-1}\left(\frac{v\sin\varphi}{v'}\right)$$



Using (2) and putting the values of v and d

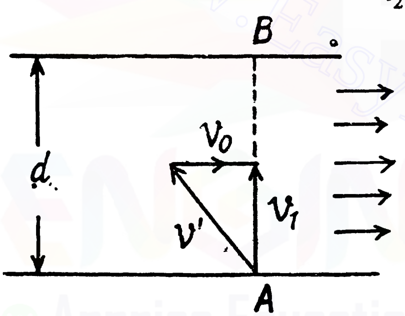
$$\theta = 19.1°$$

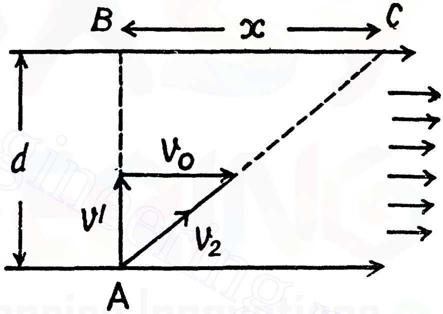
1.7 Let one of the swimmer (say 1) cross the river along AB, which is obviously the shortest path. Time taken to cross the river by the swimmer 1.

$$t_1 = \frac{d}{\sqrt{v'^2 - v_0^2}}, \text{ (where } AB = d \text{ is the width of the river)} \qquad (1)$$

For the other swimmer (say 2), which follows the quickest path, the time taken to cross the river.

$$t_2 = \frac{d}{v'} \qquad (2)$$





In the time t_2, drifting of the swimmer 2, becomes

$$x = v_0\,t_2 = \frac{v_0}{v'}d, \quad \text{(using Eq. 2)} \qquad (3)$$

If t_3 be the time for swimmer 2 to walk the distance x to come from C to B (Fig.), then

$$t_3 = \frac{x}{u} = \frac{v_0\,d}{v'\,u} \quad \text{(using Eq. 3)} \qquad (4)$$

According to the problem $t_1 = t_2 + t_3$

or, $$\frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0\,d}{v'\,u}$$

On solving we get

$$u = \frac{v_0}{\left(1 - \dfrac{v_0^2}{v'^2}\right)^{-\frac{1}{2}} - 1} = 3\ \text{km/hr.}$$

1.8 Let l be the distance covered by the boat A along the river as well as by the boat B acro the river. Let v_0 be the stream velocity and v' the velocity of each boat with respect water. Therefore time taken by the boat A in its journey

$$t_A = \frac{l}{v'+v_0} + \frac{l}{v'-v_0}$$

and for the boat B
$$t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$$

Hence,
$$\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}} \quad \left(\text{where } \eta = \frac{v'}{v}\right)$$

On substitution $t_A/t_B = 1{\cdot}8$

1.9 Let v_0 be the stream velocity and v' the velocity of boat with respect to water. A $\frac{v_0}{v'} = \eta = 2 > 0$, some drifting of boat is inevitable.

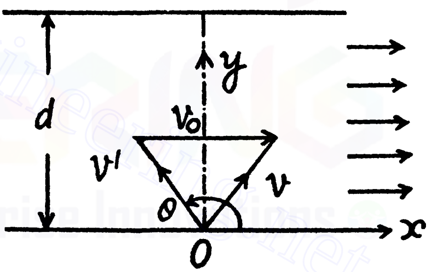
Let $\vec{v}'$ make an angle θ with flow direction. (Fig.), then the time taken to cross the rive

$$t = \frac{d}{v' \sin\theta} \quad \text{(where } d \text{ is the width of the river)}$$

In this time interval, the drifting of the boat

$$x = (v' \cos\theta + v_0)\, t$$

$$= (v'\cos\theta + v_0)\frac{d}{v'\sin\theta} = (\cot\theta + \eta\,\text{cosec}\,\theta)\, d$$



For x_{min} (minimum drifting)

$$\frac{d}{d\theta}(\cot\theta + \eta\,\text{cosec}\,\theta) = 0, \text{ which yields}$$

$$\cos\theta = -\frac{1}{\eta} = -\frac{1}{2}$$

Hence, $\theta = 120°$

1.10 The solution of this problem becomes simple in the frame attached with one of the bodies. Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematic equation for constant acceleration :

$$\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\, t + \frac{1}{2}\vec{w}_{12}\, t^2$$

So, $\vec{r}_{12} = \vec{v}_{0(12)}\, t$, (because $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

or, $|\vec{r}_{12}| = |\vec{v}_{0(12)}|\, t$ (1)

But $|\vec{v}_{01}| = |\vec{v}_{02}| = v_0$

So, from properties of triangle

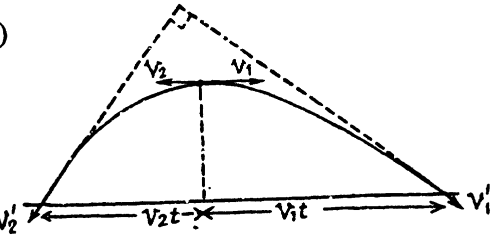
$$v_{0(12)} = \sqrt{v_0^2 + v_0^2 - 2\, v_0\, v_0 \cos(\pi/2 - \theta_0)}$$

Hence, the sought distance

$$|\vec{r}_{12}| = v_0\sqrt{2(1 - \sin\theta)}\; t = 22 \text{ m.}$$

1.11 Let the velocities of the paricles (say $\vec{v_1}'$ and $\vec{v_2}'$) becomes mutually perpendicular after time t. Then their velocitis become

$$\vec{v_1}' = \vec{v_1} + \vec{g}\,t\,;\quad \vec{v_2}' = \vec{v_2} + \vec{g}\,t \qquad (1)$$



As $\vec{v_1}' \perp \vec{v_2}'$ so, $\vec{v_1}' \cdot \vec{v_2}' = 0$

or, $(\vec{v_1} + \vec{g}\,t) \cdot (\vec{v_2} + \vec{g}\,t) = 0$

or $-v_1 v_2 + g^2 t^2 = 0$

Hence, $$t = \frac{\sqrt{v_1 v_2}}{g} \qquad (3)$$

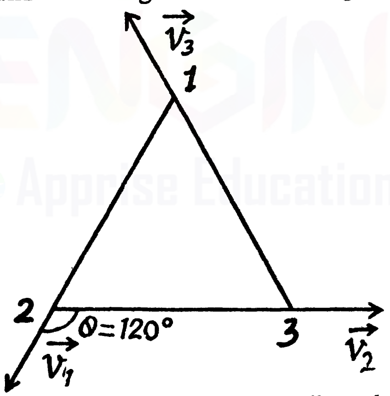
Now form the Eq. $\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\,t + \frac{1}{2}\vec{w}_{12}\,t^2$

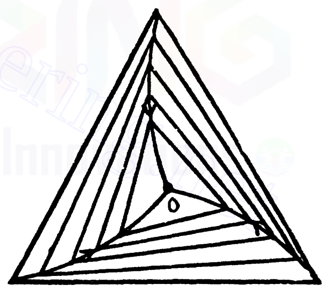
$|\vec{r}_{12}| = |\vec{v}_{0(12)}|\,t$, (because here $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

Hence the sought distance

$$|\vec{r}_{12}| = \frac{v_1 + v_2}{g}\sqrt{v_1 v_2} \quad (\text{as } |\vec{v}_{0(12)}| = v_1 + v_2)$$

1.12 From the symmetry of the problem all the three points are always located at the vertices of equilateral triangles of varying side length and finally meet at the centriod of the initial equilateral triangle whose side length is a, in the sought time interval (say t).





Let us consider an arbitrary equilateral triangle of edge length l (say).

Then the rate by which 1 approaches 2, 2 approches 3, and 3 approches 1, becomes :

$$\frac{-dl}{dt} = v - v\cos\left(\frac{2\pi}{3}\right)$$

On integrating :
$$-\int_a^0 dl = \frac{3v}{2}\int_0^t dt$$

$$a = \frac{3}{2}vt \quad \text{so} \quad t = \frac{2a}{3v}$$

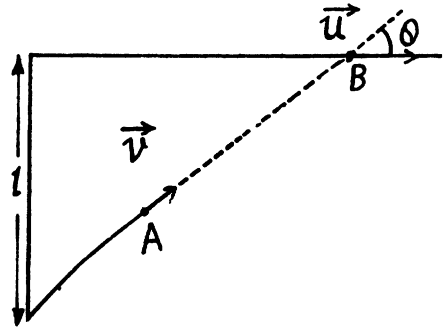
1.13 Let us locate the points A and B at an arbitrary instant of time (Fig.).
If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity $\frac{-ds}{dt} = v - u\cos\alpha$ where angle α varies with time.

On intergating,

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\, dt,$$

(where T is the sought time.)

or $$l = \int_0^T (v - u\cos\alpha)\, dt \qquad (1)$$



As both A and B cover the same distance in x-direction during the sought time interval, so the other condition which is required, can be obtained by the equation

$$\Delta x = \int v_x\, dt$$

So, $$uT = \int_0^T v\cos\alpha\, dt \qquad (2)$$

Solving (1) and (2), we get $T = \frac{ul}{v^2 - u^2}$

One can see that if $u = v$, or $u < v$, point A cannot catch B.

1.14 In the reference frame fixed to the train, the distance between the two events is obviously equal to l. Suppose the train starts moving at time $t = 0$ in the positive x direction and take the origin ($x = 0$) at the head-light of the train at $t = 0$. Then the coordinate of first event in the earth's frame is

$$x_1 = \frac{1}{2} wt^2$$

and similarly the coordinate of the second event is

$$x_2 = \frac{1}{2} w(t+\tau)^2 - l$$

The distance between the two events is obviously.

$$x_1 - x_2 = l - w\tau\,(t + \tau/2) = 0{\cdot}242 \text{ km}$$

in the reference frame fixed on the earth..

For the two events to occur at the same point in the reference frame K, moving with constant velocity V relative to the earth, the distance travelled by the frame in the time interval T must be equal to the above distance.

Thus $$V\tau = l - w\tau(t + \tau/2)$$

So, $$V = \frac{l}{\tau} - w(t + \tau/2) = 4{\cdot}03 \text{ m/s}$$

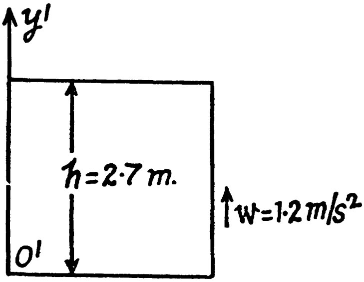
The frame K must clearly be moving in a direction opposite to the train so that if (for example) the origin of the frame coincides with the point x_1 on the earth at time t, it coincides with the point x_2 at time $t + \tau$.

1.15 (a) One good way to solve the problem is to work in the elevator's frame having the observer at its bottom (Fig.).

Let us denote the separation between floor and celing by $h = 2{\cdot}7$ m. and the acceleration of the elevator by $w = 1{\cdot}2\ \text{m/s}^2$

From the kinematical formula

$$y = y_0 + v_{0y}\,t + \frac{1}{2} w_y t^2 \qquad (1)$$



Here $y = 0, y_0 = +h, v_{0y} = 0$

and $w_y = w_{\text{bolt}\,(y)} - w_{\text{ele}\,(y)}$

$$= (-g) - (w) = -(g + w)$$

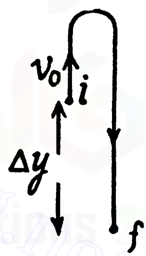
So, $$0 = h + \frac{1}{2}\{-(g+w)\}\,t^2$$

or, $$t = \sqrt{\frac{2h}{g + w}} = 0{\cdot}7\ \text{s}.$$

(b) At the moment the bolt loses contact with the elevator, it has already aquired the velocity equal to elevator, given by :

$$v_0 = (1{\cdot}2)\,(2) = 2{\cdot}4\ \text{m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt,



$$\Delta y = v_{0y}\,t + \frac{1}{2} w_y t^2$$

$$= v_0 t + \frac{1}{2}(-g)\,t^2$$

or, $$\Delta y = (2{\cdot}4)\,(0{\cdot}7) + \frac{1}{2}(-9{\cdot}8)\,(0{\cdot}7)^2 = -0{\cdot}7\ \text{m}.$$

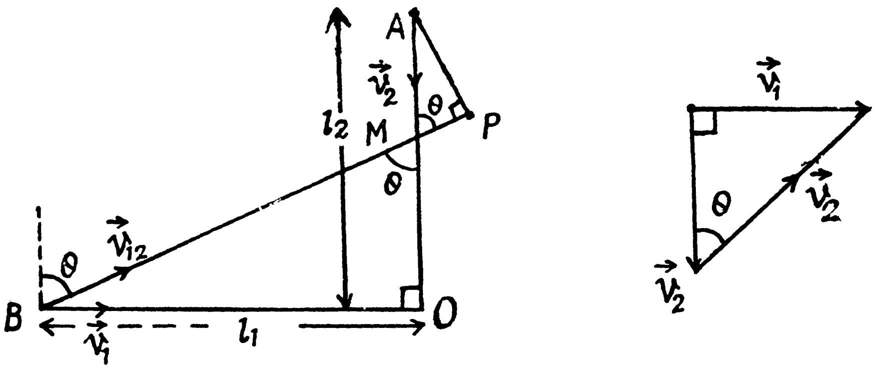
Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount $0{\cdot}7$ m (Fig.).

Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 0{\cdot}7\ \text{m} + \frac{(2{\cdot}4)^2}{(9{\cdot}8)}\ \text{m} = 1{\cdot}3\ \text{m}.$$

1.16 Let the particle 1 and 2 be at points B and A at $t = 0$ at the distances l_1 and l_2 from intersection point O.

Let us fix the inertial frame with the particle 2. Now the particle 1 moves in relative to this reference frame with a relative velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ and its trajectory is the straight line BP. Obviously, the minimum distance between the particles is equal to the length of the perpendicular AP dropped from point A on to the straight line BP (Fig.).



From Fig. (b), $v_{12} = \sqrt{v_1^2 + v_2^2}$, and $\tan\theta = \dfrac{v_1}{v_2}$ (1)

The shortest distance

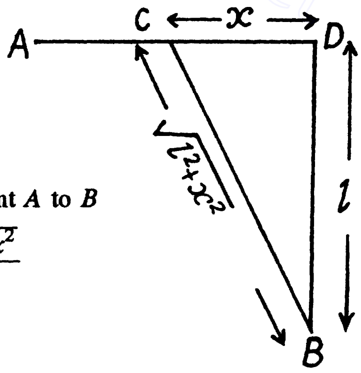
$$AP = AM \sin\theta = (OA - OM)\sin\theta = (l_2 - l_1 \cot\theta)\sin\theta$$

or $$AP = \left(l_2 - l_1 \frac{v_2}{v_1}\right)\frac{v_1}{\sqrt{v_1^2 + v_2^2}} = \frac{v_1 l_2 - v_2 l_1}{\sqrt{v_1^2 + v_2^2}} \qquad \text{(using 1)}$$

The sought time can be obtained directly from the condition that $(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$ is minimum. This gives $t = \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$.

1.17 Let the car turn off the highway at a distance x from the point D.
So, $CD = x$, and if the speed of the car in the field is v, then the time taken by the car to cover the distance $AC = AD - x$ on the highway

$$t_1 = \frac{AD - x}{\eta\, v} \qquad (1)$$



and the time taken to travel the distance CB in the field

$$t_2 = \frac{\sqrt{l^2 + x^2}}{v} \qquad (2)$$

So, the total time elapsed to move the car from point A to B

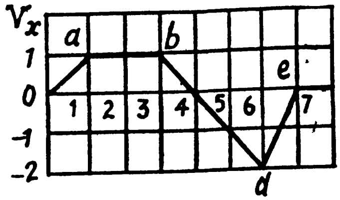
$$t = t_1 + t_2 = \frac{AD - x}{\eta\, v} + \frac{\sqrt{l^2 + x^2}}{v}$$

For t to be minimum

$$\frac{dt}{dx} = 0 \quad \text{or} \quad \frac{1}{v}\left[-\frac{1}{\eta} + \frac{x}{\sqrt{l^2 + x^2}}\right] = 0$$

or $$\eta^2 x^2 = l^2 + x^2 \quad \text{or} \quad x = \frac{l}{\sqrt{\eta^2 - 1}}$$

1.18 To plot $x(t), s(t)$ and $w_x(t)$ let us partion the given plot $v_x(t)$ into five segments (for detailed analysis) as shown in the figure.



For the part $oa: w_x = 1$ and $v_x = t = v$

Thus, $\Delta x_1(t) = \int v_x\, dt = \int_0^t dt = \frac{t^2}{2} = s_1(t)$

Putting $t = 1$, we get, $\Delta x_1 = s = \frac{1}{2}$ unit

For the part ab :

$w_x = O$ and $v_x = v =$ constant $= 1$

Thus $\Delta x_2(t) = \int v_x\, dt = \int_1^t dt = (t-1) = s_2(t)$

Putting $t = 3,\ \Delta x_2 = s_2 = 2$ unit

For the part $b4$: $w_x = 1$ and $v_x = 1-(t-3) = 4-t) = v$

Thus $\Delta x_3(t) = \int_3^t (4-t)\, dt = 4t - \frac{t^2}{2} - \frac{15}{2} = s_3(t)$

Putting $t = 4,\ \Delta x_3 = x_3 = \frac{1}{2}$ unit

For the part $4d$: $v_x = -1$ and $v_x = -(1-4) = 4-1$

So, $v = |v_x| = t-4$ for $t > 4$

Thus $\Delta x_4(t) = \int_4^t (1-t)\, dt = 4t - \frac{t^2}{2} - 8$

Putting $t = 6,\ \Delta x_4 = -1$

Similarly $s_4(t) = \int |v_x|\, dt = \int_4^t (t-4)\, dt = \frac{t^2}{2} - 4t + 8$

Putting $t = 6,\ s_4 = 2$ unit

For the part $d\,7$: $w_x = 2$ and $v_x = -2 + 2(t-6) = 2(t-7)$

$v = |v_x| = 2(7-t)$ for $t \leftarrow 7$

Now, $\Delta x(t) = \int_t^6 2\,(t-7)\, dt = t^2 - 14t + 48$

Putting $t = 4,\ \Delta x_5 = -1$

Similarly $s_5(t) = \int_t^6 2(7-t)\, dt = 14t - t^2 - 48$

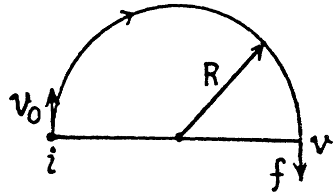
Putting $t = 7,\ s_5 = 1$

On the basis of these obtained expressions $w_x(t), x(t)$ and $s(t)$ plots can be easily plotted as shown in the figure of answersheet.

1.19 (a) Mean velocity

$$<v> = \frac{\text{Total distance covered}}{\text{Time elapsed}}$$

$$= \frac{s}{t} = \frac{\pi R}{\tau} = 50 \text{ cm/s} \quad (1)$$



(b) Modulus of mean velocity vector

$$|<\vec{v}>| = \frac{|\Delta \vec{r}|}{\Delta t} = \frac{2R}{\tau} = 32 \text{ cm/s} \quad (2)$$

(c) Let the point moves from i to f along the half circle (Fig.) and v_0 and v be the spe at the points respectively.

We have $\frac{dv}{dt} = w_t$

or, $v = v_0 + w_t t$ (as w_t is constant, according to the problem)

So, $$<v> = \frac{\int_0^t (v_0 + w_t t)\, dt}{\int_0^t dt} = \frac{v_0 + (v_0 + w_t t)}{2} = \frac{v_0 + v}{2} \quad (:$$

So, from (1) and (3)

$$\frac{v_0 + v}{2} = \frac{\pi R}{\tau} \quad (3$$

Now the modulus of the mean vector of total acceleration

$$|<\vec{w}>| = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{|\vec{v} - \vec{v}_0|}{\tau} = \frac{v_0 + v}{\tau} \text{ (see Fig.)} \quad (5$$

Using (4) in (5), we get :

$$|<\vec{w}>| = \frac{2\pi R}{\tau^2}$$

1.20 (a) we have $\vec{r} = \vec{a}\, t\,(1 - \alpha t)$

So, $\vec{v} = \frac{d\vec{r}}{dt} = \vec{a}(1 - 2\alpha t)$

and $\vec{w} = \frac{d\vec{v}}{dt} = -2\alpha \vec{a}$

(b) From the equation

$$\vec{r} = \vec{a}\, t\,(1 - \alpha t),$$

$$\vec{r} = 0, \text{ at } t = 0 \text{ and also at } t = \Delta t = \frac{1}{\alpha}$$

So, the sought time $\Delta t = \frac{1}{\alpha}$

As $\vec{v} = \vec{a}(1 - 2\alpha t)$

So, $$v = |\vec{v}| = \begin{cases} a\,(1 - 2\alpha t) & \text{for } t \le \frac{1}{2\alpha} \\ a\,(2\alpha t - 1) & \text{for } t > \frac{1}{2\alpha} \end{cases}$$

Hence, the sought distance

$$s = \int v\,dt = \int_0^{1/2\alpha} a\,(1 - 2\,\alpha\,t)\,dt + \int_{1/2\alpha}^{1/\alpha} a\,(2\,\alpha\,t - 1)\,dt$$

Simplifying, we get, $s = \dfrac{a}{2\,\alpha}$

1.21 (a) As the particle leaves the origin at $t = 0$

So, $$\Delta x = x = \int v_x\,dt \qquad (1)$$

As $$\vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right),$$

where $\vec{v_0}$ is directed towards the +ve x-axis

So, $$v_x = v_0\left(1 - \frac{t}{\tau}\right) \qquad (2)$$

From (1) and (2),

$$x = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt = v_0\,t\left(1 - \frac{t}{2\,\tau}\right) \qquad (3)$$

Hence x coordinate of the particle at $t = 6$ s.

$$x = 10 \times 6\left(1 - \frac{6}{2 \times 5}\right) = 24\text{ cm} = 0{\cdot}24\text{ m}$$

Similarly at $t = 10$ s

$$x = 10 \times 10\left(1 - \frac{10}{2 \times 5}\right) = 0$$

and at $t = 20$ s

$$x = 10 \times 20\left(1 - \frac{20}{2 \times 5}\right) = -200\text{ cm} = -2\text{ m}$$

(b) At the moments the particle is at a distance of 10 cm from the origin, $x = \pm\,10$ cm.

Putting $x = +10$ in Eq. (3)

$$10 = 10t\left(1 - \frac{t}{10}\right) \quad \text{or,} \quad t^2 - 10t + 10 = 0,$$

So, $$t = t = \frac{10 \pm \sqrt{100 - 40}}{2} = 5 \pm \sqrt{15}\text{ s}$$

Now putting $x = -10$ in Eqn (3)

$$-10 = 10\left(1 - \frac{t}{10}\right),$$

On solving, $t = 5 \pm \sqrt{35}$ s

As t cannot be negative, so,

$$t = (5 + \sqrt{35}\,)\text{ s}$$

Hence the particle is at a distance of 10 cm from the origin at three moments of time :

$$t = 5 \pm \sqrt{15} \text{ s}, \; 5 + \sqrt{35} \text{ s}$$

(c) We have $\vec{v} = \vec{v}_0\left(1 - \frac{t}{\tau}\right)$

So, $v = |\vec{v}| = \begin{cases} v_0\left(1 - \frac{t}{\tau}\right) & \text{for } t \le \tau \\ v_0\left(\frac{t}{\tau} - 1\right) & \text{for } t > \tau \end{cases}$

So $s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt$ for $t \le \tau = v_0 t\,(1 - t/2\tau)$

and $$s = \int_0^{\tau} v_0\left(1 - \frac{t}{2}\right) dt + \int_{\tau}^{t} v_0\left(\frac{t}{\tau} - 1\right) dt \text{ for } t > \tau$$

$$= v_0 \tau\,[1 + (1 - t/\tau)^2]/2 \text{ for } t > \tau \qquad \text{(A)}$$

$$s = \int_0^4 v_0\left(1 - \frac{t}{\tau}\right) dt = \int_0^4 10\left(1 - \frac{t}{5}\right) dt = 24 \text{ cm.}$$

And for $t = 8$ s

$$s = \int_0^5 10\left(1 - \frac{t}{5}\right) dt + \int_5^8 10\left(\frac{t}{5} - 1\right) dt$$

On integrating and simplifying, we get

$$s = 34 \text{ cm.}$$

On the basis of Eqs. (3) and (4), $x(t)$ and $s(t)$ plots can be drawn as shown in the answer sheet.

1.22 As particle is in unidirectional motion it is directed along the x-axis all the time. As at $t = 0$, $x = 0$

So, $\Delta x = x = s$, and $\frac{dv}{dt} = w$

Therefore, $v = \alpha\sqrt{x} = \alpha\sqrt{s}$

or, $$w = \frac{dv}{dt} = \frac{\alpha}{2\sqrt{s}}\frac{ds}{dt} = \frac{\alpha}{2\sqrt{s}}$$

$$= \frac{\alpha v}{2\sqrt{s}} = \frac{\alpha\,\alpha\sqrt{s}}{2\sqrt{s}} = \frac{\alpha^2}{2} \qquad (1)$$

As, $w = \frac{dv}{dt} = \frac{\alpha^2}{2}$

On integrating, $$\int_0^v dv = \int_0^t \frac{\alpha^2}{2} dt \quad \text{or,} \quad v = \frac{\alpha^2}{2} t \qquad (2)$$

(b) Let s be the time to cover first s m of the path. From the Eq.

$$s = \int v\, dt$$

$$s = \int_0^t \frac{\alpha^2}{2}\, dt = \frac{\alpha^2}{2}\frac{t^2}{2} \qquad \text{(using 2)}$$

or $$t = \frac{2}{\alpha}\sqrt{s} \tag{3}$$

The mean velocity of particle

$$\langle v \rangle = \frac{\int v(t)\, dt}{\int dt} = \frac{\int_0^{2\sqrt{s}/\alpha} \frac{\alpha^2}{2}\, t\, dt}{2\sqrt{s}/\alpha} = \frac{\alpha\sqrt{s}}{2}$$

1.23 According to the problem

$$-\frac{v\, dv}{ds} = a\sqrt{v} \quad \text{(as } v \text{ decreases with time)}$$

or, $$-\int_{v_0}^{0} \sqrt{v}\, dv = a\int_0^s ds$$

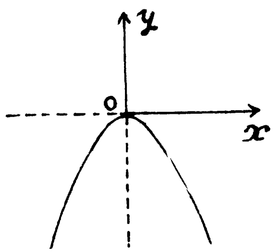
On integrating we get $s = \frac{2}{3a} v_0^{3/2}$

Again according to the problem

$$-\frac{dv}{dt} = a\sqrt{v} \quad \text{or} \quad -\frac{dv}{\sqrt{v}} = a\, dt$$

or, $$\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = a\int_0^t dt$$

Thus $$t = \frac{2\sqrt{v_0}}{a}$$



1.24 (a) As $\vec{r} = a t \vec{i} - b t^2 \vec{j}$

So, $x = a t,\ y = -b t^2$

and therefore $y = \dfrac{-b x^2}{a^2}$

which is Eq. of a parabola, whose graph is shown in the Fig.

(b) As $\vec{r} = a\,t\,\vec{i} - b\,t^2\,\vec{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = a\,\vec{i} - 2\,b\,t\,\vec{j} \qquad (1)$$

So, $v = \sqrt{a^2 (-2\,b\,t)^2} = \sqrt{a^2 + 4\,b^2\,t^2}$

Diff. Eq. (1) w.r.t. time, we get

$$\vec{w} = \frac{d\vec{v}}{dt} = -2b\,\vec{j}$$

So, $|\vec{w}| = w = 2\,b$

(c) $$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{v\,w} = \frac{(a\,\vec{i} - 2\,b\,t\,\vec{j})\cdot(-2\,b\,\vec{j})}{\left(\sqrt{a^2 + 4\,b^2\,t^2}\right)2b}$$

or, $$\cos\alpha = \frac{2\,b\,t}{\sqrt{a^2 + 4\,b^2\,t^2}},$$

so, $$\tan\alpha = \frac{a}{2\,b\,t}$$

or, $\alpha = \tan^{-1}\left(\dfrac{a}{2\,b\,t}\right)$

(d) The mean velocity vector

$$\langle \vec{v} \rangle = \frac{\int \vec{v}\,dt}{\int dt} = \frac{\int_0^t (a\,\vec{i} - 2\,b\,t\,\vec{j})\,dt}{t} = a\,\vec{i} - b\,t\,\vec{j}$$

Hence, $|\langle \vec{v} \rangle| = \sqrt{a^2 + (-b\,t)^2} = \sqrt{a^2 + b^2\,t^2}$

1.25 (a) We have

$$x = a\,t \text{ and } y = a\,t\,(1 - \alpha\,t) \qquad (1)$$

Hence, $y\,(x)$ becomes,

$$y = \frac{a\,x}{a}\left(1 - \frac{\alpha\,x}{a}\right) = x - \frac{\alpha}{a}x^2 \text{ (parabola)}$$

(b) Differentiating Eq. (1) we get

$$v_x = a \text{ and } v_y = a\,(1 - 2\,\alpha\,t) \qquad (2)$$

So, $$v = \sqrt{v_x^2 + v_y^2} = a\sqrt{1 + (1 - 2\alpha t)^2}$$

Diff. Eq. (2) with respect to time

$$w_x = 0 \text{ and } w_y = -2a\alpha$$

So, $$w = \sqrt{w_x^2 + w_y^2} = 2a\alpha$$

(c) From Eqs. (2) and (3)

We have $$\vec{v} = a\vec{i} + a(1 - 2\alpha t)\vec{j} \text{ and } \vec{w} = 2a\alpha\vec{j}$$

So, $$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\vec{v}\cdot\vec{w}}{vw} = \frac{-a(1 - 2\alpha t_0)2a\alpha}{a\sqrt{1 + (1 - 2\alpha t_0)^2}\,2a\alpha}$$

On simplifying. $$1 - 2\alpha t_0 = \pm 1$$

As, $$t_0 \neq 0,\ t_0 = \frac{1}{\alpha}$$

1.26 Differentiating motion law : $x = a\sin\omega t$, $y = a(1 - \cos\omega t)$, with respect to time, $v_x = a\omega\cos\omega t$, $v_y = a\omega\sin\omega t$

So, $$\vec{v} = a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j} \qquad (1)$$

and $$v = a\omega = \text{Const.} \qquad (2)$$

Differentiating Eq. (1) with respect to time

$$\vec{w} = \frac{d\vec{v}}{dt} = -a\omega^2\sin\omega t\,\vec{i} + a\omega^2\cos\omega t\,\vec{j} \qquad (3)$$

(a) The distance s traversed by the point during the time τ is given by

$$s = \int_0^\tau v\,dt = \int_0^\tau a\omega\,dt = a\omega\tau \quad (\text{using } 2)$$

(b) Taking inner product of $\vec{v}$ and $\vec{w}$

We get, $$\vec{v}\cdot\vec{w} = (a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j})\cdot(a\omega^2\sin\omega t(-i) + a\omega^2\cos\omega t - \vec{j})$$

So, $$\vec{v}\cdot\vec{w} = -a^2\omega^2\sin\omega t\cos\omega t + a^2\omega^3\sin\omega t\cos\omega t = 0$$

Thus, $\vec{v} \perp \vec{w}$, i.e., the angle between velocity vector and acceleration vector equals $\frac{\pi}{2}$.

1.27 Accordiing to the problem

$$\vec{w} = w(-\vec{j})$$

So, $$w_x = \frac{dv_x}{dt} = 0 \text{ and } w_y = \frac{dv_y}{dt} = -w \qquad (1)$$

Differentiating Eq. of trajectory, $y = ax - bx^2$, with respect to time

$$\frac{dy}{dt} = \frac{a\,dx}{dt} - 2bx\frac{dx}{dt} \qquad (2)$$

So, $$\left.\frac{dy}{dt}\right|_{x=0} = a\left.\frac{dx}{dt}\right|_{x=0}$$

Again differentiating with respect to time

$$\frac{d^2y}{dt^2} = \frac{a\,d^2x}{dt^2} - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\frac{d^2x}{dt^2}$$

or, $$-w = a(0) - 2b\left(\frac{dx}{dt}\right)^2 - 2bx(0) \text{ (using 1)}$$

or, $$\frac{dx}{dt} = \sqrt{\frac{w}{2b}} \text{ (using 1)}$$ (3

Using (3) in (2) $$\left.\frac{dy}{dt}\right|_{x=0} = a\sqrt{\frac{w}{2b}}$$ (4

Hence, the velocity of the particle at the origin

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2_{x=0} + \left(\frac{dy}{dt}\right)^2_{x=0}} = \sqrt{\frac{w}{2b} + a^2\frac{w}{2b}} \text{ (using Eqns (3) and (4))}$$

Hence, $$v = \sqrt{\frac{w}{2b}(1+a^2)}$$

1.28 As the body is under gravity of constant accelration $\vec{g}$, it's velocity vector and displacemen vectors are:

$$\vec{v} = \vec{v}_0 + \vec{g}t \quad (1)$$

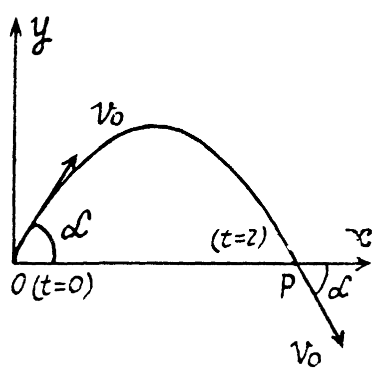
and $$\Delta\vec{r} = \vec{r} = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 \quad (\vec{r} = 0 \text{ at } t = 0) \quad (2)$$

So, $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}}{t} = \vec{v}_0 + \frac{\vec{g}t}{2} \quad (3)$$

Hence from Eq. (3), $<\vec{v}>$ over the first t seconds

$$<\vec{v}> = \vec{v}_0 + \frac{\vec{g}}{2}\tau \quad (4)$$



For evaluating t, take

$$\vec{v}\cdot\vec{v} = (\vec{v}_0 + \vec{g}t)\cdot(\vec{v}_0 + \vec{g}t) = v_0^2 + 2(\vec{v}_0\cdot\vec{g})t + g^2t^2$$

or, $$v^2 = v_0^2 + (\vec{v}_0\cdot\vec{g})t + g^2t^2$$

But we have $v = v_0$ at $t = 0$ and

Also at $t = \tau$ (Fig.) (also from energy conservation)

Hence using this propety in Eq. (5)

$$v_0^2 = v_0^2 + 2\,(\vec{v_0}\cdot\vec{g})\,\tau + g^2\tau^2$$

As $\qquad \tau \neq 0, \quad \text{so,} \quad \tau = -\dfrac{2\,(\vec{v}_o\cdot\vec{g})}{g^2}$

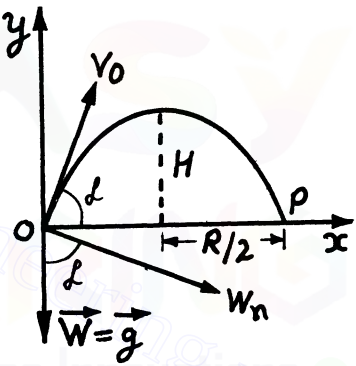
Putting this value of τ in Eq. (4), the average velocity over the time of flight

$$\langle\vec{v}\rangle = \vec{v_0} - \vec{g}\,\frac{(\vec{v_0}\cdot\vec{g})}{g^2}$$

1.29 The body thrown in air with velocity v_0 at an angle α from the horizontal lands at point P on the Earth's surface at same horizontal level (Fig.). The point of projection is taken as origin, so, $\Delta x = x$ and $\Delta y = y$

(a) From the Eq. $\quad \Delta y = v_{0y}t + \frac{1}{2}w_y t^2$

$$0 = v_0 \sin\alpha\,\tau - \frac{1}{2}g\tau^2$$



As $\tau \neq 0$, so, time of motion $\tau = \dfrac{2\,v_0\sin\alpha}{g}$

(b) At the maximum height of ascent, $v_y = 0$

so, from the Eq. $\quad v_y^2 = v_{0y}^2 + 2\,w_y\,\Delta y$

$$0 = (v_0\sin\alpha)^2 - 2\,g\,H$$

Hence maximum height $H = \dfrac{v_0^2\sin^2\alpha}{2g}$

During the time of motion the net horizontal displacement or horizontal range, will be obtained by the equation

$$\Delta x = v_{0x}\,t + \frac{1}{2}w_x\tau^2$$

or, $\qquad R = v_0\cos\alpha\,\tau - \frac{1}{2}(0)\,\tau^2 = v_0\cos\alpha\,\tau = \dfrac{v_0^2\sin 2\alpha}{g}$

when $\qquad R = H$

$$\frac{v_0^2\sin^2\alpha}{g} = \frac{v_0^2\sin^2\alpha}{2g} \quad \text{or} \quad \tan\alpha = 4, \quad \text{so,} \quad \alpha = \tan^{-1}4$$

(c) For the body, $x(t)$ and $y(t)$ are

$$x = v_0\cos\alpha\,t \qquad (1)$$

and $$y = v_0 \sin\alpha\, t - \frac{1}{2} g t^2 \qquad (2)$$

Hence putting the value of t from (1) into (2) we get,

$$y = v_0 \sin\alpha \left(\frac{x}{v_0 \cos\alpha}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = x \tan\alpha - \frac{g x^2}{2 v_0^2 \cos^2\alpha},$$

Which is the sought equation of trajectory i.e. $y(x)$

(d) As the body thrown in air follows a curve, it has some normal acceleration at all the moments of time during it's motion in air.

At the initial point ($x = 0, y = 0$), from the equation :

$$w_n = \frac{v^2}{R}, \text{ (where } R \text{ is the radius of curvature)}$$

$$g \cos\alpha = \frac{v_0^2}{R_0} \text{ (see Fig.) or } R_0 = \frac{v_0^2}{g \cos\alpha}$$

At the peak point $v_y = 0$, $v = v_x = v_0 \cos\alpha$ and the angential acceleration is zero.

Now from the Eq. $$w_n = \frac{v^2}{R}$$

$$g = \frac{v_0^2 \cos^2\alpha}{R}, \quad \text{or} \quad R = \frac{v_0^2 \cos^2\alpha}{g}$$

Note : *We may use the formula of curvature radius of a trajectory $y(x)$, to solve part (d),*

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left| d^2 y / dx^2 \right|}$$

1.30 We have, $v_x = v_0 \cos\alpha$, $v_y = v_0 \sin\alpha - gt$

As $\vec{v} \uparrow\uparrow \hat{u}_t$ all the moments of time.

Thus $$v^2 = v_t^2 - 2 g t v_0 \sin\alpha + g^2 t^2$$

Now, $$w_t = \frac{dv_t}{dt} = \frac{1}{2 v_t} \frac{d}{dt}(v_t^2) = \frac{1}{v_t}(g^2 t - g v_0 \sin\alpha)$$

$$= -\frac{g}{v_t}(v_0 \sin\alpha - g t) = -g \frac{v_y}{v_t}$$

Hence $$|w_t| = g \frac{|v_y|}{v}$$

Now $$w_n = \sqrt{w^2 - w_t^2} = \sqrt{g^2 - g^2 \frac{v_y^2}{v_t^2}}$$

or $$w_n = g \frac{v_x}{v_t} \left(\text{where } v_x = \sqrt{v_t^2 - v_y^2}\right)$$

As $\vec{v} \uparrow\uparrow \hat{v}_t$, during time of motion

$$w_v = w_t = -g\frac{v_y}{v}$$

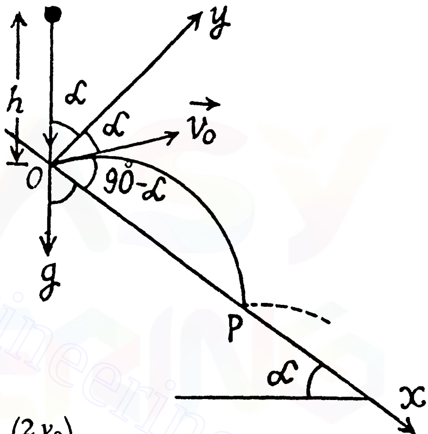
On the basis of obtained expressions or facts the sought plots can be drawn as shown in the figure of answer sheet.

1.31 The ball strikes the inclined plane (Ox) at point O (origin) with velocity $v_0 = \sqrt{2gh}$ (1)

As the ball elastically rebounds, it recalls with same velocity v_0, at the same angle α from the normal or y axis (Fig.). Let the ball strikes the incline second time at P, which is at a distance l (say) from the point O, along the incline. From the equation

$$y = v_{0y}t + \frac{1}{2}w_y t^2$$

$$0 = v_0 \cos\alpha\,\tau - \frac{1}{2}g\cos\alpha\,\tau^2$$



where τ is the time of motion of ball in air while moving from O to P.

As $\tau \neq 0$, so, $\tau = \dfrac{2v_0}{g}$ (2)

Now from the equation.

$$x = v_{0x}t + \frac{1}{2}w_x t^2$$

$$l = v_0 \sin\alpha\,\tau + \frac{1}{2}g\sin\alpha\,\tau^2$$

so, $$l = v_0 \sin\alpha\left(\frac{2v_0}{g}\right) + \frac{1}{2}g\sin\alpha\left(\frac{2v_0}{g}\right)$$

$$= \frac{4v_0^2\sin\alpha}{g} \quad \text{(using 2)}$$

Hence the sought distance, $l = \dfrac{4(2gh)\sin\alpha}{g} = 8h\sin\alpha$ (Using Eq. 1)

1.32 Total time of motion

$$\tau = \frac{2v_0\sin\alpha}{g} \quad \text{or} \quad \sin\alpha = \frac{\tau g}{2v_0} = \frac{9{\cdot}8\,\tau}{2\times 240} \qquad (1)$$

and horizontal range

$$R = v_0\cos\alpha\,\tau \quad \text{or} \quad \cos\alpha = \frac{R}{v_0\tau} = \frac{5100}{240\,\tau} = \frac{85}{4\tau} \qquad (2)$$

From Eqs. (1) and (2)

$$\frac{(9{\cdot}8)^2\tau^2}{(480)^2} + \frac{(85)^2}{(4\tau^2)^2} = 1$$

On simplifying $$\tau^4 - 2400\,\tau^2 + 1083750 = 0$$

Solving for τ^2 we get :

$$\tau^2 = \frac{2400 \pm \sqrt{1425000}}{2} = \frac{2400 \pm 1194}{2}$$

Thus $\tau = 42{\cdot}39 \text{ s} = 0{\cdot}71 \text{ min}$ and

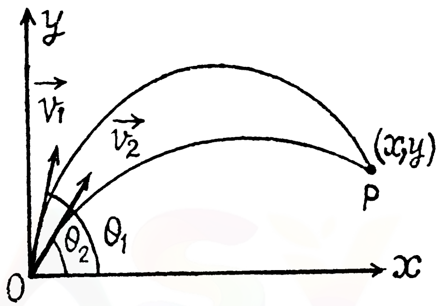
$\tau = 24{\cdot}55 \text{ s} = 0{\cdot}41 \text{ min}$ depending on the angle α.

1.33 Let the shells collide at the point $P(x, y)$. If the first shell takes t s to collide with second and Δt be the time interval between the firings, then

$$x = v_0 \cos\theta_1 \, t = v_0 \cos\theta_2 \,(t - \Delta t) \quad (1)$$

and $$y = v_0 \sin\theta_1 \, t - \frac{1}{2} g t^2$$

$$= v_0 \sin\theta_2 \,(t - \Delta t) - \frac{1}{2} g \,(t - \Delta t)^2 \quad (2)$$



From Eq. (1) $$t = \frac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1} \quad (3)$$

From Eqs. (2) and (3)

$$\Delta t = \frac{2 v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} \text{ as } \Delta t \neq 0$$

1.34 According to the problem

(a) $\frac{dy}{dt} = v_0$ or $dy = v_0 \, dt$

Integrating $$\int_0^y dy = v_0 \int_0^t dt \text{ or } y = v_0 t \quad (1)$$

And also we have $\frac{dx}{dt} = ay$ or $dx = a\,y\,dt = a\,v_0\,t\,dt$ (using 1)

So, $$\int_0^x dx = a v_0 \int_0^t t\,dt, \text{ or, } x = \frac{1}{2} a v_0 t^2 = \frac{1}{2} \frac{a y^2}{v_0} \text{ (using 1)}$$

(b) According to the problem

$$v_y = v_0 \text{ and } v_x = a\,y \quad (2)$$

So, $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + a^2 y^2}$$

Therefore $$w_t = \frac{dv}{dt} = \frac{a^2 y}{\sqrt{v_0^2 + a y^2}} \frac{dy}{dt} = \frac{a^2 y}{\sqrt{1 + (ay/v_0)^2}}$$

Diff. Eq. (2) with respect to time.

$$\frac{d v_y}{dt} = w_y = 0 \text{ and } \frac{d v_x}{dt} = w_x = a \frac{dy}{dt} = a v_0$$

So, $$w = |w_x| = a v_0$$

Hence $w_n = \sqrt{w^2 - w_t^2} = \sqrt{a^2 v_0^2 - \frac{a^4 y^2}{1 + (ay/v_0)^2}} = \frac{a v_0}{\sqrt{1 + (ay/v_0)^2}}$

1.35 (a) The velocity vector of the particle

$$\vec{v} = a\,\vec{i} + bx\,\vec{j}$$

So, $$\frac{dx}{dt} = a : \frac{dy}{dt} = bx \qquad (1)$$

From (1) $$\int_0^x dx = a \int_0^t dt \quad \text{or,} \quad x = at \qquad (2)$$

And $$dy = bx\,dt = bat\,dt$$

Integrating $$\int_0^y dy = ab \int_0^t t\,dt \quad \text{or,} \quad y = \frac{1}{2} ab\,t^2 \qquad (3)$$

From Eqs. (2) and (3), we get, $y = \frac{b}{2a} x^2$ (4)

(b) The curvature radius of trajectory $y(x)$ is :

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left| d^2 y / dx^2 \right|} \qquad (5)$$

Let us differentiate the path Eq. $y = \frac{b}{2a} x^2$ with respect to x,

$$\frac{dy}{dx} = \frac{b}{a} x \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{b}{a} \qquad (6)$$

From Eqs. (5) and (6), the sought curvature radius :

$$R = \frac{a}{b}\left[1 + \left(\frac{b}{a} x\right)^2\right]^{\frac{3}{2}}$$

1.36 In accordance with the problem

$$w_t = \vec{a} \cdot \vec{\tau}$$

But $$w_t = \frac{v\,dv}{ds} \quad \text{or} \quad v\,dv = w_t\,ds$$

So, $$v\,dv = (\vec{a} \cdot \vec{\tau})\,ds = \vec{a} \cdot d\vec{r}$$

or, $v\,dv = a\,\vec{i} \cdot d\vec{r} = a\,dx$ (because $\vec{a}$ is directed towards the x-axis)

So, $$\int_0^v v\,dv = a \int_0^x dx$$

Hence $$v^2 = 2\,ax \quad \text{or,} \quad v = \sqrt{2\,ax}$$

1.37 The velocity of the particle $v = at$

So, $$\frac{dv}{dt} = w_t = a \quad (1)$$

And $$w_n = \frac{v^2}{R} = \frac{a^2 t^2}{R} \text{ (using } v = at) \quad (2)$$

From $$s = \int v\, dt$$

$$2\pi R \eta = a \int_0^t v\, dt = \frac{1}{2} a t^2$$

So, $$\frac{4\pi\eta}{a} = \frac{t^2}{R} \quad (3)$$

From Eqs. (2) and (3) $w_n = 4\pi a \eta$

Hence $w = \sqrt{w_t^2 + w_n^2}$

$$= \sqrt{a^2 + (4\pi a \eta)^2} = a\sqrt{1 + 16\pi^2\eta^2} = 0{\cdot}8 \text{ m/s}^2$$

1.38 According to the problem

$$|w_t| = |w_n|$$

For $v(t)$, $$\frac{-dv}{dt} = \frac{v^2}{R}$$

Integrating this equation from $v_0 \le v \le v$ and $0 \le t \le t$

$$-\int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \quad \text{or,} \quad v = \frac{v_0}{\left(1 + \frac{v_0 t}{R}\right)}$$

Now for $v(s)$, $-\frac{v\,dv}{ds} = \frac{v^2}{R}$, Integrating this equation from $v_0 \le v \le v$ and $0 \le s \le s$

SO, $$\int_{v_0}^{v} \frac{dv}{v} = -\frac{1}{R}\int_0^s ds \quad \text{or,} \quad \ln\frac{v}{v_0} = -\frac{s}{R}$$

Hence $$v = v_0 e^{-s/R} \quad (2)$$

(b) The normal acceleration of the point

$$w_n = \frac{v^2}{R} = \frac{v^2 e^{-2s/R}}{R} \text{ (using 2)}$$

And as accordance with the problem

$$|w_t| = |w_n| \text{ and } w_t \hat{u}_t \perp w_n \hat{u}_n$$

so, $$w = \sqrt{2}\, w_n = \sqrt{2}\frac{v_0^2}{R} e^{-2s/R} = \sqrt{2}\frac{v^2}{R}$$

1.39 From the equation $v = a\sqrt{s}$

$$w_t = \frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2}, \text{ and}$$

$$w_n = \frac{v^2}{R} = \frac{a^2 s}{R}$$

As w_t is a positive constant, the speed of the particle increases with time, and the tangential acceleration vector and velocity vector coincides in direction.
Hence the angle between $\vec{v}$ and $\vec{w}$ is equal to between $w_t \hat{u}_t$ an $\vec{w}$, and α can be found by means of the formula : $\tan\alpha = \frac{|w_n|}{|w_t|} = \frac{a^2 s/R}{a^2/2} = \frac{2s}{R}$

1.40 From the equation $l = a \sin \omega t$

$$\frac{dl}{dt} = v = a\,\omega \cos \omega t$$

So, $$w_t = \frac{dv}{dt} = -a\,\omega^2 \sin \omega t, \text{ and} \quad (1)$$

$$w_n = \frac{v^2}{R} = \frac{a^2 \omega^2 \cos^2 \omega t}{R} \quad (2)$$

(a) At the point $l = 0$, $\sin \omega t = 0$ and $\cos \omega t = \pm 1$ so, $\omega t = 0, \pi$ etc.

Hence $$w = w_n = \frac{a^2 \omega^2}{R}$$

Similarly at $l = \pm a$, $\sin \omega t = \pm 1$ and $\cos \omega t = 0$, so, $w_n = 0$

Hence $$w = |w_t| = a\,\omega^2$$

1.41 As $w_t = a$ and at $t = 0$, the point is at rest

So, $v(t)$ and $s(t)$ are, $v = at$ and $s = \frac{1}{2}at^2$ (1)

Let R be the curvature radius, then

$$w_n = \frac{v^2}{R} = \frac{a^2 t^2}{R} = \frac{2\,as}{R} \text{ (using 1)}$$

But according to the problem

$$w_n = bt^4$$

So, $bt^4 = \frac{a^2 t^2}{R}$ or, $R = \frac{a^2}{bt^2} = \frac{a^2}{2\,bs}$ (using 1) (2)

Therefore $$w = \sqrt{w_t^2 + w_n^2} = \sqrt{a^2 + (2\,as/R)^2} = \sqrt{a^2 + \left(4\,bs^2/a^2\right)^2} \text{(using 2)}$$

Hence $$w = a\sqrt{1 + \left(4\,bs^2/a^3\right)^2}$$

1.42 (a) Let us differentiate twice the path equation $y(x)$ with respect to time.

$$\frac{dy}{dt} = 2ax\frac{dx}{dt};\ \frac{d^2y}{dt^2} = 2a\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

Since the particle moves uniformly, its acceleration at all points of the path is normal and at the point $x = 0$ it coincides with the direction of derivative d^2y/dt^2. Keeping in mind that at the point $x = 0$, $\left|\frac{dx}{dt}\right| = v$,

We get
$$w = \left|\frac{d^2y}{dt^2}\right|_{x=0} = 2av^2 = w_n$$

So,
$$w_n = 2av^2 = \frac{v^2}{R},\ \text{or}\ R = \frac{1}{2a}$$

Note that we can also calculate it from the formula of problem (1.35 b)

(b) Differentiating the equation of the trajectory with respect to time we see that

$$b^2x\frac{dx}{dt} + a^2y\frac{dy}{dt} = 0 \tag{1}$$

which implies that the vector $(b^2x\vec{i} + a^2y\vec{j})$ is normal to the velocity vector $\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$ which, of course, is along the tangent. Thus the former vactor is along the normal and the normal component of acceleration is clearly

$$w_n = \frac{b^2x\frac{d^2x}{dt^2} + a^2y\frac{d^2y}{dt^2}}{(b^4x^2 + a^4y^2)^{1/2}}$$

on using $w_n = \vec{w}\cdot\vec{n}/|\vec{n}|$. At $x = 0$, $y = \pm b$ and so at $x = 0$

$$w_n = \pm\left.\frac{d^2y}{dt^2}\right|_{x=0}$$

Differentiating (1)

$$b^2\left(\frac{dx}{dt}\right)^2 + b^2x\left(\frac{d^2x}{dt^2}\right) + a^2\left(\frac{dy}{dt}\right)^2 + a^2y\left(\frac{d^2y}{dt^2}\right) = 0$$

Also from (1)
$$\frac{dy}{dt} = 0 \text{ at } x = 0$$

So
$$\left(\frac{dx}{dt}\right) = \pm v \quad \text{(since tangential velocity is constant} = v\text{)}$$

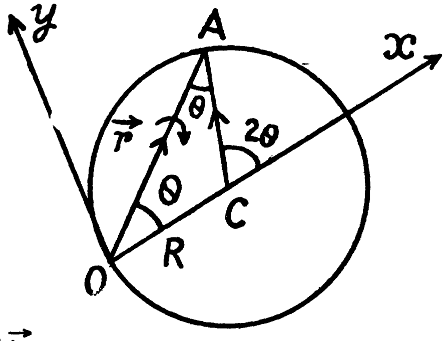
Thus
$$\left(\frac{d^2y}{dt^2}\right) = \pm\frac{b}{a^2}v^2$$

and
$$|w_n| = \frac{bv^2}{a^2} = \frac{v^2}{R}$$

This gives $R = a^2/b$.

1.43 Let us fix the co-ordinate system at the point O as shown in the figure, such that the radius vector $\vec{r}$ of point A makes an angle θ with x axis at the moment shown.

Note that the radius vector of the particle A rotates clockwise and we here take line ox as reference line, so in this case obviously the angular velocity $\omega = \left(-\frac{d\theta}{dt}\right)$ taking anticlockwise sense of angular displacement as positive.



Also from the geometry of the triangle OAC

$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)} \text{ or, } r = 2R\cos\theta.$$

Let us write,

$$\vec{r} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} = 2R\cos^2\theta\,\vec{i} + R\sin 2\theta\,\vec{j}$$

Differentiating with respect to time.

$$\frac{d\vec{r}}{dt} \text{ or } \vec{v} = 2R\,2\cos\theta(-\sin\theta)\frac{d\theta}{dt}\vec{i} + 2R\cos 2\theta\frac{d\theta}{dt}\vec{j}$$

$$\text{or, } \vec{v} = 2R\left(\frac{-d\theta}{dt}\right)[\sin 2\theta\,\vec{i} - \cos 2\theta\,\vec{j}]$$

$$\text{or, } \vec{v} = 2R\omega(\sin 2\theta\,\vec{i} - \cos^2\theta\,\vec{j})$$

So, $\quad |\vec{v}|$ or $v = 2\omega R = 0{\cdot}4\text{ m/s}.$

As ω is constant, v is also constant and $w_t = \frac{dv}{dt} = 0,$

So, $$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R = 0{\cdot}32\text{ m/s}^2$$

Alternate : From the Fig. the angular velocity of the point A, with respect to centre of the circle C becomes

$$-\frac{d(2\theta)}{dt} = 2\left(\frac{-d\theta}{dt}\right) = 2\omega = \text{constant}$$

Thus we have the problem of finding the velocity and acceleration of a particle moving along a circle of radius R with constant angular velocity 2ω.

Hence $\quad v = 2\omega R$ and

$$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R$$

1.44 Differentiating $\varphi(t)$ with respect to time

$$\frac{d\varphi}{dt} = \omega_z = 2at \qquad (1)$$

For fixed axis rotation, the speed of the point A:

$$v = \omega R = 2atR \text{ or } R = \frac{v}{2at} \qquad (2)$$

Differentiating with respect to time

$$w_t = \frac{dv}{dt} = 2aR = \frac{v}{t}, \text{ (using 1)}$$

But $$w_n = \frac{v^2}{R} = \frac{v^2}{v/2at} = 2atv \quad \text{(using 2)}$$

So, $$w = \sqrt{w_t^2 + w_n^2} = \sqrt{(v/t)^2 + (2atv)^2}$$

$$= \frac{v}{t}\sqrt{1 + 4a^2t^4}$$

1.45 The shell acquires a constant angular acceleration at the same time as it accelerates linearly. The two are related by (assuming both are constant)

$$\frac{w}{l} = \frac{\beta}{2\pi n}$$

Where w = linear acceleration and β = angular acceleration

Then, $\omega = \sqrt{2 \cdot \beta \, 2\pi n} = \sqrt{2 \cdot \frac{w}{l}(2\pi n)^2}$

But $v^2 = 2wl$, hence finally

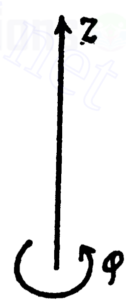
$$\omega = \frac{2\pi n v}{l}$$

1.46 Let us take the rotation axis as z-axis whose positive direction is associated with the positive direction of the cordinate φ, the rotation angle, in accordance with the right-hand screw rule (Fig.)

(a) Defferentiating $\varphi(t)$ with respect to time.

$$\frac{d\varphi}{dt} = a - 3bt^2 = \omega_z \qquad (1) \text{ and}$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega_z}{dt} = \beta_z = -6bt \qquad (2)$$



From (1) the solid comes to stop at $\Delta t = t = \sqrt{\frac{a}{3b}}$

The angular velocity $\omega = a - 3bt^2$, for $0 \le t \le \sqrt{a/3b}$

So, $$<\omega> = \frac{\int \omega \, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} (a - 3bt^2)\, dt}{\int_0^{\sqrt{a/3b}} dt} = [at - bt^3]_0^{\sqrt{a/3b}} \Big/ \sqrt{a/3b} = 2a/3$$

Similarly $\beta = |\beta_z| = 6bt$ for all values of t.

So, $$\langle \beta \rangle = \frac{\int \beta \, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} 6bt\,dt}{\int_0^{\sqrt{a/3b}} dt} = \sqrt{3ab}$$

(b) From Eq. (2) $\beta_z = -6bt$

So, $$(\beta_z)_{t = \sqrt{a/3b}} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{ab}$$

Hence $$\beta = \left| (\beta_z)_{t = \sqrt{a/3b}} \right| = 2\sqrt{3ab}$$

1.47 Angle α is related with $|w_t|$ and w_n by means of the fomula :

$$\tan\alpha = \frac{w_n}{|w_t|}, \text{ where } w_n = \omega^2 R \text{ and } |w_t| = \beta R \tag{1}$$

where R is the radius of the circle which an arbitrary point of the body circumscribes.

From the given equation $\beta = \frac{d\omega}{dt} = at$ (here $\beta = \frac{d\omega}{dt}$, as β is positive for all values of t)

Integrating within the limit $\int_0^{\omega} d\omega = a\int_0^t t\,dt$ or, $\omega = \frac{1}{2}at^2$

So, $$w_n = \omega^2 R = \left(\frac{at^2}{2}\right)^2 R = \frac{a^2t^4}{4}R$$

and $$|w_t| = \beta R = atR$$

Putting the values of $|w_t|$ and w_n in Eq. (1), we get,

$$\tan\alpha = \frac{a^2t^4R/4}{atR} = \frac{at^3}{4} \text{ or, } t = \left[\left(\frac{4}{a}\right)\tan\alpha\right]^{1/3}$$

1.48 In accordance with the problem, $\beta_z < 0$

Thus $-\frac{d\omega}{dt} = k\sqrt{\omega}$, where k is proportionality constant

or, $$-\int_{\omega_0}^{\omega}\frac{d\omega}{\sqrt{\omega}} = k\int_0^t dt \text{ or, } \sqrt{\omega} = \sqrt{\omega_0} - \frac{kt}{2} \tag{1}$$

When $\omega = 0$, total time of rotation $t = \tau = \frac{2\sqrt{\omega_0}}{k}$

Average angular velocity $\langle \omega \rangle = \dfrac{\int \omega\, dt}{\int dt} = \dfrac{\displaystyle\int_0^{2\sqrt{\omega_0}/k} \left(\omega_0 + \frac{k^2 t^2}{4} - k t \sqrt{\omega_0}\right) dt}{2\sqrt{\omega_0}/k}$

Hence $\langle \omega \rangle = \left[\omega_0 t + \dfrac{k^2 t^3}{12} - \dfrac{k}{2}\sqrt{\omega_0}\; t^2\right]_0^{2\sqrt{\omega_0}/k} \Big/ 2\dfrac{\sqrt{\omega_0}}{k} = \omega_0/3$

1.49 We have $\omega = \omega_0 - a\,\varphi = \dfrac{d\varphi}{dt}$

Integratin this Eq. within its limit for $(\varphi)\, t$

$$\int_0^{\varphi} \frac{d\varphi}{\omega_0 - k\varphi} = \int_0^{t} dt \text{ or, } \ln \frac{\omega_0 - k\varphi}{\omega_0} = -kt$$

Hence $$\varphi = \frac{\omega_0}{k}\left(1 - e^{-kt}\right) \tag{1}$$

(b) From the Eq., $\omega = \omega_0 - k\varphi$ and Eq. (1) or by differentiating Eq. (1)

$$\omega = \omega_0 e^{-kt}$$

1.50 Let us choose the positive direction of z-axis (stationary rotation axis) along the vector $\vec{\beta}_0$. In accordance with the equation

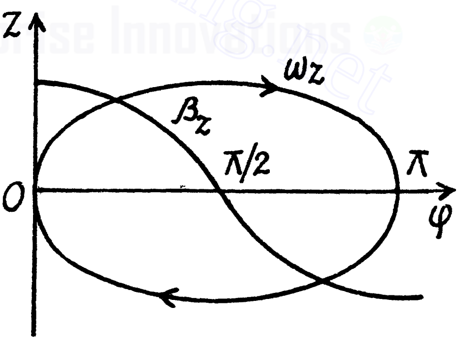
$$\frac{d\omega_z}{dt} = \beta_z \quad \text{or} \quad \omega_z \frac{d\omega_z}{d\varphi} = \beta_z$$

or, $\omega_z\, d\omega_z = \beta_z\, d\varphi = \beta \cos\varphi\, d\varphi$,

Integrating this Eq. within its limit for $\omega_z(\varphi)$

or, $$\int_0^{\omega_z} d\omega_z = \beta_0 \int_0^{\varphi} \cos\varphi\, d\varphi$$

or, $$\frac{\omega_z^2}{2} = \beta_0 \sin\varphi$$



Hence $\omega_z = \pm\sqrt{2\beta_0 \sin\varphi}$

The plot $\omega_z(\varphi)$ is shown in the Fig. It can be seen that as the angle φ grows, the vector $\vec{\omega}$ first increases, coinciding with the direction of the vector $\vec{\beta}_0$ $(\omega_z > 0)$, reaches the maximum at $\varphi = \varphi/2$, then starts decreasing and finally turns into zero at $\varphi = \pi$. After that the body starts rotating in the opposite direction in a similar fashion $(\omega_z < 0)$. As a result, the body will oscillate about the position $\varphi = \varphi/2$ with an amplitude equal to $\pi/2$.

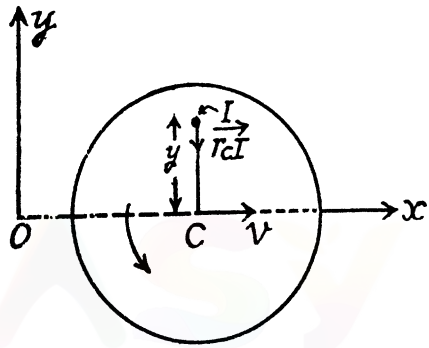
1.51 Rotating disc moves along the x-axis, in plane motion in $x-y$ plane. Plane motion of a solid can be imagined to be in pure rotation about a point (say I) at a certain instant known as instantaneous centre of rotation. The instantaneous axis whose positive sense is directed along $\vec{\omega}$ of the solid and which passes through the point I, is known as instantaneous axis of rotation.

Therefore the velocity vector of an arbitrary point (P) of the solid can be represented as :

$$\vec{v}_p = \vec{\omega} \times \vec{r}_{PI} \qquad (1)$$

On the basis of Eq. (1) for the C. M. (C) of the disc

$$\vec{v}_c = \vec{\omega} \times \vec{r}_{cI} \qquad (2)$$



According to the problem $\vec{v}_c \uparrow\uparrow \vec{i}$ and $\vec{\omega} \uparrow\uparrow \vec{k}$ i.e. $\vec{\omega} \perp x-y$ plane, so to satisy the Eqn. (2) $\vec{r}_{CI}$ is directed along $(-\vec{j})$. Hence point I is at a distance $r_{CI} = y$, above the centre of the disc along $y-$axis. Using all these facts in Eq. (2), we get

$$v_C = \omega y \text{ or } y = \frac{v_c}{\omega} \qquad (3)$$

(a) From the angular kinematical equation

$$\omega_z = \omega_{0z} + \beta_z t \qquad (4)$$

$$\omega = \beta t.$$

On the other hand $x = v t$, (where x is the x coordinate of the C.M.)

or, $$t = \frac{x}{v} \qquad (5)$$

From Eqs. (4) and (5), $\omega = \dfrac{\beta x}{v}$

Using this value of ω in Eq. (3) we get $y = \dfrac{v_c}{\omega} = \dfrac{v}{\beta x/v} = \dfrac{v^2}{\beta x}$ (hyperbola)

(b) As centre C moves with constant acceleration w, with zero initial velocity

So, $$x = \frac{1}{2} w t^2 \text{ and } v_c = w t$$

Therefore, $$v_c = w\sqrt{\frac{2x}{w}} = \sqrt{2 x w}$$

Hence $$y = \frac{v_c}{\omega} = \frac{\sqrt{2 w x}}{\omega} \text{ (parabola)}$$

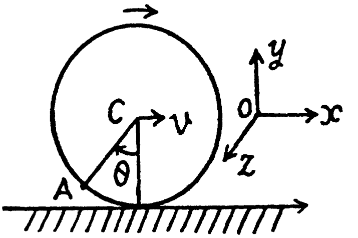
1.52 The plane motion of a solid can be imagined as the combination of translation of the C.M. and rotation about C.M.

So, we may write $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$

$$= \vec{v}_C + \vec{\omega} \times \vec{r}_{AC} \qquad (1) \text{ and}$$

$$\vec{w}_A = \vec{w}_C + \vec{w}_{AC}$$

$$= \vec{w}_C + \omega^2(-\vec{r}_{AC}) + (\vec{\beta} \times \vec{r}_{AC}) \qquad (2)$$



$\vec{r}_{AC}$ is the position of vector of A with respect to C.

In the problem $v_C = v =$ constant, and the rolling is without slipping i.e., $v_C = v = \omega R$, So, $w_C = 0$ and $\beta = 0$. Using these conditions in Eq. (2)

$$\vec{w}_A = \omega^2(-\vec{r}_{AC}) = \omega^2 R(-\hat{u}_{AC}) = \frac{v^2}{R}(-\hat{u}_{AC})$$

Here, $\hat{u}_{AC}$ is the unit vector directed along $\vec{r}_{AC}$.

Hence $w_A = \dfrac{v^2}{R}$ and $\vec{w}_A$ is directed along $(-\hat{u}_{AC})$ or directed toward the centre of the wheel.

(b) Let the centre of the wheel move toward right (positive x-axis) then for pure tolling on the rigid horizontal surface, wheel will have to rotate in clockwise sense. If ω be the angular velocity of the wheel then $\omega = \dfrac{v_C}{R} = \dfrac{v}{R}$.

Let the point A touches the horizontal surface at $t = 0$, further let us locate the point A at $t = t$,

When it makes $\theta = \omega t$ at the centre of the wheel.

From Eqn. (1) $\qquad \vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v\vec{i} + \omega(-\vec{k}) \times [R\cos\theta(-\vec{j}) + R\sin\theta(-\vec{i})]$$

or, $\qquad \vec{v}_A = v\vec{i} + \omega R[\cos\omega t(-\vec{i}) + \sin\omega t\vec{j}]$

$$= (v - \cos\omega t)\vec{i} + v\sin\omega t\vec{j} \quad (\text{as } v = \omega R)$$

So, $\quad v_A = \sqrt{(v - v\cos\omega t)^2 + (v\sin\omega t)^2}$

$$= v\sqrt{2(1-\cos\omega t)} = 2v\sin(\omega t/2)$$

Hence distance covered by the point A during $T = 2\pi/\omega$

$$s = \int v_A\, dt = \int_0^{2\pi/\omega} 2v\sin(\omega t/2)\, dt = \frac{8v}{\omega} = 8R.$$

1.53 Let us fix the co-ordinate axis xyz as shown in the fig. As the ball rolls without slipping along the rigid surface so, on the basis of the solution of problem 1.52 :

Thus $\qquad \left.\begin{array}{c} \vec{v}_0 = \vec{v}_c + \vec{\omega} \times \vec{r}_{oc} = 0 \\ v_c = \omega R \text{ and } \vec{\omega} \uparrow\uparrow (-\vec{k}) \text{ as } \vec{v}_c \uparrow\uparrow \vec{i} \end{array}\right\} \qquad (1)$

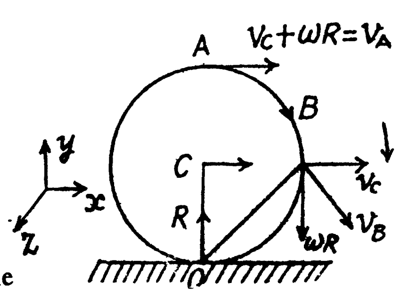
$$\vec{\omega}_c + \vec{\beta} \times \vec{r}_{oc} = 0$$

and $\quad w_c = \beta R$ and $\vec{\beta} \uparrow\uparrow (-\vec{k})$ as $\vec{w}_c \uparrow\uparrow \vec{i}$

At the position corresponding to that of Fig., in accordance with the problem,

$$w_c = w, \text{ so } v_c = wt$$

and $\quad \omega = \frac{v_c}{R} = \frac{wt}{R}$ and $\beta = \frac{w}{R}$ (using 1)



(a) Let us fix the co-ordinate system with the frame attached with the rigid surface as shown in the Fig.

As point O is the instantaneous centre of rotation of the ball at the moment shown in Fig.

so, $\quad \vec{v}_0 = 0,$

Now, $\quad \vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{j}) = (v_C + \omega R)\vec{i}$$

So, $\quad \vec{v}_A = 2 v_C \vec{i} = 2 wt \vec{i}$ (using 1)

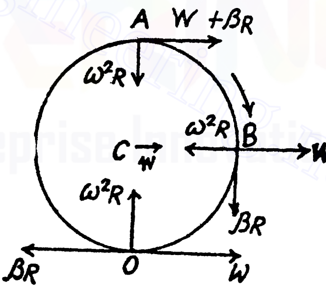
Similalry $\quad \vec{v}_B = \vec{v}_C + \vec{\omega} \times \vec{r}_{BC} = v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{i})$

$$= v_C \vec{i} + \omega R(-\vec{j}) = v_C \vec{i} + v_C(-\vec{j})$$

So, $v_B = \sqrt{2}\, v_c = \sqrt{2}\, wt$ and $\vec{v}_B$ is at an angle 45° from both $\vec{i}$ and $\vec{j}$ (Fig.)

(b) $\vec{w}_0 = \vec{w}_C + \omega^2(-\vec{r}_{oc}) + \vec{\beta} \times \vec{r}_{OC}$

$$= \omega^2(-\vec{r}_{OC}) = \frac{v_C^2}{R}(-\hat{u}_{OC}) \text{ (using 1)}$$



where $\hat{u}_{OC}$ is the unit vector along $\vec{r}_{OC}$

so, $\quad w_0 = \frac{v_0^2}{R} = \frac{w^2 t^2}{R}$ (using 2) and $\vec{w}_0$ is directed towards the centre of the ball

Now $\vec{w}_A = \vec{w}_C + \omega^2(-\vec{r}_{AC}) + \vec{\beta} \times \vec{r}_{AC}$

$$= w\vec{i} + \omega^2 R(-\vec{j}) + \beta(-\vec{k}) \times R\vec{j}$$

$$= (w + \beta R)\vec{i} + \frac{v_c^2}{R}(-\vec{j}) \text{ (using 1)} = 2w\vec{i} + \frac{w^2 t^2}{R}(-\vec{j})$$

So, $\quad w_A = \sqrt{4w^2 + \frac{w^4 t^4}{R^2}} = 2w\sqrt{1 + \left(\frac{wt^2}{2R}\right)^2}$

Similarly $\quad \vec{w}_B = \vec{w}_c + \omega^2(-\vec{r}_{BC}) + \vec{\beta} \times \vec{r}_{BC}$

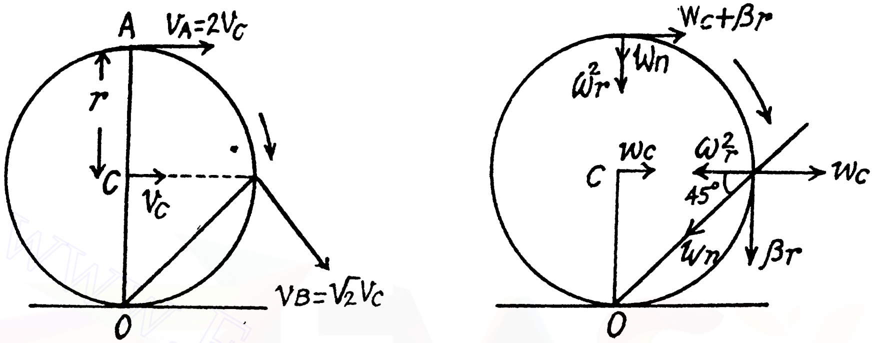
$$= w\vec{i} + \omega^2 R(-\vec{i}) + \beta(-\vec{k}) \times R(\vec{i})$$

$$= \left(v - \frac{v_C^2}{R}\right)\vec{i} + \beta R(-\vec{j}) \text{ (using 1)}$$

$$= \left(w - \frac{w^2 t^2}{R}\right)\vec{i} + w\,(-\vec{j}) \text{ (using 2)}$$

So, $$w_B = \sqrt{\left(w - \frac{w^2 t^2}{R}\right)^2 + w^2}$$

1.54 Let us draw the kinematical diagram of the rolling cylinder on the basis of the solution of problem 1.53.



As, an arbitrary point of the cylinder follows a curve, its normal acceleration and radius of curvature are related by the well known equation

$$w_n = \frac{v^2}{R}$$

so, for point A, $$w_{A\,(n)} = \frac{v_A^2}{R_A}$$

or, $$R_A = \frac{4\,v_c^2}{\omega_r^2} = 4r \text{ (because } v_c = \omega r, \text{ for pure rolling)}$$

Similarly for point B,

$$w_{B\,(n)} = \frac{v_B^2}{R_B}$$

$$\omega^2 r \cos 45° = \frac{(\sqrt{2}\,v_c)^2}{R_B},$$

or, $$R_B = 2\sqrt{2}\,\frac{v_C^2}{\omega^2 r} = 2\sqrt{2}\,r$$

1.55 The angular velocity is a vector as infinitesimal rotation commute. Then the relative angular velocity of the body 1 with respect to the body 2 is clearly.

$$\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$$

as for relative linear velocity. The relative acceleration of 1 w.r.t. 2 is

$$\left(\frac{d\vec{\omega}_1}{dt}\right)_{S'}$$

where S' is a frame corotating with the second body and S is a space fixed frame with origin coinciding with the point of intersection of the two axes,

but $$\left(\frac{d\vec{\omega}_1}{dt}\right)_S = \left(\frac{d\vec{\omega}_1}{dt}\right)_{S'} + \vec{\omega}_2 \times \vec{\omega}_1$$

Since S' rotates with angular velocity $\vec{w}_2$. However $\left(\frac{d\vec{\omega}_1}{dt}\right)_S = 0$ as the first body rotates with constant angular velocity in space, thus

$$\vec{\beta}_{12} = \vec{\omega}_1 \times \vec{\omega}_2.$$

Note that for any vector $\vec{b}$, the relation in space forced frame (k) and a frame (k') rotating with angular velocity $\vec{\omega}$ is

$$\left.\frac{d\vec{b}}{dt}\right|_K = \left.\frac{d\vec{b}}{dt}\right|_{K'} + \vec{\omega} \times \vec{b}$$

1.56 We have $\quad \vec{\omega} = at\,\vec{i} + bt^2\,\vec{j}$ (1)

So, $\quad \omega = \sqrt{(at)^2 + (bt^2)^2}$, thus, $\omega|_{t=10s} = 7.81$ rad/s

Differentiating Eq. (1) with respect to time

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = a\,\vec{i} + 2\,bt\,\vec{j} \qquad (2)$$

So, $$\beta = \sqrt{a^2 + (2\,bt)^2}$$

and $$\beta|_{t=10s} = 1{\cdot}3 \text{ rad/s}^2$$

(b) $$\cos\alpha = \frac{\vec{\omega}\cdot\vec{\beta}}{\omega\beta} = \frac{(at\,\vec{i} + bt^2\,\vec{j}\,)\cdot(a\vec{i} + 2\,bt\,\vec{j}\,)}{\sqrt{(at)^2 + (bt^2)^2}\,\sqrt{a^2 + (2bt)^2}}$$

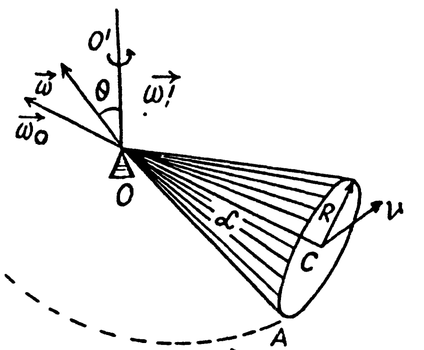
Putting the values of (a) and (b) and taking $t = 10$s, we get

$$\alpha = 17°$$

1.57 (a) Let the axis of the cone (OC) rotates in anticlockwise sense with constant angular velocity $\vec{\omega}'$ and the cone itself about it's own axis (OC) in clockwise sense with angular velocity $\vec{\omega}_0$ (Fig.). Then the resultant angular velocity of the cone.

$$\vec{\omega} = \vec{\omega}' + \vec{\omega}_0 \qquad (1)$$

As the rolling is pure the magnitudes of the vectors $\vec{\omega}'$ and $\vec{\omega}_0$ can be easily found from Fig.



$$\omega' = \frac{v}{R\cot\alpha},\quad \omega_0 = v/R \qquad (2)$$

As $\vec{\omega}' \perp \vec{\omega}_0$, from Eq. (1) and (2)

$$\omega = \sqrt{\omega'^2 + \omega_0^2}$$

$$\sqrt{\left(\frac{v}{R \cot \alpha}\right)^2 + \left(\frac{v}{R}\right)^2} = \frac{v}{R \cos \alpha} = 2{\cdot}3 \text{ rad/s}$$

(b) Vector of angular acceleration

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} \quad (\text{as } \vec{\omega}' = \text{constant.})$$

The vector $\vec{\omega}_0$ which rotates about the OO' axis with the angular velocity $\vec{\omega}'$, retains i magnitude. This increment in the time interval dt is equal to

$$|d\vec{\omega}_0| = \omega_0 \cdot \omega' \, dt \text{ or in vector form } d\vec{\omega}_0 = (\vec{\omega}' \times \vec{\omega}_0)\, dt.$$

Thus $\quad \vec{\beta} = \vec{\omega}' \times \vec{\omega}_0$ (3

The magnitude of the vector $\vec{\beta}$ is equal to

$$\beta = \omega' \omega_0 \; (\text{as } \vec{\omega}' \perp \vec{\omega}_0)$$

So, $\quad \beta = \dfrac{v}{R \cot \alpha} \dfrac{v}{R} = \dfrac{v^2}{R^2} \tan \alpha = 2{\cdot}3 \text{ rad/s}$

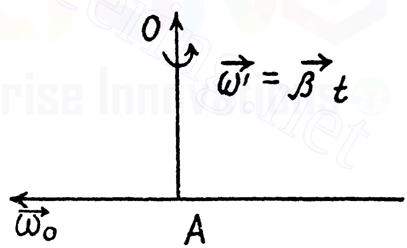
1.58 The axis AB acquired the angular velocity

$$\vec{\omega}' = \vec{\beta}_o t \qquad (1)$$

Using the facts of the solution of 1.57, the angular velocity of the body

$$\omega = \sqrt{\omega_o^2 + \omega'^2}$$

$$= \sqrt{\omega_0^2 + \beta_0^2 t^2} = 0{\cdot}6 \text{ rad/s}$$



And the angular acceleration.

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} = \frac{d\vec{\omega}'}{dt} + \frac{d\vec{\omega}_0}{dt}$$

But $\quad \dfrac{d\vec{\omega}_0}{dt} = \vec{\omega}' \times \vec{\omega}_0$, and $\dfrac{d\vec{\omega}'}{dt} = \vec{\beta}_0 t$

So, $\quad \vec{\beta} = (\vec{\beta}_0 t \times \vec{\omega}_0) + \vec{\beta}_0$

As, $\quad \vec{\beta}_0 \perp \vec{\omega}_0$ so, $\beta = \sqrt{(\omega_0 \beta_0 t)^2 + \beta_0^2} = \beta_0 \sqrt{1 + (\omega_0 t)^2} = 0{\cdot}2 \text{ rad/s}^2$

1.2 THE FUNDAMENTAL EQUATION OF DYNAMICS

1.59 Let R be the constant upward thurst on the aerostat of mass m, coming down with a constant acceleration w. Applying Newton's second law of motion for the aerostat in projection form

$$F_y = mw_y$$

$$mg - R = mw \tag{1}$$

Now, if Δm be the mass, to be dumped, then using the Eq. $F_y = mw_y$

$$R - (m - \Delta m)\, g = (m - \Delta m)\, w, \tag{2}$$

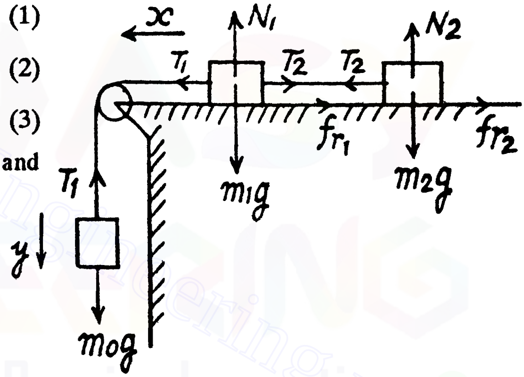
From Eqs. (1) and (2), we get, $\Delta m = \dfrac{2\,mw}{g + w}$

1.60 Let us write the fundamental equation of dynamics for all the three blocks in terms of projections, having taken the positive direction of x and y axes as shown in Fig; and using the fact that kinematical relation between the accelerations is such that the blocks move with same value of acceleration (say w)

$$m_0\, g - T_1 = m_0\, w \tag{1}$$

$$T_1 - T_2 - km_1\, g = m_1\, w \tag{2}$$

and $$T_2 - km_2\, g = m_2\, w \tag{3}$$



The simultaneous solution of Eqs. (1), (2) and (3) yields,

$$w = g\,\frac{[\,m_0 - k\,(m_1 + m_2)\,]}{m_0 + m_1 + m_2}$$

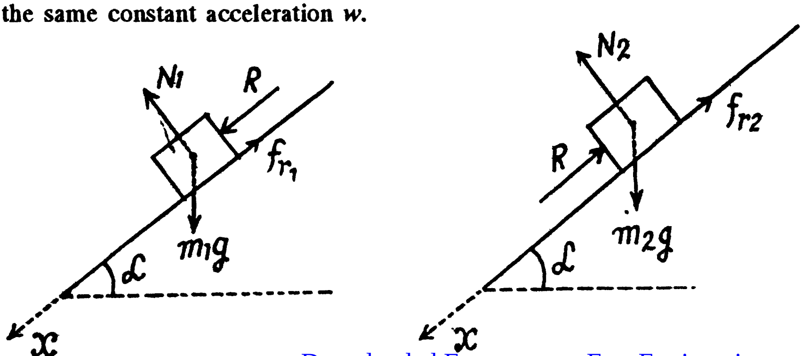
and $$T_2 = \frac{(1 + k)\,m_0}{m_0 + m_1 + m_2}\,m_2\, g$$

As the block m_0 moves down with acceleration w, so in vector form

$$\vec{w} = \frac{[m_0 - k\,(m_1 + m_2)\,]\,\vec{g}}{m_0 + m_1 + m_2}$$

1.61 Let us indicate the positive direction of x-axis along the incline (Fig.). Figures show the force diagram for the blocks.

Let, R be the force of interaction between the bars and they are obviously sliding down with the same constant acceleration w.



Newton's second law of motion in projection form along x-axis for the blocks gives :

$$m_1 g \sin \alpha - k_1 m_1 g \cos \alpha + R = m_1 w \quad (1)$$

$$m_2 g \sin \alpha - R - k_2 m_2 g \cos \alpha = m_2 w \quad (2)$$

Solving Eqs. (1) and (2) simultaneously, we get

$$w = g \sin \alpha - g \cos \alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} \text{ and}$$

$$R = \frac{m_1 m_2 (k_1 - k_2) g \cos \alpha}{m_1 + m_2} \quad (3)$$

(b) when the blocks just slide down the plane, $w = 0$, so from Eqn. (3)

$$g \sin \alpha - g \cos \alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} = 0$$

$$\text{or,} \quad (m_1 + m_2) \sin \alpha = (k_1 m_1 + k_2 m_2) \cos \alpha$$

Hence $$\tan \alpha = \frac{(k_1 m_1 + k_2 m_2)}{m_1 + m_2}$$

1.62 Case 1. When the body is launched up :

Let k be the coefficeint of friction, u the velocity of projection and l the distance traversed along the incline. Retarding force on the block $= mg \sin \alpha + k\, mg \cos \alpha$ and hence the retardation $= g \sin \alpha + k\, g \cos \alpha$.

Using the equation of particle kinematics along the incline,

$$0 = u^2 - 2 (g \sin \alpha + k\, g \cos \alpha)\, l$$

or, $$l = \frac{u^2}{2 (g \sin \alpha + k\, g \cos \alpha)} \quad (1)$$

and $$0 = u - (g \sin \alpha + k\, g \cos \alpha)\, t$$

or, $$u = (g \sin \alpha + k\, g \cos \alpha)\, t \quad (2)$$

Using (2) in (1) $l = \frac{1}{2} (g \sin \alpha + k\, g \cos \alpha)\, t^2$ (3)

Case (2). When the block comes downward, the net force on the body $= mg \sin \alpha - km\, g \cos \alpha$ and hence its acceleration $= g \sin \alpha - k\, g \cos \alpha$

Let, t be the time required then,

$$l = \frac{1}{2} (g \sin \alpha - k\, g \cos \alpha)\, t'^2 \quad (4)$$

From Eqs. (3) and (4)

$$\frac{t^2}{t'^2} = \frac{\sin \alpha - k \cos \alpha}{\sin \alpha + k \cos \alpha}$$

But $\frac{t}{t'} = \frac{1}{\eta}$ (according to the question),

Hence on solving we get

$$k = \frac{(\eta^2 - 1)}{(\eta^2 + 1)} \tan \alpha = 0{\cdot}16$$

1.63 At the initial moment, obviously the tension in the thread connecting m_1 and m_2 equals the weight of m_2.

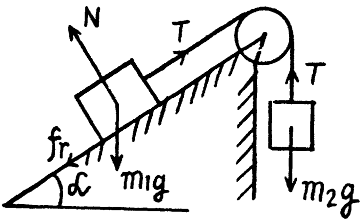
(a) For the block m_2 to come down or the block m_1 to go up, the conditions is

$$m_2 g - T \geq 0 \quad \text{and} \quad T - m_1 g \sin\alpha - fr \geq 0$$

where T is tension and f_r is friction which in the limiting case equals $km_1 g \cos\alpha$. Then

or $\quad m_2 g - m_1 \sin\alpha > k m_1 g \cos\alpha$

or $\quad \frac{m_2}{m_1} > (k\cos\alpha + \sin\alpha)$



(b) Similarly in the case

$$m_1 g \sin\alpha - m_2 g > fr_{lim}$$

or, $m_1 g \sin\alpha - m_2 g > km_1 g \cos\alpha$

or, $\quad \frac{m_2}{m_1} < (\sin\alpha - k\cos\alpha)$

(c) For this case, neither kind of motion is possible, and fr need not be limiting.

Hence, $\quad (k\cos\alpha + \sin\alpha) > \frac{m_2}{m_1} > (\sin\alpha - k\cos\alpha)$

1.64 From the conditions, obtained in the previous problem, first we will check whether the mass m_2 goes up or down.

Here, $m_2/m_1 = \eta > \sin\alpha + k\cos\alpha$, (substituting the values). Hence the mass m_2 will come down with an acceleration (say w). From the free body diagram of previous problem,

$$m_2 - g - T = m_2 w \qquad (1)$$

and $$T - m_1 g\sin\alpha - k m_1 g\cos\alpha = m_1 w \qquad (2)$$

Adding (1) and (2), we get,

$$m_2 g - m_1 g\sin\alpha - k m_1 g\cos\alpha = (m_1 + m_2) w$$

$$w = \frac{(m_2/m_1 - \sin\alpha - k\cos\alpha)\,g}{(1 + m_2/m_1)} = \frac{(\eta - \sin\alpha - k\cos\alpha)\,g}{1+\eta}$$

Substituting all the values, $w = 0{\cdot}048\,g \approx 0{\cdot}05\,g$

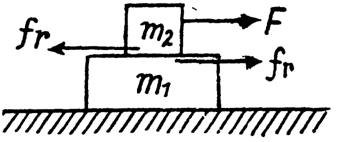
As m_2 moves down with acceleration of magnitude $w = 0.05\,g > 0$, thus in vector form acceleration of m_2:

$$\vec{w}_2 = \frac{(\eta - \sin\alpha - k\cos\alpha)\,\vec{g}}{1+\eta} = 0.05\,\vec{g}.$$

1.65 Let us write the Newton's second law in projection form along positive x-axis for the plank and the bar

$$fr = m_1 w_1, \; fr = m_2 w_2 \qquad (1)$$

At the initial moment, fr represents the static friction, and as the force F grows so does the friction force fr, but up to it's limiting value i.e. $fr = fr_{s(max)} = kN = k m_2 g$.



Unless this value is reached, both bodies moves as a single body with equal acceleration. But as soon as the force fr reaches the limit, the bar starts sliding over the plank i.e. $w_2 \geq w_1$.

Substituting here the values of w_1 and w_2 taken from Eq. (1) and taking into account that $f_r = k m_2 g$, we obtain, $(at - km_2 g)/m_2 \geq \frac{km_2}{m_1} g$, were the sign "=" corresponds to the moment $t = t_0$ (say)

Hence, $$t_0 = \frac{k g m_2 (m_1 + m_2)}{a m_1}$$

If $t \leq t_0$, then $w_1 = \frac{km_2 g}{m_1}$ (constant). and

$$w_2 = (at - km_2 g)/m_2$$

On this basis $w_1(t)$ and $w_2(t)$, plots are as shown in the figure of answersheet.

1.66 Let us designate the x-axis (Fig.) and apply $F_x = m w_x$ for body A :

$$mg \sin\alpha - k m g \cos\alpha = m w$$

or, $$w = g\sin\alpha - k g\cos\alpha$$

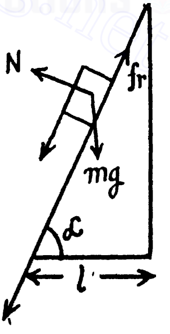
Now, from kinematical equation :

$$l \sec\alpha = 0 + (1/2) w t^2$$

or, $$t = \sqrt{2 l \sec\alpha/(\sin\alpha - k\cos\alpha) g}$$

$$= \sqrt{2 l/(\sin 2\alpha/2 - k\cos^2\alpha) g}$$

(using Eq. (1)).



for t_{min}, $$\frac{d\left(\frac{\sin 2\alpha}{2} - k\cos^2\alpha\right)}{d\alpha} = 0$$

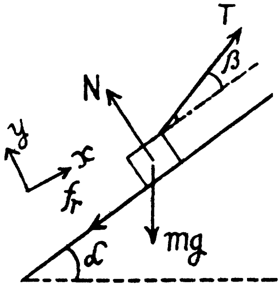
i.e. $$\frac{2\cos 2\alpha}{2} + 2k\cos\alpha\sin\alpha = 0$$

or, $$\tan 2\alpha = -\frac{1}{k} \Rightarrow \alpha = 49°$$

and putting the values of α, k and l in Eq. (2) we get $t_{min} = 1$s.

1.67 Let us fix the $x - y$ co-ordinate system to the wedge, taking the x – axis up, along the incline and the y – axis perpendicular to it (Fig.).

Now, we draw the free body diagram for the bar.



Let us apply Newton's second law in projection form along x and y axis for the bar :

$T\cos\beta - m\,g\sin\alpha - fr = 0$ (1)

$T\sin\beta + N - m\,g\cos\alpha = 0$

or, $N = m\,g\cos\alpha - T\sin\beta$ (2)

But $f_r = kN$ and using (2) in (1), we get

$$T = m\,g\sin\alpha + k\,mg\cos\alpha/(\cos\beta + k\sin\beta) \quad (3)$$

For T_{min} the value of $(\cos\beta + k\sin\beta)$ should be maximum

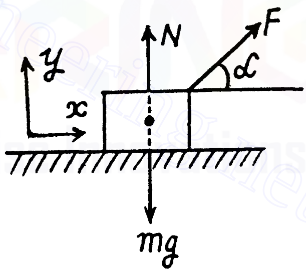
So, $$\frac{d(\cos\beta + k\sin\beta)}{d\beta} = 0 \quad \text{or} \quad \tan\beta = k$$

Putting this value of β in Eq. (3) we get,

$$T_{min} = \frac{m\,g(\sin\alpha + k\cos\alpha)}{1/\sqrt{1+k^2} + k^2/\sqrt{1+k^2}} = \frac{m\,g(\sin\alpha + k\cos\alpha)}{\sqrt{1+k^2}}$$

1.68 First of all let us draw the free body diagram for the small body of mass m and indicate x – axis along the horizontal plane and y – axis, perpendicular to it, as shown in the figure. Let the block breaks off the plane at $t = t_0$ i.e. $N = 0$

So, $N = m\,g - at_0\sin\alpha = 0$



or, $$t_0 = \frac{mg}{a\sin\alpha} \quad (1)$$

From $F_x = m\,w_x$, for the body under investigation :

$m\,dv_x/dt = at\cos\alpha$; Integrating within the limits for $v(t)$

$$m\int_0^v dv_x = a\cos\alpha\int_0 t\,dt \quad \text{(using Eq. 1)}$$

So, $$v = \frac{ds}{dt} = \frac{a\cos\alpha}{2m}t^2 \quad (2)$$

Integrating, Eqn. (2) for $s(t)$

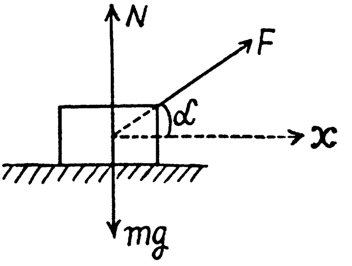
$$s = \frac{a\cos\alpha}{2m}\frac{t^3}{3} \quad (3)$$

Using the value of $t = t_0$ from Eq. (1), into Eqs. (2) and (3)

$$v = \frac{m\,g^2\cos\alpha}{2\,a\sin^2\alpha} \quad \text{and} \quad s = \frac{m^2 g^3\cos\alpha}{6\,a^2\sin^3\alpha}$$

1.69 Newton's second law of motion in projection form, along horizontal or x-axis i.e. $F_x = m\,w_x$ gives.

$$F\cos(as) = m\,v\,\frac{dv}{ds} \quad (as\ \alpha = as)$$



or, $\quad F\cos(as)\,ds = m\,v\,dv$

Integrating, over the limits for $v(s)$

$$\frac{F}{m}\int_0^{\infty}\cos(as)\,ds = \frac{v^2}{2}$$

or $\quad v = \sqrt{\dfrac{2F\sin\alpha}{m\,a}}$

$= \sqrt{2g\sin\alpha/3a}$ (using $F = \dfrac{mg}{3}$)

which is the sought relationship.

1.70 From the Newton's second law in projection from :

For the bar,

$$T - 2kmg = (2m)\,w \qquad (1)$$

For the motor,

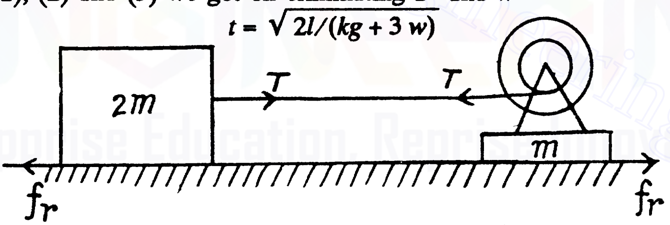
$$T - kmg = m\,w' \qquad (2)$$

Now, from the equation of kinematics in the frame of bar or motor :

$$l = \frac{1}{2}(w + w')\,t^2 \qquad (3)$$

From (1), (2) and (3) we get on eliminating T and w'

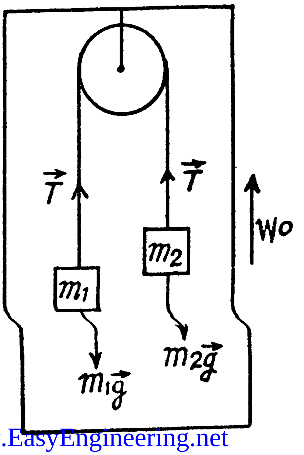
$$t = \sqrt{2l/(kg + 3w)}$$



1.71 Let us write Newton's second law in vector from $\vec{F} = m\,\vec{w}$, for both the blocks (in the frame of ground).

$$\vec{T} + m_1\vec{g} = m_1\vec{w}_1 \qquad (1)$$

$$\vec{T} + m_2\vec{g} = m_2\vec{w}_2 \qquad (2)$$



These two equations contain three unknown quantities $\vec{w}_1$, $\vec{w}_2$ and $\vec{T}$. The third equation is provided by the kinematic relationship between the accelerations :

$$\vec{w}_1 = \vec{w}_0 + \vec{w}', \quad \vec{w}_2 = \vec{w}_0 - \vec{w}' \qquad (3)$$

where $\vec{w}'$ is th acceleration of the mass m_1 with respect to the pulley or elevator car.

Summing up termwise the left hand and the right-hand sides of these kinematical equations, we get

$$\vec{w}_1 + \vec{w}_2 = 2\,\vec{w}_0 \qquad (4)$$

The simultaneous solution of Eqs.(1), (2) and (4) yields

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g} + 2\,m_2\,\vec{w}_0}{m_1 + m_2}$$

Using this result in Eq. (3), we get,

$$\vec{w}' = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0) \quad \text{and} \quad \vec{T} = \frac{2\,m_1\,m_2}{m_1 + m_2}(\vec{w}_0 - \vec{g})$$

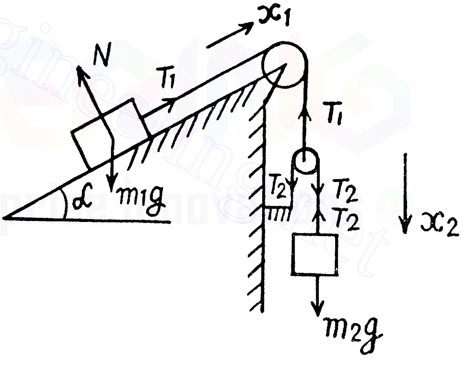
Using the results in Eq. (3) we get $\vec{w}' = \dfrac{m_1 - m_2}{m_1 + m_2}\,(\vec{g} - \vec{w}_0)$

(b) obviously the force exerted by the pulley on the celing of the car

$$\vec{F} = -2\,\vec{T} = \frac{4\,m_1\,m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0)$$

Note : *one could also solve this problem in the frame of elevator car.*

1.72 Let us write Newton's second law for both, bar 1 and body 2 in terms of projection having taken the positive direction of x_1 and x_2 as shown in the figure and assuming that body 2 starts sliding, say, upward along the incline



$$T_1 - m_1\,g\sin\alpha = m_1\,w_1 \qquad (1)$$

$$m_2\,g - T_2 = m_2\,w \qquad (2)$$

For the pulley, moving in vertical direction from the equation $F_x = m\,w_x$

$$2\,T_2 - T_1 = (m_p)\,w_1 = 0$$

(as mass of the pulley $m_p = 0$)

or $$T_1 = 2\,T_2 \qquad (3)$$

As the length of the threads are constant, the kinematical relationship of accelerations becomes

$$w = 2\,w_1 \qquad (4)$$

Simultaneous solutions of all these equations yields :

$$w = \frac{2\,g\left(2\,\dfrac{m_2}{m_1} - \sin\alpha\right)}{\left(4\,\dfrac{m_2}{m_1} + 1\right)} = \frac{2\,g\,(2\,\eta - \sin\alpha)}{(4\,\eta + 1)}$$

As $\eta > 1$, w is directed vertically downward, and hence in vector form

$$\vec{w} = \frac{2\,\vec{g}\,(2\,\eta - \sin\alpha)}{4\,\eta + 1}$$

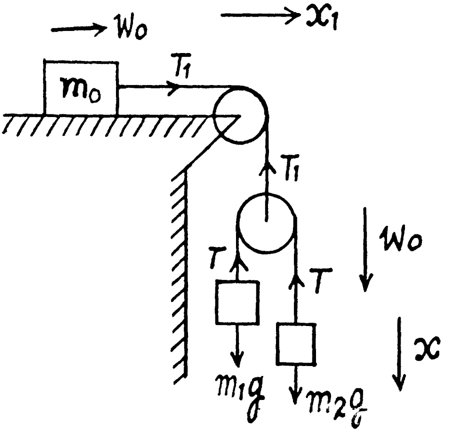
1.73 Let us write Newton's second law for masses m_1 and m_2 and moving pully in vertical direction along positive x – axis (Fig.) :

$$m_1 g - T = m_1 w_{1x} \quad (1)$$

$$m_2 g - T = m_2 w_{2x} \quad (2)$$

$$T_1 - 2T = 0 \text{ (as } m = 0)$$

or
$$T_1 = 2T \quad (3)$$



Again using Newton's second law in projection form for mass m_0 along positive x_1 direction (Fig.), we get

$$T_1 = m_0 w_0 \quad (4)$$

The kinematical relationship between the accelerations of masses gives in terms of projection on the x – axis

$$w_{1x} + w_{2x} = 2 w_0 \quad (5)$$

Simultaneous solution of the obtained five equations yields :

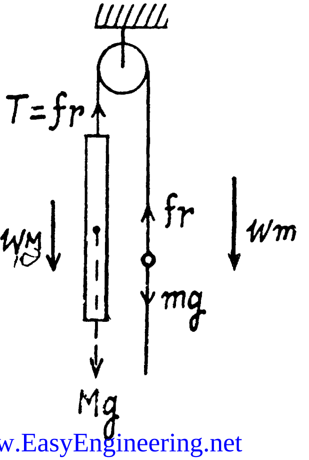
$$w_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] g}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

In vector form

$$\vec{w}_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] \vec{g}}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

1.74 As the thread is not tied with m, so if there were no friction between the thread and the ball m, the tension in the thread would be zero and as a result both bodies will have free fall motion. Obviously in the given problem it is the friction force exerted by the ball on the thread, which becomes the tension in the thread. From the condition or language of the problem $w_M > w_m$ and as both are directed downward so, relative acceleration of $M = w_M - w_m$ and is directed downward. Kinematical equation for the ball in the frame of rod in projection form along upward direction gives :

$$l = \frac{1}{2}(w_M - w_m)\, t^2 \quad (1)$$



Newton's second law in projection form along vertically down direction for both, rod and ball gives,

$$Mg - fr = M w_M \quad (2)$$

$$mg - fr = m w_m \quad (3)$$

Multiplying Eq. (2) by m and Eq. (3) by M and then subtracting Eq. (3) from (2) and after using Eq. (1) we get

$$fr = \frac{2 l M m}{(M - m) t^2}$$

1.75 Suppose, the ball goes up with accleration w_1 and the rod comes down with the acceleration w_2.

As the length of the thread is constant, $2\,w_1 = w_2$ (1)

From Newton's second law in projection form along vertically upward for the ball and vertically downward for the rod respectively gives,

$$T - m\,g = m\,w_1 \qquad (2)$$

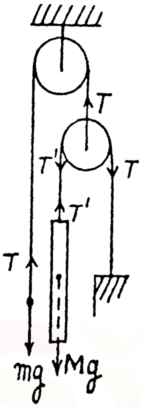
and $$M\,g - T' = M\,w_2 \qquad (3)$$

but $$T = 2\,T \quad \text{(because pulley is massless)} \qquad (4)$$

From Eqs. (1), (2), (3) and (4)

$$w_1 = \frac{(2M - m)\,g}{m + 4M} = \frac{(2 - \eta)\,g}{\eta + 4} \text{ (in upward direction)}$$

and $$w_2 = \frac{2\,(2 - \eta)\,g}{(\eta + 4)} \text{ (downwards)}$$



From kinematical equation in projection form, we get

$$l = \frac{1}{2}(w_1 + w_2)\,t^2$$

as, w_1 and w_2 are in the opposite direction.

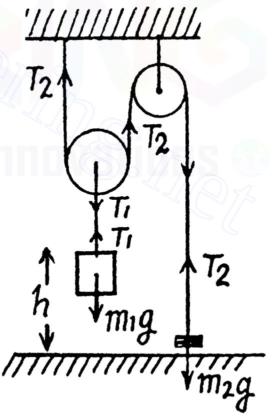
Putting the values of w_1 and w_2, the sought time becomes

$$t = \sqrt{2\,l\,(\eta + 4)/3\,(2 - \eta)\,g} = 1{\cdot}4\text{ s}$$

1.76 Using Newton's second law in projection form along x – axis for the body 1 and along negative x – axis for the body 2 respectively, we get

$$m_1\,g - T_1 = m_1\,w_1 \qquad (1)$$

$$T_2 - m_2\,g = m_2\,w_2 \qquad (2)$$



For the pulley lowering in downward direction from Newton's law along x axis,

$$T_1 - 2\,T_2 = 0 \text{ (as pulley is mass less)}$$

or, $$T_1 = 2\,T_2 \qquad (3)$$

As the length of the thread is constant so,

$$w_2 = 2\,w_1 \qquad (4)$$

The simultaneous solution of above equations yields,

$$w_2 = \frac{2\,(m_1 - 2m_2)\,g}{4\,m_2 + m_1} = \frac{2\,(\eta - 2)}{\eta + 4} \quad \left(\text{as } \frac{m_1}{m_2} = \eta\right) \qquad (5)$$

Obviously during the time interval in which the body 1 comes to the horizontal floor covering the distance h, the body 2 moves upward the distance $2h$. At the moment when the body 2 is at the height $2h$ from the floor its velocity is given by the expression :

$$v_2^2 = 2\,w_2\,(2h) = 2\left[\frac{2\,(\eta - 2)\,g}{\eta + 4}\right]2h = \frac{8\,h\,(\eta - 2)\,g}{\eta + 4}$$

After the body m_1 touches the floor the thread becomes slack or the tension in the thread zero, thus as a result body 2 is only under gravity for it's subsequent motion.

Owing to the velocity v_2 at that moment or at the height $2h$ from the floor, the body 2 further goes up under gravity by the distance,

$$h' = \frac{v_2^2}{2g} = \frac{4h(\eta - 2)}{\eta + 4}$$

Thus the sought maximum height attained by the body 2 :

$$H = 2h + h' = 2h + \frac{4h(\eta - 2)}{(\eta + 4)} = \frac{6\eta h}{\eta + 4}$$

1.77 Let us draw free body diagram of each body, i.e. of rod A and of wedge B and also draw the kinemetical diagram for accelerations, after analysing the directions of motion of A and B. Kinematical relationship of accelarations is :

$$\tan\alpha = \frac{w_A}{w_B} \tag{1}$$

Let us write Newton's second law for both bodies in terms of projections having taken positive directions of y and x axes as shown in the figure.

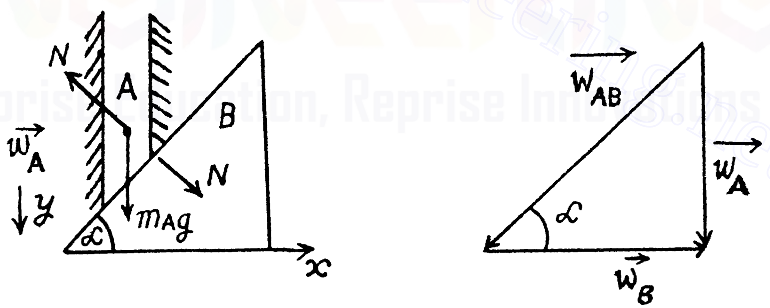
$$m_A g - N\cos\alpha = m_A w_A \tag{2}$$

and
$$N\sin\alpha = m_B w_B \tag{3}$$

Simultaneous solution of (1), (2) and (3) yields :

$$w_A = \frac{m_A g \sin\alpha}{m_A \sin\alpha + m_B \cot\alpha \cos\alpha} = \frac{g}{(1 + \eta\cot^2\alpha)} \text{ and}$$

$$w_B = \frac{w_A}{\tan\alpha} = \frac{g}{(\tan\alpha + \eta\cot\alpha)}$$



Note : *We may also solve this problem using conservation of mechanical energy instead of Newton's second law.*

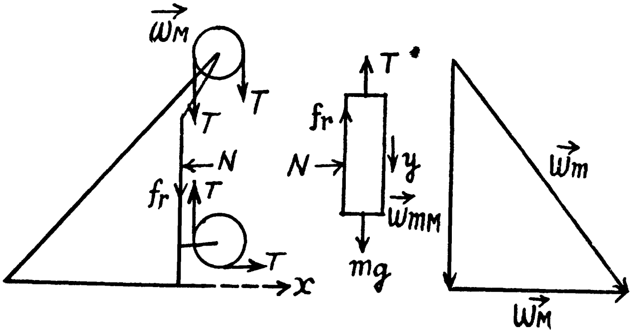
1.78 Let us draw free body diagram of each body and fix the coordinate system, as shown in the figure. After analysing the motion of M and m on the basis of force diagrams, let us draw the kinematical diagram for accelerations (Fig.).

As the length of threads are constant so,

$ds_{mM} = ds_M$ and as $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions that why

$$|\vec{w}_{mM}| = |\vec{w}_M| = w \text{ (say) and}$$

$$\vec{w}_{mM} \uparrow\uparrow \vec{v}_M \text{ and } \vec{w}_M \uparrow\uparrow \vec{v}_M$$



As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$

so, from the triangle law of vector addition

$$w_m = \sqrt{2}\, w \tag{1}$$

From the Eq. $F_x = m\, w_x$, for the wedge and block :

$$T - N = Mw, \tag{2}$$

and

$$N = mw \tag{3}$$

Now, from the Eq. $F_y = mw_y$, for the block

$$mg - T - kN = mw \tag{4}$$

Simultaneous solution of Eqs. (2), (3) and (4) yields :

$$w = \frac{mg}{(km + 2m + M)} = \frac{g}{(k + 2 + M/m)}$$

Hence using Eq. (1)

$$w_m = \frac{g\sqrt{2}}{(2 + k + M/m)}$$

1.79 Bodies 1 and 2 will remain at rest with repect to bar A for $w_{min} \le w \le w_{max}$, where w_{min} is the sought minimum acceleration of the bar. Beyond these limits there will be a relative motion between bar and the bodies. For $0 \le w \le w_{min}$, the tendency of body 1 in relation to the bar A is to move towards right and is in the opposite sense for $w \ge w_{max}$. On the basis of above argument the static friction on 2 by A is directed upward and on 1 by A is directed towards left for the purpose of calculating w_{min}.

Let us write Newton's second law for bodies 1 and 2 in terms of projection along positive x – axis (Fig.).

$T - fr_1 = m\,w$ or, $fr_1 = T - mw$ (1)

$N_2 = mw$ (2)

As body 2 has no acceleration in vertical direction, so

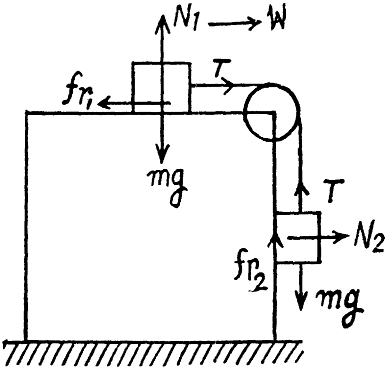
$fr_2 = mg - T$ (3)

From (1) and (3)

$(fr_1 + fr_2) = m\,(g - w)$ (4)

But $fr_1 + fr_2 \le k\,(N_1 + N_2)$

or $fr_1 + fr_2 \le k\,(mg + mw)$ (5)



From (4) and (5)

$$m(g - w) \le mk(g + w), \text{ or } w \ge \frac{g(1-k)}{(1+k)}$$

Hence
$$w_{min} = \frac{g(1-k)}{(1+k)}$$

1.80 On the basis of the initial argument of the solution of 1.79, the tendency of bar 2 with respect to 1 will be to move up along the plane.

Let us fix $(x - y)$ coordinate system in the frame of ground as shown in the figure.

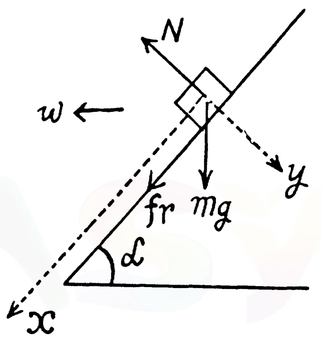
From second law of motion in projection form along y and x axes :

$m\,g\cos\alpha - N = m\,w\sin\alpha$

or, $N = m(g\cos\alpha - w\sin\alpha)$ (1)

$m\,g\sin\alpha + fr = m\,w\cos\alpha$

or, $fr = m(w\cos\alpha - g\sin\alpha)$ (2)



but $fr \le kN$, so from (1) and (2)

$(w\cos\alpha - g\sin\alpha) \le k(g\cos\alpha + w\sin\alpha)$

or, $w(\cos\alpha - k\sin\alpha) \le g(k\cos\alpha + \sin\alpha)$

or, $$w \le g\frac{(\cos\alpha + \sin\alpha)}{\cos\alpha - k\sin\alpha},$$

So, the sought maximum acceleration of the wedge :

$$w_{max} = \frac{(k\cos\alpha + \sin\alpha)g}{\cos\alpha - k\sin\alpha} = \frac{(k\cot\alpha + 1)g}{\cot\alpha - k} \text{ where } \cot\alpha > k$$

1.81 Let us draw the force diagram of each body, and on this basis we observe that the prism moves towards right say with an acceleration $\vec{w}_1$ and the bar 2 of mass m_2 moves down the plane with respect to 1, say with acceleration $\vec{w}_{21}$, then, $\vec{w}_2 = \vec{w}_{21} + \vec{w}_1$ (Fig.)

Let us write Newton's second law for both bodies in projection form along positive y_2 and x_1 axes as shown in the Fig.

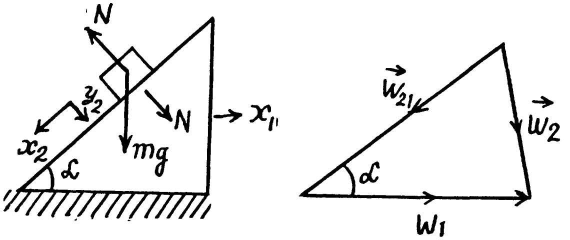
$$m_2 g\cos\alpha - N = m_2 w_{2(y_2)} = m_2\left[w_{21(y_2)} + w_{1(y_2)}\right] = m_2\left[0 + w_1\sin\alpha\right]$$

or, $$m_2 g\cos\alpha - N = m_2 w_1\sin\alpha \quad (1)$$

and $$N\sin\alpha = m_1 w_1 \quad (2)$$

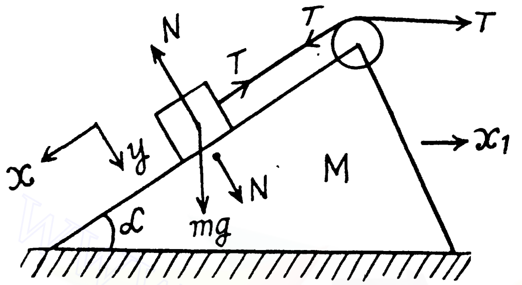
Solving (1) and (2), we get

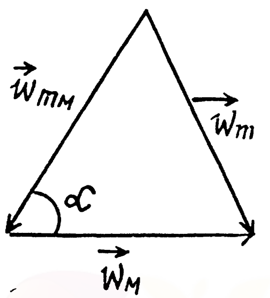
$$w_1 = \frac{m_2 g\sin\alpha\cos\alpha}{m_1 + m_2\sin^2\alpha} = \frac{g\sin\alpha\cos\alpha}{(m_1/m_2) + \sin^2\alpha}$$



1.82 To analyse the kinematic relations between the bodies, sketch the force diagram of each body as shown in the figure.

On the basis of force diagram, it is obvious that the wedge M will move towards right and the block will move down along the wedge. As the length of the thread is constant, the distance travelled by the block on the wedge must be equal to the distance travelled by the wedge on the floor. Hence $d s_{mM} = d s_M$. As $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions and acceleration that's why $\vec{w}_{mM} \uparrow\uparrow \vec{v}_{mM}$ and $\vec{w}_M \uparrow\uparrow \vec{v}_M$ and $w_{mM} = w_M = w$ (say) and accordingly the diagram of kinematical dependence is shown in figure.





As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$, so from triangle law of vector addition.

$$w_m = \sqrt{w_M^2 + w_{mM}^2 - 2\, w_{mM}\, w_M \cos\alpha} = w\sqrt{2(1-\cos\alpha)} \qquad (1)$$

From $F_x = m\, w_x$, (for the wedge),

$$T - T\cos\alpha + N\sin\alpha = M w \qquad (2)$$

For the bar m let us fix $(x-y)$ coordinate system in the frame of ground Newton's law in projection form along x and y axes (Fig.) gives

$$mg\sin\alpha - T = m\, w_{m(x)} = m\left[w_{mM(x)} + w_{M(x)}\right]$$

$$= m\left[w_{mM} + w_M\cos(\pi - \alpha)\right] = m\, w\,(1-\cos\alpha) \qquad (3)$$

$$m\, g\cos\alpha - N = m\, w_{m(y)} = m\left[w_{mM(y)} + w_{M(y)}\right] = m\,[\,0 + w\sin\alpha\,] \qquad (4)$$

Solving the above Eqs. simultaneously, we get

$$w = \frac{m\, g\sin\alpha}{M + 2m\,(1-\cos\alpha)}$$

Note : *We can study the motion of the block m in the frame of wedge also, alternately we may solve this problem using conservation of mechanical energy.*

1.83 Let us sketch the diagram for the motion of the particle of mass m along the circle of radius R and indicate x and y axis, as shown in the figure.

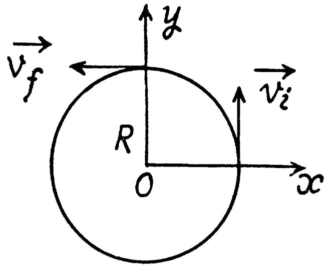
(a) For the particle, change in momentum $\Delta\vec{p} = mv(-\vec{i}) - m\,v\,(\vec{j})$

so, $\quad |\Delta\vec{p}| = \sqrt{2}\, m\, v$

and time taken in describing quarter of the circle,

$$\Delta t = \frac{\pi R}{2v}$$

Hence, $<\vec{F}> = \frac{|\Delta \vec{p}|}{\Delta t} = \frac{\sqrt{2}\, mv}{\pi R/2v} = \frac{2\sqrt{2}\, m v^2}{\pi R}$



(b) In this case

$\vec{p_i} = 0$ and $\vec{p_f} = m w_t t(-\vec{i})$,

so $|\Delta \vec{p}| = m w_t t$

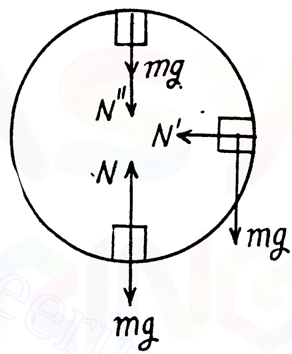
Hence, $|<\vec{F}>| = \frac{|\Delta \vec{p}|}{t} = m w_t$

1.84 While moving in a loop, normal reaction exerted by the flyer on the loop at different points and uncompensated weight if any contribute to the weight of flyer at those points.

(a) When the aircraft is at the lowermost point, Newton's second law of motion in projection form $F_n = m w_n$ gives

$$N - mg = \frac{m v^2}{R}$$

or , $N = mg + \frac{m v^2}{R} = 2{\cdot}09 \text{ kN}$



(b) When it is at the upper most point, again from $F_n = m w_n$ we get

$$N'' + mg = \frac{mv^2}{R}$$

$$N'' = \frac{m v^2}{R} - m g = 0{\cdot}7 \text{ kN}$$

(c) When the aircraft is at the middle point of the loop, again from $F_n = m w_n$

$$N' = \frac{m v^2}{R} = 1{\cdot}4 \text{ kN}$$

The uncompensated weight is mg. Thus effective weight $= \sqrt{N'^2 + m^2 g^2} = 1{\cdot}56$ kN acts obliquely.

1.85 Let us depict the forces acting on the small sphere m, (at an arbitrary position when the thread makes an angle θ from the vertical) and write equation $\vec{F} = m\vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$. From $F_t = m w_t$, we have

$$mg \sin\theta = m\frac{dv}{dt}$$

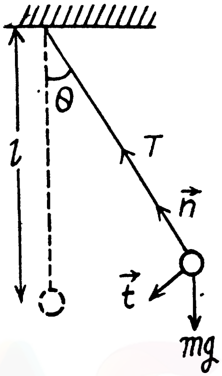
$$= m\frac{v\,dv}{ds} = m\frac{v\,dv}{l(-d\theta)}$$

(as vertical is refrence line of angular position) or $$v\,dv = -gl\sin\theta\,d\theta$$

Integrating both the sides :

$$\int_0^v v\,dv = -g\,l\int_{\pi/2}^{\theta}\sin\theta\,d\theta$$

or, $$\frac{v^2}{2} = g\,l\cos\theta$$



Hence $\frac{v^2}{l} = 2g\cos\theta = w_n$ (1)

(Eq. (1) can be easily obtained by the conservation of mechanical energy).

From $F_n = m\,w_n$

$$T - m\,g\cos\theta = \frac{m\,v^2}{l}$$

Using (1) we have

$$T = 3\,mg\cos\theta \quad (2)$$

Again from the Eq. $F_t = m\,w_t$:

$$mg\sin\theta = m\,w_t \quad\text{or}\quad w_t = g\sin\theta \quad (3)$$

Hence $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(g\sin\theta)^2 + (2g\cos\theta)^2}$ (using 1 and 3)

$$= g\sqrt{1 + 3\cos^2\theta}$$

(b) Vertical component of velocity, $v_y = v\sin\theta$

So, $v_y^2 = v^2\sin^2\theta = 2g\,l\cos\theta\sin^2\theta$ (using 1)

For maximum v_y or v_y^2, $\frac{d(\cos\theta\sin^2\theta)}{d\theta} = 0$

which yields $$\cos\theta = \frac{1}{\sqrt{3}}$$

Therefore from (2) $T = 3mg\frac{1}{\sqrt{3}} = \sqrt{3}\,mg$

(c) We have $\vec{w} = w_t\,\hat{u}_t + w_n\,\hat{u}_n$ thus $w_y = w_{t(y)} + w_{n(y)}$

But in accordance with the problem $w_y = 0$

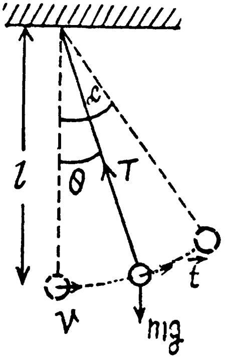
So, $$w_{t(y)} + w_{n(y)} = 0$$

or, $$g\sin\theta\sin\theta + 2g\cos^2\theta\,(-\cos\theta) = 0$$

or, $$\cos\theta = \frac{1}{\sqrt{3}} \quad\text{or,}\quad \theta = 54{\cdot}7°$$

1.86 The ball has only normal acceleration at the lowest position and only tangential acceleration at any of the extreme position. Let v be the speed of the ball at its lowest position and l be the length of the thread, then according to the problem

$$\frac{v^2}{l} = g \sin \alpha \qquad (1)$$



where α is the maximum deflection angle

From Newton's law in projection form : $F_t = m w_t$

$$-mg \sin \theta = mv \frac{dv}{l\, d\theta}$$

or, $\quad -g\, l \sin \theta\, d\theta = v\, dv$

On integrating both the sides within their limits.

$$-gl \int_0^{\alpha} \sin \theta\, d\theta = \int_v^0 v\, dv$$

or, $\quad v^2 = 2gl\,(1 - \cos \alpha) \qquad (2)$

Note : Eq. (2) can easily be obtained by the conservation of mechanical energy of the ball in the uniform field of gravity.

From Eqs. (1) and (2) with $\theta = \alpha$

$$2gl\,(1 - \cos \alpha) = lg \cos \alpha$$

or, $\quad \cos \alpha = \frac{2}{3}$ so, $\alpha = 53°$

1.87 Let us depict the forces acting on the body A (which are the force of gravity $m\vec{g}$ and the normal reaction $\vec{N}$) and write equation $\vec{F} = m\vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$ (Fig.)

From $F_t = m w_t$

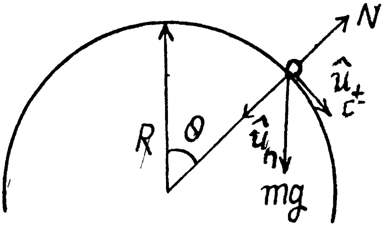
$$mg \sin \theta = m \frac{dv}{dt}$$

$$= m \frac{v\, dv}{ds} = m \frac{v\, dv}{R\, d\theta}$$

or, $\quad g R \sin \theta\, d\theta = v\, dv$

Integrating both side for obtaining $v(\theta)$

$$\int_0^{\theta} g R \sin \theta\, d\theta = \int_0^{v} v\, dv$$



or, $\quad v^2 = 2\, gR\,(1 - \cos \theta) \qquad (1)$

From $\quad F_n = m w_n$

$$mg \cos \theta - N = m \frac{v^2}{R} \qquad (2)$$

At the moment the body loses contact with the surface, $N = 0$ and therefore the Eq. (2) becomes

$$v^2 = gR \cos \theta \qquad (3)$$

where v and θ correspond to the moment when the body loses contact with the surface. Solving Eqs. (1) and (3) we obtain $\cos\theta = \frac{2}{3}$ or, $\theta = \cos^{-1}(2/3)$ and $v = \sqrt{2gR/3}$.

1.88 At first draw the free body diagram of the device as, shown. The forces, acting on the sleeve are it's weight, acting vertically downward, spring force, along the length of the spring and normal reaction by the rod, perpendicular to its length.

Let F be the spring force, and Δl be the elongation.

From, $F_n = mw_n$:

$$N\sin\theta + F\cos\theta = m\,\omega^2 r \tag{1}$$

where $r\cos\theta = (l_0 + \Delta l)$.

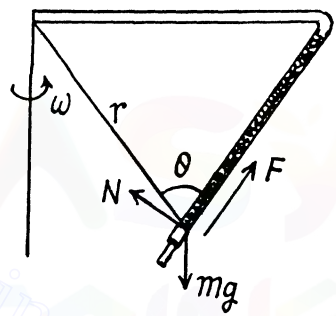
Similarly from $F_t = mw_t$

$$N\cos\theta - F\sin\theta = 0 \quad \text{or,} \quad N = F\sin\theta/\cos\theta \tag{2}$$

From (1) and (2)

$$F(\sin\theta/\cos\theta)\cdot\sin\theta + F\cos\theta = m\,\omega^2 r$$

$$= m\,\omega^2(l_0 + \Delta l)/\cos\theta$$



On putting $F = \kappa\,\Delta l$,

$$\kappa\,\Delta l\sin^2\theta + \kappa\,\Delta l\cos^2\theta = m\,\omega^2(l_0 + \Delta l)$$

on solving, we get,

$$\Delta l = m\,\omega^2\frac{l_0}{\kappa - m\,\omega^2} = \frac{l_0}{(\kappa/m\,\omega^2 - 1)}$$

and it is independent of the direction of rotation.

1.89 According to the question, the cyclist moves along the circular path and the centripetal force is provided by the frictional force. Thus from the equation $F_n = m\,w_n$

$$fr = \frac{m\,v^2}{r} \quad \text{or} \quad kmg = \frac{m\,v^2}{r}$$

or
$$k_0\left(1 - \frac{r}{R}\right)g = \frac{v^2}{r} \quad \text{or} \quad v^2 = k_0\,(r - r^2/R)\,g \tag{1}$$

For v_{max}, we should have $\frac{d\left(r - \frac{r^2}{R}\right)}{dr} = 0$

or, $$1 - \frac{2r}{R} = 0, \quad \text{so } r = R/2$$

Hence $v_{max} = \frac{1}{2}\sqrt{k_0\,gR}$

1.90 As initial velocity is zero thus

$$v^2 = 2\,w_t\,s \tag{1}$$

As $w_t > 0$ the speed of the car increases with time or distance. Till the moment, sliding starts, the static friction provides the required centripetal acceleration to the car.

Thus
$$fr = mw, \quad \text{but} \quad fr \le kmg$$

So, $$w^2 \le k^2 g^2 \text{ or, } w_t^2 + \frac{v^2}{R} \le k^2 g^2$$

or, $$v^2 \le (k^2 g^2 - w_t^2) R$$

Hence $$v_{max} = \sqrt{(k^2 g^2 - w_t^2) R}$$

so, from Eqn. (1), the sought distance $s = \frac{v_{max}^2}{2 w_t} = \frac{1}{2}\sqrt{\left(\frac{kg}{w_t}\right)^2 - 1} = 60$ m.

1.91 Since the car follows a curve, so the maximum velocity at which it can ride without sliding at the point of minimum radius of curvature is the sought velocity and obviously in this case the static friction between the car and the road is limiting.

Hence from the equation $F_n = mw$

$$kmg \ge \frac{m v^2}{R} \quad \text{or} \quad v \le \sqrt{kRg}$$

so $$v_{max} = \sqrt{k R_{min} g}\,. \tag{1}$$

We know that, radius of curvature for a curve at any point (x, y) is given as,

$$R = \left| \frac{[1 + (dy/dx)^2]^{3/2}}{(d^2 y)/dx^2} \right| \tag{2}$$

For the given curve,

$$\frac{dy}{dx} = \frac{a}{\alpha}\cos\left(\frac{x}{\alpha}\right) \text{ and } \frac{d^2 y}{dx^2} = \frac{-a}{\alpha^2}\sin\frac{x}{\alpha}$$

Substituting this value in (2) we get,

$$R = \frac{[1 + (a^2/\alpha^2)\cos^2(x/\alpha)]^{3/2}}{(a/\alpha^2)\sin(x/\alpha)}$$

For the minimum R, $\frac{x}{\alpha} = \frac{\pi}{2}$

and therefore, corresponding radius of curvature

$$R_{min} = \frac{\alpha^2}{a} \tag{3}$$

Hence from (1) and (2)

$$v_{max} = \alpha\sqrt{kg/a}$$

1.92 The sought tensile stress acts on each element of the chain. Hence divide the chain into small, similar elements so that each element may be assumed as a particle. We consider one such element of mass dm, which subtends angle $d\alpha$ at the centre. The chain moves along a circle of known radius R with a known angular speed ω and certain forces act on it. We have to find one of these forces.

From Newton's second law in projection form, $F_x = mw_x$ we get

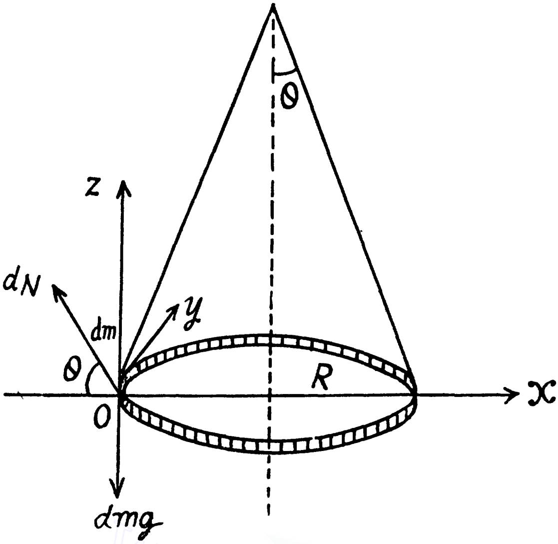
$$2T\sin(d\alpha/2) - dN\cos\theta = dm\,\omega^2 R$$

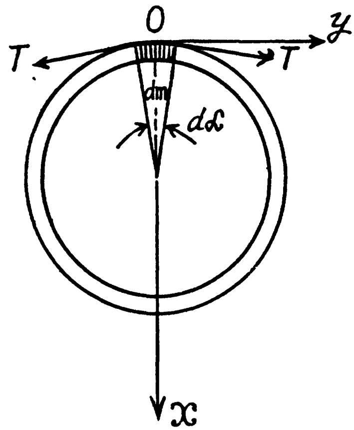
and from $F_z = mw_z$ we get

$$dN\sin\theta = g\,dm$$

Then putting $dm = m d\alpha/2\pi$ and $\sin(d\alpha/2) = d\alpha/2$ and solving, we get,

$$T = \frac{m(\omega^2 R + g\cot\theta)}{2\pi}$$





1.93 Let, us consider a small element of the thread and draw free body diagram for this element.

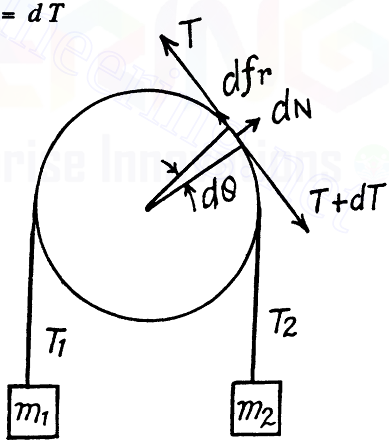
(a) Applying Newton's second law of motion in projection form, $F_n = mw_n$ for this element,

$(T + dT)\sin(d\theta/2) + T\sin(d\theta/2) - dN = dm\,\omega^2 R = 0$

or, $2T\sin(d\theta/2) = dN$, [negelecting the term$(dT\sin d\theta/2)$]

or, $$T\,d\theta = dN, \text{ as } \sin\frac{d\theta}{2} = \frac{d\theta}{2} \quad (1)$$

Also, $$d\,fr = k\,dN = (T + dT) - T = dT \quad (2)$$



From Eqs. (1) and (2),

$k\,T\,d\theta = dT$ or $\dfrac{dT}{T} = k\,d\theta$

In this case $Q = \pi$ so,

or, or, $\ln\dfrac{T_2}{T_1} = k\pi$ (3)

So, $k = \dfrac{1}{\pi}\ln\dfrac{T_2}{T_1} = \dfrac{1}{\pi}\ln\eta_0$

as $\dfrac{T_2}{T_1} = \dfrac{m_2 g}{m_1 g} = \dfrac{m_2}{m_1} = \eta_0$

(b) When $\dfrac{m_2}{m_1} = \eta$, which is greater than η_0, the blocks will move with same value of acceleration. (say w) and clearly m_2 moves downward. From Newton's second law in projection form (downward for m_2 and upward for m_1) we get :

$$m_2 g - T_2 = m_2 w \quad (4)$$

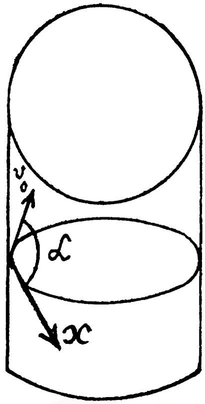
and $$T_1 - m_1 g = m_1 w \quad (5)$$

Also $$\frac{T_2}{T_1} = \eta_0 \quad (6)$$

Simultaneous solution of Eqs. (4), (5) and (6) yields :

$$w = \frac{(m_2 - \eta_0 m_1) g}{(m_2 + \eta_0 m_1)} = \frac{(\eta - \eta_0)}{(\eta + \eta_0)} g \left(\text{as } \frac{m_2}{m_1} = \eta \right)$$

1.94 The force with which the cylinder wall acts on the particle will provide centripetal force necessary for the motion of the particle, and since there is no acceleration acting in the horizontal direction, horizontal component of the velocity will remain constant througout the motion.



So $$v_x = v_0 \cos \alpha$$

Using, $F_n = m w_n$, for the particle of mass m,

$$N = \frac{m v_x^2}{R} = \frac{m v_0^2 \cos^2 \alpha}{R},$$

which is the required normal force.

1.95 Obviously the radius vector describing the position of the particle relative to the origin of coordinate is

$$\vec{r} = x\vec{i} + y\vec{j} = a \sin \omega t \, \vec{i} + b \cos \omega t \, \vec{j}$$

Differentiating twice with respect the time :

$$\vec{w} = \frac{d^2 \vec{r}}{d t^2} = -\omega^2 (a \sin \omega t \, \vec{i} + b \cos \omega t \, \vec{j}) = -\omega^2 \vec{r} \quad (1)$$

Thus $$\vec{F} = m \vec{w} = -m \omega^2 \vec{r}$$

1.96 (a) We have $\Delta \vec{p} = \int \vec{F} dt$

$$= \int_0^t m \vec{g} dt = m \vec{g} t \quad (1)$$

(b) Using the solution of problem 1.28 (b), the total time of motion, $\tau = -\frac{2(\vec{v}_0 \cdot \vec{g})}{g^2}$

Hence using $t = \tau$ in (1)

$$|\Delta \vec{p}| = mg\tau$$

$$= -2m(\vec{v}_0 \cdot \vec{g})/g \quad (\vec{v}_0 \cdot \vec{g} \text{ is } -\text{ve})$$

1.97 From the equation of the given time dependence force $\vec{F} = \vec{a} \, t(\tau - t)$ at $t = \tau$, the force vanishes,

(a) Thus $$\Delta \vec{p} = \vec{p} = \int_0^\tau \vec{F} dt$$

or, $$\vec{p} = \int_0^{\tau} \vec{a}\, t\,(\tau - t)\, dt\; \frac{\vec{a}\tau^3}{6}$$

but $$\vec{p} = m\vec{v} \text{ so } \vec{v} = \frac{\vec{a}\tau^3}{6m}$$

(b) Again from the equation $\vec{F} = m\vec{w}$

$$\vec{a}t\,(\tau - t) = m\frac{d\vec{v}}{dt}$$

or, $$\vec{a}(t\tau - t^2)\, dt = m d\vec{v}$$

Integrating within the limits for $\vec{v}(t)$,

$$\int_0^{t} \vec{a}(t\tau - t^2)\, dt = m\int_0^{\vec{v}} d\vec{v}$$

or, $$\vec{v} = \frac{\vec{a}}{m}\left(\frac{\tau t^2}{2} - \frac{t^3}{3}\right) = \frac{\vec{a}t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right)$$

Thus $$v = \frac{a t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) \text{ for } t \le \tau$$

Hence distance covered during the time interval $t = \tau$,

$$s = \int_0^{\tau} v\, dt$$

$$= \int_0^{\tau} \frac{a t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) dt = \frac{a}{m}\frac{\tau^4}{12}$$

1.98 We have $F = F_0 \sin \omega t$

or $$m\frac{d\vec{v}}{dt} = \vec{F}_0 \sin \omega t \text{ or } m\, d\vec{v} = \vec{F}_0 \sin \omega t\, dt$$

On integrating,

$$m\vec{v} = \frac{-\vec{F}_0}{\omega} \cos \omega t + C, \text{ (where } C \text{ is integration constant)}$$

When $$t = 0,\ v = 0, \text{ so } C = \frac{\vec{F}_0}{m\omega}$$

Hence, $$\vec{v} = \frac{-\vec{F}_0}{m\,\omega} \cos \omega t + \frac{\vec{F}_0}{m\,\omega}$$

As $|\cos \omega t \le 1$ so, $$v = \frac{F_0}{m\,\omega}(1 - \cos \omega t)$$

Thus $$s = \int_0^t v\,dt$$

$$= \frac{F_0 t}{m\,\omega} - \frac{F_0 \sin \omega t}{m\,\omega^2} = \frac{F_0}{m\,\omega^2}(\omega t - \sin \omega t)$$

(Figure in the answer sheet).

1.99 According to the problem, the force acting on the particle of mass m is, $\vec{F} = \vec{F}_0 \cos \omega t$

So, $$m\frac{d\vec{v}}{dt} = \vec{F}_0 \cos \omega t \quad \text{or} \quad d\vec{v} = \frac{\vec{F}_0}{m} \cos \omega t\, dt$$

Integrating, within the limits.

$$\int_0^{\vec{v}} d\vec{v} = \frac{\vec{F}_0}{m}\int_0^t \cos \omega t\, dt \quad \text{or} \quad \vec{v} = \frac{\vec{F}_0}{m\,\omega}\sin \omega t$$

It is clear from equation (1), that after starting at $t = 0$, the particle comes to rest fro the first time at $t = \frac{\pi}{\omega}$.

From Eq. (1), $v = |\vec{v}| = \frac{F_0}{m\omega}\sin \omega t$ for $t \le \frac{\pi}{\omega}$ (2)

Thus during the time interval $t = \pi/\omega$, the sought distance

$$s = \frac{F_0}{mw}\int_0^{\pi/\omega} \sin \omega t\, dt = \frac{2F}{m\,\omega^2}$$

From Eq. (1)

$$v_{max} = \frac{F_0}{m\,\omega} \text{ as } |\sin \omega t| \le 1$$

1.100 (a) From the problem $\vec{F} = -r\vec{v}$ so $m\frac{d\vec{v}}{dt} = -r\vec{v}$

Thus $$m\frac{dv}{dt} = -rv \text{ [as } d\vec{v} \uparrow\downarrow \vec{v}]$$

or, $$\frac{dv}{v} = -\frac{r}{m}dt$$

On integrating $$\ln v = -\frac{r}{m}t + C$$

But at $t = 0$, $v = v_0$, so, $C = \ln v_0$

or, $$\ln\frac{v}{v_0} = -\frac{r}{m}t \quad \text{or,} \quad v = v_0 e^{-\frac{r}{m}t}$$

Thus for $t \to \infty$, $v = 0$

(b) We have $m\frac{dv}{dt} = -rv$ so $dv = \frac{-r}{m}ds$

Integrating within the given limits to obtain $v(s)$:

or, $$\int_{v_0}^{v} dv = -\frac{r}{m}\int_0^s ds \quad \text{or} \quad v = v_0 - \frac{rs}{m} \tag{1}$$

Thus for $$v = 0,\ s = s_{total} = \frac{m v_0}{r}$$

(c) Let we have $$\frac{m\,dv}{v} = -r v \quad \text{or} \quad \frac{dv}{v} = \frac{-r}{m}dt$$

or, $$\int_0^{v_0/\eta} \frac{dv}{v} = \frac{-r}{m}\int_0^t dt, \quad \text{or, } \ln\frac{v_0}{\eta v_0} = -\frac{r}{m}t$$

So $$t = \frac{-m \ln(1/\eta)}{r} = \frac{m \ln \eta}{r}$$

Now, average velocity over this time interval,

$$\langle v \rangle = \frac{\int v\,dt}{\int dt} = \frac{\int_0^{\frac{m}{r}\ln\eta} v_0 e^{-\frac{rt}{m}}\,dt}{\frac{m}{r}\ln\eta} = \frac{v_0(\eta - 1)}{\eta \ln \eta}$$

1.101 According to the problem

$$m\frac{dv}{dt} = -k v^2 \quad \text{or,} \quad m\frac{dv}{v^2} = -k\,dt$$

Integrating, withing the limits,

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt \quad \text{or,} \quad t = \frac{m}{k}\frac{(v_0 - v)}{v_0 v} \tag{1}$$

To find the value of k, rewrite

$$mv\frac{dv}{ds} = -k v^2 \quad \text{or,} \quad \frac{dv}{v} = -\frac{k}{m}ds$$

On integrating

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}\int_0^h ds$$

So, $$k = \frac{m}{h}\ln\frac{v_0}{v} \tag{2}$$

Putting the value of k from (2) in (1), we get

$$t = \frac{h(v_0 - v)}{v_0 v \ln\frac{v_0}{v}}$$

1.102 From Newton's second law for the bar in projection from, $F_x = m\,w_x$ along x direction we get

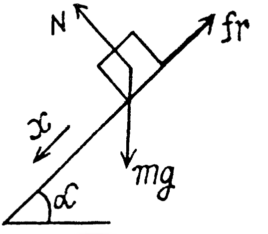
$$mg\sin\alpha - kmg\cos\alpha = mw$$

or, $$v\frac{dv}{dx} = g\sin\alpha - ax\,g\cos\alpha, \ (\text{as } k = ax),$$

or, $$v\,dv = (g\sin\alpha - ax\,g\cos\alpha)\,dx$$

or, $$\int_0^v v\,dv = g\int_0^x (\sin\alpha - x\cos\alpha)\,dx$$

So, $$\frac{v^2}{2} = g\left(\sin\alpha\, x - \frac{x^2}{2}a\cos\alpha\right) \qquad (1)$$



From (1) $v = 0$ at either

$$x = 0, \text{ or } x = \frac{2}{a}\tan\alpha$$

As the motion of the bar is unidirectional it stops after going through a distance of $\frac{2}{a}\tan\alpha$.

From (1), for v_{max},

$$\frac{d}{dx}\left(\sin\alpha x - \frac{x^2}{2}a\cos\alpha\right) = 0, \text{ which yields } x = \frac{1}{a}\tan\alpha$$

Hence, the maximum velocity will be at the distance, $x = \tan\alpha/a$

Putting this value of x in (1) the maximum velocity,

$$v_{max} = \sqrt{\frac{g\sin\alpha\tan\alpha}{a}}$$

1.103 Since, the applied force is proportional to the time and the frictional force also exists, the motion does not start just after applying the force. The body starts its motion when F equals the limiting friction.

Let the motion start after time t_0, then

$$F = at_0 = kmg \ \text{ or, } \ t_0 = \frac{kmg}{a}$$

So, for $t = \leq t_0$, the body remains at rest and for $t > t_0$ obviously

$$\frac{mdv}{dt} = a(t - t_0) \quad \text{or,} \quad m\,dv = a(t - t_0)\,dt$$

Integrating, and noting $v = 0$ at $t = t_0$, we have for $t > t_0$

$$\int_0^v m\,dv = a\int_{t_0}^t (t - t_0)\,dt \quad \text{or} \quad v = \frac{a}{2m}(t - t_0)^2$$

Thus $$s = \int v\,dt = \frac{a}{2m}\int_{t_0}^t (t - t_0)^2\,dt = \frac{a}{6m}(t - t_0)^3$$

1.104 While going upward, from Newton's second law in vertical direction :

$$m\,\frac{v\,dv}{ds} = -(mg + kv^2) \quad \text{or} \quad \frac{v\,dv}{\left(g + \dfrac{kv^2}{m}\right)} = -ds$$

At the maximum height h, the speed $v = 0$, so

$$\int_{v_0}^{0} \frac{v\,dv}{g + (kv^2/m)} = -\int_0^h ds$$

Integrating and solving, we get,

$$h = \frac{m}{2k}\ln\left(1 + \frac{kv_o^2}{mg}\right) \tag{1}$$

When the body falls downward, the net force acting on the body in downward direction equals $(mg - kv^2)$,

Hence net acceleration, in downward direction, according to second law of motion

$$\frac{v\,dv}{ds} = g - \frac{kv^2}{m} \quad \text{or,} \quad \frac{v\,dv}{g - \dfrac{kv^2}{m}} = ds$$

Thus
$$\int_0^{v'} \frac{v\,dv}{g - kv^2/m} = \int_0^h ds$$

Integrating and putting the value of h from (1), we get,

$$v' = v_0 \Big/ \sqrt{1 + k\,v_0^2/mg}.$$

1.105 Let us fix $x - y$ co-ordinate system to the given plane, taking x-axis in the direction along which the force vector was oriented at the moment $t = 0$, then the fundamental equation of dynamics expressed via the projection on x and y-axes gives,

$$F\cos\omega t = m\frac{dv_x}{dt} \tag{1}$$

and
$$F\sin\omega t = m\frac{dv_y}{dt} \tag{2}$$

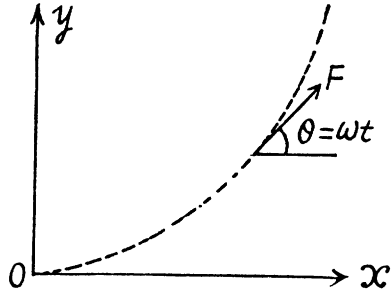
(a) Using the condition $v(0) = 0$, we obtain $v_x = \dfrac{F}{m\,\omega}\sin\omega t$ (3)

and

$$v_y = \frac{F}{m\,\omega}(1 - \cos\omega t) \tag{4}$$

Hence,
$$v = \sqrt{v_x^2 + v_y^2} = \left(\frac{2F}{m\,\omega}\right)\left|\sin\left(\frac{\omega t}{2}\right)\right|$$

(b) It is seen from this that the velocity v turns into zero after the time interval Δt, which can be found from the relation, $\omega \frac{\Delta t}{2} = \pi$. Consequentely,



the sought distance, is

$$s = \int_0^{\Delta t} v\, dt = \frac{8F}{m\omega^2}$$

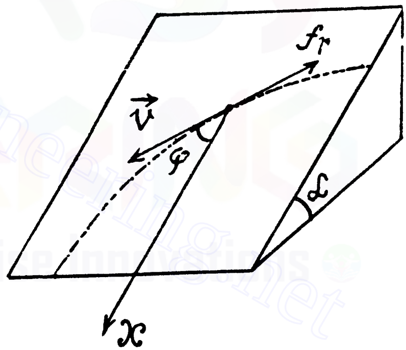
Average velocity, $< v > = \frac{\int v\, dt}{\int dt}$

So, $<v> = \int_0^{2\pi/\omega} \frac{2F}{m\omega} \sin\left(\frac{\omega t}{2}\right) dt \,/\, (2\pi\omega) = \frac{4F}{\pi m \omega}$

1.106 The acceleration of the disc along the plane is determined by the projection of the force of gravity on this plane $F_x = mg \sin\alpha$ and the friction force $fr = kmg \cos\alpha$. In our case $k = \tan\alpha$ and therefore

$$fr = F_x = mg \sin\alpha$$

Let us find the projection of the acceleration on the derection of the tangent to the trajectory and on the *x-axis* :



$m w_t = F_x \cos\varphi - fr = mg \sin\alpha (\cos\varphi - 1)$

$m w_x = F_x - fr \cos\varphi = mg \sin\alpha (1 - \cos\varphi)$

It is seen fromthis that $w_t = - w_x$, which means that the velocity v and its projection v_x differ only by a constant value C which does not change with time, i.e.

$$v = v_x + C,$$

where $v_x = v \cos\varphi$. The constant C is found from the initial condition $v = v_0$, whence $C = v_0$ since $\varphi = \frac{\pi}{2}$ initially. Finally we obtain

$$v = v_0 / (1 + \cos\varphi).$$

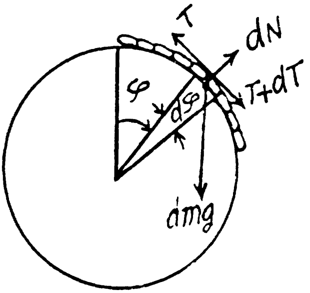
In the cource of time $\varphi \to 0$ and $v \to v_0/2$. (Motion then is unaccelerated.)

1.107 Let us consider an element of length ds at an angle φ from the vertical diameter. As the speed of this element is zero at initial instant of time, it's centripetal acceleration is zero, and hence, $dN - \lambda\, ds \cos\varphi = 0$, where λ is the linear mass density of the chain Let T and $T + dT$ be the tension at the upper and the lower ends of ds. we have from, $F_t = m w_t$

$$(T + dT) + \lambda\, ds\, g \sin\varphi - T = \lambda ds\, w_t$$

or, $$dT + \lambda R d\varphi\, g \sin\varphi = \lambda ds\, w_t$$

If we sum the above equation for all elements, the term $\int dT = 0$ because there is no tension at the free ends, so



$$\lambda g R \int_0^{l/R} \sin \varphi \, d\varphi = \lambda w_t \int ds = \lambda l w_t$$

Hence $w_t = \frac{gR}{l}\left(1 - \cos \frac{l}{R}\right)$

As $w_n = a$ at initial moment

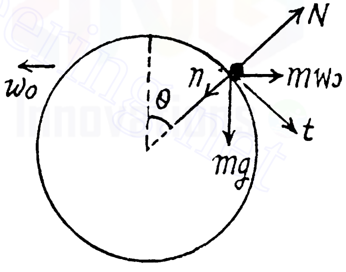
So, $w = |w_t| = \frac{gR}{l}\left(1 - \cos \frac{l}{R}\right)$

1.108 In the problem, we require the velocity of the body, realtive to the sphere, which itself moves with an acceleration w_0 in horizontal direction (say towards left). Hence it is advisible to solve the problem in the frame of sphere (non-inertial frame).

At an arbitary moment, when the body is at an angle θ with the vertical, we sketch the force diagram for the body and write the second law of motion in projection form $F_n = mw_n$

$$\text{or, } mg \cos \theta - N - mw_0 \sin \theta = \frac{mv^2}{R} \quad (1)$$

At the break off point, $N = 0$, $\theta = \theta_0$ and let $v = v_0$, so the Eq. (1) becomes,



$$\frac{v_0^2}{R} = g \cos \theta_0 - w_0 \sin \theta_0 \quad (2)$$

From, $F_t = mw_t$

$$mg \sin \theta - mw_0 \cos \theta = m \frac{v\,dv}{ds} = m \frac{v\,dv}{R\,d\theta}$$

or, $v\,dv = R\,(g \sin \theta + w_0 \cos \theta)\,d\theta$

$$\text{Integrating, } \int_0^{v_0} v\,dv = \int_0^{\theta_0} R\,(g \sin\theta + w_0 \cos\theta)\,d\theta$$

$$\frac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0 \sin \theta_0 \quad (3)$$

Note that the Eq. (3) can also be obtained by the work-energy theorem $A = \Delta T$ (in the frame of sphere)

therefore, $mgR\,(1 - \cos \theta_0) + mw_0 R \sin \theta_0 = \frac{1}{2} m v_0^2$

[here $mw_0 R \sin \theta_0$ is the work done by the pseudoforce $(-m\vec{w}_o)$]

$$\text{or, } \frac{v_0^2}{2R} = g\,(1 - \cos \theta_0) + w_0 \sin \theta_0$$

Solving Eqs. (2) and (3) we get,

$$v_0 = \sqrt{2gR/3} \text{ and } \theta_0 = \cos^{-1}\left[\frac{2 + k\sqrt{5 + 9k^2}}{3(1 + k^2)}\right], \text{ where } k = \frac{w_0}{g}$$

Hence $$\theta_0 \Big|_{w_0 = g} = 17°$$

1.109 This is not central force problem unless the path is a circle about the said point. Rather here F_t (tangential force) vanishes. Thus equation of motion becomes,

$$v_t = v_0 = \text{constant}$$

and, $$\frac{mv_0^2}{r} = \frac{A}{r^2} \text{ for } r = r_0$$

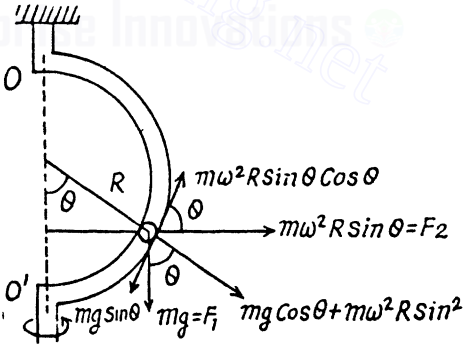
We can consider the latter equation as the equilibrium under two forces. When the motion is perturbed, we write $r = r_0 + x$ and the net force acting on the particle is,

$$-\frac{A}{(r_0 + x)^n} + \frac{mv_0^2}{r_0 + x} = \frac{-A}{r_0^n}\left(1 - \frac{nx}{r_0}\right) + \frac{m\,v_0^2}{r_0}\left(1 - \frac{x}{r_0}\right) = -\frac{m\,v_0^2}{r_0^2}(1 - n)x$$

This is opposite to the displacement x, if $n < 1 \cdot \left(\frac{mv_0^2}{r}\right.$ is an outward directed centrifugul force while $\frac{-A}{r^n}$ is the inward directed external force).

1.110 There are two forces on the sleeve, the weight F_1 and the centrifugal force F_2. We resolve both forces into tangential and normal component then the net downward tangential force on the sleeve is,

$$mg\sin\theta\left(1 - \frac{\omega^2 R}{g}\cos\theta\right)$$



This vanishes for $\theta = 0$ and for $\theta = \theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$, which is real if $\omega^2 R > g$. If $\omega^2 R < g$, then $1 - \frac{\omega^2 R}{g}\cos\theta$ is always positive for small values of θ and hence the net tangential force near $\theta = 0$ opposes any displacement away from it. $\theta = 0$ is then stable.

If $\omega^2 R > g$, $1 - \frac{\omega^2 R}{g}\cos\theta$ is negative for small θ near $\theta = 0$ and $\theta = 0$ is then unstable.

However $\theta = \theta_0$ is stable because the force tends to bring the sleeve near the equilibrium position $\theta = \theta_0$.

If $\omega^2 R = g$, the two positions coincide and becomes a stable equilibrium point.

1.111 Define the axes as shown with z along the local vertical, x due east and y due north. (We assume we are in the northern hemisphere). Then the Coriolis force has the components.

$$\vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v})$$

$$= 2m\omega\left[v_y \cos\theta - v_z \sin\theta)\vec{i} - v_x \cos\theta\,\vec{j} + v_x \cos\theta\,\vec{k}\right] = 2m\omega\,(v_y \cos\theta - v_z \sin\theta)\,\vec{i}$$

since v_x is small when the direction in which the gune is fired is due north. Thus the equation of motion (neglecting centrifugal forces) are

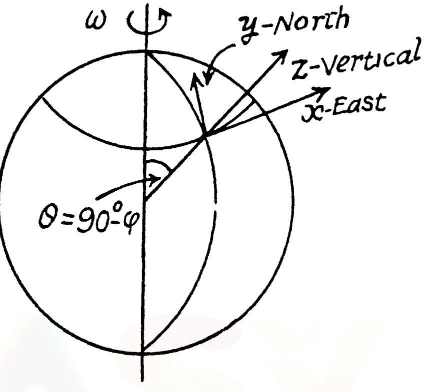
$\ddot{x} = 2m\omega\,(v_y \sin\varphi - v_z \cos\varphi), \ddot{y} = 0$ and $\ddot{z} = -g$

Integrating we get $\dot{y} = v$ (constant), $\dot{z} = -gt$

and $\dot{x} = 2\omega\, v \sin\varphi\, t + \omega g\, t^2 \cos\varphi$

Finally,

$$x = \omega v\, t^2 \sin\varphi + \frac{1}{3}\omega g t^3 \cos\varphi$$



Now $v >> gt$ in the present case. so,

$$x \approx \omega v \sin\varphi \left(\frac{s}{v}\right)^2 = \omega \sin\varphi \frac{s^2}{v}$$

$$\approx 7 \text{ cm (to the east).}$$

1.112 The disc exerts three forces which are mutually perpendicular. They are the reaction of the weight, mg, vertically upward, the Coriolis force $2mv'\,\omega$ perpendicular to the plane of the vertical and along the diameter, and $m\omega^2 r$ outward along the diameter. The resultant force is,

$$F = m\sqrt{g^2 + \omega^4 r^2 + (2v'\,\omega)^2}$$

1.113 The sleeve is free to slide along the rod AB. Thus only the centrifugal force acts on it. The equation is,

$$m\dot{v} = m\omega^2 r \quad \text{where } v = \frac{dr}{dt}.$$

$$\text{But } \dot{v} = v\frac{dv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

so,

$$\frac{1}{2}v^2 = \frac{1}{2}\omega^2 r^2 + \text{constant}$$

or,

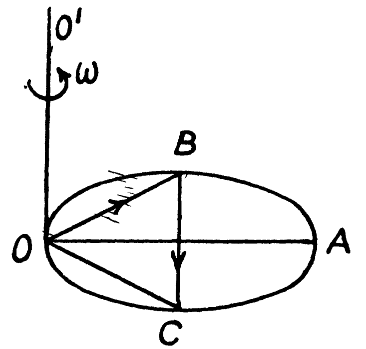
$$v^2 = v_0^2 + \omega^2 r^2$$

v_0 being the initial velocity when $r = 0$. The Coriolis force is then,

$$2m\omega\sqrt{v_0^2 + \omega^2 r^2} = 2m\omega^2 r\sqrt{1 + v_0^2/\omega^2 r^2}$$

$$= 2{\cdot}83 \text{ N on putting the values.}$$

1.114 The disc $OBAC$ is rotating with angular velocity ω about the axis OO' passing through the edge point O. The equation of motion in rotating frame is,



$$m\vec{w}' = \vec{F} + m\omega^2\vec{R} + 2m\vec{v}' \times \vec{\omega} = \vec{F} + \vec{F}_{in}$$

where $\vec{F}_{in}$ is the resultant inertial force (pseudo force) which is the vector sum of centrifugal and Coriolis forces.

(a) At A, F_{in} vanishes. Thus $0 = -2m\omega^2 R\,\hat{n} + 2mv'\omega\,\hat{n}$

where $\hat{n}$ is the inward drawn unit vector to the centre from the point in question, here A.

Thus, $$v' = \omega R$$

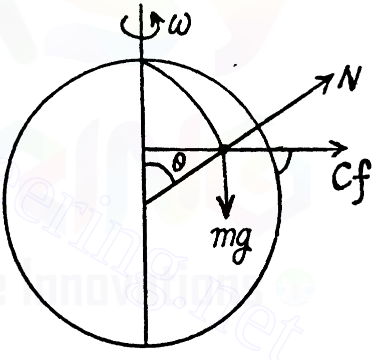
so, $$w = \frac{v'^2}{\rho} = \frac{v'^2}{R} = \omega^2 R.$$

(b) At B $$\vec{F}_{in} = m\omega^2\vec{OC} + m\omega^2\vec{BC}$$

its magnitude is $m\omega^2\sqrt{4R^2 - r^2}$, where $r = OB$.

1.115 The equation of motion in the rotating coordinate system is,

$$m\vec{w}' = \vec{F} + m\omega^2\vec{R} + 2m(\vec{v}' \times \vec{\omega})$$



Now, $\vec{v}' = R\,\dot\theta\,\vec{e}_\theta + R\sin\theta\,\dot\varphi\,\vec{e}_\varphi$

and $\vec{w}' = w'\cos\theta\,\vec{e}_r - w'\sin\theta\,\vec{e}_\theta$

$$\frac{1}{2m}\vec{F}_{cor} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_\varphi \\ 0 & R\,\dot\theta & R\sin\theta\,\dot\varphi \\ \omega\cos\theta & -\omega\sin\theta & 0 \end{vmatrix}$$

$$= \vec{e}_r(\omega R\sin^2\theta\,\dot\varphi) + \omega R\sin\theta\cos\theta\,\dot\varphi\,\vec{e}_\theta - \omega R\,\dot\theta\cos\theta\,\vec{e}_\varphi$$

Now on the sphere,

$$\vec{v}' = (-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2)\,\vec{e}_r + (R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2)\,\vec{e}_\theta + (R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi)\,\vec{e}_\varphi$$

Thus the equation of motion are,

$$m(-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2) = N - mg\cos\theta + m\omega^2 R\sin^2\theta + 2m\omega R\sin^2\theta\,\dot\varphi$$

$$m(R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2) = mg\sin\theta + m\omega^2 R\sin\theta\cos\theta + 2m\omega R\sin\theta\cos\theta\,\dot\varphi$$

$$m(R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi) = -2m\omega R\,\dot\theta\cos\theta$$

From the third equation, we get, $\dot\varphi = -\omega$

A result that is easy to understant by considering the motion in non-rotating frame. The eliminating $\dot\varphi$ we get,

$$mR\,\dot\theta^2 = mg\cos\theta - N$$
$$mR\,\ddot\theta = mg\sin\theta$$

Integrating the last equation,

$$\frac{1}{2}mR\,\dot\theta^2 = mg(1 - \cos\theta)$$

Hence $$N = (3 - 2\cos\theta)\, mg$$

So the body must fly off for $\theta = \theta_0 = \cos^{-1}\frac{2}{3}$, exactly as if the sphere were nonrotating.

Now, at this point F_{cf} = centrifugal force $= m\omega^2 R \sin\theta_0 = \sqrt{\frac{5}{9}}\, m\omega^2 R$

$$F_{cor} = \sqrt{\omega^2 R^2 \dot\theta^2 \cos^2\theta + (\omega^2 R^2)^2 \sin^2\theta} \times 2m$$

$$= \sqrt{\frac{5}{9}(\omega^2 R)^2 + \omega^2 R^2 \times \frac{4}{9} \times \frac{2g}{3R}} \times 2m = \frac{2}{3} m\omega^2 R \sqrt{5 + \frac{8g}{3\,\omega^2 R}}$$

1.116 (a) When the train is moving along a meridian only the Coriolis force has a lateral component and its magnitude (see the previous problem) is,

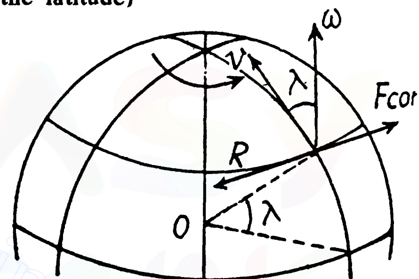
$$2m\,\omega\, v\cos\theta = 2m\,\omega\sin\lambda$$

(Here we have put $R\,\dot\theta \to v$)

So, $$F_{lateral} = 2 \times 2000 \times 10^3 \times \frac{2\pi}{86400} \times \frac{54000}{3600} \times \frac{\sqrt{3}}{2}$$

$= 3{\cdot}77$ kN, (we write λ for the latitude)

(b) The resultant of the inertial forces acting on the train is,



$$\vec{F}_{in} = -2m\omega R\,\dot\theta\cos\theta\,\vec{e}_\varphi$$
$$+ (m\omega^2 R\sin\theta\cos\theta + 2m\,\omega R\sin\theta\cos\theta\,\dot\varphi)\,\vec{e}_\theta$$
$$+ (m\omega^2 R\sin^2\theta + 2m\,\omega R\sin^2\theta\,\dot\varphi)\,\vec{e}_r$$

This vanishes if $\dot\theta = 0,\ \dot\varphi = -\frac{1}{2}\omega$

Thus $$\vec{v} = v_\varphi\,\vec{e}_\varphi,\ v_\varphi = -\frac{1}{2}\omega R\sin\theta = -\frac{1}{2}\omega R\cos\lambda$$

(We write λ for the latitude here)

Thus the train must move from the east to west along the 60th parallel with a speed,

$$\frac{1}{2}\omega R\cos\lambda = \frac{1}{4} \times \frac{2\pi}{8{\cdot}64} \times 10^{-4} \times 6{\cdot}37 \times 10^6 = 115{\cdot}8 \text{ m/s} \approx 417 \text{ km/hr}$$

1.117 We go to the equation given in 1.111. Here $v_y = 0$ so we can take $y = 0$, thus we get for the motion in the $x\,z$ plane.

$$\ddot{x} = -2\omega\, v_z\cos\theta$$

and $$\ddot{z} = -g$$

Integrating, $$z = -\frac{1}{2}gt^2$$

$$\dot{x} = \omega\, g\cos\varphi t^2$$

So $$x = \frac{1}{3}\omega\, g\cos\varphi\, t^3 = \frac{1}{3}\omega\, g\cos\varphi\left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{2\,\omega h}{3}\cos\varphi\sqrt{\frac{2h}{g}}$$

There is thus a displacement to the east of

$$\frac{2}{3} \times \frac{2\pi}{8} 64 \times 500 \times 1 \times \sqrt{\frac{400}{9{\cdot}8}} \approx 26 \text{ cm.}$$

1.3 Laws of Conservation of Energy, Momentum and Angular Momentum.

1.118 As $\vec{F}$ is constant so the sought work done

$$A = \vec{F} \cdot \Delta\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1)$$

or, $A = (3\vec{i} + 4\vec{j}) \cdot [(2\vec{i} - 3\vec{j}) - (\vec{i} + 2\vec{j})] = (3\vec{i} + 4\vec{j}) \cdot (\vec{i} - 5\vec{j}) = 17\,\text{J}$

1.119 Differentating $v(s)$ with respect to time

$$\frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2} = w$$

(As locomotive is in unidrectional motion)

Hence force acting on the locomotive $F = m\,w = \dfrac{ma^2}{2}$

Let, at $v = 0$ at $t = 0$ then the distance covered during the first t seconds

$$s = \frac{1}{2}wt^2 = \frac{1}{2}\frac{a^2}{2}t^2 = \frac{a^2}{4}t^2$$

Hence the sought work, $A = Fs = \dfrac{ma^2}{2}\dfrac{(a^2t^2)}{4} = \dfrac{m\,a^4t^2}{8}$

1.120 We have

$$T = \frac{1}{2}mv^2 = as^2 \quad \text{or,} \quad v^2 = \frac{2\,as^2}{m} \qquad (1)$$

Differentating Eq. (1) with respect to time

$$2\,vw_t = \frac{4\,as}{m}v \quad \text{or,} \quad w_t = \frac{2\,as}{m} \qquad (2)$$

Hence net acceleration of the particle

$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{\left(\frac{2\,as}{m}\right)^2 + \left(\frac{2\,as^2}{mR}\right)^2} = \frac{2\,as}{m}\sqrt{1 + (s/R)^2}$$

Hence the sought force, $F = mw = 2as\sqrt{1 + (s/R)^2}$

1.121 Let $\vec{F}$ makes an angle θ with the horizontal at any instant of time (Fig.). Newton's second law in projection form along the direction of the force, gives :

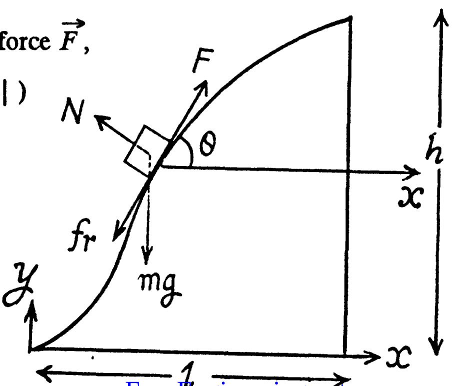
$F = kmg\cos\theta + mg\sin\theta$ (because there is no acceleration of the body.)

As $\vec{F} \uparrow\uparrow d\vec{r}$ the differential work done by the force $\vec{F}$,

$$dA = \vec{F} \cdot d\vec{r} = F\,ds, \quad (\text{where } ds = |d\vec{r}|)$$

$$= kmg\,ds\,(\cos\theta) + mg\,ds\sin\theta$$

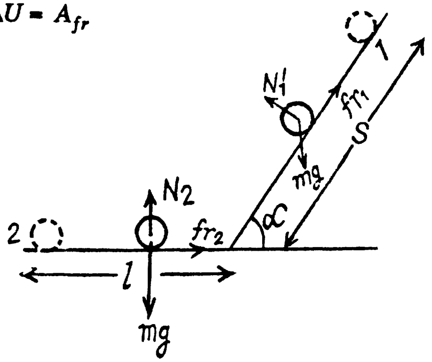
$$= kmg\,dx + mg\,dy.$$



Hence, $A = kmg\displaystyle\int_0^l dx + mg\int_0^h dy$

$$= kmg\,l + mgh = mg\,(k\,l + h).$$

1.122 Let s be the distance covered by the disc along the incline, from the Eq. of increment of M.E. of the disc in the field of gravity : $\Delta T + \Delta U = A_{fr}$



$$0 + (-mgs \sin\alpha) = -kmg \cos\alpha \, s - kmg\, l$$

or, $$s = \frac{k\,l}{\sin\alpha - k\cos\alpha} \qquad (1)$$

Hence the sought work

$$A_{fr} = -kmg\,[s\cos\alpha + l\,]$$

$$A_{fr} = -\frac{k\,lmg}{1 - k\cot\alpha} \quad \text{[Using the Eqn. (1)]}$$

On puting the values $A_{fr} = -0.05$ J

1.123 Let x be the compression in the spring when the bar m_2 is about to shift. Therefore at this moment spring force on m_2 is equal to the limiting friction between the bar m_2 and horizontal floor. Hence

$$\kappa\, x = k\, m_2\, g \quad \text{[where } \kappa \text{ is the spring constant (say)]} \qquad (1)$$

For the block m_1 from work-energy theorem : $A = \Delta T = 0$ for minimum force. (A here indudes the work done in stretching the spring.)

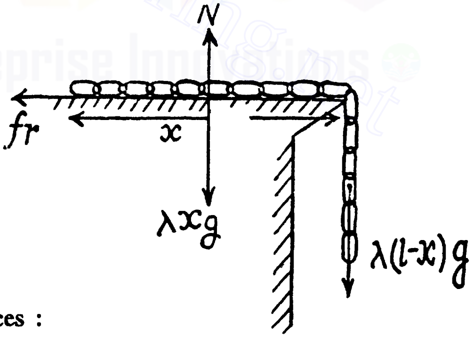
so, $$Fx - \frac{1}{2}\kappa\, x^2 - kmg\, x = 0 \quad \text{or} \quad \kappa\frac{x}{2} = F - km_1\, g \qquad (2)$$

From (1) and (2),

$$F = kg\left(m_1 + \frac{m_2}{2}\right).$$

1.124 From the initial condition of the problem the limiting fricition between the chain lying on the horizontal table equals the weight of the over hanging part of the chain, i.e.

$\lambda\eta\, lg = k\lambda(1-\eta)\, lg$ (where λ is the linear mass density of the chain)



So, $$k = \frac{\eta}{1-\eta} \qquad (1)$$

Let (at an arbitrary moment of time) the length of the chain on the table is x. So the net friction force between the chain and the table, at this moment :

$$f_r = kN = k\lambda x g \qquad (2)$$

The differential work done by the friction forces :

$$dA = \vec{f_r} \cdot d\vec{r} = -f_r\, ds = -k\lambda x g(-dx) = \lambda g\left(\frac{\eta}{1-\eta}\right) x\, dx \qquad (3)$$

(Note that here we have written $ds = -dx$., because ds is essentially a positive term and as the length of the chain decreases with time, dx is negative)

Hence, the sought work done

$$A = \int_{(1-\eta)l}^{0} \lambda g \frac{\eta}{1-\eta} x\, dx = -(1-\eta)\,\eta\frac{mgl}{2} = -1{\cdot}3 \text{ J}$$

1.125 The velocity of the body, t seconds after the begining of the motion becomes $\vec{v} = \vec{v}_0 + \vec{g}t$. The power developed by the gravity ($m\vec{g}$) at that moment, is

$$P = m\vec{g}\cdot\vec{v} = m(\vec{g}\cdot\vec{v}_0 + g^2 t) = mg(gt - v_0 \sin\alpha) \qquad (1)$$

As $m\vec{g}$ is a constant force, so the average power

$$\langle P \rangle = \frac{A}{\tau} = \frac{m\vec{g}\cdot\Delta\vec{r}}{\tau}$$

where $\Delta\vec{r}$ is the net displacement of the body during time of flight.

As, $\quad m\vec{g} \perp \Delta\vec{r}$ so $\langle P \rangle = 0$

1.126 We have $\quad w_n = \frac{v^2}{R} = at^2$, or, $v = \sqrt{aR}\,t$,

t is defined to start from the begining of motion from rest.

So, $\quad w_t = \frac{dv}{dt} = \sqrt{aR}$

Instantaneous power, $P = \vec{F}\cdot\vec{v} = m(w_t\hat{u}_t + w_n\hat{u}_t)\cdot(\sqrt{aR}\,t\,\hat{u}_t)$,

(where $\hat{u}_t$ and $\hat{u}_t$ are unit vectors along the direction of tangent (velocity) and normal respectively)

So, $\quad P = mw_t\sqrt{aR}\,t = maRt$

Hence the sought average power

$$\langle P \rangle = \frac{\int_0^t P\,dt}{\int_0^t dt} = \frac{\int_0^t maRt\,dt}{t}$$

Hence $\quad \langle P \rangle = \frac{maRt^2}{2t} = \frac{maRt}{2}$

1.127 Let the body m acquire the horizontal velocity v_0 along positive x – axis at the point O.

(a) Velocity of the body t seconds after the begining of the motion,

$$\vec{v} = \vec{v}_0 + \vec{w}t = (v_0 - kgt)\vec{i} \qquad (1)$$

Instantaneous power $P = \vec{F}\cdot\vec{v} = (-kmg\vec{i})\cdot(v_0 - kgt)\vec{i} = -kmg(v_0 - kgt)$

From Eq. (1), the time of motion $\tau = v_0/kg$

Hence sought average power during the time of motion

$$\langle P \rangle = \frac{\int_0^\tau -kmg(v_0 - kgt)\,dt}{\tau} = -\frac{kmg\,v_0}{2} = -2\text{ W (On substitution)}$$

From $F_x = mw_x$

$$-kmg = mw_x = mv_x\frac{dv_x}{dx}$$

or, $\quad v_x\,dv_x = -kg\,dx \doteq -\alpha g\,x\,dx$

To find $v(x)$, let us integrate the above equation

$$\int_{v_0}^{v} v_x\, dv_x = -\alpha g \int_0^x x\, dx \quad \text{or,} \quad v^2 = v_0^2 - \alpha g x^2 \tag{1}$$

Now, $$\vec{P} = \vec{F} \cdot \vec{v} = -m\alpha x g \sqrt{v_0^2 - \alpha g x^2} \tag{2}$$

For maximum power, $\frac{d}{dt}\left(\sqrt{v_0^2 x^2 - \lambda g x^4}\right) = 0$ which yields $x = \frac{v_0}{\sqrt{2\alpha g}}$

Putting this value of x, in Eq. (2) we get,

$$P_{max} = -\frac{1}{2} m v_0^2 \sqrt{\alpha g}$$

1.128 Centrifugal force of inertia is directed outward along radial line, thus the sought work

$$A = \int_{r_1}^{r_2} m\omega^2 r\, dr = \frac{1}{2} m\omega^2 \left(r_2^2 - r_1^2\right) = 0{\cdot}20\ \text{T} \quad \text{(On substitution)}$$

1.129 Since the springs are connected in series, the combination may be treated as a single spring of spring constant.

$$\kappa = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

From the equation of increment of M.E., $\Delta T + \Delta U = A_{ext}$

$$0 + \frac{1}{2}\kappa \Delta l^2 = A, \quad \text{or,} \quad A = \frac{1}{2}\left(\frac{\kappa \kappa_2}{\kappa_1 + \kappa_2}\right)\Delta l^2$$

1.130 First, let us find the total height of ascent. At the beginning and the end of the path of velocity of the body is equal to zero, and therefore the increment of the kinetic energy of the body is also equal to zero. On the other hand, in according with work-energy theorem ΔT is equal to the algebraic sum of the works A performed by all the forces, i.e. by the force F and gravity, over this path. However, since $\Delta T = 0$ then $A = 0$. Taking into account that the upward direction is assumed to coincide with the positive direction of the y – axis, we can write

$$A = \int_0^h (\vec{F} + m\vec{g}) \cdot d\,\vec{r} = \int_0^h (F_y - mg)\, dy$$

$$= mg \int_0^h (1 - 2\,ay)\, dy = mgh\,(1 - ah) = 0.$$

whence $h = 1/a$.

The work performed by the force F over the first half of the ascent is

$$A_F = \int_0^{h/2} F_y\, dy = 2mg \int_0^{h/2} (1 - ay)\, dy = 3\,mg/4a.$$

The corresponding increment of the potential energy is

$$\Delta U = mgh/2 = mg/2a.$$

1.131 From the equation $F_r = -\frac{dU}{dr}$ we get $F_r = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$

(a) we have at $r = r_0$, the particle is in equilibrium position. i.e. $F_r = 0$ so, $r_0 = \frac{2a}{b}$

To check, whether the position is steady (the position of stable equilibrium), we have to satisfy

$$\frac{d^2U}{dr^2} > 0$$

We have
$$\frac{d^2U}{dr^2} = \left[\frac{6a}{r^4} - \frac{2b}{r^3}\right]$$

Putting the value of $r = r_0 = \frac{2a}{b}$, we get

$\frac{d^2U}{dr^2} = \frac{b^4}{8a^3}$, (as a and b are positive constant)

So,
$$\frac{d^2U}{dr^2} = \frac{b^2}{8a^3} > 0,$$

which indicates that the potential energy of the system is minimum, hence this position is steady.

(b) We have
$$F_r = -\frac{dU}{dr} = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$$

For F_r to be maximum, $\frac{dF_r}{dr} = 0$

So, $r = \frac{3a}{b}$ and then $F_{r\,(\max)} = \frac{-b^3}{27a^2}$,

As F_r is negative, the force is attractive.

1.132 (a) We have

$$F_x = -\frac{\partial U}{dx} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{dy} = -2\beta y$$

So,
$$\vec{F} = 2\alpha x \vec{i} - 2\beta y \vec{i} \text{ and, } F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} \qquad (1)$$

For a central force, $\vec{r} \times \vec{F} = 0$

Here,
$$\vec{r} \times \vec{F} = (x\vec{i} + y\vec{j}) \times (-2\alpha x\vec{i} - 2\beta y\vec{j})$$
$$= -2\beta xy\vec{k} - 2\alpha xy(\vec{k}) \neq 0$$

Hence the force is not a central force.

(b) As $U = \alpha x^2 + \beta y^2$

So,
$$F_x = \frac{\partial U}{\partial x} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{\partial y} = -2\beta y.$$

So,
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4\alpha^2 x^2 + 4\beta^2 y^2}$$

According to the problem

$$F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} = C \text{ (constant)}$$

or, $$\alpha^2 x^2 + \beta^2 y^2 = \frac{C^2}{2}$$

or, $$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = \frac{C^2}{2\,\alpha^2\beta^2} = k \text{ (say)} \qquad (2)$$

Therefore the surfaces for which F is constant is an ellipse.

For an equipotential surface U is constant.

So, $$\alpha x^2 + \beta y^2 = C_0 \text{ (constant)}$$

or, $$\frac{x^2}{\sqrt{\beta^2}} + \frac{y^2}{\sqrt{\alpha^2}} = \frac{C_0}{\alpha\beta} = K_0 \text{ (constant)}$$

Hence the equipotential surface is also an ellipse.

1.133 Let us calculate the work performed by the forces of each field over the path from a certain point 1 (x_1, y_1) to another certain point 2 (x_2, y_2)

(i) $dA = \vec{F}\cdot d\vec{r} = ay\,\vec{i}\cdot d\vec{r} = aydx$ or, $A = a\int_{x_1}^{x_2} y\,dx$

(ii) $dA = \vec{F}\cdot d\vec{r} = (ax\vec{i} + by\vec{i})\cdot d\vec{r} = axdx + bydy$

Hence $$A = \int_{x_1}^{x_2} a\,xdx + \int_{y_1}^{y_2} bydy$$

In the first case, the integral depends on the function of type $y(x)$, i.e. on the shape of the path. Consequently, the first field of force is not potential. In the second case, both the integrals do not depend on the shape of the path. They are defined only by the coordinate of the initial and final points of the path, therefore the second field of force is potential.

1.134 Let s be the sought distance, then from the equation of increment of M.E. $\Delta T + \Delta U = A_{fr}$

$$\left(0 - \frac{1}{2}mv_0^2\right) + (+\,mg\,s\sin\alpha) = -\,kmg\cos\alpha\, s$$

or, $$s = \frac{v_0^2}{2g}\Big/(\sin\alpha + k\cos\alpha)$$

Hence $$A_{fr} = -\,kmg\cos\alpha\, s = \frac{-\,kmv_0^2}{2\,(k + \tan\alpha)}$$

1.135 Velocity of the body at height h, $v_h = \sqrt{2g\,(H-h)}$, horizontally (from the figure given in the problem). Time taken in falling through the distance h.

$t = \sqrt{\frac{2h}{g}}$ (as initial vertical component of the velocity is zero.)

Now $$s = v_h t = \sqrt{2g\,(H+h)} \times \sqrt{\frac{2h}{g}} = \sqrt{4\,(Hh - h^2)}$$

For s_{max}, $\frac{d}{ds}(Hh - h^2) = 0$, which yields $h = \frac{H}{2}$

Putting this value of h in the expression obtained for s, we get,

$$s_{max} = H$$

1.136 To complete a smooth vertical track of radius R, the minimum height at which a particle starts, must be equal to $\frac{5}{2}R$ (one can proved it from energy conservation). Thus in our problem body could not reach the upper most point of the vertical track of radius $R/2$. Let the particle A leave the track at some point O with speed v (Fig.). Now from energy conservation for the body A in the field of gravity :

$$mg\left[h - \frac{h}{2}(1 + \sin\theta)\right] = \frac{1}{2}mv^2$$

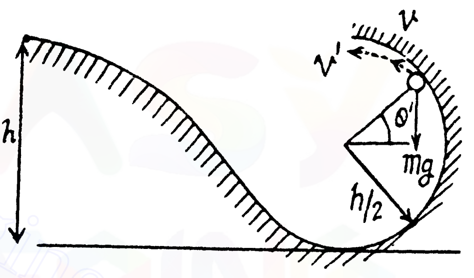
or, $\quad v^2 = gh(1 - \sin\theta) \qquad (1)$

From Newton's second law for the particle at the point O; $F_n = mw_n$,

$$N + mg\sin\theta = \frac{mv^2}{(h/2)}$$

But, at the point O the normal reaction $N = 0$

So, $\quad v^2 = \frac{gh}{2}\sin\theta \qquad (2)$



From (3) and (4), $\sin\theta = \frac{2}{3}$ and $v = \sqrt{\frac{gh}{3}}$

After leaving the track at O, the particle A comes in air and further goes up and at maximum height of it's trajectory in air, it's velocity (say v') becomes horizontal (Fig.). Hence, the sought velocity of A at this point.

$$v' = v\cos(90 - \theta) = v\sin\theta = \frac{2}{3}\sqrt{\frac{gh}{3}}$$

1.137 Let, the point of suspension be shifted with velocity v_A in the horizontal direction towards left then in the rest frame of point of suspension the ball starts with same velocity horizontally towards right. Let us work in this, frame. From Newton's second law in projection form towards the point of suspension at the upper most point (say B) :

$$mg + T = \frac{mv_B^2}{l} \quad \text{or,} \quad T = \frac{mv_B^2}{l} - mg \qquad (1)$$

Condition required, to complete the vertical circle is that $T \geq 0$. But (2)

$$\frac{1}{2}mv_A^2 = mg(2l) + \frac{1}{2}mv_B^2 \quad \text{So,} \quad v_B^2 = v_A^2 - 4gl \qquad (3)$$

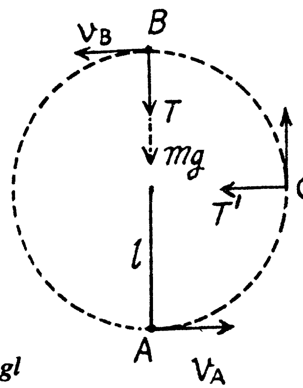
From (1), (2) and (3)

$$T = \frac{m(v_A^2 - 4gl)}{l} - mg \geq 0 \quad \text{or,} \quad v_A \geq \sqrt{5gl}$$

Thus $\quad v_{A\,(\min)} = \sqrt{5gl}$

From the equation $F_n = mw_n$ at point C

$$T' = \frac{mv_c^2}{l} \qquad (4)$$



Again from energy conservation

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_c^2 + mgl \qquad (5)$$

From (4) and (5)

$$T = 3mg$$

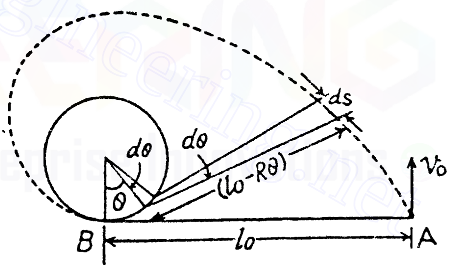
1.138 Since the tension is always perpendicular to the velocity vector, the work done by the tension force will be zero. Hence, according to the work energy theorem, the kinetic energy or velocity of the disc will remain constant during it's motion. Hence, the sought time $t = \frac{s}{v_0}$, where s is the total distance traversed by the small disc during it's motion.

Now, at an arbitary position (Fig.)

$ds = (l_0 - R\theta)\,d\theta,$

so, $$s = \int_0^{l/R} (l_0 - R\theta)\,d\theta$$

or, $$s = \frac{l_0^2}{R} - \frac{R\,l_0^2}{2R^2} = \frac{l_0^2}{2R}$$



Hence, the required time, $\quad t = \frac{l_0^2}{2Rv_0}$

It should be clearly understood that the only uncompensated force acting on the disc A in this case is the tension T, of the thread. It is easy to see that there is no point here, relative to which the moment of force T is invarible in the process of motion. Hence conservation of angular momentum is not applicable here.

1.139 Suppose that Δl is the elongation of the rubbler cord. Then from energy conservation,

$$\Delta U_{gr} + \Delta U_{el} = 0 \text{ (as } \Delta T = 0)$$

or, $$-mg(l + \Delta l) + \frac{1}{2}\kappa\,\Delta l^2 = 0$$

or, $$\frac{1}{2}\kappa\,\Delta l^2 - mg\,\Delta l - mg\,l = 0$$

or, $$\Delta l = \frac{mg \pm \sqrt{(mg)^2 + 4 \times \frac{\kappa}{2} mgl}}{2 \times \frac{\kappa}{2}} \times \frac{\kappa}{2} = \frac{mg}{\kappa}\left[1 + \sqrt{1 \pm \frac{2\kappa l}{mg}}\right]$$

Since the value of $\sqrt{1 + \frac{2\kappa l}{mg}}$ is certainly greater than 1, hence negative sign is avoided.

So, $$\Delta l = \frac{mg}{\kappa}\left(1 + \sqrt{1 + \frac{2\kappa l}{mg}}\right)$$

1.140 When the thread *PA* is burnt, obviously the speed of the bars will be equal at any instant of time until it breaks off. Let v be the speed of each block and θ be the angle, which the elongated spring makes with the vertical at the moment, when the bar A breaks off the plane. At this stage the elongation in the spring.

$$\Delta l = l_0 \sec\theta - l_0 = l_0 (\sec\theta - 1) \quad (1)$$

Since the problem is concerned with position and there are no forces other than conservative forces, the mechanical energy of the system (both bars + spring) in the field of gravity is conserved, i.e. $\Delta T + \Delta U = 0$

So, $$2\left(\frac{1}{2} mv^2\right) + \frac{1}{2}\kappa l_0^2 (\sec\theta - 1)^2 - mgl_0 \tan\theta = 0 \quad (2)$$

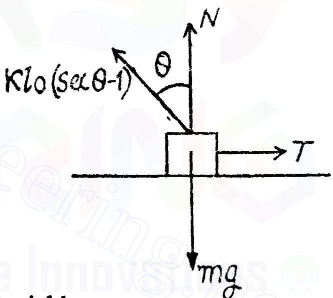
From Newton's second law in projection form along vertical direction :

$mg = N + \kappa l_0 (\sec\theta - 1)\cos\theta$

But, at the moment of break off, $N = 0$.

Hence, $\kappa l_0 (\sec\theta - 1)\cos\theta = mg$

or, $$\cos\theta = \frac{\kappa l_0 - mg}{\kappa l_0} \quad (3)$$



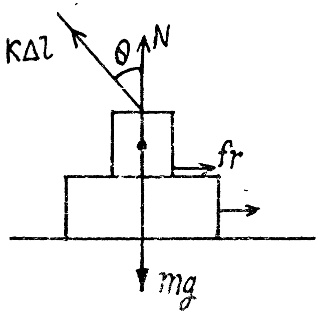
Taking $\kappa = \frac{5\,mg}{l_0}$, simultaneous solution of (2) and (3) yields :

$$v = \sqrt{\frac{19\,g\,l_0}{32}} = 1{\cdot}7 \text{ m/s}.$$

1.141 Obviously the elongation in the cord, $\Delta l = l_0 (\sec\theta - 1)$, at the moment the sliding first starts and at the moment horizontal projection of spring force equals the limiting friction.

So, $$\kappa_1 \Delta l \sin\theta = kN \quad (1)$$

(where κ_1 is the elastic constant).



From Newton's law in projection form along vertical direction :

$$\kappa_1 \Delta l \cos\theta + N = mg.$$

or, $$N = mg - \kappa_1 \Delta l \cos\theta \quad (2)$$

From (1) and (2),

$$\kappa_1 \Delta l \sin\theta = k(mg - \kappa_1 \Delta l \cos\theta)$$

or, $\kappa_1 = \dfrac{kmg}{\Delta l \sin\theta + k\,\Delta l \cos\theta}$

From the equation of the increment of mechanical energy : $\Delta U + \Delta T = A_{fr}$

or, $\left(\dfrac{1}{2}\kappa_1 \Delta l^2\right) = A_{fr}$

or, $\dfrac{kmg\,\Delta l^2}{2\,\Delta l\,(\sin\theta + k\cos\theta)} = A_{fr}$

Thus $A_{fr} = \dfrac{kmg\, l_0 (\sec\theta - 1)}{2(\sin\theta - k\cos\theta)} = 0{\cdot}09\text{ J}$ (on substitution)

1.142 Let the deformation in the spring be Δl, when the rod AB has attained the angular velocity ω. From the second law of motion in projection form $F_n = m w_n$.

$\kappa\,\Delta l = m\,\omega^2 (l_0 + \Delta l)$ or, $\Delta l = \dfrac{m\omega^2 l_0}{\kappa - m\omega^2}$

From the energy equation, $A_{ext} = \dfrac{1}{2} m v^2 + \dfrac{1}{2}\kappa\,\Delta l^2$

$$= \frac{1}{2} m\omega^2 (l_0 + \Delta l)^2 + \frac{1}{2}\kappa\,\Delta l^2$$

$$= \frac{1}{2} m\omega^2 \left(l_0 + \frac{m\omega^2 l_0}{\kappa - m\omega^2}\right)^2 + \frac{1}{2}\kappa\left(\frac{m\omega^2 l_0^2}{\kappa - m\omega^2}\right)^2$$

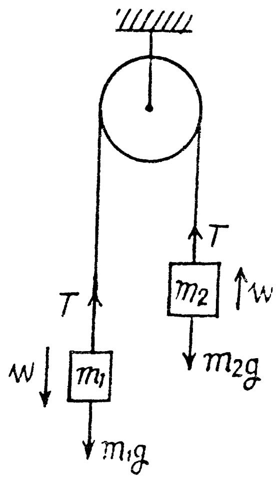
On solving $A_{ext} = \dfrac{\kappa}{2}\dfrac{l_0^2\,\eta\,(1+\eta)}{(1-\eta)^2}$, where $\eta = \dfrac{m\omega^2}{\kappa}$

1.143 We know that acceleration of centre of mass of the system is given by the expression.

$$\vec{w}_C = \frac{m_1 \vec{w}_1 + m_2 \vec{w}_2}{m_1 + m_2}$$

Since $\vec{w}_1 = -\vec{w}_2$

$$\vec{w}_C = \frac{(m_1 - m_2)\,\vec{w}_1}{m_1 + m_2} \quad (1)$$



Now from Newton's second law $\vec{F} = m\vec{w}$, for the bodies m_1 and m_2 respectively.

$$\vec{T} + m_1 \vec{g} = m_1 \vec{w}_1 \quad (2)$$

and $\vec{T} + m_2 \vec{g} = m_2 \vec{w}_2 = -m_2 \vec{w}_1 \quad (3)$

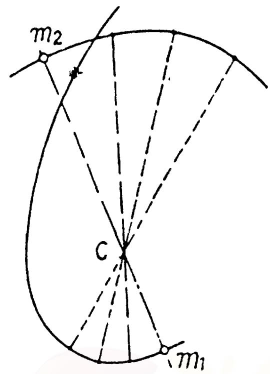
Solving (2) and (3)

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g}}{m_1 + m_2} \quad (4)$$

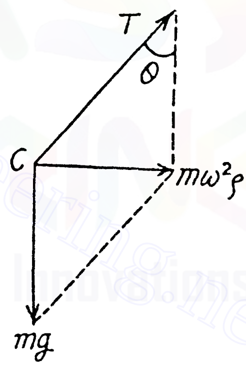
Thus from (1), (2) and (4),

$$\vec{w}_c = \frac{(m_1 - m_2)^2 \vec{g}}{(m_1 + m_2)^2}$$

1.144 As the closed system consisting two particles m_1 and of m_2 is initially at rest the C.M. of the system will remain at rest. Further as $m_2 = m_1/2$, the C.M. of the system divides the line joining m_1 and m_2 at all the moments of time in the ratio 1 : 2. In addition to it the total linear momentum of the system at all the times is zero. So, $\vec{p}_1 = -\vec{p}_2$ and therefore the velocities of m_1 and m_2 are also directed in opposite sense. Bearing in mind all these thing, the sought trajectory is as shown in the figure.



1.145 First of all, it is clear that the chain does not move in the vertical direction during the uniform rotation. This means that the vertical component of the tension T balances gravity. As for the horizontal component of the tension T, it is constant in magnitude and permanently directed toward the rotation axis. It follows from this that the C.M. of the chain, the point C, travels along horizontal circle of radius ρ (say). Therefore we have,

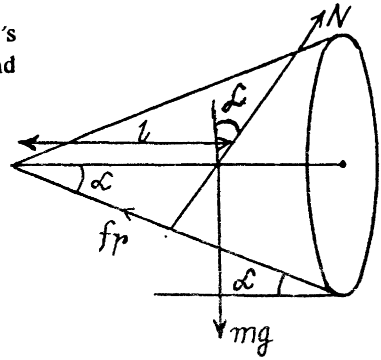


$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = m\omega^2\rho$$

Thus $\quad \rho = \dfrac{g\tan\theta}{\omega^2} = 0{\cdot}8 \text{ cm}$

and $\quad T = \dfrac{mg}{\cos\theta} = 5 \text{ N}$

1.146 (a) Let us draw free body diagram and write Newton's second law in terms of projection along vertical and horizontal direction respectively.



$$N\cos\alpha - mg + fr\sin\alpha = 0 \qquad (1)$$

$$fr\cos\alpha - N\sin\alpha = m\omega^2 l \qquad (2)$$

From (1) and (2)

$$fr\cos\alpha - \frac{\sin\alpha}{\cos\alpha}(-fr\sin\alpha + mg) = m\omega^2 l$$

So, $$fr = mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) = 6\text{N} \qquad (3)$$

(b) For rolling, without sliding,

$$fr \le kN$$

but, $N = mg\cos\alpha - m\omega^2 l\sin\alpha$

$$mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \le k(mg\cos\alpha - m\omega^2 l\sin\alpha) \text{ [Using (3)]}$$

Rearranging, we get,

$$m\omega^2 l(\cos\alpha + k\sin\alpha) \le (kmg\cos\alpha - mg\sin\alpha)$$

Thus $$\omega \le \sqrt{g(k - \tan\alpha)/(1 + k\tan\alpha)l} = 2 \text{ rad/s}$$

1.147 (a) Total kinetic energy in frame K' is

$$T = \frac{1}{2}m_1(\vec{v}_1 - \vec{V})^2 + \frac{1}{2}m_2(\vec{v}_2 - \vec{V})^2$$

This is minimum with respect to variation in $\vec{V}$, when

$$\frac{\delta T'}{\delta\vec{V}} = 0, \text{ i.e. } m_1(\vec{v}_1 - \vec{V})^2 + m_2(\vec{v}_2 - \vec{V}) = 0$$

or $$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{V}_2}{m_1 + m_2} = \vec{v}_c$$

Hence, it is the frame of C.M. in which kinetic energy of a system is minimum.

(b) Linear momentum of the particle 1 in the K' or C frame

$$\tilde{\vec{p}}_1 = m_1(\vec{v}_1 - \vec{v}_c) = \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)$$

or, $$\tilde{\vec{p}}_1 = \mu(\vec{v}_1 - \vec{v}_2), \text{ where, } \mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Similarly, $$\tilde{\vec{p}}_2 = \mu(\vec{v}_2 - \vec{v}_1)$$

So, $$|\tilde{\vec{p}}_1| = |\tilde{\vec{p}}_2| = \tilde{p} = \mu v_{rel} \text{ where, } v_{rel} = |\vec{v}_1 - \vec{v}_2| \qquad (3)$$

Now the total kinetic energy of the system in the C frame is

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{p}^2}{2m_1} + \frac{\tilde{p}^2}{2m_2} = \frac{\tilde{p}^2}{2\mu}$$

Hence $$\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$$

1.148 To find the relationship between the values of the mechanical energy of a system in the K and C reference frames, let us begin with the kinetic energy T of the system. The velocity of the i-th particle in the K frame may be represented as $\vec{v}_i = \vec{\tilde{v}}_i + \vec{v}_C$. Now we can write

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\vec{\tilde{v}}_1 + \vec{v}_C) \cdot (\vec{\tilde{v}}_i + \vec{v}_C)$$

$$= \sum \frac{1}{2} m_i \tilde{v}_i^2 + \vec{v}_C \sum m_i \vec{\tilde{v}}_1 + \sum \frac{1}{2} m_i v_C^2$$

Since in the C frame $\sum m_i \vec{\tilde{v}}_i = 0$, the previous expression takes the form

$$T = \tilde{T} + \frac{1}{2} m v_C^2 = \tilde{T} + \frac{1}{2} m V^2 \text{ (since according to the problem } v_C = V) \qquad (1)$$

Since the internal potential energy U of a system depends only on its configuration, the magnitude U is the same in all refrence frames. Adding U to the left and right hand sides of Eq. (1), we obtain the sought relationship

$$E = \tilde{E} + \frac{1}{2} m V^2$$

1.149 As initially $U = \tilde{U} = 0$, so, $\tilde{E} = \tilde{T}$

From the solution of 1.147 (b)

$$\tilde{T} = \frac{1}{2} \mu |\vec{v}_1 - \vec{v}_2|,$$

As $$\vec{v}_1 \perp \vec{v}_2$$

Thus $$\tilde{T} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$$

1.150 Velocity of masses m_1 and m_2, after t seconds are respectively.

$$\vec{v}_1{}' = \vec{v}_1 + \vec{g} t \text{ and } \vec{v}_2{}' = \vec{v}_2 + \vec{g} t$$

Hence the final momentum of the system,

$$\vec{p} = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}' = m_1 \vec{v}_1 + m_2 \vec{v}_2 + (m_1 + m_2) \vec{g} t$$

$$= \vec{p}_0 + m \vec{g} t, \text{ (where, } \vec{p}_0 = m_2 \vec{v}_1 + m_2 \vec{v}_2 \text{ and } m = m_1 + m_2)$$

And radius vector, $$\vec{r}_C = \vec{v}_C t + \frac{1}{2} \vec{w}_C t^2$$

$$\frac{(m_1 \vec{v}_1 + m_2 \vec{v}_2) t}{(m_1 + m_2)} + \frac{1}{2} \vec{g} t^2$$

$$= \vec{v}_0 t + \frac{1}{2} \vec{g} t^2, \text{ where } \vec{v}_0 = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

1.151 After releasing the bar 2 acquires the velocity v_2, obtained by the energy, conservation :

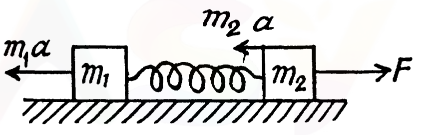
$$\frac{1}{2} m_2 v_2^2 = \frac{1}{2} \kappa x^2 \quad \text{or,} \quad v_2 = x \sqrt{\frac{\kappa}{m_2}} \qquad (1)$$

Thus the sought velocity of C.M.

$$v_{cm} = \frac{0 + m_2 x \sqrt{\frac{\kappa}{m_2}}}{m_1 + m_2} = \frac{x \sqrt{m_2 \kappa}}{(m_1 + m_2)}$$

1.152 Let us consider both blocks and spring as the physical system. The centre of mass of the system moves with acceleration $a = \frac{F}{m_1 + m_2}$ towards right. Let us work in the frame of centre of mass. As this frame is a non-inertial frame (accelerated with respect to the ground) we have to apply a pseudo force $m_1 a$ towards left on the block m_1 and $m_2 a$ towards left on the block m_2

As the center of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The elongation of the spring will be maximum or minimum at this instant. Assume that the block m_1 is displaced by the distance x_1 and the block m_2 through a distance x_2 from the initial positions.



From the energy equation in the frame of C.M.

$$\Delta \tilde{T} + U = A_{ext} \,,$$

(where A_{ext} also includes the work done by the pseudo forces)

Here,

$$\Delta \tilde{T} = 0, \; U = \frac{1}{2} k (x_1 + x_2)^2 \text{ and}$$

$$W_{ext} = \left(\frac{F - m_2 F}{m_1 + m_2} \right) x_2 + \frac{m_1 F}{m_1 + m_2} x_1 = \frac{m_1 F (x_1 + x_2)}{m_1 + m_2},$$

or,
$$\frac{1}{2} k (x_1 + x_2)^2 = \frac{m_1 (x_1 + x_2) F}{m_1 + m_2}$$

So,
$$x_1 + x_2 = 0 \text{ or, } x_1 + x_2 = \frac{2 m_1 F}{k (m_1 + m_2)}$$

Hence the maximum separation between the blocks equals : $l_0 + \frac{2 m_1 F}{k (m_1 + m_2)}$

Obviously the minimum sepation corresponds to zero elongation and is equal to l_0

1.153 (a) The initial compression in the spring Δl must be such that after burning of the thread, the upper cube rises to a height that produces a tension in the spring that is atleast equal to the weight of the lower cube. Actually, the spring will first go from its compressed

state to its natural length and then get elongated beyond this natural length. Let l be the maximum elongation produced under these circumstances.
Then

$$\kappa l = mg \tag{1}$$

Now, from energy conservation,

$$\frac{1}{2}\kappa \Delta l^2 = mg(\Delta l + l) + \frac{1}{2}\kappa l^2 \tag{2}$$

(Because at maximum elongation of the spring, the speed of upper cube becomes zero)
From (1) and (2),

$$\Delta l^2 - \frac{2mg\,\Delta l}{\kappa} - \frac{3\,m^2 g^2}{\kappa^2} = 0 \quad \text{or,} \quad \Delta l = \frac{3mg}{\kappa},\ \frac{-mg}{\kappa}$$

Therefore, acceptable solution of Δl equals $\frac{3mg}{\kappa}$

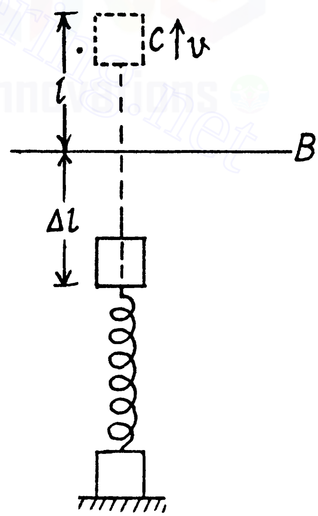
(b) Let v the velocity of upper cube at the position (say, at C) when the lower block breaks off the floor, then from energy conservation.

$$\frac{1}{2}mv^2 = \frac{1}{2}\kappa(\Delta l^2 - l^2) - mg(l + \Delta l)$$

$$\left(\text{where } l = mg/\kappa \text{ and } \Delta l = 7\frac{mg}{\kappa}\right)$$

or,
$$v^2 = 32\frac{mg^2}{\kappa} \tag{2}$$

At the position C, the velocity of C.M; $v_C = \frac{mv + 0}{2m} = \frac{v}{2}$ –Let, the C.M. of the system (spring+ two cubes) further rises up to Δy_{C2}.



Now, from energy conservation,

$$\frac{1}{2}(2m)\,v_C^2 = (2m)\,g\,\Delta r_{C2}$$

or,
$$\Delta y_{C2} = \frac{v_C^2}{2g} = \frac{v^2}{8g} = \frac{4\,mg}{\kappa}$$

But, uptil position C, the C.M. of the system has already elevated by,

$$\Delta y_{C1} = \frac{(\Delta l + l)\,m + 0}{2m} = \frac{4\,mg}{\kappa}$$

Hence, the net displacement of the C.M. of the system, in upward direction

$$\Delta y_C = \Delta y_{C1} + \Delta y_{C2} = \frac{8\,mg}{\kappa}$$

1.154 Due to ejection of mass from a moving system (which moves due to inertia) in a direction perpendicular to it, the velocity of moving system does not change. The momentum change being adjusted by the forces on the rails. Hence in our problem velocities of buggies change only due to the entrance of the man coming from the other buggy. From the

Solving (1) and (2), we get

$$v_1 = \frac{mv}{M-m} \text{ and } v_2 = \frac{Mv}{M-m}$$

As $\qquad \vec{v_1} \uparrow\downarrow \vec{v} \text{ and } \vec{v_2} \uparrow\uparrow \vec{v}$

So, $$\vec{v_1} = \frac{-m\vec{v}}{(M-m)} \text{ and } \vec{v_2} = \frac{M\vec{v}}{(M-m)}$$

1.155 From momentum conservation, for the system "rear buggy with man"

$$(M+m)\vec{v_0} = m(\vec{u}+\vec{v_R}) + M\vec{v_R} \qquad (1)$$

From momentum conservation, for the system (front buggy + man coming from rear buggy)

$$M\vec{v_0} + m(\vec{u}+\vec{v_R}) = (M+m)\vec{v_F}$$

So, $$\vec{v_F} = \frac{M\vec{v_0}}{M+m} + \frac{m}{M+m}(\vec{u}+\vec{v_R})$$

Putting the value of $\vec{v_R}$ from (1), we get

$$\vec{v_F} = \vec{v_0} + \frac{mM}{(M+m)^2}\vec{u}$$

1.156 (i) Let $\vec{v_1}$ be the velocity of the buggy after both man jump off simultaneously. For the closed system (two men + buggy), from the conservation of linear momentum,

$$M\vec{v_1} + 2m(\vec{u}+\vec{v_1}) = 0$$

or, $$\vec{v_1} = \frac{-2m\vec{u}}{M+2m} \qquad (1)$$

(ii) Let $\vec{v}'$ be the velocity of buggy with man, when one man jump off the buggy. For the closed system (buggy with one man + other man) from the conservation of linear momentum :

$$0 = (M+m)\vec{v}' + m(\vec{u}+\vec{v}') \qquad (2)$$

Let $\vec{v_2}$ be the sought velocity of the buggy when the second man jump off the buggy; then from conservation of linear momentum of the system (buggy + one man) :

$$(M+m)\vec{v}' = M\vec{v_2} + m(\vec{u}+\vec{v_2}) \qquad (3)$$

Solving equations (2) and (3) we get

$$\vec{v_2} = \frac{m(2M+3m)\vec{u}}{(M+m)(M+2m)} \qquad (4)$$

From (1) and (4)

$$\frac{v_2}{v_1} = 1 + \frac{m}{2(M+m)} > 1$$

Hence $v_2 > v_1$

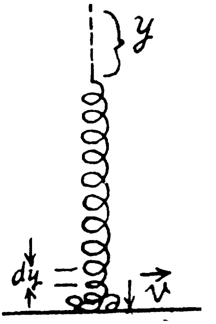
1.157 The descending part of the chain is in free fall, it has speed $v = \sqrt{2gh}$ at the instant, all its points have descended a distance y. The length of the chain which lands on the floor during the differential time interval dt following this instant is vdt.

For the incoming chain element on the floor :

From $dp_y = F_y\, dt$ (where y – axis is directed down)

$$0 - (\lambda\, v dt)\, v = F_y\, dt$$

or $$F_y = -\lambda\, v^2 = -2\,\lambda\, g\, y$$



Hence, the force exerted on the falling chain equals $\lambda\, v^2$ and is directed upward. Therefore from third law the force exerted by the falling chain on the table at the same instant of time becomes $\lambda\, v^2$ and is directed downward.

Since a length of chain of weight $(\lambda y g)$ already lies on the table the total force on the floor is $(2\lambda y g) + (\lambda y g) = (3\lambda y g)$ or the weight of a length $3y$ of chain.

1.158 Velocity of the ball, with which it hits the slab, $v = \sqrt{2\, gh}$

After first impact, $v' = ev$ (upward) but according to the problem $v' = \dfrac{v}{\eta}$, so $e = \dfrac{1}{\eta}$ (1)

and momentum, imparted to the slab,

$$= mv - (-mv') = mv\,(1 + e)$$

Similarly, velocity of the ball after second impact,

$$v'' = ev' = e^2\, v$$

And momentum imparted $= m\,(v' + v'') = m\,(1 + e)\, ev$

Again, momentum imparted during third impact,

$$= m\,(1 + e)\, e^2\, v, \text{ and so on,}$$

Hence, net momentum, imparted $= mv\,(1 + e) + mve\,(1 + e) + mve^2\,(1 + e) + \dots$

$$= mv\,(1 + e)\,(1 + e + e^2 + \dots)$$

$$= mv\,\frac{(1 + e)}{(1 - e)}, \text{ (from summation of G.P.)}$$

$$= \sqrt{2\, gh}\,\frac{\left(1 + \dfrac{1}{\eta}\right)}{\left(1 - \dfrac{1}{\eta}\right)} = m\sqrt{2\, gh}\,/\,(\eta + 1)\,/\,(\eta - 1) \text{ (Using Eq. 1)}$$

$$= 0{\cdot}2 \text{ kg m/s. (On substitution)}$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e. $\vec{r}_C =$ constant or, $\Delta\vec{r}_C = 0$ i.e. $\sum m_i\,\Delta\vec{r}_i = 0$

or, $$m\left(\Delta\vec{r}_{mM} + \Delta\vec{r}_M\right) + M\,\Delta\vec{r}_M = 0$$

Thus, $$m\,(\vec{l}\,' + \vec{l}\,) + M\,\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}\,'}{m + M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m\,[\,\vec{v}\,'(t) + \vec{v}_2(t)\,] + M\,\vec{v}_2(t)$$

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So, $$0 = m\,[\,\vec{v}\,'(t) + \vec{v}_2(t)\,] + M\,\vec{v}_2(t)$$

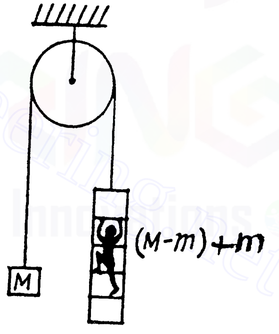
or, $$\vec{v}_2(t) = -\frac{m\,\vec{v}\,'(t)}{m+M} \qquad (1)$$

As $\vec{v}\,'(t)$ or $\vec{v}_2(t)$ is along horizontal direction, thus the sought force on the raft

$$= \frac{M\,d\vec{v}_2(t)}{dt} = -\frac{Mm}{m+M}\frac{d\vec{v}\,'(t)}{dt}$$

Note : *we may get the result of part* (a), *if we integrate Eq.* (1) *over the time of motion of man or raft.*

1.160 In the refrence frame fixed to the pulley axis the location of C.M. of the given system is described by the radius vector



$$\Delta\vec{r}_C = \frac{M\,\Delta\vec{r}_M + (M-m)\,\Delta\vec{r}_{(M-m)} + m\,\Delta\vec{r}_m}{2M}$$

But $\Delta\vec{r}_M = -\Delta\vec{r}_{(M-m)}$

and $\Delta\vec{r}_m = \Delta\vec{r}_{m(M-m)} + \Delta\vec{r}_{(M-m)}$

Thus $$\Delta\vec{r}_C = \frac{m\,\vec{l}\,'}{2M}$$

Note : *one may also solve this problem using momentum conservation.*

1.161 Velocity of cannon as well as that of shell equals $\sqrt{2\,gl\sin\alpha}$ down the inclined plane taken as the positive x – axis. From the linear impulse momentum theorem in projection form along x – axis for the system (connon + shell) i.e. $\Delta p_x = F_x\,\Delta t$:

$p\cos\alpha - M\sqrt{2\,gl\sin\alpha} = Mg\sin\alpha\,\Delta t$ (as mass of the shell is neligible)

or, $$\Delta t = \frac{p\cos\alpha - M\sqrt{2\,gl\sin\alpha}}{Mg\sin\alpha}$$

1.162 From conservation of momentum, for the system (bullet + body) along the initial direction of bullet

$$mv_0 = (m+M)\,v, \text{ or, } v = \frac{mv_0}{m+M}$$

1.163 When the disc breaks off the body M, its velocity towards right (along x-axis) equals the velocity of the body M, and let the disc's velocity in upward direction (along y-axis) at that moment be v'_y

From conservation of momentum, along x-axis for the system (disc + body)

$$mv = (m+M)\,v'_x \quad \text{or} \quad v'_x = \frac{mv}{m+M} \tag{1}$$

And from energy conservation, for the same system in the field of gravity :

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\,v'^2_x + \frac{1}{2}m\,v'^2_y + mgh',$$

where h' is the height of break off point from initial level. So,

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\frac{m^2v^2}{(M+m)^2} + \frac{1}{2}m\,v'^2_y + mgh', \qquad \text{using (1)}$$

or, $$v'^2_y = v^2 - \frac{mv^2}{(m+M)} - 2\,gh'$$

Also, if h'' is the height of the disc, from the break-off point,

then, $$v'^2_y = 2\,gh''$$

So, $$2g\,(h''+h') = v^2 - \frac{m\,v^2}{(M+m)}$$

Hence, the total height, raised from the initial level

$$= h' + h'' = \frac{M\,v^2}{2g\,(M+m)}$$

1.164 (a) When the disc slides and comes to a plank, it has a velocity equal to $v = \sqrt{2\,gh}$. Due to friction between the disc and the plank the disc slows down and after some time the disc moves in one piece with the plank with velocity v' (say).

From the momentum conservation for the system (disc + plank) along horizontal towards right :

$$mv = (m+M)\,v' \quad \text{or} \quad v' = \frac{mv}{m+M}$$

Now from the equation of the increment of total mechanical energy of a system :

$$\frac{1}{2}(M+m)\,v'^2 - \frac{1}{2}mv^2 = A_{fr}$$

or, $$\frac{1}{2}(M+m)\frac{m^2v^2}{(m+M)^2} - \frac{1}{2}mv^2 = A_{fr}$$

so, $$\frac{1}{2}v^2\left[\frac{m^2}{M+m} - m\right] = A_{fr}$$

Hence, $$A_{fr} = -\left(\frac{mM}{m+M}\right)gh = -\,\mu gh$$

$$\left(\text{where } \mu = \frac{mM}{m+M} = \text{reduced mass}\right)$$

(b) We look at the problem from a frame in which the hill is moving (together with the disc on it) to the right with speed u. Then in this frame the speed of the disc when it just gets onto the plank is, by the law of addition of velocities, $\overline{v} = u + \sqrt{2gh}$. Similarly the common speed of the plank and the disc when they move together is

$$\overline{v}' = u + \frac{m}{m+M}\sqrt{2gh}.$$

Then as above $\quad \overline{A}_{fr} = \frac{1}{2}(m+M)\overline{v}'^2 - \frac{1}{2}m\overline{v}^2 - \frac{1}{2}Mu^2$

$$= \frac{1}{2}(m+M)\left\{u^2 + \frac{2m}{m+M}u\sqrt{2gh} + \frac{m^2}{(m+M)^2}2gh\right\} - \frac{1}{2}(m+M)u^2 - \frac{1}{2}m\,2u\sqrt{2gh} - mgh$$

We see that $\overline{A}_{fr}$ is independent of u and is in fact just $-\mu g h$ as in (a). Thus the result obtained does not depend on the choice of reference frame.

Do note however that it will be in correct to apply "conservation of enegy" formula in the frame in which the hill is moving. The energy carried by the hill is not negligible in this frame. See alsó the next problem.

1.165 In a frame moving relative to the earth, one has to include the kinetic energy of the earth as well as earth's acceleration to be able to apply conservation of energy to the problem. In a reference frame falling to the earth with velocity v_0, the stone is initially going up with velocity v_0 and so is the earth. The final velocity of the stone is $0 = v_o - gt$ and that of the earth is $v_o + \frac{m}{M}gt$ (M is the mass of the earth), from Newton's third law, where t = time of fall. From conservation of energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 + mgh = \frac{1}{2}M\left(v_0 + \frac{m}{M}v_0\right)^2$$

Hence $$\frac{1}{2}v_0^2\left(m + \frac{m^2}{M}\right) = mgh$$

Negecting $\frac{m}{M}$ in comparison with 1, we get

$$v_0^2 = 2gh \text{ or } v_o = \sqrt{2gh}$$

The point is this in earth's rest frame the effect of earth's accleration is of order $\frac{m}{M}$ and can be neglected but in a frame moving with respect to the earth the effect of earth's acceleration must be kept because it is of order one (i.e. large).

1.166 From conservation of momentum, for the closed system "both colliding particles"

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$

or, $$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{1(3\vec{i} - 2\vec{j}) + 2(4\vec{j} - 6\vec{k})}{3} = \vec{i} + 2\vec{j} - 4\vec{k}$$

Hence $$|\vec{v}| = \sqrt{1 + 4 + 16}\text{ m/s} = 4{\cdot}6\text{ m/s}$$

1.167 For perfectly inelastic collision, in the C.M. frame, final kinetic energy of the colliding system (both spheres) becomes zero. Hence initial kinetic energy of the system in C.M. frame completely turns into the internal energy (Q) of the formed body. Hence

$$Q = \tilde{T}_i = \frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$$

Now from energy conservation $\Delta T = -Q = -\frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$,

In lab frame the same result is obtained as

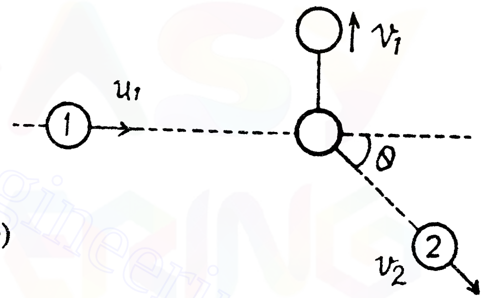
$$\Delta T = \frac{1}{2}\frac{(m_1\vec{v}_1 + m_2\vec{v}_2)^2}{m_1 + m_2} - \frac{1}{2}m_1\,|\vec{v}_1|^2 + m_2\,|\vec{v}_2|^2$$

$$= -\frac{1}{2}\mu\,|\vec{v}_1 - \vec{v}_2|^2$$

1.168 (a) Let the initial and final velocities of m_1 and m_2 are $\vec{u}_1$, $\vec{u}_2$ and $\vec{v}$, $\vec{v}_2$ respectively. Then from conservation of momentum along horizontal and vertical directions, we get :

$$m_1 u_1 = m_2 v_2 \cos\theta \qquad (1)$$

and $$m_1 v_1 = m_2 v_2 \sin\theta \qquad (2)$$



Squaring (1) and (2) and then adding them,

$$m_2^2 v_2^2 = m_1^2(u_1^2 + v_1^2)$$

Now, from kinetic energy conservation,

$$\frac{1}{2}m_1 u_1^2 = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_1 v_1^2 \qquad (3)$$

or, $$m\,(u_1^2 - v_1^2) = m_2 v_2^2 = m_2\frac{m_1^2}{m_2^2}(u_1^2 + v_1^2) \quad \text{[Using (3)]}$$

or, $$u_1^2\left(1 - \frac{m_1}{m_2}\right) = v_1^2\left(1 + \frac{m_1}{m_2}\right)$$

or, $$\left(\frac{v_1}{u_1}\right)^2 = \frac{m_2 - m_1}{m_1 + m_2} \qquad (4)$$

So, fraction of kinetic energy lost by the particle 1,

$$= \frac{\frac{1}{2}m_1 u_1^2 - \frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_1 u_1^2} = 1 - \frac{v_1^2}{u_1^2}$$

$$= 1 - \frac{m_2 - m_1}{m_1 + m_2} = \frac{2\,m_1}{m_1 + m_2} \quad \text{[Using (4)]} \qquad (5)$$

(b) When the collision occurs head on,

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \qquad (1)$$

and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left[\frac{m_1 (u_1 - v_1)^2}{m_2} \right]^2 \quad \text{[Using (5)]}$$

or, $$v_1 \left(1 + \frac{m_1}{m_2}\right) = u_1 \left(\frac{m_1}{m_2} - 1\right)$$

or, $$\frac{v_1}{u_1} = \frac{(m_1 / m_2 - 1)}{(1 + m_1 / m_2)} \quad (6)$$

Fraction of kinetic energy, lost

$$= 1 - \frac{v_1^2}{u_1^2} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4 m_1 m_2}{(m_1 + m_2)^2} \quad \text{[Using (6)]}$$

1.169 (a) When the particles fly apart in opposite direction with equal velocities (say v), then from conservatin of momentum,

$$m_1 u + 0 = (m_2 - m_1) v \quad (1)$$

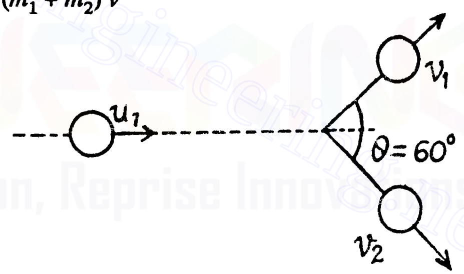
and from conservation of kinetic energy,

$$\frac{1}{2} m_1 u^2 = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2$$

or, $$m_1 u^2 = (m_1 + m_2) v^2 \quad (2)$$

From Eq. (1) and (2),

$$m_1 u^2 = (m_1 + m_2) \frac{m_1^2 u^2}{(m_2 - m_1)^2}$$



or, $$m_2^2 - 3 m_1 m_2 = 0$$

Hence $$\frac{m_1}{m_2} = \frac{1}{3} \text{ as } m_2 \neq 0$$

(b) When they fly apart symmetrically relative to the initial motion direction with the angle of divergence $\theta = 60°$,

From conservation of momentum, along horizontal and vertical direction,

$$m_1 u_1 = m_1 v_1 \cos(\theta/2) + m_2 v_2 \cos(\theta/2) \quad (1)$$

and $$m_1 v_1 \sin(\theta/2) = m_2 v_2 \sin(\theta/2)$$

or, $$m_1 v_1 = m_2 v_2 \quad (2)$$

Now, from conservation of kinetic energy,

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (3)$$

From (1) and (2),

$$m_1 u_1 = \cos(\theta/2) \left(m_1 v_1 + \frac{m_1 v_1}{m_2} m_2\right) = 2 m_1 v_1 \cos(\theta/2)$$

So, $$u_1 = 2\, v_1 \cos(\theta/2) \qquad (4)$$

From (2), (3), and (4)

$$4\, m_1 \cos^2(\theta/2)\, v_1^2 = m_1\, v_1^2 + \frac{m_2\, m_1^2\, v_1^2}{m_2^2}$$

$$\text{or,} \quad 4\cos^2(\theta/2) = 1 + \frac{m_1}{m_2}$$

or, $$\frac{m_1}{m_2} = 4\cos^2\frac{\theta}{2} - 1$$

and putting the value of θ, we get, $\dfrac{m_1}{m_2} = 2$

1.170 If (v_{1x}, v_{1y}) are the instantaneous velocity components of the incident ball and (v_{2x}, v_{2y}) are the velocity components of the struck ball at the same moment, then since there are no external impulsive forces (i.e. other than the mutual interaction of the balls) We have $$u \sin\alpha = v_{1y} \quad , \quad v_{2y} = 0$$

$$m\, u \cos\alpha = m\, v_{1x} + m\, v_{2x}$$

The impulsive force of mutual interaction satisfies

$$\frac{d}{dt}(v_{1x}) = \frac{F}{m} = -\frac{d}{dt}(v_{2x})$$

(F is along the x axis as the balls are smooth. Thus Y component of momentum is not transferred.) Since loss of K.E. is stored as deformation energy D, we have

$$D = \frac{1}{2} m\, u^2 - \frac{1}{2} m\, v_1^{\,2} - \frac{1}{2} m v_2^{\,2}$$

$$= \frac{1}{2} m u^2 \cos^2\alpha - \frac{1}{2} m v_{1x}^{\,2} - \frac{1}{2} m v_{2x}^{\,2}$$

$$= \frac{1}{2m}\left[\, m^2 u^2 \cos^2\alpha - m^2 v_{1x}^{\,2} - (\, m u \cos\alpha - m v_{1x}\,)^2 \,\right]$$

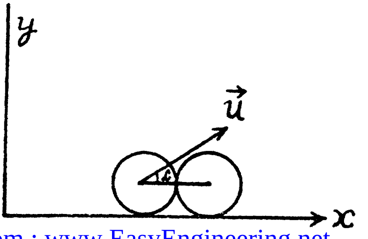
$$= \frac{1}{2m}\left[\, 2m^2 u \cos\alpha\, v_{1x} - 2m^2 v_{1x}^{\,2} \,\right] = m\,(\, v_{1x} u \cos\alpha - v_{1x}^{\,2}\,)$$

$$= m\left[\frac{u^2\cos^2\alpha}{4} - \left(\frac{u\cos\alpha}{2} - v_{1x}\right)^2\right]$$

We see that D is maximum when

$$\frac{u\cos\alpha}{2} = v_{1x}$$

and $$D_{\max} = \frac{m u^2 \cos^2\alpha}{4}$$



Then $$\eta = \frac{D_{\max}}{\frac{1}{2} m u^2} = \frac{1}{2}\cos^2\alpha = \frac{1}{4}$$

On substiuting $\alpha = 45°$

1.171 From the conservation of linear momentum of the shell just before and after its fragmentation

$$3\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \qquad (1)$$

where $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the velocities of its fragments.

From the energy conservation $\quad 3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \qquad (2)$

Now

$$\vec{\tilde{v}}_i \text{ or } \vec{v}_{iC} = \vec{v}_i - \vec{v}_C = \vec{v}_i - \vec{v} \qquad (3)$$

where $\vec{v}_C = \vec{v} =$ velocity of the C.M. of the fragments the velocity of the shell. Obviously in the C.M. frame the linear momentum of a system is equal to zero, so

$$\vec{\tilde{v}}_1 + \vec{\tilde{v}}_2 + \vec{\tilde{v}}_3 = 0 \qquad (4)$$

Using (3) and (4) in (2), we get

$$3\eta v^2 = (\vec{v} + \vec{\tilde{v}}_1)^2 + (\vec{v} + \vec{\tilde{v}}_2)^2 + (\vec{v} - \vec{\tilde{v}}_1 - \vec{\tilde{v}}_2)^2 \quad = 3v^2 + 2\tilde{v}_1^2 + 2\tilde{v}_2^2 + 2\,\vec{\tilde{v}}_1 \cdot \vec{\tilde{v}}_2$$

or,

$$2\tilde{v}_1^2 + 2\tilde{v}_1\,\tilde{v}_2\cos\theta + 2\tilde{v}_2^2 + 3\,(1-\eta)v^2 = 0 \qquad (5)$$

If we have had used $\vec{\tilde{v}}_2 = -\vec{\tilde{v}}_1 - \vec{\tilde{v}}_3$, then Eq. 5 were contain $\tilde{v}_3$ instead of $\tilde{v}_2$ and so on.

The problem being symmetrical we can look for the maximum of any one. Obviously it will be the same for each.

For $\tilde{v}_1$to be real in Eq. (5)

$$4\,\tilde{v}_2^2\cos^2\theta \geq 8(2\tilde{v}_2^2 + 3\,(1-\eta)\,v^2) \text{ or } 6(\eta - 1)v^2 \geq (4 - \cos^2\theta)\tilde{v}_2^2$$

So,

$$\tilde{v}_2 \leq v\sqrt{\frac{6\,(\eta-1)}{4-\cos^2\theta}} \quad \text{or} \quad \tilde{v}_{2(\max)} = \sqrt{2\,(\eta-1)}\;\; v$$

Hence $\quad v_{2\,(\max)} = |\vec{v} + \vec{\tilde{v}}_2|_{\max} = v + \sqrt{2\,(\eta-1)}\;v = v\left(1 + \sqrt{2\,(\eta-1)}\right) = 1\text{ km/s}$

Thus owing to the symmetry

$$v_{1(\max)} = v_{2\,(\max)} = v_{3(\max)} = v\left(1 + \sqrt{2(\eta-1)}\right) = 1\text{ km/s}$$

1.172 Since, the collision is head on, the particle 1 will continue moving along the same line as before the collision, but there will be a change in the magnitude of it's velocity vector. Let it starts moving with velocity v_1 and particle 2 with v_2 after collision, then from the conservation of momentum

$$mu = mv_1 + mv_2 \quad \text{or,} \quad u = v_1 + v_2 \qquad (1)$$

And from the condition, given,

$$\eta = \frac{\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2\right)}{\frac{1}{2}mu^2} = 1 - \frac{v_1^2 + v_2^2}{u^2}$$

or,

$$v_1^2 + v_2^2 = (1-\eta)\,u^2 \qquad (2)$$

From (1) and (2),

$$v_1^2 + (u - v_1)^2 = (1-\eta)\,u^2$$

or,

$$v_1^2 + u^2 - 2\,uv_1 + v_1^2 = (1-\eta)\,u^2$$

or, $$2v_1^2 - 2v_1 u + \eta u^2 = 0$$

So, $$v_1 = 2u \pm \frac{\sqrt{4u^2 - 8\eta u^2}}{4}$$

$$= \frac{1}{2}\left[u \pm \sqrt{u^2 - 2\eta u^2}\right] = \frac{1}{2}u\,(1 \pm \sqrt{1-2\eta}\,)$$

Positive sign gives the velocity of the 2nd particle which lies ahead. The negative sign is correct for v_1 .

So, $v_1 = \frac{1}{2}u\,(1 - \sqrt{1-2\eta}\,) = 5$ m/s will continue moving in the same direction.

Note that $v_1 = 0$ if $\eta = 0$ as it must.

1.173 Since, no external impulsive force is effective on the system "$M + m$", its total momentum along any direction will remain conserved.

So from $p_x =$ const.

$$mu = Mv_1 \cos\theta \quad \text{or,} \quad v_1 = \frac{m}{M}\frac{u}{\cos\theta} \qquad (1)$$

and from $p_y =$ const

$mv_2 = Mv_1 \sin\theta$ or, $v_2 = \frac{M}{m} v_1 \sin\theta = u \tan\theta$, [using (1)]

Final kinetic energy of the system

$$T_f = \frac{1}{2}mv_2^2 + \frac{1}{2}Mv_1^2$$

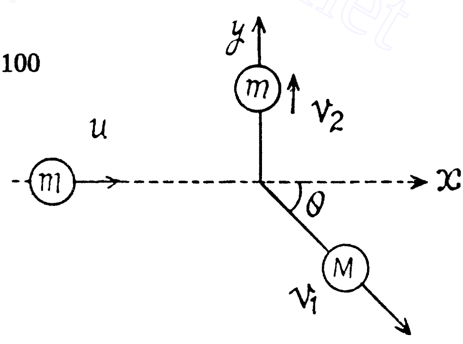
And initial kinetic energy of the system $= \frac{1}{2}mu^2$

So, $$\%\text{ change} = \frac{T_f - T_i}{T_i} \times 100$$

$$= \frac{\frac{1}{2}m\,u^2\tan^2\theta + \frac{1}{2}M\frac{m^2}{M^2}\frac{u^2}{\cos^2\theta} - \frac{1}{2}mu^2}{\frac{1}{12}mu^2} \times 100$$

$$= \frac{\frac{1}{2}u^2\tan^2\theta + \frac{1}{2}\frac{m}{M}u^2\sec^2\theta - \frac{1}{2}u^2}{\frac{1}{2}u^2} \times 100$$

$$= \left(\tan^2\theta + \frac{m}{M}\sec^2\theta - 1\right) \times 100$$



and putting the values of θ and $\frac{m}{M}$, we get % of change in kinetic energy $= -40\ \%$

1.174 (a) Let the particles m_1 and m_2 move with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively. On the basis of solution of problem 1.147 (b)

$$\tilde{p} = \mu\, v_{rel} = \mu\,\left|\vec{v_1} - \vec{v_2}\right|$$

As $$\vec{v}_1 \perp \vec{v}_2$$

So, $$\tilde{p} = \mu\sqrt{v_1^2 + v_2^2} \quad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

(b) Again from 1.147 (b)

$$\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$$

So, $$\tilde{T} = \frac{1}{2}\mu\left(v_1^2 + v_2^2\right)$$

1.175 From conservation of momentum

$$\vec{p}_1 = \vec{p}_1{}' + \vec{p}_2{}'$$

so $$\left(\vec{p}_1 - \vec{p}_1{}'\right)^2 = p_1^2 - 2p_1\, p_1' \cos\theta_1 + p_1'^{\,2} = p_2'^{\,2}$$

From conservation of energy

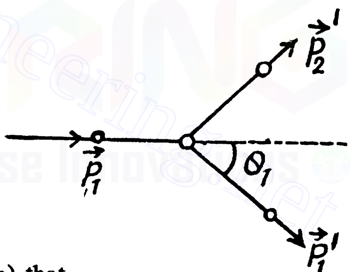
$$\frac{p_1^2}{2m_1} = \frac{p_1'^{\,2}}{2\,m_1} + \frac{p_2'^{\,2}}{2m_2}$$

Eliminating p_2' we get

$$0 = p_1'^{\,2}\left(1 + \frac{m_2}{m_1}\right) - 2p_1' p_1 \cos\theta_1 + p_1^2\left(1 - \frac{m_2}{m_1}\right)$$

This quadratic equation for p_1' has a real solution in terms of p_1 and $\cos\theta_1$ only if

$$4\cos^2\theta_1 \geq 4\left(1 - \frac{m_2^2}{m_1^2}\right)$$



or $$\sin^2\theta_1 \leq \frac{m_2^2}{m_1^2}$$

or $\sin\theta_1 \leq +\dfrac{m_2}{m_1}$ or $\sin\theta_1 \geq -\dfrac{m_2}{m_1}$

This clearly implies (since only + sign makes sense) that

$$\sin\theta_{1\,\max} = \frac{m_2}{m_1}.$$

1.176 From the symmetry of the problem, the velocity of the disc A will be directed either in the initial direction or opposite to it just after the impact. Let the velocity of the disc A after the collision be v' and be directed towards right after the collision. It is also clear from the symmetry of problem that the discs B and C have equal speed (say v'') in the directions, shown. From the condition of the problem,

$$\cos\theta = \frac{\eta\frac{d}{2}}{d} = \frac{\eta}{2} \text{ so, } \sin\theta = \sqrt{4 - \eta^2}\,/\,2 \tag{1}$$

For the three discs, system, from the conservation of linear momentum in the symmetry direction (towards right)

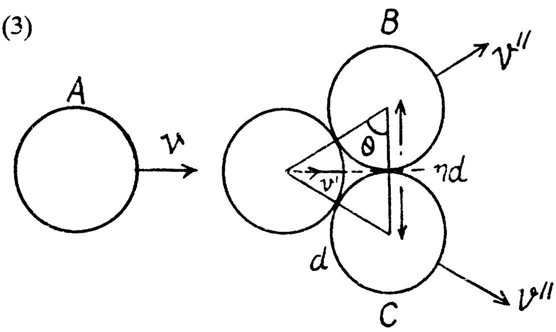
$$mv = 2m\,v''\sin\theta + m\,v' \quad \text{or,} \quad v = 2\,v''\sin\theta + v' \tag{2}$$

From the definition of the coefficeint of restitution, we have for the discs A and B (or C)

$$e = \frac{v'' - v' \sin \theta}{v \sin \theta - 0}$$

But $e = 1$, for perfectly elastic collision,

So, $v \sin \theta = v'' - v' \sin \theta$ (3)



From (2) and (3),

$$v' = \frac{v\,(1 - 2\sin^2\theta)}{(1 + 2\sin^2\theta)}$$

$$= \frac{v\,(\eta^2 - 2)}{6 - \eta^2} \quad \{\text{using (1)}\}$$

Hence we have,

$$v' = \frac{v\,(\eta^2 - 2)}{6 - \eta^2}$$

Therefore, the disc A will recoil if $\eta < \sqrt{2}$ and stop if $\eta = \sqrt{2}$.

Note : *One can write the equations of momentum conservation along the direction perpendicular to the initial direction of disc A and the consevation of kinetic energy instead of the equation of restitution.*

1.177 (a) Let a molecule comes with velocity $\vec{v_1}$ to strike another stationary molecule and just after collision their velocities become $\vec{v}'_1$ and $\vec{v}'_2$ respectively. As the mass of the each molecule is same, conservation of linear momentum and conservation of kinetic energy for the system (both molecules) respectively gives :

$$\vec{v_1} = \vec{v}'_1 + \vec{v}'_2$$

and $$v_1^2 = v'^2_1 + v'^2_2$$

From the property of vector addition it is obvious from the obtained Eqs. that

$$\vec{v}'_1 \perp \vec{v}'_2 \quad \text{or} \quad \vec{v}'_1 \cdot \vec{v}'_2 = 0$$

(b) Due to the loss of kinetic energy in inelastic collision $v_1^2 > v'^2_1 + v'^2_2$

so, $\vec{v}'_1 \cdot \vec{v}'_2 > 0$ and therefore angle of divergence $< 90°$.

1.178 Suppose that at time t, the rocket has the mass m and the velocity $\vec{v}$, relative to the reference frame, employed. Now consider the inertial frame moving with the velocity that the rocket has at the given moment. In this reference frame, the momentum increament that the rocket & ejected gas system acquires during time dt is,

$$d\vec{p} = m\,d\vec{v} + \mu\,dt\,\vec{u} = \vec{F}\,dt$$

or, $$m\frac{d\vec{v}}{dt} = \vec{F} - \mu\,\vec{u}$$

or, $$m\,\vec{w} = \vec{F} - \mu\,\vec{u}$$

1.179 According to the question, $\vec{F} = 0$ and $\mu = -dm/dt$ so the equation for this system becomes,

$$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $d\vec{v} \uparrow\downarrow \vec{u}$ so, $m\,dv = -u\,dm.$

Integrating within the limits :

$$\frac{1}{u}\int_0^v dv = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = \ln\frac{m_0}{m}$$

Thus, $v = u\ln\dfrac{m_0}{m}$

As $d\vec{v} \uparrow\downarrow \vec{u},$ so in vector form $\vec{v} = -\vec{u}\ln\dfrac{m_0}{m}$

1.180 According to the question, $\vec{F}$ (external force) = 0

So, $$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $$d\vec{v} \uparrow\downarrow \vec{u},$$

so, in scalar form, $$m\,dv = -u\,dm$$

or, $$\frac{wdt}{u} = -\frac{dm}{m}$$

Integrating within the limits for $m(t)$

$$\frac{wt}{u} = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or,} \quad \frac{v}{u} = -\ln\frac{m}{m_0}$$

Hence, $$m = m_0\,e^{-(wt/u)}$$

1.181 As $\vec{F} = 0$, from the equation of dynamics of a body with variable mass;

$$m\frac{d\vec{v}}{dt} = \vec{u}\frac{dm}{dt} \quad \text{or,} \quad d\vec{v} = \vec{u}\frac{dm}{m} \qquad (1)$$

Now $d\vec{v} \uparrow\downarrow \vec{u}$ and since $\vec{u} \perp \vec{v},$ we must have $|d\vec{v}| = v_0\,d\alpha$ (because v_0 is constant) where $d\alpha$ is the angle by which the spaceship turns in time dt.

So, $$-u\frac{dm}{m} = v_0\,d\alpha \quad \text{or,} \quad d\alpha = -\frac{u}{v_0}\frac{dm}{m}$$

or, $$\alpha = -\frac{u}{v_0}\int_{m_0}^{m}\frac{dm}{m} = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right)$$

1.182 We have $\frac{dm}{dt} = -\mu$ or, $dm = -\mu\, dt$

Integrating $$\int_{m_0}^{m} dm = -\mu \int_0^t dt \quad \text{or,} \quad m = m_0 - \mu t$$

As $\vec{u} = 0$ so, from the equation of variable mass system :

$$(m_0 - \mu t)\frac{d\vec{v}}{dt} = \vec{F} \quad \text{or,} \quad \frac{d\vec{v}}{dt} = \vec{w} = \vec{F}/(m_0 - \mu t)$$

or, $$\int_0^{\vec{v}} d\vec{v} = \vec{F}\int_0^t \frac{dt}{(m_0 - \mu t)}$$

Hence $$\vec{v} = \frac{\vec{F}}{\mu}\ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

1.183 Let the car be moving in a reference frame to which the hopper is fixed and at any instant of time, let its mass be m and velocity $\vec{v}$.

Then from the general equation, for variable mass system.

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}$$

We write the equation, for our system as,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \vec{v}\frac{dm}{dt} \quad \text{as,} \quad \vec{u} = -\vec{v} \qquad (1)$$

So $$\frac{d}{dt}(m\vec{v}) = \vec{F}$$

and $$\vec{v} = \frac{\vec{F}t}{m} \text{ on integration.}$$

But $$m = m_0 + \mu t$$

so, $$\vec{v} = \frac{\vec{F}t}{m_0\left(1 + \frac{\mu t}{m_0}\right)}$$

Thus the sought acceleration, $$\vec{w} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m_0\left(1 + \frac{\mu t}{m_0}\right)^2}$$

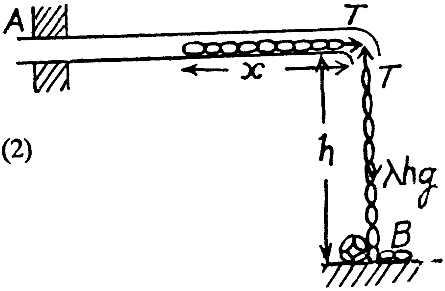
1.184 Let the length of the chain inside the smooth horizontal tube at an arbitrary instant is x. From the equation,

$$m\vec{w} = \vec{F} + \vec{u}\frac{dm}{dt}$$

as $\vec{u} = 0$, $\vec{F} \uparrow\uparrow \vec{w}$, for the chain inside the tube

$$\lambda x w = T \text{ where } \lambda = \frac{m}{l} \qquad (1)$$

Similarly for the overhanging part,



$$\vec{u} = 0$$

Thus $\quad mw = F$

or $\quad \lambda h w = \lambda h g - T \qquad (2)$

From (1) and (2),

$$\lambda (x + h) w = \lambda h g \quad \text{or,} \quad (x + h) v \frac{dv}{ds} = hg$$

or, $$(x + h) v \frac{dv}{(-dx)} = gh,$$

[As the length of the chain inside the tube decreases with time, $ds = -dx$.]

or, $$v\, dv = -g h \frac{dx}{x + h}$$

Integrating, $$\int_0^v v\, dv = -g h \int_{(l-h)}^{0} \frac{dx}{x + h}$$

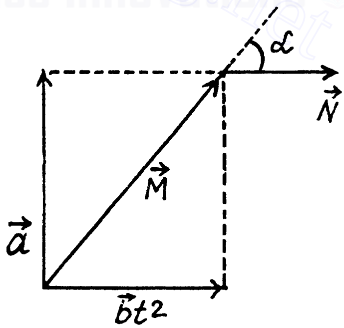
or, $$\frac{v^2}{2} = gh \ln \left(\frac{l}{h}\right) \quad \text{or} \quad v = \sqrt{2gh \ln \left(\frac{l}{h}\right)}$$

1.185 Force moment relative to point O ;

$$\vec{N} = \frac{d\vec{M}}{dt} = 2\vec{b}t$$

Let the angle between $\vec{M}$ and $\vec{N}$,

$\alpha = 45°$ at $t = t_0$,



Then $$\frac{1}{\sqrt{2}} = \frac{\vec{M} \cdot \vec{N}}{|\vec{M}||\vec{N}|} = \frac{(\vec{a} + \vec{b} t_0^2) \cdot (2\vec{b} t_0)}{\sqrt{a^2 + b^2 t_0^4}\ 2bt_0}$$

$$= \frac{2b^2 t_0^3}{\sqrt{a^2 + b^2 t_0^4}\ 2bt_0} = \frac{b t_0^2}{\sqrt{a^2 + b^2 t_0^4}}$$

So, $\quad 2 b^2 t_0^4 = a^2 + b^2 t_0^4$ or, $t_0 = \sqrt{\frac{a}{b}}$ (as t_0 cannot be negative)

It is also obvious from the figure that the angle α is equal to 45° at the moment t_0, when $a = b t_0^2$, i.e. $t_0 = \sqrt{a/b}$ and $\vec{N} = 2\sqrt{\frac{a}{b}}\, \vec{b}$.

1.186 $\vec{M}(t) = \vec{r} \times \vec{p} = \left(\vec{v}_0 t + \frac{1}{2}\vec{g}t^2\right) \times m\left(\vec{v}_0 + \vec{g}t\right)$

$$= mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(-\vec{k}) + \frac{1}{2}mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(\vec{k})$$

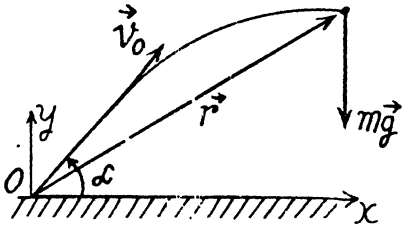
$$= \frac{1}{2}mv_0 gt^2 \cos\alpha\,(-\vec{k}):$$

Thus $M(t) = \dfrac{mv_0 g t^2 \cos\alpha}{2}$

Thus angular momentum at maximum height

i.e. at $t = \dfrac{\tau}{2} = \dfrac{v_0 \sin\alpha}{g}$,

$M\left(\dfrac{\tau}{2}\right) = \left(\dfrac{m v_0^3}{2g}\right)\sin^2\alpha \cos\alpha = 37\text{ kg} - \text{m}^2/\text{s}$

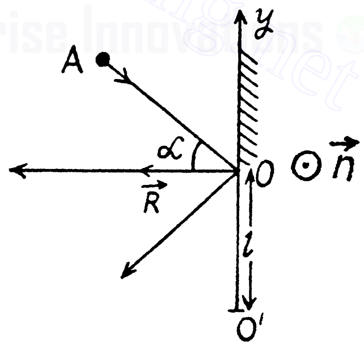


Alternate :

$$\vec{M}(0) = 0 \text{ so, } \vec{M}(t) = \int_0^t \vec{N}\,dt = \int_0^t (\vec{r} \times m\vec{g})$$

$$= \int_0^t \left[\left(\vec{v}_0 t + \frac{1}{2}\vec{g}t^2\right) \times m\vec{g}\right] dt = \left(\vec{v}_0 \times m\vec{g}\right)\frac{t^2}{2}$$

1.187 (a) The disc experiences gravity, the force of reaction of the horizontal surface, and the force $\vec{R}$ of reaction of the wall at the moment of the impact against it. The first two forces counter-balance each other, leaving only the force $\vec{R}$. It's moment relative to any point of the line along which the vector $\vec{R}$ acts or along normal to the wall is equal to zero and therefore the angular momentum of the disc relative to any of these points does not change in the given process.



(b) During the course of collision with wall the position of disc is same and is equal to $\vec{r}_{oo'}$ Obviously the increment in linear momentum of the ball $\Delta\vec{p} = 2mv\cos\alpha\,\hat{n}$

Here, $\Delta\vec{M} = \vec{r}_{oo'} \times \Delta\vec{p} = 2\,mv\,l\cos\alpha\,\hat{n}$ and directed normally emerging from the plane of figure

Thus $|\Delta\vec{M}| = 2\,mv\,l\cos\alpha$

1.188 (a) The ball is under the influence of forces $\vec{T}$ and $m\vec{g}$ at all the moments of time, while moving along a horizontal circle. Obviously the vertical component of $\vec{T}$ balance $m\vec{g}$ and so the net moment of these two about any point becoems zero. The horizontal component of $\vec{T}$, which provides the centripetal acceleration to ball is already directed toward the centre (C) of the horizontal circle, thus its moment about the point C equals zero at all the moments of time. Hence the net moment of the force acting on the ball about point C equals zero and that's why the angular mommetum of the ball is conserved about the horizontal circle.

(b) Let α be the angle which the thread forms with the vertical.

Now from equation of particle dynamics :

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 l \sin\alpha$

Hence on solving $\cos\alpha = \frac{g}{\omega^2 l}$ (1)

As $|\vec{M}|$ is constant in magnitude so from figure.

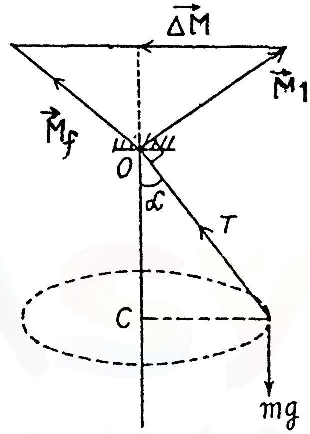
$|\Delta\vec{M}| = 2M\cos\alpha$ where

$M = |\vec{M}_i| = |\vec{M}_f|$

$= |\vec{r}_{bo} \times m\vec{v}| = mvl \left(\text{as } \vec{r}_{bo} \perp \vec{v}\right)$

Thus $|\Delta\vec{M}| = 2mvl\cos\alpha = 2m\omega l^2 \sin\alpha\cos\alpha$

$$= \frac{2mgl}{\omega}\sqrt{1 - \left(\frac{g}{\omega^2 t}\right)^2} \text{ (using 1).}$$



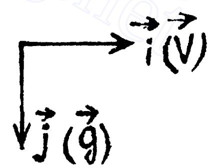
1.189 During the free fall time $t = \tau = \sqrt{\frac{2h}{g}}$, the reference point O moves in hoizontal direction (say towards right) by the distance $V\tau$. In the translating frame as $\vec{M}(O) = 0$, so

$$\Delta\vec{M} = \vec{M}_f = \vec{r}$$

$$= (-V\tau\vec{i} + h\vec{j}) \times m[g\tau\vec{j} - V\vec{i}]$$

$$= -mVg\tau^2\vec{h} + mVh(+\vec{k})$$

$$= -mVg\left(\frac{2h}{g}\right)\vec{k} + mVh(+\vec{k}) = -mVh\vec{k}$$



Hence $|\Delta\vec{M}| = mVh$

1.190 The Coriolis force is $(2m\vec{v}' \times \vec{\omega})$.

Here $\vec{\omega}$ is along the z-axis (vertical). The moving disc is moving with velocity v_0 which is constant. The motion is along the x-axis say. Then the Coriolis force is along y-axis and has the magnitude $2m v_0 \omega$. At time t, the distance of the centre of moving disc from O is $v_0 t$ (along x-axis). Thus the torque N due to the coriolis force is

$N = 2m v_0 \omega \cdot v_0 t$ along the z-axis.

Hence equating this to $\frac{dM}{dt}$

$$\frac{dM}{dt} = 2m\, v_0^2\, \omega t \quad \text{or} \quad M = mv_0^2\, \omega t^2 + \text{constant.}$$

The constant is irrelevant and may be put equal to zero if the disc is originally set in motion from the point O.

This discussion is approximate. The Coriolis force will cause the disc to swerve from straight line motion and thus cause deviation from the above formula which will be substantial for large t.

1.191 If $\dot{r} =$ radial velocity of the particle then the total energy of the particle at any instant is

$$\frac{1}{2} m\,\dot{r}^2 + \frac{M^2}{2mr^2} + kr^2 = E \tag{1}$$

where the second term is the kinetic energy of angular motion about the centre O. Then the extreme values of r are determined by $\dot{r} = 0$ and solving the resulting quadratic equation

$$k(r^2)^2 - Er^2 + \frac{M^2}{2m} = 0$$

we get

$$r^2 = \frac{E \pm \sqrt{E^2 - \frac{2M^2k}{m}}}{2k}$$

From this we see that

$$E = k(r_1^2 + r_2^2) \tag{2}$$

where r_1 is the minimum distance from O and r_2 is the maximum distance. Then

$$\frac{1}{2}mv_2^2 + 2kr_2^2 = k\,(r_1^2 + r_2^2)$$

Hence, $$m = \frac{2kr^2}{v_2^2}$$

Note : Eq. (1) can be derived from the standard expression for kinetic energy and angular momentum in plane poler coordinates :

$$T = \frac{1}{2} m\,\dot{r}^2 + \frac{1}{2} m\,r^2\,\dot{\theta}^2$$

M = angular momentum = $mr^2\,\dot{\theta}$

1.192 The swinging sphere experiences two forces : The gravitational force and the tension of the thread. Now, it is clear from the condition, given in the problem, that the moment of these forces about the vertical axis, passing through the point of suspension $N_z = 0$. Consequently, the angular momentum M_z of the sphere relative to the given axis (z) is constant.

Thus $$mv_0\,(l \sin\theta) = mv\,l \tag{1}$$

where m is the mass of the sphere and v is it s velocity in the position, when the thread forms an angle $\frac{\pi}{2}$ with the vertical. Mechanical energy is also conserved, as the sphere is under the influence if only one other force, i.e. tension, which does not perform any work, as it is always perpendicular to the velocity.

So, $$\frac{1}{2}mv_0^2 + mg\,l\cos\theta = \frac{1}{2}mv^2 \qquad (2)$$

From (1) and (2), we get,

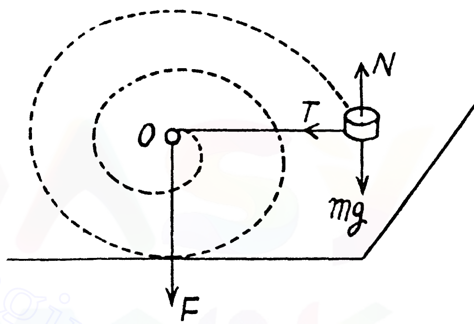
$$v_0 = \sqrt{2gl/\cos\theta}$$

1.193 Forces, acting on the mass m are shown in the figure. As $\vec{N} = m\vec{g}$, the net torque of these two forces about any fixed point must be equal to zero. Tension T, acting on the mass m is a central force, which is always directed towards the centre O. Hence the moment of force T is also zero about the point O and therefore the angular momentum of the particle m is conserved about O.

Let, the angular velocity of the particle be ω, when the separation between hole and particle m is r, then from the conservation of momentum about the point O, :

$$m(\omega_0 r_0) r_0 = m(\omega r) r,$$

or $$\omega = \frac{\omega_0 r_0^2}{r^2}$$



Now, from the second law of motion for m,

$$T = F = m\omega^2 r$$

Hence the sought tension;

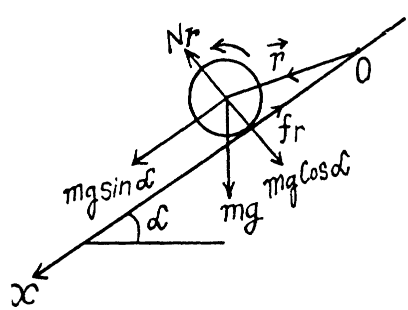
$$F = \frac{m\omega_0^2 r_0^4 r}{r^4} = \frac{m\omega_0^2 r_0^4}{r^3}$$

1.194 On the given system the weight of the body m is the only force whose moment is effective about the axis of pulley. Let us take the sense of $\vec{\omega}$ of the pulley at an arbitrary instant as the positive sense of axis of rotation (z-axis)

As $$M_z(0) = 0, \text{ so, } \Delta M_z = M_z(t) = \int N_z\,dt$$

So, $$M_z(t) = \int_0^t mgR\,dt = mgRt$$

1.195 Let the point of contact of sphere at initial moment ($t = 0$) be at O. At an arbitrary moment, the forces acting on the sphere are shown in the figure. We have normal reaction $N_r = mg\sin\alpha$ and both pass through same line and the force of static friction passes through the point O, thus the moment about point O becomes zero. Hence $mg\sin\alpha$ is the only force which has effective torque about point O, and is given by $|\vec{N}| = mgR\sin\alpha$ normally emerging from the plane of figure.



As $\vec{M}(t=0) = 0$, so, $\Delta\vec{M} = \vec{M}(t) = \int \vec{N}\,dt$

Hence, $M(t) = Nt = mgR\sin\alpha t$

1.196 Let position vectors of the particles of the system be $\vec{r_i}$ and $\vec{r_i}'$ with respect to the points O and O' respectively. Then we have,

$$\vec{r_i} = \vec{r_i}' + \vec{r_0} \qquad (1)$$

where $\vec{r_O}$ is the radius vector of O' with respect to O.

Now, the angular momentum of the system relative to the point O can be written as follows;

$$\vec{M} = \sum \left(\vec{r_i} \times \vec{p_i}\right) = \sum \left(\vec{r_i}' \times \vec{p_i}\right) + \sum \left(\vec{r_0} \times \vec{p_i}\right) \qquad \text{[using (1)]}$$

or, $$\vec{M} = \vec{M}' + \left(\vec{r_0} \times \vec{p}\right), \text{ where, } \vec{p} = \sum \vec{p_i} \qquad (2)$$

From (2), if the total linear momentum of the system, $\vec{p} = 0$, then its angular momentum does not depend on the choice of the point O.

Note that in the C.M. frame, the system of particles, as a whole is at rest.

1.197 On the basis of solution of problem 1.196, we have concluded that; "in the C.M. frame, the angular momentum of system of particles is independent of the choice of the point, relative to which it is determined" and in accordance with the problem, this is denoted by $\vec{M}$.

We denote the angular momentum of the system of particles, relative to the point O, by $\vec{M}$. Since the internal and proper angular momentum $\tilde{\vec{M}}$, in the C.M. frame, does not depend on the choice of the point O', this point may be taken coincident with the point O of the K-frame, at a given moment of time. Then at that moment, the radius vectors of all the particles, in both reference frames, are equal $(\vec{r_i}' = \vec{r_i})$ and the velocities are related by the equation,

$$\vec{v_i} = \tilde{\vec{v}}_i + \vec{v_c}, \qquad (1)$$

where $\vec{v_c}$ is the velocity of C.M. frame, relative to the K-frame. Consequently, we may write,

$$\vec{M} = \sum m_i \left(\vec{r_i} \times \vec{v_i}\right) = \sum m_i \left(\vec{r_i}' \times \tilde{\vec{v}}_i\right) + \sum m_i \left(\vec{r_i} \times \vec{v_c}\right)$$

or, $$\vec{M} = \tilde{\vec{M}} + m\left(\vec{r_c} \times \vec{v_c}\right), \text{ as } \sum m_i \vec{r_i} = m\vec{r_c}, \text{ where } m = \sum m_i .$$

or, $$\vec{M} = \tilde{\vec{M}} + (\vec{r_c} \times m\vec{v_c}) = \tilde{\vec{M}} + (\vec{r_c} \times \vec{p})$$

1.198 From conservation of linear momentum along the direction of incident ball for the system consists with colliding ball and phhere

$$mv_0 = mv' + \frac{m}{2} v_1 \qquad (1)$$

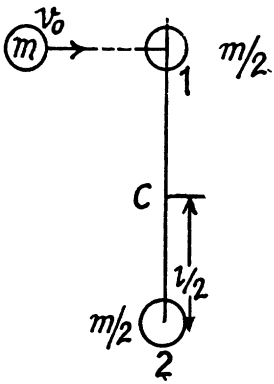
where v' and v_1 are the velocities of ball and sphere 1 respectively after collision. (Remember that the collision is head on).

As the collision is perfectly elastic, from the definition of co-efficeint of restitution,

$$1 = \frac{v' - v_1}{0 - v_0} \quad \text{or,} \quad v' - v_1 = -v_0 \qquad (2)$$

Solving (1) and (2), we get,

$v_1 = \frac{4\,v_0}{3}$, directed towards right.



In the C.M. frame of spheres 1 and 2 (Fig.)

$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$ and $|\vec{\tilde{p}}_1| = |\vec{\tilde{p}}_2| = \mu\,|\vec{v}_1 - \vec{v}_2|$

Also, $\vec{r}_{1C} = -\vec{r}_{2C}$, thus $\vec{\tilde{M}} = 2\,[\vec{r}_{1C} \times \vec{\tilde{p}}_1]$

As $\vec{r}_{1C} \perp \vec{\tilde{p}}_1$, so, $\vec{\tilde{M}} = 2\left[\frac{l}{2}\;\frac{m/2}{2}\;\frac{4\,v_0}{3}\;\hat{n}\right]$

(where $\hat{n}$ is the unit vector in the sense of $\vec{r}_{1C} \times \vec{\tilde{p}}_1$)

Hence $\tilde{M} = \frac{m\,v_0\,l}{3}$

1.199 In the C.M. frame of the system (both the discs + spring), the linear momentum of the discs are related by the relation, $\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$, at all the moments of time.

where, $$\tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \mu\,v_{rel}$$

And the total kinetic energy of the system,

$$T = \frac{1}{2}\mu\,v_{rel}^2 \text{ [See solution of 1.147 (b)]}$$

Bearing in mind that at the moment of maximum deformation of the spring, the projection of $\vec{v}_{rel}$ along the length of the spring becomes zero, i.e. $v_{rel\,(x)} = 0$.

The conservation of mechanical energy of the considered system in the C.M. frame gives.

$$\frac{1}{2}\left(\frac{m}{2}\right)v_0^2 = \frac{1}{2}\kappa\,x^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_{rel\,(y)}^2 \quad (1)$$

Now from the conservation of angular momentum of the system about the C.M.,

$$\frac{1}{2}\left(\frac{l_0}{2}\right)\left(\frac{m}{2}\,v_0\right) = 2\left(\frac{l_0 + x}{2}\right)\frac{m}{2}\,v_{rel\,(y)}$$

or, $$v_{rel\,(y)} = \frac{v_0\,l_0}{(l_0 + x)} = v_0\left(1 + \frac{x}{l_0}\right)^{-1} \approx v_0\left(1 - \frac{x}{l_0}\right), \text{ as } x << l_0 \quad (2)$$

Using (2) in (1), $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{x}{l_0}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{2x}{l_0} + \frac{x^2}{l_0^2}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{m\,v_0^2\,x}{l_0} \approx \kappa\,x^2, \text{ [neglecting } x^2/l_0^2\text{]}$$

As $$x \neq 0, \text{ thus } x = \frac{m\,v_0^2}{\kappa\,l_0}$$

1.4 UNIVERSAL GRAVITATION

1.200 We have

$$\frac{Mv^2}{r} = \frac{\gamma M m_s}{r^2} \quad \text{or} \quad r = \frac{\gamma m_s}{v^2}$$

Thus

$$\omega = \frac{v}{r} = \frac{v}{\gamma m_s / v^2} = \frac{v^3}{\gamma m_s}$$

(Here m_s is the mass of the Sun.)

So $$T = \frac{2\pi \gamma m_s}{v^3} = \frac{2\pi \times 6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(34{\cdot}9 \times 10^3)^3} = 1{\cdot}94 \times 10^7 \text{ sec} = 225 \text{ days.}$$

(The answer is incorrectly written in terms of the planetary mass M)

1.201 For any planet

$$M R \omega^2 = \frac{\gamma M m_s}{R^2} \quad \text{or} \quad \omega = \sqrt{\frac{\gamma m_s}{R^3}}$$

So,

$$T = \frac{2\pi}{\omega} = 2\pi R^{3/2} / \sqrt{\gamma m_s}$$

(a) Thus

$$\frac{T_J}{T_E} = \left(\frac{R_J}{R_E}\right)^{3/2}$$

So

$$\frac{R_J}{R_E} = (T_J / T_E)^{2/3} = (12)^{2/3} = 5 \cdot 24.$$

(b)

$$V_J^2 = \frac{\gamma m_s}{R_J}, \quad \text{and} \quad R_J = \left(T \frac{\sqrt{\gamma m_s}}{2\pi}\right)^{2/3}$$

So

$$V_J^2 = \frac{(\gamma m_s)^{2/3} (2\pi)^{2/3}}{T^{2/3}} \quad \text{or,} \quad V_J = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3}$$

where $T = 12$ years. $m_s =$ mass of ths Sun.

Putting the values we get $V_J = 12{\cdot}97$ km/s

$$\text{Acceleration} = \frac{v_J^2}{R_J} = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3} \times \left(\frac{2\pi}{T\sqrt{\gamma m_s}}\right)^{2/3}$$

$$= \left(\frac{2\pi}{T}\right)^{4/3} (\gamma m_s)^{1/3}$$

$$= 2{\cdot}15 \times 10^{-4} \ \text{km/s}^2$$

1.202 Semi-major axis= $(r+R)/2$

It is sufficient to consider the motion be along a circle of semi-major axis $\frac{r+R}{2}$ for T does not depend on eccentricity.

Hence
$$T = \frac{2\pi\left(\frac{r+R}{2}\right)^{3/2}}{\sqrt{\gamma m_s}} = \pi\sqrt{(r+R)^3/2\gamma m_s}$$

(again m_s is the mass of the Sun)

1.203 We can think of the body as moving in a very elongated orbit of maximum distance R and minimum distance 0 so semi major axis = $R/2$. Hence if τ is the time of fall then

$$\left(\frac{2\tau}{T}\right)^2 = \left(\frac{R/2}{R}\right)^3 \quad \text{or} \quad \tau^2 = T^2/32$$

or
$$\tau = T/4\sqrt{2} = 365/4\sqrt{2} = 64{\cdot}5 \text{ days.}$$

1.204 $T = 2\pi R^{3/2}/\sqrt{\gamma m_s}$

If the distances are scaled down, $R^{3/2}$ decreases by a factor $\eta^{3/2}$ and so does m_s. Hence T does not change.

1.205 The double star can be replaced by a single star of mass $\frac{m_1 m_2}{m_1+m_2}$ moving about the centre of mass subjected to the force $\gamma m_1 m_2/r^2$. Then

$$T = \frac{2\pi r^{3/2}}{\sqrt{\gamma m_1 m_2 \Big/ \frac{m_1 m_2}{m_1+m_2}}} = \frac{2\pi r^{3/2}}{\sqrt{\gamma M}}$$

So
$$r^{3/2} = \frac{T}{2\pi}\sqrt{\gamma M}$$

or,
$$r = \left(\frac{T}{2\pi}\right)^{2/3}(\gamma M)^{1/3} = \sqrt[3]{\gamma M (T/2\pi)^2}$$

1.206 (a) The gravitational potential due to m_1 at the point of location of m_2 :

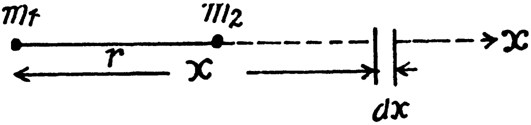
$$V_2 = \int_r^\infty \vec{G}\cdot d\vec{r} = \int_r^\infty -\frac{\gamma m_1}{x^2}dx = -\frac{\gamma m_1}{r}$$

So,
$$U_{21} = m_2 V_2 = -\frac{\gamma m_1 m_2}{r}$$

Similarly
$$U_{12} = -\frac{\gamma m_1 m_2}{r}$$

Hence

$$U_{12} = U_{21} = U = -\frac{\gamma m_1 m_2}{r}$$



(b) Choose the location of the point mass as the origin. Then the potential energy dU of an element of mass $dM = \frac{M}{l}dx$ of the rod in the field of the point mass is

$$dU = -\gamma m \frac{M}{l}dx \frac{1}{x}$$

where x is the distance between the element and the point. (Note that the rod and the point mass are on a straight line.) If then a is the distance of the nearer end of the rod from the point mass.

$$U = -\gamma \frac{mM}{l} \int_a^{a+l} \frac{dx}{x} = -\gamma m \frac{M}{l} \ln\left(1 + \frac{l}{a}\right)$$

The force of interaction is

$$F = -\frac{\partial U}{\partial a}$$

$$= \gamma \frac{mM}{l} \times \frac{1}{1 + \frac{l}{a}}\left(-\frac{l}{a^2}\right) = -\frac{\gamma mM}{a(a+l)}$$

Minus sign means attraction.

1.207 As the planet is under central force (gravitational interaction), its angular momentum is conserved about the Sun (which is situated at one of the focii of the ellipse)

So, $$m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1^2 = \frac{v_2^2 r_2^2}{r_1^2} \tag{1}$$

From the conservation of mechanical energy of the system (Sun + planet),

$$-\frac{\gamma m_s m}{r_1} + \frac{1}{2} m v_1^2 = -\frac{\gamma m_s m}{r_2} + \frac{1}{2} m v_2^2$$

or, $$-\frac{\gamma m_s}{r_1} + \frac{1}{2} v_2^2 \frac{r_2^2}{r_1^2} = -\left(\frac{\gamma m_s}{r_2}\right) + \frac{1}{2} v_2^2 \quad \text{[Using (1)]}$$

Thus, $$v_2 = \sqrt{2 \gamma m_s r_1 / r_2 (r_1 + r_2)} \tag{2}$$

Hence $$M = m v_2 r_2 = m \sqrt{2 \gamma m_s r_1 r_2 / (r_1 + r_2)}$$

1.208 From the previous problem, if r_1, r_2 are the maximum and minimum distances from the sun to the planet and v_1, v_2 are the corresponding velocities, then, say,

$$E = \frac{1}{2} m v_2^2 - \frac{\gamma m m_s}{r_2}$$

$$= \frac{\gamma m m_s}{r_1 + r_2} \cdot \frac{r_1}{r_2} - \frac{\gamma m m_s}{r_2} = - \frac{\gamma m m_s}{r_1 + r_2} = - \frac{\gamma m m_s}{2a} \text{ [Using Eq. (2) of 1.207]}$$

where $2a$ = major axis $= r_1 + r_2$. The same result can also be obtained directly by writing an equation analogous to Eq (1) of problem 1.191.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m m_s}{r}$$

(Here M is angular momentum of the planet and m is its mass). For extreme position $\dot{r} = 0$ and we get the quadratic

$$Er^2 + \gamma m m_s r - \frac{M^2}{2m} = 0$$

The sum of the two roots of this equation are

$$r_1 + r_2 = - \frac{\gamma m m_s}{E} = 2a$$

Thus $$E = - \frac{\gamma m m_s}{2a} = \text{ constant}$$

1.209 From the conservtion of angular momentum about the Sun.

$$m v_0 r_0 \sin \alpha = m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1 r_1 = v_2 r_2 = v_0 r_2 \sin \alpha \qquad (1)$$

From conservation of mechanical energy,

$$\frac{1}{2} m v_0^2 - \frac{\gamma m_s m}{r_0} = \frac{1}{2} m v_1^2 - \frac{\gamma m_s m}{r_1}$$

or, $$\frac{v_0^2}{2} - \frac{\gamma m_s}{r_0} = \frac{v_0^2 r_0^2 \sin^2 \alpha}{2 r_1^2} - \frac{\gamma m_s}{r_1} \text{ (Using 1)}$$

or, $$\left(v_0^2 - \frac{2 \gamma m_s}{r_0} \right) r_1^2 + 2 \gamma m_s r_1 - v_0^2 r_0^2 \sin \alpha = 0$$

So, $$r_1 = \frac{-2 \gamma m_s \pm \sqrt{4 \gamma^2 m_s^2 + 4 \left(v_0^2 r_0^2 \sin^2 \alpha \right) \left(v_0^2 - \frac{2 \gamma m_s}{r_0} \right)}}{2 \left(v_0^2 - \frac{2 \gamma m_s}{r_0} \right)}$$

$$= \frac{1 \pm \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2 \alpha}{\gamma m_s} \left(\frac{2}{r_0} - \frac{v_0^2}{r m_s} \right)}}{\left(\frac{2}{r_0} - \frac{v_0}{\gamma m_s} \right)} = \frac{r_0 \left[1 \pm \sqrt{1 - (2 - \eta) \eta \sin^2 \alpha} \right]}{(2 - \eta)}$$

where $\eta = v_0^2 r_0 / \gamma m_s$, ($m_s$ is the mass of the Sun).

1.210 At the minimum separation with the Sun, the cosmic body's velocity is perpendicular to its position vector relative to the Sun. If r_{min} be the sought minimum distance, from conservation of angular momentum about the Sun (C).

$$m v_0 l = m v r_{min} \text{ or, } v = \frac{v_0 l}{r_{min}} \qquad (1)$$

From conservation of mechanical energy of the system (sun + cosmic body),

$$\frac{1}{2} m v_0^2 = -\frac{\gamma m_s m}{r_{min}} + \frac{1}{2} m v^2$$

So,
$$\frac{v_0^2}{2} = -\frac{\gamma m_s}{r_{min}} + \frac{v_0^2}{2 r_{min}^2} \quad \text{(using 1)}$$

or,
$$v_0^2 r_{min}^2 + 2\gamma m_s r_{min} - v_0^2 l^2 = 0$$

So,
$$r_{min} = \frac{-2\gamma m_s \pm \sqrt{4\gamma^2 m^2 + 4 v_0^2 v_0^2 l^2}}{2 v_0^2} = \frac{-\gamma m_s \pm \sqrt{\gamma^2 m_s^2 + v_0^4 l^2}}{v_0^2}$$

Hence, taking positive root

$$r_{min} = (\gamma m_s / v_0^2)\left[\sqrt{1 + (l v_0^2/\gamma m_s)^2} - 1\right]$$

1.211 Suppose that the sphere has a radius equal to a. We may imagine that the sphere is made up of concentric thin spherical shells (layers) with radii ranging from 0 to a, and each spherical layer is made up of elementry bands (rings). Let us first calculate potential due to an elementry band of a spherical layer at the point of location of the point mass m (say point P) (Fig.). As all the points of the band are located at the distance l from the point P, so,

$$\partial \varphi = -\frac{\gamma \, \partial M}{l} \quad \text{(where mass of the band)} \qquad (1)$$

$$\partial M = \left(\frac{dM}{4\pi a^2}\right)(2\pi a \sin\theta)(a \, d\theta)$$

$$= \left(\frac{dM}{2}\right)\sin\theta \, d\theta \qquad (2)$$

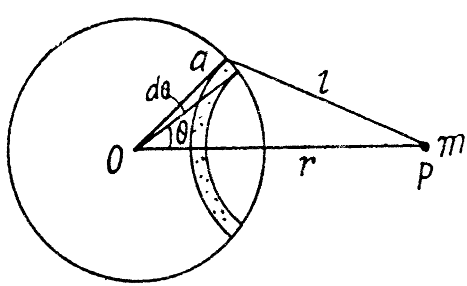
And $\quad l^2 = a^2 + r^2 - 2 ar\cos\theta \qquad (3)$

Differentiating Eq. (3), we get

$l dl = ar \sin\theta \, d\theta \qquad (4)$

Hence using above equations

$$\partial \varphi = -\left(\frac{\gamma \, dM}{2ar}\right) dl \qquad (5)$$



Now integrating this Eq. over the whole spherical layer

$$d\varphi = \int \partial\varphi = -\frac{\gamma\, dM}{2ar} \int_{r-a}^{r+a}$$

So $$d\varphi = -\frac{\gamma\, dM}{r} \qquad (6)$$

Equation (6) demonstrates that the potential produced by a thin uniform spherical layer outside the layer is such as if the whole mass of the layer were concentrated at it's centre; Hence the potential due to the sphere at point P;

$$\varphi = \int d\varphi = -\frac{\gamma}{r}\int dM = -\frac{\gamma M}{r} \qquad (7)$$

This expression is similar to that of Eq. (6)

Hence thte sought potential energy of gravitational interaction of the particle m and the sphere,

$$U = m\varphi = -\frac{\gamma M m}{r}$$

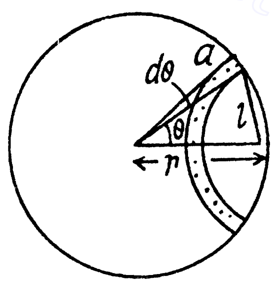
(b) Using the Eq., $$G_r = -\frac{\partial\varphi}{\partial r}$$

$$G_r = -\frac{\gamma M}{r^2} \qquad \text{(using Eq. 7)}$$

So $$\vec{G} = -\frac{\gamma M}{r^3}\vec{r} \text{ and } \vec{F} = m\vec{G} = -\frac{\gamma m M}{r^3}\vec{r} \qquad (8)$$

1.212 (The problem has already a clear hint in the answer sheet of the problem book). Here we adopt a different method.

Let m be the mass of the spherical layer, wich is imagined to be made up of rings. At a point inside the spherical layer at distance r from the centre, the gravitational potential due to a ring element of radius a equals,



$$d\varphi = -\frac{\gamma m}{2ar}\, dl \quad \text{(see Eq. (5) of solution of 1.211)}$$

So, $$\varphi = \int d\varphi = -\frac{\gamma m}{2ar}\int_{a-r}^{a+r} dl = -\frac{\gamma m}{a} \qquad (1)$$

Hence $$G_r = -\frac{\partial\varphi}{\partial r} = 0\,.$$

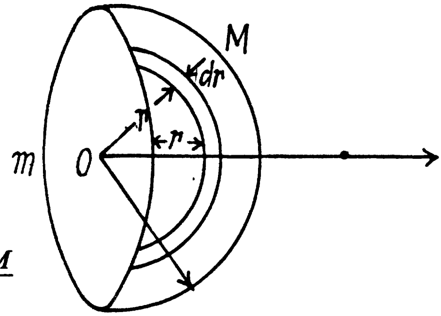
Hence gravitational field strength as well as field force becomes zero, inside a thin sphereical layer.

1.213 One can imagine that the uniform hemisphere is made up of thin hemispherical layers of radii ranging from 0 to R. Let us consider such a layer (Fig.). Potential at point O, due to this layer is,

$$d\varphi = -\frac{\gamma dm}{r} = -\frac{3\gamma M}{R^3} r dr, \text{ where } dm = \frac{M}{(2/3)\pi R^3}\left(\frac{4\pi r^2}{2}\right) dr$$

(This is because all points of each hemispherical shell are equidistant from O.)

Hence, $\varphi = \int d\varphi = -\frac{3\gamma M}{R^3}\int_0^R r dr = -\frac{3\gamma M}{2R}$



Hence, the work done by the gravitational field force on the particle of mass m, to remove it to infinity is given by the formula

$A = m\varphi$, since $\varphi = 0$ at infinity.

Hence the sought work,

$$A_{0\to\infty} = -\frac{3\gamma mM}{2R}$$

(The work done by the external agent is $-A$.)

1.214 In the solution of problem 1.211, we have obtained φ and G due to a uniform shpere, at a distance r from it's centre outside it. We have from Eqs. (7) and (8) of 1.211,

$$\varphi = -\frac{\gamma M}{r} \text{ and } \vec{G} = -\frac{\gamma M}{r^3}\vec{r} \qquad (A)$$

Accordance with the Eq. (1) of the solution of 1.212, potential due to a spherical shell of radius a, at any point, inside it becomes

$$\varphi = \frac{\gamma M}{a} = \text{Const. and } G_r = -\frac{\partial \varphi}{\partial r} = 0 \qquad (B)$$

For a point (say P) which lies inside the uniform solid sphere, the potential φ at that point may be represented as a sum.

$$\varphi_{inside} = \varphi_1 + \varphi_2$$

where φ_1 is the potential of a solid sphere having radius r and φ_2 is the potential of the layer of radii r and R. In accordance with equation (A)

$$\varphi_1 = -\frac{\gamma}{r}\left(\frac{M}{(4/3)\pi R^3}\frac{4}{3}\pi r^3\right) = -\frac{\gamma M}{R^3} r^2$$

The potential φ_2 produced by the layer (thick shell) is the same at all points inside it. The potential φ_2 is easiest to calculate, for the point positioned at the layer's centre. Using Eq. (B)

$$\varphi_2 = -\gamma \int_r^R \frac{dM}{r} = -\frac{3}{2}\frac{\gamma M}{R^3}(R^2 - r^2)$$

where $dM = \frac{M}{(4/3)\pi R^3} 4\pi r^2 dr = \left(\frac{3M}{R^3}\right) r^2 dr$

is the mass of a thin layer between the radii r and $r + dr$.

Thus $$\varphi_{inside} = \varphi_1 + \varphi_2 = \left(\frac{\gamma M}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad (C)$$

From the Eq. $$G_r = \frac{-\partial \varphi}{\partial r}$$

$$G_r = \frac{\gamma M r}{R^3}$$

or $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma \frac{4}{3}\pi \rho \vec{r}$$

(where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$, is the density of the sphere) (D)

The plots $\varphi(r)$ and $G(r)$ for a uniform sphere of radius R are shown in figure of answersheet.

Alternate : Like Gauss's theorem of electrostatics, one can derive Gauss's theorem for gravitation in the form $\oint \vec{G}\cdot d\vec{S} = -4\pi\gamma\, m_{inclosed}$. For calculation of $\vec{G}$ at a point inside the sphere at a distance r from its centre, let us consider a Gaussian surface of radius r, Then,

$$G_r\, 4\pi r^2 = -4\pi\gamma\left(\frac{M}{R^3}\right)r^3 \quad \text{or,} \quad G_r = -\frac{\gamma M}{R^3}r$$

Hence, $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma\frac{4}{3}\pi\rho\vec{r}\left(\text{as } \rho = \frac{M}{(4/3)\pi R^3}\right)$$

So, $$\varphi = \int_r^\infty G_r\, dr = \int_r^R -\frac{\gamma M}{R^3}r\, dr + \int_R^\infty -\frac{\gamma M}{r^2}dr$$

Integrating and summing up, we get,

$$\varphi = -\frac{\gamma M}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

And from Gauss's theorem for outside it :

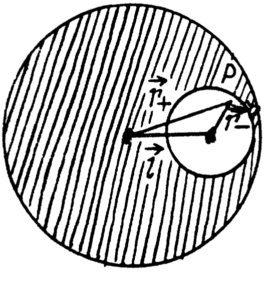
$$G_r\, 4\pi r^2 = -4\pi\gamma M \quad \text{or} \quad G_r = -\frac{\gamma M}{r^2}$$

Thus $$\varphi(r) = \int_r^\infty G_r\, dr = -\frac{\gamma M}{r}$$

1.215 Treating the cavity as negative mass of density $-\rho$ in a uniform sphere density $+\rho$ and using the superposition principle, the sought field strength is :

$$\vec{G} = \vec{G}_1 + \vec{G}_2$$

or $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\,\vec{r}_+ + -\frac{4}{3}\gamma\pi(-\rho)\,\vec{r}_-$$



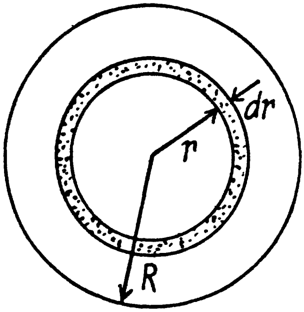
(where $\vec{r}_+$ and $\vec{r}_-$ are the position vectors of an orbitrary point P inside the cavity with respect to centre of sphere and cavity respectively.)

Thus $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\left(\vec{r}_+ - \vec{r}_-\right) = -\frac{4}{3}\pi\gamma\rho\,\vec{l}$$

1.216 We partition the solid sphere into thin spherical layers and consider a layer of thickness dr lying at a distance r from the centre of the ball. Each spherical layer presses on the layers within it. The considered layer is attracted to the part of the sphere lying within it (the outer part does not act on the layer). Hence for the considered layer

$$dp\,4\pi r^2 = dF$$

or, $$dP\,4\pi r^2 = \frac{\gamma\left(\frac{4}{3}\pi r^3 \rho\right)(4\pi r^2 dr\,\rho)}{r^2}$$



(where ρ is the mean density of sphere)

or, $$dp = \frac{4}{3}\pi\gamma\rho^2 r\,dr$$

Thus $$p = \int_r^R dp = \frac{2\pi}{3}\gamma\rho^2(R^2 - r^2)$$

(The pressure must vanish at $r = R$.)

or, $$p = \frac{3}{8}\left(1 - (r^2/R^2)\right)\gamma M^2/\pi R^4, \text{ Putting } \rho = M/(4/3)\pi R^3$$

Putting $r = 0$, we have the pressure at sphere's centre, and treating it as the Earth where mean density is equal to $\rho = 5{\cdot}5 \times 10^3\ \text{kg/m}^3$ and $R = 64 \times 10^2$ km

we have, $p = 1{\cdot}73 \times 10^{11}$ Pa or $1{\cdot}72 \times 10^6$ atms.

1.217 (a) Since the potential at each point of a spherical surface (shell) is constant and is equal to $\varphi = -\frac{\gamma m}{R}$, [as we have in Eq. (1) of solution of problem 1.212]

We obtain in accordance with the equation

$$U = \frac{1}{2}\int dm\,\varphi = \frac{1}{2}\varphi\int dm$$

$$= \frac{1}{2}\left(-\frac{\gamma m}{R}\right)m = -\frac{\gamma m^2}{2R}$$

(The factor $\frac{1}{2}$ is needed otherwise contribution of different mass elements is counted twice.)

(b) In this case the potential inside the sphere depends only on r (see Eq. (C) of the solution of problem 1.214)

$$\varphi = -\frac{3\gamma m}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

Here dm is the mass of an elementry spherical layer confined between the radii r and $r + dr$:

$$dm = (4\pi r^2 dr\,\rho) = \left(\frac{3m}{R^3}\right)r^2 dr$$

$$U = \frac{1}{2}\int dm\, \varphi$$

$$= \frac{1}{2}\int_0^R \left(\frac{3\,m}{R^3}\right) r^2\, dr \left\{-\frac{3\,\gamma\, m}{2\,R}\left(1 - \frac{r^2}{3\,R^2}\right)\right\}$$

After integrating, we get

$$U = -\frac{3}{5}\frac{\gamma\, m^2}{R}$$

1.218 Let $\omega = \sqrt{\frac{\gamma M_E}{r^3}}$ = circular frequency of the satellite in the outer orbit,

$\omega_0 = \sqrt{\frac{\gamma M_E}{(r - \Delta r)^3}}$ = circular frequency of the satellite in the inner orbit.

So, relative angular velocity = $\omega_0 \pm \omega$ where − sign is to be taken when the satellites are moving in the same sense and + sign if they are moving in opposite sense.
Hence, time between closest approaches

$$= \frac{2\pi}{\omega_0 \pm \omega} = \frac{2\pi}{\sqrt{\gamma M_E}\,/\,r^{3/2}}\,\frac{1}{\frac{3\,\Delta r}{2\,r} + \delta} = \begin{cases} 4{\cdot}5 \text{ days } (\delta = 0) \\ 0{\cdot}80 \text{ hour } (\delta = 2)\end{cases}$$

where δ is 0 in the first case and 2 in the second case.

1.219 $\omega_1 = \frac{\gamma M}{R^2} = \frac{6{\cdot}67 \times 10^{-11} \times 5{\cdot}96 \times 10^{24}}{(6{\cdot}37 \times 10^6)^2} = 9{\cdot}8 \text{ m/s}^2$

$$\omega_2 = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \left(\frac{2 \times 22}{24 \times 3600 \times 7}\right)^2 6{\cdot}37 \times 10^6 = 0{\cdot}034 \text{ m/s}^2$$

and $\omega_3 = \frac{\gamma M_S}{R^2_{mean}} = \frac{6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(149{\cdot}50 \times 10^6 \times 10^3)^2} = 5{\cdot}9 \times 10^{-3} \text{ m/s}^2$

Then $\omega_1 : \omega_2 : \omega_3 = 1 : 0{\cdot}0034 : 0{\cdot}0006$

1.220 Let h be the sought height in the first case. so

$$\frac{99}{100}\, g = \frac{\gamma M}{(R + h)^2}$$

$$= \frac{\gamma M}{R^2\left(1 + \frac{h}{R}\right)^2} = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

or $$\frac{99}{100} = \left(1 + \frac{h}{R}\right)^{-2}$$

From the statement of the problem, it is obvious that in this case $h << R$

Thus $$\frac{99}{100} = \left(1 - \frac{2h}{R}\right) \quad \text{or} \quad h = \frac{R}{200} = \left(\frac{6400}{200}\right) \text{km} = 32 \text{ km}$$

In the other case if h' be the sought height, than

$$\frac{g}{2} = g\left(1 + \frac{h'}{R}\right)^{-2} \quad \text{or} \quad \frac{1}{2} = \left(1 + \frac{h'}{R}\right)^{-2}$$

From the language of the problem, in this case h' is not very small in comparision with R. Therefore in this case we cannot use the approximation adopted in the previous case.

Here, $\left(1 + \frac{h'}{R}\right)^2 = 2$ So, $\frac{h'}{R} = \pm\sqrt{2} - 1$

As $-ve$ sign is not acceptable

$$h' = (\sqrt{2} - 1) R = (\sqrt{2} - 1)\, 6400 \text{ km} = 2650 \text{ km}$$

1.221 Let the mass of the body be m and let it go upto a height h.
From conservation of mechanical energy of the system

$$-\frac{\gamma M m}{R} + \frac{1}{2} m v_0^2 = \frac{-\gamma M m}{(R+h)} + 0$$

Using $\frac{\gamma M}{R^2} = g$, in above equation and on solving we get,

$$h = \frac{R v_0^2}{2 g R - v_0^2}$$

1.222 Gravitational pull provides the required centripetal acceleration to the satelite. Thus if h be the sought distance, we have

so, $$\frac{m v^2}{(R+h)} = \frac{\gamma m M}{(R+h)^2} \quad \text{or,} \quad (R+h) v^2 = \gamma M$$

or, $$R v^2 + h v^2 = g R^2, \quad \text{as} \quad g = \frac{\gamma M}{R^2}$$

Hence $$h = \frac{g R^2 - R v^2}{v^2} = R\left[\frac{g R}{v^2} - 1\right]$$

1.223 A satellite that hovers above the earth's equator and corotates with it moving from the west to east with the diurnal angular velocity of the earth appears stationary to an observer on the earth. It is called geostationary. For this calculation we may neglect the annual motion of the earth as well as all other influences. Then, by Newton's law,

$$\frac{\gamma M m}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r$$

where M = mass of the earth, T = 86400 seconds = period of daily rotation of the earth and r = distance of the satellite from the centre of the earth. Then

$$r = \sqrt[3]{\gamma M \left(\frac{T}{2\pi} \right)^2}$$

Substitution of $M = 5{\cdot}96 \times 10^{24}$ kg gives

$$r = 4{\cdot}220 \times 10^4 \text{ km}$$

The instantaneous velocity with respect to an inertial frame fixed to the centre of the earth at that moment will be

$$\left(\frac{2\pi}{T} \right) r = 3{\cdot}07 \text{ km/s}$$

and the acceleration will be the centripetal acceleration.

$$\left(\frac{2\pi}{T} \right)^2 r = 0{\cdot}223 \text{ m/s}^2$$

1.224 We know from the previous problem that a satellite moving west to east at a distance $R = 2{\cdot}00 \times 10^4$ km from the centre of the earth will be revolving round the earth with an angular velocity faster than the earth's diurnal angualr velocity. Let

$$\omega = \text{angular velocity of the satellite}$$

$$\omega_0 = \frac{2\pi}{T} = \text{anuglar velocity of the earth. Then}$$

$$\omega - \omega_0 = \frac{2\pi}{\tau}$$

as the relative angular velocity with respect to earth. Now by Newton's law

$$\frac{\gamma M}{R^2} = \omega^2 R$$

So,

$$M = \frac{R^3}{\gamma} \left(\frac{2\pi}{\tau} + \frac{2\pi}{T} \right)^2$$

$$= \frac{4\pi^2 R^3}{\gamma T^2} \left(1 + \frac{T}{\tau} \right)^2$$

Substitution gives

$$M = 6{\cdot}27 \times 10^{24} \text{ kg}$$

1.225 The velocity of the satellite in the inertial space fixed frame is $\sqrt{\frac{\gamma M}{R}}$ east to west. With respect to the Earth fixed frame, from the $\vec{v}_1{}' = \vec{v} - (\vec{w} \times \vec{r})$ the velocity is

$$v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7{\cdot}03 \text{ km/s}$$

Here M is the mass of the earth and T is its period of rotation about its own axis.

It would be $-\frac{2\pi R}{T}+\sqrt{\frac{\gamma M}{R}}$, if the satellite were moving from west to east.

To find the acceleration we note the formula

$$m\vec{w}' = \vec{F} + 2m(\vec{v}' \times \vec{\omega}) + m\omega^2 \vec{R}$$

Here $\vec{F} = -\frac{\gamma M m}{R^3}\vec{R}$ and $\vec{v}' \perp \vec{\omega}$ and $\vec{v}' \times \vec{\omega}$ is directed towards the centre of the Earth.

Thus $$w' = \frac{\gamma M}{R^2} + 2\left(\frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}}\right)\frac{2\pi}{T} - \left(\frac{2\pi}{T}\right)^2 R$$

toward the earth's rotation axis

$$= \frac{\gamma M}{R^2} + \frac{2\pi}{T}\left[\frac{2\pi R}{T} + 2\sqrt{\frac{\gamma M}{R}}\right] = 4{\cdot}94 \text{ m/s}^2 \text{ on substitution.}$$

1.226 From the well known relationship between the velocities of a particle w.r.t a space fixed frame (K) rotating frame (K') $\vec{v} = \vec{v}' + (\vec{w} \times \vec{r})$

$$v'_1 = v - \left(\frac{2\pi}{T}\right)R$$

Thus kinetic energy of the satellite in the earth's frame

$$T_1' = \frac{1}{2}m v'^2_1 = \frac{1}{2}m\left(v - \frac{2\pi R}{T}\right)^2$$

Obviously when the satellite moves in opposite sense comared to the rotation of the Earth its velocity relative to the same frame would be

$$v_2' = v + \left(\frac{2\pi}{T}\right)R$$

And kinetic energy

$$T_2' = \frac{1}{2}m v'^2_2 = \frac{1}{2}m\left(v + \frac{2\pi R}{T}\right)^2 \qquad (2)$$

From (1) and (2)

$$T' = \frac{\left(v + \frac{2\pi R}{T}\right)^2}{\left(v - \frac{2\pi R}{T}\right)^2} \qquad (3)$$

Now from Newton's second law

$$\frac{\gamma M m}{R^2} = \frac{m v^2}{R} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{R}} = \sqrt{gR} \qquad (4)$$

Using (4) and (3)

$$\frac{T_2'}{T_1'} = \frac{\left(\sqrt{gR} + \frac{2\pi R}{T}\right)^2}{\left(\sqrt{gR} - \frac{2\pi R}{T}\right)^2} = 1{\cdot}27 \quad \text{nearly (Using Appendices)}$$

1.227 For a satellite in a circular orbit about any massive body, the following relation holds between kinetic, potential & total energy :

$$T = -E, \; U = 2E \tag{1}$$

Thus since total mechanical energy must decrease due to resistance of the cosmic dust, the kintetic energy will increase and the satellite will 'fall', We see then, by work energy theorm

$$dT = -dE = -dA_{fr}$$

So, $\quad mvdv = \alpha v^2 vdt \quad$ or, $\quad \frac{\alpha dt}{m} = \frac{dv}{v^2}$

Now from Netow's law at an arbitray radius r from the moon's centre.

$$\frac{v^2}{r} = \frac{\gamma M}{r^2} \text{ or } v = \sqrt{\frac{\gamma M}{r}}$$

(M is the mass of the moon.) Then

$$v_i = \sqrt{\frac{\gamma M}{\eta R}}, \; v_f = \sqrt{\frac{\gamma M}{R}}$$

where R= moon's radius. So

$$\int_{v_1}^{v_f} \frac{dv}{v^2} = \frac{\alpha}{m} \int_0^{\tau} dt = \frac{\alpha \tau}{m}$$

or, $$\tau = \frac{m}{\alpha}\left(\frac{1}{v_i} - \frac{1}{v_f}\right) = \frac{m}{\alpha \sqrt{\gamma \frac{M}{R}}}(\sqrt{\eta} - 1) = \frac{m}{\alpha \sqrt{gR}}(\sqrt{\eta} - 1)$$

where g is moon's gravity. The averaging implied by Eq. (1) (for noncircular orbits) makes the result approximate.

1.228 From Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv_0^2}{R} \text{ or } v_0 = \sqrt{\frac{\gamma M}{R}} = 1{\cdot}67 \text{ km/s} \tag{1}$$

From conservation of mechanical energy

$$\frac{1}{2}mv_e^2 - \frac{\gamma Mm}{R} = 0 \text{ or, } v_e = \sqrt{\frac{2\gamma M}{R}} = 2{\cdot}37 \text{ km/s}^2 \tag{2}$$

In Eq. (1) and (2), M and R are the mass of the moon and its radius. In Eq. (1) if M and R represent the mass of the earth and its radius, then, using appendices, we can easily get

$$v_0 = 7{\cdot}9 \text{ km/s} \quad \text{and} \quad v_e = 11{\cdot}2 \text{ km/s}.$$

1.229 In a parabolic orbit, $E = 0$

So $$\frac{1}{2} m v_i^2 - \frac{\gamma M m}{R} = 0 \text{ or, } v_i = \sqrt{2}\sqrt{\frac{\gamma M}{R}}$$

where M = mass of the Moon, R = its radius. (This is just the escape velocity.)
On the other hand in orbit

$$m v_f^2 R = \frac{\gamma M m}{R^2} \text{ or } v_f = \sqrt{\frac{\gamma M}{R}}$$

Thus $$\Delta v = (1 - \sqrt{2})\sqrt{\frac{\gamma M}{R}} = -0{\cdot}70 \text{ km/s}.$$

1.230 From 1.228 for the Earth surface

$$v_0 = \sqrt{\frac{\gamma M}{R}} \text{ and } v_e = \sqrt{\frac{2\gamma M}{R}}$$

Thus the sought additional velocity

$$\Delta v = v_e - v_0 = \sqrt{\frac{\gamma M}{R}}(\sqrt{2} - 1) = gR(\sqrt{2} - 1)$$

This 'kick' in velocity must be given along the direction of motion of the satellite in its orbit.

1.231 Let r be the sought distance, then

$$\frac{\gamma \eta M}{(nR - r)^2} = \frac{\gamma M}{r^2} \text{ or } \eta r^2 = (nR - r)^2$$

or $$\sqrt{\eta}\, r = (nR - r) \text{ or } r = \frac{nR}{\sqrt{\eta} + 1} = 3{\cdot}8 \times 10^4 \text{ km}.$$

1.232 Between the earth and the moon, the potential energy of the spaceship will have a maximum at the point where the attractions of the earth and the moon balance each other. This maximum P.E. is approximately zero. We can also neglect the contribution of either body to the p.E. of the spaceship sufficiently near the other body. Then the minimum energy that must be imparted to the spaceship to cross the maximum of the P.E. is clearly (using E to denote the earth)

$$\frac{\gamma M_E m}{R_E}$$

With this energy the spaceship will cross over the hump in the P.E. and coast down the hill of p.E. towards the moon and crashland on it. What the problem seeks is the minimum energy reguired for softlanding. That reguies the use of rockets to loving about the braking of the spaceship and since the kinetic energy of the gases ejected from the rocket will always be positive, the total energy required for softlanding is greater than that required for crashlanding. To calculate this energy we assume that the rockets are used fairly close to the moon when the spaceship has nealy attained its terminal velocity on the moon $\sqrt{\frac{2\gamma M_O}{R_0}}$ where M_0 is the mass of the moon and R_0 is its radius. In general $dE = vdp$ and since the speed of the ejected gases is not less than the speed of the rocket, and momentum transfered to the ejected gases must equal the momentum of the spaceship the energy E of the gass ejected is not less than the kinetic energy of spaceship

$$\frac{\gamma M_0 m}{R_0}$$

Addding the two we get the minimum work done on the ejected gases to bring about the softlanding.

$$A_{\min} \approx \gamma m \left(\frac{M_E}{R_E} + \frac{M_0}{R_0}\right)$$

On substitution we get $1 \cdot 3 \times 10^8$ kJ.

1.233 Assume first that the attraction of the earth can be neglected. Then the minimum velocity, that must be imparted to the body to escape from the Sun's pull, is, as in 1·230, equal to

$$(\sqrt{2} - 1)\, v_1$$

where $v_1^2 = \gamma M_s / r$, r = radius of the earth's orbit, M_s = mass of the Sun.

In the actual case near the earth, the pull of the Sun is small and does not change much over distances, which are several times the radius of the Earth. The velocity v_3 in question is that which overcomes the earth's pull with sufficient velocity to escape the Sun's pull. Thus

$$\frac{1}{2} m v_3^2 - \frac{\gamma M_E}{R} = \frac{1}{2} m (\sqrt{2} - 1)^2 v_1^2$$

where R = radius of the earth, M_E = mass of the earth.

Writing $v_1^2 = \gamma M_E / R$, we get

$$v_3 = \sqrt{2 v_2^2 + (\sqrt{2} - 1)^2 v_1^2} = 16 \cdot 6 \text{ km/s}$$

1.5 DYNAMICS OF A SOLID BODY

1.234 Since, motion of the rod is purely translational, net torque about the C.M. of the rod should be equal to zero.

Thus $$F_1 \frac{l}{2} = F_2 \left(\frac{l}{2} - a\right) \text{ or, } \frac{F_1}{F_2} = 1 - \frac{a}{l/2} \quad (1)$$

For the translational motion of rod.

$$F_2 - F_1 = m w_c \text{ or } 1 - \frac{F_1}{F_2} = \frac{m w_c}{F_2} \quad (2)$$

From (1) and (2)

$$\frac{a}{l/2} = \frac{m w_c}{F_2} \text{ or, } l = \frac{2aF_2}{m w_c} = 1 \text{ m}$$

1.235 Sought moment $\vec{N} = \vec{r} \times \vec{F} = (a\vec{i} + b\vec{j}) \times (A\vec{i} + B\vec{j})$

$$= aB\vec{k} + Ab(-\vec{k}) = (aB - Ab)\vec{k}$$

and arm of the force $$l = \frac{N}{F} = \frac{aB - Ab}{\sqrt{A^2 + B^2}}$$

1.236 Relative to point O, the net moment of force :

$$\vec{N} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (a\vec{i} \times A\vec{j}) + (B\vec{j} \times B\vec{i})$$

$$= ab\vec{k} + AB(-\vec{k}) = (ab - AB)\vec{k} \quad (1)$$

Resultant of the external force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = A\vec{j} + B\vec{i} \quad (2)$$

As $\vec{N} \cdot \vec{F} = 0$ (as $\vec{N} \perp \vec{F}$) so the sought arm l of the force $\vec{F}$

$$l = N/F = \frac{ab - AB}{\sqrt{A^2 + B^2}}$$

1.237 For coplanar forces, about any point in the same plane, $\sum \vec{r}_i \times \vec{F}_i = \vec{r} \times \vec{F}_{net}$

(where $\vec{F}_{net} = \sum \vec{F}_i$ = resultant force) or, $\vec{N}_{net} = \vec{r} \times \vec{F}_{net}$

Thus length of the arm, $l = \dfrac{N_{net}}{F_{net}}$

Here obviously $|\vec{F}_{net}| = 2F$ and it is directed toward right along AC. Take the origin at C. Then about C,

$\vec{N} = \left(\sqrt{2}\, a F + \dfrac{a}{\sqrt{2}} F - \sqrt{2}\, a F\right)$ directed normally into the plane of figure.

(Here a = side of the square.)

Thus $\vec{N} = F \dfrac{a}{\sqrt{2}}$ directed into the plane of the figure.

Hence $$l = \frac{F(a/\sqrt{2})}{2F} = \frac{a}{2\sqrt{2}} = \frac{a}{2} \sin 45°$$

Thus the point of application of force is at the mid point of the side BC.

1.238 (a) Consider a strip of length dx at a perpendicular distance x from the axis about which we have to find the moment of inertia of the rod. The elemental mass of the rod equals

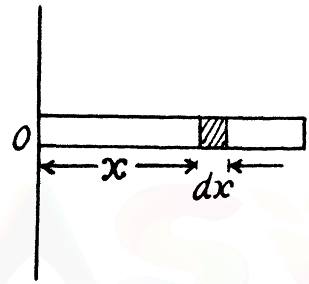
$$dm = \frac{m}{l}dx$$

Moment of inertia of this element about the axis

$$dI = dm\,x^2 = \frac{m}{l}dx \cdot x^2$$

Thus, moment of inertia of the rod, as a whole about the given axis

$$I = \int_0^l \frac{m}{l}x^2\,dx = \frac{m\,l^2}{3}$$



(b) Let us imagine the plane of plate as xy plane taking the origin at the intersection point of the sides of the plate (Fig.).

Obviously $\quad I_x = \int dm\,y^2$

$$= \int_0^a \left(\frac{m}{ab}b\,dy\right)y^2$$

$$= \frac{m\,a^2}{3}$$

Similarly $\quad I_y = \frac{m\,b^2}{3}$

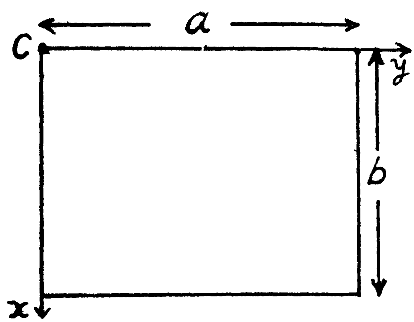
Hence from perpendicular axis theorem

$$I_z = I_x + I_y = \frac{m}{3}\left(a^2 + b^2\right),$$

which is the sought moment of inertia.

1.239 (a) Consider an elementry disc of thickness dx. Moment of inertia of this element about the z-axis, passing through its C.M.

$$dI_z = \frac{(dm)R^2}{2} = \rho S\,dx\,\frac{R^2}{2}$$



where ρ = density of the material of the plate and S = area of cross section of the plate.

Thus the sought moment of inertia

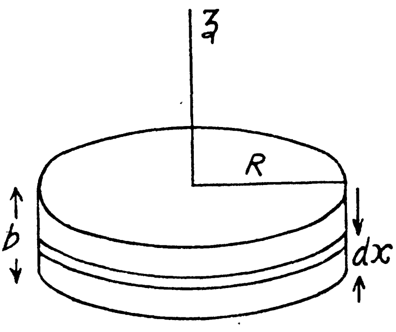
$$I_z = \frac{\rho S R^2}{2}\int_0^b dx = \frac{R^2}{2}\rho S b$$

$$= \frac{\pi}{2}\rho\,b\,R^4\left(\text{as } S = \pi R^2\right)$$

putting all the vallues we get, $I_z = 2 \cdot \text{gm} \cdot \text{m}^2$

(b) Consider an element disc of radius r and thickness dx at a distance x from the point O. Then $r = x \tan\alpha$ and volume of the disc

$$= \pi x^2 \tan^2 \alpha \, dx$$



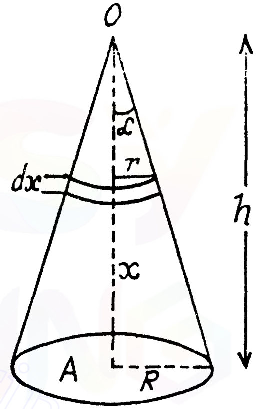
Hence, its mass $dm = \pi x^2 \tan\alpha \, dx \cdot \rho$ (where ρ = density of the cone $= m/\frac{1}{3}\pi R^2 h$)

Moment of inertia of this element, about the axis OA,

$$dI = dm \frac{r^2}{2}$$

$$= (\pi x^2 \tan^2\alpha \, dx) \frac{x^2 \tan^2\alpha}{2}$$

$$= \frac{\pi \rho}{2} x^4 \tan^4\alpha \, dx$$



Thus the sought moment of inertia $I = \frac{\pi \rho}{2} \tan^4 \alpha \int_0^h x^4 \, dx$

$$= \frac{\pi \rho R^4 \cdot h^5}{10 h^4} \left(\text{as } \tan\alpha = \frac{R}{h} \right)$$

Hence $\qquad I = \frac{3m R^2}{10} \left(\text{putting } \rho = \frac{3m}{\pi R^2 h} \right)$

1.240 (a) Let us consider a lamina of an arbitrary shape and indicate by 1,2 and 3, three axes coinciding with x, y and z – axes and the plane of lamina as $x - y$ plane.

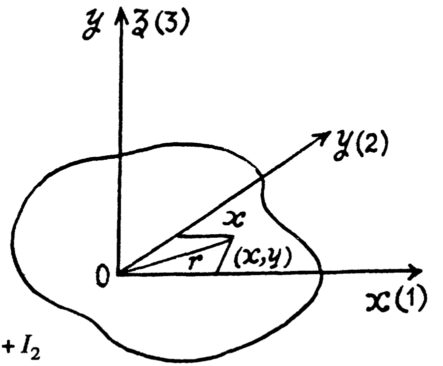
Now, moment of inertia of a point mass about x – axis, $dI_x = dm \, y^2$

Thus moment of inertia of the lamina about this axis, $I_x = \int dm y^2$

Similarly, $I_y = \int dm x^2$

and $I_z = \int dm r^2$

$= \int dm \, (x^2 + y^2)$ as $r = \sqrt{x^2 + y^2}$



Thus, $\qquad I_z = I_x + I_y$ or, $I_3 = I_1 + I_2$

(b) Let us take the plane of the disc as $x - y$ plane and origin to the centre of the disc (Fig.) From the symmetry $I_x = I_y$. Let us consider a ring element of radius r and thickness dr, then the moment of inertia of the ring element about the y – axis.

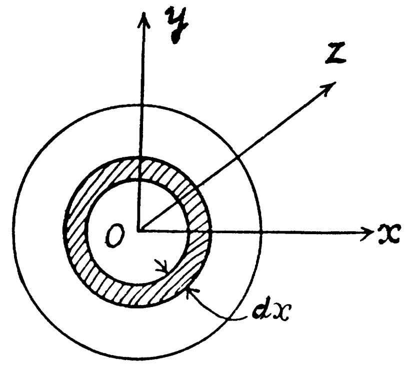
$$dI_z = dm r^2 = \frac{m}{\pi R^2}(2\pi r dr)\, r^2$$

Thus the moment of inertia of the disc about $z-$ axis

$$I_z = \frac{2m}{R^2}\int_0^R r^3\, dr = \frac{mR^2}{2}$$

But we have $I_z = I_x + I_y = 2I_x$

Thus $I_x = \frac{I_z}{2} = \frac{mR^2}{4}$

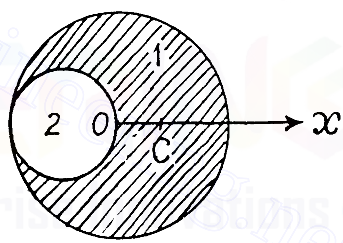


1.241 For simplicity let us use a mathematical trick. We consider the portion of the given disc as the superposition of two complete discs (without holes), one of positive density and radius R and other of negative density but of same magnitude and radius $R/2$.

As (area) α (mass), the respective masses of the considered discs are $(4m/3)$ and $(-m/3)$ respectively, and these masses can be imagined to be situated at their respective centers (C.M). Let us take point O as origin and point $x-$ axis towards right. Obviously the C.M. of the shaded position of given shape lies on the $x-$ axis. Hence the C.M. (C) of the shaded portion is given by

$$x_c = \frac{(-m/3)(-R/2) + (4m/3)0}{(-m/3) + 4m/3} = \frac{R}{6}$$

Thus C.M. of the shape is at a distance $R/6$ from point O toward $x-$ axis



Using parallel axis theorem and bearing in mind that the moment of inertia of a complete homogeneous disc of radius m_0 and radius r_0 equals $\frac{1}{2} m_0 r_0^2$. The moment of inetia of the small disc of mass $(-m/3)$ and radius $R/2$ about the axis passing through point C and perpendicular to the plane of the disc

$$I_{2C} = \frac{1}{2}\left(-\frac{m}{3}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{m}{3}\right)\left(\frac{R}{2} + \frac{R}{6}\right)^2$$

$$= -\frac{mR^2}{24} - \frac{4}{27} mR^2$$

Similarly $I_{1C} = \frac{1}{2}\left(\frac{4m}{3}\right)R^2 + \left(\frac{4m}{3}\right)\left(\frac{R}{6}\right)^2$

$$= \frac{2}{3} mR^2 + \frac{mR^2}{27}$$

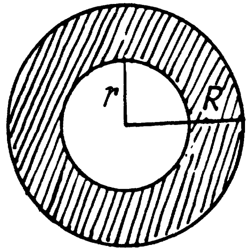
Thus the sought moment of inertia,

$$I_C = I_{1C} + I_{2C} = \frac{15}{24} m R^2 - \frac{3}{27} mR^2 = \frac{37}{72} mR^2$$

1.242 Moment of inertia of the shaded portion, about the axis passing through it's certre,

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3 \rho\right) r^3$$

$$= \frac{2}{5}\frac{4}{3}\pi\rho\left(R^5 - r^5\right)$$



Now, if $R = r + dr$, the shaded portion becomes a shell, which is the required shape to calculate the moment of inertia.

Now, $$I = \frac{2}{5} - \frac{4}{3}\pi\rho\left\{(r + dr)^5 - r^5\right\}$$

$$= \frac{2}{5}\frac{4}{3}\pi\rho\left(r^5 + 5r^4\, dr + \ldots\ldots - r^5\right)$$

Neglecting higher terms.

$$= \frac{2}{3}\left(4\pi r^2\, dr\rho\right) r^2 = \frac{2}{3} mr^2$$

1.243 (a) Net force which is effective on the system (cylinder M + body m) is the weight of the body m in a uniform gravitational field, which is a constant. Thus the initial acceleration of the body m is also constant.

From the conservation of mechanical energy of the said system in the uniform field of gravity at time $t = \Delta t$: $\Delta T + \Delta U = 0$

or $$\frac{1}{2} mv^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 - mg\,\Delta h = 0$$

or, $$\frac{1}{4}(2m + M)v^2 - mg\,\Delta h = 0 \;\; [\text{ as } v = \omega R \text{ at all times }] \quad (1)$$

But $$v^2 = 2w\,\Delta h$$

Hence using it in Eq. (1), we get

$$\frac{1}{4}(2m + M)\, 2w\,\Delta h - mg\,\Delta h = 0 \text{ or } w = \frac{2mg}{(2m + M)}$$

From the kinematical relationship, $\beta = \dfrac{w}{R} = \dfrac{2mg}{(2m + M)R}$

Thus the sought angular velocity of the cylinder

$$\omega(t) = \beta t = \frac{2mg}{(2m + M)R}\, t = \frac{gt}{(1 + M/2m)R}$$

(b) Sought kinetic energy.

$$T(t) = \frac{1}{2} mv^2 + \frac{1}{2}\frac{Ml^2}{2}\omega^2 = \frac{1}{4}(2m + M)R^2\omega^2$$

1.244 For equilibrium of the disc and axle

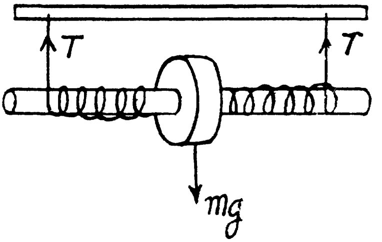
$$2T = mg \text{ or } T = mg/2$$

As the disc unwinds, it has an angular acceleration β given by

$$I\beta = 2Tr \text{ or } \beta = \frac{2Tr}{I} = \frac{mgr}{I}$$

The corresponding linear acceleration is

$$r\beta = w = \frac{mgr^2}{I}$$



Since the disc remains stationary under the combined action of this acceleration and the acceleration $(-w)$ of the bar which is transmitted to the axle, we must have

$$w = \frac{mgr^2}{I}$$

1.245 Let the rod be deviated through an angle φ from its initial position at an arbitrary instant of time, measured relative to the initial position in the positive direction. From the equation of the increment of the mechanical energy of the system.

$$\Delta T = A_{ext}$$

or,
$$\frac{1}{2} I\omega^2 = \int N_z \, d\varphi$$

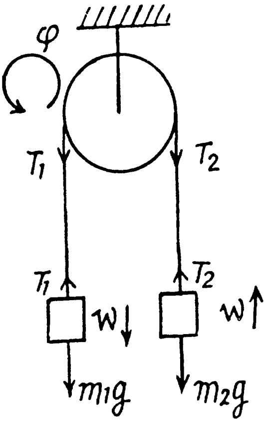
or,
$$\frac{1}{2}\frac{Ml^2}{3}\omega^2 = \int_0^{\varphi} Fl\cos\varphi \, d\varphi = Fl\sin\varphi$$

Thus,
$$\omega = \sqrt{\frac{6F\sin\varphi}{Ml}}$$

1.246 First of all, let us sketch free body diagram of each body. Since the cylinder is rotating and massive, the tension will be different in both the sections of threads. From Newton's law in projection form for the bodies m_1 and m_2 and noting that $w_1 = w_2 = w = \beta R$, (as no thread slipping), we have $(m_1 > m_2)$

$$m_1 g - T_1 = m_1 w = m_1 \beta R$$

and $T_2 - m_2 g = m_2 w$ (1)



Now from the equation of rotational dynamics of a solid about stationary axis of rotation. i.e. $N_z = I\beta_z$, for the cylinder.

or, $(T_1 - T_2)R = I\beta = mR^2\beta/2$ (2)

Similtaneous solution of the above equations yields :

$$\beta = \frac{(m_1 - m_2)g}{R\left(m_1 + m_2 + \frac{m}{2}\right)} \text{ and } \frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

1.247 As the system $(m + m_1 + m_2)$ is under constant forces, the acceleration of body m_1 an m_2 is constant. In addition to it the velocities and accelerations of bodies m_1 and m_2 ai equal in magnitude (say v and w) because the length of the thread is constant.
From the equation of increament of mechanical energy i.e. $\Delta T + \Delta U = A_{fr}$, at time t whe block m_1 is distance h below from initial position corresponding to $t = 0$,

$$\frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} - m_2 gh = -km_1 gh \qquad (1$$

(as angular velocity $\omega = v/R$ for no slipping of thread.)

But $$v^2 = 2wh$$

So using it in (1), we get

$$w = \frac{2(m_2 - km_1)g}{m + 2(m_1 + m_2)} \qquad (2$$

Thus the work done by the friction force on m_1

$$A_{fr} = -km_1 gh = -km_1 g\left(\frac{1}{2}wt^2\right)$$

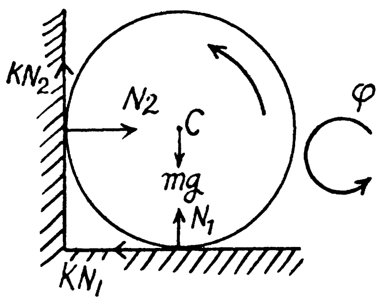
$$= -\frac{km_1(m_1 - km_1)g^2 t^2}{m + 2(m_1 + m_2)} \qquad \text{(using 2).}$$

1.248 In the problem, the rigid body is in translation equlibrium but there is an angular retardation. We first sketch the free body diagram of the cylinder. Obviously the friction forces, acting on the cylinder, are kinetic. From the condition of translational equlibrium for the cylinder,

$$mg = N_1 + kN_2; \quad N_2 = kN_1$$

Hence, $$N_1 = \frac{mg}{1+k^2};\ N_2 = k\frac{mg}{1+k^2}$$

For pure rotation of the cylinder about its rotation axis, $N_z = I\beta_z$



or, $$-kN_1 R - kN_2 R = \frac{mR^2}{2}\beta_z$$

or, $$-\frac{kmgR(1+k)}{1+k^2} = \frac{mR^2}{2}\beta_z$$

or, $$\beta_z = -\frac{2k(1+k)g}{(1+k^2)R}$$

Now, from the kinematical equation,

$$\omega^2 = \omega_0^2 + 2\beta_z \Delta\varphi \text{ we have,}$$

$$\Delta\varphi = \frac{\omega_0^2(1+k^2)R}{4k(1+k)g}, \qquad \text{because } \omega = 0$$

Hence, the sought number of turns,

$$n = \frac{\Delta\varphi}{2\pi} = \frac{\omega_0^2 (1 + k^2) R}{8\pi k (1 + k) g}$$

1.249 It is the moment of friction force which brings the disc to rest. The force of friction is applied to each section of the disc, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section.

To find N_z, where z is the axis of rotation of the disc let us partition the disc into thin rings (Fig.). The force of friction acting on the considered element

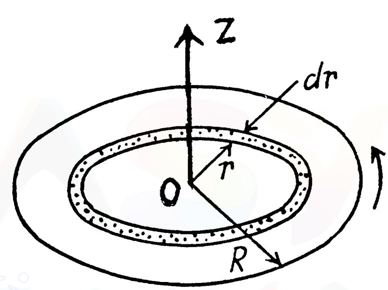
$dfr = k(2\pi r\, dr\, \sigma) g$, (where σ is the density of the disc)

The moment of this force of friction is

$$dN_z = -r\, dfr = -2\pi k \sigma g r^2 dr$$

Integrating with respect to r from zero to R, we get

$$N_z = -2\pi k \sigma g \int_0^R r^2 dr = -\frac{2}{3}\pi k \sigma g R^3.$$



For the rotation of the disc about the stationary axis z, from the equation $N_z = I\beta_z$

$$-\frac{2}{3}\pi k \sigma g R^3 = \frac{(\pi R^2 \sigma) R^2}{2} \beta_z \quad \text{or} \quad \beta_z = -\frac{4kg}{3R}$$

Thus from the angular kinematical equation $\omega_z = \omega_{0z} + \beta_z t$

$$0 = \omega_0 + \left(-\frac{4kg}{3R}\right) t \quad \text{or} \quad t = \frac{3R\,\omega_0}{4kg}$$

1.250 According to the question,

$$I\frac{d\omega}{dt} = -k\sqrt{\omega} \quad \text{or,} \quad I\frac{d\omega}{\sqrt{\omega}} = -k\,dt$$

Integrating, $\sqrt{\omega} = -\dfrac{kt}{2I} + \sqrt{\omega_0}$

or, $\omega = \dfrac{k^2 t^2}{4I^2} - \dfrac{\sqrt{\omega_0}\, kt}{I} + \omega_0$, (Noting that at $t = 0$, $\omega = \omega_0$.)

Let the flywheel stops at $t = t_0$ then from Eq. (1), $t_0 = \dfrac{2I\sqrt{\omega_0}}{k}$

Hence sought average angular velocity

$$\langle \omega \rangle = \frac{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}} \left(\frac{k^2 t^2}{4I^2} - \frac{\sqrt{\omega_0}\, kt}{I} + \omega_0\right) dt}{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

1.251 Let us use the equation $\frac{dM_z}{dt} = N_z$ relative to the axis through O (1)

For this purpose, let us find the angular momentum of the system M_z about the given rotation axis and the corresponding torque N_z. The angular momentum is

$$M_z = I\omega + mvR = \left(\frac{m_0}{2} + m\right) R^2 \omega$$

[where $I = \frac{m_0}{2} R^2$ and $v = \omega R$ (no cord slipping)]

So, $$\frac{dM_z}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\beta_z \quad (2)$$

The downward pull of gravity on the overhanging part is the only external force, which exerts a torque about the z-axis, passing through O and is given by,

$$N_z = \left(\frac{m}{l}\right) xgR$$

Hence from the equation $\frac{dM_z}{dt} = N_z$

$$\left(\frac{MR^2}{2} + mR^2\right)\beta_z = \frac{m}{l} xgR$$

Thus, $$\beta_z = \frac{2mgx}{lR(M + 2m)} > 0$$

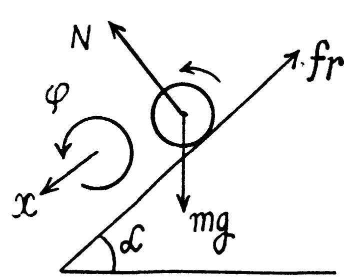
Note : *We may solve this problem using conservation of mechanical energy of the system (cylinder + thread) in the uniform field of gravity.*

1.252 (a) Let us indicate the forces acting on the sphere and their points of application. Choose positive direction of x and φ (rotation angle) along the incline in downward direction and in the sense of $\vec{\omega}$ (for undirectional rotation) respectively. Now from equations of dynamics of rigid body i.e. $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ we get :

$$mg \sin\alpha - f_r = mw \quad (1)$$

and $$f_r R = \frac{2}{5} mR^2 \beta \quad (2)$$

But $$f_r \le kmg \cos\alpha \quad (3)$$



In addition, the absence of slipping provides the kinematical realtionship between the accelerations :

$$w = \beta R \quad (4)$$

The simultaneous solution of all the four equations yields :

$$k \cos\alpha \ge \frac{2}{7} \sin\alpha, \text{ or } k \ge \frac{2}{7} \tan\alpha$$

(b) Solving Eqs. (1) and (2) [of part (a)], we get :

$$w_c = \frac{5}{7} g \sin \alpha.$$

As the sphere starts at $t = 0$ along positive x axis, for pure rolling

$$v_c(t) = w_c t = \frac{5}{7} g \sin \alpha\, t \qquad (5)$$

Hence the sought kinetic energy

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2}\frac{2}{5} mR^2 \omega^2 = \frac{7}{10} m v_c^2 \text{ (as } \omega = v_c/R\text{)}$$

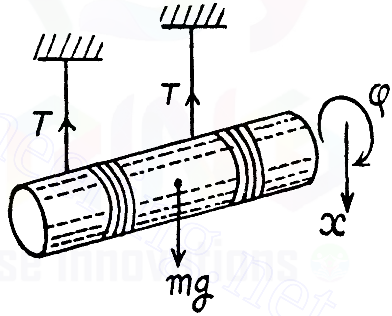
$$= \frac{7}{10} m \left(\frac{5}{7} g \sin \alpha\, t\right)^2 = \frac{5}{14} m g^2 \sin^2 \alpha\, t^2$$

1.253 (a) Let us indicate the forces and their points of application for the cylinder. Choosing the positive direction for x and φ as shown in the figure, we write the equation of motion of the cylinder axis and the equation of moments in the C.M. frame relative to that axis i.e. from equation $F_x = m w_c$ and $N_z = I_c \beta_z$.

$$mg - 2T = m w_c ; \; 2TR = \frac{mR^2}{2} \beta$$

As there is no slipping of thread on the cylinder

$$w_c = \beta R$$



From these three equations

$$T = \frac{mg}{6} = 13 \text{ N}, \; \beta = \frac{2}{5}\frac{g}{R} = 5 \times 10^2 \text{ rad/s}^2$$

(b) we have $\beta = \frac{2}{3}\frac{g}{R}$

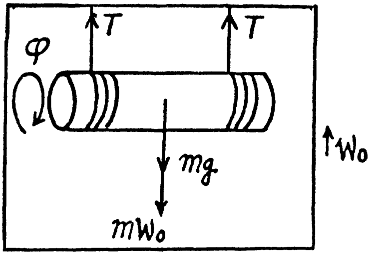
So, $w_c = \frac{2}{3} g > 0$ or, in vector form $\vec{w}_c = \frac{2}{3}\vec{g}$

$$P = \vec{F} \cdot \vec{v} = \vec{F} \cdot (\vec{w}_c t)$$

$$= m\vec{g} \cdot \left(\frac{2}{3}\vec{g} t\right) = \frac{2}{3} m g^2 t$$

1.254 Let us depict the forces and their points of application corresponding to the cylinder attached with the elevator. Newton's second law for solid in vector form in the frame of elevator, gives :

$$2\vec{T} + m\vec{g} + m(-\vec{w}_0) = m\vec{w}' \qquad (1)$$



The equation of moment in the C.M. frame relative to the cylinder axis i.e. from $N_z = I_c \beta_z$ ·

$$2TR = \frac{mR^2}{2} \beta = \frac{mR^2}{2} \frac{w'}{R}$$

[as thread does not slip on the cylinder, $w' = \beta R$]

or, $$T = \frac{mw'}{4}$$

As (1) $\vec{T} \uparrow\downarrow \vec{w}$

so in vector form

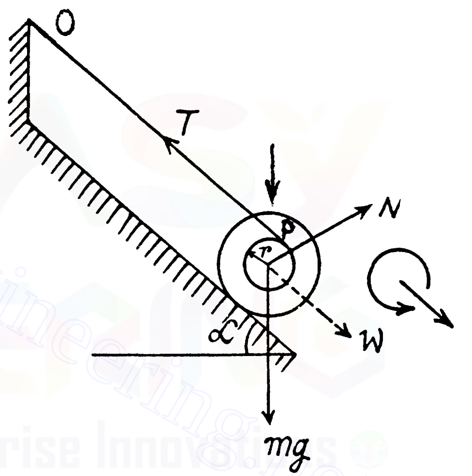
$$\vec{T} = -\frac{m\vec{w}}{4} \tag{2}$$

Solving Eqs. (1) and (2), $\vec{w}' = \frac{2}{3}(\vec{g} - \vec{w}_0)$ and sought force

$$\vec{F} = 2\vec{T} = \frac{1}{3}m(\vec{g} - \vec{w}_0).$$

1.255 Let us depict the forces and their points of application for the spool. Choosing the positive direction for x and φ as shown in the fig., we apply $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ and get

$$mg \sin\alpha - T = mw;\ Tr = I\beta$$



"Notice that if a point of a solid in plane motion is connected with a thread, the projection of velocity vector of the solid's point of contact along the length of the thread equals the velocity of the other end of the thread (if it is not slacked)"

Thus in our problem, $v_p = v_0$ but $v_0 = 0$, hence point P is the instantaneous centre of rotation of zero velocity for the spool. Therefore $v_c = \omega r$ and subsequently $w_c = \beta r$.

Solving the equations simultaneously, we get

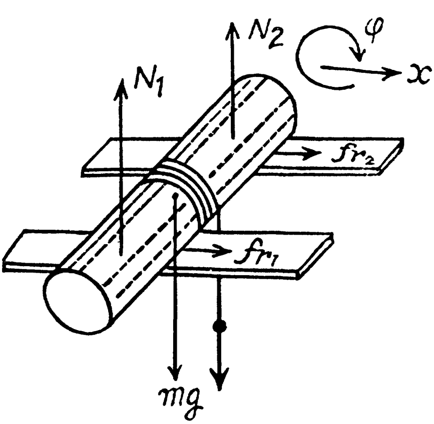
$$w = \frac{g \sin\alpha}{1 + \frac{I}{mr^2}} = 1{\cdot}6\ \text{m/s}^2$$

1.256 Let us sketch the force diagram for solid cylinder and apply Newton's second law in projection form along x and y axes (Fig.) :

$$fr_1 + fr_2 = mw_c \tag{1}$$

and $$N_1 + N_2 - mg - F = 0$$

or $$N_1 + N_2 = mg + F \tag{2}$$



Now choosing positive direction of φ as shown in the figure and using $N_{cz} = I_c \beta_z$, we get

$$FR - (fr_1 + fr_2)R = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w_c}{R} \tag{3}$$

[as for pure rolling $w_c = \beta R$]. In addition to,

$$fr_1 + fr_2 \le k(N_1 + N_2) \tag{4}$$

Solving the Eqs., we get

$$F \le \frac{3\,kmg}{(2-3k)}, \quad \text{or} \quad F_{max} = \frac{3\,k\,mg}{2-3\,k}$$

and
$$w_{c\,(max)} = \frac{k\,(N_1+N_2)}{m}$$

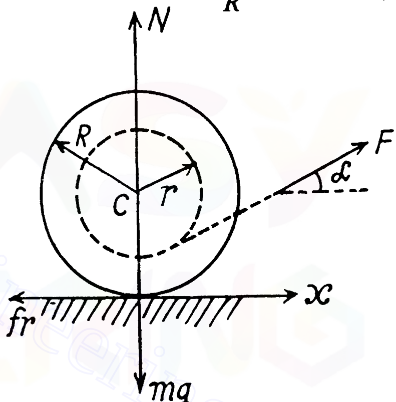
$$= \frac{k}{m}[mg + F_{max}] = \frac{k}{m}\left[mg + \frac{3\,kmg}{2-3k}\right] = \frac{2\,kg}{2-3k}$$

1.257 (a) Let us choose the positive direction of the rotation angle φ, such that w_{cx} and β_z have identical signs (Fig.). Equation of motion, $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$ gives :

$$F\cos\alpha - fr = mw_{cx} : fr\,R - Fr = I_c\,\beta_z = \gamma m R^2\,\beta_z$$

In the absence of the slipping of the spool $w_{cx} = \beta_z R$

From the three equations $w_{cx} = w_c = \dfrac{F\,[\cos\alpha - (r/R)\,]}{m\,(1+\gamma)}$, where $\cos\alpha > \dfrac{r}{R}$ (1)



(b) As static friction (fr) does not work on the spool, from the equation of the increment of mechanical energy $A_{ext} = \Delta T$.

$$A_{ext} = \frac{1}{2}mv_c^2 + \frac{1}{2}\gamma\,mR^2\,\frac{v_c^2}{R^2} = \frac{1}{2}m\,(1+\gamma)\,v_c^2$$

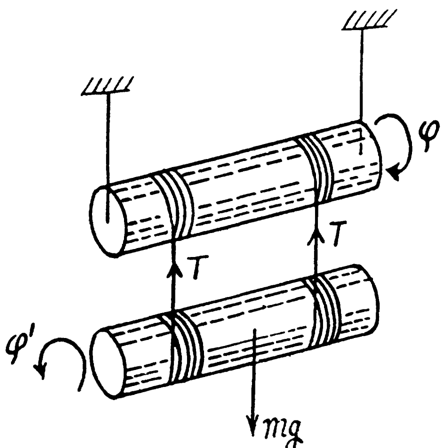
$$= \frac{1}{2}m\,(1+\gamma)\,2w_c x = \frac{1}{2}m\,(1+\gamma)\,2\,w_c\left(\frac{1}{2}w_c t^2\right)$$

$$= \frac{F^2\left(\cos\alpha - \dfrac{r}{R}\right)^2 t^2}{2\;m\,(1+\gamma)}$$

Note that at $\cos\alpha = r/R$, *there is no rolling and for* $\cos\alpha < r/R$, $w_{cx} < 0$, *i.e. the spool will move towards negative x-axis and rotate in anticlockwise sense.*

1.258 For the cylinder from the equation $N_z = I\,\beta_z$ about its stationary axis of rotation.

$$2\,Tr = \frac{mr^2}{2}\beta \quad \text{or} \quad \beta = \frac{4T}{mr} \qquad (1)$$



For the rotation of the lower cylinder from the equation $N_{cz} = I_c\,\beta_z$

$$2Tr = \frac{mr^2}{2}\beta' \quad \text{or,} \quad \beta' = \frac{4T}{mr} = \beta$$

Now for the translational motion of lower cylinder from the Eq. $F_x = mw_{cx}$:

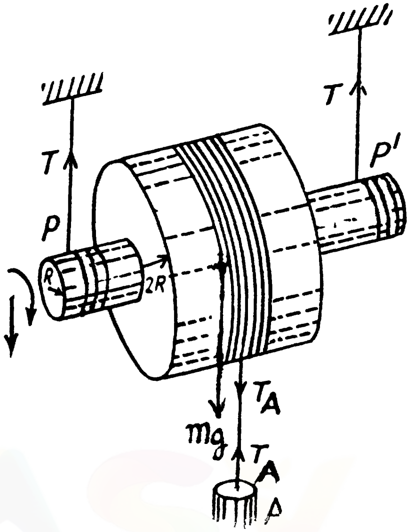
$$mg - 2\,T = mw_c \qquad (2)$$

As there is no slipping of threads on the cylinders :

$$w_c = \beta' r + \beta r = 2\,\beta r \qquad (3)$$

Simultaneous solution of (1), (2) and (3) yields

$$T = \frac{mg}{10}.$$



1.259 Let us depict the forces acting on the pulley and weight A, and indicate positive direction for x and φ as shown in the figure. For the cylinder from the equation $F_x = m\, w_x$ and $N_{cz} = I_c\, \beta_z$, we get

$$Mg + T_A - 2T = M w_c \qquad (1)$$

and $$2TR + T_A(2R) = I\beta = \frac{I w_c}{R} \qquad (2)$$

For the weight A from the equation

$$F_x = mw_x$$

$$mg - T_A = mw_A \qquad (3)$$

As there is no slipping of the threads on the pulleys.

$$w_A = w_c + 2\beta R = w_c + 2w_c = 3w_c \qquad (4)$$

Simultaneous solutions of above four equations gives :

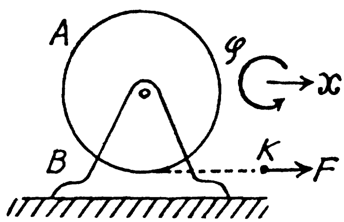
$$w_A = \frac{3(M + 3m)g}{\left(M + 9m + \frac{I}{R^2}\right)}$$

1.260 (a) For the translational motion of the system $(m_1 + m_2)$, from the equation : $F_x = mw_{cx}$

$$F = (m_1 + m_2)\, w_c \quad \text{or,} \quad w_c = F/(m_1 + m_2) \qquad (1)$$

Now for the rotational motion of cylinder from the equation : $N_{cx} = I_c\, \beta_z$

$$Fr = \frac{m_1 r^2}{2}\beta \quad \text{or} \quad \beta r = \frac{2F}{m_1} \qquad (2)$$



But $\qquad w_K = w_c + \beta r$, So

$$w_K = \frac{F}{m_1 + m_2} + \frac{2F}{m_1} = \frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)} \qquad (3)$$

(b) From the equation of increment of mechanical energy : $\Delta T = A_{ext}$

Here $\qquad \Delta T = T(t)$, so, $T(t) = A_{ext}$

As force F is constant and is directed along x-axis the sought work done.

$$A_{ext} = Fx$$

(where x is the displacement of the point of application of the force F during time interval t)

$$= F\left(\frac{1}{2} w_K t^2\right) = \frac{F^2 t^2 (3 m_1 + 2 m_2)}{2 m_1 (m_1 + m_2)} = T(t)$$

(using Eq. (3)

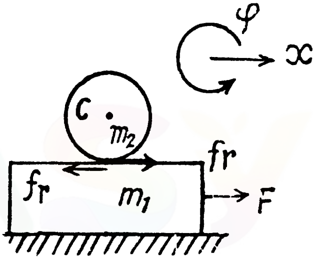
Alternate : $T(t) = T_{translation}(t) + T_{rotation}(t)$

$$= \frac{1}{2}(m_1 + m_2)\left(\frac{Ft}{(m_1 + m_2)}\right)^2 + \frac{1}{2}\frac{m_1 r^2}{2}\left(\frac{2Ft}{m_1 r}\right)^2 = \frac{F^2 t^2 (3 m_1 + 2 m_2)}{2 m_1 (m_1 + m_2)}$$

1.261 Choosing the positive direction for x and φ as shown in Fig, let us we write the equation of motion for the sphere $F_x = m w_{cx}$ and $N_{cz} = I_c \beta_z$

$$fr = m_2 w_2 ;\ fr\, r = \frac{2}{5} m_2 r^2 \beta$$

(w_2 is the acceleration of the C.M. of sphere.)



For the plank from the Eq. $F_x = m w_x$

$$F - f_r = m_1 w_1$$

In addition, the condition for the absence of slipping of the sphere yields the kinematical relation between the accelerations :

$$w_1 = w_2 + \beta r$$

Simultaneous solution of the four equations yields :

$$w_1 = \frac{F}{\left(m_1 + \frac{2}{7} m_2\right)} \quad \text{and} \quad w_2 = \frac{2}{7} w_1$$

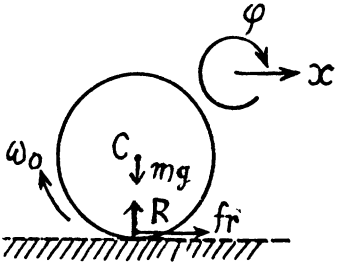
1.262 (a) Let us depict the forces acting on the cylinder and their point of applications for the cylinder and indicate positive direction of x and φ as shown in the figure. From the equations for the plane motion of a solid $F_x = m w_{cx}$ and $N_{cz} = I_c \beta_z$:

$$k\, mg = m\, w_{cx} \quad \text{or} \quad w_{cx} = kg \qquad (1)$$

$$- kmg\, R = \frac{mR^2}{2} \beta^z \quad \text{or} \quad \beta_z = -2\frac{kg}{R} \qquad (2)$$

Let the cylinder starts pure rolling at $t = t_0$ after releasing on the horizontal floor at $t = 0$.

From the angular kinematical equation



$$\omega_z = \omega_{oz} + \beta_z t,$$

or $$\omega = \omega_0 - 2\frac{kg}{R} t \qquad (3)$$

From the equation of the linear kinematics,

$$v_{cx} = v_{ocx} + w_{cx} t$$

or $$v_c = 0 + kg\, t_0 \qquad (4)$$

But at the moment $t = t_0$, when pure rolling starts $v_c = \omega R$

so, $$kg\, t_0 = \left(\omega_0 - 2\frac{kg}{R} t_0\right) R$$

Thus $$t_0 = \frac{\omega_0 R}{3\, kg}$$

(b) As the cylinder pick, up speed till it starts rolling, the point of contact has a purely translatory movement equal to $\frac{1}{2} w_c\ t_0^2$ in the forward directions but there is also a backward movement of the point of contact of magnitude $(\omega_0 \tau_0 - \frac{1}{2} \beta\, t_0^2) R$. Because of slipping the net displacement is backwards. The total work done is then,

$$A_{fr} = kmg\left[\frac{1}{2} w_c\ t_0^2 - (\omega_0 t_0 + \frac{1}{2}\beta t_0^2) R\right]$$

$$= kmg\left[\frac{1}{2} kg\, t_0^2 - \frac{1}{2}\left(-\frac{2kg}{R}\right) t_0^2\, R - \omega_0\, t_0\, R\right]$$

$$= kmg\, \frac{\omega_0 R}{3kg}\left[\frac{\omega_0 R}{6} + \frac{\omega_0 R}{3} - \omega_0 R\right] = -\frac{m\omega_0^2 R^2}{6}$$

The same result can also be obtained by the work-energy theorem, $A_{fr} = \Delta T$.

1.263 Let us write the equation of motion for the centre of the sphere at the moment of breaking-off:

$$mv^2/(R + r) = mg\cos\theta,$$

where v is the velocity of the centre of the sphere at that moment, and θ is the corresponding angle (Fig.). The velocity v can be found from the energy conservation law :

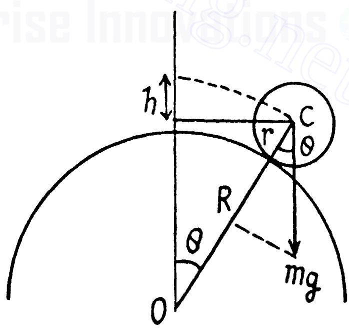
$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2,$$

where I is the moment of inertia of the sphere relative to the axis passing through the sphere's centre. i.e. $I = \frac{2}{5} mr^2$. In addition,

$$v = \omega r\ ;\ h = (R + r)(1 - \cos\theta).$$

From these four equations we obtain

$$\omega = \sqrt{10\, g\, (R + r)\, 17\, r^2}.$$



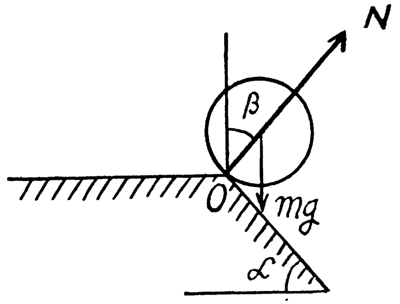
1.264 Since the cylinder moves without sliding, the centre of the cylinder rotates about the point O, while passing through the common edge of the planes. In other words, the point O becomes the foot of the instantaneous axis of rotation of the cylinder.

It at any instant during this motion the velocity of the C.M. is v_1 when the angle (shown in the figure) is β, we have

$$\frac{m\, v_1^2}{R} = mg\cos\beta - N,$$

where N is the normal reaction of the edge

or, $$v_1^2 = gR\cos\beta - \frac{NR}{m} \qquad (1)$$



From the energy conservation law,

$$\frac{1}{2}I_0\frac{v_1^2}{R^2} - \frac{1}{2}I_0\frac{v_0^2}{R^2} = mgR(1-\cos\beta)$$

But $$I_0 = \frac{mR^2}{2} + mR^2 = \frac{3}{2}mR^2,$$

(from the parallel axis theorem)

Thus, $$v_1^2 = v_0^2 + \frac{4}{3}gR(1-\cos\beta) \qquad (2)$$

From (1) and (2)

$$v_0^2 = \frac{gR}{3}(7\cos\beta - 4) - \frac{NR}{m}$$

The angle β in this equation is clearly smaller than or equal to α so putting $\beta = \alpha$ we get

$$v_0^2 = \frac{gR}{3}(7\cos\alpha - 4) - \frac{N_0R}{M}$$

where N_0 is the corresponding reaction. Note that $N \geq N_0$. No jumping occurs during this turning if $N_0 > 0$. Hence, v_0 must be less than

$$v_{max} = \sqrt{\frac{gR}{3}(7\cos\alpha - 4)}$$

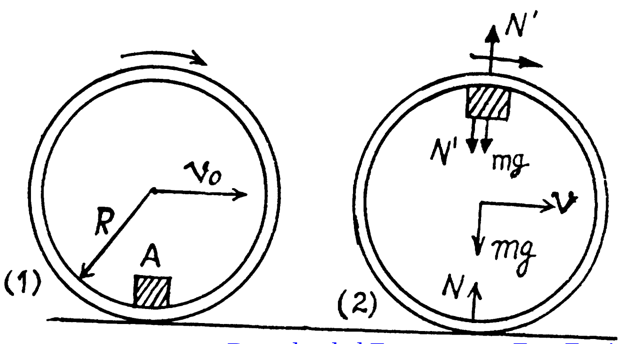
1.265 Clearly the tendency of bouncing of the hoop will be maximum when the small body A, will be at the highest point of the hoop during its rolling motion. Let the velocity of C.M. of the hoop equal v at this position. The static friction does no work on the hoop, so from conservation of mechanical energy; $E_1 = E_2$

or, $$0 + \frac{1}{2}mv_0^2 + \frac{1}{2}mR^2\left(\frac{v_0}{R}\right)^2 - mgR = \frac{1}{2}m(2v)^2 + \frac{1}{2}mv^2 + \frac{1}{2}mR^2\left(\frac{v}{R}\right)^2 + mgR$$

or, $$3v^2 = v_0^2 - 2gR \qquad (1)$$

From the equation $F_n = mw_n$ for body A at final position 2 :

$$mg + N' = m\omega^2R = m\left(\frac{v}{R}\right)^2 R \qquad (2)$$



As the hoop has no acceleration in vertical direction, so for the hoop,

$$N + N' = mg \tag{3}$$

From Eqs. (2) and (3),

$$N = 2\,mg - \frac{m\,v^2}{R} \tag{4}$$

As the hoop does not bounce, $N \geq 0$ (5)

So from Eqs. (1), (4) and (5),

$$\frac{8\,g\,R - v_0^2}{3\,R} \geq 0 \quad \text{or} \quad 8\,g\,R \geq v_0^2$$

Hence $\qquad v_0 \leq \sqrt{8\,g\,R}$

1.266 Since the lower part of the belt is in contact with the rigid floor, velocity of this part becomes zero. The crawler moves with velocity v, hence the velocity of upper part of the belt becomes $2v$ by the rolling condition and kinetic energy of upper part $= \frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = mv^2$, which is also the sought kinetic energy, assuming that the length of the belt is much larger than the radius of the wheels.

1.267 The sphere has two types of motion, one is the rotation about its own axis and the other is motion in a circle of radius R. Hence the sought kinetic energy

$$T = \frac{1}{2} I_1\,\omega_1^2 + \frac{1}{2} I_2\,\omega_2^2 \tag{1}$$

where I_1 is the moment of inertia about its own axis, and I_2 is the moment of inertia about the vertical axis, passing through O,

But, $I_1 = \frac{2}{5} mr^2$ and $I_2 = \frac{2}{5} mr^2 + mR^2$ (using parallel axis theorem,) (2)

In addition to

$$\omega_1 = \frac{v}{r} \text{ and } \omega_2 = \frac{v}{R} \tag{3}$$

Using (2) and (3) in (1), we get $T' = \frac{7}{10}\,mv^2\left(1 + \frac{2r^2}{7R^2}\right)$

1.268 For a point mass of mass dm, looked at from C rotating frame, the equation is

$$dm\,\vec{w}' = \vec{f} + dm\,\omega^2\,\vec{r}' + 2\,dm\,(\vec{v}' \times \vec{\omega})$$

where $\vec{r}'$ = radius vector in the rotating frame with respect to rotation axis and $\vec{v}'$ = velocity in the same frame. The total centrifugal force is clearly

$$\vec{F}_{cf} = \sum dm\,\omega^2\,\vec{r}' = m\omega^2\,\vec{R}_c$$

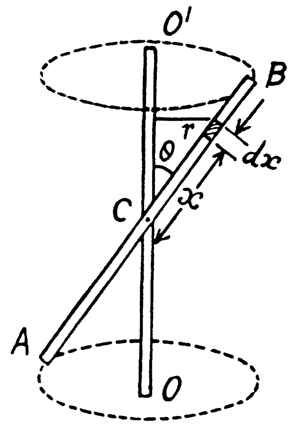
$\vec{R}_c$ is the radius vector of the C.M. of the body with respect to rotation axis, also

$$\vec{F}_{cor} = 2m\,\vec{v}_c' \times \vec{\omega}$$

where we have used the definitions

$$m\,\vec{R}_c = \sum dm\,\vec{r}' \text{ and } m\,\vec{v}_c' = \sum dm\,\vec{v}'$$

1.269 Consider a small element of length dx at a distance x from the point C, which is rotating in a circle of radius $r = x \sin\theta$



Now, mass of the element $= \left(\frac{m}{l}\right) dx$

So, centrifugal force acting on this element $= \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta$ and moment of this force about C,

$$|dN| = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta \cdot x \cos\theta$$

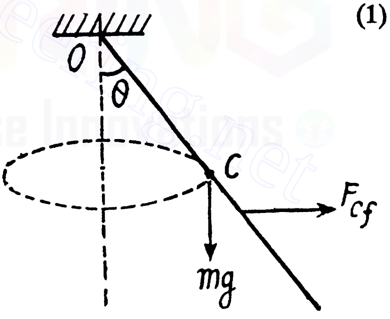
$$= \frac{m\,\omega^2}{2l} \sin 2\theta\, x^2\, dx$$

and hence, total moment

$$N = 2\int_0^{l/2} \frac{m\,\omega^2}{2l} \sin 2\theta\; x^2\, dx = \frac{1}{24} m\,\omega^2\, l^2 \sin 2\theta,$$

1.270 Let us consider the system in a frame rotating with the rod. In this frame, the rod is at rest and experiences not only the gravitational force $m\vec{g}$ and the reaction force $\vec{R}$, but also the centrifugal force $\vec{F}_{cf}$.

In the considered frame, from the condition of equilibrium i.e. $N_{0z} = 0$



or, $$N_{cf} = mg\,\frac{l}{2} \sin\theta \qquad (1)$$

where N_{cf} is the moment of centrifugal force about O. To calculate N_{cf}, let us consider an element of length dx, situated at a distance x from the point O. This element is subjected to a horizontal pseudo force $\left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta$.

The moment of this pseudo force about the axis of rotation through the point O is

$$dN_{cf} = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin\theta\, x \cos\theta$$

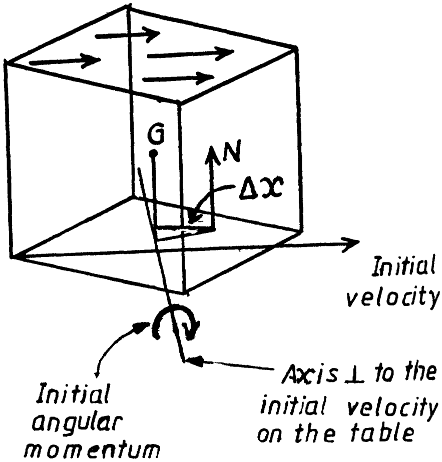
$$= \frac{m\,\omega^2}{l} \sin\theta \cos\theta\; x^2\, dx$$

So $$N_{cf} = \int_0^{l} \frac{m\,\omega^2}{l} \sin\theta\cos\theta\, x^2\, dx = \frac{m\,\omega^2\, l^2}{3} \sin\theta\cos\theta \qquad (2)$$

It follows from Eqs. (1) and (2) that,

$$\cos\theta = \left(\frac{3g}{2\,\omega^2 l}\right) \text{ or } \theta = \cos^{-1}\left(\frac{3g}{2\,\omega^2 l}\right) \qquad (3)$$

1.271 When the cube is given an initial velocity on the table in some direction (as shown) it acquires an angular momentum about an axis on the table perpendicular to the initial velocity and (say) just below the C.G.. This angular momentum will disappear when the cube stops and this can only by due to a torque. Frictional forces cannot do this by themselves because they act in the plain containing the axis. But if the force of normal reaction act eccentrically (as shown), their torque can bring about the vanishing of the angular momentum. We can calculate the distance Δx between the point of application of the normal reaction and the C.G. of the cube as follows. Take the moment about C.G. of all the forces. This must vanish because the cube does not turn or tumble on the table. Then if the force of friction is fr



$$fr\,\frac{a}{2} = N\,\Delta x$$

But $N = mg$ and $fr = kmg$, so

$$\Delta x = ka/2$$

1.272 In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence, it follows that

$$\frac{1}{2}\frac{Ml^2}{3}\omega_0^2 = \frac{1}{2}m(\omega^2 l^2 + v'^2) + \frac{1}{2}\frac{Ml^2}{3}\omega^2$$

(ω is the final angular velocity of the rod)

and $$\frac{Ml^2}{3}\omega_0 = \frac{Ml^2}{3}\omega + ml^2\omega$$

From these equations we obtain

$$\omega = \omega_0 \Big/ \left(1 + \frac{3M}{M}\right) \text{ and}$$

$$v' = \omega_0 l \Big/ \sqrt{1 + 3m/M}$$

1.273 Due to hitting of the ball, the angular impulse received by the rod about the C.M. is equal to $p\frac{1}{2}$. If ω is the angular velocity acquired by the rod, we have

$$\frac{ml^2}{12}\omega = \frac{pl}{2} \quad \text{or} \quad \omega = \frac{6p}{ml} \qquad (1)$$

In the frame of C.M., the rod is rotating about an axis passing through its mid point with the angular velocity ω. Hence the force exerted by one half on the other = mass of one half × acceleration of C.M. of that part, in the frame of C.M.

$$= \frac{m}{2}\left(\omega^2\frac{l}{4}\right) = m\frac{\omega^2 l}{8} = \frac{9p^2}{2ml} = 9\text{ N}$$

1.274 (a) In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence it follows that

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2$$

and $$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{Ml^2}{3}\omega$$

From these equations we obtain

$$v' = \left(\frac{3m-4M}{3m+4M}\right)v. \quad \text{and} \quad \omega = \frac{4v}{l\,(1+4\,m/3M)}$$

As $\vec{v}' \uparrow\uparrow \vec{v}$, so in vector form $\vec{v}' = \left(\frac{3m-4M}{3m+4M}\right)\vec{v}$

(b) Obviously the sought force provides the centripetal acceleration to the C.M. of the rod and is

$$F_n = mw_{cn}$$

$$= M\omega^2\frac{l}{2} = \frac{8Mv^2}{l\,(1+4M/3m)^2}$$

1.275 (a) About the axis of rotation of the rod, the angular momentum of the system is conserved. Thus if the velocity of the flying bullet is v.

$$mvl = \left(ml^2 + \frac{Ml^2}{3}\right)\omega$$

$$\omega = \frac{mv}{\left(m+\frac{M}{3}\right)l} \approx \frac{3mv}{Ml} \quad \text{as} \quad m << M \qquad (1)$$

Now from the conservation of mechanical energy of the system (rod with bullet) in the uniform field of gravity

$$\frac{1}{2}\left(ml^2 + \frac{Ml^2}{3}\right)\omega^2 = (M+m)\,g\,\frac{l}{2}(1-\cos\alpha) \qquad (2)$$

[because C.M. of rod raises by the height $\frac{l}{2}(1-\cos\alpha)$]

Solving (1) and (2), we get

$$v = \left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\,\sin\frac{\alpha}{2} \quad \text{and} \quad \omega = \sqrt{\frac{6g}{l}}\,\sin\frac{\alpha}{2}$$

(b) Sought $\Delta p = \left[m(\omega l) + M\left(\omega\frac{l}{2}\right)\right] - mv$

where ωl is the veloccity of the bullet and $\omega\frac{l}{2}$ equals the velocity of C.M. of the rod after the impact. Putting the value of v and ω we get

$$\Delta p \approx \frac{1}{2}mv = M\sqrt{\frac{gl}{6}}\,\sin\frac{\alpha}{2}$$

This is caused by the reaction at the hinge on the upper end.

(c) Let the rod starts swinging with angular velocity ω', in this case. Then, like part (a)

$$mvx = \left(\frac{Ml^2}{3} + mx^2\right)\omega' \quad \text{or} \quad \omega' \approx \frac{3mvx}{Ml^2}$$

Final momentum is

$$p_f = m\,x\,\omega' + \int_0^l y\,\omega' \frac{M}{l}\,dy \approx \frac{M}{2}\omega'\,l \approx \frac{3}{2}m\,v\frac{x}{l}$$

So, $$\Delta p = p_f - p_i \approx m\,v\left(\frac{3\,x}{2l} - 1\right)$$

This vanishes for $$x \approx \frac{2}{3}l$$

1.276 (a) As force F on the body is radial so its angular momentum about the axis becomes zero and the angular momentum of the system about the given axis is conserved. Thus

$$\frac{MR^2}{2}\omega_0 + m\,\omega_0 R^2 = \frac{MR^2}{2}\omega \quad \text{or} \quad \omega = \omega_0\left(1 + \frac{2m}{M}\right)$$

(b) From the equation of the increment of the mechanical energy of the system :

$$\Delta T = A_{ext}$$

$$\frac{1}{2}\frac{MR^2}{2}\omega^2 - \frac{1}{2}\left(\frac{MR^2}{2} + m\,R^2\right)\omega_0^2 = A_{ext}$$

Putting the value of ω from part (a) and solving we get

$$A_{ext} = \frac{m\,\omega_0^2 R^2}{2}\left(1 + \frac{2m}{M}\right)$$

1.277 (a) Let z be the rotation axis of disc and φ be its rotation angle in accordance with right-hand screw rule (Fig.). (φ and φ' are to be measured in the same sense algebraically.) As M_z of the system (disc + man) is conserved and $M_{z(initial)} = 0$, we have at any instant,

$$0 = \frac{m_2 R^2}{2}\frac{d\varphi}{dt} + m_1\left[\left(\frac{d\varphi'}{dt}\right)R + \left(\frac{d\varphi}{dt}\right)R\right]R$$

or, $$d\varphi = \left[-\frac{m_1}{m_1 + (m_2/2)}\right]d\varphi'$$

On integrating $$\int_0^{\varphi} d\varphi = -\int_0^{\varphi'}\left(\frac{m_1}{m_1 + (m_2/2)}\right)d\varphi'$$

or, $$\varphi = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\varphi' \qquad (1)$$

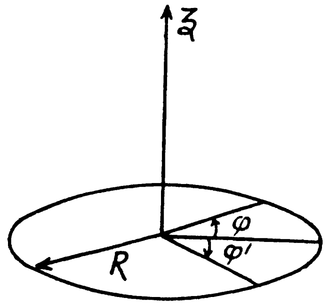
This gives the total angle of rotation of the disc.

(b) From Eq. (1)

$$\frac{d\varphi}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{d\varphi'}{dt} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{v'(t)}{R}$$

Differentiating with respect to time

$$\frac{d^2\varphi}{dt^2} = -\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$



Thus the sought force moment from the Eq. $N_z = I\beta_z$

$$N_z = \frac{m_2 R^2}{2}\frac{d^2\varphi}{dt^2} = -\frac{m_2 R^2}{2}\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Hence $$N_z = -\frac{m_1 m_2 R}{2m_1 + m_2}\frac{dv'(t)}{dt}$$

1.278 (a) Frome the law of conservation of angular momentum of the system relative to vertical axis z, it follows that:

$$I_1\omega_{1z} + I_2\omega_{2z} = (I_1 + I_2)\omega_z$$

Hence $$\omega_z = (I_1\omega_{1z} + I_2\omega_{2z}) / (I_1 + I_2) \quad (1)$$

Not that for $\omega_z > 0$, the corresponding vector $\vec{\omega}$ coincides with the poitive direction to the z axis, and vice versa. As both discs rotates about the same vertical axis z, thus in vector form.

$$\vec{\omega} = I_1\vec{\omega}_1 + I_2\vec{\omega}_2 / (I_1 + I_2)$$

However, the problem makes sense only if $\vec{\omega}_1 \uparrow\uparrow \vec{\omega}_2$ or $\vec{\omega}_1 \uparrow\downarrow \vec{\omega}_2$

(b) From the equation of increment of mechanical energy of a system: $A_{fr} = \Delta T$.

$$= \frac{1}{2}(I_1 + I_2)\omega_z^2 - \frac{1}{2}I_1\omega_{1z}^2 + \frac{1}{2}I_2\omega_{2z}^2$$

Using Eq. (1)

$$A_{fr} = -\frac{I_1 I_2}{2(I_1 + I_2)}(\omega_{1z} - \omega_{2z})^2$$

1.279 For the closed system (disc + rod), the angular momentum is conserved about any axis. Thus from the conservation of angular momentum of the system about the rotation axis of rod passing through its C.M. gives :

$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{\eta m l^2}{12}\omega \quad (1)$$

(v' is the final velocity of the disc and ω angular velocity of the rod)

For the closed system linear momentum is also conserved. Hence

$$mv = mv' + \eta\, mv_c \qquad (2)$$

(where v_c is the velocity of C.M. of the rod)

From Eqs (1) and (2) we get

$$v_c = \frac{l\,\omega}{3} \quad \text{and} \quad v - v' = \eta v_c$$

Applying conservation of kinetic energy, as the collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\eta m v_c^2 + \frac{1}{2}\,\frac{\eta m l^2}{12}\,\omega^2 \qquad (3)$$

or $\quad v^2 - v'^2 = 4\eta v_c^2$ and hence $v + v' = 4v_c$

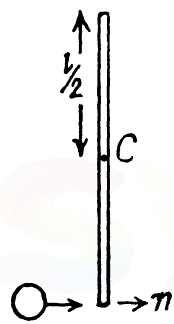
Then

$$v' = \frac{4-\eta}{4+\eta}\,v \text{ and } \omega = \frac{12\,v}{(4+\eta)\,l}$$

Vectorially, noting that we have taken $\vec{v}'$ parallel to $\vec{v}$

$$\vec{u}' = \left(\frac{4-\eta}{4+\eta}\right)\vec{v}$$

So, $\vec{u}' = 0$ for $\eta = 4$ and $\vec{u}' \downarrow\uparrow \vec{v}$ for $\eta > 4$



1.280 See the diagram in the book (Fig. 1.72)

(a) When the shaft BB' is turned through 90° the platform must start turning with angular velocity Ω so that the angular momentum remains constant. Here

$$(I + I_0)\,\Omega = I_0\,\omega_0 \quad \text{or,} \quad \Omega = \frac{I_0\,\omega_0}{I + I_0}$$

The work performed by the motor is therefore

$$\frac{1}{2}(I + I_0)\,\Omega^2 = \frac{1}{2}\,\frac{I_0^2\,\omega_0^2}{I + I_0}$$

If the shaft is turned through 180°, angular velocity of the sphere changes sign. Thus from conservation of angular momentum,

$$I\,\Omega - I_0\,\omega_0 = I_0\,\omega_0$$

(Here $-I_0\,\omega_0$ is the complete angular momentum of the sphere i. e. we assume that the angular velocity of the sphere is just $-\omega_0$). Then

$$\Omega = 2I_0\,\frac{\omega_0}{I}$$

and the work done must be,

$$\frac{1}{2}I\,\Omega^2 + \frac{1}{2}I_0\,\omega_0^2 - \frac{1}{2}I_0\,\omega_0^2 = \frac{2I_0^2\,\omega_0^2}{I}$$

(b) In the case (a), first part, the angular momentum vector of the sphere is precessing with angular velocity Ω. Thus a torque,

$$I_0\,\omega_0\,\Omega = \frac{I_0^2\,\omega_0^2}{I + I_0} \text{ is needed.}$$

1.281 The total centrifugal force can be calculated by,

$$\int_0^{l_0} \frac{m}{l_0}\,\omega^2\,x\,dx = \frac{1}{2}\,m\,l_0\,\omega^2$$

Then for equilibrium,

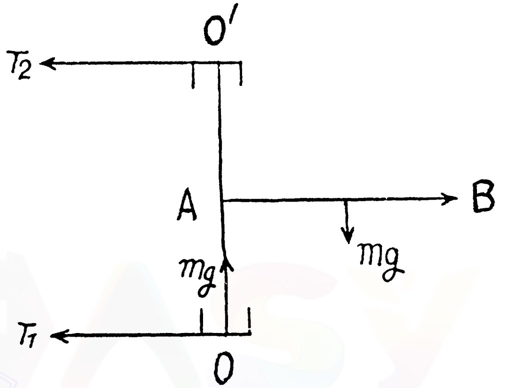
$$(T_2 - T_1)\,\frac{l}{2} = mg\,\frac{l_0}{2}$$

and, $T_2 + T_1 = \frac{1}{2}\,m\,l_0\,\omega^2$

Thus T_1 vanishes, when

$$\omega^2 = \frac{2g}{l},\quad \omega = \sqrt{\frac{2g}{l}} = 6 \text{ rad/s}$$

Then $T_2 = mg\,\frac{l_0}{l} = 25$ N



1.282 See the diagram in the book (Fig. 1.71).

(a) The angular velocity $\vec{\omega}$ about OO' can be resolved into a component parallel to the rod and a component $\omega \sin\theta$ perpendicular to the rod through C. The component parallel to the rod does not contribute so the angular momentum

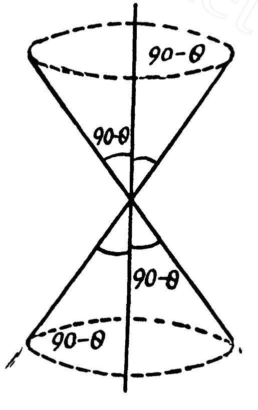
$$M = I\,\omega\sin\theta = \frac{1}{12}\,m\,l^2\,\omega\sin\theta$$

Also, $$M_z = M\sin\theta = \frac{1}{12}\,m\,l^2\,\omega\sin^2\theta$$

This can be obtained directly also,

(b) The modulus of $\vec{M}$ does not change but the modulus of the change of $\vec{M}$ is $|\,\Delta\vec{M}\,|$.

$$|\,\Delta\vec{M}\,| = 2M\sin(90 - \theta) = \frac{1}{12}\,m\,l^2\,\omega\sin 2\theta$$



(c) Here $M_\perp = M\cos\theta = I\,\omega\sin\theta\cos\theta$

Now $$\left|\frac{d\vec{M}}{dt}\right| = I\,\omega\sin\theta\cos\theta\,\frac{\omega\,dt}{dt} = \frac{1}{24}\,ml^2\,\omega^2\sin^2\theta$$

as $\vec{M}$ precesses with angular velocity ω.

1.283 Here $M = I\omega$ is along the symmetry axis. It has two components, the part $I\omega\cos\theta$ is constant and the part $M_\perp = I\omega\sin\theta$ presesses, then

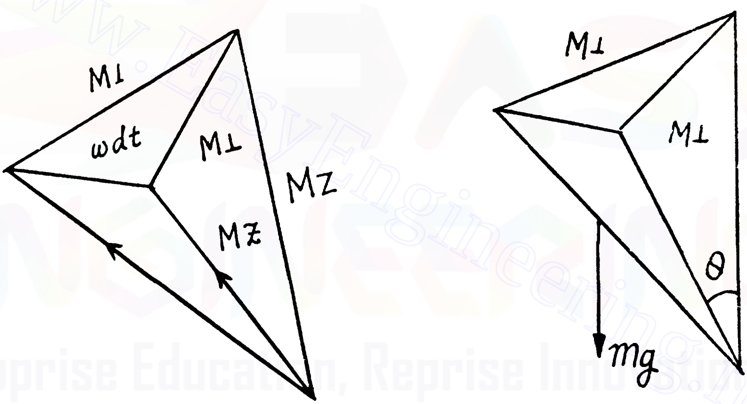
$$\left|\frac{d\vec{M}}{dt}\right| = I\omega\sin\theta\,\omega' = mgl\sin\theta$$

or, $$\omega' = \text{precession frequency} = \frac{mgl}{I\omega} = 0{\cdot}7\ \text{rad/s}$$

(b) This force is the centripetal force due to precession. It acts inward and has the magnitude

$$|\vec{F}| = \left|\sum m_i\omega'^2\vec{\rho_i}\right| = m\omega'^2 l\sin\theta = 12\ \text{mN}.$$

$\vec{\rho_i}$ is the distance of the i th element from the axis. This is the force that the table will exert on the top. See the diagram in the answer sheet



1.284 See the diagram in the book (Fig. 1.73).

The moment of inertia of the disc about its symmentry axis is $\frac{1}{2}mR^2$. If the angular velocity of the disc is ω then the angular momentum is $\frac{1}{2}mR^2\omega$. The precession frequency being $2\pi n$,

we have $$\left|\frac{d\vec{M}}{dt}\right| = \frac{1}{2}mR^2\omega \times 2\pi n$$

This must equal $m(g+w)l$, the effective gravitational torques (g being replaced by $g+w$ in the elevator). Thus,

$$\omega = \frac{(g+w)l}{\pi R^2 n} = 300\ \text{rad/s}$$

1.285 The effective g is $\sqrt{g^2+w^2}$ inclined at angle $\tan^{-1}\frac{w}{g}$ with the vertical. Then with reference to the new " vertical" we proceed as in problem 1·283. Thus

$$\omega' = \frac{ml\sqrt{g^2+w^2}}{I\omega} = 0{\cdot}8 \text{ rad/s.}$$

The vector $\vec{\omega}'$ forms an angle $\theta = \tan^{-1}\frac{w}{g} = 6°$ with the normal vertical.

1.286 The moment of inertia of the sphere is $\frac{2}{5}mR^2$ and hence the value of angular momentum is $\frac{2}{5}mR^2\omega$. Since it precesses at speed ω' the torque required is

$$\frac{2}{5}mR^2\omega\,\omega' = F'l$$

So, $$F' = \frac{2}{5}mR^2\omega\,\omega'/l = 300 \text{ N}$$

(The force F' must be vertical.)

1.287 The moment of inertia is $\frac{1}{2}mr^2$ and angular momentum is $\frac{1}{2}mr^2\omega$. The axle oscillates about a horizontal axis making an instantaneous angle.

$$\varphi = \varphi_m \sin\frac{2\pi t}{T}$$

This means that there is a variable precession with a rate of precession $\frac{d\varphi}{dt}$. The maximum value of this is $\frac{2\pi\varphi_m}{T}$. When the angle between the axle and the axis is at its maximum value, a torque $I\omega\Omega$

$$= \frac{1}{2}mr^2\omega\,\frac{2\pi\varphi_m}{T} = \frac{\pi mr^2\omega\varphi_m}{T} \text{ acts on it.}$$

The corresponding gyroscopic force will be $\frac{\pi mr^2\omega\varphi_m}{lT} = 90$ N

1.288 The revolutions per minute of the flywheel being n, the angular momentum of the flywheel is $I\times 2\pi n$. The rate of precession is $\frac{v}{R}$

Thus $N = 2\pi INV/R = 5{\cdot}97$ kN. m.

1.289 As in the previous problem a couple $2\pi Inv/R$ must come in play. This can be done if a force, $\frac{2\pi Inv}{Rl}$ acts on the rails in opposite directions in addition to the centrifugal and other forces. The force on the outer rail is increased and that on the inner rail decreased. The additional force in this case has the magnitude 1·4 kN. m.

1.6 ELASTIC DEFORMATIONS OF A SOLID BODY

1.290 Variation of length with temperature is given by

$$l_t = l_0(1+\alpha\Delta t) \text{ or } \frac{\Delta l}{l_0} = \alpha\,\Delta t = \varepsilon \qquad (1)$$

But
$$\varepsilon = \frac{\sigma}{E},$$

Thus $\sigma = \alpha\Delta t E$, which is the sought stress of pressure.

Putting the value of α and E from Appendix and taking $\Delta t = 100°C$, we get

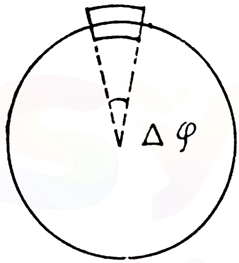
$$\sigma = 2{\cdot}2\times10^3 \text{ atm.}$$

1.291 (a) Consider a transverse section of the tube and concentrate on an element which subtends an angle $\Delta\varphi$ at the centre. The forces acting on a portion of length Δl on the element are

(1) tensile forces side ways of magnitude $\sigma\Delta r\Delta l$.

The resultant of these is

$$2\sigma\Delta r\Delta l\sin\frac{\Delta\varphi}{2} \approx \sigma\Delta r\Delta l\Delta\varphi$$



radially towards the cente.

(2) The force due to fluid pressure $= pr\Delta\varphi\Delta l$

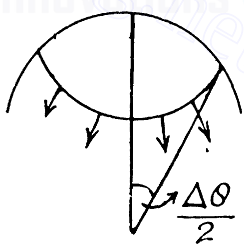
Since these balance, we get $p_{max} \approx \sigma_m \frac{\Delta r}{r}$

where σ_m is the maximum tensile force.

Putting the values we get $p_{max} = 19{\cdot}7$ atmos.

(b) Consider an element of area $dS = \pi(r\,\Delta\theta/2)^2$ about z-axis chosen arbitrarily. There are tangential tensile forces all around the ring of the cap. Their resultant is

$$\sigma\left[2\pi\left(r\frac{\Delta\theta}{2}\right)\Delta r\right]\sin\frac{\Delta\theta}{2}$$



Hence in the limit

$$p_m\,\pi\left(\frac{r\Delta\theta}{2}\right)^2 = \sigma_m\,\pi\left(\frac{r\Delta\theta}{2}\right)\Delta r\,\Delta\theta$$

or $$p_m = \frac{2\sigma_m\,\Delta r}{r} = 39{\cdot}5 \text{ atmos.}$$

1.292 Let us consider an element of rod at a distance x from its rotation axis (Fig.). From Newton's second law in projection form directed towards the rotation axis

$$-dT = (dm)\,\omega^2 x = \frac{m}{l}\,\omega^2 x\,dx$$

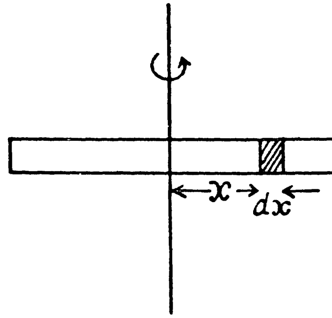
On integrating

$$-T = \frac{m\omega^2}{l}\frac{x^2}{2} + C\text{ (constant)}$$

But at $x = \pm \frac{l}{2}$ or free end, $T = 0$

Thus $0 = \frac{m\omega^2}{2}\frac{l^2}{4} + C$ or $C = -\frac{m\omega^2 l}{8}$

Hence $T = \frac{m\omega^2}{2}\left(\frac{l}{4} - \frac{x^2}{l}\right)$



Thus $T_{max} = \frac{m\omega^2 l}{8}$ (at mid point)

Condition required for the problem is

$$T_{max} = S\sigma_m$$

So, $\frac{m\omega^2 l}{8} = S\sigma_m$ or $\omega = \frac{2}{l}\sqrt{\frac{2\sigma_m}{\rho}}$

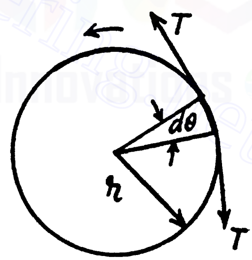
Hence the sought number of r p s

$$n = \frac{\omega}{2\pi} = \frac{1}{\pi l}\sqrt{\frac{2\sigma_m}{\rho}}$$ [using the table $n = 0{\cdot}8 \times 10^2$rps]

1.293 Let us consider an element of the ring (Fig.). From Newton's law $F_n = mw_n$ for this element, we get,

$$Td\theta = \left(\frac{m}{2\pi}d\theta\right)\omega^2 r$$ [see solution of 1.93or 1.92]

So, $T = \frac{m}{2\pi}\omega^2 r$



Condition for the problem is :

$\frac{T}{\pi r^2} \le \sigma_m$ or, $\frac{m\omega^2 r}{2\pi^2 r^2} \le \sigma_m$

or, $\omega_{max}^2 = \frac{2\pi^2 \sigma_m r}{\pi r^2 (2\pi r\rho)} = \frac{\sigma_m}{\rho r^2}$

Thus sought number of rps

$$n = \frac{\omega_{max}}{2\pi} = \frac{1}{2\pi r}\sqrt{\frac{\sigma_m}{\rho}}$$

Using the table of appendices $n = 23$rps

1.294 Let the point O desend by the distance x (Fig.). From the condition of equilibrium of point O.

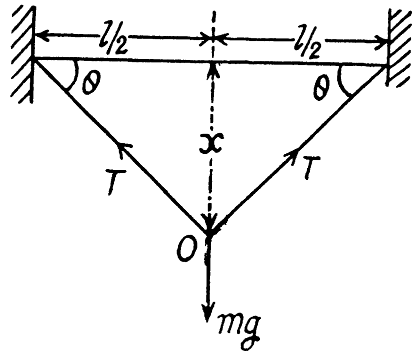
$$2T\sin\theta = mg \text{ or } T = \frac{mg}{2\sin\theta} = \frac{mg}{2x}\sqrt{(l/2)^2 + x^2} \quad (1)$$

Now, $$\frac{T}{\pi(d/2)^2} = \sigma = \varepsilon E \text{ or } T = \varepsilon E\pi\frac{d^2}{4} \quad (2)$$

(σ here is stress and ε is strain.)

In addition to it,

$$\varepsilon = \frac{\sqrt{(l/2)^2 + x^2} - \frac{l}{2}}{l/2} = \sqrt{1 + \left(\frac{2x}{l}\right)^2} - 1 \quad (3)$$



From Eqs. (1), (2) and (3)

$$x - \frac{x}{\sqrt{1 + \left(\frac{2x}{l}\right)^2}} = \frac{mgl}{\pi E d^2} \quad \text{as } x << l$$

So, $\dfrac{4x^3}{2l^2} \approx \dfrac{mgl}{\pi E d^2}$

or, $x = l\left(\dfrac{mg}{2\pi E d^2}\right)^{1/3} = 2{\cdot}5 \text{ cm}$

1.295 Let us consider an element of the rod at a distance x from the free end (Fig.). For the considered element '$T - T$' are internal restoring forces which produce elongation and dT provides the acceleration to the element. For the element from Newton's law :

$$dT = (dm)\, w = \left(\frac{m}{l} dx\right) \frac{F_o}{m} = \frac{F_o}{l} dx$$

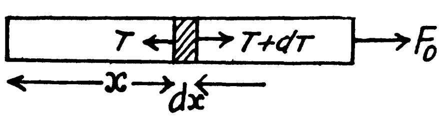
As free end has zero tension, on integrating the above expression,

$$\int_o^T dT = \frac{F_o}{l} \int_o^x dx \quad \text{or} \quad T = \frac{F_o}{l} x$$

Elongation in the considered element of lenght dx :

$$\partial \xi = \frac{\sigma}{E}(x) \; dx = \frac{T}{SE} dx = \frac{F_{xo} x dx}{SEL}$$

Thus total elengation $\xi = \dfrac{F_o}{SEl} \displaystyle\int_x^l x\, dx = \dfrac{F_o l}{2SE}$



Hence the sought strain

$$\sigma = \frac{\xi}{l} = \frac{F_o}{2SE}$$

1.296 Let us consider an element of the rod at a distance r from it's rotation axis. As the element rotates in a horizontal circle of radius r, we have from Newton's second law in projection form directed toward the axis of rotation :

$$T - (T + dT) = (dm)\, \omega^2 r$$

or, $$-dT = \left(\frac{m}{l} dr\right) \omega^2 r = \frac{m}{l} \omega^2 r\, dr$$

At the free end tension becomes zero. Integrating the above experession we get, thus

$$-\int_T^0 dT = \frac{m}{l}\omega^2 \int_r^l r\,dr$$

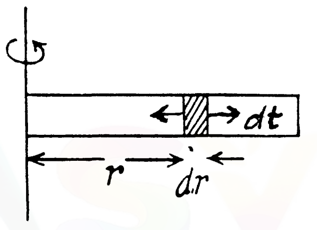
Thus $$T = \frac{m\omega^2}{l}\left(\frac{l^2 - r^2}{2}\right) = \frac{m\,\omega^2\, l}{2}\left(1 - \frac{r^2}{l^2}\right)$$

Elongation in elemental length dr is given by :

$$\partial\,\xi = \frac{\sigma(r)}{E}\,dr = \frac{T}{SE}\,dr$$

(where S is the cross sectional area of the rod and T is the tension in the rod at the considered element)

or, $$\partial\,\xi = \frac{m\,\omega^2\, l}{2\,SE}\left(1 - \frac{r^2}{l^2}\right)dr$$



Thus the sought elongation

$$\xi = \int d\,\xi = \frac{m\,\omega^2\, l}{2\,SE}\int_0^l \left(1 - \frac{r^2}{l^2}\right)dr$$

or, $$\xi = \frac{m\,\omega^2\, l}{2\,SE}\,\frac{2l}{3} = \frac{(Sl\,\rho)}{3\,SE}\,\omega^2\, l^3$$

$$= \frac{1}{3}\frac{\rho\,\omega^2\, l^3}{E}$$ (where ρ is the density of the copper.)

1.297 Volume of a solid cylinder

$$V = \pi\, r^2\, l$$

So, $$\frac{\Delta V}{V} = \frac{\pi\, 2\, r\,\Delta\, r\, l}{\pi\, r^2\, l} + \frac{\pi\, r^2\,\Delta\, l}{\pi\, r^2\, l} = \frac{2\,\Delta\, r}{r} + \frac{\Delta\, l}{l} \qquad (1)$$

But longitudinal strain $\Delta l/l$ and accompanying lateral strain $\Delta\, r/r$ are related as

$$\frac{\Delta\, r}{r} = -\mu\,\frac{\Delta\, l}{l} \qquad (2)$$

Using (2) in (1), we get :

$$\frac{\Delta V}{V} = \frac{\Delta l}{l}\,(1 - 2\,\mu) \qquad (3)$$

But $$\frac{\Delta l}{l} = \frac{-F/\pi\, r^2}{E}$$

(Because the increment in the length of cylinder Δl is negative)

So, $$\frac{\Delta V}{V} = \frac{-F}{\pi\, r^2\, E}\,(1 - 2\,\mu)$$

Thus, $$\Delta V = \frac{-Fl}{E}(1-2\mu)$$

Negative sign means that the volume of the cylinder has decreased.

1.298 (a) As free end has zero tension, thus the tension in the rod at a vestical distance y from its lower end

$$T = \frac{m}{l} g y \quad (1)$$

Let ∂l be the elongation of the element of length dy, then

$$\partial l = \frac{\sigma(y)}{E} dy$$

$$= \frac{T}{SE} dy = \frac{mgydy}{SlE} = \rho\, gydy/E$$ (where ρ is the density of the copper)

Thus the sought elongation

$$\Delta l = \int \partial l = \rho g \int_0^l \frac{y\,dy}{E} = \frac{1}{2}\rho g l^2/E \quad (2)$$

(b) If the longitudinal (tensile) strain is $\varepsilon = \frac{\Delta l}{l}$, the accompanying lateral (compressive) strain is given by

$$\varepsilon' = \frac{\Delta r}{r} = -\mu\varepsilon \quad (3)$$

Then since $V = \pi r^2 l$ we have

$$\frac{\Delta V}{V} = \frac{2\Delta r}{r} + \frac{\Delta l}{l}$$

$$= (1-2\mu)\frac{\Delta l}{l} \quad \text{[Using (3)]}$$

where $\frac{\Delta l}{l}$ is given in part (a), μ is the Poisson ratio for copper.

1.299 Consider a cube of unit length before pressure is applied. The pressure acts on each face. The pressures on the opposite faces constitute a tensile stress producing longitudianl compression and lateral extension. The compressions is $\frac{p}{E}$ and the lateral extension is $\mu\frac{p}{E}$ The net result is a compression

$$\frac{p}{E}(1-2\mu) \quad \text{in each side.}$$

Hence $\frac{\Delta V}{V} = -\frac{3p}{E}(1-2\mu)$ because from symmetry $\frac{\Delta V}{V} = 3\frac{\Delta l}{l}$

(b) Let us consider a cube under an equal compressive stress σ, acting on all its faces.

Then, $$\text{volume strain} = -\frac{\Delta V}{V} = \frac{\sigma}{k}, \qquad (1)$$

where k is the bulk modulus of elasticity.

So $$\frac{\sigma}{k} = \frac{3\sigma}{E}(1-2\mu)$$

or, $$E = 3k(1-2\mu) = \frac{3}{\beta}(1-2\mu)\left(\text{as } k = \frac{1}{\beta}\right)$$

$\mu \le \frac{1}{2}$ if E and β are both to remain positive.

1.300 A beam clamped at one end and supporting an applied load at the free end is called a cantilever. The theory of cantilevers is discussed in advanced text book on mechanics. The key result is that elastic forces in the beam generate a couple, whose moment, called the moment of resistances, balances the external bending moment due to weight of the beam, load etc. The moment of resistance, also called internal bending moment (I.B.M) is given by

$$\text{I.B.M.} = EI/R$$

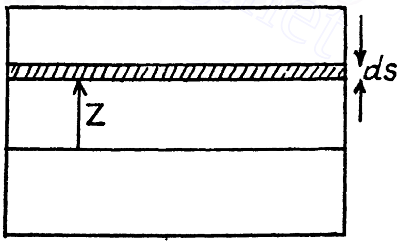
Here R is the radius of curvature of the beam at the representative point (x, y). I is called the geometrical moment of inertia

$$I = \int z^2\, dS$$

of the cross section relative to the axis passing through the netural layer which remains unstretched. (Fig.1.). The section of the beam beyond P exerts the bending moment $N(x)$ and we have,

$$\frac{EI}{R} = N(x)$$

If there is no load other than that due to the weight of the beam, then

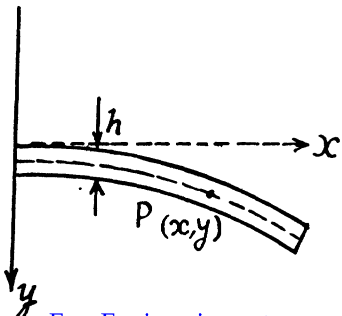


$$N(x) = \frac{1}{2}\rho g(l-x)^2 bh$$

where ρ = density of steel.

Hence, at $x = 0$

$$\left(\frac{I}{R}\right)_0 = \frac{\rho g l^2 b h}{2EI}$$



Here b = width of the beam perpendicular to paper.

Also, $$I = \int_{-h/2}^{h/2} z^2\, b\,dz = \frac{bh^3}{12}.$$

Hence, $$\left(\frac{1}{R}\right)_0 = \frac{6\rho g l^2}{Eh^2} = (0.121 \text{ km})^{-1}$$

1.301 We use the equation given above and use the result that when y is small

$$\frac{1}{R} \approx \frac{d^2 y}{dx^2}. \quad \text{Thus,} \quad \frac{d^2 y}{dx^2} = \frac{N(x)}{EI}$$

(a) Here $N(x) = N_0$ is a constant. Then integration gives,

$$\frac{dy}{dx} = \frac{N_0 x}{EI} + C_1$$

But $\left(\frac{dy}{dx}\right) = 0$ for $x = 0$, so $C_1 = 0$. Integrating again,

$$y = \frac{N_0 x^2}{2EI}$$

where we have used $y = 0$ for $x = 0$ to set the constant of integration at zero. This is the equation of a parabola. The sag of the free end is

$$\lambda = y(x = l) = \frac{N_0 l^2}{2EI}$$

(b) In this case $N(x) = F(l - x)$ because the load F at the extremity is balanced by a similar force at F directed upward and they constitute a couple. Then

$$\frac{d^2 y}{dx^2} = \frac{F(l-x)}{EI}.$$

Integrating, $$\frac{dy}{dx} = \frac{F(lx - x^2/2)}{EI} + C_1$$

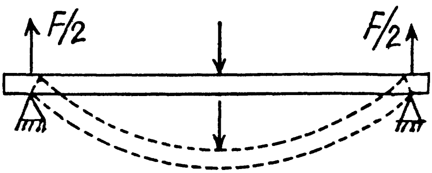
As before $C_1 = 0$. Integrating again, using $y = 0$ for $x = 0$

$$y = \frac{F\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)}{EI} \quad \text{here } \lambda = \frac{Fl^3}{3EI}$$

Here for a square cross section

$$I = \int_{-a/2}^{a/2} z^2 a \, dz = a^4/12.$$

1.302 One can think of it as analogous to the previous case but with a beam of length $l/2$ loaded upward by a force $F/2$.



Thus $$\lambda = \frac{F l^3}{48 EI},$$

On using the last result of the previous problem.

1.303 (a) In this case $N(x) = \frac{1}{2}\rho g b h (l - x)^2$ where $b =$ width of the girder.

Also $I = b h^3/12$. Then,

$$\frac{E\,b\,h^2}{12}\frac{d^2y}{dx^2}=\frac{\rho\,g\,b\,h}{2}(l^2-2l\,x+x^2).$$

Integrating, $$\frac{dy}{dx}=\frac{6\,\rho g}{E\,h^2}\left(l^2x-lx^2+\frac{x^3}{3}\right)$$

using $\frac{dy}{dx}=0$ for $x=0$. Again integrating

$$y=\frac{6\,\rho g}{Eh^2}\left(\frac{l^2x^2}{2}-\frac{lx^3}{3}+\frac{x^4}{12}\right)$$

Thus $$\lambda=\frac{6\,\rho\,g\,l^4}{Eh^2}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{12}\right)$$

$$=\frac{6\,\rho\,g\,l^4}{Eh^2}\,\frac{3}{12}=\frac{3\,\rho\,g\,l^4}{2\,Eh^2}$$

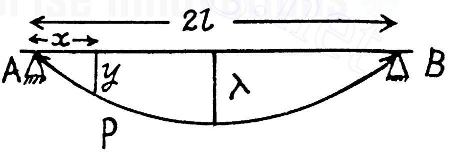
(b) As before, $EI\frac{d^2y}{dx^2}=N(x)$ where $N(x)$ is the bending moment due to section PB.

This bending moment is clearly

$$N=\int_x^{2l}w\,d\xi\,(\xi-x)-wl\,(2l-x)$$

$$=w\left(2l^2-2xl+\frac{x^2}{2}\right)-wl\,(2l-x)=w\left(\frac{x^2}{2}-xl\right)$$

(Here $w=\rho\,g\,b\,h$ is weight of the beam per unit length)



Now integrating, $EI\frac{dy}{dx}=w\left(\frac{x^3}{6}-\frac{x^2\,l}{2}\right)+c_0$

or since $\frac{dy}{dx}=0$ for $x=l$, $c_0=w\,l^3/3$

Integrating again, $E\,I\,y=w\left(\frac{x^4}{24}-\frac{x^3\,l}{6}\right)+\frac{wl^3\,x}{3}+c_1$

As $y=0$ for $x=0$, $c_1=0$. From this we find

$$\lambda=y\,(x=l)=\frac{5\,w\,l^4}{24}\Big/EI=\frac{5\,\rho g\,l^4}{2\,E\,h^2}$$

1.304 The deflection of the plate can be noticed by going to a co- rotating frame. In this frame each element of the plate experiences a pseudo force proportional to its mass. These forces have a moment which constitutes the bending moment of the problem. To calculate this moment we note that the acceleration of an element at a distance ξ from the axis is $a=\xi\,\beta$ and the moment of the forces exerted by the section between x and l is

$$N = \rho\, l\, h\, \beta \int_x^l \xi^2\, d\xi = \frac{1}{3} \rho\, l\, h\, \beta\, (l^3 - x^3).$$

From the fundamental equation

$$E I \frac{d^2 y}{dx^2} = \frac{1}{3} \rho\, l\, h\, \beta\, (l^3 - x^3).$$

The moment of inertia $I = \int_{-h/2}^{+h/2} z^2\, l\, dz = \frac{l h^3}{12.}$

Note that the neutral surface (i.e. the surface which contains lines which are neither stretched nor compressed) is a vertical plane here and z is perpendicular to it.

$$\frac{d^2 y}{dx^2} = \frac{4 \rho \beta}{E h^2} (l^3 - x^3). \text{ Integrating}$$

$$\frac{dy}{dx} = \frac{4 \rho \beta}{E h^2} \left(l^3 x - \frac{x^4}{4} \right) + c_1$$

Since $\frac{dy}{dx} = 0$, for $x = 0$, $c_1 = 0$. Integrating again,

$$y = \frac{4 \rho \beta}{E h^2} \left(\frac{l^3 x^2}{2} - \frac{x^5}{20} \right) + c_2$$

$c_2 = 0$ because $y = 0$ for $x = 0$

Thus $$\lambda = y\,(x = l) = \frac{9 \rho \beta l^5}{5 E h^2}$$

1.305 (a) Consider a hollow cylinder of length l, outer radius $r + \Delta r$ inner radius r, fixed at one end and twisted at the other by means of a couple of moment N. The angular displacement φ, at a distance l from the fixed end, is proportional to both l and N. Consider an element of length dx at the twisted end. It is moved by an angle φ as shown. A vertical section is also shown and the twisting of the parallelopipe of length l and area $\Delta r\, dx$ under the action of the twisting couple can be discussed by elementary means. If f is the tangential force generated then shearing stress is $f/\Delta r\, dx$ and this must equal

$$G\theta = G \frac{r \varphi}{l}, \text{ since } \theta = \frac{r \varphi}{l}.$$

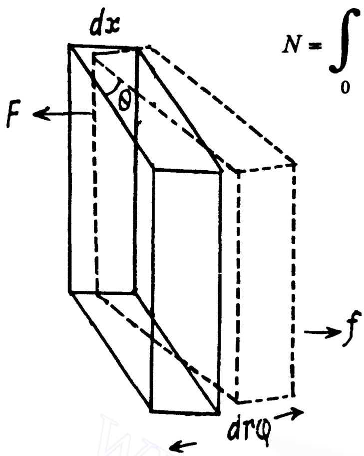
Hence, $$f = G\, \Delta r\, dx \frac{r \varphi}{l}.$$

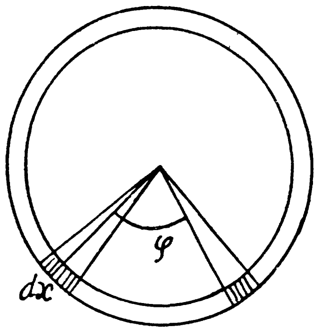
The force f has moment fr about the axis and so the total moment is

$$N = G\, \Delta r \frac{\varphi}{l} r^2 \int dx = \frac{2 \pi r^3 \Delta r\, \varphi}{l} G$$

(b) For a solid cylinder we must integrate over r. Thus

$$N = \int_0^r \frac{2\pi r^3\, dr\, \varphi\, G}{l} = \frac{\pi r^4 G \varphi}{2l}$$





1.306 Clearly $N = \int_{d_1/2}^{d_2/2} \frac{2\pi r^3\, d r\, \varphi\, G}{l} = \frac{\pi}{32\, l} G\, \varphi\, (d_2^4 - d_1^4)$

using $$G = 81 \text{ G Pa} = 8{\cdot}1 \times 10^{10} \frac{\text{N}}{\text{m}^2}$$

$$d_2 = 5 \times 10^{-2}\text{ m},\; d_1 = 3 \times 10^{-2}\text{ m}$$

$$\varphi = 2{\cdot}0° = \frac{\pi}{90} \text{ radians},\; l = 3 \text{ m}$$

$$N = \frac{\pi \times 8{\cdot}1 \times \pi}{32 \times 3 \times 90}(625 - 81) \times 10^2 \text{ N·m}$$

$$= 0{\cdot}5033 \times 10^3 \text{ N·m} \approx 0{\cdot}5 \text{ k N·m}$$

1.307 The maximum power that can be transmitted by means of a shaft rotating about its axis is clearly $N\omega$ where N is the moment of the couple producing the maximum permissible torsion, φ. Thus

$$P = \frac{\pi r^4 G \varphi}{2l} \cdot \omega = 16{\cdot}9 \text{ kw}$$

1.308 Consider an elementary ring of width dr at a distant r from the axis. The part outside exerts a couple $N + \frac{dN}{dr} dr$ on this ring while the part inside exerts a couple N in the opposite direction. We have for equilibrium

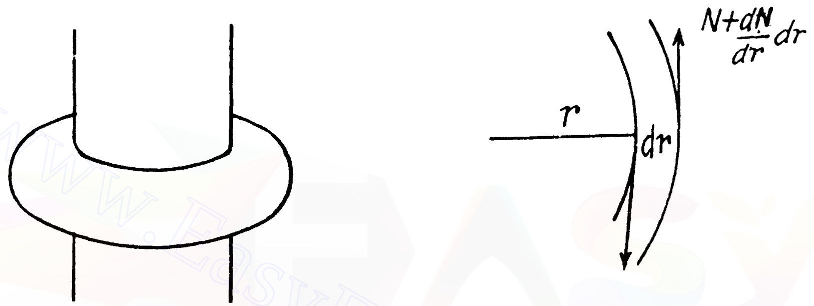
$$\frac{dN}{dr} dr = -dI\beta$$

where dI is the moment of inertia of the elementary ring, β is the angular acceleration and minus sign is needed because the couple $N(r)$ decreases, with distance vanshing at the outer radius, $N(r_2) = 0$. Now

$$dI = \frac{m}{\pi (r_2^2 - r_1^2)} 2 \pi r \, dr \; r^2$$

Thus
$$dN = \frac{2m \beta}{(r_1^2 - r_1^2)} r^3 \, dr$$

or,
$$N = \frac{1}{2} \frac{m \beta}{(r_2^2 - r_1^2)} (r_2^4 - r^4), \text{ on integration}$$



1.309 We assume that the deformation is wholly due to external load, neglecting the effect of the weight of the rod (see next problem). Then a well known formula says,

elastic energy per unit volume

$$= \frac{1}{2} \text{stress} \times \text{strain} = \frac{1}{2} \sigma \varepsilon$$

This gives $\frac{1}{2} \frac{m}{\rho} E \varepsilon^2 \approx 0{\cdot}04$ kJ for the total deformation energy.

1.310 When a rod is deformed by its own weight the stress increases as one moves up, the stretching force being the weight of the portion below the element considered.

The stress on the element dx is

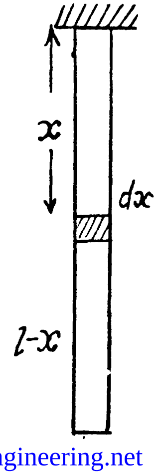
$$\rho \pi r^2 (l - x) g / \pi r^2 = \rho g (l - x)$$

The extension of the element is

$$\Delta \, dx = d \, \Delta x = \rho g (l - x) \, dx / E$$

Integrating $\Delta l = \frac{1}{2} \rho g l^2 / E$ is the extension of the whole rod. The elastic energy of the element is

$$\frac{1}{2} \rho g (l - x) \frac{\rho g (l - x)}{E} \pi r^2 \, dx$$



Integrating

$$\Delta U = \frac{1}{6} \pi r^2 \rho^2 g^2 l^3 / E = \frac{2}{3} \pi r^2 l E \left(\frac{\Delta l}{l} \right)^2$$

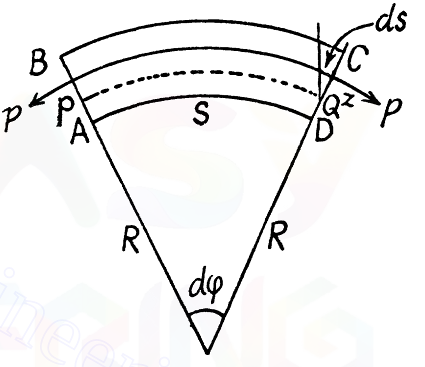
1.311 The work done to make a loop out of a steel band appears as the elastic energy of the loop and may be calculated from the same.

If the length of the band is l, the radius of the loop $R = \frac{l}{2\pi}$. Now consider an element $ABCD$ of the loop. The elastic energy of this element can be calculated by the same sort of arguments as used to derive the formula for internal bending moment. Consider a fibre at a distance z from the neutral surface PQ. This fibre experiences a force p and undergoes an extension ds where $ds = Z\,d\varphi$, while $PQ = s = R\,d\varphi$. Thus strain $\frac{ds}{s} = \frac{Z}{R}$. If α is the cross sectional area of the fibre, the elastic energy associated with it is

$$\frac{1}{2} E \left(\frac{Z}{R}\right)^2 R\,d\varphi\;\alpha$$

Summing over all the fibres we get

$$\frac{E\,I\,\varphi}{2R} \sum \alpha Z^2 = \frac{E\,I\,d\varphi}{2R}$$



For the whole loop this gives,

using $\int d\varphi = 2\pi$,

$$\frac{E\,I\,\pi}{R} = \frac{2\,E\,I\,\pi^2}{l}$$

Now $$I = \int_{-\delta/2}^{\delta/2} Z^2\,h\,dZ = \frac{h\,\delta^3}{12}$$

So the energy is $$\frac{1}{6}\,\frac{\pi^2 E\,h\,\delta^3}{l} = 0{\cdot}08\ \text{kJ}$$

1.312 When the rod is twisted through an angle θ, a couple $N(\theta) = \frac{\pi r^4 G}{2l}\theta$ appears to resist this. Work done in twisting the rod by an angle φ is then

$$\int_0^{\varphi} N(\theta)\,d\theta = \frac{\pi r^4 G}{4l}\varphi^2 = 7\ \text{J} \quad \text{on putting the values.}$$

1.313 The energy between radii r and $r + dr$ is, by differentiation, $\frac{\pi r^3\,dr}{l} G\,\varphi^2$

Its density is $$\frac{\pi r^3\,dr}{2\pi r\,dr\,l}\,\frac{G\,\varphi^2}{l} = \frac{1}{2}\,\frac{G\,\varphi^2 r^2}{l^2}$$

1.314 The energy density is as usual $1/2$ stress $\times$ strain. Stress is the pressure $\rho g h$. Strain is $\beta \times \rho gh$ by defination of β. Thus

$$u = \frac{1}{2}\beta(\rho gh)^2 = 23{\cdot}5\ \text{kJ/m}^3 \quad \text{on putting the values.}$$

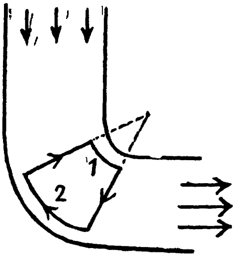
1.7 HYDRODYNAMICS

1.315 Between 1 and 2 fluid particles are in nearly circular motion and therefore have centripetal acceleration. The force for this acceleration, like for any other situation in an ideal fluid, can only come from the pressure variation along the line joining 1 and 2. This requires that pressure at 1 should be greater than the pressure at 2 i.e.

$$p_1 > p_2$$

so that the fluid particles can have required acceleration. If there is no turbulence, the motion can be taken as irrotational. Then by considering

$$\oint \vec{v} \cdot d\vec{l} = 0$$



along the circuit shown we infer that

$$v_2 > v_1$$

(The portion of the circuit near 1 and 2 are streamlines while the other two arms are at right angle to streamlines)

In an incompressible liquid we also have div $\vec{v} = 0$

By electrostatic analogy we then find that the density of streamlines is proportional to the velocity at that point.

1.316 From the conservation of mass

$$v_1 S_1 = v_2 S_2 \tag{1}$$

But $S_1 < S_2$ as shown in the figure of the problem, therefore

$$v_1 > v_2$$

As every streamline is horizontal between 1 & 2, Bernoull's theorem becomes

$$p + \frac{1}{2} \rho v^2 = \text{ constant, which gives}$$

$$p_1 < p_2 \text{ as } v_1 > v_2$$

As the difference in height of the water column is Δh, therefore

$$p_2 - p_1 = \rho g \Delta h \tag{2}$$

From Bernoull's theorem between points 1 and 2 of a streamline

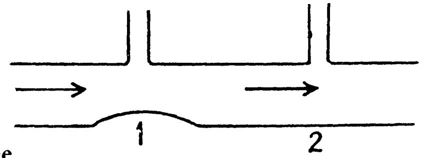
$$p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

or, $$p_2 - p_1 = \frac{1}{2} \rho (v_1^2 - v_2^2)$$

or $$\rho g \Delta h = \frac{1}{2} \rho (v_1^2 - v_2^2) \quad (3) \quad \text{(using Eq. 2)}$$

using (1) in (3), we get

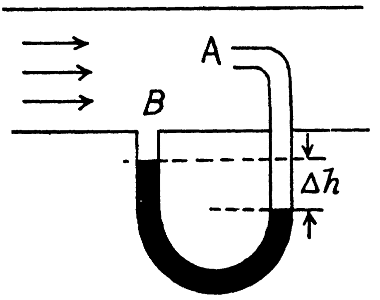
$$v_1 = S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$



Hence the sought volume of water flowing per see

$$Q = v_1 S_1 = S_1 S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$

1.317 Applying Bernoulli's theorem for the point A and B,



$$p_A = p_B + \frac{1}{2}\rho v^2 \quad \text{as,} \quad v_A = 0$$

or, $$\frac{1}{2}\rho v^2 = p_A - p_B = \Delta h \rho_0 g$$

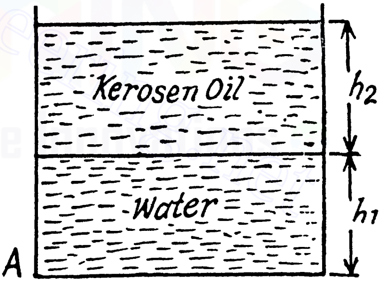
So, $$v = \sqrt{\frac{2 \Delta h \rho_0 g}{\rho}}$$

Thus, rate of flow of gas, $Q = S v = S\sqrt{\dfrac{2 \Delta h \rho_0 g}{\rho}}$

The gas flows over the tube past it at B. But at A the gas becomes stationary as the gas will move into the tube which already contains gas.

In applying Bernoulli's theorem we should remember that $\frac{p}{\rho} + \frac{1}{2}v^2 + gz$ is constant along a streamline. In the present case, we are really applying Bernoulli's theorem somewhat indirectly. The streamline at A is not the streamline at B. Nevertheless the result is correct. To be convinced of this, we need only apply Bernoull's theorem to the streamline that goes through A by comparing the situation at A with that above B on the same level. In steady conditions, this agrees with the result derived because there cannot be a transverse pressure differential.

1.318 Since, the density of water is greater than that of kerosene oil, it will collect at the bottom. Now, pressure due to water level equals $h_1 \rho_1 g$ and pressure due to kerosene oil level equals $h_2 \rho_2 g$. So, net pressure becomes $h_1 \rho_1 g + h_2 \rho_2 g$.

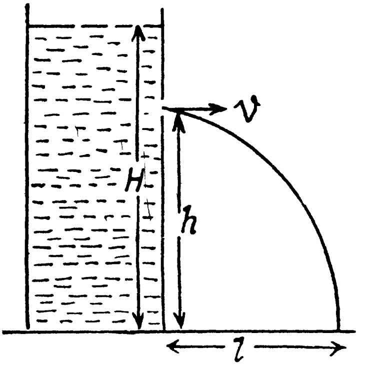


From Bernoulli's theorem, this pressure energy will be converted into kinetic energy while flowing through the whole A.

i.e. $$h_1 \rho_1 g + h_2 \rho_2 g = \frac{1}{2}\rho_1 v^2$$

Hence $$v = \sqrt{2\left(h_1 + h_2 \frac{\rho_2}{\rho_1}\right) g} = 3 \text{ m/s}$$

1.319 Let, H be the total height of water column and the hole is made at a height h from the bottom.



Then from Bernoulli's theorem

$$\frac{1}{2}\rho v^2 = (H-h)\rho g$$

or, $v = \sqrt{(H-h)\ 2g}$, which is directed horizontally.

For the horizontal range, $l = v\ t$

$$= \sqrt{2g(H-h)} \cdot \sqrt{\frac{2h}{g}} = 2\sqrt{(Hh - h^2)}$$

Now, for maximum l, $\frac{d\,(Hh - h^2)}{dh} = 0$

which yields $$h = \frac{H}{2} = 25 \text{ cm.}$$

1.320 Let the velocity of the water jet, near the orifice be v', then applying Bernoullis theorem,

$$\frac{1}{2}\rho\, v'^2 = h_0\, \rho\, g + \frac{1}{2}\rho\, v^2$$

or, $$v' = \sqrt{v^2 - 2g\, h_0} \qquad (1)$$

Here the pressure term on both sides is the same and equal to atmospheric pressure. (In the problem book Fig. should be more clear.)

Now, if it rises upto a height h, then at this height, whole of its kinetic energy will be converted into potential energy. So,

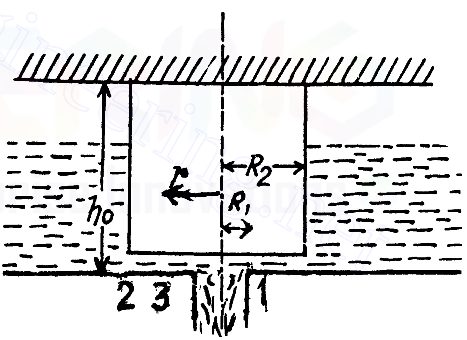
$$\frac{1}{2}\rho\, v'^2 = \rho\, gh \quad \text{or} \quad h = \frac{v'^2}{2g}$$

$$= \frac{v^2}{2g} - h_0 = 20 \text{ cm, [using Eq. (1)]}$$

1.321 Water flows through the small clearance into the orifice. Let d be the clearance. Then from the equation of continuity

$$(2\pi R_1 d)\, v_1 = (2\pi\, r\, d)\, v = (2\pi R_2 d)\, v_2$$

or $$v_1 R_1 = v\, r = v_2 R_2 \qquad (1)$$



where v_1, v_2 and v are respectively the inward radial velocities of the fluid at 1, 2 and 3.

Now by Bernoulli's theorem just before 2 and just after it in the clearance

$$p_0 + h\,\rho g = p_2 + \frac{1}{2}\rho v_2^2 \qquad (2)$$

Applying the same theorem at 3 and 1 we find that this also equals

$$p + \frac{1}{2}\rho v^2 = p_0 + \frac{1}{2}\rho v_1^2 \qquad (3)$$

(since the pressure in the orifice is p_0)

From Eqs. (2) and (3) we also hence

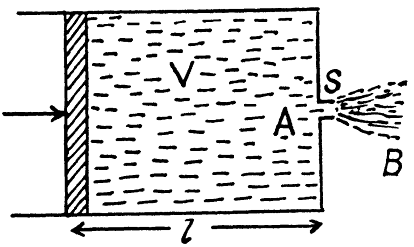
$$v_1 = \sqrt{2gh} \qquad (4)$$

and $$p = p_0 + \frac{1}{2}\rho v_1^2 \left(1 - \left(\frac{v}{v_1}\right)^2\right)$$

$$= p_0 + h\rho g \left(1 - \left(\frac{R_1}{r}\right)^2\right) \quad \text{[Using (1) and (4)]}$$

1.322 Let the force acting on the piston be F and the length of the cylinder be l.

Then, work done $= Fl$ (1)



Applying Bernoulli's theorem for points A and B, $p = \frac{1}{2}\rho v^2$ where ρ is the density and v is the velocity at point B. Now, force on the piston,

$$F = pA = \frac{1}{2}\rho v^2 A \qquad (2)$$

where A is the cross section area of piston.

Also, discharge through the orifice during time interval $t = Svt$ and this is equal to the volume of the cylinder, i.e.,

$$V = Svt \quad \text{or} \quad v = \frac{V}{St} \qquad (3)$$

From Eq. (1), (2) and (3) work done

$$= \frac{1}{2}\rho v^2 A\, l = \frac{1}{2}\rho A \frac{V^2}{(St)^2} l = \frac{1}{2}\rho V^3/S^3 t^2 \quad (\text{as } Al = V)$$

1.323 Let at any moment of time, water level in the vessel be H then speed of flow of water through the orifice, at that moment will be

$$v = \sqrt{2gH} \qquad (1)$$

In the time interval dt, the volume of water ejected through orifice,

$$dV = s\, v\, dt \qquad (2)$$

On the other hand, the volume of water in the vessel at time t equals

$$V = SH$$

Differentiating (3) with respect to time,

$$\frac{dV}{dt} = S\frac{dH}{dt} \quad \text{or} \quad dV = S\, dH \qquad (4)$$

Eqs. (2) and (4)

$$S\, dH = s\, v\, dt \quad \text{or} \quad dt = \frac{S}{s}\frac{dH}{\sqrt{2gH}} \text{ ,from (2)}$$

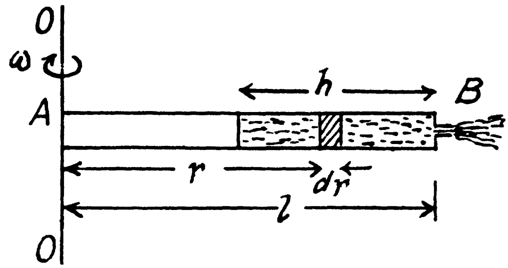
Integrating, $$\int_0^t dt = \frac{S}{s\sqrt{2h}} \int_h^0 \frac{dh}{\sqrt{H}}$$

Thus, $$t = \frac{S}{s}\sqrt{\frac{2h}{g}}$$

1.324 In a rotating frame (with constant angular velocity) the Eulerian equation is

$$-\vec{\nabla} p + \rho \vec{g} + 2\rho(\vec{v}' \times \vec{\omega}) + \rho\omega^2 \vec{r} = \rho \frac{d\vec{v}'}{dt}$$

In the frame of rotating tube the liquid in the "column" is practically static because the orifice is sufficiently small. Thus the Eulerian Eq. in projection form along $\vec{r}$ (which is the position vector of an arbitrary liquid element of lenth dr relative to the rotation axis) reduces to



$$\frac{-dp}{dr} + \rho\,\omega^2 r = 0$$

or, $$dp = \rho\,\omega^2 r dr$$

so, $$\int_{p_0}^{p} dp = \rho\omega^2 \int_{(l-h)}^{r} r dr$$

Thus $$p(r) = p_0 + \frac{\rho\omega^2}{2}\left[r^2 - (l-h)^2\right] \qquad (1)$$

Hence the pressure at the end B just before the orifice i.e.

$$p(l) = p_0 + \frac{\rho\,\omega^2}{2}(2\,l\,h - h^2) \qquad (2)$$

Then applying Bernoull's theorem at the orifice for the points just inside and outside of the end B

$p_0 + \frac{1}{2}\rho\,\omega^2\,(2\,l\,h - h^2) = p_0 + \frac{1}{2}\rho\,v^2$ (where v is the sought velocity)

So, $$v = \omega h\sqrt{\frac{2l}{h} - 1}$$

1.325 The Euler's equation is $\rho\frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla}p = -\vec{\nabla}(p + \rho\,gz)$, where z is vertically upwards.

Now $$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}. \qquad (1)$$

But $$(\vec{v}\cdot\vec{\nabla})\,\vec{v} = \vec{\nabla}\left(\frac{1}{2}v^2\right) - \vec{v}\times \text{Curl}\,\vec{v} \qquad (2)$$

we consider the steady (i.e. $\partial\vec{v}/\partial t = 0$) flow of an incompressible fluid then $\rho =$ constant. and as the motion is irrotational Curl $\vec{v} = 0$

So from (1) and (2) $$\rho\,\vec{\nabla}\left(\frac{1}{2}v^2\right) = -\vec{\nabla}(p + \rho\,gz)$$

or, $$\vec{\nabla}\left(p + \frac{1}{2}\rho\,v^2 + \rho gz\right) = 0$$

Hence $$p + \frac{1}{2}\rho\,v^2 + \rho\,gz = \text{constant.}$$

1.326 Let the velocity of water, flowing through A be v_A and that through B be v_B, then discharging rate through $A = Q_A = S\,v_A$ and similarly through $B = S\,v_B$.

Now, force of reaction at A,

$$F_A = \rho\,Q_A\,v_A = \rho\,S\,v_B^2$$

Hence, the net force,

$$F = \rho S (v_A^2 - v_B^2) \quad \text{as} \quad \vec{F}_A \uparrow\downarrow \vec{F}_B \qquad (1)$$

Applying Bernoulli's theorem to the liquid flowing out of A we get

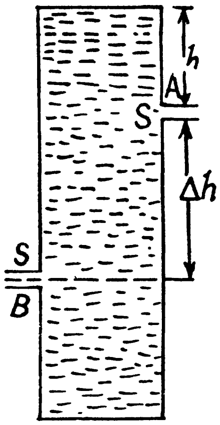
$$\rho_0 + \rho g h = \rho_0 + \frac{1}{2} \rho v_A^2$$

and similarly at B

$$\rho_0 + \rho g (h + \Delta h) = \rho_0 + \frac{1}{2} \rho v_B^2$$

Hence $\quad (v_B^2 - v_A^2) \frac{\rho}{2} = \Delta h \rho g$

Thus $\quad F = 2 \rho g S \Delta h = 0{\cdot}50\text{ N}$



1.327 Consider an element of height dy at a distance y from the top. The velocity of the fluid coming out of the element is

$$v = \sqrt{2 g y}$$

The force of reaction dF due to this is $dF = \rho\, dA\, v^2$, as in the previous problem, $= \rho\, (b\, dy)\, 2 g y$

Integrating $\quad F = \rho g b \int_{h-1}^{h} 2y\, dy$

$$= \rho g b\, [h^2 - (h - l)^2] = \rho g b l\, (2h - l)$$

(The slit runs from a depth $h - l$ to a depth h from the top.)

1.328 Let the velocity of water flowing through the tube at a certain instant of time be u, then $u = \frac{Q}{\pi r^2}$, where Q is the rate of flow of water and πr^2 is the cross section area of the tube.

From impulse momentum theorem, for the stream of water striking the tube corner, in x–direction in the time interval dt,

$$F_x\, dt = -\rho Q u\, dt \quad \text{or} \quad F_x = -\rho Q u$$

and similarly, $F_y = \rho Q u$

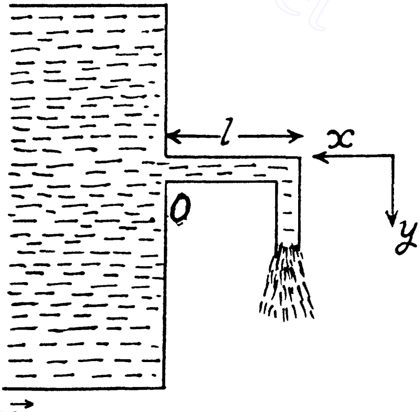
Therefore, the force exerted on the water stream by the tube,

$$\vec{F} = -\rho Q u \vec{i} + \rho Q u \vec{j}$$

According to third law, the reaction force on the tube's wall by the stream equals $(-\vec{F})$

$$= \rho Q u \vec{i} - \rho Q u \vec{j}.$$

Hence, the sought moment of force about 0 becomes



$$\vec{N} = l(-\vec{i}) \times (\rho Q u \vec{i} - \rho Q u \vec{j}) = \rho Q u l \vec{k} = \frac{\rho Q^2}{\pi r^2} l \vec{k}$$

and $\quad |\vec{N}| = \frac{\rho Q^2 l}{\pi r^2} = 0{\cdot}70\text{ N·m}$

1.329 Suppose the radius at A is R and it decreases uniformaly to r at B where $S = \pi R^2$ and $s = \pi r^2$. Assume also that the semi vectical angle at 0 is α. Then

$$\frac{R}{L_2} = \frac{r}{L_1} = \frac{y}{x}$$

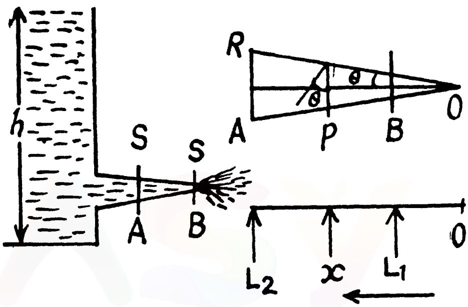
So
$$y = r + \frac{R-r}{L_2 - L_1}(x - L_1)$$

where y is the radius at the point P distant x from the vertex O. Suppose the velocity with which the liquid flows out is V at A, v at B and u at P. Then by the equation of continuity

$$\pi R^2 V = \pi r^2 v = \pi y^2 u$$

The velocity v of efflux is given by

$$v = \sqrt{2gh}$$



and Bernoulli's theorem gives

$$p_p + \frac{1}{2}\rho u^2 = p_0 + \frac{1}{2}\rho v^2$$

where p_p is the pressure at P and p_0 is the atmospheric pressure which is the pressure just outside of B. The force on the nozzle tending to pull it out is then

$$F = \int (p_\rho - p_0)\sin\theta\; 2\pi y ds$$

We have subtracted p_0 which is the force due to atmosphenic pressure the factor $\sin\theta$ gives horizontal component of the force and ds is the length of the element of nozzle surface, $ds = dx \sec\theta$ and

$$\tan\theta = \frac{R-r}{L_2 - L_1}$$

Thus

$$F = \int_{L_1}^{L_2} \frac{1}{2}(v^2 - u^2)\rho\; 2\pi y \;\frac{R-r}{L_2 - L_1}\, dx$$

$$= \pi\rho \int_r^R v^2\left(1 - \frac{r^4}{y^4}\right) y\, dy$$

$$= \pi\rho v^2 \frac{1}{2}\left(R^2 - r^2 + \frac{r^4}{R^2} - r^2\right) = \rho g h\left(\frac{\pi(R^2 - r^2)^2}{R^2}\right)$$

$$= \rho g h\; (S - s)^2/S = 6{\cdot}02\,\text{N} \quad \text{on putting the values.}$$

Note : If we try to calculate F from the momentum change of the liquid flowing out w will be wrong even as regards the sign of the force.

There is of course the effect of pressure at S and s but quantitative derivation of F fron Newton's law is difficult.

1.330 The Euler's equation is $\rho \frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla} p$ in the space fixed frame where $\vec{f} = -\rho g \vec{k}$ downward. We assume incompressible fluid so ρ is constant.

Then $\vec{f} = -\vec{\nabla}(\rho g z)$ where z is the height vertically upwards from some fixed origin. We go to rotating frame where the equation becomes

$$\rho \frac{d\vec{v}'}{dt} = -\vec{\nabla}(p + \rho g z) + \rho \omega^2 \vec{r} + 2\rho (\vec{v}' \times \vec{\omega})$$

the additional terms on the right are the well known coriolis and centrifugal forces. In the frame rotating with the liquid $\vec{v}' = 0$ so

$$\vec{\nabla}\left(p + \rho g z - \frac{1}{2}\rho \omega^2 r^2\right) = 0$$

or
$$p + \rho g z - \frac{1}{2}\rho \omega^2 r^2 = \text{constant}$$

On the free surface $p =$ constant, thus

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}$$

If we choose the origin at point $r = 0$ (i.e. the axis) of the free surface then "cosntant" = 0 and

$$z = \frac{\omega^2}{2g} r^2 \quad \text{(The paraboloid of revolution)}$$

At the bottom $z =$ constant

So
$$p = \frac{1}{2}\rho \omega^2 r^2 + \text{constant}$$

If $p = p_0$ on the axis at the bottom, then

$$p = p_0 + \frac{1}{2}\rho \omega^2 r^2.$$

1.331 When the disc rotates the fuild in contact with, corotates but the fluid in contact with the walls of the cavity does not rotate. A velocity gradient is then set up leading to viscous forces. At a distance r from the axis the linear velocity is ωr so there is a velocity gradient $\frac{\omega r}{h}$ both in the upper and lower clearance. The corresponding force on the element whose radial width is dr is

$$\eta\, 2\pi r dr\ \frac{\omega r}{h} \quad \left(\text{from the formular } F = \eta A \frac{dv}{dx}\right)$$

The torque due to this force is

$$\eta\ 2\pi r dr\ \frac{\omega r}{h}\ r$$

and the net torque considering both the upper and lower clearance is

$$2\int_0^R \eta 2\pi r^3\, dr \frac{\omega}{h}$$
$$= \pi R^4 \omega \eta / h$$

So power developed is

$$P = \pi R^4 \omega^2 \eta / h = 9 \cdot 05 \text{ W (on putting the values).}$$

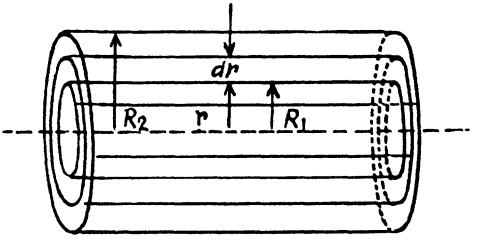
(As instructed end effects i.e. rotation of fluid in the clearance $r > R$ has been neglected.)

1.332 Let us consider a coaxial cylinder of radius r and thickness dr, then force of friction or viscous force on this elemental layer, $F = 2\pi r l \eta \frac{dv}{dr}$.

This force must be constant from layer to layer so that steady motion may be possible.

or, $$\frac{F\,dr}{r} = 2\pi l \eta\, dv. \qquad (1)$$

Integrating,



$$F\int_{R_2}^{r}\frac{dr}{r} = 2\pi l\eta\int_0^v dv$$

or, $$F\ln\left(\frac{r}{R_2}\right) = 2\pi l \eta v \qquad (2)$$

Putting $r = R_1$, we get

$$F\ln\frac{R_1}{R_2} = 2\pi l \eta v_0$$

From (2) by (3) we get,

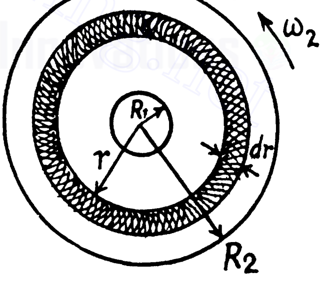
$$v = v_0\frac{\ln r/R_2}{\ln R_1/R_2}$$

Note : The force F is supplied by the agency which tries to carry the inner cylinder with velocity v_0 .

1.333 (a) Let us consider an elemental cylinder of radius r and thickness dr then from Newton's formula

$$F = 2\pi r l \eta r\frac{d\omega}{dr} = 2\pi l \eta r^2\frac{d\omega}{dr}$$

and moment of this force acting on the element,



$$N = 2\pi r^2 l\eta\frac{d\omega}{dr} r = 2\pi r^3 l\eta\frac{d\omega}{dr}$$

or, $$2\pi l\eta\, d\omega = N\frac{dr}{r^3} \qquad (2)$$

As in the previous problem N is constant when conditions are steady

Integrating, $$2\pi l\eta\int_0^{\omega} d\omega = N\int_{R_1}^{r}\frac{dr}{r^3}$$

or, $$2\pi l\eta\omega = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right] \qquad (3)$$

Putting $r = R_2$, $\omega = \omega_2$, we get

$$2\pi l\eta\omega_2 = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right] \qquad (4)$$

From (3) and (4),

$$\omega = \omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right]$$

(b) From Eq. (4),

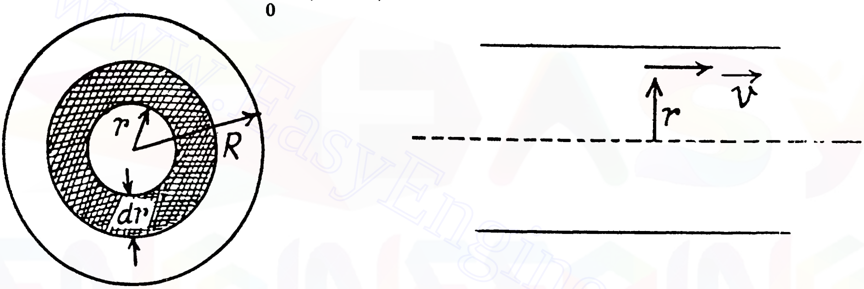
$$N_1 = \frac{N}{l} = 4\pi\eta\,\omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

1.334 (a) Let dV be the volume flowing per second through the cylindrical shell of thickness dr then,

$$dV = -(2\pi r\,dr)\,v_0\left(1 - \frac{r^2}{R^2}\right) = 2\pi v_0\left(r - \frac{r^3}{R^2}\right)dr$$

and the total volume,

$$V = 2\pi v_0 \int_0^R \left(r - \frac{r^3}{R^2}\right)dr = 2\pi v_0 \frac{R^2}{4} = \frac{\pi}{2} R^2 v_0$$



(b) Let, dE be the kinetic energy, within the above cylindrical shell. Then

$$dT = \frac{1}{2}(dm)\,v^2 = \frac{1}{2}(2\pi r\,l\,d\,r\rho)\,v^2$$

$$= \frac{1}{2}(2\pi\,l\,\rho)\,r\,dr\,v_0^2\left(1 - \frac{r^2}{R^2}\right) = \pi\,l\,\rho\,v_0\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right]dr$$

Hence, total energy of the fluid,

$$T = \pi\,l\,\rho\,v_0^2 \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr = \frac{\pi R^2 \rho\,l\,v_0^2}{6}$$

(c) Here frictional force is the shearing force on the tube, exerted by the fluid, which equals $-\eta S \frac{dv}{dt}$.

Given, $$v = v_0\left(1 - \frac{r^2}{R^2}\right)$$

So, $$\frac{dv}{dr} = -2 v_0 \frac{r}{R^2}$$

And at $$r = R,\ \frac{dv}{dr} = -\frac{2 v_0}{R}$$

Then, viscous force is given by, $F = -\eta\,(2\pi Rl)\left(\frac{dv}{dr}\right)_{r=R}$

$$= -2\pi R\eta l\left(-\frac{2v_0}{R}\right) = 4\pi\eta v_0 l$$

(d) Taking a cylindrical shell of thickness dr and radius r viscous force,

$$F = -\eta\,(2\pi rl)\frac{dv}{dr},$$

Let Δp be the pressure difference, then net force on the element $= \Delta p\,\pi r^2 + 2\pi\eta\, l\, r\frac{dv}{dr}$

But, since the flow is steady, $F_{net} = 0$

or, $$\Delta p = \frac{-2\pi l\eta r\frac{dv}{dr}}{\pi r^2} = \frac{-2\pi l\,\eta r\left(-2v_0\frac{r}{R^2}\right)}{\pi r^2} = 4\eta v_0 l/R^2$$

1.335 The loss of pressure head in travelling a distance l is seen from the middle section to be $h_2 - h_1 = 10$ cm. Since $h_2 - h_1 = h_1$ in our problem and $h_3 - h_2 = 15\text{ cm} = 5 + h_2 - h_1$, we see that a pressure head of 5 cm remains incompensated and must be converted into kinetic energy, the liquid flowing out. Thus

$$\frac{\rho v^2}{2} = \rho g\,\Delta h \quad \text{where } \Delta h = h_3 - h_2$$

Thus $$v = \sqrt{2g\Delta h} = 1 \text{ m/s}$$

1.336 We know that, Reynold's number (R_e) is defined as, $R_e = \rho v l/\eta$, where v is the velocity l is the characteristic length and η the coefficient of viscosity. In the case of circular cross section the chracteristic length is the diameter of cross-section d, and v is taken as average velocity of flow of liquid.

Now, R_{e_1} (Reynold's number at x_1 from the pipe end) $= \frac{\rho d_1 v_1}{m\eta}$ where v_1 is the velocity at distance x_1

and similarly, $$R_{e_2} = \frac{\rho d_2 v_2}{\eta} \quad \text{so} \quad \frac{R_{e_1}}{R_{e_2}} = \frac{d_1 v_1}{d_2 v_2}$$

From equation of continuity, $A_1 v_1 = A_2 v_2$

or, $$\pi r_1^2 v_1 = \pi r_2^2 v_2 \quad \text{or} \quad d_1 v_1 r_1 = d_2 v_2 r_2$$

$$\frac{d_1 v_1}{d_2 v_2} = \frac{r_2}{r_1} = \frac{r_0 e^{-\alpha x_2}}{r_0 e^{-\alpha x_1}} = e^{-\alpha\Delta x} \quad (\text{as } x_2 - x_1 = \Delta x)$$

Thus $$\frac{R_{e_2}}{R_{e_1}} = e^{\alpha\Delta x} = 5$$

1.337 We know that Reynold's number for turbulent flow is greater than that on laminar flow.

Now, $$(R_e)_l = \frac{\rho v d}{\eta} = \frac{2\rho_1 v_1 r_1}{\eta_1} \quad \text{and} \quad (R_e)_t = \frac{2\rho_2 v_2 r_2}{\eta}$$

But, $(R_e)_t \geq (R_e)_l$

so $$v_{2_{max}} = \frac{\rho_1 v_1 r_1 \eta_2}{\rho_2 r_2 \eta_1} = 5\ \mu\text{ m/s on putting the values.}$$

1.338 We have $R = \dfrac{v \rho_0 d}{\eta}$ and v is given by

$$6 \pi \eta r v = \frac{4\pi}{3} r^2 (\rho - \rho_0) g$$

(ρ = density of lead, ρ_0 = density of glycerine.)

$$v = \frac{2}{9\eta}(\rho - \rho_0) g r^2 = \frac{1}{18\eta}(\rho - \rho_0) g d^2$$

Thus $$\frac{1}{2} = \frac{1}{18\eta^2}(\rho - \rho_o) g \rho_0 d^3$$

and $d = [9\eta^2/\rho_0(\rho - \rho_0) g]^{1/3} = 5{\cdot}2$ mm on putting the values.

1.339 $$m\frac{dv}{dt} = mg - 6\pi\eta rv$$

or $$\frac{dv}{dt} + \frac{6\pi\eta r}{m} v = g$$

or $$\frac{dv}{dt} + kv = g,\ k = \frac{6\pi\eta r}{m}$$

or $$e^{kt}\frac{dv}{dt} + ke^{kt} v = ge^{kt} \quad \text{or} \quad \frac{d}{dt} e^{kt} v = ge^{kt}$$

or $$v e^{kt} = \frac{g}{k} e^{kt} + C \quad \text{or} \quad v = \frac{g}{k} + C e^{-kt} \quad \text{(where } C \text{ is const.)}$$

Since $$v = 0 \text{ for } t = 0,\ 0 = \frac{g}{k} + C$$

So $$C = -\frac{g}{k}$$

Thus $$v = \frac{g}{k}(1 - e^{-kt})$$

The steady state velocity is $\dfrac{g}{k}$.

v differs from $\dfrac{g}{k}$ by n where $e^{-kt} = n$

or $$t = \frac{1}{k}\ln n$$

Thus $$\frac{1}{k} = -\frac{\frac{4\pi}{3} r^3 P}{6\pi\eta r} = -\frac{4 r^2 \rho}{18\eta} = -\frac{d^2 \rho}{18\eta}$$

We have neglected buoyancy in olive oil.

1.8 RELATIVISTIC MECHANICS

1.340 From the formula for length contraction

$$\left(l_0 - l_0\sqrt{1-\frac{v^2}{c^2}}\right) = \eta\, l_0$$

So, $$1-\frac{v^2}{c^2} = (1-\eta)^2 \quad \text{or} \quad v = c\sqrt{\eta\,(2-\eta)}$$

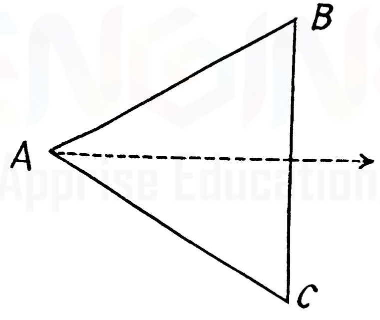
1.341 (a) In the frame in which the triangle is at rest the space coordinates of the vertices are (000), $\left(a\frac{\sqrt{3}}{2}, +\frac{a}{2}, 0\right)\left(a\frac{\sqrt{3}}{2}, -\frac{a}{2}, 0\right)$, all measured at the same time t. In the moving frame the corresponding coordinates at time t' are

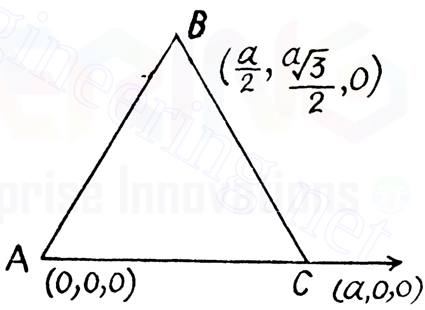
$A:(vt', 0, 0), B:\left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2}+vt', \frac{a}{2}, 0\right)$ and $C:\left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2}+vt', -\frac{a}{2}, 0\right)$

The perimeter P is then

$$P = a + 2a\left(\frac{3}{4}(1-\beta^2)+\frac{1}{4}\right)^{1/2} = a\left(1+\sqrt{4-3\beta^2}\right)$$

(b) The coordinates in the first frame are shown at time t. The coordinates in the moving frame are,





$A:(vt', 0,0), B:\left(\frac{a}{2}\sqrt{1-\beta^2}+vt',\ a\frac{\sqrt{3}}{2}, 0\right),\ C:\left(a\sqrt{1-\beta^2}+vt', 0, 0\right)$

The perimeter P is then

$$P = a\sqrt{1-\beta^2}+\frac{a}{2}\left[1-\beta^2+3\right]^{1/2}\times 2 = a\left(\sqrt{1-\beta^2}+\sqrt{4-\beta^2}\right) \text{ here } \beta = \frac{V}{c}$$

1.342 In the rest frame, the coordinates of the ends of the rod in terms of proper length l_0

$$A:(0,0,0)\ B:(l_0\cos\theta_0, l_0\sin\theta_0, 0)$$

at time t. In the laboratory frame the coordinates at time t' are

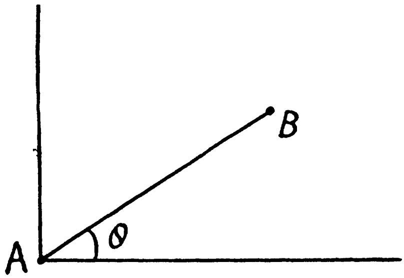
$$A:(vt', 0, 0), B:\left(l_0\cos\theta_0\sqrt{1-\beta^2}+vt', l_0\sin\theta_0, 0\right)$$

Therefore we can write,

$$l\cos\theta_0 = l_0\cos\theta_0\sqrt{1-\beta^2} \text{ and } l\sin\theta = l_0\sin\theta_0$$

Hence $l_0^2 = (l^2)\left(\frac{\cos^2\theta + (1-\beta^2)\sin^2\theta}{1-\beta^2}\right)$

or, $= \sqrt{\frac{1-\beta^2\sin^2\theta}{1-\beta^2}}$



1.343 In the frame K in which the cone is at rest the coordinates of A are (0,0,0) and of B are $(h, h\tan\theta, 0)$. In the frame K', which is moving with velocity v along the axis of the cone, the coordinates of A and B at time t' are

$A : (-vt', 0,0), B : \left(h\sqrt{1-\beta^2} - vt', h\tan\theta, 0\right)$

Thus the taper angle in the frame K' is

$$\tan\theta' = \frac{\tan\theta}{\sqrt{1-\beta^2}}\left(= \frac{y'_B - y'_A}{x'_B - x'_A}\right)$$

and the lateral surface area is,

$$S = \pi h'^2 \sec\theta' \tan\theta'$$

$$= \pi h^2(1-\beta^2)\frac{\tan\theta}{\sqrt{1-\beta^2}}\sqrt{1+\frac{\tan^2\theta}{1-\beta^2}} = S_0\sqrt{1-\beta^2\cos^2\theta}$$

Here $S_0 = \pi h^2 \sec\theta\tan\theta$ is the lateral surface area in the rest frame and

$$h' = h\sqrt{1-\beta^2}, \ \beta = v/c.$$

1.344 Because of time dilation, a moving clock reads less time. We write,

$$t - \Delta t = t\sqrt{1-\beta^2}, \ \beta = \frac{v}{c}$$

Thus, $$1 - \frac{2\Delta t}{t} + \left(\frac{\Delta t}{t}\right)^2 = 1 - \beta^2$$

or, $$v = c\sqrt{\frac{\Delta t}{t}\left(2 - \frac{\Delta t}{t}\right)}$$

1.345 In the frame K the length l of the rod is related to the time of flight Δt by

$$l = v\,\Delta t$$

In the reference frame fixed to the rod (frame K')the proper length l_0 of the rod is given by

$$l_0 = v\,\Delta t'$$

But $$l_0 = \frac{l}{\sqrt{1-\beta^2}} = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}, \ \beta = \frac{v}{c}$$

Thus, $$v\,\Delta t' = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}$$

So $$1-\beta^2 = \left(\frac{\Delta t}{\Delta t'}\right)^2 \quad \text{or} \quad v = c\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

and $$l_0 = c\sqrt{(\Delta t')^2-(\Delta t)^2} = c\,\Delta t'\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

1.346 The distance travelled in the laboratory frame of reference is $v\Delta t$ where v is the velocity of the particle. But by time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \quad \text{So} \quad v = c\sqrt{1-(\Delta t_0/\Delta t)^2}$$

Thus the distance traversed is

$$c\Delta t\sqrt{1-(\Delta t_0/\Delta t)^2}$$

1.347 (a) If τ_0 is the proper life time of the muon the life time in the moving frame is

$$\frac{\tau_0}{\sqrt{1-v^2/c^2}} \quad \text{and hence} \quad l = \frac{v\,\tau_0}{\sqrt{1-v^2/c^2}}$$

Thus $$\tau_0 = \frac{l}{v}\sqrt{1-v^2/c^2}$$

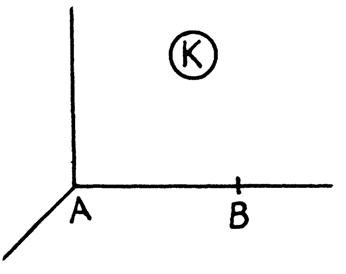
(The words "from the muon's stand point" are not part of any standard terminology)

1.348 In the frame K in which the particles are at rest, their positions are A and B whose coordinates may be taken as,

$$A:(0,0,0),\ B = (l_0\,,0,0)$$

In the frame K' with respect to which K is moving with a velocity v the coordinates of A and B at time t' in the moving frame are

$$A = (vt', 0,0)\ B = \left(l_0\sqrt{1-\beta^2}+vt', 0,0\right),\ \beta = \frac{v}{c}$$



Suppose B hits a stationary target in K' after time t'_B while A hits it after time $t_B+\Delta t$. Then,

$$l_0\sqrt{1-\beta^2}+vt'_B = v\,(t'_B+\Delta t)$$

So, $$l_0\frac{v\,\Delta t}{\sqrt{1-v^2/c^2}}$$

1.349 In the reference frame fixed to the ruler the rod is moving with a velocity v and suffers Lorentz contraction. If l_0 is the proper length of the rod, its measured length will be

$$\Delta x_1 = l_0\sqrt{1-\beta^2}\,,\ \beta = \frac{v}{c}$$

In the reference frame fixed to the rod the ruler suffers Lorentz contraction and we must have

$$\Delta x_2 \sqrt{1-\beta^2} = l_0 \text{ thus } l_0 = \sqrt{\Delta x_1 \Delta x_2}$$

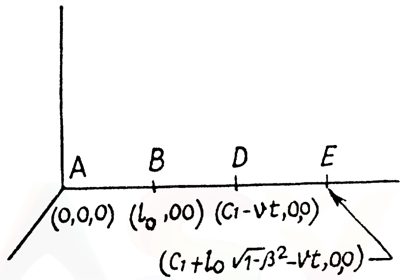
and
$$1-\beta^2 = \frac{\Delta x_1}{\Delta x_2} \text{ or } v = c\sqrt{1-\frac{\Delta x_1}{\Delta x_2}}$$

1.350 The coordinates of the ends of the rods in the frame fixed to the left rod are shown. The points B and D coincides when

$$l_0 = c_1 - vt_0 \text{ or } t_0 = \frac{c_1 - l_0}{v}$$

The points A and E coincide when

$$0 = c_1 + l_0\sqrt{1-\beta^2} - vt_1, \; t_1 = \frac{c_1 + l_0\sqrt{1-\beta^2}}{v}$$



Thus $\Delta t = t_1 - t_0 = \frac{l_0}{v}\left(1 + \sqrt{1-\beta^2}\right)$

or $\left(\frac{v\,\Delta t}{l_0} - 1\right)^2 = 1-\beta^2 = 1-\frac{v^2}{c^2}$

From this
$$v = \frac{2c^2 \Delta t/l_0}{1 + c^2 \Delta t^2/l_0^2} = \frac{2l_0/\Delta t}{1+(l_0/c\,\Delta t)^2}$$

1.351 In K_0, the rest frame of the particles, the events corresponding to the decay of the particles are,

$$A : (0,0,0,0) \text{ and } (0, l_0, 0,0) = B$$

In the reference frame K, the corresponding coordintes are by Lorentz transformation

$$A : (0,0,0,0), B : \left(\frac{vl_0}{c^2\sqrt{1-\beta^2}}, \frac{l_0}{\sqrt{1-\beta^2}}, 0,0\right)$$

Now
$$l_0\sqrt{1-\beta^2} = l$$

by Lorentz Fitzgerald contraction formula. Thus the time lag of the decay time of B is

$$\Delta t = \frac{vl_0}{c^2\sqrt{1-\beta^2}} = \frac{v\,l}{c^2(1-\beta^2)} = \frac{vl}{c^2-v^2}$$

B decays later (B is the forward particle in the direction of motion)

1.352 (a) In the reference frame K with respect to which the rod is moving with velocity v, the coordinates of A and B are

$$A : t, x_A + v(t - t_A), 0, 0$$

$$B : t, x_B + v(t - t_B), 0, 0$$

Thus $l = x_A - x_B - v(t_A - t_B) = l_0\sqrt{1-\beta^2}$

So $$l_0 = \frac{x_A - x_B - v(t_A - t_B)}{\sqrt{1 - v^2/c^2}}$$

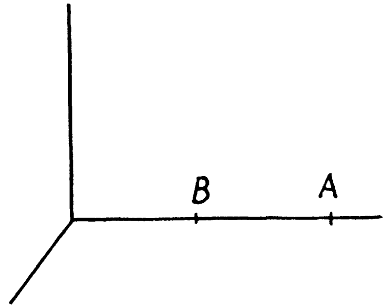
(b) $\pm l_0 - v(t_A - t_B) = l = l_0\sqrt{1 - v^2/c^2}$

(since $x_A - x_B$ can be either $+l_0$ or $-l_0$.)

Thus $v(t_A - t_B) = \left(\pm 1 - \sqrt{1 - v^2/c_2}\right) l_o$

i.e. $$t_A - t_B = \frac{l_0}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$$

or $$t_B - t_A = \frac{l_0}{v}\left(1 + \sqrt{1 - v^2/c^2}\right)$$



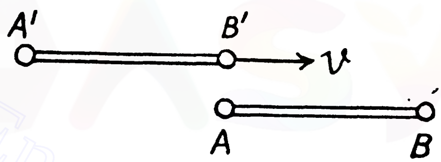
1.353 At the instant the picture is taken the coordintes of A, B, A', B' in the rest frame of AB are

$A : (0, 0, 0, 0)$

$B : (0, l_0, 0, 0)$

$B' : (0, 0, 0, 0)$

$A' : (0_1 - l_0\sqrt{1 - v^2/c^2}, 0, 0)$



In this frame the coordinates of B' at other times are $B' : (t, vt, 0, 0)$. So B' is opposite to B at time $t(B) = \frac{l_0}{v}$. In the frame in which B', A' is at rest the time corresponding this is by Lorentz tranformation.

$$t^0(B') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{l_0}{v} - \frac{vl_0}{c^2}\right) = \frac{l_0}{v}\sqrt{1 - v^2/c^2}$$

Similarly in the rest frame of A, B, te coordinates of A at other times are

$$A' : \left(t, -l_0\sqrt{1 - \frac{v^2}{c^2}} + vt, 0, 0\right)$$

A' is opposite to A at time $t(A) = \frac{l_0}{v}\sqrt{1 - \frac{v^2}{c^2}}$

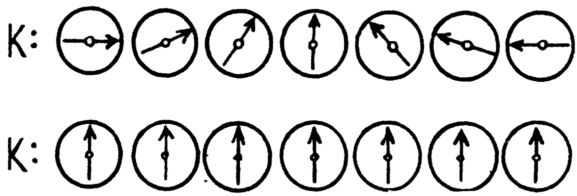
The corresponding time in the frame in which A', B' are at rest is

$$t(A') = \gamma\, t(A) = \frac{l_0}{v}$$

1.354 By Lorentz transformation $t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$

So at time $\qquad t = 0, \; t' = \dfrac{vx}{c^2}\dfrac{1}{\sqrt{1 - v^2/c^2}}$

If $x > 0 \; t' < 0$, if $x < 0$, $t' > 0$ and we get the diagram given below "in terms of the K–clock".



The situation in terms of the K' clock is reversed.

1.355 Suppose $x(t)$ is the locus of points in the frame K at which the readings of the clocks of both reference system are permanently identical, then by Lorentz transformation

$$t' = \frac{1}{\sqrt{1 - V^2/c^2}}\left(t - \frac{V x(t)}{c^2}\right) = t$$

So differentiating $x(t) = \dfrac{c^2}{V}\left(1 - \sqrt{1 - \dfrac{V^2}{c^2}}\right) = \dfrac{c}{\beta}\left(1 - \sqrt{1 - \beta^2}\right), \; \beta = \dfrac{V}{c}$

Let $\qquad \beta = \tan h\theta, \; 0 \le \theta < \infty$, Then

$$x(t) = \frac{c}{\tan h\theta}\left(1 - \sqrt{1 - \tan h^2\theta}\right) = c\,\frac{\cos h\theta}{\sin h\,\theta}\left(1 - \frac{1}{\cos h\,\theta}\right)$$

$$= c\,\frac{\cos h\,\theta - 1}{\sin h\,\theta} = c\sqrt{\frac{\cos h\,\theta - 1}{\cos h\,\theta + 1}} = c \tan h\frac{\theta}{2} \le v$$

·(tan $h\,\theta$ is a monotonically increasing function of θ)

1.356 We can take the coordinates of the two events to be

$$A : (0, 0, 0, 0) \quad B : (\Delta t, a, 0, 0)$$

For B to be the effect and A to be cause we must have $\Delta t > \dfrac{|a|}{c}$.

In the moving frame the coordinates of A and B become

$A : (0, 0, 0, 0), B : \left[\gamma\left(\Delta t - \dfrac{aV}{c^2}\right), \gamma(a - V\Delta t), 0, 0\right]$ where $\gamma = \dfrac{1}{\sqrt{1 - \left(\dfrac{V^2}{c^2}\right)}}$

Since

$$(\Delta t')^2 - \frac{a'^2}{c^2} = \gamma^2\left[\left(\Delta t - \frac{aV}{c^2}\right)^2 - \frac{1}{c^2}(a - V\Delta t)^2\right] = (\Delta t)^2 - \frac{a^2}{c^2} > 0$$

we must have $\Delta t' > \dfrac{|a'|}{c}$

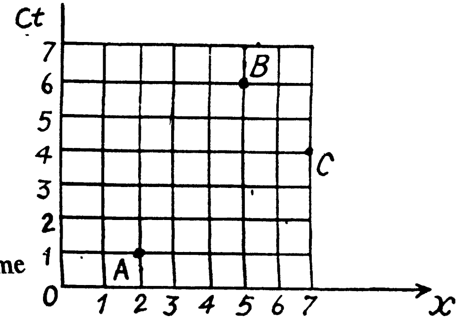
1.357 (a) The four-dimensional interval between A and B (assuming $\Delta y = \Delta z = 0$) is :

$$5^2 - 3^2 = 16 \text{ units}$$

Therefore the time interval between these two events in the reference frame in which the events occurred at the same place is

$$c\,(t'_B - t'_A) = \sqrt{16} = 4\text{ m}$$

or $$t'_B - t'_A = \frac{4}{c} = \frac{4}{3} \times 10^{-8}\text{ s}$$



(b) The four dimensional interval between A and C is (assuming $\Delta y = \Delta z = 0$)

$$3^2 - 5^2 = -16$$

So the distance between the two events in the frame in which they are simultaneous is 4 units = 4m.

1.358 By the velocity addition formula

$$v_x' = \frac{v_x - V}{1 - \dfrac{V v_x}{c^2}}, \quad v'_y = \frac{v_y \sqrt{1 - V^2/c^2}}{1 - \dfrac{v_x V}{c^2}}$$

and $$v' = \sqrt{v'^2_x + v'^2_y} = \frac{\sqrt{(v_x - V)^2 + v_y^2\,(1 - V^2/c^2)}}{1 - \dfrac{v_x V}{c^2}}$$

1.359 (a) By definition the velocity of apporach is

$$v_{\text{approach}} = \frac{dx_1}{dt} - \frac{dx_2}{dt} = v_1 - (-v_2) = v_1 + v_2$$

in the reference frame K.

(b) The relative velocity is obtained by the transformation law

$$v_r = \frac{v_1 - (-v_2)}{1 - \dfrac{v_1(-v_2)}{c^2}} = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

1.360 The velocity of one of the rods in the reference frame fixed to the other rod is

$$V = \frac{v + v}{1 + \dfrac{v^2}{c^2}} = \frac{2v}{1 + \beta^2}$$

The length of the moving rod in this frame is

$$l = l_0 \sqrt{1 - \frac{4\,v^2/c^2}{(1 + \beta^2)^2}} = l_0\,\frac{1 - \beta^2}{1 + \beta^2}$$

1.361 The approach velocity is defined by

$$\vec{V}_{\text{approach}} = \frac{d\vec{r}_1}{dt} - \frac{d\vec{r}_2}{dt} = \vec{V}_1 - \vec{V}_2$$

in the laboratory frame. So $V_{\text{approach}} = \sqrt{v_1^2 + v_2^2}$

On the other hand, the relative velocity can be obtained by using the velocity addition formula and has the components

$$\left[-v_1,\ v_2\sqrt{1-\left(\frac{v_1^2}{c^2}\right)}\right] \text{ so } V_r = \sqrt{v_1^2+v_2^2-\frac{v_1 v_2^2}{c^2}}$$

1.362 The components of the velocity of the unstable particle in the frame K are

$$\left(V,\ v'\sqrt{1-\frac{V^2}{c^2}},\ 0\right)$$

so the velocity relative to K is

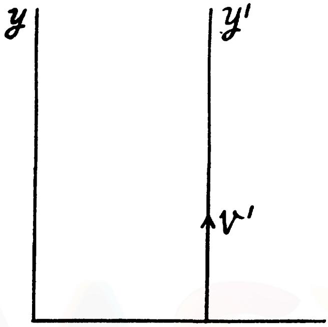
$$\sqrt{V^2+v'^2-\frac{v'^2 V^2}{c^2}}$$

The life time in this frame dilates to

$$\Delta t_0 \Big/ \sqrt{1-\frac{V^2}{c^2}-\frac{v'^2}{c^2}+\frac{v'^2 V^2}{c^4}}$$

and the distance traversed is

$$\Delta t_0 \frac{\sqrt{V^2+v'^2-(v'^2 V^2)/c^2}}{\sqrt{1-V^2/c^2}\,\sqrt{1-v'^2/c^2}}$$

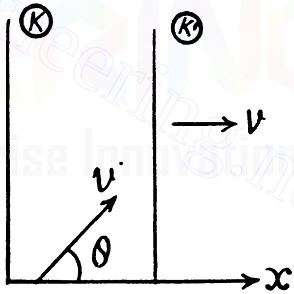


1.363 In the frame K' the components of the velocity of the particle are

$$v'_x = \frac{v\cos\theta - V}{1-\dfrac{v V\cos\theta}{c^2}}$$

$$v'_y = \frac{v\sin\theta\sqrt{1-V^2/c^2}}{1-\dfrac{v V}{c^2}\cos\theta}$$

Hence, $\tan\theta' = \dfrac{v'_y}{v'_x} = \dfrac{v\sin\theta}{v\cos\theta - V}\sqrt{(1-V^2)/c^2}$



1.364 In K' the coordinates of A and B are

$$A:(t',0,-v't',0);\ B:(t',l,-v't',0)$$

After performing Lorentz transformation to the frame K *we get*

$$A: t=\gamma t' \quad B: t=\gamma\left(t'+\frac{Vl}{c^2}\right)$$

$$x=\gamma V t' \quad x=\gamma(l+Vt')$$

$$y=v't' \quad y=-v't'$$

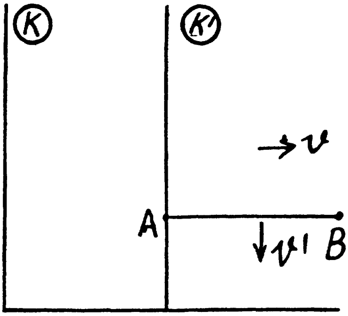
$$z=0 \quad z=0$$

By translating $t' \to t' - \dfrac{Vl}{c^2}$, we can write

the coordinates of B as $B: t=\gamma t'$

$$x = \gamma l \left(1 - \frac{V^2}{c^2}\right) + V t' \gamma = l \sqrt{1 - \frac{v^2}{c^2}} + V t' \gamma$$

$$y = -v' \left(t' - \frac{Vl}{c^2}\right), \; z = 0$$



Thus $$\Delta x = l \sqrt{1 - \left(\frac{V^2}{c^2}\right)}, \; \Delta y = \frac{v' V l}{c^2}$$

Hence $$\tan \theta' = \frac{v' V}{c^2 \sqrt{1 - \frac{v' V}{c^2}}}$$

1.365 t ---------- $t + dt$ Ⓚ

$\vec{v}$ $\quad \vec{v} + \vec{w} dt$

In K the velocities at time t and $t + dt$ are respectively v and $v + w dt$ along x – axis which is parallel to the vector $\vec{V}$. In the frame K' moving with velocity $\vec{V}$ with respect to K, the velocities are respectively,

$$\frac{v - V}{1 - \frac{vV}{c^2}} \text{ and } \frac{v + w\,dt - V}{1 - (v + w\,dt)\frac{V}{c^2}}$$

The latter velocity is written as

$$\frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt}{1 - v\frac{V}{c^2}} + \frac{v - V}{\left(1 - \frac{vV}{c^2}\right)} \frac{wV}{c^2} dt = \frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{vV}{c^2}\right)^2}$$

Also by Lorentz transformation

$$dt' = \frac{dt - V\,dx/c^2}{\sqrt{1 - V^2/c^2}} = dt\,\frac{1 - vV/c^2}{\sqrt{1 - V^2/c^2}}$$

Thus the acceleration in the K' frame is

$$w' = \frac{dv'}{dt'} = \frac{w}{\left(1 - \frac{vV}{c^2}\right)^3}\left(1 - \frac{V^2}{c^2}\right)^{3/2}$$

(b) In the K frame the velocities of the particle at the time t and $t + dt$ are repectively

$$(0, v, 0) \text{ and } (0, v + w dt, 0)$$

where $\vec{V}$ is along x–axis. In the K' frame the velocities are

$$\left(-V, v\sqrt{1 - V^2/c^2}, 0\right)$$

and $$\left(-V, (v + w dt)\sqrt{1 - V^2/c^2}, 0\right) \text{ respectively}$$

Thus the acceleration

$$w' = \frac{w\,dt\sqrt{(1 - V^2/c^2)}}{dt'} = w\left(1 - \frac{V^2}{c^2}\right) \text{ along the } y\text{-axis.}$$

We have used $dt' = \dfrac{dt}{\sqrt{1 - V^2/c^2}}$

1.366 In the instantaneous rest frame $v = V$ and

$$w' = \frac{w}{\left(1 - \dfrac{V^2}{c^2}\right)^{3/2}} \text{ (from 1.365a)}$$

So,

$$= \frac{dv}{\left(1 - \dfrac{V^2}{c^2}\right)^{3/2}} = w'\,dt$$

w' is constant by assumption. Thus integration gives

$$v = \frac{w'\,t}{\sqrt{1 + \left(\dfrac{w'\,t}{c}\right)^2}}$$

Integrating once again $x = \dfrac{c^2}{w'}\left(\sqrt{1 + \left(\dfrac{w'\,t}{c}\right)^2} - 1\right)$

1.367 The boost time τ_0 in the reference frame fixed to the rocket is related to the time τ elapsed on the earth by

$$\tau_0 = \int_0^\tau \sqrt{1 - \frac{v^2}{c^2}}\,dt = \int_0^\tau \left[1 - \frac{\left(\dfrac{w'\,t}{c}\right)^2}{1 + \left(\dfrac{w'\,t}{c}\right)^2}\right]^{1/2} dt$$

$$= \int_0^\tau \frac{dt}{\sqrt{1 + \left(\dfrac{w'\,t}{c}\right)^2}} = \frac{c}{w'}\int_0^{(w'\tau)/c} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{c}{w'}\ln\left[\frac{w'\tau}{c} + \sqrt{1 + \left(\frac{w'\tau}{c}\right)^2}\right]$$

1.368 $m = \dfrac{m_0}{\sqrt{1 - \beta^2}}$

For $\beta \to 1$, $\dfrac{m}{m_0} \to \dfrac{1}{\sqrt{2(1 - \beta)}} = \dfrac{1}{\sqrt{2}\,\eta}$

1.369 We define the density ρ in the frame K in such a way that $\rho\,dx\,dy\,dz$ is the rest mass dm_0 of the element. That is $\rho\,dx\,dy\,dz = \rho_0\,dx_0\,dy_0\,dz_0$, where ρ_0 is the proper density dx_0, dy_0, dz_0 are the dimensions of the element in the rest frame K_0. Now

$$dy = dy_0,\ dz = dz_0,\ dx = dx_0\sqrt{1 - \frac{v^2}{c^2}}$$

if the frame K is moving with velocity, v relative to the frame K_0. Thus

$$\rho = \frac{\rho_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Defining η by $\rho = \rho_0 (1 + \eta)$

We get $\quad 1 + \eta = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ or, $\dfrac{v^2}{c^2} = 1 - \dfrac{1}{(1+\eta)^2} = \dfrac{\eta(2+\eta)}{(1+\eta)^2}$

or $$v = c\sqrt{\frac{\eta(2+\eta)}{(1+\eta)^2}} = \frac{c\sqrt{\eta(2+\eta)}}{1+\eta}$$

1.370 We have

$$\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = p \quad \text{or,} \quad \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{m_0^2 + \frac{p^2}{c^2}}$$

or $$1 - \frac{v^2}{c^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2} = 1 - \frac{p^2}{p^2 + m_0^2 c^2}$$

or $$v = \frac{c_p}{\sqrt{p^2 + m_0^2 c^2}} = \frac{c}{\sqrt{1 + \left(\frac{m_0 c}{p}\right)^2}}$$

So $$\frac{c - v}{c} = \left[1 - \left(1 + \left(\frac{m_0 c}{p}\right)^2\right)^{-1/2}\right] \times 100\ \% \approx \frac{1}{2}\left(\frac{m_0 c}{p}\right)^2 \times 100\ \%$$

1.371 By definition of η,

$$\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \eta\, m_0 v \quad \text{or} \quad 1 - \frac{v^2}{c^2} = \frac{1}{\eta^2}$$

or $$v = c\sqrt{1 - \frac{1}{\eta^2}} = \frac{c}{\eta}\sqrt{\eta^2 - 1}$$

1.372 The work done is equal to change in kinetic energy which is different in the two cases Classically i.e. in nonrelativistic mechanics, the change in kinetic energy is

$$\frac{1}{2} m_0 c^2 \left((0{\cdot}8)^2 - (0{\cdot}6)^2\right) = \frac{1}{2} m_0 c^2\, 0{\cdot}28 = 0{\cdot}14\, m_0 c^2$$

Relativistically it is,

$$\frac{m_0 c^2}{\sqrt{1 - (0{\cdot}8)^2}} - \frac{m_0 c^2}{\sqrt{1 - (0{\cdot}6)^2}} = \frac{m_0 c^2}{0{\cdot}6} - \frac{m_0 c^2}{0{\cdot}8} = m_0 c^2 (1{\cdot}666 - 1{\cdot}250)$$

$$= 0{\cdot}416\, m_0 c^2 = 0{\cdot}42\, m_0 c^2$$

1.373 $$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2 m_0 c^2$$

or $$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \quad \text{or} \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

or $$\frac{v}{c} = \frac{\sqrt{3}}{2} \cdot \text{i.e.} \quad v = c\,\frac{\sqrt{3}}{2}$$

1.374 Relativistically

$$\frac{T}{m_0 c^2} = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$$

So $$\beta_{rel}^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\beta_{rel}^2\right)^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\frac{2T}{m_0 c^2}\right)^2$$

Thus $$\beta_{rel} = \left[\frac{2T}{m_0 c^2} - 3\frac{T^2}{m_0^2 c^4}\right]^{1/2} = \sqrt{\frac{2T}{m_0 c^2}}\left(1 - \frac{3}{4}\frac{T}{m_0 c^2}\right)$$

But Classically, $\beta_{cl} = \sqrt{\frac{2T}{m_0 c^2}}$ so $\frac{\beta_{rel} - \beta_{cl}}{\beta_{cl}} = \frac{3}{4}\frac{T}{m_0 c^2} = \varepsilon$

Hence if $$\frac{T}{m_0 c^2} < \frac{4}{3}\varepsilon$$

the velocity β is given by the classical formula with an error less than ε.

1.375 From the formula

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

we find $$E^2 = c^2 p^2 + m_0^2 c^4 \quad \text{or} \quad (m_0 c^2 + T)^2 = c^2 p^2 + m_0^2 c^4$$

or $$T(2 m_0 c^2 + T) = c^2 p^2 \quad \text{i.e.} \quad p = \frac{1}{c}\sqrt{T(2 m_0 c^2 + T)}$$

1.376 Let the total force exerted by the beam on the target surface be F and the power liberated there be P. Then, using the result of the previous problem we see

$$F = Np = \frac{N}{c}\sqrt{T(T + 2m_0 c^2)} = \frac{I}{ec}\sqrt{T(T + 2m_0 c^2)}$$

since $I = Ne$, N being the number of particles striking the target per second. Also,

$$P = N\left(\frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2\right) = \frac{I}{e}T$$

These will be, respectively, equal to the pressure and power developed per unit area of the target if I is current density.

1.377 In the frame fixed to the sphere :- The momentum transferred to the eastically scatterred particle is

$$\frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

The density of the moving element is, from 1.369, $n\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

and the momentum transferred per unit time per unit area is

$$p = \text{the pressure} = \frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}\,n\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot v = \frac{2\,mn\,v^2}{1-\frac{v^2}{c^2}}$$

In the frame fixed to the gas :- When the sphere hits a stationary particle, the latter recoils with a velocity

$$= \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2\,v}{1+\frac{v^2}{c^2}}$$

The momentum transferred is $\dfrac{\dfrac{m\cdot 2v}{1+v^2/c^2}}{\sqrt{1-\dfrac{4v^2/c^2}{(1-v^2/c^2)^2}}} = \dfrac{2\,mv}{1-\frac{v^2}{c^2}}$

and the pressure is $\dfrac{2\,mv}{1-\frac{v^2}{c^2}}\cdot n\cdot v = \dfrac{2\,mnv^2}{1-\frac{v^2}{c^2}}$

1.378 The equation of motion is

$$\frac{d}{dt}\left(\frac{m_0\,v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

Integrating $= \dfrac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{\beta}{\sqrt{1-\beta^2}} = \dfrac{F_t}{m_0\,c}$, using $v=0$ for $t=0$

$$\frac{\beta^2}{1-\beta^2} = \left(\frac{Ft}{m_0\,c}\right)^2 \text{ or, } \beta^2 = \frac{(Ft)^2}{(Ft)^2+(m_0\,c)^2} \text{ or, } v = \frac{Fct}{\sqrt{(m_0\,c)^2+(Ft)^2}}$$

$$\text{or } \quad x = \int\frac{Fct\;dt}{\sqrt{F^2\,t^2+m_0^2\,c^2}} = \frac{c}{F}\int\frac{\xi\,d\xi}{\sqrt{\xi^2+(m_0\,c)^2}} = \frac{c}{F}\sqrt{F^2\,t^2+m_0^2\,c^2} + \text{constant}$$

or using $x=0$ at $t=0$, we get, $x = \sqrt{c^2\,t^2+\left(\dfrac{m_0\,c^2}{F}\right)^2} - \dfrac{m_0\,c^2}{F}$

1.379 $x = \sqrt{a^2 + c^2 t^2}$, so $\dot{x} = v = \frac{c^2 t}{a^2 + c^2 t^2}$

or, $$\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c^2 t}{a}. \quad \text{Thus} \quad \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m_0 c^2}{a} = F$$

1.380 $$\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \frac{\dot{\vec{v}}}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 \frac{\vec{v}}{c^2} \vec{v} \cdot \dot{\vec{v}} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

Thus $$\vec{F}_{\perp} = m_0 \frac{\vec{w}}{\sqrt{1 - \beta^2}}, \quad \vec{w} = \dot{\vec{v}}, \quad \vec{w} \perp \vec{v}$$

$$\vec{F}_{\parallel} = m_0 \frac{\vec{w}}{(1 - \beta^2)^{3/2}}, \quad \vec{w} = \dot{\vec{v}}, \quad \vec{w} \parallel \vec{v}$$

1.381 By definition,

$$E = m_0 \frac{c^2}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \frac{m_0 c^3 \, dt}{ds}, \quad p_x = m_0 \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c \, m_0 \, dx}{ds}$$

where $ds^2 = c^2 dt^2 - dx^2$ is the invariant interval $(dy = dz = 0)$

Thus, $$p'_x = c m_0 \frac{dx'}{ds} = c m_0 \gamma \frac{(dx - V dt)}{ds} = \frac{p_x - VE/c^2}{\sqrt{1 - V^2/c^2}}$$

$$E' = m_0 c^3 \frac{dt'}{ds} - c^3 m_0 \gamma \frac{\left(dt - \frac{V d_x}{c^2}\right)}{ds} = \frac{E - V p_x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

1.382 For a photon moving in the x direction

$$\varepsilon = c p_x, \; p_y = p_z = 0,$$

In the moving frame, $\varepsilon' = \frac{1}{\sqrt{1 - \beta^2}}\left(\varepsilon - V \frac{\varepsilon}{c}\right) = \varepsilon \sqrt{\frac{1 - V/c}{1 + V/c}}$

Note that $\quad \varepsilon' = \frac{\varepsilon}{2}$ if, $\frac{1}{4} = \frac{1 - \beta}{1 + \beta}$ or $\beta = \frac{3}{5}$, $V = \frac{3c}{5}$.

1.383 As before

$$E = m_0 c^3 \frac{dt}{ds}, \; p_x = m_0 c \frac{dx}{ds}.$$

Similarly $$p_y = m_0 c \frac{dy}{ds},\ p_z = m_0 c \frac{dz}{ds}$$

Then $$E^2 - c^2 p^2 = E^2 - c^2 \left(p_x^2 + p_y^2 + p_x^2\right)$$

$$= m_0^2 c^4 \frac{(c^2 dt^2 - dx^2 - dy^2 - dz^2)}{ds^2} = m_0^2 c^4 \text{ is invariant}$$

1.384 (b) & (a) In the CM frame, the total momentum is zero, Thus

$$\frac{V}{c} = \frac{c p_{1x}}{E_1 + E_2} = \frac{\sqrt{T(T + 2m_0 c^2)}}{T + 2m_0 c^2} = \sqrt{\frac{T}{T + 2m_0 c^2}}$$

where wc have used the result of probiem (1.375)

Then

$$\frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - \frac{T}{T + 2m_0 c^2}}} = \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}}$$

Total energy in the CM frame is

$$\frac{2m_0 c^2}{\sqrt{1 - V^2/c^2}} = 2m_0 c^2 \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}} = \sqrt{2\, m_0 c^2 (T + 2m_0 c^2)} = \tilde{T} + 2m_0 c^2$$

So $$\tilde{T} = 2m_0 c^2 \left(\sqrt{1 + \frac{T}{2m_0 c^2}} - 1\right)$$

Also $2\sqrt{c^2 \tilde{p}^2 + m_0^2 c^4} = \sqrt{2m_0 c^2 (T + 2m_0 c^2)}$, $4 c^2 \tilde{p}^2 = 2m_0 c^2 T$, or $\tilde{p} = \sqrt{\frac{1}{2} m_0 T}$

1.385 $M_0 c^2 = \sqrt{E^2 - c^2 p^2}$

$$\sqrt{(2m_0 c^2 + T)^2 - T(2m_0 c^2 + T)} = \sqrt{2m_0 c^2 (2m_0 c^2 + T)} = c\sqrt{2m_0 (2m_0 c^2 + T)}$$

Also $$cp = \sqrt{T(T + 2m_0 c^2)},\quad v = \frac{c^2 p}{E} = c\sqrt{\frac{T}{T + 2m_0 c^2}}$$

1.386 Let T' = kinetic energy of a proton striking another stationary particle of the same rest mass. Then, combined kinetic energy in the CM frame

$$= 2m_0 c^2 \left(\sqrt{1 + \frac{T'}{2m_0 c^2}} - 1\right) = 2T,\quad \left(\frac{T}{m_0 c^2} + 1\right)^2 = 1 + \frac{T'}{2m_0 c^2}$$

$$\frac{T'}{2m_0 c^2} = \frac{T(2m_0 c^2 + T)}{m_0^2 c^4},\quad T' = \frac{2T(T + 2m_o c^2)}{m_0 c^2}$$

1.387 We have

$$E_1 + E_2 + E_3 = m_0 c^2, \; \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

Hence $$(m_0 c^2 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (E_2 + E_3)^2 - \left(\vec{p}_2 + \vec{p}_3\right)^2 c^2$$

The L.H.S. $$= (m_0^2 c^4 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (m_0^2 + m_1^2) c^4 - 2 m_0 c^2 E_1$$

The R.H.S. is an invariant. We can evaluate it in any frame. Choose the CM frame of the particles 2 and 3.

In this frame R.H.S. $= (E'_2 + E'_3)^2 = (m_2 + m_3)^2 c^4$

Thus $$(m_0^2 + m_1^2) c^4 - 2 m_0 c^2 E_1 = (m_2 + m_3)^2 c^4$$

or $$2 m_0 c^2 E_1 \le \left\{ m_0^2 + m_1^2 - (m_2 + m_3)^2 \right\} c^4, \quad \text{or} \quad E_1 \le \frac{m_0^2 + m_1^2 - (m_2 + m_3)^2}{2 m_0} c^2$$

1.388 The velocity of ejected gases is u realtive to the rocket. In an earth centred frame it is

$$\frac{v - u}{1 - \frac{vu}{c^2}}$$

in the direction of the rocket. The momentum conservation equation then reads

$$(m + dm)(v + dv) + \frac{v - u}{1 - \frac{uv}{c^2}} (-dm) = mv$$

or $$m dv - \left(\frac{v - u}{1 - \frac{uv}{c^2}} - v \right) dm = 0$$

Here $-dm$ is the mass of the ejected gases. so

$$m dv - \frac{-u + \frac{uv^2}{c^2}}{1 - \frac{uv}{c^2}} dm = 0, \quad \text{or} \quad m dv + u \left(1 - \frac{v^2}{c^2} \right) dm = 0$$

(neglecting $1 - \frac{uv}{c^2}$ since u is non- relativistic.)

Integrating $$\left(\beta = \frac{v}{c} \right), \int \frac{d\beta}{1 - \beta^2} + \frac{u}{c} \int \frac{dm}{m} = 0, \; \ln \frac{1 + \beta}{1 - \beta} + \frac{u}{c} \ln m = \text{constant}$$

The constant $= \frac{u}{c} \ln m_0$ since $\beta = 0$ initially.

Thus $$\frac{1 - \beta}{1 + \beta} = \left(\frac{m}{m_0} \right)^{u/c} \quad \text{or} \quad \beta = \frac{1 - \left(\frac{m}{m_0} \right)^{u/c}}{1 + \left(\frac{m}{m_0} \right)^{u/c}}$$

PART TWO

THERMODYNAMICS AND MOLECULAR PHYSICS

2.1 EQUATION OF THE GAS STATE • PROCESSES

2.1 Let m_1 and m_2 be the masses of the gas in the vessel before and after the gas is released. Hence mass of the gas released,

$$\Delta m = m_1 - m_2$$

Now from ideal gas equation

$$p_1 V = m_1 \frac{R}{M} T_0 \text{ and } p_2 V = m_2 \frac{R}{M} T_0$$

as V and T are same before and after the release of the gas.

so, $$(p_1 - p_2) V = (m_1 - m_2) \frac{R}{M} T_0 = \Delta m \frac{R}{M} T_0$$

or, $$\Delta m = \frac{(p_1 - p_2) V M}{R T_0} = \frac{\Delta p \, V M}{R T_0} \qquad (1)$$

We also know $p = \rho \frac{R}{M} T$ so, $\frac{M}{R T_0} = \frac{\rho}{p_0}$ (2)

(where $p_0 =$ standard atmospheric pressure and $T_0 = 273$ K)

From Eqs. (1) and (2) we get

$$\Delta m = \rho V \cdot \frac{\Delta p}{p_0} = 1{\cdot}3 \times 30 \times \frac{0{\cdot}78}{1} = 30 \text{ g}$$

2.2 Let m_1 be the mass of the gas enclosed.

Then, $$p_1 V = \nu_1 R T_1$$

When heated, some gas, passes into the evacuated vessel till pressure difference becomes Δp. Let p'_1 and p'_2 be the pressure on the two sides of the valve. Then $p'_1 V = \nu'_1 R T_2$ and

$$p'_2 V = \nu'_2 R T_2 = (\nu_1 - \nu'_1) R T_2$$

$$p'_2 V = \left(\frac{p_1 V}{R T_1} - \frac{p'_1 V}{R T_2}\right) \quad \text{or} \quad p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_1}{T_2}\right) T_2$$

But, $$p'_1 - p'_2 = \Delta p$$

So, $$p'_2 = \left(\frac{p_1}{T_1} - \frac{p'_2 + \Delta p}{T_2}\right) T_2$$

$$= \frac{p_1 T_2}{T_1} - p'_2 - \Delta p$$

or, $$p_2' = \frac{1}{2}\left(\frac{p_1 T_2}{T_1} - \Delta p\right) = 0{\cdot}08 \text{ atm}$$

2.3 Let the mixture contain ν_1 and ν_2 moles of H_2 and H_e respectively. If molecular weights of H_2 and H_e are M_1 and M_2, then respective masses in the mixture are equal to

$$m_1 = \nu_1 M_1 \text{ and } m_2 = \nu_2 M_2$$

Therefore, for the total mass of the mixture we get,

$$m = m_1 + m_2 \quad \text{or} \quad m = \nu_1 M_1 + \nu_2 M_2 \tag{1}$$

Also, if ν is the total number of moles of the mixture in the vessels, then we know,

$$\nu = \nu_1 + \nu_2 \tag{2}$$

Solving (1) and (2) for ν_1 and ν_2, we get,

$$\nu_1 = \frac{(\nu M_2 - m)}{M_2 - M_1}, \qquad \nu_2 = \frac{m - \nu M_1}{M_2 - M_1}$$

Therefore, we get $m_1 = M_1 \cdot \dfrac{(\nu M_2 - m)}{M_2 - M_1}$ and $m_2 = M_2 \dfrac{(m - \nu M_1)}{M_2 - M_1}$

or, $$\frac{m_1}{m_2} = \frac{M_1}{M_2} \frac{(\nu M_2 - m)}{(m - \nu M_1)}$$

One can also express the above result in terms of the effective molecular weight M of the mixture, defined as,

$$M = \frac{m}{\nu} = m \frac{R T}{pV}$$

Thus, $$\frac{m_1}{m_2} = \frac{M_1}{M_2} \cdot \frac{M_2 - M}{M - M_1} = \frac{1 - M/M_2}{M/M_1 - 1}$$

Using the data and table, we get :

$$M = 3{\cdot}0 \text{ g and, } \frac{m_1}{m_2} = 0{\cdot}50$$

2.4 We know, for the mixture, N_2 and CO_2 (being regarded as ideal gases, their mixture too behaves like an ideal gas)

$$pV = \nu RT, \quad \text{so} \quad p_0 V = \nu RT$$

where, ν is the total number of moles of the gases (mixture) present and V is the volume of the vessel. If ν_1 and ν_2 are number of moles of N_2 and CO_2 respectively present in the mixture, then

$$\nu = \nu_1 + \nu_2$$

Now number of moles of N_2 and CO_2 is, by definition, given by

$$\nu_1 = \frac{m_1}{M_1} \text{ and, } \nu_2 = \frac{m_2}{M_2}$$

where, m_1 is the mass of N_2 (Moleculer weight = M_1) in the mixture and m_2 is the mass of CO_2 (Molecular weight = M_2) in the mixture.

Therefore density of the mixture is given by

$$\rho = \frac{m_1 + m_2}{V} = \frac{m_1 + m_2}{(\nu RT/P_0)}$$

$$= \frac{p_0}{RT} \cdot \frac{m_1 + m_2}{\nu_1 + \nu_2} = \frac{p_0 (m_1 + m_2) M_1 M_2}{RT (m_1 M_2 + m_2 M_1)}$$

$$= 1{\cdot}5 \text{ kg/m}^3 \text{ on substitution}$$

2.5 (a) The mixture contains ν_1, ν_2 and ν_3 moles of O_2, N_2 and CO_2 respectively. Then the total number of moles of the mixture

$$\nu = \nu_1 + \nu_2 + \nu_3$$

We know, ideal gas equation for the mixture

$$pV = \nu RT \quad \text{or} \quad p = \frac{\nu RT}{V}$$

or, $$p = \frac{(\nu_1 + \nu_2 + \nu_3) RT}{V} = 1{\cdot}968 \text{ atm on substitution}$$

(b) Mass of oxygen (O_2) present in the mixture : $m_1 = \nu_1 M_1$

Mass of nitrogen (N_2) present in the mixture : $m_2 = \nu_2 M_2$

Mass of carbon dioxide (CO_2) present in the mixture : $m_3 = \nu_3 M_3$

So, mass of the mixture

$$m = m_1 + m_2 + m_3 = \nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3$$

Moleculer mass of the mixture : $M = \dfrac{\text{mass of the mixture}}{\text{total number of moles}}$

$$= \frac{\nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3}{\nu_1 + \nu_2 + \nu_3} = 36{\cdot}7 \text{ g/mol. on substitution}$$

2.6 Let p_1 and p_2 be the pressure in the upper and lower part of the cylinder respectively at temperature T_0. At the equilibrium position for the piston :

$$p_1 S + mg = p_2 S \quad \text{or,} \quad p_1 + \frac{mg}{S} = p_2 \text{ (} m \text{ is the mass of the piston.)}$$

But $p_1 = \dfrac{R\,T_0}{\eta\, V_0}$ (where V_0 is the initial volume of the lower part)

So, $$\frac{RT_0}{\eta V_0} + \frac{mg}{S} = \frac{RT_0}{V_0} \quad \text{or,} \quad \frac{mg}{S} = \frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) \tag{1}$$

Let T' be the sought temperature and at this temperature the volume of the lower part becomes V', then according to the problem the volume of the upper part becomes $\eta' V'$

Hence, $$\frac{mg}{S} = \frac{RT'}{V'}\left(1 - \frac{1}{\eta'}\right) \tag{2}$$

From (1) and (2).

$$\frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) = \frac{RT'}{V'}\left(1 - \frac{1}{\eta'}\right) \quad \text{or,} \quad T' = \frac{T_0\left(1 - \frac{1}{\eta}\right)V'}{V_0\left(1 - \frac{1}{\eta'}\right)}$$

As, the total volume must be constant,

$$V_0(1+\eta) = V'(1+\eta') \quad \text{or,} \quad V' = \frac{V_0(1+\eta)}{(1+\eta')}$$

Putting the value of V' in Eq. (3), we get

$$T' = \frac{T_0\left(1 - \frac{1}{\eta}\right)V_0\frac{(1+\eta)}{(1+\eta')}}{V_0\left(1 - \frac{1}{\eta'}\right)}$$

$$= \frac{T_0(\eta^2 - 1)\eta'}{(\eta'^2 - 1)\eta} = 0{\cdot}42\text{ k K}$$

2.7 Let ρ_1 be the density after the first stroke. The the mass remains constant

$$V\rho = (V + \Delta V)\rho_1, \quad \text{or,} \quad \rho_1 = \frac{V\rho}{(V + \Delta V)}$$

Similarly, if ρ_2 is the density after second stroke

$$V\rho_1 = (V + \Delta V)\rho_2 \quad \text{or,} \quad \rho_2 = \left(\frac{V}{V + \Delta V}\right)\rho_1 = \left(\frac{V}{V + \Delta V}\right)^2 \rho_0$$

In this way after nth stroke.

$$\rho_n = \left(\frac{V}{V + \Delta V}\right)^n \rho_0$$

Since pressure α density,

$$p_n = \left(\frac{V}{V + \Delta V}\right)^n p_0 \quad \text{(because temperature is constant.)}$$

It is required by $\dfrac{p_n}{p_0}$ to be $\dfrac{1}{\eta}$

so, $$\frac{1}{\eta} = \left(\frac{V}{V + \Delta V}\right)^n \quad \text{or,} \quad \eta = \left(\frac{V + \Delta V}{V}\right)^n$$

Hence $$n = \frac{\ln \eta}{\ln\left(1 + \dfrac{\Delta V}{V}\right)}$$

2.8 From the ideal gas equation $p = \dfrac{m}{M}\dfrac{RT}{V}$

$$\frac{dp}{dt} = \frac{RT}{MV}\frac{dm}{dt} \tag{1}$$

In each stroke, volume v of the gas is ejected, where v is given by

$$v = \frac{V}{m_N}\left[m_{N-1} - m_N\right]$$

In case of continuous ejection, if (m_{N-1}) corresponds to mass of gas in the vessel at time t, then m_N is the mass at time $t + \Delta t$, where Δt, is the time in which volume v of the gas has come out. The rate of evacuation is therefore $\dfrac{v}{\Delta t}$ i.e.

$$C = \frac{v}{\Delta t} = -\frac{V}{m(t + \Delta t)} \cdot \frac{m(t + \Delta t) - m(t)}{\Delta t}$$

In the limit $\Delta t - 0$, we get

$$C = \frac{V}{m}\frac{dm}{dt} \tag{2}$$

From (1) and (2)

$$\frac{dp}{dt} = -\frac{C}{V}\frac{mRT}{MV} = -\frac{C}{V}p \quad \text{or} \quad \frac{dp}{p} = -\frac{C}{V}dt$$

Integrating $$\int_p^{p_0} \frac{dp}{p} = -\frac{C}{V}\int_t^0 dt \quad \text{or} \quad \ln\frac{p}{p_0} = -\frac{C}{V}t$$

Thus $$p = p_0 e^{-Ct/V}$$

2.9 Let ρ be the instantaneous density, then instantaneous mass $= V_\rho$. In a short interval dt the volume is increased by Cdt.

So, $$V\rho = (V + Cdt)(\rho + d\rho)$$

(because mass remains constant in a short interval dt)

so, $$\frac{d\rho}{\rho} = -\frac{C}{V}dt$$

Since pressure α density $\frac{dp}{p} = -\frac{C}{V}dt$

or $$\int_{p_1}^{p_2} -\frac{dp}{p} = \frac{C}{V}t,$$

or $$t = \frac{V}{C}\ln\frac{p_1}{p_2} = \frac{V}{C}\ln\frac{1}{\eta}\ 1{\cdot}0 \text{ min}$$

2.10 The physical system consists of one mole of gas confined in the smooth vertical tube. Let m_1 and m_2 be the masses of upper and lower pistons and S_1 and S_2 are their respective areas.

For the lower piston

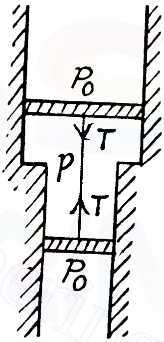
$$p S_2 + m_2 g = p_0 S_2 + T,$$

or, $$T = (p - p_0) S_2 + m_2 g \qquad (1)$$

Similarly for the upper piston

$$p_0 S_1 + T + m_1 g = p S_1,$$

or, $$T = (p - p_0) S_1 - m_1 g \qquad (2)$$



From (1) and (2)

$$(p - p_0)(S_1 - S_2) = (m_1 + m_2) g$$

or, $$(p - p_0)\Delta S = mg$$

so, $$p = \frac{mg}{\Delta S} + p_0 = \text{constant}$$

From the gas law, $p V = \nu R T$

$$p \Delta V = \nu R \Delta T \text{ (because } p \text{ is constant)}$$

So, $$\left(p_0 + \frac{mg}{\Delta S}\right)\Delta S\, l = R\Delta T,$$

Hence, $$\Delta T = \frac{1}{R}(p_0 \Delta S + mg)\, l = 0{\cdot}9 \text{ K}$$

2.11 (a) $p = p_0 - \alpha V^2 = p_0 - \alpha\left(\frac{RT}{p}\right)^2$

(as, $V = RT/p$ for one mole of gas)

Thus, $$T = \frac{1}{R\sqrt{\alpha}}p\sqrt{p_0 - p} = \frac{1}{R\sqrt{\alpha}}\sqrt{p_0 p^2 - p^3} \qquad (1)$$

For T_{max}, $\frac{d}{dp}\left(p_0 p^2 - p^3\right)$ must be zero which yields, $$p = \frac{2}{3} p_0 \tag{2}$$

Hence, $$T_{max} = \frac{1}{R\sqrt{\alpha}} \cdot \frac{2}{3} p_0 \sqrt{p_0 - \frac{2}{3} p_0} = \frac{2}{3}\left(\frac{p_0}{R}\right)\sqrt{\frac{p_0}{3\alpha}}$$

(b) $p = p_0 e^{-\beta V} = p_0 e^{-\beta RT/p}$

so $$\frac{\beta RT}{p} = \ln \frac{p_o}{p}, \quad \text{and } T = \frac{p}{\beta R} \ln \frac{p_0}{p} \tag{1}$$

For T_{max} the condition is $\frac{dT}{dp} = 0$, which yields

$$p = \frac{p_0}{e}$$

Hence using this value of p in Eq. (1), we get

$$T_{max} = \frac{p_o}{e\beta R}$$

2.12 $T = T_0 + \alpha V^2 = T_0 + \alpha \frac{R^2 T^2}{p^2}$

(as, $V = RT/p$ for one mole of gas)

So, $$p = \sqrt{\alpha}\, RT (T - T_0)^{1/2} \tag{1}$$

For p_{min}, $\frac{dp}{dT} = 0$, which gives

$$T = 2T_0 \tag{2}$$

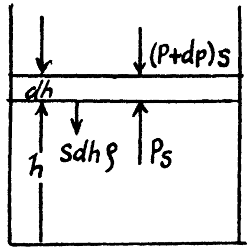
From (1) and (2), we get,

$$p_{min} = \sqrt{\alpha}\, R\, 2T_0 (2T_0 - T_0)^{-1/2} = 2R\sqrt{\alpha T_0}$$

2.13 Consider a thin layer at a height h and thickness dh. Let p and $dp + p$ be the pressure on the two sides of the layer. The mass of the layer is $Sdh\rho$. Equating vertical downward force to the upward force acting on the layer.

$$Sdh\,\rho g + (p + dp) S = pS$$

So, $$\frac{dp}{dh} = -\rho g \tag{1}$$



But, $p = \frac{\rho}{M} RT$, we have $dp = \frac{\rho R}{M} dT$,

or, $-\frac{\rho R}{M} dT = \rho g\, dh$

So, $\frac{dT}{dh} = -\frac{gM}{R} = -34$ K/km

That means, temperature of air drops by 34°C at a height of 1 km above bottom.

2.14 We have, $\frac{dp}{dh} = -\rho g$ (See 2.13) (1)

But, from $p = C\rho^n$ (where C is, *a* const) $\frac{dp}{d\rho} = Cn\,\rho^{n-1}$ (2)

We have from gas low $p = \rho \frac{R}{M} T$, so using (2)

$$C\rho^n = \rho \frac{R}{M} \cdot T, \quad \text{or} \quad T = \frac{M}{R} C \rho^{n-1}$$

Thus, $$\frac{dT}{d\rho} = \frac{M}{R} \cdot C(n-1)\,\rho^{n-2} \tag{3}$$

But, $$\frac{dT}{dh} = \frac{dT}{d\rho} \cdot \frac{d\rho}{dp} \cdot \frac{dp}{dh}$$

So, $$\frac{dT}{dh} = \frac{M}{R} C(n-1)\,\rho^{n-2} \frac{1}{C\,n\,\rho^{n-1}} (-\rho g) = \frac{-Mg(n-1)}{nR}$$

2.15 We have, $dp = -\rho\, g\, dh$ and from gas law $\rho = \frac{M}{RT} p$ (1)

Thus $$\frac{dp}{p} = -\frac{Mg}{RT} dh$$

Integrating, we get

or, $$\int_{p_o}^{p} \frac{dp}{p} = -\frac{Mg}{RT} \int_0^h dh \quad \text{or,} \quad \ln \frac{p}{p_0} = -\frac{Mg}{RT} h,$$

(where p_0 is the pressure at the surface of the Earth.)

$$p = p_0\, e^{-Mgh/RT},$$

[Under standard condition, $p_0 = 1$ atm, $T = 273$ K

Pressure at a height of 5 atm $= 1 \times e^{-28 \times 9{\cdot}81 \times 5000/8314 \times 273} = 0{\cdot}5$ atm.

Pressure in a mine at a depth of 5 km $= 1 \times e^{-28 \times 9{\cdot}81 \times (-5000)/8314 \times 273} = 2$ atm.]

2.16 We have $dp = -\rho g\, dh$ but from gas law $p = \frac{\rho}{M} RT$,

Thus $$dp = \frac{d\rho}{M} RT \text{ at const. temperature}$$

So, $$\frac{d\rho}{\rho} = \frac{gM}{RT} dh$$

Integrating within limits $$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = \int_0^h \frac{gM}{RT} dh$$

or, $$\ln \frac{\rho}{\rho_0} = -\frac{gM}{RT} h$$

So, $\rho = \rho_0 e^{-Mgh/RT}$ and $h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0}$

(a) Given $T = 273°K$, $\frac{\rho_0}{\rho} = e$

Thus $$h = -\frac{RT}{Mg} \ln e^{-1} = 8 \text{ km}.$$

(b) $T = 273° K$ and

$$\frac{\rho_0 - \rho}{\rho_0} = 0.01 \quad \text{or} \quad \frac{\rho}{\rho_0} = 0.99$$

Thus $$h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0} = 0.09 \text{ km on substitution}$$

2.17 From the Barometric formula, we have

$$p = p_0 e^{-Mgh/RT}$$

and from gas law $$\rho = \frac{pM}{RT}$$

So, at constant temperature from these two Eqs.

$$\rho = \frac{Mp_0}{RT} e^{-Mgh/RT} = \rho_0 e^{-Mgh/RT} \qquad (1)$$

Eq. (1) shows that density varies with height in the same manner as pressure. Let us consider the mass element of the gas contained in the column.

$$dm = \rho (Sdh) = \frac{Mp_0}{RT} e^{-Mgh/RT} Sdh$$

Hence the sought mass,

$$m = \frac{Mp_0S}{RT} \int_0^h e^{-Mgh/RT} dh = \frac{p_0 S}{g} \left(1 - e^{-Mgh/RT}\right)$$

2.18 As the gravitational field is constant the centre of gravity and the centre of mass are same. The location of C.M.

$$h = \frac{\int_0^\infty h\, dm}{\int_0^\infty dm} = \frac{\int_0^\infty h \rho\, dh}{\int_0^\infty \rho\, dh}$$

But from Barometric formula and gas law $\rho = \rho_0 e^{-Mgh/RT}$

So,
$$h = \frac{\int_0^{\alpha} h \left(e^{-Mg\,h/RT}\right) dh}{\int_0^{\alpha} \left(e^{-Mg\,h/RT}\right) dh} = \frac{RT}{Mg}$$

2.19 (a) We know that the variation of pressure with height of a fluid is given by :

$$dp = -\rho\, g\, dh$$

But from gas law $\quad p = \frac{\rho}{M} RT \quad$ or, $\quad \rho = \frac{pM}{RT}$

From these two Eqs.

$$dp = -\frac{p\,Mg}{RT}\, dh \qquad (1)$$

or,
$$\frac{dp}{p} = \frac{-M\,g\,dh}{R\,T_0\,(1-ah)}$$

Integrating,
$$\int_{p_0}^{p} \frac{dp}{p} = \frac{-Mg}{R\,T_0} \int_0^h \frac{dh}{(1-ah)}, \text{ we get}$$

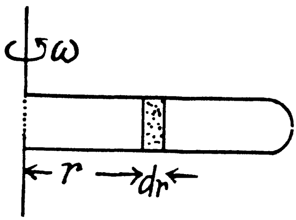
$$\ln \frac{p}{p_0} = \ln\,(1-ah)^{Mg/aRT_0}$$

Hence, $\quad p = p_0\,(1-ah)^{Mg/aRT_0}$. Obviously $h < \frac{1}{a}$

(b) Proceed up to Eq. (1) of part (a), and then put $T = T_0\,(1 + a\,h)$ and proceed further in the same fashion to get

$$p = \frac{p_0}{(1+ah)^{Mg/aRT_0}}$$

2.20 Let us consider the mass element of the gas (thin layer) in the cylinder at a distance r from its open end as shown in the figure.



Using Newton's second law for the element

$$F_n = m w_n:$$

$$(p + dp)\,S - pS = (\rho\, S\, dr)\,\omega^2 r$$

or, $\quad dp = \rho\,\omega^2 r\,dr = \frac{pM}{RT}\,\omega^2 r\,dr$

So, $$\frac{dp}{p} = \frac{M\omega^2}{RT} r\,dr \quad \text{or,} \quad \int_{p_0}^{p} \frac{dp}{p} = \frac{M\omega^2}{RT} \int_0^r r\,dr,$$

Thus, $$\ln \frac{p}{p_0} = \frac{M\omega^2}{2RT} r^2 \quad \text{or,} \quad p = p_0\, e^{M\omega^2 r^2/2RT}$$

2.21 For an ideal gas law

$$p = \frac{\rho}{M} R\,T$$

So, $$p = 0{\cdot}082 \times 300 \times \frac{500}{44} \text{ atms} = 279{\cdot}5 \text{ atmosphere}$$

For Vander Waal gas Eq.

$$\left(p + \frac{\nu^2 a}{V^2}\right)(V - \nu\,b) = \nu\,R\,T, \quad \text{where } V = \nu V_M$$

or, $$p = \frac{\nu\,R\,T}{V - \nu\,b} - \frac{a\nu^2}{V^2} = \frac{m\,R\,T/M}{V - \frac{mb}{M}} - \frac{am^2}{V^2 M^2}$$

$$= \frac{\rho\,R\,T}{M - \rho\,b} - \frac{a\,\rho^2}{M^2} = 79{\cdot}2 \text{ atm}$$

2.22 (a) $$p = \left[\frac{R\,T}{V_M - b} - \frac{a}{V_M^2}\right](1 + \eta) = \frac{R\,T}{V_M}$$

(The pressure is less for a Vander Waal gas than for an ideal gas)

or, $$\frac{a\,(1+\eta)}{V_M^2} = R\,T\left[\frac{-1}{V_M} + \frac{1+\eta}{V_M - b}\right] = R\,T\,\frac{\eta\,V_M + b}{V_M\,(V_M - b)}$$

or, $$T = \frac{a\,(1+\eta)\,(V_M - b)}{R\,V_M(\eta\,V_M + b)}, \quad \text{(here } V_M \text{ is the molar volume.)}$$

$$= \frac{1{\cdot}35 \times 1{\cdot}1 \times (1 - 0{\cdot}039)}{0{\cdot}082 \times (0{\cdot}139)} \approx 125 \text{ K}$$

(b) The corresponding pressure is

$$p = \frac{RT}{V_M - b} - \frac{a}{V_M^2} = \frac{a\,(1+\eta)}{V_M\,(\eta\,V_M + b)} - \frac{a}{V_M^2}$$

$$= \frac{a}{V_M^2}\,\frac{(V_M + \eta\,V_M - \eta\,V_M - b)}{(\eta\,V_M + b)} = \frac{a}{V_M^2}\,\frac{(V_M - b)}{(V_M + b)}$$

$$= \frac{1{\cdot}35}{1} \times \frac{0{\cdot}961}{0{\cdot}139} \approx 9{\cdot}3 \text{ atm}$$

2.23 $p_1 = RT_1 \dfrac{1}{V-b} - \dfrac{a}{V^2}$, $p_2 = R\,T_2 \dfrac{1}{V-b} - \dfrac{a}{V^2}$

So,
$$p_2 - p_1 = \frac{R\,(T_2 - T_1)}{V-b}$$

or,
$$V - b = \frac{R\,(T_2 - T_1)}{p_2 - p_1} \quad \text{or,} \quad b = V - \frac{R\,(T_2 - T_1)}{p_2 - p_1}$$

Also,
$$p_1 = T_1 \frac{p_2 - p_1}{T_2 - T_1} - \frac{a}{V^2},$$

$$\frac{a}{V^2} = \frac{T_1\,(p_2 - p_1)}{T_2 - T_1} - p_1 = \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

or,
$$a = V^2 \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

Using $T_1 = 300$ K, $p_1 = 90$ atms, $T_2 = 350$ K, $p_2 = 110$ atm, $V = 0{\cdot}250$ litre

$$a = 1{\cdot}87 \text{ atm. litre}^2/\text{mole}^2,\; b = 0{\cdot}045 \text{ litre/mole}$$

2.24 $p = \dfrac{R\,T}{V-b} - \dfrac{a}{V^2} - V\left(\dfrac{\partial p}{\partial V}\right)_T = \dfrac{R\,T\,V}{(V-b)^2} - \dfrac{2a}{V^2}$

or,
$$\kappa = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

$$= \left[\frac{R\,T\,V^3 - 2a\,(V-b)^2}{V^2\,(V-b)^2}\right]^{-1} = \frac{V^2\,(V-b)}{\left[RTV^3 - 2a\,(V-b)^2\right]}$$

2.25 For an ideal gas $\kappa_0 = \dfrac{V}{RT}$

Now $\kappa = \dfrac{(V-b)^2}{RTV} \left\{1 - \dfrac{2a\,(V-b)^2}{RTV^3}\right\}^{-1} = \kappa_0 \left(1 - \dfrac{b}{V}\right)^2 \left\{1 - \dfrac{2a}{RTV}\left(1 - \dfrac{b}{V}\right)^2\right\}^{-1}$

$$= \kappa_0 \left\{1 - \frac{2b}{V} + \frac{2a}{RTV}\right\}, \text{ to leading order in } a, b$$

Now
$$\kappa > \kappa_0 \quad \text{if} \quad \frac{2a}{RTV} > \frac{2b}{V} \quad \text{or} \quad T < \frac{a}{bR}$$

If a, b do not vary much with temperature, then the effect at high temperature is clearly determined by b and its effect is repulsive so compressibility is less.

2.2 THE FIRST LAW OF THERMODYNAMICS. HEAT CAPACITY

2.26 Internal energy of air, treating as an ideal gas

$$U = \frac{m}{M} C_V T = \frac{m}{M} \frac{R}{\gamma - 1} T = \frac{pV}{\gamma - 1} \qquad (1)$$

Using $C_V = \frac{R}{\gamma - 1}$, since $C_p - C_V = R$ and $\frac{C_p}{C_V} = \gamma$

Thus at constant pressure $U =$ constant, because the volume of the room is a constant.

Puting the value of $p = p_{atm}$ and V in Eq. (1), we get $U = 10$ MJ.

2.27 From energy conservation

$$U_i + \frac{1}{2} (\nu M) v^2 = U_f$$

or,
$$\Delta U = \frac{1}{2} \nu M v^2 \qquad (1)$$

But from $U = \nu \frac{RT}{\gamma - 1}$, $\Delta U = \frac{\nu R}{\gamma - 1} \Delta T$ (from the previous problem) (2)

Hence from Eqs. (1) and (2).

$$\Delta T = \frac{M v^2 (\gamma - 1)}{2R}$$

2.28 On opening the valve, the air will flow from the vessel at heigher pressure to the vessel at lower pressure till both vessels have the same air pressure. If this air pressure is p, the total volume of the air in the two vessels will be $(V_1 + V_2)$. Also if ν_1 and ν_2 be the number of moles of air initially in the two vessels, we have

$$p_1 V_1 = \nu_1 R T_1 \text{ and } p_2 V_2 = \nu_2 R T_2 \qquad (1)$$

After the air is mixed up, the total number of moles are $(\nu_1 + \nu_2)$ and the mixture is at temperature T.

Hence
$$p (V_1 + V_2) = (\nu_1 + \nu_2) RT \qquad (2)$$

Let us look at the two portions of air as one single system. Since this system is contained in a thermally insulated vessel, no heat exchange is involved in the process. That is, total heat transfer for the combined system $Q = 0$

Moreover, this combined system does not perform mechanical work either. The walls of the containers are rigid and there are no pistons etc to be pushed, looking at the total system, we know $A = 0$.

Hence, internal energy of the combined system does not change in the process. Initially energy of the combined system is equal to the sum of internal energies of the two portions of air :

$$U_i = U_1 + U_2 = \frac{\nu_1 R T_1}{\gamma - 1} + \frac{\nu_2 R T_2}{\gamma - 1} \qquad (3)$$

Final internal energy of $(n_1 + n_2)$ moles of air at temperature T is given by

$$U_f = \frac{(\nu_1 + \nu_2) RT}{\gamma - 1} \quad (4)$$

Therefore, $U_i = U_f$ implies :

$$T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = \frac{p_1 V_1 + p_2 V_2}{(p_1 V_1/T_1) + (p_2 V_2/T_2)} = T_1 T_2 \frac{p_1 V_1 + p_2 V_2}{p_1 V_1 T_2 + p_2 V_2 T_2}$$

From (2), therefore, final pressure is given by :

$$p = \frac{\nu_1 + \nu_2}{V_1 + V_2} RT = \frac{R}{V_1 + V_2} (\nu_1 T_1 + \nu_2 T_2) = \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2}$$

This process in an example of free adiabatic expansion of ideal gas.

2.29 By the first law of thermodynamics,

$$Q = \Delta U + A$$

Here $A = 0$, as the volume remains constant,

So, $$Q = \Delta U = \frac{\nu R}{\gamma - 1} \Delta T$$

From gas law, $$p_0 V = \nu R T_0$$

So, $$\Delta U = \frac{p_0 V \Delta T}{T_0 (\gamma - 1)} = -0{\cdot}25 \text{ kJ}$$

Hence amount of heat lost $= -\Delta U = 0.25$ kJ

2.30 By the first law of thermodynamics $Q = \Delta U + A$

But $$\Delta U = \frac{p \Delta V}{\gamma - 1} = \frac{A}{\gamma - 1} \text{ (as } p \text{ is constant)}$$

$$Q = \frac{A}{\gamma - 1} + A = \frac{\gamma \cdot A}{\gamma - 1} = \frac{1{\cdot}4}{1{\cdot}4 - 1} \times 2 = 7 \text{ J}$$

2.31 Under isobaric process $A = p\Delta V = R \Delta T$ (as $\nu = 1$) $= 0{\cdot}6$ kJ

From the first law of thermodynamics

$$\Delta U = Q - A = Q - R \Delta T = 1 \text{ kJ}$$

Again increment in internal energy $\Delta U = \frac{R \Delta T}{\gamma - 1}$, for $\nu = 1$

Thus $$Q - R \Delta T = \frac{R \Delta T}{\gamma - 1} \text{ or } \gamma = \frac{Q}{Q - R \Delta T} = 1{\cdot}6$$

2.32 Let $\nu = 2$ moles of the gas. In the first phase, under isochoric process, $A_1 = 0$, therefore from gas law if pressure is reduced n times so that temperature i.e. new temperature becomes T_0/n.

Now from first law of thermodynamics

$$Q_1 = \Delta U_1 = \frac{\nu R \Delta T}{\gamma - 1}$$

$$= \frac{\nu R}{\gamma - 1}\left(\frac{T_o}{n} - T_0\right) = \frac{\nu R T_0 (1 - n)}{n(\gamma - 1)}$$

During the second phase (under isobaric process),

$$A_2 = p\Delta V = \nu R \,\Delta T$$

Thus from first law of thermodynamics :

$$Q_2 = \Delta U_2 + A_2 = \frac{\nu R \,\Delta T}{\gamma - 1} + \nu R \Delta T$$

$$= \frac{\nu R \left(T_0 - \frac{T_0}{n}\right)\gamma}{\gamma - 1} = \frac{\nu R T_0 (n - 1)\gamma}{n(\gamma - 1)}$$

Hence the total amount of heat absorbed

$$Q = Q_1 + Q_2 = \frac{\nu R T_0 (1 - n)}{n(\gamma - 1)} + \frac{\nu R T_0 (n - 1)\gamma}{n(\gamma - 1)}$$

$$= \frac{\nu R T_0 (n - 1)\gamma}{n(\gamma - 1)}(-1 + \gamma) = \nu R T_0\left(1 - \frac{1}{n}\right)$$

2.33 Total no. of moles of the mixture $\nu = \nu_1 + \nu_2$

At a certain temperature, $U = U_1 + U_2$ or $\nu C_V = \nu_1 C_{V_1} + \nu_2 C_{V_2}$

Thus $$C_V = \frac{\nu_1 C_{V_1} + \nu_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}\right)}{\nu}$$

Similarly $$C_p = \frac{\nu_1 C_{p_1} + \nu_2 C_{p_2}}{\nu}$$

$$= \frac{\nu_1 \gamma_1 C_{V_1} + \nu_2 \gamma_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \frac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \frac{\gamma_2 R}{\gamma_2 - 1}\right)}{\nu}$$

Thus $$\gamma = \frac{C_p}{C_V} = \frac{\nu_1 \frac{\gamma_1}{\gamma_1 - 1} R + \nu_2 \frac{\gamma_2}{\gamma_2 - 1} R}{\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}}$$

$$= \frac{\nu_1 \gamma_1 (\gamma_2 - 1) + \nu_2 \gamma_2 (\gamma_1 - 1)}{\nu_1 (\gamma_2 - 1) + \nu_2 (\gamma_1 - 1)}$$

2.34 From the previous problem

$$C_V = \frac{\nu_1 \frac{R}{\gamma_1 - 1} + \nu_2 \frac{R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 15{\cdot}2 \text{ J/mole.K}$$

and $$C_p = \frac{\nu_1 \frac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \frac{\gamma_2 R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 23{\cdot}85 \text{ J/mole. K}$$

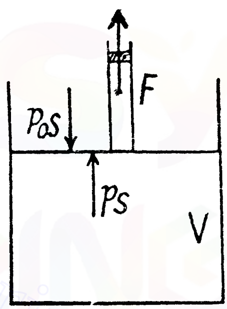
Now molar mass of the mixture $(M) = \dfrac{\text{Total mass}}{\text{Total number of moles}} = \dfrac{20+7}{\frac{1}{2}+\frac{1}{4}} = 36$

Hence $$c_V = \frac{C_V}{M} = 0{\cdot}42 \text{ J/g}\cdot\text{K} \text{ and } c_p = \frac{C_p}{M} = 0{\cdot}66 \text{ J/g}\cdot\text{K}$$

2.35 Let S be the area of the piston and F be the force exerted by the external agent. Then, $F + pS = p_0 S$ (Fig.) at an arbitrary instant of time. Here p is the pressure at the instant the volume is V. (Initially the pressure inside is p_0)

$$A \text{ (Work done by the agent)} = \int_{V_0}^{\eta V_0} F\,dx$$

$$= \int_{V_0}^{\eta V_0} (p_0 - p)\, S \cdot dx = \int_{V_0}^{\eta V_0} (p_0 - p)\, dV$$



$$= p_0(\eta - 1)V_0 - \int_{V_0}^{\eta V_0} p\,dV = p_0(\eta - 1)V_0 - \int_{V_0}^{\eta V_0} \nu RT \cdot \frac{dV}{V}$$

$$= (\eta - 1)p_0 V_0 - nRT \ln \eta = (\eta - 1)\nu RT - \nu RT \ln \eta$$

$$= \nu RT(\eta - 1 - \ln \eta) = RT(\eta - 1 - \ln \eta) \text{ (For } \nu = 1 \text{ mole)}$$

2.36 Let the agent move the piston to the right by x. In equilibirium position,

$$p_1 S + F_{agent} = p_2 S, \quad \text{or,} \quad F_{agent} = (p_2 - p_1) S$$

Work done by the agent in an infinitesmal change dx is

$$F_{agent} \cdot dx = (p_2 - p_1) S dx = (p_2 - p_1)\, dV$$

By applying $pV =$ constant, for the two parts,

$$p_1 (V_0 + Sx) = p_0 V_0 \text{ and } p_2 (V_0 - Sx) = p_0 V_0$$

So, $$p_2 - p_1 = \frac{p_0 V_0 2Sx}{V_0^2 - S^2 x^2} = \frac{2 p_0 V_0 V}{V_0^2 - V^2} \text{ (where } Sx = V)$$

When the volume of the left end is η times the volume of the right end

$$(V_0 + V) = \eta (V_0 - V), \quad \text{or,} \quad V = \frac{\eta - 1}{\eta + 1} V_0$$

$$A = \int_0^V (p_2 - p_1)\, dV = \int_0^V \frac{2p_0 V_0 V}{V_0^2 - V^2}\, dV = -p_0 V_0 \left[\ln (V_0^2 - V^2)\right]_0^V$$

$$= -p_0 V_0 \left[\ln (V_0^2 - V^2) - \ln V_0^2\right]$$

$$= -p_0 V_0 \left[\ln \left\{V_0^2 - \left(\frac{\eta - 1}{\eta + 1}^2\right) V_0^2\right\} - \ln V_0^2\right]$$

$$= -p_0 V_0 \left(\ln \frac{4\eta}{(\eta + 1)^2}\right) = p_0 V_0 \ln \frac{(\eta + 1)^2}{4\eta}$$

2.37 In the isothermal process, heat transfer to the gas is given by

$$Q_1 = \nu R T_0 \ln \frac{V_2}{V_1} = \nu R T_0 \ln \eta \quad \left(\text{For } \eta = \frac{V_2}{V_1} = \frac{p_1}{p_2}\right)$$

In the isochoric process, $A = 0$

Thus heat transfer to the gas is given by

$$Q_2 = \Delta U = \nu C_V \Delta T = \frac{\nu R}{\gamma - 1} \Delta T \quad \left(\text{for } C_V = \frac{R}{\gamma - 1}\right)$$

But $\quad \frac{p_2}{p_1} = \frac{T_0}{T}$, or, $T = T_0 \frac{p_1}{p_2} = \eta T_0 \quad \left(\text{for } \eta = \frac{p_1}{p_2}\right)$

or, $\quad \Delta T = \eta T_0 - T_0 = (\eta - 1) T_0$. so, $Q_2 = \frac{\nu R}{\gamma - 1} \cdot (\eta - 1) T_0$

Thus, net heat transfer to the gas

$$Q = \nu R T_0 \ln \eta + \frac{\nu R}{\gamma - 1} \cdot (\eta - 1) T_0$$

or, $\quad \frac{Q}{\nu R T_0} = \ln \eta + \frac{\eta - 1}{\gamma - 1}$, or, $\frac{Q}{\nu R T_0} - \ln \eta = \frac{\eta - 1}{\gamma - 1}$

or, $\quad \gamma = 1 + \frac{\eta - 1}{\frac{Q}{\nu R T_0} - \ln \eta} = 1 + \frac{6 - 1}{\left(\frac{80 \times 10^3}{3 \times 8{\cdot}314 \times 273}\right) - \ln 6} = 1{\cdot}4$

2.38 (a) From ideal gas law $p = \left(\frac{\nu R}{V}\right) T = kT \left(\text{where } k = \frac{\nu R}{V}\right)$

For isochoric process, obviously $k =$ constant, thus $p = kT$, represents a straight line passing through the origin and its slope becomes k.

For isobaric process $p =$ constant, thus on $p - T$ curve, it is a horizontal straight line parallel to T – axis, if T is along horizontal (or x – axis)

For isothermal process, $T =$ constant, thus on $p - T$ curve, it represents a vertical straight line if T is taken along horizontal (or x – axis)

For adiabatic process $T^\gamma p^{1-\gamma} =$ constant

After diffrentiating, we get $(1 - \gamma) p^{-\gamma} dp \cdot T^\gamma + \gamma p^{1-\gamma} \cdot T^{\gamma - 1} \cdot dT = 0$

$$\frac{dp}{dT} = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{p^{1-\gamma}}{p^{-\gamma}}\right)\left(\frac{T^{\gamma-1}}{T^{\gamma}}\right) = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{T}$$

The approximate plots of isochoric, isobaric, isothermal, and adiabatic processess are drawn in the answersheet.

(b) As p is not considered as variable, we have from ideal gas law

$$V = \frac{\nu R}{p}T = k'T\left(\text{where } k' = \frac{\nu R}{p}\right)$$

On $V-T$ co-ordinate system let us, take T along x – axis.

For isochoric process $V =$ constant, thus $k' =$ constant and $V = k'T$ obviously represents a straight line pasing through the origin of the co-ordinate system and k' is its slope.

For isothermal process $T =$ constant. Thus on the stated co- ordinate system it represents a straight line parallel to the V – axis.

For adiabatic process $TV^{\gamma-1} =$ constant

After differentiating, we get $(\gamma-1)V^{\gamma-2}dV\cdot T + V^{\gamma-1}dT = 0$

$$\frac{dV}{dT} = -\left(\frac{1}{\gamma-1}\right)\cdot\frac{V}{T}$$

The approximate plots of isochoric, isobaric, isothermal and adiabatic processess are drawn in the answer sheet.

2.39 According to $T-p$ relation in adiabatic process, $T^{\gamma} = kp^{\gamma-1}$ (where $k =$ constant)

and $\left(\frac{T_2}{T_1}\right)^{\gamma} = \left(\frac{p_2}{p_1}\right)^{\gamma-1}$ So, $\frac{T^{\gamma}}{T_0^{\gamma}} = \eta^{\gamma-1}\left(\text{for } \eta = \frac{p_2}{p_1}\right)$

Hence $T = T_0\cdot\eta^{\frac{\gamma-1}{\gamma}} = 290\times 10^{(1\cdot4-1)/1\cdot4} = 0{\cdot}56 \text{ kK}$

(b) Using the solution of part (a), sought work done

$$A = \frac{\nu R\,\Delta T}{\gamma-1} = \frac{\nu RT_0}{\gamma-1}\left(\eta^{(\gamma-1)/\gamma}-1\right) = 5{\cdot}61 \text{ kJ} \quad \text{(on substitution)}$$

2.40 Let (p_0, V_0, T_0) be the initial state of the gas.

We know $A_{adia} = \frac{-\nu R\Delta T}{\gamma-1}$ (work done by the gas)

But from the equation $TV^{\gamma-1} =$ constant, we get $\Delta T = T_0\left(\eta^{\gamma-1}-1\right)$

Thus $$A_{\text{adia}} = \frac{-\nu RT_0\left(\eta^{\gamma-1}-1\right)}{\gamma-1}$$

On the other hand, we know $A_{iso} = \nu RT_0 \ln\left(\frac{1}{\eta}\right) = -\nu R\,T_0 \ln \eta$ (work done by the gas)

Thus $$\frac{A_{\text{adia}}}{A_{\text{iso}}} = \frac{\eta^{\gamma-1}-1}{(\gamma-1)\ln\eta} = \frac{5^{0\cdot4}-1}{0\cdot4\times\ln 5} = 1{\cdot}4$$

2.41 Since here the piston is conducting and it is moved slowly the temperature on the two sides increases and maintained at the same value.

Elementary work done by the agent = Work done in compression - Work done in expansion i.e. $dA = p_2\,dV - p_1\,dV = (p_2 - p_1)\,dV$

where p_1 and p_2 are pressures at any instant of the gas on expansion and compression side respectively.

From the gas law $p_1(V_0 + Sx) = \nu RT$ and $p_2(V_0 - Sx) = \nu RT$, for each section

(x is the displacement of the piston towards section 2)

So, $$p_2 - p_1 = \nu RT \frac{2\,Sx}{V_0^2 - S^2x^2} = \nu RT \cdot \frac{2V}{V_0^2 - V^2} \text{ (as } Sx = V)$$

So $$dA = \nu RT \frac{2V}{V_0^2 - V^2}\,dV$$

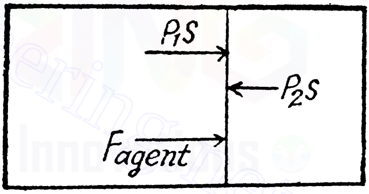
Also, from the first law of thermodynamics

$$dA = -dU = -2\nu \frac{R}{\gamma - 1}\,dT \quad (\text{as } dQ = 0)$$

So, work done on the gas $= -dA = 2\nu \cdot \dfrac{R}{\gamma - 1}\,dT$

Thus $$2\nu \frac{R}{\gamma - 1}\,dT = \nu RT \frac{2V \cdot dV}{V_0^2 - V^2},$$

or, $$\frac{dT}{T} = \gamma - 1 \; \frac{V\,dV}{V_0^2 - V^2}$$



When the left end is η times the volume of the right end.

$$(V_0 + V) = \eta(V_0 - V) \quad \text{or} \quad V = \frac{\eta - 1}{\eta + 1} V_0$$

On integrating $$\int_{T_0}^{T} \frac{dT}{T} = (\gamma - 1)\int_0^V \frac{V\,dV}{V_0^2 - V^2}$$

or $$\ln\frac{T}{T_0} = (\gamma - 1)\left[-\frac{1}{2}\ln(V_0^2 - V^2)\right]_0^V$$

$$= -\frac{\gamma - 1}{2}\left[\ln(V_0^2 - V^2) - \ln V_0^2 - V^2) - \ln V_0^2\right]$$

$$= \frac{\gamma - 1}{2}\left[\ln V_0^2 - \ln V_0^2\left\{1 - \left(\frac{\eta - 1}{\eta + 1}\right)^2\right\}\right] = \frac{\gamma - 1}{2}\ln\frac{(\eta + 1)^2}{4\eta}$$

Hence $$T = T_0\left(\frac{(\eta + 1)^2}{4\eta}\right)^{\frac{\gamma - 1}{2}}$$

2.42 From energy conservation as in the derivation of Bernoullis theorem it reads

$$\frac{p}{\rho}+\frac{1}{2}v^2+gz+u+Q_d = \text{constant} \qquad (1)$$

In the Eq. (1) u is the internal energy per unit mass and in this case is the thermal energy per unit mass of the gas. As the gas vessel is thermally insulated $Q_d = 0$, also in our case.

Just inside the vessel $u = \frac{C_V T}{M} = \frac{RT}{M(\gamma-1)}$ also $\frac{p}{\rho} = \frac{RT}{M}$. Inside the vessel $v = 0$ also. Just outside $p = 0$, and $u = 0$. Ingeneral gz is not very significant for gases.

Thus applying Eq. (1) just inside and outside the hole, we get

$$\frac{1}{2}v^2 = \frac{p}{\rho}+u$$

$$= \frac{RT}{M}+\frac{RT}{M(\gamma-1)} = \frac{\gamma RT}{M(\gamma-1)}$$

Hence $\quad v^2 = \frac{2\gamma RT}{M(\gamma-1)}$ or, $v = \sqrt{\frac{2\gamma RT}{M(\gamma-1)}} = 3.22$ km/s.

Note : The velocity here is the velocity of hydrodynamic flow of the gas into vaccum. This requires that the diameter of the hole is not too small ($D >$ mean free path l). In the opposite case ($D << l$) the flow is called effusion. Then the above result does not apply and kinetic theory methods are needed.

2.43 The differential work done by the gas

$$dA = pdV = \frac{vRT^2}{a}\left(-\frac{a}{T^2}\right)dT = -vRdT$$

$$\left(\text{as } pV = vRT \text{ and } V = \frac{a}{T}\right)$$

So, $$A = -\int_T^{T+\Delta T} vRdT = -vR\Delta T$$

From the first law of thermodynamics

$$Q = \Delta U + A = \frac{vR}{\gamma-1}\Delta T - vR\Delta T$$

$$= vR\Delta T\cdot\frac{2-\gamma}{\gamma-1} = R\,\Delta T\cdot\frac{2-\gamma}{\gamma-1} \quad (\text{for } v = 1 \text{ mole})$$

2.44 According to the problem : $A \alpha U$ or $dA = aU$ (where a is proportionality constant)

or, $$pdV = \frac{a\,v\,R\,dT}{\gamma-1} \qquad (1)$$

From ideal gas law, $pV = vRT$, on differentiating

$$pdV + Vdp = vRdT \qquad (2)$$

Thus from (1) and (2)

$$pdV = \frac{a}{\gamma - 1}(pdV + Vdp)$$

or, $pdV\left(\frac{a}{\gamma - 1} - 1\right) + \frac{a}{\gamma - 1} V\,dp = 0$

or, $pdV(k-1) + kVdp = 0$ (where $k = \frac{a}{\gamma - 1} =$ another constant)

or, $pdV\frac{k-1}{k} + Vdp = 0$

or, $pdVn + Vdp = 0$ (where $\frac{k-1}{k} = n =$ ratio)

Dividing both the sides by pV

$$n\frac{dV}{V} + \frac{dp}{p} = 0$$

On integrating $n \ln V + \ln p = \ln C$ (where C is constant)

or, $\ln(pV^n) = \ln C$ or, $pV^n = C$ (const.)

2.45 In the polytropic process work done by the gas

$$A = \frac{\nu R\,[T_i - T_f]}{n-1}$$

(where T_i and T_f are initial and final temperature of the gas like in adiabatic process)

and $$\Delta U = \frac{\nu R}{\gamma - 1}(T_f - T_i)$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= \frac{\nu R}{\gamma - 1}(T_f - T_i) + \frac{\nu R}{n-1}(T_i - T_f)$$

$$= (T_f - T_i)\,\nu R\left[\frac{1}{\gamma - 1} - \frac{1}{n-1}\right] = \frac{\nu R\,[n-\gamma]}{(n-1)(\gamma-1)}\Delta T$$

According to definition of molar heat capacity when number of moles $\nu = 1$ and $\Delta T = 1$ then $Q =$ Molar heat capacity.

Here, $$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} < 0 \quad \text{for} \quad 1 < n < \gamma$$

2.46 Let the process be polytropic according to the law $pV^n =$ constant

Thus, $$p_f V_f^n = p_i V_i^n \quad \text{or,} \quad \left(\frac{p_i}{p_f}\right) = \beta$$

So, $$\alpha^n = \beta \quad \text{or} \quad \ln\beta = n\ln\alpha \quad \text{or} \quad n = \frac{\ln\beta}{\ln\alpha}$$

In the polytropic process molar heat capacity is given by

$$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} = \frac{R}{\gamma-1} - \frac{R}{n-1}$$

$$= \frac{R}{\gamma-1} - \frac{R\ln\alpha}{\ln\beta - \ln\alpha}, \quad \text{where } n = \frac{\ln\beta}{\ln\alpha}$$

So,
$$C_n = \frac{8{\cdot}314}{1{\cdot}66-1} - \frac{8{\cdot}314\ln 4}{\ln 8 - \ln 4} = -42 \text{ J/mol.K}$$

2.47 (a) Increment of internal energy for ΔT, becomes

$$\Delta U = \frac{\nu R\Delta T}{\gamma-1} = \frac{R\,\Delta T}{\gamma-1} = -324 \text{ J (as } \nu = 1 \text{ mole)}$$

From first law of thermodynamics

$$Q = \Delta U + A = \frac{R\,\Delta T}{\gamma-1} - \frac{R\Delta T}{n-1} = 0{\cdot}11 \text{ kJ}$$

(b) Sought work done, $A_n = \int p\,dV = \int_{V_i}^{V_f} \frac{k}{V^n}\,dV$

$$(\text{where } pV^n = k = p_i V_i^n = p_f V_f^n)$$

$$= \frac{k}{1-n}\left(V_f^{1-n} - V_i^{1-n}\right) = \frac{\left(p_f V_f^n V_f^{1-n} - p_i V_i^n V_i^{1-n}\right)}{1-n}$$

$$= \frac{p_f V_f - p_i V_i}{1-n} = \frac{\nu R(T_f - T_i)}{1-n}$$

$$= \frac{\nu R\Delta T}{n-1} = -\frac{R\Delta T}{n-1} = 0{\cdot}43 \text{ kJ (as } \nu = 1 \text{ mole)}$$

2.48 Law of the process is $p = \alpha V$ or $pV^{-1} = \alpha$

so the process is polytropic of index $n = -1$

As $p = \alpha V$ so, $p_i = \alpha V_0$ and $p_f = \alpha\eta V_0$

(a) Increment of the internal energy is given by

$$\Delta U = \frac{\nu R}{\gamma-1}[T_f - T_i] = \frac{p_f V_f - p_i V_i}{\gamma-1}$$

(b) Work done by the gas is given by

$$A = \frac{p_i V_i - p_f V_f}{n-1} = \frac{\alpha V_0^2 - \alpha\eta V_0 \cdot \eta V_0}{-1-1}$$

$$= \frac{\alpha V_0^2(1-\eta^2)}{-2} = \frac{1}{2}\alpha V_0^2(\eta^2-1)$$

(c) Molar heat capacity is given by

$$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} = \frac{R(-1-\gamma)}{(-1-1)(\gamma-1)} = \frac{R}{2}\frac{\gamma+1}{\gamma-1}$$

2.49 (a) $\Delta U = \frac{\nu R}{\gamma - 1}\Delta T$ and $Q = \nu C_n \Delta T$

where C_n is the molar heat capacity in the process. It is given that $Q = -\Delta U$

So, $$C_n \Delta T = \frac{R}{\gamma - 1}\Delta T, \quad \text{or} \quad C_n = -\frac{R}{\gamma - 1}$$

(b) By the first law of thermodynamics, $dQ = dU + dA$,

or, $$2\,dQ = dA \text{ (as } dQ = -dU\text{)}$$

$$2\nu C_n\, dT = p dV, \text{ or, } \frac{2R\nu}{\gamma - 1}dT + pdV = 0$$

So, $$\frac{2RV}{\gamma - 1}dT + \frac{\nu RT}{V}dV = 0, \quad \text{or,} \quad \frac{2}{(\gamma - 1)}\frac{dT}{T} + \frac{dV}{V} = 0$$

or, $$\frac{dT}{T} + \frac{\gamma - 1}{2}\frac{dV}{V} = 0, \quad \text{or,} \quad TV^{(\gamma - 1)/2} = \text{constant.}$$

(c) We know $C_n = \frac{(n - \gamma) R}{(n - 1)(\gamma - 1)}$

But from part (a), we have $C_n = -\frac{R}{\gamma - 1}$

Thus $$-\frac{R}{\gamma - 1} = \frac{(n - \gamma) R}{(n - 1)(\gamma - 1)} \text{ which yields}$$

$$n = \frac{1 + \gamma}{2}$$

From part (b); we know $TV^{(\gamma - 1)/2} = \text{constant}$

So, $$\frac{T_0}{T} = \left(\frac{V}{V_0}\right)^{(\gamma - 1)/2} = \eta^{(\gamma - 1)/2} \text{ (where } T \text{ is the final temperature)}$$

Work done by the gas for one mole is given by

$$A = R\frac{(T_0 - T)}{n - 1} = \frac{2RT_0[1 - \eta^{(1 - \gamma)/2}]}{\gamma - 1}$$

2.50 Given $p = aT^{\alpha}$ (for one mole of gas)

So, $$pT^{-\alpha} = a \quad \text{or,} \quad p\left(\frac{pV}{R}\right)^{-\alpha} = a,$$

or, $$p^{1 - \alpha}V^{-\alpha} = aR^{-\alpha} \quad \text{or,} \quad pV^{\alpha/(\alpha - 1)} = \text{constant}$$

Here polytropic exponent $n = \frac{\alpha}{\alpha - 1}$

(a) In the polytropic process for one mole of gas :

$$A = \frac{R\Delta T}{1 - n} = \frac{R\Delta T}{\left(1 - \frac{\alpha}{\alpha - 1}\right)} = R\Delta T(1 - \alpha)$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} - \frac{R}{n - 1} = \frac{R}{\gamma - 1} - \frac{R}{\frac{\alpha}{\alpha - 1} - 1} = \frac{R}{\gamma - 1} + R(1 - \alpha)$$

2.51 Given $U = aV^{\alpha}$

or, $\nu C_V T = a V^{\alpha}$, or, $\nu C_V \frac{pV}{\nu R} = a V^{\alpha}$

or, $aV^{\alpha} \cdot \frac{R}{C_V} \cdot \frac{1}{pV} = 1$, or, $V^{\alpha - 1} \cdot p^{-1} = \frac{C_V}{Ra}$

or $pV^{1-\alpha} = \frac{Ra}{C_V} = \text{constant} = a(\gamma - 1) \left[\text{as } C_V = \frac{R}{\gamma - 1} \right]$

So polytropric index $n = 1 - \alpha$.

(a) Work done by the gas is given by

$$A = \frac{-\nu R \Delta T}{n-1} \text{ and } \Delta U = \frac{\nu R \Delta T}{\gamma - 1}$$

Hence $A = \frac{-\Delta U(\gamma - 1)}{n - 1} = \frac{\Delta U(\gamma - 1)}{\alpha}$ (as $n = 1 - \alpha$)

By the first law of thermodynamics, $Q = \Delta U + A$

$$= \Delta U + \frac{\Delta U(\gamma - 1)}{\alpha} = \Delta U \left[1 + \frac{\gamma - 1}{\alpha} \right]$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} - \frac{R}{n - 1} = \frac{R}{\gamma - 1} - \frac{R}{1 - \alpha - 1}$$

$$= \frac{R}{\gamma - 1} + \frac{R}{\alpha} \text{ (as } n = 1 - \alpha)$$

2.52 (a) By the first law of thermodynamics

$$dQ = dU + dA = \nu C_V dT + p dV$$

Molar specific heat according to definition

$$C = \frac{dQ}{\nu dT} = \frac{C_V dT + p dV}{\nu dT}$$

$$= \frac{\nu C_V dT + \frac{\nu RT}{V} dV}{\nu dT} = C_V + \frac{RT}{V} \frac{dV}{dT},$$

We have $T = T_0 e^{\alpha V}$

After differentiating, we get $dT = \alpha T_0 e^{\alpha V} \cdot dV$

So, $\frac{dV}{DT} = \frac{1}{\alpha T_0 e^{\alpha V}},$

Hence $C = C_V + \frac{RT}{V} \cdot \frac{1}{\alpha T_0 e^{\alpha V}} = C_V + \frac{RT_0 e^{\alpha V}}{\alpha V T_0 e^{\alpha v}} = C_V + \frac{R}{\alpha V}$

(b) Process is $p = p_0 e^{\alpha V}$

$$p = \frac{RT}{V} = p_0 e^{\alpha V}$$

$$\text{or,}\quad T = \frac{p_0}{R} e^{\alpha V} \cdot V$$

So, $C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + p_0 e^{\alpha V} \cdot \frac{R}{p_0 e^{\alpha V}(1+\alpha V)} = C_V + \frac{R}{1+\alpha V}$

2.53 Using 2.52

(a) $C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + \frac{p dV}{dT}$ (for one mole of gas)

We have $p = p_0 + \frac{\alpha}{V}$, or, $\frac{RT}{V} = p_0 + \frac{\alpha}{V}$, or, $RT = p_0 V + \alpha$

Therefore $\quad RdT = p_0 dV$, So, $\frac{dV}{dT} = \frac{R}{p_0}$

Hence $\quad C = C_V + \left(p_0 + \frac{\alpha}{V}\right) \cdot \frac{R}{p_0} = \frac{R}{\gamma - 1} + \left(1 + \frac{\alpha}{p_0 V}\right) R$

$$= \left(R + \frac{R}{\gamma - 1}\right) + \frac{\alpha R}{p_0 V} = \frac{\gamma R}{\gamma - 1} + \frac{\alpha R}{p_0 V}$$

(b) Work done is given by

$$A = \int_{V_1}^{V_2} \left(p_0 + \frac{\alpha}{V}\right) dV = p_0 (V_2 - V_1) + \alpha \ln \frac{V_2}{V_1}$$

$$\Delta U = C_V (T_2 - T_1) = C_V \left(\frac{p_2 V_2}{R} - \frac{p_1 V_1}{R}\right) \text{ (for one mole)}$$

$$= \frac{R}{(\gamma - 1) R} (p_2 V_2 - p_1 V_1)$$

$$= \frac{1}{\gamma - 1}\left[\left(p_0 + \alpha V_2\right) V_2 - \left(p_0 + \frac{\alpha}{V_1}\right) V_1\right] = \frac{p_0 (V_2 - V_1)}{\gamma - 1}$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= p_0 (V_2 - V_1) + \alpha \ln \frac{V_2}{V_1} + \frac{p_0 (V_2 - V_1)}{(\gamma - 1)}$$

$$= \frac{\gamma p_0 (V_2 - V_1)}{\gamma - 1} + \alpha \ln \frac{V_2}{V_1}$$

2.54 (a) Heat capacity is given by

$$C = C_V + \frac{RT}{V}\frac{dV}{dT} \text{ (see solution of 2.52)}$$

We have $\quad T = T_0 + \alpha V$ or, $V = \frac{T}{\alpha} - \frac{T_0}{\alpha}$

After differentiating, we get, $\frac{dV}{dT} = \frac{1}{\alpha}$

Hence $$C = C_V + \frac{RT}{V}\cdot\frac{1}{\alpha} = \frac{R}{\gamma-1} + \frac{R(T_0+\alpha V)}{V}\cdot\frac{1}{\alpha}$$

$$= \frac{R}{\gamma-1} + R\left(\frac{T_0}{\alpha V}+1\right) = \frac{\gamma R}{\gamma-1} + \frac{RT_0}{\alpha V} = C_V + \frac{RT}{\alpha V} = C_p + \frac{RT_0}{\alpha V}$$

(b) Given $T = T_0 + \alpha V$

As $T = \frac{pV}{R}$ for one mole of gas

$$p = \frac{R}{V}(T_0 + \alpha V) = \frac{RT}{V} = \alpha R$$

Now $$A = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2}\left(\frac{RT_0}{v} + \alpha R\right)dV \text{ (for one mole)}$$

$$= RT_0 \ln\frac{V_2}{V_1} + \alpha (V_2 - V_1)$$

$$\Delta U = C_V(T_2 - T_1)$$

$$= C_V[T_0 + \alpha V_2 - T_0 \alpha V_1] = \alpha C_V (V_2 - V_1)$$

By the first law of thermodynamics $Q = \Delta U + A$

$$= \frac{\alpha R}{\gamma - 1}(V_2 - V_1) + RT_0 \ln\frac{V_2}{V_1} + \alpha R(V_2 - V_1)$$

$$= \alpha R(V_2 - V_1)\left[1 + \frac{1}{\gamma - 1}\right] + RT_0 \ln\frac{V_2}{V_1}$$

$$= \alpha C_p (V_2 - V_1 + RT_0 \ln\frac{V_2}{V_1}$$

$$= \alpha C_p (V_2 - V_1) + RT_0 \ln\frac{V_2}{V_1}$$

2.55 Heat capacity is given by $C = C_V + \frac{RT}{V}\frac{dV}{dT}$

(a) Given $C = C_V + \alpha T$

So, $C_V + \alpha T = C_V + \frac{RT}{V}\frac{dV}{dT}$ or, $\frac{\alpha}{R}dT = \frac{dV}{V}$

Integrating both sides, we get $\frac{\alpha}{R}T = \ln V + \ln C_0 = \ln VC_0$, C_0 is a constant.

Or, $$V\cdot C_0 = e^{\alpha T/R} \quad \text{or} \quad V\cdot e^{\alpha T/R} = \frac{1}{C_0} = \text{constant}$$

(b) $C = C_V + \beta V$

and $$C = C_V + \frac{RT}{V}\frac{dV}{dT} \text{ so, } C_V \frac{RT}{V}\frac{dV}{dT} = C_V + \beta V$$

or, $$\frac{RT}{V}\frac{dV}{dT} = \beta V \text{ or, } \frac{dV}{V^2} = \frac{\beta}{R}\frac{dT}{T} \text{ or, } V^{-2} = \frac{dT}{T}$$

Integrating both sides, we get $\frac{R}{\beta}\frac{V^{-1}}{\beta - 1} = \ln T + \ln C_0 = \ln T{\cdot}C_0$

So, $\ln T \cdot C_0 = -\frac{R}{\beta V}$ $T \cdot C_0 = e^{-R/\beta V}$ or, $T e^{-R/\beta V} = \frac{1}{C_0} =$ constant

(c) $C = C_V + ap$ and $C = C_V + \frac{RT}{V}\frac{dV}{dT}$

So, $$C_V + ap = C_V + \frac{RT}{V}\frac{dV}{dT} \text{ so, } ap = \frac{RT}{V}\frac{dV}{dT}$$

or, $$a\frac{RT}{V} = \frac{RT}{V}\frac{dV}{dT} \text{ (as } p = \frac{RT}{V} \text{ for one mole of gas)}$$

or, $$\frac{dV}{dT} = a \text{ or, } dV = a dT \text{ or, } dT = \frac{dV}{a}$$

So, $T = \frac{V}{a} +$ constant or $V - aT =$ constant

2.56 (a) By the first law of thermodynamics $A = Q - \Delta U$

or, $$= C dT - C_V dT = (C - C_V)\, dT \text{ (for one mole)}$$

Given $$C = \frac{\alpha}{T}$$

So, $$A = \int_{T_0}^{\eta T_0} \left(\frac{\alpha}{T} - C_V\right) dT = \alpha \ln \frac{\eta T_0}{T_0} - C_V(\eta T_0 - T_0)$$

$$= \alpha \ln \eta - C_V T_0 (\eta - 1) = \alpha \ln \eta + \frac{RT}{\gamma - 1}(\eta - 1)$$

(b) $C = +\frac{dQ}{dT} = \frac{RT}{V}\frac{dV}{dT} + C_V$

Given $$C = \frac{\alpha}{T}, \text{ so } C_V + \frac{RT}{V}\frac{dV}{dT} = \frac{\alpha}{T}$$

or, $$\frac{R}{\gamma - 1}\frac{1}{RT} + \frac{dV}{V} = \frac{\alpha}{RT^2} dT$$

or, $$\frac{dV}{V} = \frac{\alpha}{RT^2} dT - \frac{1}{\gamma - 1} \cdot \frac{dT}{T}$$

or, $$(\gamma - 1)\frac{dV}{V} = \frac{\alpha(\gamma - 1)}{RT^2} dT - \frac{dT}{T}$$

Integrating both sides, we get

or, $$(\gamma - 1) \ln V = -\frac{\alpha(\gamma - 1)}{RT} - \ln T + \ln K$$

or, $$\ln V^{\gamma - 1}\frac{T}{K} = \frac{-\alpha(\gamma - 1)}{RT}$$

$$\ln V^{\gamma - 1} \cdot \frac{pV}{RK} = \frac{-\alpha(\gamma - 1)}{pV}$$

or, $$\frac{pV^{\gamma}}{RK} = e^{-\alpha(\gamma - 1)/pV}$$

or, $$pV^{\gamma} e^{\alpha(\gamma - 1)/pV} = RK = \text{constant}$$

2.57 The work done is

$$A = \int_{V_1}^{V_2} p\, dV = \int_{V_1}^{V_2} \left(\frac{RT}{v - b} - \frac{a}{V^2}\right) dV$$

$$= RT \ln \frac{V_2 - b}{V_1 - b} + a\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

2.58 (a) The increment in the internal energy is

$$\Delta U = \int_{V_1}^{V_1} \left(\frac{\partial U}{\partial V}\right)_T dV$$

But from second law

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

On the other hand $$p = \frac{RT}{V - b} - \frac{a}{V^2}$$

or, $$T\left(\frac{\partial p}{\partial T}\right)_V = \frac{RT}{V - b} \text{ and } \left(\frac{\partial U}{\partial V}\right)_t = \frac{a}{V^2}$$

So, $$\Delta U = a\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

(b) From the first law

$$Q = A + \Delta U = RT \ln \frac{V_2 - b}{V_1 - b}$$

2.59 (a) From the first law for an adiabatic

$$dQ = dU + p\,dV = 0$$

From the previous problem

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = C_V dT + \frac{a}{V^2} dV$$

So, $$0 = C_V dT + \frac{RT\, dV}{V - b}$$

This equation can be integrated if we assume that C_V and b are constant then

$$\frac{R}{C_V}\frac{dV}{V-b}+\frac{dT}{T}=0, \quad \text{or,} \quad \ln T+\frac{R}{C_V}\ln(V-b)=\text{constant}$$

or,
$$T(V-b)^{R/C_V}=\text{constant}$$

(b) We use

$$dU=C_V\,dT+\frac{a}{V^2}dV$$

Now,
$$dQ=C_V\,dT+\frac{RT}{V-b}dV$$

So along constant p,
$$C_p=C_V+\frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p$$

Thus
$$C_p-C_V=\frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p, \quad \text{But } p=\frac{RT}{V-b}-\frac{a}{V^2}$$

On differentiating,
$$0=\left(-\frac{RT}{(V-b)^2}+\frac{2a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_p+\frac{R}{V-b}$$

or,
$$T\left(\frac{\partial V}{\partial T}\right)_p=\frac{RT/V-b}{\dfrac{RT}{(V-b)^2}-\dfrac{2a}{V^3}}=\frac{V-b}{1-\dfrac{2a(V-b)^2}{RTV^3}}$$

and
$$C_p-C_V=\frac{R}{1-\dfrac{2a(V-b)^2}{RTV^3}}$$

2.60 From the first law

$Q=U_f-U_i+A=0$, as the vessels are themally insulated.

As this is free expansion, $A=0$, so, $U_f=U_i$

But
$$U=\nu C_V T-\frac{a\nu^2}{V}$$

So,
$$C_V(T_f-T_i)=\left(\frac{a}{V_1+V_2}-\frac{a}{V_1}\right)\nu=\frac{-aV_2\nu}{V_1(V_1+V_2)}$$

or,
$$\Delta T=\frac{-a(\gamma-1)V_2\nu}{RV_1(V_1+V_2)}$$

Substitution gives $\Delta T=-3$ K

2.61 $Q=U_f-U_i+A=U_f-U_i$, (as $A=0$ in free expansion).

So at constant temperature.

$$Q=\frac{-a\nu^2}{V_2}-\left(-\frac{a\nu^2}{V_1}\right)=a\nu^2\frac{V_2-V_1}{V_1\cdot V_2}$$

$= 0{\cdot}33$ kJ from the given data.

2.3 KINETIC THEORY OF GASES. BOLTZMANN'S LAW AND MAXWELL'S DISTRIBUTION

2.62 From the formula $p = nkT$

$$n = \frac{p}{kT} = \frac{4 \times 10^{-15} \times 1{\cdot}01 \times 10^5}{1{\cdot}38 \times 10^{-23} \times 300} \text{ per m}^3$$

$$= 1 \times 10^{11} \text{ per m}^3 = 10^5 \text{ per c.c}$$

Mean distance between molecules

$$(10^{-5} \text{ c.c.})^{1/3} = 10^{1/3} \times 10^{-2} \text{ cm} = 0{\cdot}2 \text{ mm.}$$

2.63 After dissociation each N_2 molecule becomes two N-atoms and so contributes, 2×3 degrees of freedom. Thus the number of moles becomes

$$\frac{m}{M}(1 + \eta) \quad \text{and} \quad p = \frac{mRT}{MV}(1 + \eta)$$

Here M is the molecular weight in grams of N_2.

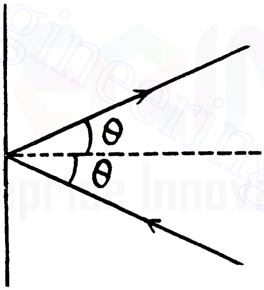
2.64 Let n_1 = number density of He atoms, n_2 = number density of N_2 molecules

Then $$\rho = n_1 m_1 + n_2 m_2$$

where m_1 = mass of He atom, m_2 = mass of N_2 molecule also $p = (n_1 + n_2) kT$

From these two equations we get

$$n_1 = \left(\frac{p}{kT} - \frac{\rho}{m_2}\right) \Big/ \left(1 - \frac{m_1}{m_2}\right)$$



2.65 $$p = \frac{nv \times 2\, mv \cos\theta \times dA \cos\theta}{dA}$$

$$= 2\, m\, n\, v^2 \cos^2\theta$$

2.66 From the formula

$$v = \sqrt{\frac{\gamma p}{\rho}}, \quad \gamma = \frac{\rho V^2}{p}$$

If i = number of degrees of freedom of the gas then

$$C_p = C_V + RT \text{ and } C_V = \frac{i}{2} RT$$

$$\gamma = \frac{C_p}{C_v} = 1 + \frac{2}{i} \quad \text{or} \quad i = \frac{2}{\gamma - 1} = \frac{2}{\frac{\rho v^2}{p} - 1}$$

2.67 $$v_{\text{sound}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}, \text{ and } v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}$$

so, $$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{\gamma}{3}} = \sqrt{\frac{i+2}{3i}}$$

(a) For monoatomic gases $i = 3$

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{5}{9}} = 0{\cdot}75$$

(b) For rigid diatomic molecules $i = 5$

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{7}{15}} = 0{\cdot}68$$

2.68 For a general noncollinear, nonplanar molecule

mean energy $= \frac{3}{2}kT$ (translational) $+ \frac{3}{2}kT$ (rotational) $+ (3N-6)kT$ (vibrational)

$= (3N-3)kT$ per molecule

For linear molecules, mean energy $= \frac{3}{2}kT$ (translational)

$+ kT$ (rotational) $+ (3N-5)kT$ (vibrational)

$= \left(3N - \frac{5}{2}\right)kT$ per molecule

Translational energy is a fraction $\frac{1}{2(N-1)}$ and $\frac{1}{2N - \frac{5}{3}}$ in the two cases.

2.69 (a) A diatomic molecule has 3 translational, 2 rotational and one vibrational degrees of freedom. The corresponding energy per mole is

$$\frac{3}{2}RT, \text{ (for translational)} + 2 \times \frac{1}{2}RT, \text{ (for rotational)}$$

$$+ 1 \times RT, \text{ (for vibrational)} = \frac{7}{2}RT$$

Thus, $\quad C_V = \frac{7}{2}R,\quad$ and $\quad \gamma = \frac{C_p}{C_V} = \frac{9}{7}$

(b) For linear $N-$ atomic molecules energy per mole

$$= \left(3N - \frac{5}{2}\right)RT \text{ as before}$$

So, $\quad C_V = \left(3N - \frac{5}{2}\right)R\quad$ and $\quad \gamma = \frac{6N-3}{6N-5}$

(c) For noncollinear $N-$ atomic molecules

$C_V = 3(N-1)R$ as before (2.68) $\quad \gamma = \frac{3N-2}{3N-3} = \frac{N-2/3}{N-1}$

2.70 In the isobaric process, work done is

$$A = pdv = RdT \text{ per mole.}$$

On the other hand heat transferred $Q = C_p\,dT$

Now $C_p = (3N-2)R$ for non-collinear molecules and $C_p = \left(3N - \frac{3}{2}\right)R$ for linear molecules

Thus $$\frac{A}{Q} = \begin{cases} \dfrac{1}{3N-2} & \text{non collinear} \\ \dfrac{1}{3N-\frac{3}{2}} & \text{linear} \end{cases}$$

For monoatomic gases, $c_p = \frac{5}{2}$ and $\frac{A}{Q} = \frac{2}{5}$

2.71 Given specific heats c_p, c_v (per unit mass)

$$M(c_p - c_v) = R \quad \text{or,} \quad M = \frac{R}{c_p - c_v}$$

Also $$\gamma = \frac{c_p}{c_v} = \frac{2}{i} + 1, \quad \text{os,} \quad i = \frac{2}{\frac{c_p}{c_v} 1} = \frac{2c_v}{c_p - c_v}$$

2.72 (a) $C_p = 29 \frac{J}{°\text{K mole}} = \frac{29}{8 \cdot 3} \text{R}$

$$C_v = \frac{20 \cdot 7}{8 \cdot 3} R, \quad \gamma = \frac{29}{20 \cdot 7} = 1 \cdot 4 = \frac{7}{5}$$

$$i = 5$$

(b) In the process $pT =$ const.

$$\frac{T^2}{V} = \text{const}, \quad \text{So} \quad 2\frac{dT}{T} - \frac{dV}{V} = 0$$

Thus $$C\,dT = C_V\,dT + p\,dV = C_V\,dT + \frac{RT}{V}dV = C_V\,dT + \frac{2RT}{T}dT$$

or $$C = C_V + 2R = \left(\frac{29}{8 \cdot 3}\right) R \quad \text{So} \quad C_V = \frac{12 \cdot 4}{8 \cdot 3} R = \frac{3}{2} R$$

Hence $i = 3$ (monoatomic)

2.73 Obviously

$$\frac{1}{R} C_V = \frac{3}{2}\gamma_1 + \frac{5}{2}\gamma_2$$

(Since a monoatomic gas has $C_V = \frac{3}{2}R$ and a diatomic gas has $C_V = \frac{5}{2}R$. [The diatomic molecule is rigid so no vibration])

$$\frac{1}{R} C_p = \frac{3}{2}\gamma_1 + \frac{5}{2}\gamma_2 + \gamma_1 + \gamma_2$$

Hence $$\gamma = \frac{C_p}{C_V} = \frac{5\gamma_1 + 7\gamma_2}{3\gamma_1 + 5\gamma_2}$$

2.74 The internal energy of the molecules are

$$U = \frac{1}{2} m N < (\vec{u} - \vec{v})^2 > = \frac{1}{2} m N < u^2 - v^2 >$$

where $\vec{v}$ = velocity of the vessel, N = number of molecules, each of mass m. When the vessel is stopped, internal energy becomes $\frac{1}{2} mN <u^2>$

So there is an increase in internal energy of $\Delta U = \frac{1}{2} m N v^2$. This will give rise to a rise in temperature of

$$\Delta T = \frac{\frac{1}{2} m N v^2}{\frac{i}{2} R}$$

$$= \frac{m N v^2}{iR}$$

there being no flow of heat. This change of temperature will lead to an excess pressure

$$\Delta p = \frac{R \Delta T}{V} = \frac{m N v^2}{iV}$$

and finally $$\frac{\Delta p}{p} = \frac{M v^2}{i R T} = 2 \cdot 2 \%$$

where M = molecular weight of N_2, i = number of degrees of freedom of N_2

2.75 (a) From the equipartition theorem

$$\bar{\varepsilon} = \frac{3}{2} k T = 6 \times 10^{-21} \text{ J}; \text{ and } v_{rms} = \sqrt{\frac{3 k T}{m}} = \sqrt{\frac{3 R T}{M}} = 0 \cdot 47 \text{ km/s}$$

(b) In equilibrium the mean kinetic energy of the droplet will be equal to that of a molecule.

$$\frac{1}{2} \frac{\pi}{6} d^3 \rho v_{rms}^2 = \frac{3}{2} k T \quad \text{or} \quad v_{rms} = 3 \sqrt{\frac{2 k T}{\pi d^3 \rho}} = 0 \cdot 15 \text{ m/s}$$

2.76 Here $i = 5, C_V = \frac{5}{2} R, \gamma = \frac{7}{5}$ given

$$v'_{rms} = \sqrt{\frac{3 R T'}{M}} = \frac{1}{\eta} v_{rms} = \frac{1}{\eta} \sqrt{\frac{3 R T}{M}} \quad \text{or} \quad T' = \frac{1}{\eta^2} T$$

Now in an adiabatic process

$$TV^{\gamma - 1} = T V^{2/i} = \text{constant} \quad \text{or} \quad V T^{i/2} = \text{constant}$$

$$V' \left(\frac{1}{\eta^2} T\right)^{i/2} = V T^{i/2} \quad \text{or} \quad V' \eta^{-i} = V \quad \text{or} \quad V' = \eta^i V$$

The gas must be expanded η^i times, i.e 7·6 times.

2.77 Here $$C_V = \frac{5}{2} \frac{m}{M} R \ (i = 5 \text{ here})$$

m= mass of the gas, M= molecular weight. If v_{rms} increases η times, the temperature will have increased η^2times. This will require (neglecting expansion of the vessels) a heat flow of amount

$$\frac{5}{2} \frac{m}{M} R (\eta^2 - 1) T = 10 \text{ kJ.}$$

2.78 The root mean square angular velocity is given by

$$\frac{1}{2} I\,\omega^2 = 2\times\frac{1}{2} kT \text{ (2 degrees of rotations)}$$

or $$\omega = \sqrt{\frac{2kT}{I}} = 6{\cdot}3\times 10^{12}\ \text{rad/s}$$

2.79 Under compression, the temperature will rise

$$TV^{\gamma-1} = \text{constant},\quad TV^{2/i} = \text{constant}$$

or, $$T'(\eta^{-1}V_0)^{2/i} = T_0 V_0^{2/i} \quad\text{or,}\quad T' = \eta^{+2/i}\,T_0$$

So mean kinetic energy of rotation per molecule in the compressed state

$$= kT' = kT_0\,\eta^{2/i} = 0{\cdot}72\times 10^{-20}\ \text{J}$$

2.80 No. of collisions $= \frac{1}{4} n \langle v\rangle = \nu$

Now, $$\frac{\nu'}{\nu} = \frac{n'}{n}\frac{\langle v'\rangle}{\langle v\rangle} = \frac{1}{\eta}\sqrt{\frac{T'}{T}}$$

(When the gas is expanded η times, n decreases by a factor η). Also

$T'(\eta V)^{2/i} = TV^{2/i}$ or $T' = \eta^{2/i}T$ so, $\frac{\nu'}{\nu} = \frac{1}{\eta}\eta^{-1/i} = \eta^{\frac{-i-1}{i}}$

i.e. collisions decrease by a factor $\eta^{\frac{i+1}{i}}$, $i = 5$ here.

2.81 In a polytropic process $pV^n =$ constant., where n is called the polytropic index. For this process

$$pV^n = \text{constant or } TV^{n-1} = \text{constant}$$

$$\frac{dT}{T} + (n-1)\frac{dV}{V} = 0$$

Then $$dQ = C\,dT = dU + p\,dV = C_V\,dT + p\,dV$$

$$= \frac{i}{2}R\,dT + \frac{RT}{V}dV = \frac{i}{2}R\,dT - \frac{1}{n-1}R\,dT = \left(\frac{i}{2} - \frac{1}{n-1}\right)R\,dT$$

Now $$C = R \quad\text{so}\quad \frac{i}{2} - \frac{1}{n-1} = 1$$

or, $$\frac{1}{n-1} = \frac{i}{2} - 1 = \frac{i-2}{2} \quad\text{or}\quad n = \frac{i}{i-2}$$

Now $$\frac{\nu'}{\nu} = \frac{n'}{n}\frac{\langle v'\rangle}{\langle v\rangle} = \frac{1}{\eta}\sqrt{\frac{T'}{T}} = \frac{1}{\eta}\left(\frac{V}{V'}\right)^{\frac{n-1}{2}}$$

$$= \frac{1}{\eta}\left(\frac{1}{\eta}\right)^{\frac{1}{i-2}} = \left(\frac{1}{\eta}\right)^{\frac{i-1}{i-2}} = \eta^{\frac{-i-1}{i-2}}\ \text{times} = \frac{1}{2{\cdot}52}\ \text{times}$$

2.82 If α is the polytropic index then

$$pV^{\alpha} = \text{constant}, \; TV^{\alpha-1} = \text{constant}.$$

Now
$$\frac{\nu'}{\nu} = \frac{n'}{n}\frac{<v'>}{<v>} = \frac{V}{V'}\sqrt{\frac{T'}{T}} = \frac{V\,T^{-1/2}}{V'\,T'^{-1/2}} = 1$$

Hence
$$\frac{1}{\alpha-1} = -\frac{1}{2} \quad \text{or} \quad \alpha = -1$$

Then
$$C = \frac{iR}{2} + \frac{R}{2} = 3R$$

2.83 $v_p = \sqrt{\dfrac{2kT}{m}} = \sqrt{\dfrac{2RT}{M}} = \sqrt{\dfrac{2p}{\rho}} = 0{\cdot}45 \text{ km/s},$

$$v_{av} = \sqrt{\frac{8}{\pi}\frac{p}{\rho}} = \cdot 51 \text{ km/s} \quad \text{and} \quad v_{rms} = \sqrt{\frac{3p}{\rho}} = 0{\cdot}55 \text{ km/s}$$

2.84 (a) The formula is

$$df(u) = \frac{4}{\sqrt{\pi}}u^2 e^{-u^2}du, \quad \text{where} \quad u = \frac{v}{v_p}$$

Now Prob $\left(\dfrac{|v - v_p|}{v_p} < \delta\eta\right) = \displaystyle\int_{1-\delta\eta}^{1+\delta\eta} df(u)$

$$= \frac{4}{\sqrt{\pi}}e^{-1} \times 2\,\delta\eta = \frac{8}{\sqrt{\pi}\,e}\delta\eta = 0{\cdot}0166$$

(b) Prob$\left(\left|\dfrac{v - v_{rms}}{v_{rms}}\right| < \delta\eta\right) = \text{Prob}\left(\left|\dfrac{v}{v_p} - \dfrac{v_{rms}}{v_p}\right| < \delta\eta\,\dfrac{v_{rms}}{v_p}\right)$

$$= \text{Prob}\left(\left|u - \sqrt{\frac{3}{2}}\right| < \sqrt{\frac{3}{2}}\,\delta\eta\right)$$

$$= \int_{\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}}\delta\eta}^{\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}}\delta\eta} \frac{4}{\sqrt{\pi}}u^2 e^{-u^2}du$$

$$= \frac{4}{\sqrt{\pi}} \times \frac{3}{2}e^{-3/2} \times 2\sqrt{\frac{3}{2}}\,\delta\eta = \frac{12\sqrt{3}}{\sqrt{2\pi}}e^{-3/2}\delta\eta = 0{\cdot}0185$$

2.85 (a) $v_{rms} - v_p = (\sqrt{3} - \sqrt{2})\sqrt{\frac{kT}{m}} = \Delta v,$

$$T = \frac{m}{k}\left(\frac{\Delta v}{(\sqrt{3}-\sqrt{2})}\right)^2 / \text{K} = 384\ \text{K}$$

(b) Clearly v is the most probable speed at this temperature. So

$$\sqrt{\frac{2kT}{m}} = v \quad \text{or} \quad T = \frac{mv^2}{2k} = 342\ \text{K}$$

2.86 (a) We have,

$$\frac{v_1^2}{v_p^2} e^{-v_1^2/v_p^2} = \frac{v_2^2}{v_p^2} e^{-v_2^2/v_p^2} \quad \text{or} \quad \left(\frac{v_1}{v_2}\right)^2 = e^{v_1^2 - v_2^2/v_p^2} \quad \text{or} \quad v_p^2 = \frac{2kT}{m} = \frac{v_1^2 - v_2^2}{(\ln v_1^2/v_2^2)}$$

So
$$T = \frac{m(v_1^2 - v_2^2)}{2k \ln \frac{v_1^2}{v_2^2}} = 330\ \text{K}$$

(b) $F(v) = \frac{4}{\sqrt{\pi}} \frac{v^2}{v_p^2} e^{-v^2/v_p^2} \times \frac{1}{v_p}\left(\frac{1}{v_p} \text{ comes from } F(v)\,dv = d\,f(u),\ du = \frac{dv}{v_p}\right)$

Thus $\frac{v^2}{v_{p_1}^3} e^{-v^2/v^2_{p_1}} = v^2/v_{p_2}\, e^{-v_2/v_{2_2}},\ v_{p_1}^2 = \frac{2kT_0}{m},\ v_{p_2}^2 = \frac{2kT_0}{m}\eta$ now

$$e^{-\frac{mv^2}{2kT_0}\left(1-\frac{1}{\eta}\right)} = \frac{1}{\eta^{3/2}} \quad \text{or} \quad \frac{mv^2}{2kT_0}\left(1 - \frac{1}{\eta}\right) = \frac{3}{2}\ln\eta$$

Thus
$$v = \sqrt{\frac{3kT_0}{m}}\sqrt{\frac{\ln\eta}{1 - 1/\eta}}$$

2.87 $v_{pN} = \sqrt{\frac{2kT}{m_N}} = \sqrt{\frac{2RT}{M_N}},\quad v_{p_0} = \sqrt{\frac{2RT}{M_0}}$

$$v_{p_N} - v_{p_0} = \Delta v = \sqrt{\frac{2RT}{M_N}}\left(1 - \sqrt{\frac{M_N}{M_0}}\right)$$

$$T = \frac{M_N(\Delta v)^2}{2R\left(1 - \sqrt{\frac{M_N}{M_0}}\right)^2} = \frac{m_N(\Delta v)^2}{2k\left(1 - \sqrt{\frac{m_N}{M_0}}\right)^2} = 363\ \text{K}$$

2.88 $\frac{v^2}{v_{p_H}^3} e^{-v^2/v^2_{p_H}} = \frac{v^2}{v_{p_{He}}^3} e^{-v^2/v^2_{p_{He}}}$ or $e^{v^2\left(\frac{m_{He}}{2kT} - \frac{m_H}{2kT}\right)} = \left(\frac{m_{He}}{m_H}\right)^{3/2}$

$v^2 = 3kT\frac{\ln m_{He}/m_H}{m_{H_e} - m_H}$, Putting the values we get $v = 1{\cdot}60$ km/s

2.89 $dN(v) = \frac{N\,4}{\sqrt{\pi}} \frac{v^2\,dv}{v_p^3} e^{-v^2/v_p^2}$

For a given range v to $v + dv$ (i.e. given v and dv) this is maximum when

$$\frac{\delta}{\delta v_p} \frac{dN(v)}{N v^2 dv} = 0 = \left(-3v_p^{-4} + \frac{2v^2}{v_p^6}\right) e^{-v^2/v_p^2}$$

or, $$v^2 = \frac{3}{2} v_p^2 = \frac{3kT}{m}. \quad \text{Thus} \quad T = \frac{1}{3}\frac{mv^2}{k}$$

2.90 $d^3 v = 2\pi v_\perp dv_\perp dv_x$

Thus $$dn(v) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}\left(v_x^2 + v_\perp^2\right)} dv_x\, 2\pi v_\perp\, dv_\perp$$

2.91 $\langle v_x \rangle = 0$ by symmetry

$$\langle |v_x| \rangle = \int_{-\infty}^{\infty} |v_x| e^{-\frac{mv_x^2}{2kT}} dv_x \Big/ \int_0^{\infty} e^{\frac{-mv_x^2}{2kT}} dv_x = \int_0^{\infty} v_x e^{\frac{-mv_x^2}{2kT}} dv_x \Big/ \int_0^{\infty} e^{\frac{-mv^2}{2kT}} dv_x$$

$$= \sqrt{\frac{2kT}{m}} \int_0^{\infty} ue^{-u^2} du \Big/ \int_0^{\infty} e^{-u^2} du$$

$$= \sqrt{\frac{2kT}{m}} \int_0^{\infty} \frac{1}{2} e^{-x} dx \Big/ \int_0^{\infty} e^{-x} \frac{dx}{2\sqrt{x}}$$

$$= \sqrt{\frac{2kT}{m}}\, \Gamma(1) \Big/ \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2kT}{m\pi}}$$

2.92 $$\langle v_x^2 \rangle = \int_0^{\infty} v_x^2 e^{-\frac{mv_x^2}{2kt}} dv_x \Big/ \int_0^{\infty} e^{-\frac{mv_x^2}{2kt}} dv_x$$

$$= \frac{2kT}{m} \int_0^{\infty} xe^{-x} \frac{dx}{2\sqrt{x}} \Big/ \int_0^{\infty} e^{-x} \frac{dx}{2\sqrt{x}}$$

$$= \frac{2kT}{m} \Gamma\left(\frac{3}{2}\right) \Big/ \Gamma\left(\frac{1}{2}\right) = \frac{kT}{m}$$

2.93 Here νdA = No. of molecules hitting an area dA of the wall per second

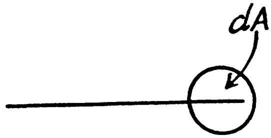
$$= \int_0^{\infty} dN(v_x)\, v_x\, dA$$

or, $$v = \int_0^\infty n \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}} v_x \, dv_x$$

$$= \int_0^\infty \frac{n}{\sqrt{\pi}} \left(\frac{2kT}{m}\right)^{1/2} e^{-u^2} u \, du$$

$$= \frac{1}{2} n \sqrt{\frac{2kT}{m\pi}} = n \sqrt{\frac{kT}{2m\pi}} = \frac{1}{4} n \langle v \rangle,$$

$$\left(\text{where } \langle v \rangle = \sqrt{\frac{8kT}{m\pi}}\right)$$



2.94 Let, $dn(v_x) = n \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} dv_x$

be the number of molecules per unit volume with x component of velocity in the range v_x to $v_x + dv_x$

Then $$p = \int_0^\infty 2 m v_x \cdot v_x \, dn(v_x)$$

$$= \int_0^\infty 2 m v_x^2 \, n \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} dv_x$$

$$= 2mn \frac{1}{\sqrt{\pi}} \frac{2kT}{m} \int_0^\infty u^2 e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} nkT \cdot \int_0^\infty x \, e^{-x} \frac{dx}{2\sqrt{x}} = nkT$$

2.95 $$\left\langle \frac{1}{v} \right\rangle = \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} 4\pi v^2 \, dv \frac{1}{v}$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi \frac{1}{2} \frac{2kT}{m} \int_0^\infty e^{-x} dx$$

$$= 2\left(\frac{m}{2\pi kT}\right)^{1/2} = \left(\frac{2m}{\pi kT}\right)^{1/2} = \left(\frac{16}{\pi^2} \frac{m\pi}{8kT}\right)^{1/2} = \frac{4}{\pi \langle v \rangle}$$

2.96 $$dN(v) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2\, dv = dN(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon}\, d\varepsilon$$

or, $$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2RT}\, 4\pi v^2 \frac{dv}{d\varepsilon}$$

Now, $$\varepsilon = \frac{1}{2} mv^2 \quad \text{so} \quad \frac{dv}{d\varepsilon} = \frac{1}{mv}$$

or, $$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\varepsilon/kT}\, 4\pi \sqrt{\frac{2\varepsilon}{m}}\, \frac{1}{m}$$

$$= N\frac{2}{\sqrt{\pi}}(kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}$$

i.e. $$dN(\varepsilon) = N\frac{2}{\sqrt{\pi}}(kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}\, d\varepsilon$$

The most probable kinetic energy is given from

$$\frac{d}{d\varepsilon}\frac{dN(\varepsilon)}{d\varepsilon} = 0 \quad \text{or,} \quad \frac{1}{2}\varepsilon^{-1/2} e^{-\varepsilon/kT} - \frac{\varepsilon^{1/2}}{kT} e^{\varepsilon/kT} = 0 \quad \text{or} \quad \varepsilon = \frac{1}{2}kT = \varepsilon_{pr}$$

The corresponding velocity is $v = \sqrt{\dfrac{kT}{m}} \neq v_{pr}$

2.97 The mean kinetic energy is

$$\langle\varepsilon\rangle = \int_0^\infty \varepsilon^{3/2} e^{-\varepsilon/kT}\, d\varepsilon \Big/ \int_0^\infty \varepsilon^{1/2} e^{-\varepsilon/kT}\, d\varepsilon = kT\frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{3}{2}kT$$

Thus

$$\frac{\delta N}{N} = \int_{\frac{3}{2}kT(1-\delta\eta)}^{\frac{3}{2}(1+\delta\eta)kT} \frac{2}{\sqrt{\pi}}(kT)^{-3/2} e^{-\varepsilon/kT} \varepsilon^{1/2}\, d\varepsilon$$

$$= \frac{2}{\sqrt{\pi}} e^{-3/2}\left(\frac{3}{2}\right)^{3/2} 2\,\delta\eta = 3\sqrt{\frac{6}{\pi}}\, e^{3/2}\,\delta\eta$$

If $\delta\eta = 1\%$ this gives 0·9 %

2.98 $$\frac{\Delta N}{N} = \frac{2}{\sqrt{\pi}}(kT)^{-3/2} \int_{\varepsilon_0}^{\infty} \sqrt{\varepsilon}\, e^{-\varepsilon/kT}\, d\varepsilon$$

$$\approx \frac{2}{\sqrt{\pi}}(kT)^{-3/2} \sqrt{\varepsilon_0} \int_{\varepsilon_0}^{\infty} e^{-\varepsilon/kT}\, d\varepsilon \quad (\varepsilon_0 >> kT)$$

$$= \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \sqrt{\varepsilon_0}\, kT\, e^{-\varepsilon_0/kT} = 2\sqrt{\frac{\varepsilon_0}{\pi kT}}\, e^{-\varepsilon_0/kT}$$

(In evaluating the integral, we have taken out $\sqrt{\varepsilon}$ as $\sqrt{\varepsilon_0}$ since the integral is dominated by the lower limit.)

2.99 (a) $F(v) = A v^3 e^{-mv^2/2kT}$

For the most probable value of the velocity

$$\frac{dF(v)}{dv} = 0 \quad \text{or} \quad 3A v^2 e^{-mv^2/2kT} - A v^3 \frac{2mv}{2kT} e^{-mv^2/2kT} = 0$$

So,
$$v_{pr} = \sqrt{\frac{3kT}{m}}$$

This should be compared with the value $v_{pr} = \sqrt{\frac{2kT}{m}}$ for the Maxwellian distribution.

(b) In terms of energy, $\varepsilon = \frac{1}{2} mv^2$

$$F(\varepsilon) = A v^3 e^{-mv^2/2kT} \frac{dv}{d\varepsilon}$$

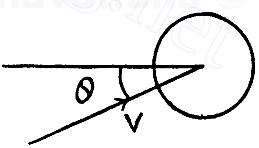
$$= A \left(\frac{2\varepsilon}{m}\right)^{3/2} e^{-\varepsilon/kT} \frac{1}{\sqrt{2m\varepsilon}} = A \frac{2\varepsilon}{m^2} e^{-\varepsilon/kT}$$

From this the probable energy comes out as follows : $F'(\varepsilon) = 0$ implies

$$\frac{2A}{m^2}\left(e^{-\varepsilon/kT} - \frac{\varepsilon}{kT} e^{-\varepsilon/kT}\right) = 0, \quad \text{or,} \quad \varepsilon_{pr} = kT$$

2.100 The number of molecules reaching a unit area of wall at angle between θ and $\theta + d\theta$ to its normal per unit time is

$$dv = \int_{v=0}^{v=\infty} dn(v) \frac{d\Omega}{4\pi} v \cos\theta$$



$$= \int_0^\infty n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3\, dv \sin\theta \cos\theta\, d\theta \times 2\pi$$

$$= n \left(\frac{2kT}{m\pi}\right)^{1/2} \int_0^\infty e^{-x} x\,dx\, \sin\theta\cos\theta\, d\theta = n\left(\frac{2kT}{m\pi}\right)^{1/2} \sin\theta\cos\theta\, d\theta$$

2.101 Similarly the number of molecules reaching the wall (per unit area of the wall with velocities in the interval v to $v + dv$ per unit time is

$$dv = \int_{\theta=0}^{\theta=\pi/2} dn(v) \frac{d\Omega}{4\pi} v \cos 0$$

$$= \int_{\theta=0}^{\theta=\pi/2} n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, v^3\, dv \sin\theta \cos\theta\, d\theta \times 2\pi$$

$$= n\pi \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, v^3\, dv$$

2.102 If the force exerted is F then the law of variation of concentration with height reads

$n(z) = n_0 e^{-Fz/kT}$ So, $\eta = e^{F\Delta h/kT}$ or $F = \frac{kT \ln \eta}{\Delta h} = 9 \times 10^{-20}$ N

2.103 Here $F = \frac{\pi}{6} d^3 \Delta\rho\, g = \frac{RT \ln \eta}{N_a h}$ or $N_a = \frac{6RT \ln \eta}{\pi d^3 g \Delta\rho h}$

In the problem, $\frac{\eta}{\eta_0} = 1{\cdot}39$ here

$T = 290$K, $\eta = 2$, $h = 4 \times 10^{-5}$ m, $d = 4 \times 10^{-7}$ m, $g = 9{\cdot}8$ m/s^2, $\Delta\rho = 0{\cdot}2 \times 10^3$ kg/m^3 and $R = 8{\cdot}31$ J/k

Hence, $N_a = \frac{6 \times 8{\cdot}31 \times 290 \times \ln 2}{\pi \times 64 \times 9{\cdot}8\, 200 \times 4} \times 10^{26} = 6{\cdot}36 \times 10^{23}$ mole^{-1}

2.104 $\eta = \frac{\text{concetration of } H_2}{\text{concentration of } N_2} = \eta_0 \frac{e^{-M_{H_2} gh/RT}}{e^{-M_{N_2} - gh/RT}} = \eta_0 e^{(M_{N_2} - M_{H_2})\, gh/RT}$

So more N_2 at the bottom, $\left(\frac{\eta}{\eta_0} = 1.39 \text{ here}\right)$

2.105 $n_1(h) = n_1 e^{-m_1 gh/kt}$, $n_2(h) = n_2 e^{-m_2 gh/kT}$

They are equal at a height h where $\frac{n_1}{n_2} = e^{gh(m_1 - m_2)/kT}$

$$\text{or} \quad h = \frac{kT}{g} \frac{\ln n_1 - \ln n_2}{m_1 - m_2}$$

2.106 At a temperature T the concentration $n(z)$ varies with height according to

$$n(z) = n_0 e^{-mgz/kT}$$

This means that the cylinder contains $\int_0^\infty n(z)\, dz$

$$= \int_0^\infty n_0 e^{-mgz/kT}\, dz = \frac{n_0 kT}{mg}$$

particles per unit area of the base. Clearly this cannot change. Thus $n_0 kT = p_0 =$ pressure at the bottom of the cylinder must not change with change of temperature.

2.107

$$\langle U \rangle = \frac{\int_0^\infty mgz\, e^{-mgz/kt}\, dz}{\int_0^\infty e^{-mgz/kT}\, dz} = kT \frac{\int_0^\infty x\, e^{-x}\, dx}{\int_0^\infty e^{-x}\, dx} = kT \frac{\Gamma(2)}{\Gamma(1)} = kT$$

When there are many kinds of molecules, this formula holds for each kind and the average energy

$$\langle U \rangle = \frac{\sum f_i kT}{\sum f_i} = kT$$

where f_i α fractional concentration of each kind at the ground level.

2.108 The constant acceleration is equivalent to a pseudo force wherein a concentration gradient is set up. Then

$$e^{-M_A w l/RT} = 1 - \eta$$

or $$w = -\frac{RT \ln(1-\eta)}{M_A l} = \frac{\eta RT}{M_A l} = 70 \text{ g}$$

2.109 In a centrifuge rotating with angular velocity ω about an axis, there is a centrifugal acceleration $\omega^2 r$ where r is the radial distance from the axis. In a fluid if there are suspended colloidal particles they experience an additional force. If m is the mass of each particle then its volume is $\frac{m}{\rho}$ and the excess force on this particle is

$\frac{m}{\rho}(\rho - \rho_0)\,\omega^2 r$ outward corresponding to a potential energy $-\frac{m}{2\rho}(\rho - \rho_0)\,\omega^2 r^2$

This gives rise to a concentration variation

$$n(r) = n_0 \exp\left(+\frac{m}{2\rho kT}(\rho - \rho_0)\,\omega^2 r^2\right)$$

Thus $$\frac{n(r_2)}{n(r_1)} = \eta = \exp\left(+\frac{M}{2\rho RT}(\rho - \rho_0)\,\omega^2 \left(r_2^2 - r_1^2\right)\right)$$

where $\frac{m}{k} = \frac{M}{R}$, $M = N_A m$ is the molecular weight

Thus $$M = \frac{2\rho RT \ln \eta}{(\rho - \rho_0)\,\omega^2 (r_2^2 - r_1^2)}$$

2.110 The potential energy associated with each molecule is : $-\frac{1}{2} m\, \omega^2 r^2$

and there is a concentration variation

$$n(r) = n_0 \exp\left(\frac{m\,\omega^2 r^2}{2kT}\right) = n_0 \exp\left(\frac{M\,\omega^2 r^2}{2RT}\right)$$

Thus $$\eta = \exp\left(\frac{M\,\omega^2 l^2}{2RT}\right) \quad \text{or} \quad \omega = \sqrt{\frac{2RT}{Ml^2}\ln \eta}$$

Using $M = 12 + 32 = 44$ gm, $l = 100$ cm, $R = 8{\cdot}31 \times 10^7 \frac{\text{erg}}{°\text{K}}$, $T = 300$, we get $\omega = 280$ radians per second.

2.111 Here $n(r) = n_0 \exp\left(-\frac{ar^2}{kT}\right)$

(a) The number of molecules located at the distance between r and $r + dr$ is

$$4\pi r^2\, dr\, n(r) = 4\pi n_0 \exp\left(-\frac{ar^2}{kT}\right) r^2\, dr$$

(b) r_{pr} is given by $\frac{d}{dr} r^2 n(r) = 0$ or, $2r - \frac{2ar^3}{kT} = 0$ or $r_{pr} = \sqrt{\frac{kT}{a}}$

(c) The fraction of molecules lying between r and $r + dr$ is

$$\frac{dN}{N} = \frac{4\pi r^2\, dr\, n_0 \exp(-ar^2/kT)}{\int_0^\infty 4\pi r^2\, dr\, n_0 \exp(-ar^2/kT)}$$

$$\int_0^\infty 4\pi r^2\, dr \exp\left(-\frac{ar^2}{kT}\right) = \left(\frac{kT}{a}\right)^{3/2} 4\pi \int_0^\infty x \frac{dx}{2\sqrt{x}} \exp(-x)$$

$$= \left(\frac{kT}{a}\right)^{3/2} 2\pi\, \Gamma\left(\frac{3}{2}\right) = \left(\frac{\pi kT}{a}\right)^{3/2}$$

Thus $$\frac{dN}{N} = \left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2\, dr \exp\left(\frac{-ar^2}{kT}\right)$$

(d) $$dN = N\left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2\, dr \exp\left(\frac{-ar^2}{kT}\right),$$

So $$n(r) = N\left(\frac{a}{\pi kT}\right)^{1/2} \exp\left(\frac{-ar^2}{kT}\right)$$

When T decreases η times $n(0) = n_0$ will increase $\eta^{3/2}$ times.

2.112 Write $U = ar^2$ or $r = \sqrt{\frac{U}{a}}$, so $dr = \sqrt{\frac{1}{a}} \frac{dU}{2\sqrt{U}} = \frac{dU}{2\sqrt{a\,U}}$

so $$dN = n_0 4\pi \frac{U}{a} \frac{dU}{2\sqrt{aU}} \exp\left(\frac{U}{kT}\right)$$

$$= 2\pi n_0 a^{-3/2}\, U^{1/2} \exp\left(\frac{-U}{kT}\right) dU$$

The most probable value of U is given by

$$\frac{d}{dU}\left(\frac{dN}{dU}\right) = 0 = \left(\frac{1}{2\sqrt{u}} - \frac{U^{1/2}}{kT}\right) \exp\left(\frac{-U}{kT}\right) \quad \text{or,} \quad U_{pr} = \frac{1}{2} kT$$

From 2.111 (b), the potential energy at the most probable distance is kT.

2.4 THE SECOND LAW OF THERMODYNAMICS. ENTROPY

2.113 The efficiency is given by

$$\eta = \frac{T_1 - T_2}{T_1}, \; T_1 > T_2$$

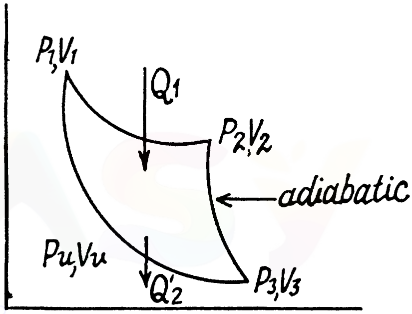
Now in the two cases the efficiencies are

$$\eta_h = \frac{T_1 + \Delta T - T_2}{T_1 + \Delta T}, \quad T_1 \text{ increased}$$

$$\eta_l = \frac{T_1 - T_2 + \Delta T}{T_1}, \; T_2 \text{ decreased}$$

Thus $\qquad \eta_h < n_l$

2.114 For H_2, $\gamma = \frac{7}{5}$



$$p_1 V_1 = p_2 V_2, \; p_3 V_3 = p_4 V_4$$

$$p_2 V_2^\gamma = p_3 V_3^\gamma, \; p_1 V_1^\gamma = p_4 V_4^\gamma$$

Define n by $V_3 = n V_2$

Then $p_3 = p_2 n^{-\gamma}$ so

$$p_4 V_4 = p_3 V_3 = p_2 V_2 n^{1-\gamma} = p_1 V_1 n^{1-\gamma}$$

$$p_4 V_4^\gamma = p_1 V_1^\gamma \text{ so } V_4^{1-\gamma} = V_1^{1-\gamma} n^{1-\gamma} \text{ or } V_4 = n V_1$$

Also $\qquad Q_1 = p_2 V_2 \ln \frac{V_2}{V_1}, \; Q'_2 = p_3 V_3 \ln \frac{V_3}{V_4} n^{1-\gamma} = p_2 v_2 \ln \frac{V_3}{v_4}$

Finally $\qquad \eta = 1 - \frac{Q_2}{Q_1} = 1 - n^{1-\gamma} = 0{\cdot}242$

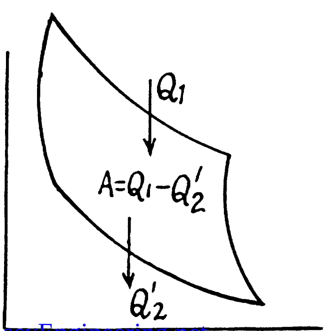
(b) Define n *by* $p_3 = \frac{p_2}{n}$

$$p_2 V_2^\gamma = \frac{p_2}{n} V_3^\gamma \quad \text{or} \quad V_3 = n^{1/\gamma} V_2$$

So we get the formulae here by $n \to n^{1/\gamma}$ in the previous case.

$$\eta = 1 - n^{(1/\gamma) - 1} = 1 - n^{-\frac{2}{7}} \approx 0{\cdot}18$$

2.115 Used as a refrigerator, the refrigerating efficiency of a heat engine is given by



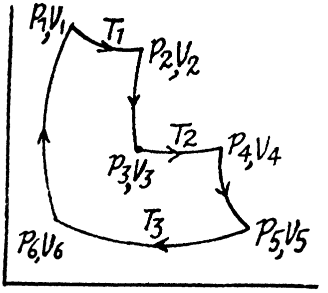
$$\varepsilon = \frac{Q_2'}{A} = \frac{Q_2'}{Q_1 - Q_2} = \frac{Q_2'/Q_1}{1 - \frac{Q_2'}{Q_1}} = \frac{1-\eta}{\eta} = 9 \text{ here,}$$

where η is the efficiency of the heat engine.

2.116 Given $V_2 = n V_1$, $V_4 = n V_3$

$Q_1 =$ Heat taken at the upper temperature

$$= RT_1 \ln n + R T_2 \ln n = R(T_1 + T_2) \ln n$$



Now $\quad T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1}$ or $V_3 = \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} V_2$

Similarly $\quad V_5 = \left(\frac{T_2}{T_3}\right)^{\frac{1}{\gamma-1}} V_4$, $V_6 = \left(\frac{T_1}{T_3}\right)^{\frac{1}{\gamma-1}} V_1$

Thus $Q_2 =$ heat ejected at the lower temperature $= -RT_3 \ln \frac{V_6}{V_5}$

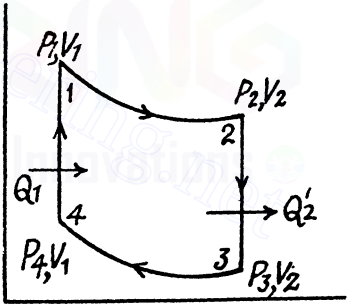
$$= -R T_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{V_1}{V_4} = -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{V_2}{n^2 V_3}$$

$$= -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{\frac{1}{\gamma-1}} \frac{1}{n^2} \left(\frac{T_1}{T_2}\right)^{-\frac{1}{\gamma-1}} = 2 R T_3 \ln n$$

Thus $$\eta = 1 - \frac{2T_3}{T_1 + T_2}$$

2.117 $Q'_2 = C_V (T_2 - T_3) = \frac{C_V}{R} V_2 (p_2 - p_3)$

$$Q_1 = \frac{C_V}{R} V_1 (p_1 - p_4)$$



Thus $\quad \eta = 1 - \frac{V_2 (p_2 - p_3)}{V_1 (p_1 - p_4)}$

On the other hand,

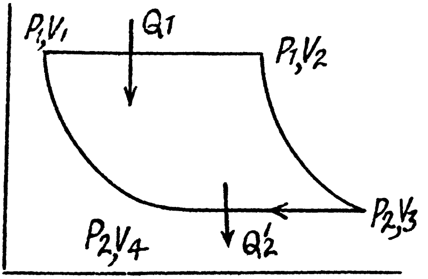
$6p_1 V_1^\gamma = p_2 V_2^\gamma$, $p_3 V_2^\gamma = p_4 V_1^\gamma$ also $V_2 = n V_1$

Thus $\quad p_1 = p_2 n^\gamma$, $p_4 = p_3 n^\gamma$

and $\eta = 1 - n^{1-\gamma}$, with $\gamma = \frac{7}{5}$ for N_2 this is $\eta = 0{\cdot}602$

2.118 $Q_1 = \frac{C_p}{R} p_1 (V_2 - V_1)$, $Q'_2 = \frac{C_p}{R} p_2 (V_3 - V_4)$

So $\quad \eta = 1 - \frac{p_2 (V_3 - V_4)}{p_1 (V_2 - V_1)}$



Now $p_1 = n p_2$, $p_1 V_2^\gamma$ or $V_3 = n^{\frac{1}{\gamma}} V_2$

$p_2 V_4^\gamma = p_1 V_1^\gamma$ or $V_4 = n^{\frac{1}{\gamma}} V_1$

so $\quad \eta = 1 - \frac{1}{n} \cdot n^{\frac{1}{\gamma}} = 1 - n^{\frac{1}{\gamma} - 1}$

2.119 Since the absolute temperature of the gas rises n times both in the isochoric heating and in the isobaric expansion

$$p_1 = np_2 \text{ and } V_2 = n V_1 . \text{ Heat taken is}$$

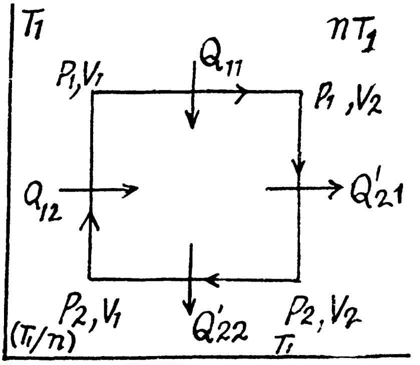
$$Q_1 = Q_{11} + Q_{12}$$

where $Q_{11} = C_p (n-1) T_1$ and $Q_{12} = C_V T_1 \left(1 - \frac{1}{n}\right)$

Heat rejected is

$$Q'_2 = Q'_{21} + Q'_{22} \text{ where}$$

$$Q'_{21} = C_V T_1 (n-1), \; Q'_{22} = C_p T_1 \left(1 - \frac{1}{n}\right)$$



$$\text{Thus } \eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_V (n-1) + C_p \left(1 - \frac{1}{n}\right)}{C_p (n-1) + C_V \left(1 - \frac{1}{n}\right)}$$

$$= 1 - \frac{n - 1 + \gamma \left(1 - \frac{1}{n}\right)}{\gamma (n-1) + \left(1 - \frac{1}{n}\right)} = 1 - \frac{1 + \frac{\gamma}{n}}{\gamma + \frac{1}{n}} = 1 - \frac{n + \gamma}{1 + n\gamma}$$

2.120 (a) Here $p_2 = n p_1$, $p_1 V_1 = p_0 V_0$,

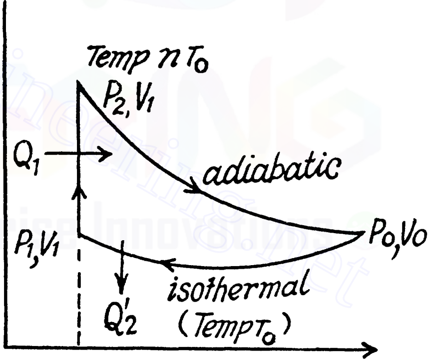
$$n p_1 V_1^\gamma = p_0 V_0^\gamma$$

$$Q'_2 = RT_0 \ln \frac{V_0}{V_1}, \; Q_1 = C_V T_0 (n-1)$$

But $n V_1^{\gamma - 1} = V_0^{\gamma - 1}$ or, $V_1 = V_0 n^{\frac{-1}{\gamma - 1}}$

$$Q'_2 = RT_0 \ln n^{\frac{1}{\gamma - 1}} = \frac{RT_0}{\gamma - 1} \ln n$$

Thus $\eta = 1 - \frac{\ln n}{n-1}$, on using $C_V = \frac{R}{\gamma - 1}$



(b) Here $V_2 = nV_1$, $p_1 V_1 = p_0 V_0$

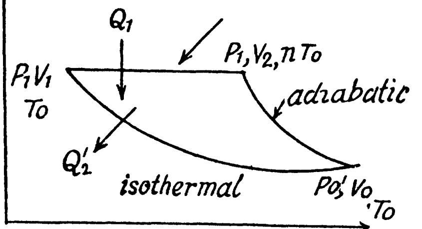
and $\quad p_1 (n V_1)^\gamma = p_0 V_0^\gamma$

i.e. $n^\gamma V_1^{\gamma - 1} = V_0^{\gamma - 1}$ or $V_1 = n^{-\frac{\gamma}{\gamma - 1}} V_0$

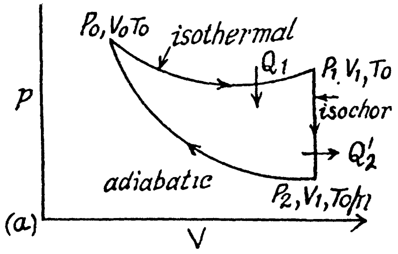
Also $Q_1 = C_p T_0 (n-1)$, $Q'_2 = RT_0 \ln \frac{V_0}{V_1}$

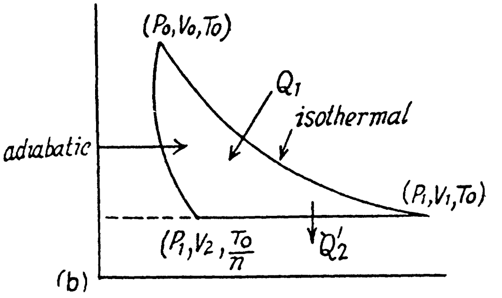
or $Q'_2 = RT_0 \ln n^{\frac{\gamma}{\gamma - 1}} = \frac{R\gamma}{\gamma - 1} T_0 \ln n = C_p T_0 \ln n$

Thus $$\eta = 1 - \frac{\ln n}{n-1}$$



2.121 Here the isothermal process proceeds at the maximum temperature instead of at the minimum temperature of the cycle as in 2.120.





(a) Here $p_1 V_1 = p_0 V_0$, $p_2 = \frac{p_1}{n}$

$$p_2 V_1^\gamma = p_0 V_0^\gamma \text{ or } p_1 V_1^\gamma = np_0 V_0^\gamma$$

i.e. $$V_1^{\gamma-1} = nV_0^{\gamma-1} \text{ or } V_1 = V_0 n^{\frac{1}{\gamma-1}}$$

$$Q'_2 = C_V T_0 \left(1 - \frac{1}{n}\right), \; Q_1 = RT_0 \ln \frac{V_1}{V_0} = \frac{RT_0}{\gamma - 1} \ln n = C_V T_0 \ln n.$$

Thus $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{n-1}{n \ln n}$$

(b) Here $V_2 = \frac{V_1}{n}$, $p_0 V_0 = p_1 V_1$

$$p_0 V_0^\gamma = p_1 V_2^\gamma = p_1 n^{-\gamma} V_1^\gamma \; = V_0^{\gamma-1} n^{-\gamma} V_1^{\gamma-1} \text{ or } V_1 = n^{(\gamma/\gamma-1)} V_0$$

$$Q'_2 = C_p T_0 \left(1 - \frac{1}{n}\right), \; Q_1 = RT_0 \ln \frac{V_1}{V_0} = \frac{R\gamma}{\gamma - 1} T_0 \ln n = C_p T_0 \ln n$$

Thus $$\eta = 1 - \frac{n-1}{n \ln n}$$

2.122 The section from (p_1, V_1, T_0) to $(p_2, V_2, T_0/n)$ is a polytropic process of index α. We shall assume that the corresponding specific heat C is + ve.

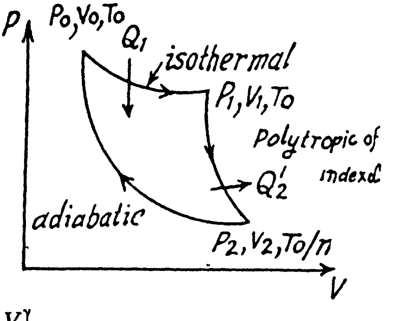
Here, $dQ = CdT = C_V dT + pdV$

Now pV^α = constant or $TV^{\alpha-1}$ = constant.

so $pdV = \frac{RT}{V} dV = -\frac{R}{\alpha - 1} dT$

Then $C = C_V - \frac{R}{\alpha - 1} = R\left(\frac{1}{\gamma - 1} - \frac{1}{\alpha - 1}\right)$

We have $p_1 V_1 = RT_0 = p_2 V_2 = \frac{RT_0}{n} = \frac{p_1 V_1}{n}$



$$p_0 V_0 = p_1 V_1 = n p_2 V_2, \; p_0 V_0^\gamma = p_2 V_2^\gamma,$$

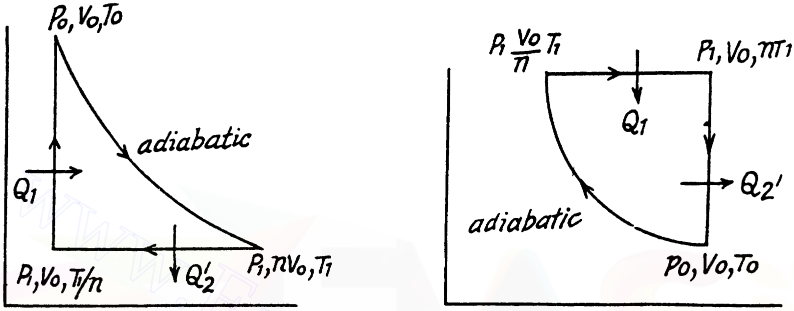
$$p_1 V_1^{\alpha} = p_2 V_2^{\alpha} \quad \text{or} \quad V_0^{\gamma-1} = \frac{1}{n} V_2^{\gamma-1} \quad \text{or} \quad V_2 = V_0\, n^{\frac{1}{\gamma-1}}$$

$$V_1^{\alpha-1} = \frac{1}{n} V_2^{\alpha-1} \quad \text{or} \quad V_1 = n^{-\frac{1}{\alpha-1}} V_2 = n^{\frac{1}{\gamma-1} - \frac{1}{\alpha-1}} V_0$$

Now $Q'_2 = C T_0 \left(1 - \frac{1}{n}\right)$, $Q_1 = R T_0 \ln \frac{V_1}{V_0} = R T_0 \left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1}\right) \ln n = C T_0 \ln n$

Thus
$$\eta = 1 - \frac{n-1}{n \ln n}$$

2.123



(a) Here $Q'_2 = C_p \left(T_1 - \frac{T_1}{n}\right) = C_p T_1 \left(1 - \frac{1}{n}\right)$, $Q_1 = C_V \left(T_0 - \frac{T_1}{n}\right)$

Along the adiabatic line

$$T_0 V_0^{\gamma-1} = T_1 (n V_0)^{\gamma-1} \quad \text{or,} \quad T_0 = T_1 n^{\gamma-1}.$$

so
$$Q_1 = C_V \frac{T_1}{n} (n^{\gamma} - 1). \quad \text{Thus} \quad \eta = 1 - \frac{\gamma (n-1)}{n^{\gamma-1}}$$

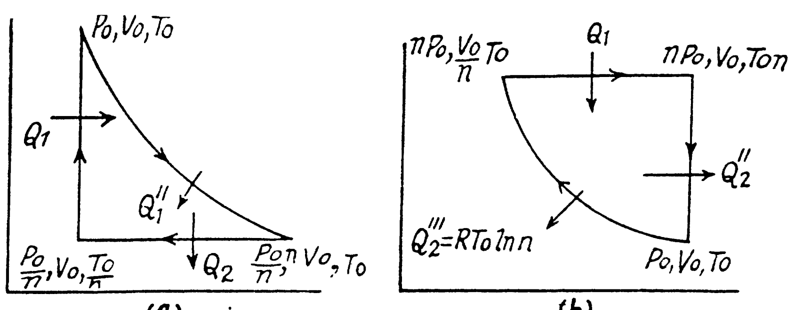
(b) Here $Q'_2 = C_V (n T_1 - T_0)$, $Q_1 = C_p \cdot T_1 (n-1)$

Along the adiabatic line $T V^{\gamma-1} =$ constant

$$T_0 V_0^{\gamma-1} = T_1 \left(\frac{V_0}{n}\right)^{\gamma-1} \quad \text{or} \quad T_1 = n^{\gamma-1} T_0$$

Thus
$$\eta = 1 - \frac{n^{\gamma} - 1}{\gamma n^{\gamma-1} (n-1)}$$

2.124



(a) $Q'_2 = C_p T_0 \left(1 - \frac{1}{n}\right)$, $Q''_1 = RT_0 \ln n$, $Q'_1 = C_V T_0 \left(1 - \frac{1}{n}\right)$, $Q_1 = Q'_1 + Q''_1$

So
$$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_p \left(1 - \frac{1}{n}\right)}{C_V \left(1 - \frac{1}{n}\right) + R \ln n}$$

$$= 1 - \frac{\gamma}{1 + \frac{R}{C_V} \frac{n \ln n}{n-1}} = 1 - \frac{\gamma (n-1)}{n - 1 + (\gamma - 1)\, n \ln n}$$

(b) $Q_1 = C_p T_0 (n-1)$, $Q''_2 = C_V T_0 (n-1)$ $\quad Q'''_2 = RT_0 \ln n$, $Q'_2 = Q''_2 + Q'''_2$

So
$$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{n - 1 + (\gamma - 1) \ln n}{\gamma (n-1)}$$

2.125 We have

$Q'_1 = \tau R T_0 \ln \nu$, $Q''_2 = C_V T_0 (\tau - 1)$ $Q_1 = Q'_1 + Q''_1$ and

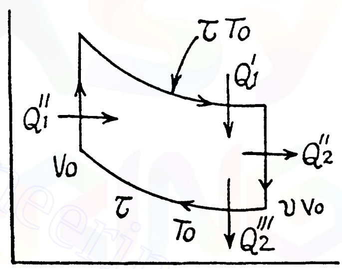
$Q'''_2 = RT_0 \ln \nu$, $Q''_1 = C_V T_0 (\tau - 1)$

as well as $Q_1 = Q_1' + Q_1''$ and

$Q_2' = Q_2'' + Q_2'''$

So $\eta = 1 - \frac{Q'_2}{Q_1} + 1 = \frac{C_V(\tau - 1) + R \ln \nu}{C_V (\tau - 1) + \tau R \ln \nu}$

$$= 1 - \frac{\frac{\tau - 1}{\gamma - 1} + \ln \nu}{\frac{\tau - 1}{\gamma - 1} + \tau \ln \nu} = \frac{(\tau - 1) \ln \nu}{\tau \ln \nu + \frac{\tau - 1}{\gamma - 1}}$$



2.126 Here $Q_1'' = C_p T_0 (\tau - 1)$, $Q'1'' = \tau R T_0 \ln n$ and

$Q_2'' = C^P T_0 (\tau - 1)$, $Q_2''' = RT_0 \ln n$

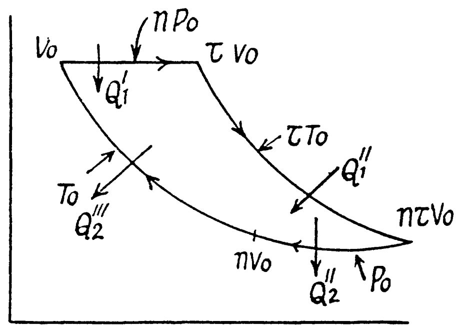
in addition to we have

$Q_1 = Q_1' + Q_1''$ and

$Q_2' = Q_2'' + Q_2'''$

So $\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_p (\tau - 1) + R \ln n}{C_p (\tau - 1) + \tau R \ln n}$

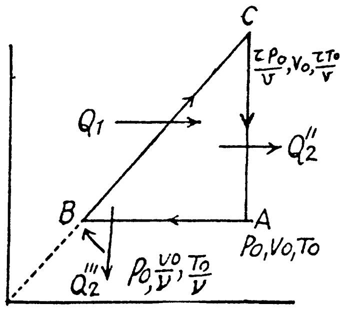
$$= 1 - \frac{\tau - 1 + \left(1 - \frac{1}{\gamma}\right) \ln n}{\tau - 1 + \left(1 - \frac{1}{\gamma}\right) \tau \ln n}$$



$$= 1 - \frac{\tau - 1 + \left(1 - \frac{1}{\gamma}\right) \ln n}{\tau - 1 + \left(1 - \frac{1}{\gamma}\right) \tau \ln n} = \frac{(\tau - 1) \ln n}{\tau \ln n + \frac{\gamma (\tau - 1)}{\gamma - 1}}$$

2.127 Because of the linearity of the section BC whose equation is

$$\frac{p}{p_0} = \frac{\nu V}{V_0} (\Rightarrow p = \alpha V)$$



We have $\frac{\tau}{\nu} = \nu$ or $\nu = \sqrt{\tau}$

Here $Q''_2 = C_V T_0 (\sqrt{\tau} - 1)$,

$$Q'''_2 = C_p T_0 \left(1 - \frac{1}{\sqrt{\tau}}\right) = C_p \frac{T_0}{\sqrt{\tau}} (\sqrt{\tau} - 1)$$

Thus $Q'_2 = Q''_2 + Q'''_2 = \frac{R T_0}{\gamma - 1} (\sqrt{\tau} - 1) \left(1 + \frac{\gamma}{\sqrt{\tau}}\right)$

Along BC, the specific heat C is given by

$$C dT = C_V dT + p dV = C_V dT + d\left(\frac{1}{2} \alpha V^2\right) = \left(C_V + \frac{1}{2} R\right) dT$$

Thus $$Q_1 = \frac{1}{2} R T_0 \frac{\gamma + 1}{\gamma - 1} \frac{\tau - 1}{\sqrt{\tau}}$$

Finally $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - 2 \frac{\sqrt{\tau} + \gamma}{\sqrt{\tau} + 1} \frac{1}{\gamma + 1} = \frac{(\gamma - 1)(\sqrt{\tau} - 1)}{(\gamma + 1)(\sqrt{\tau} + 1)}$$

2.128 We write Claussius inequality in the form

$$\int \frac{đ_1 Q}{T} - \int \frac{đ_2 Q}{T} \le 0$$

where $đQ$ is the heat transeferred to the system but $đ_2 Q$ is heat rejected by the system, both are $+ve$ and this explains the minus sign before $đ_2 Q$,

In this inequality $T_{max} > T > T_{min}$ and we can write

$$\int \frac{đ_1 Q}{T_{max}} - \int \frac{đ_2 Q}{T_{min}} < 0$$

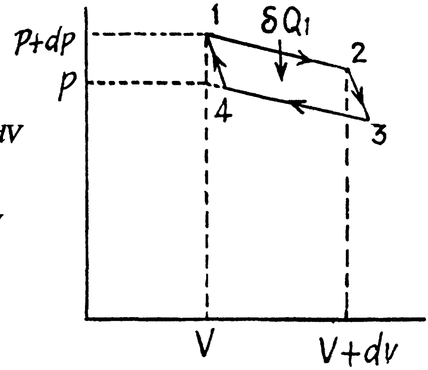
Thus $$\frac{Q_1}{T_{max}} < \frac{Q'_2}{T_{min}} \quad \text{or} \quad \frac{T_{min}}{T_{max}} < \frac{Q'_2}{Q_1}$$

or $$\eta = 1 - \frac{Q'_2}{Q_1} < 1 - \frac{T_{min}}{T_{max}} = \eta_{carnot}$$

2.129 We consider an infinitesimal carnot cycle with isothermal process at temperatures $T + dT$ and T.

Let δA be the work done in the cycle and δQ, be the heat received at the higher temperature. Then by Carnot's theorem

$$\frac{\delta A}{\delta Q_1} = \frac{dT}{T}$$



On the other hand $\delta A = dp\, dV = \left(\frac{\partial p}{\partial T}\right)_V dT\, dV$

while $\delta Q_1 = dU_{12} + p dV = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] dV$

Hence $\left(\frac{\partial U}{\partial V}\right)_T + p = T\left(\frac{\partial p}{\partial T}\right)_V$

2.130 (a) In an isochoric process the entropy change will be

$$\Delta S = \int_{T_i}^{T_f} \frac{C_V\, dT}{T} = C_V \ln \frac{T_f}{T_i} = C_V \ln n = \frac{R \ln n}{\gamma - 1}$$

For carbon dioxide $\gamma = 1{\cdot}30$

so, $\Delta S = 19{\cdot}2$ Joule/°K – mole

(b) For an isobaric process,

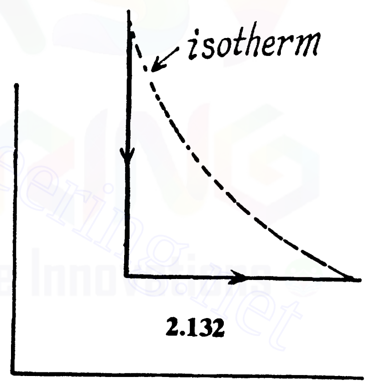
$$\Delta S = C_p \ln \frac{T_f}{T_i} = C_p \ln n = \frac{\gamma R \ln n}{\gamma - 1}$$

$= 25$ Joule/°K – mole

2.131 In an isothermal expansion

$$\Delta S = \nu R \ln \frac{V_f}{V_i}$$

so, $\frac{V_f}{V_i} = e^{\Delta S/\nu R} = 2{\cdot}0$ times



2.132

2.132 The entropy change depends on the final & initial states only, so we can calculate it directly along the isotherm, it is $\Delta S = 2R \ln n = 20$ J/°K (assuming that the final volume is n times the inital volume)

2.133 If the initial temperature is T_0 and volume is V_0 then in adiabatic expansion.

$$T V^{\gamma - 1} = T_0 V_0^{\gamma - 1}$$

so, $$T = T_0 n^{1-\gamma} = T_1 \text{ where } n = \frac{V_1}{V_0}$$

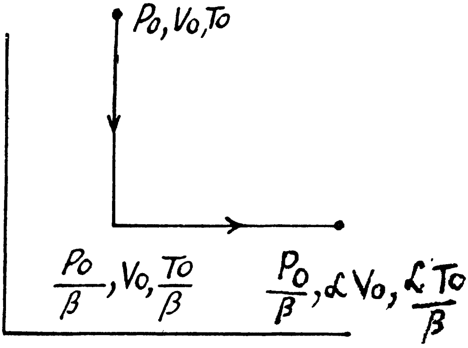
V_1 being the volume at the end of the adiabatic process. There is no entropy change in this process. Next the gas is compressed isobarically and the net entropy change is

$$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{T_f}{T_1}$$

But $$\frac{V_1}{T_1} = \frac{V_0}{T_f}, \quad \text{or} \quad T_f = T_1 \frac{V_0}{V_1} = T_0 n^{-\gamma}$$

So $$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{1}{n} = -\frac{m}{M} C_p \ln n = -\frac{m}{M} \frac{R\gamma}{\gamma - 1} \ln n = -9{\cdot}7 \text{ J/K}$$

2.134 The entropy change depends on the initial and final state only so can be calculated for any process whatsoever.



We choose to evaluate the entropy change along the pair of lines shown above. Then

$$\Delta S = \int_{T_0}^{\frac{T_0}{\beta}} \frac{\nu C_V \, dT}{T} + \int_{\frac{T_0}{\beta}}^{\frac{\alpha T_0}{\beta}} \nu C_p \frac{dT}{T}$$

$$= (-C_V \ln \beta + C_p \ln \alpha)\, \nu = \frac{\nu R}{\gamma - 1} (\gamma \ln \alpha - \ln \beta) \approx -11 \frac{\text{Joule}}{°\text{K}}$$

2.135 To calculate the required entropy difference we only have to calculate the entropy difference for a process in which the state of the gas in vessel 1 is changed to that in vessel 2.

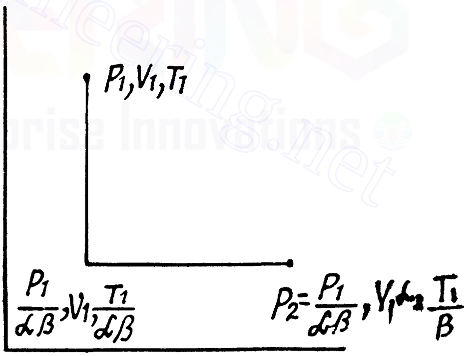
$$\Delta S = \nu \left(\int_{T_1}^{\frac{T_1}{\alpha\beta}} C_V \frac{dT}{T} + \int_{\frac{T_1}{\alpha\beta}}^{\frac{T_1}{\beta}} C_p \frac{dT}{T} \right)$$

$$= \nu \,(C_p \ln \alpha - C_V \ln \alpha\beta)$$

$$= \nu \left(R \ln \alpha - \frac{R}{\gamma - 1} \ln \beta\right) = \nu R \left(\ln \alpha - \frac{\ln \beta}{\gamma - 1}\right)$$

With $\gamma = \frac{5}{3}$, $\alpha = 2$ and $\beta = 1{\cdot}5$, $\nu = 1{\cdot}2$,

this gives $\Delta S = 0{\cdot}85$ Joule/°K



2.136 For the polytropic process with index n

$$p V^n = \text{constant}$$

Along this process (See 2.122)

$$C = R \left(\frac{1}{\gamma - 1} - \frac{1}{n - 1}\right) = \frac{n - \gamma}{(\gamma - 1)(n - 1)} \cdot R$$

So $$\Delta S = \int_{T_0}^{\tau T_0} C \frac{dT}{T} = \frac{n - \gamma}{(\gamma - 1)(n - 1)} R \ln \tau$$

2.137 The process in question may be written as

$$\frac{p}{p_0} = \alpha \frac{V}{V_0}$$

where α is a constant and p_0, V_0 are some reference values. For this process (see 2.127) the specific heat is

$$C = C_V + \frac{1}{2}R = R\left(\frac{1}{\gamma - 1} + \frac{1}{2}\right) = \frac{1}{2}R\frac{\gamma + 1}{\gamma - 1}$$

Along the line volume increases α times then so does the pressure. The temperature must then increase α^2times. Thus

$$\Delta S = \int_{T_0}^{\alpha^2 T_0} \nu C \frac{dT}{T} = \frac{\nu R}{2}\frac{\gamma + 1}{\gamma - 1}\ln \alpha^2 = \nu R \frac{\gamma + 1}{\gamma - 1}\ln \alpha$$

if $\quad \nu = 2,\ \gamma = \frac{5}{3},\ \alpha = 2, \Delta S = 46{\cdot}1$ Joule/°K

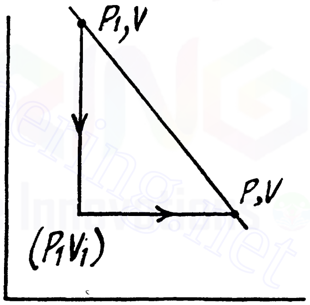
2.138 Let (p_1, V_1) be a reference point on the line

$$p = p_0 - \alpha V$$

and let (p, V) be any other point.

The entropy difference

$$\Delta S = S(p, V) - S(p_1, V_1)$$



$$= C_V \ln \frac{p}{p_1} + C_p \ln \frac{V}{V_1} = C_V \ln \frac{p_0 - \alpha V}{p_1} + C_p \ln \frac{V}{V_1}$$

For an exetremum of ΔS

$$\frac{\partial \Delta S}{\partial V} = \frac{-\alpha C_V}{p_0 - \alpha V} + \frac{C_p}{V} = 0$$

or $\quad C_p(p_0 - \alpha V) - \alpha V C_V = 0$

or $\quad \gamma(p_0 - \alpha V) - \alpha V = 0 \quad$ or $\quad V = V_m = \frac{\gamma p_0}{\alpha(\gamma + 1)}$

This gives a maximum of ΔS because $\frac{\partial^2 \Delta S}{\partial V^2} < 0$

(**Note :-** a maximum of ΔS is a maximum of $S(p, V)$)

2.139 Along the process line : $S = aT + C_V \ln T$

or the specific heat is : $C = T\frac{dS}{dT} = aT + C_V$

On the other hand : $dQ = CdT = C_V dT + pdV$ for an ideal gas.

Thus, $\quad pdV = \frac{RT}{V}dV = aT\,dT$

or $$\frac{R}{a}\frac{dV}{V} = dT \quad \text{or,} \quad \frac{R}{a}\ln V + \text{constant} = T$$

Using $T = T_0$ when $V = V_0$, we get, $T = T_0 + \frac{R}{a}\ln\frac{V}{V_0}$

2.140 For a Vander Waal gas

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

The entropy change along an isotherm can be calculated from

$$\Delta S = \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_T dV$$

It follows from (2.129) that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V - b}$$

assuming a, b to be known constants.

Thus $$\Delta S = R\ln\frac{V_2 - b}{V_1 - b}$$

2.141 We use, $$\Delta S = \int_{V_1, T_1}^{V_2, T_2} dS\,(V, T) = \int_{T_1}^{T_2} \left(\frac{\partial S}{\partial T}\right)_{V_1} dT + \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_{T = T_2} dV$$

$$= \int_{T_1}^{T_2} \frac{C_V\,dT}{T} + \int_{V_1}^{V_2} \frac{R}{V - b}\,dV = C_V \ln\frac{T_2}{T_1} + R\ln\frac{V_2 - b}{V_1 - b}$$

assuming C_V, a, b to be known constants.

2.142 We can take $S \to 0$ as $T \to 0$ Then

$$S = \int_0^T C\frac{dT}{T} = \int_0^T aT^2\,dT = \frac{1}{3}aT^3$$

2.143 $$\Delta S = \int_{T_1}^{T_2} \frac{C\,dT}{T} = \int_{T_1}^{T_2} \frac{m(a + bT)}{T}\,dT = mb(T_2 - T_1) + ma\ln\frac{T_2}{T_1}$$

2.144 Here $T = a\,S^n$ or $S = \left(\frac{T}{a}\right)^{\frac{1}{n}}$

Then
$$C = T \frac{1}{n} \frac{T^{\frac{1}{n}-1}}{a^{1/n}} = \frac{S}{n}$$

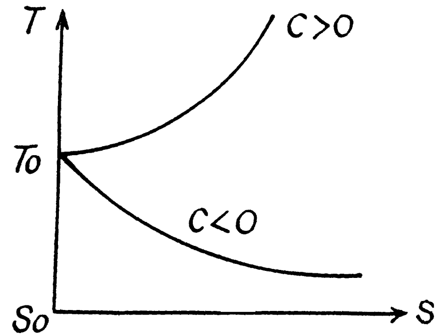
Clearly
$$C < 0 \quad \text{if } n < 0.$$

2.145 We know,

$$S - S_0 = \int_{T_0}^{T} \frac{CdT}{T} = C \ln \frac{T}{T_0}$$

assuming C to be a known constant.

Then $\quad T = T_0 \exp \left(\frac{S - S_0}{C}\right)$



2.146 (a) $C = T \frac{dS}{dT} = -\frac{\alpha}{T}$

(b) $Q = \int_{T_1}^{T_2} CdT = \alpha \ln \frac{T_1}{T_2}$

(c) $W = \Delta Q - \Delta U \qquad = \alpha \ln \frac{T_1}{T_2} + C_V (T_1 - T_2)$

Since for an ideal gas C_V is constant and $\Delta U = C_V (T_2 - T_1)$

(U does not depend on V)

2.147 (a) We have from the definition

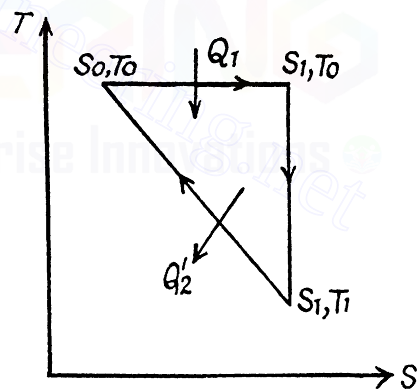
$$Q = \int T dS = \text{ area under the curve}$$

$$Q_1 = T_0 (S_1 - S_0)$$

$$Q'_2 = \frac{1}{2} (T_0 + T_1)(S_1 - S_0)$$

Thus, using $T_1 = \frac{T_0}{n}$,

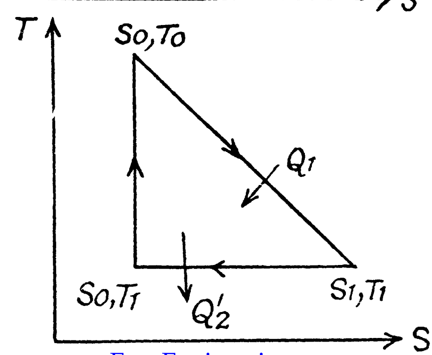
$$\eta = 1 - \frac{T_0 + T_1}{2\,T_0} = 1 - \frac{1 + \frac{1}{n}}{2} = \frac{n-1}{2n}$$



(b) Here $Q_1 = \frac{1}{2} (S_1 - S_0)(T_1 + T_0)$

$$Q'_2 = T_1 (S_1 - S_0)$$

$$\eta = 1 - \frac{2T_1}{T_1 + T_0} = \frac{T_0 - T_1}{T_0 - T_1} = \frac{n-1}{n+1}$$



2.148 In this case, called free expansion no work is done and no heat is exchanged. So internal energy must remain unchanged $U_f = U_i$. For an ideal gas this implies constant temperature $T_f = T_i$. The process is irreversible but the entropy change can be calculated by considering a reversible isothermal process. Then, as before

$$\Delta S = \int \frac{dQ}{T} = \int_{V_1}^{V_2} \frac{pdV}{T} = \nu R \ln n = 20{\cdot}1 \text{ J/K}$$

2.149 The process consists of two parts. The first part is free expansion in which $U_f = U_i$. The second part is adiabatic compression in which work done results in change of internal energy. Obviously,

$$0 = U_F - U_f + \int_{V_f}^{V_0} pdV, \; V_f = 2V_0$$

Now in the first part $p_f = \frac{1}{2}p_0$, $V_f = 2V_0$, because there is no change of temperature.

In the second part, $p\,V^\gamma = \frac{1}{2}p_0\,(2V_0)^\gamma = 2^{\gamma-1}p_0\,V_0^\gamma$

$$\int_{2V_0}^{V_0} pdV = \int_{2V_0}^{V_0} \frac{2^{\gamma-1}p_0\,V_0^\gamma}{V^\gamma}dV = \left[\frac{2^{\gamma-1}p_0\,V_0^\gamma}{-\gamma+1}V^{1-\gamma}\right]_{2V_0}^{V_0}$$

$$= 2^{\gamma-1}p_0\,V_0^\gamma\,V_0^{-\gamma+1}\,\frac{2^{-\gamma+1}-1}{\gamma-1} = -\frac{(2^{\gamma-1}-1}{\gamma-1}RT$$

Thus $$\Delta U = U_F - U_i = \frac{R\,T_0}{\gamma-1}(2^{\gamma-1}-1)$$

The entropy change $\Delta S = \Delta S_I + \Delta S_{II}$

$\Delta S_I = R\ln 2$ and $\Delta S_{II} = 0$ as the process is reversible adiabatic. Thus $\Delta S = R\ln 2$.

2.150 In all adiabatic processes

$$Q = U_f - U_i + A = 0$$

by virtue of the first law of thermodynamics. Thus,

$$U_f = U_i - A$$

For a slow process, $A' = \int_{V_0}^{V} pdV$ where for a quasistatic adiabatic process $pV^\gamma =$ constant.

On the other hand for a fast process the external work done is $A'' < A'$. In fact $A'' = 0$ for free expansion. Thus U'_f (slow) $< U''_f$ (fast)

Since U depends on temperature only, $T_f < T'_f$

Consequently, $p''_f > p'_f$

(From the ideal gas equation $pV = RT$)

2.151 Let $V_1 = V_0$, $V_2 = n\,V_0$

Since the temperature is the same, the required entropy change can be calculated by considering isothermal expansion of the gas in either parts into the whole vessel.

Thus
$$\Delta S = \Delta S_I + \Delta S_{II} = \nu_1 R \ln \frac{V_1 + V_2}{V_1} + \nu_2 R \ln \frac{V_1 + V_2}{V_2}$$

$$= \nu_1 R \ln (1+n) + \nu_2 R \ln \frac{1+n}{n} = 5{\cdot}1 \text{ J/K}$$

2.152 Let c_1 = specific heat of copper specific heat of water = c_2

Then
$$\Delta S = \int_{7+273}^{T_0} \frac{c_2 m_2\, dT}{T} - \int_{T_0}^{97+273} \frac{m_1 c_1\, dT}{T} = m_2 c_2 \ln \frac{T_0}{280} - m_1 c_1 \ln \frac{370}{T_0}$$

T_0 is found from

$$c_2 m_2 (T_0 - 280) = m_1 c_1 (370 - T_0) \quad \text{or} \quad T_0 = \frac{280\, m_2 c_2 + 370\, m_1 c_1}{c_2 m_2 + m_1 c_1}$$

using
$$c_1 = 0{\cdot}39 \text{ J/g °K}, \; c_2 = 4{\cdot}18 \text{ J/g °K},$$

$$T_0 = 300\text{°K} \text{ and } \Delta S = 28{\cdot}4 - 24{\cdot}5 = 3{\cdot}9 \text{ J/°K}$$

2.153 For an ideal gas the internal energy depends on temperature only. We can consider the process in question to be one of simultaneous free expansion. Then the total energy $U = U_1 + U_2$. Since

$U_1 = C_V T_1$, $U_2 = C_V T_2$, $U = 2C_V \frac{T_1 + T_2}{2}$ and $(T_1 + T_2)/2$ is the final temperature. The entropy change is obtained by considering isochoric processes because in effect, the gas remains confined to its vessel.

$$\Delta S = \int_{T_1}^{(T_1+T_2)/2} \frac{C_V\, dT}{T} - \int_{(T_1+T_2)/2}^{T_2} C_V \frac{dT}{T} = C_V \ln \frac{(T_1 + T_2)^2}{4\, T_1 T_2}$$

Since
$$(T_1 + T_2)^2 = (T_1 - T_2)^2 + 4\, T_1 T_2, \; \Delta S > 0$$

2.154 (a) Each atom has a probability $\frac{1}{2}$ to be in either campartment. Thus

$$p = 2^{-N}$$

(b) Typical atomic velocity at room temperature is $\sim 10^5$ cm/s so it takes an atom 10^{-5} sec to cross the vessel. This is the relevant time scale for our problem. Let $T = 10^{-5}$ sec, then in time t there will be t/T crossing or arrangements of the atoms. This will be large enough to produce the given arrangement if

$$\frac{t}{\tau} 2^{-N} \sim 1 \quad \text{or} \quad N \sim \frac{\ln t/\tau}{\ln 2} \sim 75$$

2.155 The statistical weight is

$$N_{C_{N/2}} = \frac{N!}{N/2\,!\,\frac{N}{2}!} = \frac{10\times 9\times 8\times 7\times 6}{8\times 4\times 3\times 2} = 252$$

The probability distribution is

$$N_{C_{N/2}}\, 2^{-N} = 252\times 2^{-10} = 24{\cdot}6\,\%$$

2.156 The probabilites that the half A contains n molecules is

$$N_{C_n}\times 2^{-N} = \frac{N\,!}{n\,!\,(N-n)\,!}\,2^{-N}$$

2.157 The probability of one molecule being confined to the marked volume is

$$p = \frac{V}{V_0}$$

We can choose this molecule in many (N_{C_1}) ways. The probability that n molecules get confined to the marked volume is cearly

$$N_{C_n}\,p^n\,(1-p)^{N-n} = \frac{N\,!}{n\,!\,(N-n)\,!}\,p^n\,(1-p)^{N-n}$$

2.158 In a sphere of diameter d there are

$$N = \frac{\pi\, d^3}{6}\, n_0 \quad \text{molecules}$$

where n_0 = Loschmidt's number = No. of molecules per unit volume (1 cc) under NTP. The relative fluctuation in this number is

$$\frac{\partial N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \eta$$

or $\dfrac{1}{\eta^2} = \dfrac{\pi}{6}\, d^3\, n_0$ or $d^3 = \dfrac{6}{\pi\, n_0\, \eta^2}$ or $d = \left(\dfrac{6}{\pi\, \eta^2\, n_0}\right)^{1/3} = 0{\cdot}41\ \mu\text{m}$

The average number of molecules in this sphere is $\dfrac{1}{\eta^2} = 10^6$

2.159 For a monoatomic gas $C_V = \dfrac{3}{2}R$ per mole

The entropy change in the process is

$$\Delta S = S - S_0 = \int_{T_0}^{T_0+\Delta T} C_V\,\frac{dT}{T} = \frac{3}{2}R\ln\left(1+\frac{\Delta T}{T_0}\right)$$

Now from the Boltzmann equation

$$S = k\ln\Omega$$

$$\frac{\Omega}{\Omega_0} = e^{(S-S_0)/k} = \left(1+\frac{\Delta T}{T_0}\right)^{\frac{3N_A}{2}} = \left(1+\frac{1}{300}\right)^{\frac{3\times 6}{2}\times 10^{23}} = 10^{13}\times 10^{21}$$

Thus the statistical weight increases by this factor.

2.5 LIQUIDS. CAPILLARY EFFECTS

2.160 (a) $\Delta p = \alpha\left(\frac{1}{d/2}+\frac{1}{d/2}\right) = \frac{4\alpha}{d}$

$$= \frac{4\times 490\times 10^{-3}}{1{\cdot}5\times 10^{-6}}\frac{\text{N}}{\text{m}^2} = 1{\cdot}307\times 10^{6}\frac{\text{N}}{\text{m}^2} = 13 \text{ atmosphere}$$

(b) The soap bubble has two surfaces

so
$$\Delta p = 2\,\alpha\left(\frac{1}{d/2}+\frac{1}{d/2}\right) = \frac{8\alpha}{d}$$

$$= \frac{8\times 45}{3\times 10^{-3}}\times 10^{-3} = 1{\cdot}2\times 10^{-3} \text{ atomsphere.}$$

2.161 The pressure just inside the hole will be less than the outside pressure by $4\,\alpha/d$. This can support a height h of Hg where

$$\rho\, g\, h = \frac{4\alpha}{d} \quad \text{or} \quad h = \frac{4\alpha}{\rho\, g\, d}$$

$$= \frac{4\times 490\times 10^{-3}}{13{\cdot}6\times 10^{3}\times 9{\cdot}8\times 70\times 10^{-6}} = \frac{200}{13{\cdot}6\times 70} \approx {\cdot}21 \text{ m of Hg}$$

2.162 By Boyle's law

$$\left(p_0+\frac{8\alpha}{d}\right)\frac{4\,\pi}{3}\left(\frac{d}{2}\right)^3 = \left(\frac{p_0}{n}+\frac{8\alpha}{\eta d}\right)\frac{4\pi}{3}\left(\frac{\eta\, d}{2}\right)^3$$

or
$$p_0\left(1-\frac{\eta^3}{n}\right) = \frac{8\alpha}{d}(\eta^2-1)$$

Thus
$$\alpha = \frac{1}{8}p_0\, d\left(1-\frac{\eta^3}{n}\right)(\eta^2-1)$$

2.163 The pressure has terms due to hydrostatic pressure and capillarity and they add

$$p = p_0+\rho g h+\frac{4\alpha}{d}$$

$$= \left(1+\frac{5\times 9{\cdot}8\times 10^3}{10^5}+\frac{4\times {\cdot}73\times 10^{-3}}{4\times 10^{-6}}\times 10^{-5}\right)\text{atoms} = 2{\cdot}22 \text{ atom.}$$

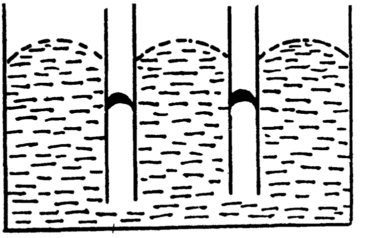
2.164 By Boyle's law

$$\left(p_0+h\,g\,\rho+\frac{4\alpha}{d}\right)\frac{\pi}{6}d^3 = \left(p_0+\frac{4\alpha}{nd}\right)\frac{\pi}{6}n^3\,d^3$$

or
$$\left[hg\rho - p_0\,(n^3-1)\right] = \frac{4\alpha}{d}(n^2-1)$$

or
$$h = \left[p_0\,(n^3-1)+\frac{4\alpha}{d}(n^2-1)\right]\Big/ g\,\rho = 4{\cdot}98 \text{ meter of water}$$

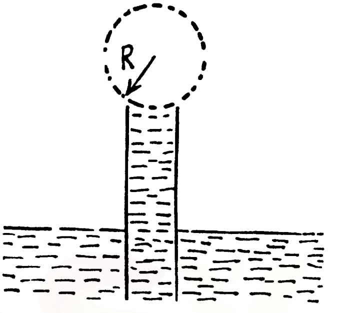
2.165 Clearly



$$\Delta h\,\rho\,g = 4\,\alpha\,|\cos\theta|\left(\frac{1}{d_1}-\frac{1}{d_2}\right)$$

$$\Delta h = \frac{4\,\alpha\,|\cos\theta|\,(d_2-d_1)}{d_1\,d_2\,\rho g} = 11 \text{ mm}$$

2.166 In a capillary with diameter $d = 0{\cdot}5$ mm water will rise to a height



$$\frac{2\alpha}{\rho\,g\,r} = \frac{4\alpha}{\rho\,g\,d}$$

$$= \frac{4\times73\times10^{-3}}{10^3\times9{\cdot}8\times0{\cdot}5\times10^{-3}} = 59{\cdot}6 \text{ mm}$$

Since this is greater than the height (= 25 mm) of the tube, a meniscus of radius R will be formed at the top of the tube, where

$$R = \frac{2\alpha}{\rho\,gh} = \frac{2\times73\times10^{-3}}{10^3\times9{\cdot}8\times25\times10^{-3}} \approx 0{\cdot}6 \text{ mm}$$

2.167 Initially the pressure of air in the cppillary is p_0 and it's length is l. When submerged under water, the pressure of air in the portion above water must be $p_0 + 4\frac{\alpha}{d}$, since the level of water inside the capillary is the same as the level outside. Thus by Boyle's law

$$\left(p_0+\frac{4\alpha}{d}\right)(l-x) = p_0\,l$$

or
$$\frac{4\alpha}{d}(l-x) = p_0\,x \quad \text{or} \quad x = \frac{l}{1+\frac{p_0\,d}{4\alpha}}$$

2.168 We have by Boyle's law

$$\left(p_0-\rho gh+\frac{4\,\alpha\cos\theta}{d}\right)(l-h) = p_0\,l$$

or,
$$\frac{4\,\alpha\cos\theta}{d} = \rho\,gh+\frac{p_0\,h}{l-h}$$

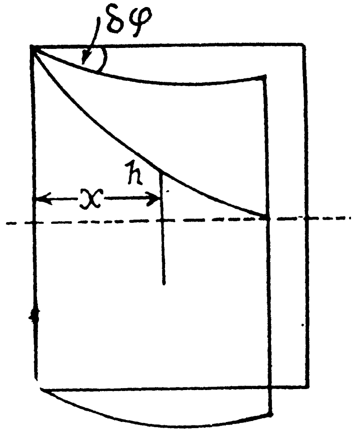
Hence,
$$\alpha = \left(\rho\,gh+\frac{p_0\,h}{l-h}\right)\frac{d}{4\cos\theta}$$

2.169 Suppose the liquid rises to a height h. Then the total energy of the liquid in the capillary is

$$E(h) = \frac{\pi}{4}(d_2^2-d_1^2)\,h\times\rho g\times\frac{h}{2}-\pi\,(d_2-d_1)\,\alpha h$$

Minimising E we get

$$h = \frac{4\alpha}{\rho g\,(d_2-d_1)} = 6 \text{ cm.}$$



2.170 Let h be the height of the water level at a distance x from the edge. Then the total energy of water in the wedge above the level outside is.

$$E = \int x\,\delta\varphi \cdot dx \cdot h \cdot \rho\, g\frac{h}{2} - 2\int dx \cdot h \cdot \alpha \cos\theta$$

$$= \int dx \frac{1}{2} x\,\rho\, g\,\delta\varphi\left(h^2 - 2\frac{2\,\alpha\cos\theta}{x\,\rho\, g\,\delta\varphi}h\right)$$

$$= \int dx \frac{1}{2} x\,\rho g\,\delta\varphi\left[\left(h - \frac{2\,\alpha\cos\theta}{x\,\rho\, g\,\delta\varphi}\right)^2 - \frac{4\,\alpha^2\cos^2\theta}{x^2\rho^2 g^2\,\delta\varphi^2}\right]$$

This is minimum when $h = \dfrac{2\,\alpha\cos\theta}{x\,\rho\, g\,\delta\varphi}$

2.171 From the equation of continuity

$$\frac{\pi}{4}d^2 \cdot v = \frac{\pi}{4}\left(\frac{d}{n}\right)^2 \cdot V \quad \text{or} \quad V = n^2 v.$$

We then apply Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2}v^2 + \Phi = \text{constant}$$

The pressure p differs from the atmospheric pressure by capillary effects. At the upper section

$$p = p_0 + \frac{2\alpha}{d}$$

neglecting the curvature in the vertical plane. Thus,

$$\frac{p_0 + \frac{2\alpha}{d}}{\rho} + \frac{1}{2}v^2 + gl = \frac{p_0 + \frac{2\,n\,\alpha}{d}}{\rho} + \frac{1}{2}n^4 v^2$$

or
$$v = \sqrt{\frac{2\,gl - \frac{4\,\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

Finally, the liquid coming out per second is,

$$V = \frac{1}{4}\pi d^2\sqrt{\frac{2\,gl - \frac{4\alpha}{\rho d}(n-1)}{n^4 - 1}}$$

2.172 The radius of curvature of the drop is R_1 at the upper end of the drop and R_2 at the lower end. Then the pressure inside the drop is $p_0 + \dfrac{2\alpha}{R_1}$ at the top end and $p_0 + \dfrac{2\alpha}{R_2}$ at the bottom end. Hence

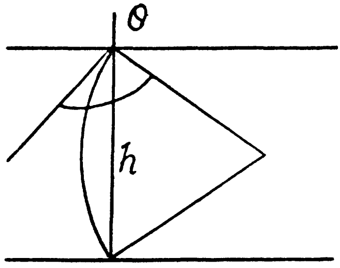
$$p_0 + \frac{2\alpha}{R_1} = p_0 + \frac{2\alpha}{R_2} + \rho\, gh \quad \text{or} \quad \frac{2\,\alpha\,(R_2 - R_1)}{R_1 R_2} = \rho\, gh$$

To a first approximation $R_1 \approx R_2 \approx \dfrac{h}{2}$ so $R_2 - R_1 \approx \dfrac{1}{8}\rho\, gh^3/\alpha. \approx 0{\cdot}20$ mm

if $\qquad h = 2{\cdot}3$ mm, $\alpha = 73$ mN/m

2.173 We must first calculate the pressure difference inside the film from that outside. This is

$$p = \alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$



Here $2\,r_1\,|\cos\theta| = h$ and $r_2 \sim -R$ the radius of the tablet and can be neglected. Thus the total force exerted by mercury drop on the upper glass plate is

$$\frac{2\pi R^2\alpha\,|\cos\theta|}{h} \text{ typically}$$

We should put h/n for h because the tablet is compressed n times. Then since Hg is nearly, incompressible, $\pi R^2 h$ = constants so $R \to R\sqrt{n}$. Thus,

$$\text{total force} = \frac{2\pi R^2\alpha\,|\cos\theta|}{h}\,n^2$$

Part of the force is needed to keep the Hg in the shape of a table rather than in the shape of infinitely thin sheet. This part can be calculated being putting $n = 1$ above. Thus

$$mg + \frac{2\pi R^2\alpha\,|\cos\theta|}{h} = \frac{2\pi R^2\alpha\,|\cos\theta|}{h}\,n^2$$

or
$$m = \frac{2\pi R^2\alpha\,|\cos\theta|}{hg}(n^2 - 1) = 0{\cdot}7\text{ kg}$$

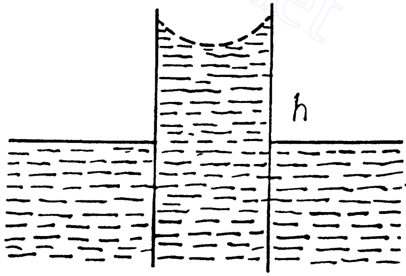
2.174 The pressure inside the film is less than that outside by an amount $\alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ where r_1 and r_2 are the principal radii of curvature of the meniscus. One of these is small being given by $h = 2\,r_1\cos\theta$ while the other is large and will be ignored. Then $F \approx \frac{2A\cos\theta}{h}\alpha$ where A = area of the water film between the plates.

Now $A = \frac{m}{\rho h}$ so $F = \frac{2m\alpha}{\rho h^2}$ when θ (the angle of contact) = 0

2.175 This is analogous to the previous problem except that : $A = \pi R^2$

So
$$F = \frac{2\pi R^2\alpha}{h} = 0{\cdot}6\text{ kN}$$

2.176 The energy of the liquid between the plates is



$$E = l\,d\,h\,\rho\,g\,\frac{h}{2} - 2\,\alpha\,l\,h = \frac{1}{2}\rho\,g\,l\,d\,h^2 - 2\,\alpha\,l\,h$$

$$= \frac{1}{2}\rho\,g\,l\,d\left(h - \frac{2\alpha}{\rho g d}\right)^2 - \frac{2\alpha^2 l}{\rho g d}$$

This energy is minimum when, $h = \frac{2\alpha}{\rho g d}$ and

the minimum potential energy is then $E_{min} = -\frac{2\alpha^2 l}{\rho g d}$

The force of attraction between the plates can be obtained from this as

$$F = \frac{-\partial E_{min}}{\partial d} = -\frac{2\alpha^2 l}{\rho g d^2} \text{ (minus sign means the force is attractive.)}$$

Thus
$$F = -\frac{\alpha l h}{d} = 13\text{ N}$$

2.177 Suppose the radius of the bubble is x at some instant. Then the pressure inside is $p_0 + \frac{4\alpha}{x}$. The flow through the capillary is by Poiseuille's equation,

$$Q = \frac{\pi r^4}{8\eta l}\frac{4\alpha}{x} = -4\pi^2\frac{dx}{dt}$$

Integrating $\frac{\pi r^4 \alpha}{2\eta l}t = \pi(R^4 - x^4)$ where we have used the fact that $t = 0$ where $x = R$.

This gives $t = \frac{2\eta l R^4}{\alpha r^4}$ as the life time of the bubble corresponding to $x = 0$

2.178 If the liquid rises to a height h, the energy of the liquid column becomes

$$E = \rho g \pi r^2 h \cdot \frac{h}{2} - 2\pi r h \alpha = \frac{1}{2}\rho g \pi \left(rh - 2\frac{\alpha}{\rho g}\right)^2 - \frac{2\pi\alpha^2}{\rho g}$$

This is minimum when $rh = \frac{2\alpha}{\rho g}$ and that is relevant height to which water must rise.

At this point, $$E_{min} = -\frac{2\pi\alpha^2}{\rho g}$$

Since $E = 0$ in the absence of surface tension a heat $Q = \frac{2\pi\alpha^2}{\rho g}$ must have been liberated.

2.179 (a) The free energy per unit area being α,

$$F = \pi\alpha d^2 = 3\ \mu J$$

(b) $F = 2\pi\alpha d^2$ because the soap bubble has two surfaces. Substitution gives $F = 10\ \mu J$

2.180 When two mercury drops each of diameter d merge, the resulting drop has diameter d_1

where $$\frac{\pi}{6}d_1^3 = \frac{\pi}{6}d^3 \times 2 \quad \text{or,} \quad d_1 = 2^{1/3}d$$

The increase in free energy is

$$\Delta F = \pi 2^{2/3} d^2\alpha - 2\pi d^2\alpha = 2\pi d^2\alpha(2^{-1/3} - 1) = -1{\cdot}43\ \mu J$$

2.181 Work must be done to stretch the soap film and compress the air inside. The former is simply $2\alpha \times 4\pi R^2 = 8\pi R^2\alpha$, there being two sides of the film. To get the latter we note that the compression is isothermal and work done is

$$-\int_{V_i = V_0}^{V_f = V} p\,dV \quad \text{where} \quad V_0 p_0 = \left(p_0 + \frac{4\alpha}{R}\right)\cdot V,\ V = \frac{4\pi}{3}R^3$$

or $$V_0 = \frac{pV}{p_0},\ p = p_0 + \frac{4\alpha}{R}$$

and minus sign is needed becaue we are calculating work done on the system. Thus since pV remains constants, the work done is

$$pV\ln\frac{V_0}{V} = pV\ln\frac{p}{p_0}$$

So $$A' = 8\pi R^2\alpha + pV\ln\frac{p}{p_0}$$

2.182 When heat is given to a soap bubble the temperature of the air inside rises and the bubble expands but unless the bubble bursts, the amount of air inside does not change. Further we shall neglect the variation of the surface tension with temperature. Then from the gas equations

$$\left(p_0 + \frac{4\alpha}{r}\right)\frac{4\pi}{3}r^3 = \nu R T, \quad \nu = \text{Constant}$$

Differentiating

$$\left(p_0 + \frac{8\alpha}{3r}\right) 4\pi r^2\, dr = \nu R\, dT$$

or
$$dV = 4\pi r^2\, dr = \frac{\nu R dT}{p_0 + \frac{8\alpha}{3r}}$$

Now from the first law

$$\text{đ}\, Q = \nu C dT = \nu C_V dT + \frac{\nu R\, dT}{p_0 + \frac{8\alpha}{3r}} \cdot \left(p_0 + \frac{4\alpha}{r}\right)$$

or
$$C = C_V + R\,\frac{p_0 + \frac{4\alpha}{r}}{p_0\, \frac{8\alpha}{3r}}$$

using
$$C_p = C_V + R\,, \quad C = C_p + \frac{\frac{1}{2}R}{1 + \frac{3\,p_0\, r}{8\alpha}}$$

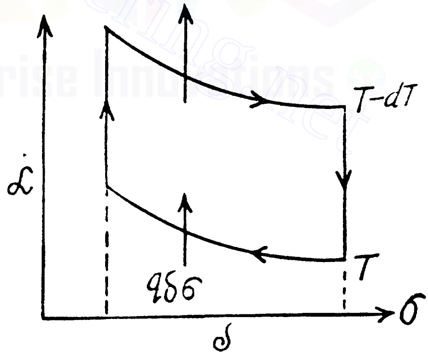
2.183 Consider an infinitesimal Carnot cycle with isotherms at $T - dT$ and T. Let A be the work done during the cycle. Then

$$A = [\alpha\,(T - dT) - \alpha\,(T)\,]\,\delta\,\sigma = -\frac{d\alpha}{dT}\, dT\,\delta\sigma$$

Where $\delta\sigma$ is the change in the area of film (we are considering only one surface).

Then $\eta = \frac{A}{Q_1} = \frac{dT}{T}$ by Carnot therom.

or
$$\frac{-\frac{d\alpha}{dT}\, dT\,\delta\,\sigma}{q\,\delta\,\sigma} = \frac{dT}{T} \quad \text{or} \quad q = -T\,\frac{d\alpha}{dT}$$



2.184 As before we can calculate the heat required. It, is taking into account two sides of the soap film

$$\delta q = -T\,\frac{d\alpha}{dT}\,\delta\,\sigma \times 2$$

Thus
$$\Delta S = \frac{\delta q}{T} = -2\,\frac{d\alpha}{dT}\,\delta\,\sigma$$

Now $\quad \Delta F = 2\,\alpha\,\delta\,\sigma$ so, $\Delta U = \Delta F + T\,\Delta S = 2\left(\alpha - T\,\frac{d\alpha}{dT}\right)\delta\sigma$

2.6 PHASE TRANSFORMATIONS

2.185 The condensation takes place at constant pressure and temperature and the work done is

$$p\,\Delta V$$

where ΔV is the volume of the condensed vapour in the vapour phase. It is

$$p\,\Delta V = \frac{\Delta m}{M}RT = 120{\cdot}6\text{ J}$$

where $M = 18$ gm is the molecular weight of water.

2.186 The specific volume of water (the liquid) will be written as V'_l. Since $V'_v >> V'_l$, most of the weight is due to water. Thus if m_l is mass of the liquid and m_v that of the vapour then

$$m = m_l + m_v$$

$$V = m_l V'_l + m_v V'_v \quad \text{or} \quad V - mV'_l = m_v (V'_v - V'_l)$$

So $m_v = \dfrac{V - mV'_l}{V'_v - V'_1} = 20$ gm in the present case. Its volume is $m_v V'_v = 1{\cdot}01$

2.187 The volume of the condensed vapour was originally $V_0 - V$ at temperature $T = 373$ K. Its mass will be given by

$p(V_0 - V) = \dfrac{m}{M}RT$ or $m = \dfrac{Mp(V_0 - V)}{RT} = 2$ gm where $p =$ atmospheric pressure

2.188 We let $V'_l =$ specific volume of liquid. $V'_v = NV'_l =$ specific volume of vapour.

Let $V =$ Original volume of the vapour. Then

$$M\frac{pV}{RT} = m_l + m_v = \frac{V}{NV'_l} \quad \text{or} \quad \frac{V}{n} = (m_l + Nm_v)V'_l$$

So $\quad (N-1)m_l V'_l = V\left(1 - \frac{1}{n}\right) = \frac{V}{n}(n-1) \quad \text{or} \quad \eta = \frac{m_l V'_l}{V/n} = \frac{n-1}{N-1}$

In the case when the final volume of the substance corresponds to the midpoint of a horizontal portion of the isothermal line in the p, v diagram, the final volume must be $(1+N)\dfrac{V'_l}{2}$ per unit mass of the substance. Of this the volume of the liquid is $V'_1/2$ per unit total mass of the substance.

Thus $$\eta = \frac{1}{1+N}$$

2.189 From the first law of thermodynamics

$$\Delta U + A = Q = mq$$

where q is the specific latent heat of vaporization

Now $$A = p(V'_v - V'_l)m = m\frac{RT}{M}$$

Thus $$\Delta U = m\left(q - \frac{RT}{M}\right)$$

For water this gives $\approx 2{\cdot}08 \times 10^6$ Joules.

2.190 Some of the heat used in heating water to the boiling temperature

$T = 100°C = 373$ K. The remaining heat

$$= Q - mc\,\Delta T$$

($c =$ specific heat of water, $\Delta T = 100\,K$) is used to create vapour. If the piston rises to a height h then the volume of vapour will be $\approx sh$(neglecting water). Its mass will be $\frac{p_0\,sh}{RT} \times M$ and heat of vapourization will be $\frac{p_0\,sh\,Mq}{RT}$. To this must be added the work done in creating the saturated vapour $\approx p_0\,sh$. Thus

$$Q - mc\,\Delta T \approx p_0\,S\,h\left(1 + \frac{qM}{RT}\right) \quad \text{or} \quad h = \frac{Q - mc\,\Delta T}{p_0\,s\left(1 + \frac{qM}{RT}\right)} = 20 \text{ cm}$$

2.191 A quantity $\frac{mc\,(T - T_0)}{q}$ of saturated vapour must condense to heat the water to boiling point $T = 373°$K

(Here $c =$ specific heat of water, $T_0 = 295$ K $=$ initial water temperature).

The work done in lowering the piston will then be

$$\frac{mc\,(T - T_0)}{q} \times \frac{RT}{M} = 25 \text{ J},$$

since work done per unit mass of the condensed vapour is $p\,V = \frac{RT}{M}$

2.192 Given $\Delta P = \frac{\rho_v}{\rho_l}\frac{2\alpha}{r} = \frac{\rho_v}{\rho_l} \times \frac{4\alpha}{d} = \eta\,p_{vap} = \eta\,\frac{\frac{m}{M}RT}{V_{vap}} = \frac{\eta\,RT}{M}\rho_v$

or
$$d = \frac{4\,\alpha\,M}{\rho_1\,RT\,\eta}$$

For water $\alpha = 73$ dynes/cm, $M = 18$gm, $\rho_l =$ gm/cc, $T = 300$ K, and with $\eta \approx 0{\cdot}01$, we get

$$d \approx 0{\cdot}2\,\mu\,\text{m}$$

2.193 In equilibrium the number of "liquid" molecules evaporoting must equals the number of "vapour" molecules condensing. By kinetic theory, this number is

$$\eta \times \frac{1}{4} n \langle v \rangle = \eta \times \frac{1}{4} n \times \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$

Its mass is

$$\mu = m \times \eta \times n \times \sqrt{\frac{kT}{2\pi m}} = \eta\,n\,k\,T\sqrt{\frac{m}{2\pi kt}}$$

$$= \eta\,p_0\sqrt{\frac{M}{2\pi RT}} = 0{\cdot}35 \text{ g/cm}^2\cdot\text{s}.$$

where p_0 is atmospheric pressure and $T = 373$ K and $M =$ molecular weight of water.

2.194 Here we must assume that μ is also the rate at which the tungsten filament loses mass when in an atmosphere of its own vapour at this temperature and that η (of the previous problem) $\approx$ 1. Then

$$p = \mu \sqrt{\frac{2\pi RT}{M}} = 0{\cdot}9 \text{ n Pa}$$

from the previous problem where $p =$ pressure of the saturated vapour.

2.195 From the Vander Waals equation

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

where $V =$ Volume of one gm mole of the substances.

For water $V =$ 18 c.c. per mole $= 1{\cdot}8 \times 10^{-2}$ litre per mole

$$a = 5{\cdot}47 \text{ atmos} \cdot \frac{\text{litre}^2}{\text{mole}^2}$$

If molecular attraction vanished the equation will be

$$p' = \frac{RT}{V-b}$$

for the same specific volume. Thus

$$\Delta p = \frac{a}{V^2} = \frac{5{\cdot}47}{1{\cdot}8 \times 1{\cdot}8} \times 10^4 \text{ atmos} \approx 1{\cdot}7 \times 10^4 \text{ atmos}$$

2.196 The internal pressure being $\frac{a}{V^2}$, the work done in condensation is

$$\int_{V_l}^{V_g} \frac{a}{V^2} dV = \frac{a}{V_l} - \frac{a}{V_g} \approx \frac{a}{V_l}$$

This by assumption is Mq, M being the molecular weight and V_l, V_g being the molar volumes of the liquid and gas.

Thus
$$p_i = \frac{a}{V_l^2} = \frac{Mq}{V_l} = \rho q$$

where ρ is the density of the liquid. For water $p_i \approx 3{\cdot}3 \times 10^{13}$ atm

2.197 The Vandar Waal's equation can be written as (for one mole)

$$p(V) = \frac{RT}{V-b} - \frac{a}{V^2}$$

At the critical point $\left(\frac{\partial p}{\partial V}\right)_T$ and $\left(\frac{\partial^2 p}{\partial V^2}\right)_T$ vanish. Thus

$$0 = \frac{RT}{(V-b)^2} + \frac{2a}{V^3} \quad \text{or} \quad \frac{RT}{(V-b)^2} = \frac{2a}{V^3}$$

$$0 = \frac{2RT}{(V-b)^3} - \frac{6a}{V^4} \quad \text{or} \quad \frac{RT}{(V-b)^3} = \frac{3a}{V^4}$$

Solving these simultaneously we get on division

$$V - b = \frac{2}{3} V, \quad V = 3b = V_{MCr}$$

This is the critical molar volume. Putting this back

$$\frac{R T_{Cr}}{4 b^2} = \frac{2a}{27 b^3} \quad \text{or} \quad T_{Cr} = \frac{8a}{27 bR}$$

Finally $$p_{Cr} = \frac{RT_{Cr}}{V_{MCr} - b} - \frac{a}{V_{MCr}^2} = \frac{4a}{27 b^2} - \frac{a}{9 b^2} = \frac{a}{27 b^2}$$

From these we see that $$\frac{P_{Cr} V_{MCr}}{RT_{Cr}} = \frac{a/9b}{8a/27 b} = \frac{3}{8}$$

2.198 $$\frac{P_{Cr}}{RT_{Cr}} = \frac{a/27 b^2}{8 a/27 b} = \frac{1}{8 b}$$

Thus $$b = R \frac{T_{Cr}}{8 p_{Cr}} = \frac{0{\cdot}082 \times 304}{73 \times 8} = 0 {\cdot}043 \text{ litre/mol}$$

and $$\frac{(RT_{Cr})^2}{p_{Cr}} = \frac{64a}{27} \quad \text{or} \quad a = \frac{27}{64}(RT_{Cr})^2/p_{Cr} = 3{\cdot}59 \frac{\text{atm·litre}^2}{(\text{mol})^2}$$

2.199 Specific volume is molar volume divided by molecular weight. Thus

$$V'_{Cr} = \frac{V_{MCr}}{M} = \frac{3 RT_{Cr}}{8 M p_{Cr}} = \frac{3 \times {\cdot}082 \times 562}{8 \times 78 \times 47} \frac{\text{litre}}{g} = 4{\cdot}71 \frac{cc}{g}$$

2.200 $$\left(p + \frac{a}{V_M^2}\right)(V_m - b) = RT$$

or $$\frac{p + \frac{a}{V_m^2}}{P_{Cr}} \times \frac{V_m - b}{V_{MCr}} = \frac{8}{3} \frac{T}{T_{Cr}}$$

or $$\left(\pi + \frac{a}{p_{Cr} V_m^2}\right) \times \left(\nu - \frac{b}{V_{MCr}}\right) = \frac{8}{3}\tau,$$

where $$\pi = \frac{p}{p_{Cr}}, \ \nu = \frac{V_m}{V_{MCr}}, \ \tau = \frac{T}{T_{Cr}}$$

or $$\left(\pi + \frac{27 b^2}{V_M^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau, \quad \text{or} \quad \left(\pi + \frac{3}{\nu^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau$$

When $$\pi = 12 \text{ and } \nu = \frac{1}{2}, \ \tau = \frac{3}{8} \times 24 \times \frac{1}{6} = \frac{3}{2}$$

2.201 (a) The ciritical Volume V_{MCr} is the maximum volume in the liquid phase and the minimum volume in the gaseous. Thus

$$V_{max} = \frac{1000}{18} \times 3 \times {\cdot}030 \text{ litre} = 5 \text{ litre}$$

(b) The critical pressure is the maximum possible pressure in the vapour phase in equilibrium with liquid phase. Thus

$$p_{\max} = \frac{a}{27\,b^2} = \frac{5\cdot 47}{27 \times \cdot 03 \times \cdot 03} = 225 \text{atmosphere}$$

2.202 $T_{Cr} = \frac{8}{27}\frac{a}{bR} = \frac{8}{27} \times \frac{3\cdot 62}{\cdot 043 \times \cdot 082} \approx 304 \text{ K}$

$$\rho_{Cr} = \frac{M}{3b} = \frac{44}{3 \times 43} \text{gm/c.c.} = 0\cdot 34 \text{ gm/c.c.}$$

2.203 The vessel is such that either vapour or liquid of mass m occupies it at critical point. Then its volume will be

$$v_{Cr} = \frac{m}{M} V_{MCr} = \frac{3}{8}\frac{RT_{Cr}}{p_{Cr}}\frac{m}{M}$$

The corresponding volume in liquid phase at room temperature is

$$V = \frac{m}{\rho}$$

where ρ = density of liquid ether at room tmeperature. Thus

$$\eta = \frac{V}{V_{Cr}} = \frac{8\,Mp_{Cr}}{3\,RT_{Cr}\,\rho} \approx 0\cdot 254$$

using the given data (and ρ = 720gm per litre)

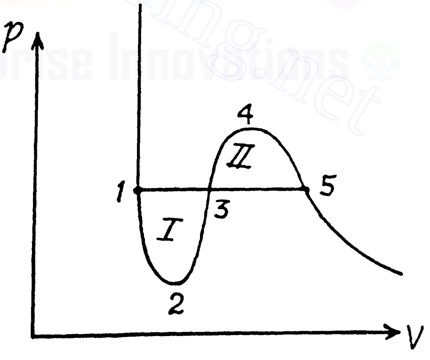
2.204 We apply the relation (T = constant)

$$T\oint dS = \oint dU + \oint p\,dV$$

to the cycle 1234531.

Here $\oint dS = \oint dU = 0$

So $\oint p\,dV = 0$



This implies that the areas I and II are equal. This reasoning is inapplicable to the cycle 1231, for example. This cycle is irreversible because it involves the irreversible transition from a single phase to a two-phase state at the point 3.

2.205 When a portion of supercool water turns into ice some heat is liberated, which should heat it upto ice point. Neglecting the variation of specific heat of water, the fraction of water turning inot ice is clearly

$$f = \frac{c\,|t|}{q} = 0\cdot 25$$

where c = specific heat of water and q = latent heat of fusion of ice, Clearly $f = 1$ at $t = -80°C$

2.206 From the Claussius-Clapeyron (C–C)equations

$$\frac{dT}{dp} = \frac{T\left(V'_2 - V'_1\right)}{q_{12}}$$

$q_{12}$is the specific latent heat absorbed in 1 → 2 (1 = solid, 2 = liquid)

$$\Delta T = \frac{T\left(V'_w - V'_{ice}\right)}{q_{12}}\,\Delta p = -\frac{273 \times \cdot 091}{333} \times 1\,\frac{\text{atm} \times \text{cm}^3 \times \text{K}}{\text{joule}}$$

$$1\,\frac{\text{atm} \times \text{cm}^3}{\text{Joule}} \approx \frac{10^5\,\dfrac{\text{N}}{\text{m}^2} \times 10^{-6}\,\text{m}^3}{\text{Joule}} = 10^{-1},\ \Delta T = -\cdot 0075\text{ K}$$

2.207 Here 1 = liquid, 2 = Steam

$$\Delta T = \frac{T\left(V'_s - V'_{liq}\right)}{q_{12}}\,\Delta P$$

or $$V'_s \approx \frac{q_{12}}{T}\frac{\Delta T}{\Delta p} = \frac{2250}{373} \times \frac{0\cdot 9}{3\cdot 2} \times 10^{-3}\,\text{m}^3/\text{g} = 1\cdot 7\,\text{m}^3/\text{kg}$$

2.208 From C-C equations

$$\frac{dp}{dT} = \frac{q_{12}}{T\left(V'_2 - V'_1\right)} \approx \frac{q_{12}}{T\,V'_2}$$

Assuming the saturated vapour to be ideal gas

$$\frac{1}{V'_2} = \frac{mp}{RT},\ \text{Thus}\quad \Delta p = \frac{Mq}{RT^2}\,p\,\Delta T$$

and $$p \approx p_0\left(1 + \frac{Mq}{RT^2}\,\Delta T\right) = 1\cdot 04\ \text{atmosphere}$$

2.209 From C-C equation, neglecting the voolume of the liquid

$$\frac{dp}{dT} = \frac{q_{12}}{TV'_2} = \frac{Mq}{RT^2}\,p,\ (q = q_{12})$$

or $$\frac{dp}{p} = \frac{Mq}{RT}\frac{dT}{T}$$

Now $$pV = \frac{m}{M}RT\ \text{ or }\ m = \frac{MpV}{RT}\ \text{for a perfect gas}$$

So $$\frac{dm}{m} = \frac{dp}{p} - \frac{dT}{T}\ (V\text{ is Const} = \text{specific volume})$$

$$= \left(\frac{Mq}{RT} - 1\right)\frac{dT}{T} = \left(\frac{18 \times 2250}{8\cdot 3 \times 373} - 1\right) \times \frac{1\cdot 5}{373} \approx 4\cdot 85\ \%$$

2.210 From C-C equation

$$\frac{dp}{dT} \approx \frac{q}{T V'_2} = \frac{Mq}{R T^2} p$$

Integrating $\ln p =$ constant $- \dfrac{Mq}{RT}$

So
$$p = p_0 \exp\left[\frac{Mq}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

This is reasonable for $|T - T_0| << T_0$, and far below critical temperature.

2.211 As before (2.206) the lowering of melting point is given by

$$\Delta T = -\frac{T \Delta V'}{q} p$$

The superheated ice will then melt in part. The fraction that will melt is

$$\eta = \frac{C T \Delta V'}{q^2} p \approx \cdot 03$$

2.212 (a) The equations of the transition lines are

$$\log p = 9{\cdot}05 - \frac{1800}{T} : \text{ Solid gas}$$

$$= 6{\cdot}78 - \frac{1310}{T} : \text{ Liquid gas}$$

At the triple point they intersect. Thus

$$2{\cdot}27 = \frac{490}{T_{tr}} \quad \text{or} \quad T_{tr} = \frac{490}{2{\cdot}27} = 216 \text{ K}$$

corresponding p_{tr} is 5.14 atmosphere.

In the formula $\log p = a - \dfrac{b}{T}$, we compare b with the corresponding term in the equation in 2.210. Then

$$\ln p = a \times 2{\cdot}303 - \frac{2{\cdot}303\, b}{T} \quad \text{So, } 2{\cdot}303 = \frac{Mq}{R}$$

or,
$$q_{sublimation} = \frac{2{\cdot}303 \times 1800 \times 8{\cdot}31}{44} = 783 \text{ J/gm}$$

$$q_{liquid-gas} = \frac{2{\cdot}303 \times 1310 \times 8{\cdot}31}{44} = 570 \text{ J/gm}$$

Finally $q_{solid-liquid} = 213$ J/gm on subtraction

2.213
$$\Delta S = \int_{T_1}^{T_2} mc \frac{dT}{T} + \frac{mq}{T_2} = m\left(c \ln \frac{T_2}{T_1} + \frac{q}{T_2}\right)$$

$$= 10^3 \left(4{\cdot}18 \ln \frac{373}{283} + \frac{2250}{373}\right) = 7{\cdot}2 \text{ kJ/K}$$

2.214 $\Delta S = \frac{q_m}{T_1} + c \ln \frac{T_2}{T} + \frac{q_v}{T_2}$

$$= \frac{333}{273} + 4{\cdot}18 \ln \frac{373}{283} \approx 8{\cdot}56 \text{ J/°K}$$

2.215 c = specific heat of copper = $0{\cdot}39 \frac{\text{J}}{\text{g} \cdot \text{K}}$ Suppose all ice does not melt, then

heat rejected = $90 \times 0{\cdot}39\,(90 - 0) = 3159$ J

heat gained by ice = $50 \times 2{\cdot}09 \times 3 + x \times 333$

Thus $x = 8{\cdot}5$ gm

The hypothesis is correct and final temperature will be $T = 273$K.

Hence change in entropy of copper piece

$$= mc \ln \frac{273}{363} = -10 \text{ J/K.}$$

2.216 (a) Here $t_2 = 60$°C. Suppose the final temperature is t °C. Then

heat lost by water = $m_2\, c\,(t_2 - t)$

heat gained by ice = $m_1\, q_m + m_1\, c\,(t - t_1)$, if all ice melts

In this case $m_1\, q_m = m_2 \times 4{\cdot}18\,(60 - t)$, for $m_1 = m_2$

So the final temperature will be 0°C and only some ice will melt.

Then $100 \times 4{\cdot}18\,(60) = m'_1 \times 333$

$m'_1 = 75{\cdot}3$ gm = amount of ice that will melt

Finally $\Delta S = 75{\cdot}3 \times \frac{333}{273} + 100 \times 4{\cdot}18 \ln \frac{273}{333}$

$$\Delta S = \frac{m'_1\, q_m}{T_1} + m_2\, c \ln \frac{T_1}{T_2}$$

$$= m_2\, c\, \frac{(T_2 - T_1)}{T_1} - m_2 \ln \frac{T_2}{T_1}$$

$$= m_2\, C \left[\frac{T_2}{T_1} - 1 - \ln \frac{T_2}{T_1}\right] = 8{\cdot}8 \text{ J/K}$$

(b) If $m_2\, c\, t_2 > m_1\, q_m$ then all ice will melt as one can check and the final temperature can be obtained like this

$$m_2\, c\,(T_2 - T) = m_1\, q_m + m_1\, c\,(T - T_1)$$

$$(m_2\, T_2 + m_1\, T_1)\, c - m_1\, q_m = (m_1 + m_2)\, cT$$

or $$T = \frac{m_2\, T_2 + m_1\, T_1 - \frac{m_1\, q_m}{c}}{m_1 + m_2} = 280 \text{ K}$$

and $$\Delta S = \frac{m_1\, q}{T_1} + c\left(m_1 \ln \frac{T}{T_1} - m_2 \ln \frac{T_2}{T}\right) = 19 \text{ J/K}$$

2.217 $\Delta S = -\frac{m\,q_1}{T_2} - mc\ln\frac{T_2}{T_1} + \frac{M\,q_{ice}}{T_1}$

where $$M\,q_{ice} = m\left(q_2 + c\,(T_2 - T_1)\right)$$

$$= mq_2\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + mc\left(\frac{T_2}{T_1} - 1 - \frac{T_2}{T_1}\right)$$

$$= 0{\cdot}2245 + 0{\cdot}2564 \approx 0{\cdot}48\ \text{J/K}$$

2.218 When heat dQ is given to the vapour its temperature will change by dT, pressure by dp and volume by dV, it being assumed that the vapour remains saturated.

Then by C-C equation

$$\frac{dp}{dT} = \frac{q}{TV'}\,(V'_{\text{vapour}} >> V'_{\text{Liq}}),\quad \text{or}\quad dp = \frac{q}{TV'}\,dT$$

on the other hand, $pV' = \frac{RT}{M}$

So $$pdV' + V'\,dp = \frac{RdT}{M},$$

Hence $$pdV' = \left(\frac{R}{M} - \frac{q}{T}\right)dT$$

finally $$dQ = CdT = dU + pdV'$$

$$= C_V\,dT + \left(\frac{T}{M} - \frac{q}{T}\right)dT = C_p\,dT - \frac{q}{T}\,dT$$

(C_p, C_V refer to unit mass here). Thus

$$C = C_p - \frac{q}{T}$$

For water $C_p = \frac{R\gamma}{\gamma - 1}\cdot\frac{1}{M}$ with $\gamma = 1{\cdot}32$ and $M = 18$

So $$C_p = 1{\cdot}90\ \text{J/gm K}$$

and $$C = -4{\cdot}13\ \text{J/gm}^\circ\text{K} = -74\ \text{J/mole K}$$

2.219 The required entropy change can be calculated along a process in which the water is heated from T_1 to T_2 and then allowed to evaporate. The entropy change for this is

$$\Delta S = C_p\ln\frac{T_2}{T_1} + \frac{qM}{T_2}$$

where q = specific latent heat of vaporization.

2.7 TRANSPORT PHENOMENA

2.220 (a) The fraction of gas molecules which traverses distances exceeding the mean free path without collision is just the probability to traverse the distance $s = \lambda$ without collision.

Thus $$P = e^{-1} = \frac{1}{e} = 0{\cdot}37$$

(b) This probability is

$$P = e^{-1} - e^{-2} = 0{\cdot}23$$

2.221 From the formula

$$\frac{1}{\eta} = e^{-\Delta l/\lambda} \quad \text{or} \quad \lambda = \frac{\Delta l}{\ln \eta}$$

2.222 (a) Let $P(t) =$ probability of no collision in the interval $(0, t)$. Then

$$P(t + dt) = P(t)(1 - \alpha\, dt)$$

or $$\frac{dP}{dt} = -\alpha P(t) \quad \text{or} \quad P(t) = e^{-\alpha t}$$

where we have used $P(0) = 1$

(b) The mean interval between collision is also the mean interval of no collision. Then

$$\langle t \rangle = \frac{\int_0^\infty t e^{-\alpha t} dt}{\int_0^\infty e^{-\alpha t} dt} = \frac{1}{\alpha} \frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{\alpha}$$

2.223 (a) $\lambda = \dfrac{1}{\sqrt{2}\,\pi d^2 n} = \dfrac{kT}{\sqrt{2}\,\pi d^2 p}$

$$= \frac{1{\cdot}38 \times 10^{-23} \times 273}{\sqrt{2}\,\pi (0{\cdot}37 \times 10^{-9})^2 \times 10^5} = 6{\cdot}2 \times 10^{-8}\ \text{m}$$

$$\tau = \frac{\lambda}{\langle v \rangle} = \frac{6{\cdot}2 \times 10^{-8}}{454} s = {\cdot}136\ \text{n s}$$

$\lambda = 6.2 \times 10^6$ m

(b) $\eta = 1{\cdot}36 \times 10^4$ s $= 3{\cdot}8$ hours

2.224 The mean distance between molecules is of the order

$$\left(\frac{22{\cdot}4 \times 10^{-3}}{6{\cdot}0 \times 10^{23}}\right)^{1/3} = \left(\frac{224}{6}\right)^{1/3} \times 10^{-9}\ \text{meters} \approx 3{\cdot}34 \times 10^{-9}\ \text{meters}$$

This is about 18.5 times smaller than the mean free path calculated in 2.223 (a) above.

2.225 We know that the Vander Waal's constant b is four times the molecular volume. Thus

$$b = 4 N_A \frac{\pi}{6} d^3 \quad \text{or} \quad d = \left(\frac{3b}{2\pi N_A}\right)^{1/3}$$

Hence $$\lambda = \left(\frac{kT_0}{\sqrt{2}\,\pi p_0}\right)\left(\frac{2\pi N_A}{3b}\right)^{2/3}$$

2.226 The volocity of sound in N_2 is

$$\sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

so,
$$\frac{1}{\nu} = \sqrt{\frac{\gamma RT_0}{M}} = \frac{RT_0}{\sqrt{2}\,\pi d^2 p_0 N_A}$$

or,
$$\nu = \pi d^2 p_0 N_A \sqrt{\frac{2\gamma}{MRT_0}}$$

2.227 (a) $\lambda > l$ if $p < \frac{kT}{\sqrt{2}\,\pi d^2 l}$

Now $\frac{kT}{\sqrt{2}\,\pi d^2 l}$ for O_2 of O is 0·7 Pa.

(b) The corresponding n is obtained by dividing by kT and is $1{\cdot}84 \times 10^{20}$ per $m^3 = 1{\cdot}84^{14}$per c.c. and the corresponding mean distance is $\frac{l}{n^{1/3}}$.

$$= \frac{10^{-2}}{(0{\cdot}184)^{1/3} \times 10^5} = 1{\cdot}8 \times 10^{-}\ m \approx 0{\cdot}18\ \mu\text{m}.$$

2.228 (a) $\nu = \frac{1}{\tau} = \frac{1}{\lambda/\langle v\rangle} = \frac{\langle v\rangle}{\lambda}$

$$= \sqrt{2}\,\pi d^2 n \langle v\rangle = {\cdot}74 \times 10^{10}\ \text{s}^{-1} \text{ (see 2.223)}$$

(b) Total number of collisions is

$$\frac{1}{2} n\nu \approx 1{\cdot}0 \times 10^{29}\ \text{s cm}^{-3}$$

Note, the factor $\frac{1}{2}$. When two molecules collide we must not count it twice.

2.229 (a) $\lambda = \frac{1}{\sqrt{2}\,\pi d^2 n}$

d is a constant and n is a constant for an isochoric process so λ is constant for an isochoric process.

$$\nu = \frac{\langle v\rangle}{\lambda} = \frac{\sqrt{\frac{8RT}{M\pi}}}{\lambda}\ \alpha \sqrt{T}$$

(b) $\lambda = \frac{1}{\sqrt{2}\,\pi d^2} \frac{kT}{p}\ \alpha\ T$ for an isobaric process.

$$\nu = \frac{\langle v\rangle}{\lambda}\ \alpha\ \frac{\sqrt{T}}{T} = \frac{1}{\sqrt{T}} \text{ for an isobaric process.}$$

2.230 (a) In an isochoric process λ is constant and

$$\nu\ \alpha \sqrt{T}\ \alpha \sqrt{pV}\ \alpha \sqrt{p}\ \alpha \sqrt{n}$$

(b) $\lambda = \frac{kT}{\sqrt{2}\,\pi d^2 p}$ must decrease n times in an isothermal process and ν must increase n times because $\langle v\rangle$ is constant in an isothermal process.

2.231 (a) $\lambda \propto \frac{1}{n} \Rightarrow \frac{1}{N/V} = \frac{V}{N}$

Thus $$\lambda \propto V \text{ and } \nu \propto \frac{T^{1/2}}{V}$$

But in an adiabatic process $\left(\gamma = \frac{7}{5} \text{ here}\right)$

$$TV^{\gamma - 1} = \text{constant so } TV^{2/5} = \text{ constant}$$

or $$T^{1/2} \propto V^{-1/5} \text{ Thus } \nu \propto V^{-6/5}$$

(b) $\lambda \propto \frac{T}{p}$

But $p\left(\frac{T}{p}\right)^{\gamma} = \text{constant or } \frac{T}{p} \propto p^{-1/\gamma} \text{ or } T \propto p^{1-1/\gamma}$

Thus $$\lambda \propto p^{-1/\gamma} = p^{-5/7}$$

$$\nu = \frac{\langle v \rangle}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1/2 + \frac{1}{2\gamma}} = p^{\frac{\gamma+1}{2\gamma}} = p^{6/7}$$

(c) $\lambda \propto V$

But $$TV^{2/5} = \text{constant or } V \propto T^{-5/2}$$

Thus $$\lambda \propto T^{-5/2}$$

$$\nu \propto \frac{T^{1/2}}{V} \propto T^{3}$$

2.232 In the polytropic process of index n

$$pV^{n} = \text{constant, } TV^{n-1} = \text{constant and } p^{1-n}T^{n} = \text{constant}$$

(a) $\lambda \propto V$

$$\nu \propto \frac{T^{1/2}}{V} = V^{\frac{1-n}{2}} V^{-1} = V^{\frac{-n+1}{2}}$$

(b) $\lambda \propto \frac{T}{p},\ T^{n} \propto p^{n-1} \text{ or } T \propto p^{1-\frac{1}{n}}$

so $$\lambda \propto p^{-1/n}$$

$$\nu = \frac{\langle v \rangle}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1-\frac{1}{2}+\frac{1}{2n}} = p^{\frac{n+1}{2n}}$$

(c) $\lambda \propto \frac{T}{p},\ p \propto T^{\frac{n}{n-1}}$

$$\lambda \propto T^{1-\frac{n}{n\,1}} = T^{-\frac{1}{n-1}} = T^{\frac{1}{1-n}}$$

$$\nu \propto \frac{p}{\sqrt{T}} \propto T^{\frac{n}{n-1}-\frac{1}{2}} = T^{\frac{n+1}{2(n-1)}}$$

2.233 (a) The number of collisions between the molecules in a unit volume is

$$\frac{1}{2}n\,v = \frac{1}{\sqrt{2}}\pi\,d^2\,n^2 <v> \alpha\,\frac{\sqrt{T}}{V^2}$$

This remains constant in the poly process pV^{-3} = constant
Using (2.122) the molar specific heat for the polytropic process

$$pV^{\alpha} = \text{ constant},$$

is
$$C = R\left(\frac{1}{\gamma-1} - \frac{1}{\alpha-1}\right)$$

Thus
$$C = R\left(\frac{1}{\gamma-1} + \frac{1}{4}\right) = R\left(\frac{5}{2} + \frac{1}{4}\right) = \frac{11}{4}R$$

It can also be written as $\frac{1}{4}R\,(1+2\,i)$ where $i = 5$

(b) In this case $\frac{\sqrt{T}}{V}$ = constant and so pV^{-1} = constant

so
$$C = R\left(\frac{1}{\gamma-1} + \frac{1}{2}\right) = R\left(\frac{5}{2} + \frac{1}{2}\right) = 3R$$

It can also be written as $\frac{R}{2}(i+1)$

2.234 We can assume that all molecules, incident on the hole, leak out. Then,

$$-dN = -d\,(nV) = \frac{1}{4}n <v> S\,dt$$

or
$$dn = -n\frac{dt}{4v/S <v>} = -n\frac{dt}{\tau}$$

Integrating $\quad n = n_0\,e^{-t/\tau}$. Hence $<v> = \sqrt{\frac{8RT}{\pi M}}$

2.235 If the temperature of the compartment 2 is η times more than that of compartment 1, it must contain $\frac{1}{\eta}$ times less number of molecules since pressure must be the same when the big hole is open. If M = mass of the gas in 1 than the mass of the gas in 2 must be $\frac{M}{\eta}$. So immediately after the big hole is closed.

$$n_1^0 = \frac{M}{mV},\; n_2^0 = \frac{M}{mV\eta}$$

where m = mass of each molecule and n_1^0, n_2^0 are concentrations in 1 and 2. After the big hole is closed the pressures will differ and concentration will become n_1 and n_2 where

$$n_1 + n_2 = \frac{M}{m\,V\eta}(1+\eta)$$

On the other hand

$$n_1 <v_1> = n_2 <v_2> \quad \text{i.e. } n_1 = \sqrt{\eta}\,n_2$$

Thus $$n_2'(1+\sqrt{\eta}) = \frac{m}{mV\eta}(1+\eta) = n_2^0(1+\eta)$$

So $$n_2 = n_2^0 \frac{1+\eta}{1+\sqrt{\eta}}$$

2.236 We know

$$\eta = \frac{1}{3} \langle v \rangle \lambda \rho = \frac{1}{3} \langle v \rangle \frac{1}{\sqrt{2}\,\pi d^2} m \, \alpha \sqrt{T}$$

Thus η changing α times implies T changing α^2times.

On the other hand

$$D = \frac{1}{3} \langle v \rangle \lambda = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}} \frac{kT}{\sqrt{2}\,\pi d^2 p}$$

Thus D changing β times means $\dfrac{T^{3/2}}{p}$ changing β times

So p must change $\dfrac{\alpha^3}{\beta}$ times

2.237 $D \, \alpha \, \dfrac{\sqrt{T}}{n} \propto V\sqrt{T}\,,\ \eta \propto \sqrt{T}$

(a) D will increase n times

η will remain constant if T is constant

(b) $D \, \alpha \, \dfrac{T^{3/2}}{p} \, \alpha \, \dfrac{(p\,V)^{3/2}}{p} = p^{1/2}\, V^{3/2}$

$$\eta \, \alpha \, \sqrt{pV}$$

Thus D will increase $n^{3/2}$ times, η will increase $n^{1/2}$ times, if p is constant

2.238 $D \, \alpha \, V\sqrt{T}\,,\ \eta \, \alpha \, \sqrt{T}$

In an adiabatic process

$$TV^{\gamma-1} = \text{constant, or } T \propto V^{1-\gamma}$$

Now V is decreased $\dfrac{1}{n}$ times. Thus

$$D \, \alpha \, V^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{\frac{3-\gamma}{2}} = \left(\frac{1}{n}\right)^{4/5}$$

$$\eta \, \alpha \text{ of } V^{\frac{1-\gamma}{2}} = \left(\frac{1}{n}\right)^{-1/5} = n^{1/5}$$

So D decreases $n^{4/5}$times and η increase $n^{1/5}$times.

2.239 (a) $D \, \alpha \, V\sqrt{T} \, \alpha \, \sqrt{pV^3}$

Thus D remains constant in the process $pV^3 =$ constant

So polytropic index $n = 3$

(b) $\eta \, \alpha \, \sqrt{T} \, \alpha \, \sqrt{pV}$

So η remains constant in the isothermal process

$$pV = \text{constant},\ n = 1,\ \text{here}$$

(c) Heat conductivity $\kappa = \eta C_V$

and C_V is a constant for the ideal gas

Thus $\qquad n = 1$ here also,

2.240 $\eta = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}}\frac{m}{\sqrt{2}\pi d^2} = \frac{2}{3}\sqrt{\frac{mkT}{\pi^3}}\frac{1}{d^2}$

$$\text{or} \quad d = \left(\frac{2}{3\eta}\right)^{1/2}\left(\frac{mkT}{\pi^3}\right)^{1/4} = \left(\frac{2}{3\times 18\cdot 9}\times 10^6\right)^{1/2}\left(\frac{4\times 8\cdot 31\times 273\times 10^{-3}}{\pi^3\times 36\times 10^{46}}\right)^{1/4}$$

$$= 10^{-10}\left(\frac{2}{3\times 18\cdot 9}\right)^{1/2}\left(\frac{4\times 83\cdot 1\times 273}{\pi^3\times \cdot 36}\right)^{1/4} \approx 0\cdot 178 \text{ nm}$$

2.241 $\kappa = \frac{1}{3}<v>\lambda\rho c_V$

$$= \frac{1}{3}\sqrt{\frac{8kT}{m\pi}}\frac{1}{\sqrt{2}\pi d^2 n}mn\frac{C_V}{M}$$

$\left(C_V \text{ is the specific heat capacity which is } \frac{C_V}{M}\right)$. Now C_V is the same for all monoatomic gases such as He and A. Thus

$$\kappa \propto \frac{1}{\sqrt{M}\,d^2}$$

$$\text{or} \quad \frac{\kappa_{He}}{\kappa_A} = 8\cdot 7 = \frac{\sqrt{M_A}\,d_A^2}{\sqrt{M_{He}}\,d_{He}^2} = \sqrt{10}\frac{d_A^2}{d_{He}^2}$$

$$\frac{d_A}{d_{He}} = \sqrt{\frac{8\cdot 7}{\sqrt{10}}} = 1\cdot 658 \approx 1\cdot 7$$

2.242 In this case

$$N_1\frac{r_2^2 - r_1^2}{r_1^2 r_2^2} = 4\pi\eta\omega$$

$$\text{or} \quad N_1\frac{2R\,\Delta R}{R^4} \approx 4\pi\eta\omega \quad \text{or} \quad N_1 = \frac{2\pi\eta\omega R^3}{\Delta R}$$

To decrease N_1, n times η must be decreased n times. Now η does not depend on pressure until the pressure is so low that the mean free path equals, say, $\frac{1}{2}\Delta R$. Then the mean free path is fixed and η decreases with pressure. The mean free path equals $\frac{1}{2}\Delta R$ when

$$\frac{1}{\sqrt{2}\pi d^2 n_0} = \Delta R \quad (n_0 = \text{concentration})$$

Corresponding pressure is $p_0 = \dfrac{\sqrt{2}\,k\,T}{\pi\,d^2\,\Delta R}$

The sought pressure is n times less

$$p = \frac{\sqrt{2}\,k\,T}{\pi\,d^2\,n\,\Delta R} = 70{\cdot}7 \times \frac{10^{-23}}{10^{-18} \times 10^{-3}} \approx 0{\cdot}71 \text{ Pa}$$

The answer is qualitative and depends on the choice $\frac{1}{2}\Delta R$ for the mean free path.

2.243 We neglect the moment of inertia of the gas in a shell. Then the moment of friction forces on a unit length of the cylinder must be a constant as a function of r.

So, $$2\pi r^3 \eta \frac{d\omega}{dr} = N_1 \quad \text{or} \quad \omega(r) = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right)$$

and $$\omega = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad \text{or} \quad \eta = \frac{N_1}{4\pi\omega}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

2.244 We consider two adjoining layers. The angular velocity gradient is $\frac{\omega}{h}$. So the moment of the frictional force is

$$N = \int_0^a r \cdot 2\pi r\,dr \cdot \eta\, r\,\frac{\omega}{h} = \frac{\pi\,\eta\,a^4\,\omega}{2h}$$

2.245 In the ultrararefied gas we must determine η by taking $\lambda = \frac{1}{2}h$. Then

$$\eta = \frac{1}{3}\sqrt{\frac{8kT}{m\pi}} \times \frac{1}{2}h \times \frac{mp}{kT} = \frac{1}{3}\sqrt{\frac{2M}{\pi RT}}\,hp$$

so, $$N = \frac{1}{3}\,\omega\,a^4\,p\sqrt{\frac{\pi M}{2RT}}$$

2.246 Take an infinitesimal section of length dx and apply Poiseuilles equation to this. Then

$$\frac{dV}{dt} = \frac{-\pi a^4}{8\eta}\frac{\partial p}{\partial x}$$

From the formula $$pV = RT \cdot \frac{m}{M}$$

$$pdV = \frac{RT}{M}\,dm$$

or $$\frac{dm}{dt} = \mu = -\frac{\pi a^4 M}{8\,\eta\,RT}\frac{pdp}{dx}$$

This equation implies that if the flow is isothermal then $p\frac{dp}{dx}$ must be a constant and so equals $\dfrac{|p_2^2 - p_1^2|}{2\,l}$ in magnitude.

Thus, $$\mu = \frac{\pi a^4 M}{16\,\eta\,R\,T}\frac{|p_2^2 - p_1^2|}{l}$$

2.247 Let $T =$ temperature of the interface.

Then heat flowing from left = heat flowing into right in equilibrium.

Thus, $\kappa_1 \dfrac{T_1 - T}{l_1} = \kappa_2 \dfrac{T - T_2}{l_2}$ or $T = \dfrac{\left(\dfrac{\kappa_1 T_1}{l_1} + \dfrac{\kappa_2 T_2}{l_2}\right)}{\left(\dfrac{\kappa_1}{l_1} + \dfrac{\kappa_2}{l_2}\right)}$

2.248 We have

$$\kappa_1 \frac{T_1 - T}{l_1} = \kappa_2 \frac{T - T_2}{l_2} = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or using the previous result

$$\frac{\kappa_1}{l_1}\left(T_1 - \frac{\dfrac{\kappa_1 T_1}{l_1} + \dfrac{\kappa_2 T_2}{l_2}}{\dfrac{\kappa_1}{l_1} + \dfrac{\kappa_1}{l_2}}\right) = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or $$\frac{\kappa_1}{l_1} \frac{\dfrac{\kappa_2}{l_2}(T_1 - T_2)}{\dfrac{\kappa_1}{l_1} + \dfrac{\kappa_2}{l_2}} = \kappa \frac{T_1 - T_2}{l_1 + l_2} \quad \text{or} \quad \kappa = \frac{l_1 + l_2}{\dfrac{l_1}{\kappa_1} + \dfrac{l_2}{\kappa_2}}$$

2.249 By definition the heat flux (per unit area) is

$$\dot{Q} = -K \frac{dT}{dx} = -\alpha \frac{d}{dx} \ln T = \text{constant} = +\alpha \frac{\ln T_1/T_2}{l}$$

Integrating $$\ln T = \frac{x}{l} \ln \frac{T_2}{T_1} + \ln T_1$$

where $T_1 =$ temperature at the end $x = 0$

So $$T = T_1 \left(\frac{T_2}{T_1}\right)^{x/l} \quad \text{and} \quad \dot{Q} = \frac{\alpha \ln T_1/T_2}{l}$$

2.250 Suppose the chunks have temperatures T_1, T_2 at time t and $T_1 - dT_1$, $T_2 + dT_2$ at time $dt + t$.

Then $$C_1\, dT_1 = C_2\, dT_2 = \frac{\kappa S}{l}(T_1 - T_2)\, dt$$

Thus $$d\,\Delta T = -\frac{\kappa S}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \Delta T\, dt \text{ where } \Delta T = T_1 - T_2$$

Hence $$\Delta T = (\Delta T)_0\, e^{-t/\tau} \text{ where } \frac{1}{\tau} = \frac{\kappa s}{l}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

2.251 $\dot{Q} = \kappa \frac{\partial T}{\partial x} = -A\sqrt{T}\frac{\partial T}{\partial x}$

$$= -\frac{2}{3}A\frac{\partial T^{3/2}}{\partial x}, \ (A = \text{constant})$$

$$= \frac{2}{3}A\frac{(T_1^{3/2} - T_2^{3/2})}{l}$$

Thus $$T^{3/2} = \text{constant} - \frac{x}{l}\left(T_1^{3/2} - T_2^{3/2}\right)$$

or using $$T = T_1 \text{ at } x = 0$$

$$T^{3/2} = T_1^{3/2} + \frac{x}{l}\left(T_2^{3/2} - T_1^{3/2}\right) \text{ or } \left(\frac{T}{T_1}\right)^{3/2} = 1 + \frac{x}{l}\left(\left(\frac{T_2}{T}\right)^{3/2} - 1\right)$$

$$T = T_1\left[1 + \frac{x}{l}\left\{\left(\frac{T_2}{T_1}\right)^{3/2} - 1\right\}\right]^{2/3}$$

2.252 $\kappa = \frac{1}{3}\sqrt{\frac{8RT}{\pi M}}\frac{1}{\sqrt{2}\,\pi d^2 n}mn\frac{R\frac{i}{2}}{M} = \frac{R^{3/2}\,i\,T^{3/2}}{3\pi^{3/2}d^2\sqrt{M}\,N_A}$

Then from the previous problem

$$q = \frac{2i\,R^{3/2}(T_2^{3/2} - T_1^{3/2})}{9\pi^{3/2}d^2\sqrt{M}\,N_A\,l}, \ i = 3 \text{ here.}$$

2.253 At this pressure and average temparature = 27°C = 300K = $T = \frac{(T_1 + T_2)}{2}$

$\lambda = \frac{1}{\sqrt{2}\,\pi d^2}\frac{\kappa T}{p} = 2330 \times 10^{-5}\text{m} = 23{\cdot}3\text{mm} >> 5{\cdot}0\text{mm} = l$

The gas is ultrathin and we write $\lambda = \frac{1}{2}l$ here

Then $$q = \kappa\frac{dT}{dx} = \kappa\frac{T_2 - T_1}{l}$$

where $$\kappa = \frac{1}{3}<v>\times\frac{1}{2}l\times\frac{MP}{RT}\times\frac{R}{\gamma - 1}\times\frac{1}{M} = \frac{p<v>}{6T(\gamma - 1)}l$$

and $$q = \frac{p<v>}{6T(\gamma - 1)}(T_2 - T_1)$$

where $<v> = \sqrt{\frac{8RT}{M\pi}}$. We have used $T_2 - T_1 << \frac{T_2 + T_1}{2}$ here.

2.254 In equilibrium $2\pi r \kappa \dfrac{dT}{dr} = -A =$ constant. So $T = B - \dfrac{A}{2\pi\kappa} \ln r$

But $T = T_1$ when $r = R_1$ and $T = T_2$ when $r = R_2$.

From this we find $T = T_1 + \dfrac{T_2 - T_1}{\ln \dfrac{R_2}{R_1}} \ln \dfrac{r}{r_1}$

2.255 In equilibrium $4\pi r^2 \kappa \dfrac{dT}{dr} = -A =$ constant

$$T = B + \frac{A}{4\pi\kappa} \frac{1}{r}$$

Using $T = T_1$ when $r = R_1$ and $T = T_2$ when $r = R_2$,

$$T = T_1 + \frac{T_2 - T_1}{\dfrac{1}{R_2} - \dfrac{1}{R_1}} \left(\frac{1}{r} - \frac{1}{R_1} \right)$$

2.256 The heat flux vector is $-\kappa$ grad T and its divergence equals w. Thus

$$\nabla^2 T = -\frac{w}{\kappa}$$

or $$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = -\frac{w}{\kappa} \text{ in cylindrical coordinates.}$$

or $$T = B + A \ln r - \frac{\omega}{2x} r^2$$

Since T is finite at $r = 0, A = 0$. Also $T = T_0$ at $r = R$

so $$B = T_0 + \frac{w}{4\kappa} R^2$$

Thus $$T = T_0 + \frac{w}{4\kappa} (R^2 - r^2)$$

r here is the distance from the axis of wire (axial radius).

2.257 Here again

$$\nabla^2 T = -\frac{w}{\kappa}$$

So in spherical polar coordinates,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = -\frac{w}{\kappa} \text{ or } r^2 \frac{\partial T}{\partial r} = -\frac{w}{3\kappa} r^3 + A$$

or $$T = B - \frac{A}{r} - \frac{w}{6\kappa} r^2$$

Again $$A = 0 \text{ and } B = T_0 + \frac{w}{6\kappa} R^2$$

so finally $$T = T_0 + \frac{w}{6\kappa} (R^2 - r^2)$$

PART THREE

ELECTRODYNAMICS

3.1 CONSTANT ELECTRIC FIELD IN VACUUM

3.1 F_{el} (for electorns) $= \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ and $F_{gr} = \dfrac{\gamma m^2}{r^2}$

Thus $$\frac{F_{el}}{F_{gr}} \text{ (for electrons)} = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$$

$$= \frac{(1{\cdot}602\times 10^{-19}\text{ C})^2}{\left(\dfrac{1}{9\times 10^9}\right)\times 6{\cdot}67\times 10^{-11}\text{ m}^3/(\text{kg}\cdot\text{s}^2)\times(9{\cdot}11\times 10^{-31}\text{ kg})^2} = 4\times 10^{42}$$

Similarly $$\frac{F_{el}}{F_{gr}} \text{ (for proton)} = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$$

$$= \frac{(1{\cdot}602\times 10^{-19}\text{ C})^2}{\left(\dfrac{1}{9\times 10^9}\right)\times 6{\cdot}67\times 10^{-11}\text{ m}^3/(\text{kg}\cdot\text{s}^2)\times(1{\cdot}672\times 10^{-27}\text{ kg})^2} = 1\times 10^{36}$$

For $F_{el} = F_{gr}$

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{\gamma m^2}{r^2} \quad\text{or}\quad \frac{q}{m} = \sqrt{4\pi\varepsilon_0\gamma}$$

$$= \sqrt{\frac{6{\cdot}67\times 10^{-11}\text{ m}^3(\text{kg}-\text{s}^2)}{9\times 10^9}} = 0{\cdot}86\times 10^{-10}\text{ C/kg}$$

3.2 Total number of atoms in the sphere of mass 1 gm $= \dfrac{1}{63{\cdot}54}\times 6{\cdot}023\times 10^{23}$

So the total nuclear charge $\lambda = \dfrac{6{\cdot}023\times 10^{23}}{63{\cdot}54}\times 1{\cdot}6\times 10^{-19}\times 29$

Now the charge on the sphere = Total nuclear charge – Total electronic charge

$$= \frac{6{\cdot}023 \times 10^{23}}{63{\cdot}54} \times 1{\cdot}6 \times 10^{-19} \times \frac{29 \times 1}{100} = 4{\cdot}298 \times 10^2 \text{ C}$$

Hence force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{[4{\cdot}398 \times 10^2]^2}{1^2} \text{N} = 9 \times 10^9 \times 10^4 \times 19{\cdot}348 \text{ N} = 1{\cdot}74 \times 10^{15} \text{ N}$$

3.3 Let the balls be deviated by an angle θ, from the vertical when separtion between them equals x.

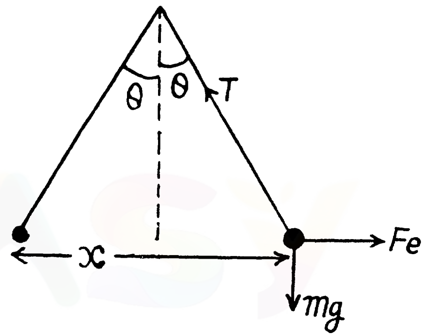
Applying Newton's second law of motion for any one of the sphere, we get,

$$T\cos\theta = mg \qquad (1)$$

and $$T\sin\theta = F_e \qquad (2)$$

From the Eqs. (1) and (2)

$$\tan\theta = \frac{F_e}{mg} \qquad (3)$$



But from the figure

$$\tan\theta = \frac{x}{2\sqrt{l^2 - \left(\frac{x}{2}\right)^2}} \approx \frac{x}{2l} \text{ as } x << l \qquad (4)$$

From Eqs. (3) and (4)

$$F_e = \frac{mg\,x}{2l} \quad \text{or} \quad \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mg\,x}{2l}$$

Thus $$q^2 = \frac{2\pi\varepsilon_0 mg\,x^3}{l} \qquad (5)$$

Differentiating Eqn. (5) with respect to time

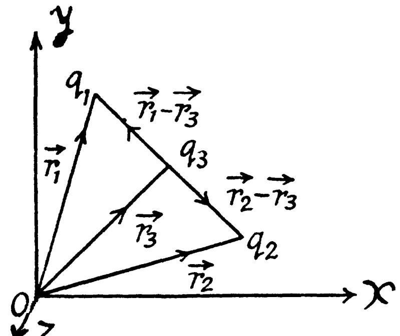
$$2q\frac{dq}{dt} = \frac{2\pi\varepsilon_0 mg}{l} 3x^2 \frac{dx}{dt}$$

According to the problem $\frac{dx}{dt} = v = a/\sqrt{x}$ (approach velocity is $\frac{dx}{dt}$)

so, $$\left(\frac{2\pi\varepsilon_0 mg}{l} x^3\right)^{1/2} \frac{dq}{dt} = \frac{3\pi\varepsilon_0 mg}{l} x^2 \frac{a}{\sqrt{x}}$$

Hence, $$\frac{dq}{dt} = \frac{3}{2} a \sqrt{\frac{2\pi\varepsilon_0 mg}{l}}$$

3.4 Let us choose coordinate axes as shown in the figure and fix three charges, q_1, q_2 and q_3 having position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively.



Now, for the equilibrium of q_3

$$\frac{+q_2 q_3 (\vec{r}_2 - \vec{r}_3)}{|\vec{r}_2 - \vec{r}_3|^3} + \frac{q_1 q_3 (\vec{r}_1 - \vec{r}_3)}{|\vec{r}_1 - \vec{r}_3|^3} = 0$$

or, $$\frac{q_2}{|\vec{r_2}-\vec{r_3}|^2} = \frac{q_1}{|\vec{r_1}-\vec{r_3}|^2}$$

because $$\frac{\vec{r_2}-\vec{r_3}}{|\vec{r_2}-\vec{r_3}|} = -\frac{\vec{r_1}-\vec{r_3}}{|\vec{r_1}-\vec{r_3}|}$$

or, $$\sqrt{q_2}\,(\vec{r_1}-\vec{r_3}) = \sqrt{q_1}\,(\vec{r_3}-\vec{r_2})$$

or, $$\vec{r_3} = \frac{\sqrt{q_2}\,\vec{r_1}+\sqrt{q_1}\,\vec{r_2}}{\sqrt{q_1}+\sqrt{q_2}}$$

Also for the equilibrium of q_1,

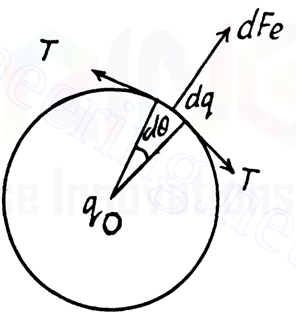
$$\frac{q_3(\vec{r_3}-\vec{r_1})}{|\vec{r_3}-\vec{r_1}|^3} + \frac{q_2(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3} = 0$$

or, $$q_3 = \frac{-q_2}{|\vec{r_2}-\vec{r_1}|^2}\,|\vec{r_1}-\vec{r_3}|^2$$

Substituting the value of $\vec{r_3}$, we get,

$$q_3 = \frac{-q_1 q_2}{(\sqrt{q_1}+\sqrt{q_2})^2}$$

3.5 When the charge q_0 is placed at the centre of the ring, the wire get stretched and the extra tension, produced in the wire, will balance the electric force due to the charge q_0. Let the tension produced in the wire, after placing the charge q_0, be T. From Newton's second law in projection form $F_n = mw_n$.



$$T\,d\theta - \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi r}\,r\,d\theta\right) = (dm)\,0,$$

or, $$T = \frac{q\,q_0}{8\pi^2\varepsilon_0 r^2}$$

3.6 Sought field strength

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r_0}|^2}$$

$= 4{\cdot}5$ kV/m on putting the values.

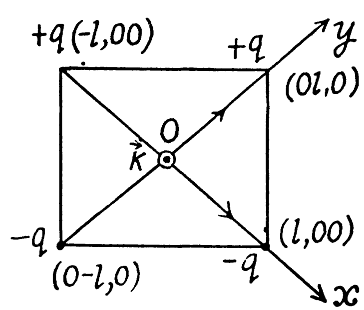
3.7 Let us fix the coordinate system by taking the point of intersection of the diagonals as the origin and let $\vec{k}$ be directed normally, emerging from the plane of figure.
Hence the sought field strength :

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{l\vec{i} + x\vec{k}}{(l^2 + x^2)^{3/2}} + \frac{-q}{4\pi\varepsilon_0}\frac{l(-\vec{i}\,) + x\,\vec{k}}{(l^2 + x^2)^{3/2}}$$

$$+ \frac{-q}{4\pi\varepsilon_0}\frac{l\vec{j} + x\vec{k}}{(l^2 + x^2)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{l(-\vec{j}\,) + x\,\vec{k}}{(l^2 + x^2)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(l^2 + x^2)^{3/2}}[2\,l\vec{i} - 2l\vec{j}\,]$$

Thus $E = \dfrac{ql}{\sqrt{2}\,\pi\,\varepsilon_0\,(l^2 + x^2)^{3/2}}$

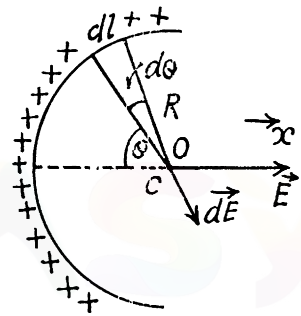


3.8 From the symmetry of the problem the sought field.

$$E = \int dE_x$$

where the projection of field strength along x – axis due to an elemental charge is

$$dE_x = \frac{dq\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{qR\cos\theta\,d\theta}{4\pi^2\varepsilon_0 R^3}$$



Hence

$$E = \frac{q}{4\pi^2\varepsilon_0 R^2}\int_{\pi/2}^{\pi/2}\cos\theta\,d\theta\;\frac{q}{2\pi^2\varepsilon_0 R^2}$$

3.9 From the symmetry of the condition, it is clear that, the field along the normal will be zero

i.e. $E_n = 0$ and $E = E_l$

Now $$dE_l = \frac{dq}{4\pi\varepsilon_0(R^2 + l^2)}\cos\theta$$

But $$dq = \frac{q}{2\pi R}dx \text{ and } \cos\theta = \frac{l}{(R^2 + l^2)^{1/2}}$$

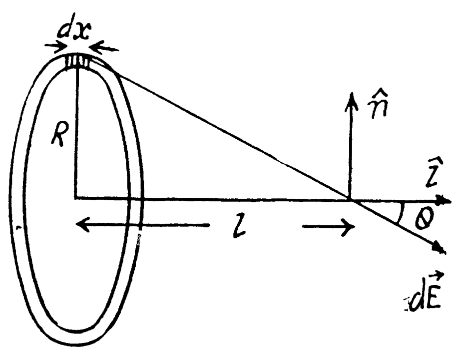
Hence

$$E = \int dE_l = \int_0^{2\pi R}\frac{ql}{2\pi R}\cdot\frac{dx}{4\pi\varepsilon_0(R^2 + l^2)^{3/2}}$$

or $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{ql}{(l^2 + R^2)^{3/2}}$

and for $l >> R$, the ring behaves like a point charge, reducing the field to the value,

$$E \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{l^2}$$



For E_{max}, we should have $\frac{dE}{dl} = 0$

So, $(l^2 + R^2)^{3/2} - \frac{3}{2} l (l^2 + R^2)^{1/2} 2l = 0$ or $l^2 + R^2 - 3l^2 = 0$

Thus $l = \frac{R}{\sqrt{2}}$ and $E_{max} = \frac{q}{6\sqrt{3}\,\pi\,\varepsilon_0 R^2}$

3.10 The electric potential at a distance x from the given ring is given by,

$$\varphi(x) = \frac{q}{4\pi\varepsilon_0 x} - \frac{q}{4\pi\varepsilon_0 (R^2 + x^2)^{1/2}}$$

Hence, the field strength along x-axis (which is the net field strength in our case),

$$E_x = -\frac{d\varphi}{dx} = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2} - \frac{qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[\left(1 + \frac{R^2}{x^2}\right)^{3/2} - 1\right]}{x^2 (R^2 + x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[1 + \frac{3}{2}\frac{R^2}{x^2} + \frac{3}{8}\frac{R^4}{x^4} + \dots\right]}{x^2 (R^2 + x^2)^{3/2}}$$

Neglecting the higher power of R/x, as $x >> R$.

$$E = \frac{3qR^2}{8\pi\varepsilon_0 x^4}.$$

Note : Instead of $\varphi(x)$, we may write $E(x)$ directly using 3.9

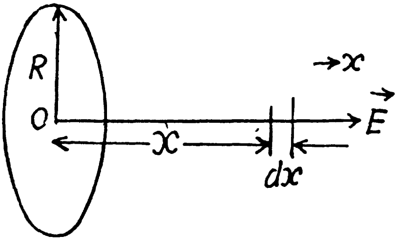
3.11 From the solution of 3.9, the electric field strength due to ring at a point on its axis (say x-axis) at distance x from the centre of the ring is given by :

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

And from symmetry $\vec{E}$ at every point on the axis is directed along the x–axis (Fig.).

Let us consider an element (dx) on thread which carries the charge $(\lambda\, dx)$. The electric force experienced by the element in the field of ring.

$$dF = (\lambda\, dx) E(x) = \frac{\lambda qx\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$



Thus the sought interaction

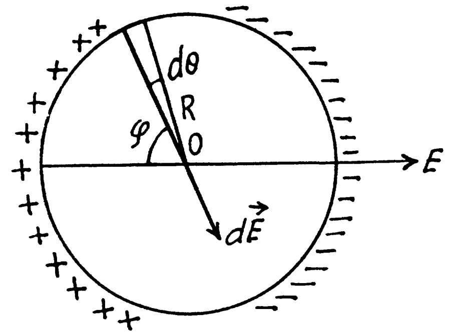
$$F = \int_0^\infty \frac{\lambda q x\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

On integrating we get, $F = \frac{\lambda q}{4\pi\varepsilon_0 R}$

3.12 (a) The given charge distribution is shown in Fig. The symmetry of this distribution implies that vector $\vec{E}$ at the point O is directed to the right, and its magnitude is equal to the sum of the projection onto the direction of $\vec{E}$ of vectors $d\vec{E}$ from elementary charges dq. The projection of vector $d\vec{E}$ onto vector $\vec{E}$ is

$$dE\cos\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\cos\varphi,$$

where $dq = \lambda R\, d\varphi = \lambda_0\, R\cos\varphi\, d\varphi$.



Integrating (1) over φ between 0 and 2π we find the magnitude of the vector E:

$$E = \frac{\lambda_0}{4\pi\varepsilon_0 R}\int_0^{2\pi}\cos^2\varphi\, d\varphi = \frac{\lambda_0}{4\varepsilon_0 R}.$$

It should be noted that this integral is evaluated in the most simple way if we take into account that $\langle\cos^2\varphi\rangle = 1/2$. Then

$$\int_0^{2\pi}\cos^2\varphi\, d\varphi = \langle\cos^2\varphi\rangle\, 2\pi = \pi.$$

(b) Take an element S at an azimuthal angle φ from the x-axis, the element subtending an angle $d\varphi$ at the centre.
The elementary field at P due to the element is

$$\frac{\lambda_0\cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0(x^2+R^2)}$$ along SP with components

$$\frac{\lambda_0\cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0(x^2+R^2)} \times \{\cos\theta \text{ along } OP, \sin\theta \text{ along } OS\}$$

where $$\cos\theta = \frac{x}{(x^2+R^2)^{1/2}}$$

The component along OP vanishes on integration as $\int_0^{2\pi}\cos\varphi\, d\varphi = 0$

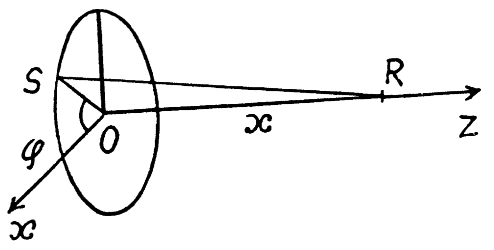
The component alon OS can be broken into the parts along OX and OY with

$$\frac{\lambda_0 R^2\cos\varphi\, d\varphi}{4\pi\varepsilon_0(x^2+R^2)^{3/2}} \times \{\cos\varphi \text{ along } OX, \sin\varphi \text{ along } OY\}$$

On integration, the part along OY vanishes.
Finally

$$E = E_x = \frac{\lambda_0 R^2}{4\varepsilon_0(x^2+R^2)^{3/2}}$$

For $x >> R$



$$E_x = \frac{p}{4\pi\varepsilon_0 x^3} \text{ where } p = \lambda_0\pi R^2$$

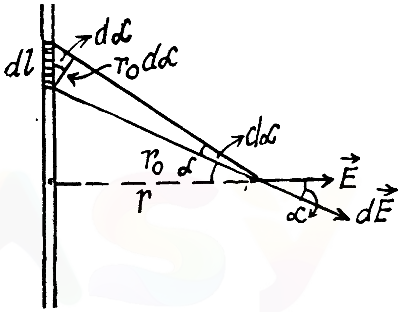
3.13 (a) It is clear from symmetry considerations that vector $\vec{E}$ must be directed as shown in the figure. This shows the way of solving this problem : we must find the component dE_r of the field created by the element dl of the rod, having the charge dq and then integrate the result over all the elements of the rod. In this case

$$dE_r = d\,E\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dl}{r_0^2}\cos\alpha,$$

where $\lambda = \frac{q}{2a}$ is the linear charge density. Let us reduce this equation of the form convenient for integration. Figure shows that $dl\cos\alpha = r_0\,d\alpha$ and $r_0 = \frac{r}{\cos\alpha}$;

Consequently,

$$dE_r = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,r_0\,d\alpha}{r_0^2} = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\alpha\,d\alpha$$



This expression can be easily integrated :

$$E = \frac{\lambda}{4\pi\varepsilon_0 r}\,2\int_0^{\alpha_0}\cos\alpha\,d\alpha = \frac{\lambda}{4\pi\varepsilon_0 r}\,2\sin\alpha_0$$

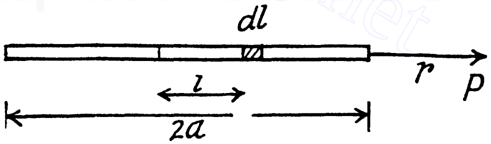
where α_0 is the maximum value of the angle α,

$$\sin\alpha_0 = a\Big/\sqrt{a^2+r^2}$$

Thus, $E = \frac{q/2a}{4\pi\varepsilon_0 r}\,2\frac{a}{\sqrt{a^2+r^2}} = \frac{q}{4\pi\varepsilon_0 r\sqrt{a^2+r^2}}$

Note that in this case also $E \approx \frac{q}{4\pi\varepsilon_0 r^2}$ *for* $r >> a$ *as of the field of a point charge.*

(b) Let, us consider the element of length dl at a distance l from the centre of the rod, as shown in the figure.



Then field at P, due to this element.

$$dE = \frac{\lambda\,dl}{4\pi\varepsilon_0 (r-l)^2},$$

if the element lies on the side, shown in the diagram, and $dE = \frac{\lambda\,dl}{4\pi\varepsilon_0 (r+l)^2}$, if it lies on other side.

Hence $$E = \int dE = \int_0^a \frac{\lambda\,dl}{4\pi\varepsilon_0 (r-l)^2} + \int_0^a \frac{\lambda\,dl}{4\pi\varepsilon_0 (r+l)^2}$$

On integrating and putting $\lambda = \frac{q}{2a}$, we get, $E = \frac{q}{4\pi\varepsilon_0}\frac{1}{(r^2-a^2)}$

For $r >> a$, $E \approx \frac{q}{4\pi\varepsilon_0 r^2}$

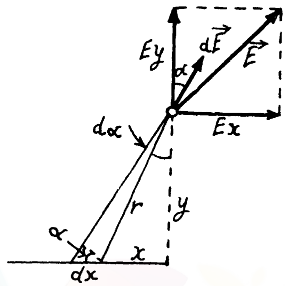
3.14 The problem is reduced to finding E_x and E_y viz. the projections of $\vec{E}$ in Fig, where it is assumed that $\lambda > 0$.

Let us start with E_x. The contribution to E_x from the charge element of the segment dx is

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx}{r^2}\sin\alpha \qquad (1)$$

Let us reduce this expression to the form convenient for integration. In our case, $dx = r\,d\alpha/\cos\alpha$, $r = y/\cos\alpha$. Then

$$dE_x = \frac{\lambda}{4\pi\varepsilon_0 y}\sin\alpha\,d\alpha.$$



Integrating this expression over α between 0 and $\pi/2$, we find

$$E_x = \lambda/4\pi\varepsilon_0 y.$$

In order to find the projection E_y it is sufficient to recall that dE_y differs from dE_x in that $\sin\alpha$ in (1) is simply replaced by $\cos\alpha$.

This gives

$dE_y = (\lambda\cos\alpha\,d\alpha)/4\pi\varepsilon_0 y$ and $E_y = \lambda/4\pi\varepsilon_0 y$.

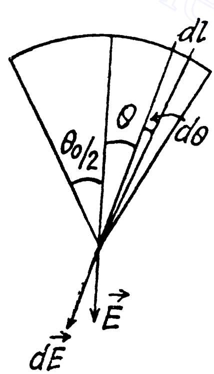
We have obtained an interesting result :

$E_x = E_y$ independently of y,

i.e. $\vec{E}$ is oriented at the angle of 45° to the rod. The modulus of $\vec{E}$ is

$$E = \sqrt{E_x^2 + E_y^2} = \lambda\sqrt{2}/4\pi\varepsilon_0 y.$$

3.15 (a) Using the solution of 3.14, the net electric field strength at the point O due to straight parts of the thread equals zero. For the curved part (arc) let us derive a general expression i.e. let us calculate the field strength at the centre of arc of radius R and linear charge density λ and which subtends angle θ_0 at the centre.



From the symmetry the sought field strength will be directed along the bisector of the angle θ_0 and is given by

$$E = \int_{-\theta_0/2}^{+\theta_0/2} \frac{\lambda\,(R\,d\theta)}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{\lambda}{2\pi\varepsilon_0 R}\sin\frac{\theta_0}{2}$$

In our problem $\theta_0 = \pi/2$, thus the field strength due to the turned part at the point

$E_0 = \dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}$ which is also the sought result.

(b) Using the solution of 3.14 (a), net field strength at O due to stright parts equals $\sqrt{2}\left(\dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}\right) = \dfrac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically down. Now using the solution of 3.15

(a), field strength due to the given curved part (semi-circle) at the point O becomes $\frac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically upward. Hence the sought net field strengh becomes zero.

3.16 Given charge distribution on the surface $\sigma = \vec{a}\cdot\vec{r}$ is shown in the figure. Symmetry of this distribution implies that the sought $\vec{E}$ at the centre O of the sphere is opposite to $\vec{a}$

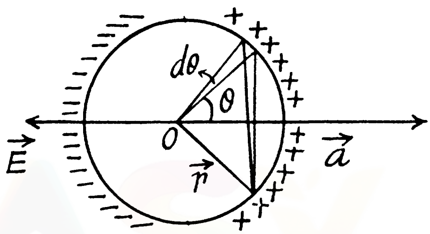
$dq = \sigma\,(2\pi r\sin\theta)\,r\,d\theta = (\vec{a}\cdot\vec{r})\,2\pi r^2\sin\theta\,d\theta = 2\pi a r^3\sin\theta\cos\theta\,d\theta$

Again from symmetry, field strength due to any ring element $d\vec{E}$ is also opposite to $\vec{a}$ i.e. $d\vec{E}\uparrow\downarrow\vec{a}$. Hence

$$d\vec{E} = \frac{dq\,r\cos\theta}{4\pi\varepsilon_0(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}}\,\frac{-\vec{a}}{a}\ \text{(Using the result of 3.9)}$$

$$= \frac{(2\pi a r^3\sin\theta\cos\theta\,d\theta)\,r\cos\theta}{4\pi\varepsilon_0 r^3}\,\frac{(-\vec{a})}{a}$$

$$= \frac{-\vec{a}\,r}{2\varepsilon_0}\sin\theta\cos^2 d\theta$$



Thus

$$\vec{E} = \int d\vec{E} = \frac{(-\vec{a})\,r}{2\varepsilon_0}\int_0^{\pi}\sin\theta\cos^2\theta\,d\theta$$

Integrating, we get $\vec{E} = -\frac{\vec{a}\,r}{2\varepsilon_0}\,\frac{2}{3} = -\frac{\vec{a}\,r}{3\varepsilon_0}$

3.17 We start from two charged spherical balls each of radius R with equal and opposite charge densities $+\rho$ and $-\rho$. The centre of the balls are at $+\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$ respectively so the equation of their surfaces are $\left|\vec{r} - \frac{\vec{a}}{2}\right| = R$ or $r - \frac{a}{2}\cos\theta \approx R$ and $r + \frac{a}{2}\cos\theta \approx R$, considering a to be small. The distance between the two surfaces in the radial direction at angle θ is $|\,a\cos\theta\,|$ and does not depend on the azimuthal angle. It is seen from the diagram that the surface of the sphere has in effect a surface density $\sigma = \sigma_0\cos\theta$ when

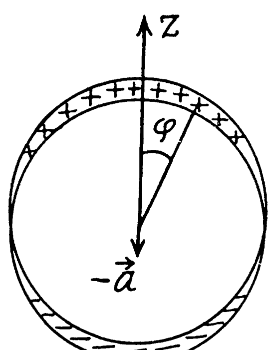
$$\sigma_0 = \rho a.$$

Inside any uniformly charged spherical ball, the field is radial and has the magnitude given by Gauss's theorm

$$4\pi r^2 E = \frac{4\pi}{3}r^3\rho/\varepsilon_0$$

or

$$E = \frac{\rho r}{3\varepsilon_0}$$



In vector notation, using the fact the V must be measured from the centre of the ball, we get, for the present case

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - \frac{\vec{a}}{2}\right) - \frac{\rho}{3\varepsilon_0}\left(\vec{r} + \frac{\vec{a}}{2}\right)$$

$$= -\rho \vec{a}/3\varepsilon_0 = \frac{\sigma_0}{3\varepsilon_0}\vec{k}$$

When $\vec{k}$ is the unit vector along the polar axis from which θ is measured.

3.18 Let us consider an elemental spherical shell of thickness dr. Thus surface charge density of the shell $\sigma = \rho\, dr = (\vec{a} \cdot \vec{r})\, dr$.

Thus using the solution of 3.16, field strength due to this sperical shell

$$d\vec{E} = -\frac{\vec{a}\, r}{3\,\varepsilon_0}\, dr$$

Hence the sought field strength

$$\vec{E} = -\frac{\vec{a}}{3\,\varepsilon_0}\int_0^R r\, dr = -\frac{\vec{a} R^2}{6\,\varepsilon_0}.$$

3.19 From the solution of 3.14 field strength at a perpendicular distance $r < R$ from its left end

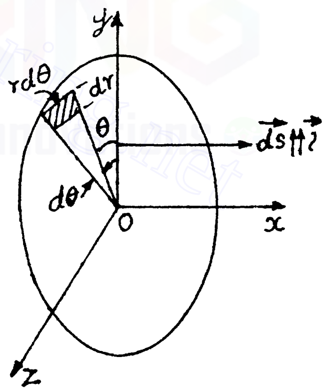
$$\vec{E}(r) = \frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)$$

Here $\hat{e}_r$ is a unit vector along radial direction.

Let us consider an elemental surface, $dS = dy\, dz = dz\,(r\, d\theta)$ a figure. Thus flux of $\vec{E}(r)$ over the element $d\vec{S}$ is given by

$$d\Phi = \vec{E} \cdot d\vec{S} = \left[\frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)\right] \cdot dr\,(r\, d\theta)\,\vec{i}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}\, dr\, d\theta \left(\text{as } \vec{e}_r \perp \vec{i}\right)$$



The sought flux, $\Phi = -\dfrac{\lambda}{4\pi\varepsilon_0}\displaystyle\int_0^R dr \int_0^{2\pi} d\theta = -\dfrac{\lambda R}{2\,\varepsilon_0}.$

If we have taken $d\vec{S} \uparrow\uparrow (-\vec{i})$, then Φ were $\dfrac{\lambda R}{2\,\varepsilon_0}$

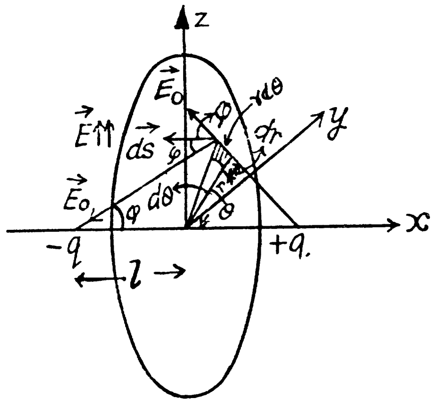
Hence $\qquad |\Phi| = \dfrac{\lambda R}{2\,\varepsilon_0}$

3.20 Let us consider an elemental surface area as shown in the figure. Then flux of the vector $\vec{E}$ through the elemental area,

$$d\Phi = \vec{E} \cdot d\vec{S} = E\, dS = 2E_0 \cos\varphi\, dS \;(\text{as } \vec{E} \uparrow\uparrow d\vec{S})$$

$$= \frac{2q}{4\pi\varepsilon_0 (l^2 + r^2)} \frac{l}{(l^2 + r^2)^{1/2}} (r\, d\theta)\, dr = \frac{2ql\, r\, dr\, d\theta}{4\pi\varepsilon_0 (r^2 + l^2)^{3/2}}$$

where $E_0 = \dfrac{q}{4\pi\varepsilon_0(l^2+r^2)}$ is magnitude of field strength due to any point charge at the point of location of considered elemental area.



$$\text{Thus } \Phi = \frac{2ql}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}}\int_0^{2\pi} d\theta$$

$$= \frac{2ql\times 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}} = \frac{q}{\varepsilon_0}\left[1-\frac{l}{\sqrt{l^2+R^2}}\right]$$

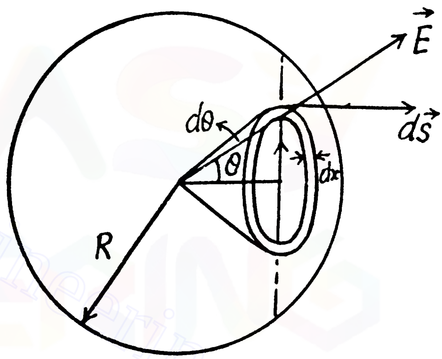
It can also be solved by considering a ring element or by using solid angle.

3.21 Let us consider a ring element of radius x and thickness dx, as shown in the figure. Now, flux over the considered element,

$$d\Phi = \vec{E}\cdot\vec{dS} = E_r\,dS\cos\theta$$

But $E_r = \dfrac{\rho r}{3\varepsilon_0}$ from Gauss's theorem,

and $\quad dS = 2\pi x\,dx$, $\cos\theta = \dfrac{r_0}{r}$



$$\text{Thus} \quad d\Phi = \frac{\rho r}{3\varepsilon_0}2\pi x\,dx\frac{r_0}{r} = \frac{\rho r_0}{3\varepsilon_0}2\pi x\,dx$$

Hence sought flux

$$\Phi = \frac{2\pi\rho r_0}{3\varepsilon_0}\int_0^{\sqrt{R^2-r_0^2}} x\,dx = \frac{2\pi\rho r_0}{3\varepsilon_0}\frac{(R^2-r_0^2)}{2} = \frac{\pi\rho r_0}{3\varepsilon_0}(R^2-r_0^2)$$

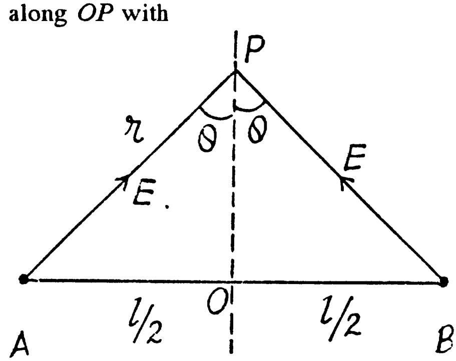
3.22 The field at P due to the threads at A and B are both of magnitude $\dfrac{\lambda}{2\pi\varepsilon_0(x^2+l^2/4)^{1/2}}$

and directed along AP and BP. The resultant is along OP with

$$E = \frac{2\lambda\cos\theta}{2\pi\varepsilon_0(\pi^2+\pi^{1/2})^{1/2}} = \frac{\lambda x}{\pi\varepsilon_0(x^2+l^2/4)}$$

$$= \frac{\lambda}{\pi\varepsilon_0\left[x+\dfrac{l^2}{4x}-2\cdot\dfrac{l}{2\sqrt{x}}\cdot\sqrt{x}+l\right]}$$

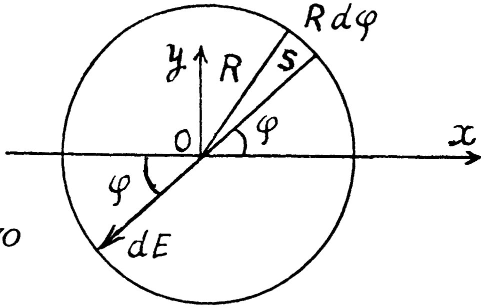
$$= \frac{\lambda}{\pi\varepsilon_0\left[\left(\sqrt{x}-\dfrac{l}{2\sqrt{x}}\right)^2+l\right]}$$



This is maximum when $x = l/2$ and then $E = E_{max} = \dfrac{\lambda}{\pi\varepsilon_0 l}$

3.23 Take a section of the cylinder perpendicular to its axis through the point where the electric field is to be calculated. (All points on the axis are equivalent.) Consider an element S with azimuthal angle φ. The length of the element is $R\,d\varphi$, R being the radius of cross section of the cylinder. The element itself is a section of an infinite strip. The electric field at O due to this strip is

$$\frac{\sigma_0 \cos\varphi\,(R\,d\varphi)}{2\pi\varepsilon_0 R} \text{ along } SO$$



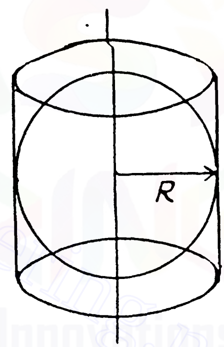
This can be resolved into

$$\frac{\sigma_0 \cos\varphi\, d\varphi}{2\pi\varepsilon_0}\begin{cases}\cos\varphi \text{ along } OX \text{ towards } O\\ \sin\varphi \text{ along } YO\end{cases}$$

On integration the component along YO vanishes. What remains is

$$\int_0^{2\pi}\frac{\sigma_0\cos^2\varphi\,d\varphi}{2\pi\varepsilon_0} = \frac{\sigma_0}{2\varepsilon_0} \text{ along } XO \text{ i.e. along the direction } \varphi = \pi.$$

3.24 Since the field is axisymmetric (as the field of a uniformly charged filament), we conclude that the flux through the sphere of radius R is equal to the flux through the lateral surface of a cylinder having the same radius and the height $2R$, as arranged in the figure.



Now, $\quad \Phi = \oint \vec{E}\cdot d\vec{S} = E_r S$

But $\quad E_r = \dfrac{a}{R}$

Thus $\quad \Phi = \dfrac{a}{R}S = \dfrac{a}{R}2\pi R\cdot 2R = 4\pi a R$

3.25 (a) Let us consider a sphere of radius $r < R$ then charge, inclosed by the considered sphere,

$$q_{inclosed} = \int_0^r 4\pi r^2\,dr\,\rho = \int_0^r 4\pi r^2 \rho_0\left(1-\frac{r}{R}\right)dr \qquad (1)$$

Now, applying Gauss' theorem,

$E_r\,4\pi r^2 = \dfrac{q_{inclosed}}{\varepsilon_0}$, (where E_r is the projection of electric field along the radial line.)

$$= \frac{\rho_0}{\varepsilon_0}\int_0^r 4\pi r^2\left(1-\frac{r}{R}\right)dr$$

or, $$E_r = \frac{\rho_0}{3\varepsilon_0}\left[r^2 - \frac{3r^2}{4R}\right]$$

And for a point. outside the sphere $r > R$.

$$q_{inclosed} = \int_0^R 4\pi r^2 \, d r \, \rho_0 \left(1 - \frac{r}{R}\right) \text{(as there is no charge outside the ball)}$$

Again from Gauss' theorem,

$$E_r \, 4\pi r^2 = \int_0^R \frac{4\pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right)}{\varepsilon_0}$$

or,
$$E_r = \frac{\rho_0}{r^2 \varepsilon_0} \left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{\rho_0 R^3}{12 r^2 \varepsilon_0}$$

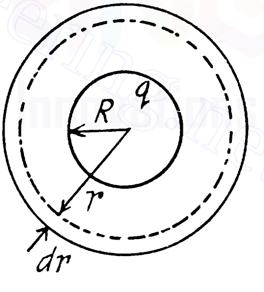
(b) As magnitude of electric field decreases with increasing r for $r > R$, field will be maximum for $r < R$. Now, for E_r to be maximum,

$$\frac{d}{dr}\left(r - \frac{3r^2}{4R}\right) = 0 \quad \text{or} \quad 1 - \frac{3r}{2R} = 0 \quad \text{or} \quad r = r_m = \frac{2R}{3}$$

Hence
$$E_{max} = \frac{\rho_0 R}{9 \varepsilon_0}$$

3.26 Let the charge carried by the sphere be q, then using Gauss' theorem for a spherical surface having radius $r > R$, we can write.

$$E \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0} + \frac{1}{\varepsilon_0} \int_R^r \frac{\alpha}{r} 4\pi r^2 \, dr$$



On integrating we get,

$$E \, 4\pi r^2 = \frac{(q - 2\pi\alpha R^2)}{\varepsilon_0} + \frac{4\pi\alpha r^2}{2\varepsilon_0}$$

The intensity E does not depend on r when the experession in the parentheses is equal to zero. Hence

$$q = 2\pi\alpha R^2 \text{ and } E = \frac{\alpha}{2\varepsilon_0}$$

3.27 Let us consider a spherical layer of radius r and thickness dr, having its centre coinciding with the centre of the system. Then using Gauss' theorem for this surface,

$$E_r \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \int_0^r \frac{\rho \, dV}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \int_0^r \rho_0 \, e^{-\alpha r^3} 4\pi r^2 \, dr$$

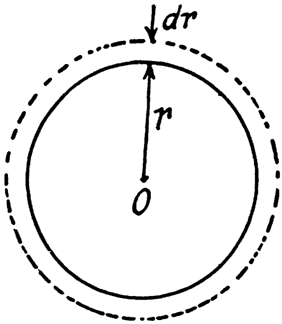
After integration

$$E_r 4\pi r^2 = \frac{\rho 4\pi}{3\varepsilon_0 \alpha}[1 - e^{-\alpha r^3}]$$

or, $$E_r = \frac{\rho_0}{3\varepsilon\alpha r^2}[1 - e^{-\alpha r^3}]$$

Now when $\alpha r^3 << 1$, $E_r \approx \frac{\rho_0 r}{3\varepsilon_0}$

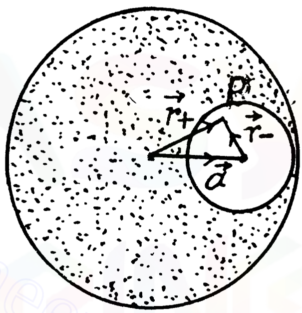
And when $\alpha r^3 >> 1$, $E_r \approx \frac{\rho_0}{3\varepsilon_0 \alpha r^2}$



3.28 Using Gauss theorem we can easily show that the electric field strength within a uniformly charged sphere is $\vec{E} = \left(\frac{\rho}{3\varepsilon_0}\right)\vec{r}$

The cavity, in our problem, may be considered as the superposition of two balls, one with the charge density ρ and the other with – ρ.

Let P be a point inside the cavity such that its position vector with respect to the centre of cavity be $\vec{r}_-$ and with respect to the centre of the ball $\vec{r}_+$. Then from the principle of superposition, field inside the cavity, at an arbitrary point P,



$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\varepsilon_0}\vec{a}$$

Note : Obtained expression for $\vec{E}$ shows that it is valid regardless of the ratio between the radii of the sphere and the distance between their centres.

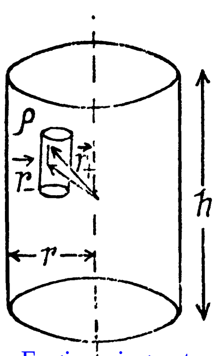
3.29 Let us consider a cylinderical Gaussian surface of radius r and height h inside an infinitely long charged cylinder with charge density ρ. Now from Gauss theorem :

$$E_r 2\pi r h = \frac{q_{inclosed}}{\varepsilon_0}$$

(where E_r is the field inside the cylinder at a distance r from its axis.)

or, $E_r 2\pi rh = \frac{\rho\pi r^2 h}{\varepsilon_0}$ or $E_r = \frac{\rho r}{2\varepsilon_0}$

Now, using the method of 3.28 field at a point P, inside the cavity, is



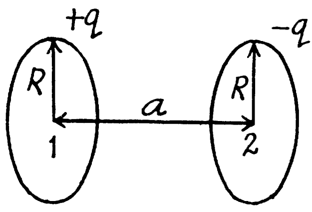
$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{2\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{2\varepsilon_0}\vec{a}$$

3.30 The arrangement of the rings are as shown in the figure. Now, potential at the point 1, φ_1 = potential at 1 due to the ring 1 + potential at 1 due to the ring 2.

$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$

Similarly, the potential at point 2,

$$\varphi_2 = \frac{-q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$



Hence, the sought potential difference,

$$\varphi_1 - \varphi_2 = \Delta\varphi = 2\left(\frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}\right)$$

$$= \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1+(a/R)^2}}\right)$$

3.31 We know from Gauss theorem that the electric field due to an infinietly long straight wire, at a perpendicular distance r from it equals, $E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$. So, the work done is

$$\int_1^2 E_r\,dr = \int_x^{\eta x} \frac{\lambda}{2\pi\varepsilon_0 r}\,dr$$

(where x is perpendicular distance from the thread by which point 1 is removed from it.)

Hence $$\Delta\varphi_{12} = \frac{\lambda}{2\pi\varepsilon_0}\ln\eta$$

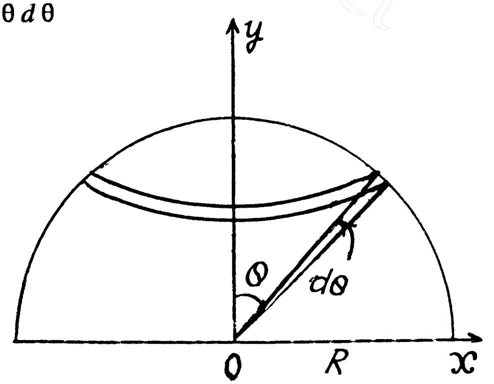
3.32 Let us consider a ring element as shown in the figure. Then the charge, carried by the element, $dq = (2\pi R\sin\theta)R\,d\theta\,\sigma$,

Hence, the potential due to the considered element at the centre of the hemisphere,

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{2\pi\sigma R\sin\theta\,d\theta}{4\pi\varepsilon_0} = \frac{\sigma R}{2\varepsilon_0}\sin\theta\,d\theta$$

So potential due to the whole hemisphere

$$\varphi = \frac{R\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\,d\theta = \frac{\sigma R}{2\varepsilon_0}$$

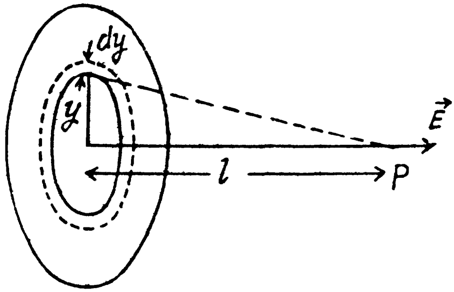


Now from the symmetry of the problem, net electric field of the hemisphere is directed towards the negative y-axis. We have

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq\cos\theta}{R^2} = \frac{\sigma}{2\varepsilon_0}\sin\theta\cos\theta\,d\theta$$

Thus $$E = E_y' = \frac{\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta\,d\theta = \frac{\sigma}{4\varepsilon_0},\ \text{along } YO$$

3.33 Let us consider an elementary ring of thickness dy and radius y as shown in the figure. Then potential at a point P, at distance l from the centre of the disc, is



$$d\varphi = \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}}$$

Hence potential due to the whole disc,

$$\varphi = \int_0^R \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}} = \frac{\sigma l}{2\varepsilon_0}\left(\sqrt{1 + (R/l)^2} - 1\right)$$

From symmetry

$$E = E_l = -\frac{d\varphi}{dl}$$

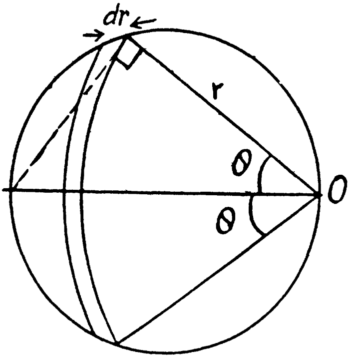
$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2l}{2\sqrt{R^2 + l^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right]$$

when $l \to 0$, $\varphi = \frac{\sigma R}{2\varepsilon_0}$, $E = \frac{\sigma}{2\varepsilon_0}$ and when $l >> R$,

$$\varphi \approx \frac{\sigma R^2}{4\varepsilon_0 l},\quad E = \frac{\sigma R^2}{4\varepsilon_0 l^2}$$

3.34 By definition, the potential in the case of a surface charge distribution is defined by integral $\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r}$. In order to simplify integration, we shall choose the area element dS in the form of a part of the ring of radius r and width dr in (Fig.). Then $dS = 2\theta\, r\, dr$, $r = 2R\cos\theta$ and $dr = -2R\sin\theta\, d\theta$. After substituting these expressions into integral $\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r}$, we obtain the expression for φ at the point O:

$$\varphi = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\pi/2}^{0} \theta \sin\theta\, d\theta.$$



We integrate by parts, denoting $\theta = u$ and $\sin\theta\, d\theta = dv$:

$$\int \theta\sin\theta\, d\theta = -\theta\cos\theta$$

$$+\int \cos\theta\, d\theta = -\theta\cos\theta + \sin\theta$$

which gives -1 after substituting the limits of integration. As a result, we obtain

$$\varphi = \sigma R/\pi\varepsilon_0.$$

3.35 In accordance with the problem $\varphi = \vec{a} \cdot \vec{r}$

Thus from the equation : $\vec{E} = -\vec{\nabla}\varphi$

$$\vec{E} = -\left[\frac{\partial}{\partial x}(a_x x)\,\vec{i} + \frac{\partial}{\partial_y}(a_y y)\,\vec{j} + \frac{\partial}{\partial_z}(a_z z)\,\vec{k}\right] = -[a_x \vec{i} + a_y \vec{j} + {}_z \vec{k}\,] = -\vec{a}$$

3.36 (a) Given, $\varphi = a(x^2 - y^2)$

So, $$\vec{E} = -\vec{\nabla}\varphi = -2a(x\vec{i} - y\vec{j}\,)$$

The sought shape of field lines is as shown in the figure (a) of answersheet assuming $a > 0$:

(b) Since $\varphi = axy$

So, $$\vec{E} = -\vec{\nabla}\varphi = -ay\,\vec{i} - ax\,\vec{j}$$

Plot as shown in the figure (b) of answersheet.

3.37 Given, $\varphi = a(x^2 + y^2) + bz^2$

So, $$\vec{E} = -\vec{\nabla}\varphi = -[2\,ax\,\vec{i} + 2\,ay\,\vec{j} + 2\,bz\,\vec{k}]$$

Hence $$|\vec{E}| = 2\sqrt{a^2(x^2 + y^2) + b^2 z^2}$$

Shape of the equipotential surface :

Put $$\vec{\rho} = x\,\vec{i} + y\,\vec{j} \quad \text{or} \quad \rho^2 = x^2 + y^2$$

Then the equipotential surface has the equation

$$a\,\rho^2 + b\,z^2 = \text{constant} = \varphi$$

If $a > 0,\ b > 0$ then $\varphi > 0$ and the equation of the equipotential surface is

$$\frac{\rho}{\varphi/a} + \frac{z^2}{\varphi/b} = 1$$

which is an ellipse in ρ, z coordinates. In three dimensions the surface is an ellipsoid of revolution with semi- axis $\sqrt{\varphi/a}$, $\sqrt{\varphi/a}$, $\sqrt{\varphi/b}$.

If $a > 0,\ b < 0$ then φ can be ≥ 0. If $\varphi > 0$ then the equation is

$$\frac{\rho^2}{\varphi/a} - \frac{z^2}{\varphi/|b|} = 1$$

This is a single cavity hyperboloid of revolution about z axis. If $\varphi = 0$ then

$$a\rho^2 - |b|\,z^2 = 0$$

or $$z = \pm\sqrt{\frac{a}{|b|}}\,\rho$$

is the equation of a right circular cone.

If $\varphi < 0$ then the equation can be written as

$$|b|\,z^2 - a\,\rho^2 = |\varphi|$$

or $$\frac{z^2}{|\varphi|\,/|b|} - \frac{\rho^2}{|\varphi|\,/a} = 1$$

This is a two cavity hyperboloid of revolution about z-axis.

3.38 From Gauss' theorem intensity at a point, inside the sphere at a distance r from the centre is given by, $E_r = \dfrac{\rho\, r}{3\,\varepsilon_0}$ and outside it, is given by $E_r = \dfrac{1}{4\,\pi\,\varepsilon_0}\dfrac{q}{r^2}$.

(a) Potential at the centre of the sphere,

$$\varphi_0 = \int_0^\infty \vec{E}\cdot d\vec{r} = \int_0^R \frac{\rho r}{3\,\varepsilon_0}\,dr + \int_R^\infty \frac{q}{4\,\pi\,\varepsilon_0\, r^2}\,dr = \frac{\rho}{3\,\varepsilon}\,\frac{R^2}{2} + \frac{q}{4\,\pi\,\varepsilon_0\, R}$$

as $$= \frac{q}{8\,\pi\,\varepsilon_0\, R} + \frac{q}{4\,\pi\,\varepsilon_0\, R} = \frac{3q}{8\,\pi\,\varepsilon_0\, R} \quad \left(\text{as } \rho = \frac{3\,q}{4\,\pi\, R^3}\right)$$

(b) Now, potential at any point, inside the sphere, at a distance r from it s centre.

$$\varphi(r) = \int_r^R \frac{\rho}{3\,\varepsilon_0}\, r\, dr \int_r^\infty \frac{q}{4\,\pi\,\varepsilon_0}\,\frac{dr}{r^2}$$

On integration : $\varphi(r) = \dfrac{3\,q}{8\,\pi\,\varepsilon_0\, R}\left[1 - \dfrac{r^2}{3\, R^2}\right] = \varphi_0\left[1 - \dfrac{r^2}{3\, R^2}\right]$

3.39 Let two charges $+q$ and $-q$ be separated by a distance l. Then electric potential at a point at distance $r >> l$ from this dipole,

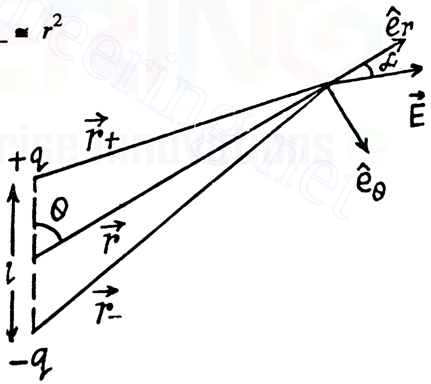
$$\varphi(r) = \frac{+q}{4\,\pi\,\varepsilon_0\, r_+} + \frac{-q}{4\,\pi\,\varepsilon_0\, r_-} = \frac{q}{4\,\pi\,\varepsilon_0}\left(\frac{r_- - r_+}{r_+\, r_-}\right) \qquad (1)$$

But $\quad r_- - r_+ = l\cos\theta$ and $r_+\, r_- = r^2$

From Eqs. (1) and (2),

$$\varphi(r) = \frac{q\, l\cos\theta}{4\,\pi\,\varepsilon_0\, r^2} = \frac{p\cos\theta}{4\,\pi\,\varepsilon_0\, r^2}\ \varphi = \frac{\vec{p}\cdot\vec{r}}{4\,\pi\,\varepsilon_0\, r^3},$$

where p is magnitude of electric moment vector.



Now, $E_r = -\dfrac{\partial\varphi}{\partial r} = \dfrac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\, r^3}$

and $E_\theta = -\dfrac{\partial\varphi}{r\,\partial\,\theta} = \dfrac{p\sin\theta}{4\,\pi\,\varepsilon_0\, r^3}$

So $\quad E = \sqrt{E_r^2 + E_0^2} = \dfrac{p}{4\,\pi\,\varepsilon_0\, r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$

3.40 From the results, obtained in the previous problem,

$$E_r = \frac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\, r^3} \text{ and } E_\theta = \frac{p\sin\theta}{4\,\pi\,\varepsilon_0\, r^3}$$

From the given figure, it is clear that,

$$E_z = E_r\cos\theta - E_\theta\sin\theta = \frac{p}{4\,\pi\,\varepsilon_0\, r^3}(3\cos^2\theta - 1)$$

and $$E_\perp = E_r \sin\theta + E_0 \cos\theta = \frac{3\,p \sin\theta \cos\theta}{4\pi\varepsilon_0 r^3}$$

When $\vec{E} \perp \vec{p}$, $|\vec{E}| = E_\perp$ and $E_z = 0$

So $$3\cos^2\theta = 1 \text{ and } \cos\theta = \frac{1}{\sqrt{3}}$$

Thus $\vec{E}_\perp \vec{p}$ at the points located on the lateral surface of the cone, having its axis, coinciding with the direction of z–axis and semi vertex angle $\theta = \cos^{-1} 1/\sqrt{3}$.

3.41 Let us assume that the dipole is at the centre of the one equipotential surface which is spherical (Fig.). On an equipotential surface the net electric field strength along the tangent of it becomes zero. Thus

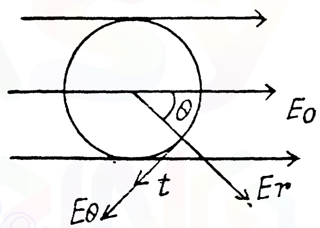
$$-E_0 \sin\theta + E_\theta = 0 \quad \text{or} \quad -E_0 \sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} = 0$$

Hence $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

Alternate : Potential at the point, near the dipole is given by,

$$\varphi = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} - \vec{E}_0 \cdot \vec{r} + \text{constant},$$

$$= \left(\frac{p}{4\pi\varepsilon_0 r^3} - E_0\right)\cos\theta + \text{Const}$$



For φ to be constant,

$$\frac{p}{4\pi\varepsilon_0 r^3} - E_0 = 0 \quad \text{or,} \quad \frac{p}{4\pi\varepsilon_0 r^3} = E_0$$

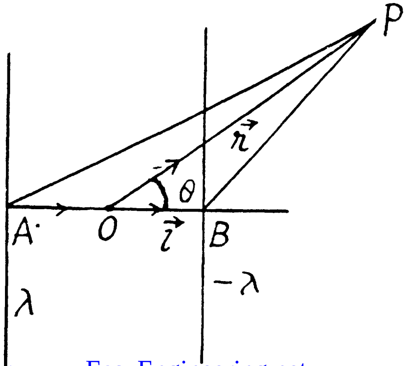
Thus $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

3.42 Let P be a point, at distace $r >> l$ and at an angle to θ the vector $\vec{l}$ (Fig.).

Thus $$\vec{E} \text{ at } P = \frac{\lambda}{2\pi\varepsilon_0} \frac{\vec{r} + \frac{\vec{l}}{2}}{\left|\vec{r} + \frac{\vec{l}}{2}\right|^2} - \frac{\lambda}{2\pi\varepsilon_0} \frac{\vec{r} - \frac{\vec{l}}{2}}{\left|\vec{r} - \frac{\vec{l}}{2}\right|^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{\vec{r} + \vec{l}/2}{r^2 + \frac{l^2}{4} + r\,l\cos\theta} - \frac{\vec{r} - \vec{l}/2}{r^2 + \frac{l^2}{4} - r\,l\cos\theta}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\vec{l}}{r^2} - \frac{2\,l\,\vec{r}}{r^3}\cos\theta\right)$$



Hence $$E = |\vec{E}| = \frac{\lambda\,l}{2\pi\varepsilon_0 r^2}, \quad r >> l$$

Also,
$$\varphi = \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}+\vec{l}/2| - \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}-\vec{l}/2]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2+rl\cos\theta+l^2/4}{r^2-rl\cos\theta+l^2/4} = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0 r},\ r>>l$$

3.43 The potential can be calculated by superposition. Choose the plane of the upper ring as $x = l/2$ and that of the lower ring as $x = -l/2$.

Then
$$\varphi = \frac{q}{4\pi\varepsilon_0[R^2+(x-l/2)^2]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+(x+l/2)^2]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0[R^2+x^2-lx]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+x^2+lx]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1+\frac{lx}{2(R^2+x^2)}\right) - \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1-\frac{lx}{2(R^2+x^2)}\right)$$

$$= \frac{q\,lx}{4\pi\varepsilon_0(R^2+x^2)^{3/2}}$$

For
$$|x|>>R,\ \varphi = \frac{q\,l}{4\pi x^2}$$

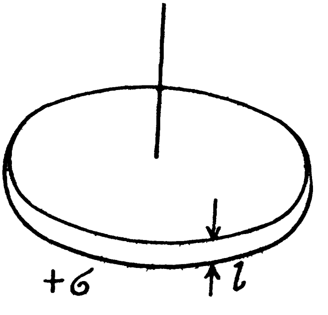
The electric field is $E = -\frac{\partial\varphi}{\partial x}$

$$= -\frac{ql}{4\pi\varepsilon_0(R^2+x^2)^{3/2}} + \frac{3}{2}\frac{ql}{(R^2+x^2)^{5/2}\,4\pi\varepsilon_0}\times 2x = \frac{ql(2x^2-R^2)}{4\pi\varepsilon_0(R^2+x^2)^{5/2}}$$

For
$$|x|>>R,\ E = \frac{ql}{2\pi\varepsilon_0 x^3}.$$
The plot is as given in the book.

3.44 The field of a pair of oppositely charged sheets with holes can by superposition be reduced to that of a pair of uniform opposite charged sheets and discs with opposite charges. Now the charged sheets do not contribute any field outside them. Thus using the result of the previous problem

$$\varphi = \int_0^R \frac{(-\sigma)\,l\,2\pi r\,dr\,x}{4\pi\varepsilon_0(r^2+x^2)^{3/2}}$$



$$= -\frac{\sigma xl}{4\varepsilon_0}\int_{x^2}^{R^2+x^2}\frac{dy}{y^{3/2}} = \frac{\sigma xl}{2\varepsilon_0\sqrt{R^2+x^2}}$$

$$E_x = -\frac{\partial\varphi}{\partial x} = -\frac{\sigma l}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+x^2}} - \frac{x^2}{(R^2+x^2)^{3/2}}\right] = -\frac{\sigma l R^2}{2\varepsilon_0(R^2+x^2)^{3/2}}.$$

The plot is as shown in the answersheet.

3.45 For $x > 0$ we can use the result as given above and write

$$\varphi \approx \pm \frac{\sigma l}{2\varepsilon_0}\left(1 - \frac{|x|}{(R^2 + x^2)^{1/2}}\right)$$

for the solution that vanishes at α. There is a discontinuity in potential for $|x| = 0$. The solution for negative x is obtained by $\sigma \to -\sigma$. Thus

$$\varphi = -\frac{\sigma l x}{2\varepsilon_0 (R + x^2)^{1/2}} + \text{constant}$$

Hence ignoring the jump

$$E = -\frac{\partial\varphi}{\partial x} = \frac{\sigma l R^2}{2\varepsilon_0 (R^2 + x^2)^{3/2}}$$

for large $|x|$ $\quad \varphi \approx \pm \dfrac{p}{4\pi\varepsilon_0 x^2}$ and $E \approx \dfrac{p}{2\pi\varepsilon_0 |x|^3}$ (where $p = \pi R^2 \sigma l$)

3.46 Here $E_r = \dfrac{\lambda}{2\pi\varepsilon_0 r}$, $E_\theta = E_\varphi = 0$ and $\vec{F} = p\dfrac{\partial \vec{E}}{\partial l}$

(a) $\vec{p}$ along the thread.

$\vec{E}$ does not change as the point of observation is moved along the thread.

$$\vec{F} = 0$$

(b) $\vec{p}$ along $\vec{r}$,

$$\vec{F} = F_r \vec{e}_r = -\frac{\lambda p}{2\pi\varepsilon_0 r^2}\vec{e}_r = -\frac{\lambda \vec{p}}{2\pi\varepsilon_0 r^2}\left(\text{On using } \frac{\partial}{\partial r}\vec{e}_r = 0\right)$$

(c) $\vec{p}$ along $\vec{e}_\theta$

$$\vec{F} = p\frac{\partial}{r\,\partial\theta}\frac{\lambda}{2\pi\varepsilon_0 r}e_r$$

$$= \frac{p\lambda}{2\pi\varepsilon_0 r^2}\frac{\partial \vec{e}_r}{\partial\theta} = \frac{p\lambda}{2\pi\varepsilon_0 r^2}\vec{e}_\theta = \frac{\vec{p}\lambda}{2\pi\varepsilon_0 r^2}.$$

3.47 Force on a dipole of moment p is given by,

$$F = \left|\varphi\frac{\partial \vec{E}}{\partial l}\right|$$

In our problem, field, due to a dipole at a distance l, where a dipole is placed,

$$|\vec{E}| = \frac{p}{2\pi\varepsilon_0 l^3}$$

Hence, the force of interaction,

$$F = \frac{3p^2}{2\pi\varepsilon_0 l^4} = 2{\cdot}1 \times 10^{-16}\,\text{N}$$

3.48 $-d\varphi = \vec{E}\cdot d\vec{r} = a(y\,dx + x\,dy) = a\,d(xy)$

On integrating, $\quad \varphi = -a\,xy + C$

3.49 $-d\varphi = \vec{E}\cdot d\vec{r} = [2a\,xy\,\vec{i} + 2(x^2 - y^2)\vec{j}]\cdot[dx\,\vec{i} + dy\,\vec{j}]$

or, $\quad d\varphi = 2a\,x\,y\,dx + a(x^2 - y^2)\,dy = ad(x^2 y) - ay^2\,dy$

On integrating, we get,

$$\varphi = ay\left(\frac{y^2}{3} - x^2\right) + C$$

3.50 Given, again

$$-d\varphi = \vec{E}\cdot d\vec{r} = (ay\vec{i} + (ax + bz)\vec{j} + by\,\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dx\,\vec{k})$$

$$= a\,(y\,dx + ax\,dy) + b\,(z\,dy + y\,dz) = ad\,(x\,y) + bd\,(yz)$$

On integrating,

$$\varphi = -(a\,xy + b\,yz) + C$$

3.51 Field intensity along x-axis.

$$E_x = -\frac{\partial\varphi}{\partial x} = 3\,a\,x^2 \qquad (1)$$

Then using Gauss's theorem in differential from

$$\frac{\partial E_x}{\partial x} = \frac{\rho\,(x)}{\varepsilon_0} \quad \text{so,}\ \ \rho\,(x) = 6a\,\varepsilon_0\,x.$$

3.52 In the space between the plates we have the Poisson equation

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\rho_0}{\varepsilon_0}$$

or,
$$\varphi = -\frac{\rho_0}{2\,\varepsilon_0}x^2 + A\,x + B$$

where ρ_0 is the constant space charge density between the plates.

We can choose $\varphi\,(0) = 0$ so $B = 0$

Then
$$\varphi\,(d) = \Delta\,\varphi = Ad - \frac{\rho_0\,d^2}{2\,\varepsilon_0} \quad \text{or,}\quad A = \frac{\Delta\,\varphi}{d} + \frac{\rho_0\,d}{2\,\varepsilon_0}$$

Now
$$E = -\frac{\partial\,\varphi}{\partial\,x} = \frac{\rho_0}{\varepsilon_0}x - A = 0 \quad \text{for } x = 0$$

if
$$A = \frac{\Delta\,\varphi}{d} + \frac{\rho_0\,d}{2\,\varepsilon_0} = 0$$

then
$$\rho_0 = -\frac{2\,\varepsilon_0\,\Delta\,\varphi}{d^2}$$

Also
$$E\,(d) = \frac{\rho_0\,d}{\varepsilon_0}.$$

3.53 Field intensity is along radial line and is

$$E_r = -\frac{\partial\,\varphi}{\partial r} = -2\,a\,r \qquad (1)$$

From the Gauss' theorem,

$$4\,\pi\,r^2\,E_r = \int\frac{dq}{\varepsilon_0},$$

where dq is the charge contained between the sphere of radii r and $r + dr$.

Hence
$$4\,\pi\,r^2\,E_r = 4\,\pi\,r^2 \times (-2ar) = \frac{4\pi}{\varepsilon_0}\int_0^r r'^2\,\rho\,(r')\,dr' \qquad (2)$$

Differentiating (2) $\rho = -6\,\varepsilon_0\,a$

3.2 CONDUCTORS AND DIELECTRICS IN AN ELECTRIC FIELD

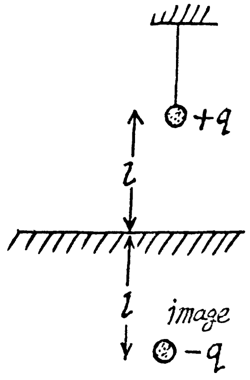
3.54 When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force, exerted, on the ball due to the extension in the spring.

Thus $F_{el} = F_{spr}$

or, $\dfrac{q^2}{4\pi\varepsilon_0 (2l)^2} = \kappa x$, (The force on the charge q might be considered as arised from attraction by the electrical image)

or, $q = 4l\sqrt{\pi\varepsilon_0 \kappa x}$,

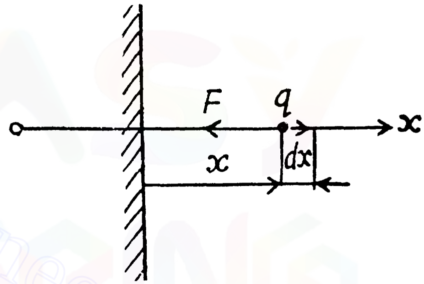
sought charge on the sphere.



3.55 By definition, the work of this force done upon an elementry displacement dx (Fig.) is given by

$$dA = F_x\, dx = -\frac{q^2}{4\pi\varepsilon_0 (2x)^2}\, dx,$$

where the expression for the force is obtained with the help of the image method. Integrating this equation over x between l and ∞, we find

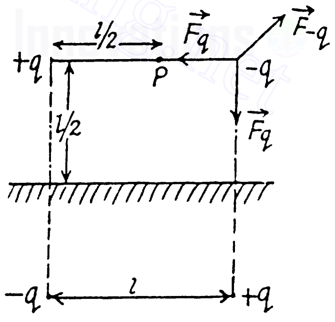


$$A = -\frac{q^2}{16\pi\varepsilon_0}\int_l^{\infty}\frac{dx}{x^2} = -\frac{q^2}{16\pi\varepsilon_0 l}.$$

3.56 (a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,

$$|\vec{F}| = \sqrt{2}\,\frac{q^2}{4\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}\,l)^2}$$

$$= \frac{q^2}{8\pi\varepsilon_0 l^2}(2\sqrt{2} - 1)$$



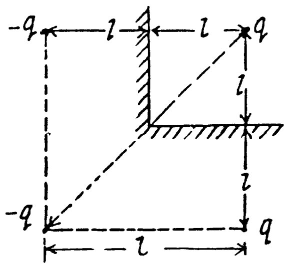
(b) Also, from the figure, magnitude of electrical field strength at P

$$E = 2\left(1 - \frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 l^2}$$

3.57 Using the concept of electrical image, it is easily seen that the force on the charge q is,

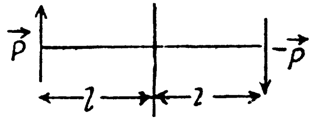
$$F = \frac{\sqrt{2}\,q^2}{4\pi\varepsilon_0 (2l)^2} + \frac{(-q)^2}{4\pi\varepsilon_0 (2\sqrt{2}\,l)^2}$$

$$= \frac{(2\sqrt{2} - 1)\,q^2}{32\pi\varepsilon_0 l^2} \quad \text{(It is attractive)}$$



3.58 Using the concept of electrical image, force on the dipole $\vec{p}$,

$\vec{F} = p \frac{\partial \vec{E}}{\partial l}$, where $\vec{E}$ is field at the location of $\vec{p}$ due to $(-\vec{p})$

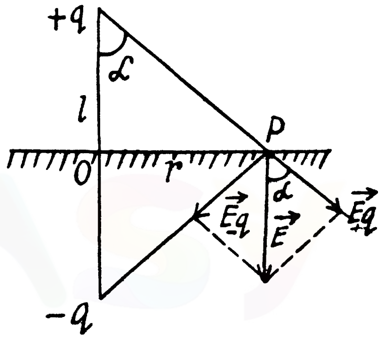


or, $|\vec{F}| = \left|\frac{\partial \vec{E}}{\partial l}\right| p = \frac{3p^2}{32\pi\varepsilon_0 l^4}$

as, $|\vec{E}| = \frac{p}{4\pi\varepsilon_0 (2l)^3}$

3.59 To find the surface charge density, we must know the electric field at the point P (Fig.) which is at a distance r from the point O.

Using the image mirror method, the field at P,



$$E = 2E\cos\alpha = 2\frac{q}{4\pi\varepsilon_0 x^2}\frac{l}{x} = \frac{ql}{2\pi\varepsilon_0 (l^2+r^2)^{3/2}}$$

Now from Gauss' theorem the surface charge density on conductor is connected with the electric field near its surface (in vaccum) through the relation $\sigma = \varepsilon_0 E_n$, where E_n is the projection of $\vec{E}$ onto the outward normal $\vec{n}$ (with respect to the conductor).

As our field strength $\vec{E} \uparrow\downarrow \vec{n}$, so

$$\sigma = -\varepsilon_0 E = -\frac{ql}{2\pi(l^2+r^2)^{3/2}}$$

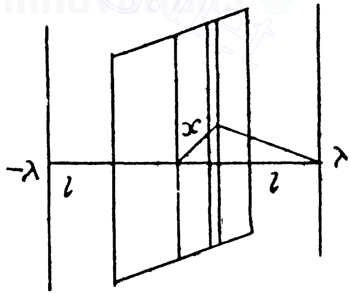
3.60 (a) The force F_1 on unit length of the thread is given by

$$F_1 = \lambda E_1$$

where E_1 is the field at the thread due to image charge :

$$E_1 = \frac{-\lambda}{2\pi\varepsilon_0 (2l)}$$

Thus $$F_1 = \frac{-\lambda^2}{4\pi\varepsilon_0 l}$$



minus singn means that the force is one of attraction.

(b) There is an image thread with charge density $-\lambda$ behind the conducting plane. We calculate the electric field on the conductor. It is

$$E(x) = E_n(x) = \frac{\lambda l}{\pi\varepsilon_0 (x^2+l^2)}$$

on considering the thread and its image.

Thus

$$\sigma(x) = \varepsilon_0 E_n = \frac{\lambda l}{\pi(x^2+l^2)}$$

3.61 (a) At O,

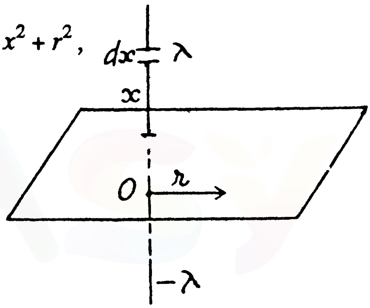
$$E_n(O) = 2\int_l^\infty \frac{\lambda\, dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{2\pi\varepsilon_0 l}$$

So $\quad \sigma(O) = \varepsilon_0 E_n = \frac{\lambda}{2\pi l}$

(b) $$E_n(r) = 2\int_l^\infty \frac{\lambda\, dx}{4\pi\varepsilon_0 (x^2+r^2)} \frac{x}{(x^2+r^2)^{1/2}} = \frac{\lambda}{2\pi\varepsilon_0}\int_l^\infty \frac{x\, dx}{(x^2+r^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{l^2+r^2}^\infty \frac{dy}{y^{3/2}}, \text{ on putting } y = x^2+r^2,$$

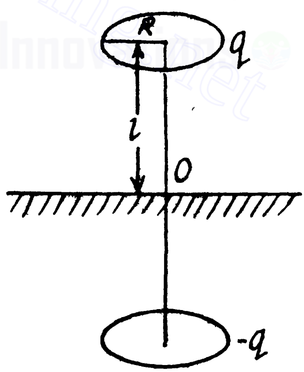
$$= \frac{\lambda}{2\pi\varepsilon_0\sqrt{l^2+r^2}}$$



Hence $\quad \sigma(r) = \varepsilon_0 E_n = \frac{\lambda}{2\pi\sqrt{l^2+r^2}}$

3.62 It can be easily seen that in accordance with the image method, a charge $-q$ must be located on a similar ring but on the other side of the conducting plane. (Fig.) at the same perpendicular distance. From the solution of 3.9 net electric field at O,

$\vec{E} = 2\frac{ql}{4\pi\varepsilon_0 (R^2+l^2)^{3/2}}(-\vec{n})$ where $\vec{n}$ is outward normal with respect to the conducting plane.



Now $\quad E_n = \frac{\sigma}{\varepsilon_0}$

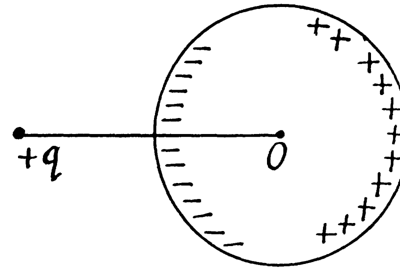
Hence $\quad \sigma = \frac{-ql}{2\pi(R^2+l^2)^{3/2}}$

where minus sign indicates that the induced carge is opposite in sign to that of charge $q > 0$.

3.63 Potential φ is the same for all the points of the sphere. Thus we calculate its value at the centre O of the sphere. Thus we can calculate its value at the centre O of the sphere, because only for this point, it can be calculated in the most simple way.

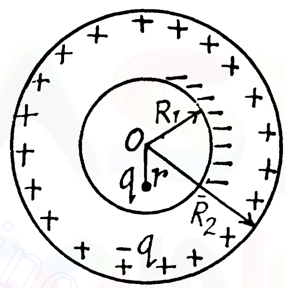
$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l} + \varphi' \tag{1}$$

where the first term is the potential of the charge q, while the second is the potential due to the charges induced on the surface of the sphere. But since all induced charges are at the same distance equal to the radius of the circle from the point C and the total induced charge is equal to zero, $\varphi' = 0$, as well. Thus equation (1) is reduced to the form,



$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}$$

3.64 As the sphere has conducting layers, charge $-q$ is induced on the inner surface of the sphere q and consequently charge $+q$ is induced on the outer layer as the sphere as a whole is uncharged.



Hence, the potential at O is given by,

$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{(-q)}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_0 R_2}$$

It should be noticed that the potential can be found in such a simple way only at O, since all the induced charges are at the same distance from this point, and their distribution, (which is unknown to us), does not play any role.

3.65 Potential at the inside sphere,

$$\varphi_a = \frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 b}$$

Obviously $\qquad \varphi_a = 0 \text{ for } q_2 = -\frac{b}{a}q_1 \qquad (1)$

When $\qquad r \geq b,$

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 r} = \frac{q_1}{4\pi\varepsilon_0}\left(1 - \frac{b}{a}\right)\Big/ r, \text{ using Eq. (1).}$$

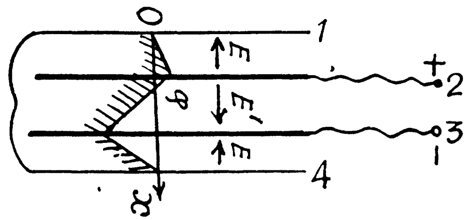
And when $r \leq b$

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 b} = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right)$$

3.66 (a) As the metallic plates 1 and 4 are isolated and conncted by means of a conductor, $\varphi_1 = \varphi_4$. Plates 2 and 3 have the same amount of positive and negative charges and due to induction, plates 1 and 4 are respectively negatively and positively charged and in addition to it all the four plates are located a small but at equal distance d relative to each other, the magnitude of electric field strength between 1 - 2 and 3 - 4 are both equal in magnitude and direction (say $\vec{E}$). Let $\vec{E'}$ be the field strength between the plates 2 and 3, which is directed form 2 to 3. Hence $\vec{E'} \uparrow\downarrow \vec{E}$ (Fig.).

According to the problem

$$E' d = \Delta\varphi = \varphi_2 - \varphi_3 \qquad (1)$$



In addition to

$$\varphi_1 - \varphi_4 = 0 = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_4)$$

or, $$0 = -Ed + \Delta\varphi - Ed$$

or, $$\Delta\varphi = 2Ed \text{ or } E = \frac{\Delta\varphi}{2d}$$

Hence $$E = \frac{E'}{2} = \frac{\Delta\varphi}{2d} \qquad (2)$$

(b) Since $E \alpha \sigma$, we can state that according to equation (2) for part (a) the charge on the plate 2 is divided into two parts; such that 1 / 3 rd of it lies on the upper side and 2 / 3 rd on its lower face.

Thus charge density of upper face of plate 2 or of plate 1 or plate 4 and lower face of $3\sigma = \varepsilon_0 E = \frac{\varepsilon_0 \Delta\varphi}{2d}$ and charge density of lower face of 2 or upper face of 3

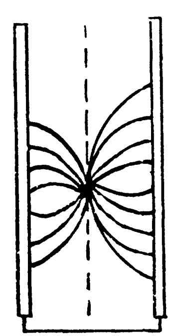
$$\sigma' = \varepsilon_0 E' = \varepsilon_0 \frac{\Delta\varphi}{d}$$

Hence the net charge density of plate 2 or 3 becomes $\sigma + \sigma' = \frac{3\,\varepsilon_0\,\Delta\,\varphi}{2\,d}$, which is obvious from the argument.

3.67 The problem of point charge between two conducting planes is more easily tackled (if we want only the total charge induced on the planes) if we replace the point charge by a uniformly charged plane sheet.

Let σ be the charge density on this sheet and E_1, E_2 outward electric field on the two sides of this sheet.

Then $$E_1 + E_2 = \frac{\sigma}{\varepsilon_0}$$



The conducting planes will be assumed to be grounded. Then $E_1 x = E_2 (l - x)$.

Hence $$E_1 = \frac{\sigma}{l\,\varepsilon_0}(l - x),\ E_2 = \frac{\sigma}{l\,\varepsilon_0} x$$

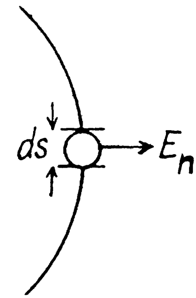
This means that the induced charge density on the plane conductors are

$$\sigma_1 = -\frac{\sigma}{l}(l - x),\ \sigma_2 = -\frac{\sigma}{l} x$$

Hence $q_1 = -\frac{q}{l}(l - x),\ q_2 = -\frac{q}{l} x$

3.68 Near the conductor $E = E_n = \dfrac{\sigma}{\varepsilon_0}$

This field can be written as the sum of two parts E_1 and E_2. E_1 is the electric field due to an infinitesimal area dS.

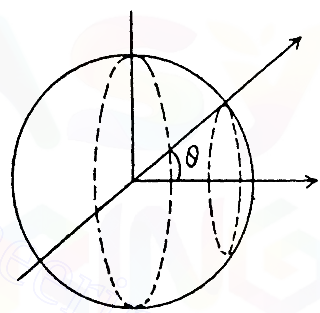


Very near it $E_1 = \pm \dfrac{\sigma}{2\,\varepsilon_0}$

The remaining part contributes $E_2 = \dfrac{\sigma}{2\,\varepsilon_0}$ on both sides. In calculating the force on the element dS we drop E_1 (because it is a self-force.) Thus

$$\frac{dF}{dS} = \sigma \cdot \frac{\sigma}{2\,\varepsilon_0} = \frac{\sigma^2}{2\,\varepsilon_0}$$

3.69 The total force on the hemisphere is



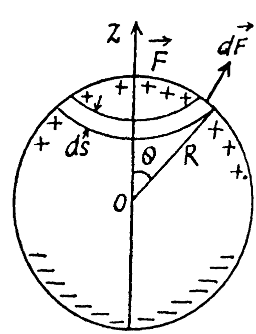
$$F = \int_0^{\pi/2} \frac{\sigma^2}{2\,\varepsilon_0} \cdot \cos\theta \cdot 2\,\pi\, R \sin\theta\, R\, d\theta$$

$$= \frac{2\,\pi\, R^2\,\sigma^2}{2\,\varepsilon_0} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2\,\pi\, R^2}{2\,\varepsilon_0} \times \frac{1}{2} \times \left(\frac{q}{4\,\pi\, R^2}\right)^2 = \frac{q^2}{32\,\pi\,\varepsilon_0\, R}.$$

3.70 We know that the force acting on the area element dS of a conductor is,

$$d\vec{F} = \frac{1}{2}\,\sigma\,\vec{E}\; dS \qquad (1)$$



It follows from symmetry considerations that the resultant force $\vec{F}$ is directed along the z-axis, and hence it can be represented as the sum (integral) of the projection of elementary forces (1) onto the z-axis :

$$dF_z = dF \cos\theta \qquad (2)$$

For simplicity let us consider an element area $dS = 2\,\pi\, R \sin\theta\, R\, d\theta$(Fig.). Now considering that $E = \sigma/\varepsilon_0$. Equation (2) takes the from

$$dF_z = \frac{\pi\,\sigma^2\, R^2}{\varepsilon_0} \sin\theta \cos\theta\, d\theta$$

$$= -\left(\frac{\pi\,\sigma_0^2\, R^2}{\varepsilon_0}\right) \cos^3\theta\, d\cos\theta$$

Integrating this expression over the half sphere (i.e. with respect to cos θ between 1 and 0),

we obtain $$F = F_z = \frac{\pi \sigma_0^2 R^2}{4 \varepsilon_0}$$

3.71 The total polarization is $P = (\varepsilon - 1)\, \varepsilon_0 E$. This must equals $\frac{n_0 p}{N}$ where n_0 is the concerntation of water molecules. Thus

$$N = \frac{n_0 p}{(\varepsilon - 1)\, \varepsilon_0 E} = 2{\cdot}93 \times 10^3 \text{ on putting the values}$$

3.72 From the general formula

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{3\vec{p}\cdot\vec{r}\vec{r} - \vec{p}r^2}{r^5}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{l^3}, \text{ where } r = l \text{ and } \vec{r} \uparrow\uparrow \vec{p}$$

This will cause the induction of a dipole moment.

$$\vec{p}_{ind} = \beta \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{l^3} \times \varepsilon_0$$

Thus the force,

$$\vec{F} = \frac{\beta}{4\pi} \frac{2p}{l^3} \frac{\partial}{\partial l} \frac{1}{4\pi\varepsilon_0} \frac{2p}{l^3} = \frac{3\beta p^2}{4\pi^2 \varepsilon_0 l^7}$$

3.73 The electric field E at distance x from the centre of the ring is,

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

The induced dipole moment is $p = \beta \varepsilon_0 E = \dfrac{q\beta x}{4\pi (R^2 + x^2)^{3/2}}$

The force on this molecule is

$$F = p\frac{\partial}{\partial x} E = \frac{q\beta x}{4\pi (R^2 + x^2)^{3/2}} \frac{q}{4\pi\varepsilon_0} \frac{\partial}{\partial x} \frac{x}{(R^2 + x^2)^{3/2}} = \frac{q^2 \beta}{16\pi^2 \varepsilon_0} \frac{x(R^2 - 2x^2)}{(R^2 + x^2)^4}$$

This vanishes for $x = \dfrac{\pm R}{\sqrt{2}}$ (apart from $x = 0$, $x = \infty$)

It is maximum when

$$\frac{\partial}{\partial x} \frac{x(R^2 - x^2 \times 2)}{(R^2 + x^2)^4} = 0$$

or, $$(R^2 - 2x^2)(R^2 + x^2) - 4x^2 (R^2 + x^2) - 8x^2 (R^2 - 2x^2) = 0$$

or, $$R^4 - 13x^2 R^2 + 10x^4 = 0 \text{ or, } x^2 = \frac{R^2}{20}\left(13 \pm \sqrt{129}\right)$$

or, $x = \dfrac{R}{\sqrt{20}} \sqrt{13 \pm \sqrt{129}}$ (on either side), Plot of $F_x(x)$ is as shown in the answersheet.

3.74 Inside the ball

$$\vec{D}(\vec{r}) = \frac{q}{4\pi}\frac{\vec{r}}{r^3} = \varepsilon\,\varepsilon_0\,\vec{E}.$$

Also $$\varepsilon_0\vec{E}+\vec{P}=\vec{D} \quad\text{or}\quad \vec{P}=\frac{\varepsilon-1}{\varepsilon}\vec{D}=\frac{\varepsilon-1}{\varepsilon}\frac{q}{4\pi}\frac{\vec{r}}{r^3}$$

Also, $$q' = -\oint \vec{P}\cdot d\vec{S} = -\frac{\varepsilon-1}{\varepsilon}\frac{q}{4\pi}\int d\Omega = -\frac{\varepsilon-1}{\varepsilon}q$$

3.75 $D_{diel} = \varepsilon\,\varepsilon_0\,E_{diel} = D_{conductor} = \sigma$ or, $E_{diel} = \dfrac{\sigma}{\varepsilon\,\varepsilon_0}$

$$P_n = (\varepsilon-1)\,\varepsilon_0\,E_{diel} = \frac{\varepsilon-1}{\varepsilon}\sigma$$

$$\sigma' = -P_n = -\frac{\varepsilon-1}{\varepsilon}\sigma$$

This is the surface density of bound charges.

3.76 From the solution of the previous problem q'_{in} = charge on the interior surface of the conductor

$$= -(\varepsilon-1)/\varepsilon\int\sigma\,dS = -\frac{\varepsilon-1}{\varepsilon}q$$

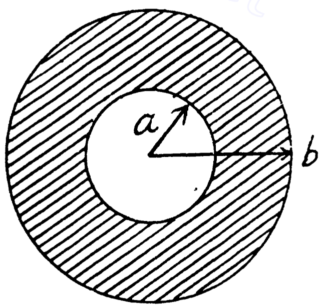
Since the dielectric as a whole is neutral there must be a total charge equal to $q'_{outer} = +\dfrac{\varepsilon-1}{\varepsilon}q$ on the outer surface of the dielectric.

3.77 (a) Positive extraneous charge is distributed uniformly over the internal surface layer. Let σ_0 be the surface density of the charge.

Clearly, $E = 0$, for $r < a$

For $a < r$

$\varepsilon_0 E \times 4\pi r^2 = 4\pi a^2\sigma_0$ by Gauss theorem.



or, $$E = \frac{\sigma_0}{\varepsilon_0\,\varepsilon}\left(\frac{a}{r}\right)^2,\quad a < r < b$$

For $r > b$, similarly

$$E = \frac{\sigma_0}{\varepsilon_0}\left(\frac{a}{r}\right)^2,\quad r > b$$

Now, $$E = -\frac{\partial\varphi}{\partial r}.$$

So by integration from infinity where $\varphi(\infty) = 0$,

$$\varphi = \frac{\sigma_0\,a^2}{\varepsilon_0\,r}\quad r > b$$

$a < r < b \quad \varphi = \dfrac{\sigma_0 a^2}{\varepsilon \varepsilon r} + B$, B is a constant

or by continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{r} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}$, $a < r < b$

For $r < a$. $\varphi = A =$ Constant

By continuity, $\varphi = \dfrac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\dfrac{1}{a} - \dfrac{1}{b}\right) + \dfrac{\sigma_0 a^2}{\varepsilon_0 b}$

(b) Positive extraneous charge is distributed uniformly over the internal volume of the dielectric

Let $\rho_0 =$ Volume density of the charge in the dielectric, for $a < r < b$.

$$E = 0, \quad r < a$$

$$\varepsilon_0 \varepsilon \, 4\pi r^2 E = \frac{4\pi}{3}(r^3 - a^3)\rho_0, \; (a < r < b)$$

or,
$$E = \frac{\rho_0}{3\varepsilon_0 \varepsilon}\left(r - \frac{a^3}{r^2}\right)$$

$$E = \frac{4\pi}{3}(b^3 - a^3)\rho_0 \Big/ \varepsilon_0 4\pi r^2, \quad r > b$$

or,
$$E = \frac{(b^3 - a^3)\rho_0}{3\varepsilon_0 r^2} \quad \text{for } r > b$$

By integration,

$$\varphi = \frac{(b^3 - a^3)\rho_0}{3\varepsilon_0 r} \quad \text{for } r > b$$

or,
$$\varphi = B - \frac{\rho_0}{3\varepsilon_0 \varepsilon}\left(\frac{r^2}{2} + \frac{a^3}{r}\right), \quad a < r < b$$

By continuity

$$\frac{b^3 - a^3}{3\varepsilon_0 b}\rho_0 = B - \frac{\rho_0}{3\varepsilon_0 \varepsilon}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$$

or,
$$B = \frac{\rho_0}{3\varepsilon_0 \varepsilon}\left\{\frac{\varepsilon(b^3 - a^3)}{b} + \left(\frac{b^2}{2} + \frac{a^3}{b}\right)\right\}$$

Finally
$$\varphi = B - \frac{\rho_0}{3\varepsilon_0 \varepsilon}\left(\frac{a^2}{2} + a^2\right) = B - \frac{\rho_0 a^2}{2\varepsilon_0 \varepsilon}, \quad r < a$$

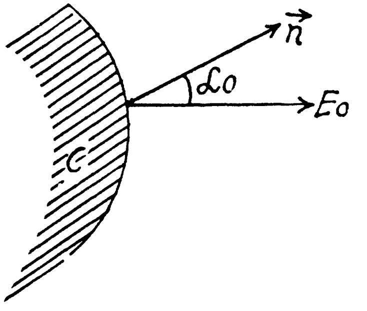
On the basis of obtained expressions $E(r)$ and $(\varphi)(r)$ can be plotted as shown in the answer-sheet.

3.78 Let the field in the dielectric be $\vec{E}$ making an angle α with $\vec{n}$. Then we have the boundary conditions,

$E_0 \cos \alpha_0 = \varepsilon E \cos \alpha$ and $E_0 \sin c_0 = E \sin \alpha$

So $E = E_0 \sqrt{\sin^2 \alpha_0 + \frac{1}{\varepsilon^2} \cos^2 \alpha_0}$ and $\tan \alpha = \varepsilon \tan \alpha_0$

In the dielectric the normal component of the induction vector is

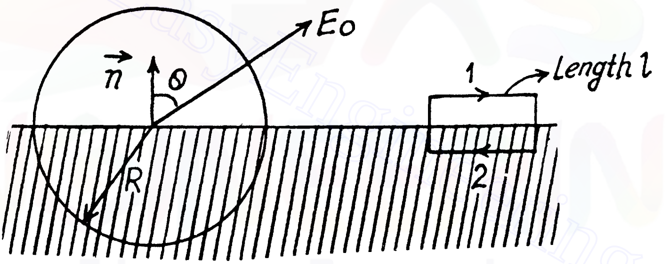


$D_n = \varepsilon_0 \varepsilon E_n = \varepsilon_0 \varepsilon E \cos \alpha = \varepsilon_0 E_0 \cos \alpha_0$

$\sigma' = P_n = D_n - \varepsilon_0 E_n = \left(1 - \frac{1}{\varepsilon}\right) \varepsilon_0 E_0 \cos \alpha_0$

or, $\sigma' = \frac{\varepsilon - 1}{\varepsilon} \varepsilon_0 E_0 \cos \alpha_0$

3.79 From the previous problem, $\sigma' = \varepsilon_0 \frac{\varepsilon - 1}{\varepsilon} E_0 \cos \theta$



(a) Then $\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} Q = \pi R^2 E_0 \cos \theta \frac{\varepsilon - 1}{\varepsilon}$

(b) $\oint \vec{D} \cdot d\vec{l} = (D_{1t} - D_{2t})\, l = (\varepsilon_0 E_0 \sin \theta - \varepsilon \varepsilon_0 E_0 \sin \theta) = -(\varepsilon - 1)\, \varepsilon_0 E_0 l \sin \theta$

3.80 (a) $\text{div} \vec{D} = \frac{\partial D_x}{\partial x} = \rho$ and $D = \rho l$

$$E_x = \frac{\rho l}{\varepsilon \varepsilon_0},\ l < d \text{ and } E_x = \frac{\rho d}{\varepsilon_0} \text{ constant for } l > d$$

$\varphi(x) = -\frac{\rho l^2}{2\varepsilon \varepsilon_0},\ l < d$ and $\varphi(x) = A - \frac{\rho l d}{\varepsilon_0},\ l > d$ then $\varphi(x) = \frac{\rho d}{\varepsilon_0}\left(d - \frac{d}{2\varepsilon} - l\right)$,

by continuity.

On the basis of obtained expressions $E_x(x)$ and $\varphi(x)$ can be plotted as shown in the figure of answersheet.

(b) $\rho' = -\text{div}\,\vec{P} = -\text{div}\,(\varepsilon-1)\,\varepsilon_0\vec{E} = -\rho\dfrac{(\varepsilon-1)}{\varepsilon}$

$\sigma' = P_{1n} - P_{2n}$, where n is the normal from 1 to 2.

$= P_{1n}$, ($\vec{P}_2 = 0$ as 2 is vacuum.)

$= (\rho d - \rho\, d/\varepsilon) = \rho\, d\dfrac{\varepsilon-1}{\varepsilon}$

3.81 $\text{div}\vec{D} = \dfrac{l}{r^2}\dfrac{\partial}{\partial r}r^2 D_r = \rho$

$r^2 D_r = \rho\dfrac{r^3}{3} + A \quad D_r = \dfrac{1}{3}\rho r + \dfrac{A}{r^2},\ r < R$

$A = 0$ as $D_r \neq \infty$ at $r = 0$,Thus, $E_r = \dfrac{\rho r}{3\,\varepsilon\,\varepsilon_0}$

For $\quad r > R,\ D_r = \dfrac{B}{r^2}$

By continuity of D_r at $r = R$; $B = \dfrac{\rho R^3}{3}$

so, $\quad E_r = \dfrac{\rho R^3}{3\,\varepsilon_0 r^2},\ r > R$

$$\varphi = \frac{\rho R^3}{3\,\varepsilon_0 r},\ r > R \text{ and } \varphi = -\frac{\rho r^2}{6\,\varepsilon\,\varepsilon_0} + C,\ r < R$$

$$C = +\frac{\rho R^2}{3\,\varepsilon_0} + \frac{\rho R^2}{6\,\varepsilon\,\varepsilon_0}, \text{ by continuity of } \varphi.$$

See answer sheet for graphs of $E(r)$ and $\varphi(r)$

(b) $\rho' = \text{div}\,\vec{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left\{\dfrac{r^3}{3}\rho\left(1-\dfrac{1}{\varepsilon}\right)\right\} = -\dfrac{\rho(\varepsilon-1)}{\varepsilon}$

$\sigma' = P_{1r} - P_{2r} = P_{1r} = \dfrac{1}{3}\rho R\left(1-\dfrac{1}{\varepsilon}\right)$

3.82 Because there is a discontinuity in polarization at the boundary of the dielectric disc, a bound surface charge appears, which is the source of the electric field inside and outside the disc.

We have for the electric field at the origin.

$$\vec{E} = -\int\frac{\sigma'\,dS}{4\pi\,\varepsilon_0 r^3}\vec{r},$$

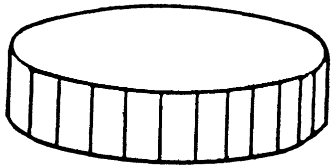
where $\vec{r}$ = radius vector to the origin from the element dS.

$\sigma' = P_n = P\cos\theta$ on the curved surface

($P_n = 0$ on the flat surface.)

Here θ = angle between $\vec{r}$ and $\vec{P}$

By symmetry, $\vec{E}$ will be parallel to $\vec{P}$. Thus



$$E = -\int_0^{2\pi} \frac{P\cos\theta\, R\, d\theta \cdot \cos\theta}{4\pi\varepsilon_0 R^2} \cdot d$$

where, $\quad r = R$ if $d << R$.

So, $\quad E = -\dfrac{Pd}{4\varepsilon_0 R}$ and $\vec{E} = -\dfrac{\vec{P}d}{4\varepsilon_0 R}$

3.83. Since there are no free extraneous charges anywhere

$$\text{div}\,\vec{D} = \frac{\partial D_x}{\partial x} = 0 \quad \text{or,} \quad D_x = \text{Constant}$$

But $\quad D_x = 0$ at ∞, so, $D_x = 0$, every where.

Thus, $\quad \vec{E} = -\dfrac{\vec{P}_0}{\varepsilon_0}\left(1 - \dfrac{x^2}{d^2}\right)$ or, $E_x = -\dfrac{P_0}{\varepsilon_0}\left(1 - \dfrac{x^2}{d^2}\right)$

So, $\quad \varphi = \dfrac{P_0 x}{\varepsilon_0} - \dfrac{P_0 x^3}{3\varepsilon_0 d^2} + \text{constant}$

Hence,

$$\varphi(+d) - \varphi(-d) = \frac{2P_0 d}{\varepsilon_0} - \frac{2P_0 d^3}{3d^2\varepsilon_0} = \frac{4P_0 d}{3\varepsilon_0}$$

3.84 (a) We have $D_1 = D_2$, or, $\varepsilon E_2 = E_1$

Also, $\quad E_1\dfrac{d}{2} + E_2\dfrac{d}{2} = E_0 d$ or, $E_1 + E_2 = 2E_0$

Hence, $\quad E_2 = \dfrac{2E_0}{\varepsilon + 1}$ and $E_1 = \dfrac{2\varepsilon E_0}{\varepsilon + 1}$ and $D_1 = D_2 = \dfrac{2\varepsilon\varepsilon_0 E_0}{\varepsilon + 1}$

(b) $D_1 = D_2$, or, $\varepsilon E_2 = E_1 = \dfrac{\sigma}{\varepsilon_0} = E_0$

Thus, $\quad E_1 = E_0,\ E_2 = \dfrac{E_0}{\varepsilon}$ and $D_1 = D_2 = \varepsilon_0 E_0$

3.85 (a) Constant voltage acros the plates;

$$E_1 = E_2 = E_0\,,\ D_1 = \varepsilon_0 E_0,\ D_2 = \varepsilon_0\,\varepsilon\,E_0.$$

(b) Constant charge across the plates;

$$E_1 = E_2\,,\ D_1 = \varepsilon_0 E_1,\ D_2 = \varepsilon\,\varepsilon_0 E_2 = \varepsilon\,D_1$$

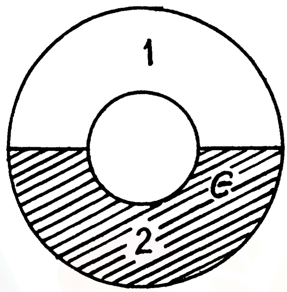
$$E_1(1+\varepsilon) = 2E_0 \quad \text{or} \quad E_1 = E_2 = \frac{2E_0}{\varepsilon+1}$$

3.86 At the interface of the dielectric and vacuum,

$$E_{1t} = E_{2t}$$

The electric field must be radial and

$$E_1 = E_2 = \frac{A}{\varepsilon_0\,\varepsilon\,r^2},\ a < r < b$$



Now, $q = \frac{A}{R^2}(2\pi R^2) + \frac{A}{\varepsilon R^2}(2\pi R^2)$

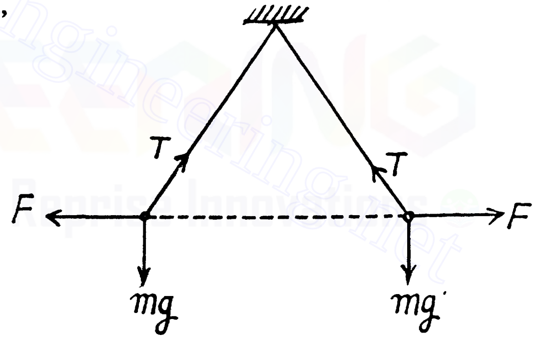
$$= A\left(1+\frac{1}{\varepsilon}\right)2\pi$$

or, $E_1 = E_2 = \dfrac{q}{2\pi\,\varepsilon_0\,r^2\,(1+\varepsilon)}$

3.87 In air the forces are as shown. In K–oil,

$F \to F' = F/\varepsilon$ and $mg \to mg\left(1-\frac{\rho_0}{\rho}\right).$

Since the inclinations do not change



$$\frac{1}{\varepsilon} = 1 - \frac{\rho_0}{\rho}$$

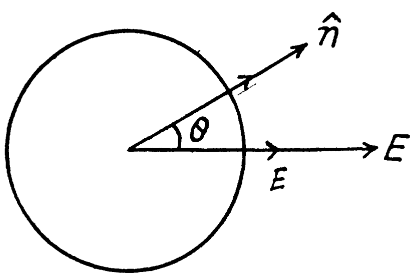
or, $\dfrac{\rho_0}{\rho} = 1 - \dfrac{1}{\varepsilon} = \dfrac{\varepsilon-1}{\varepsilon}$

or, $\rho = \rho_0\dfrac{\varepsilon}{\varepsilon-1}$

where ρ_0 is the density of K–oil and ρ that of the material of which the balls are made.

3.88 Within the ball the electric field can be resolved into normal and tangential components.

$$E_n = E\cos\theta,\ E_t = E\sin\theta$$



Then, $D_n = \varepsilon\,\varepsilon_0 E\cos\theta$

and $P_n = (\varepsilon-1)\ \varepsilon_0 E\cos\theta$

or, $\sigma' = (\varepsilon-1)\,\varepsilon_0 E\cos\theta$

so, $\sigma_{max} = (\varepsilon-1)\,\varepsilon_0 E,$

and total charge of one sign,

$$q' = \int_0^1 (\varepsilon - 1)\, \varepsilon_0 E \cos\theta \, 2\pi R^2 \, d(\cos\theta) = \pi R^2 \varepsilon_0 (\varepsilon - 1) E$$

(Since we are interested in the total charge of one sign we must intergrate cos θ from 0 to 1 only).

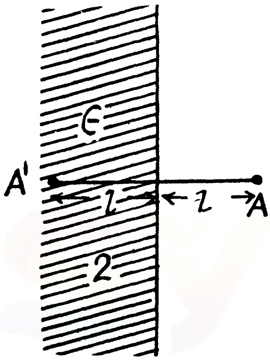
3.89 The charge is at A in the medium 1 and has an image point at A' in the medium 2. The electric field in the medium 1 is due to the actual charge q at A and the image charge q' at A'. The electric field in 2 is due to a corrected charge q' at A. Thus on the boundary between 1 and 2,

$$E_{1n} = \frac{q'}{4\pi\varepsilon_0 r^2}\cos\theta - \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{2n} = \frac{-q''}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{1t} = \frac{q'}{4\pi\varepsilon_0 r^2}\sin\theta + \frac{q}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$E_{2t} = \frac{q''}{4\pi\varepsilon_0 r^2}\sin\theta$$



The boundary conditions are

$$D_{1n} = D_{2n} \text{ and } E_{1t} = E_{2t}$$

$$\varepsilon q'' = q - q'$$

$$q'' = q + q'$$

So,
$$q'' = \frac{2q}{\varepsilon + 1}, \quad q' = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

(a) The surface density of the bound charge on the surface of the dielectric

$$\sigma' = P_{2n} = D_{2n} - \varepsilon_0 E_{2n} = (\varepsilon - 1)\varepsilon_0 E_{2n}$$

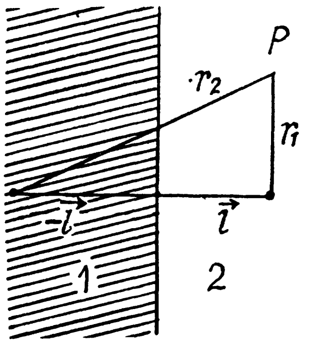
$$= -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{2\pi r^2}\cos\theta = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{ql}{2\pi r^3}$$

(b) Total bound charge is,
$$-\frac{\varepsilon - 1}{\varepsilon + 1} q \int_0^\infty \frac{l}{2\pi (l^2 + x^2)^{3/2}} 2\pi x \, dx = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

3.90 The force on the point charge q is due to the bound charges. This can be calculated from the field at this charge after extracting out the self field. This image field is

$$E_{\text{image}} = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{4\pi\varepsilon_0 (2l)^2}$$

Thus,
$$F = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q^2}{16\pi\varepsilon_0 l^2}$$



3.91 $E_p = \dfrac{q\,\vec{r}_1}{4\pi\varepsilon_0 r_1^3} + \dfrac{q'\vec{r}_2}{4\pi r_2^3\varepsilon_0}$, P in 1

$$E_p = \frac{q''\,\vec{r}_1}{4\pi\varepsilon_0 r_1^3},\ P \text{ in } 2$$

where $q'' = \dfrac{2q}{\varepsilon + 1}$, $q' = q'' - q$

In the limit $\vec{l} \to 0$

$$\vec{E}_p = \frac{(q + q')\,\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{q\,\vec{r}}{2\pi\varepsilon_0(1+\varepsilon)\,r^3}, \text{ in either part.}$$

Thus,
$$E_p = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)\,r^2}$$

$$\varphi = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)\,r}$$

$$D = \frac{q}{2\pi\varepsilon_0(1+\varepsilon)\,r^2} \times \begin{cases} 1 & \text{in vacuum} \\ \varepsilon & \text{in dielectric} \end{cases}$$

3.92 $\vec{E}_p = \dfrac{q\,\vec{r}_2}{4\pi\varepsilon_0\varepsilon r_2^3} + \dfrac{q'\,\vec{r}_1}{4\pi\varepsilon_0 r_1^3}$; P in 2

$$\vec{E}_p = \frac{q''\,\vec{r}_2}{4\pi\varepsilon_0 r_2^3};\ P \text{ in } 1$$

Using the boundary conditions,

$$E_{1n} = \varepsilon E_{2n},\ E_{1t} = E_{2t}$$

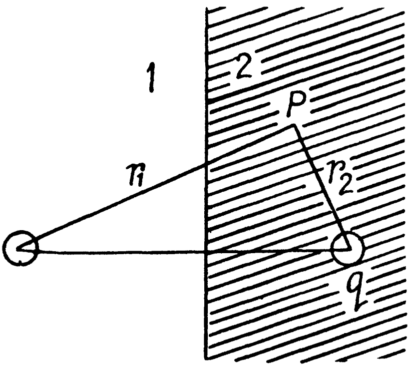
This implies

$q - \varepsilon q' = q''$ and $q + \varepsilon q' = \varepsilon q''$

So, $q'' = \dfrac{2q}{\varepsilon + 1}$, $q' = \dfrac{\varepsilon - 1}{\varepsilon + 1}\dfrac{q}{\varepsilon}$

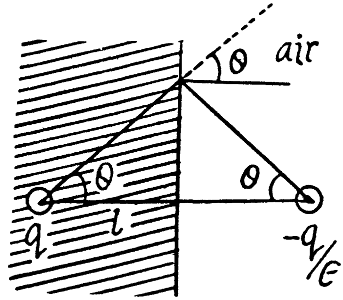
Then, as earlier,

$$\sigma' = \frac{ql}{2\pi r^3}\cdot\left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)\cdot\frac{1}{\varepsilon}$$



3.93 To calculate the electric field, first we note that an image charge will be needed to ensure that the electric field on the metal boundary is normal to the surface.

The image charge must have magnitude $-\frac{q}{\varepsilon}$ so that the tangential component of the electric field may vanish. Now,



$$E_n = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\varepsilon r^2}\right) 2\cos\theta = \frac{ql}{2\pi\varepsilon_0 \varepsilon r^3}$$

Then $P_n = D_n - \varepsilon_0 E_n = \frac{(\varepsilon - 1)\, ql}{2\pi\varepsilon r^3} = \sigma'$

This is the density of bound charge on the surface.

3.94 Since the condenser plates are connected,

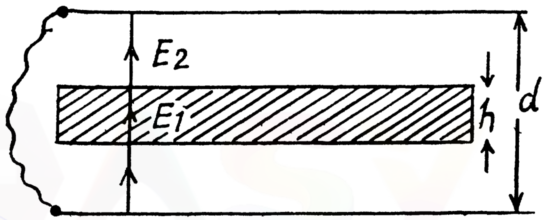
$$E_1 h + E_2 (d - h) = 0$$

and $\quad P + \varepsilon_0 E_1 = \varepsilon_0 E_2$

or, $\quad E_1 + \frac{P}{\varepsilon_0} = E_2$

Thus, $E_2 d - \frac{Ph}{\varepsilon_0} = 0$, or, $E_2 = \frac{Ph}{\varepsilon_0 d}$

$$E_1 = -\frac{P}{\varepsilon_0}\left(1 - \frac{h}{d}\right)$$



3.95 Given $\vec{P} = \alpha \vec{r}$, where $\vec{r}$ = distance from the axis. The space density of charges is given by, $\rho' = -\operatorname{div} \vec{P} = -2\alpha$

On using. $\quad \operatorname{div} \vec{r} = \frac{1}{r}\frac{\partial}{\partial r}(\vec{r}\cdot\vec{r}) = 2$

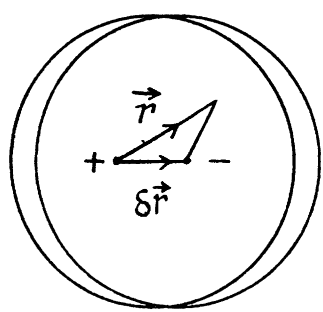
3.96 In a uniformly charged sphere,

$$E_r = \frac{\rho_0 r}{3\varepsilon_0} \quad \text{or,} \quad \vec{E} = \frac{\rho_0}{3\varepsilon_0}\vec{r}$$

The total electric field is

$$\vec{E} = \frac{1}{3\varepsilon_0}\rho_0 \vec{r} - \frac{1}{3\varepsilon_0}(\vec{r} - \delta\vec{r})\,\rho_0$$

$$= \frac{1}{3\varepsilon_0}\rho_0 \delta\vec{r} = -\frac{\vec{P}}{3\varepsilon_0}$$



where $\rho\,\delta\vec{r} = -\vec{P}$ (dipole moment is defined with its direction being from the $-ve$ charge to $+ve$ charge.)

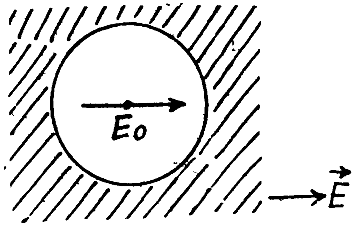
The potential outside is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} - \frac{Q}{|\vec{r} - \delta\vec{r}|}\right), = \frac{\vec{p}_0\cdot\vec{r}}{4\pi\varepsilon_0 r^3}, \quad r > R$$

where $\vec{p}_0 = -\frac{4\pi}{3}R^3 \rho_0\, \partial\vec{r}$ is the total dipole moment.

3.97 The electric field $\vec{E}_0$ in a spherical cavity in a uniform dielectric of permittivity ε is related to the far away field $\vec{E}$, in the following manner. Imagine the cavity to be filled up with the dielectric. Then there will be a uniform field $\vec{E}$ everywhere and a polarization $\vec{P}$, given by,

$$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$



Now take out the sphere making the cavity, the electric field inside the sphere will be $-\dfrac{\vec{P}}{3\varepsilon_0}$

By superposition. $\vec{E}_0 - \dfrac{\vec{P}}{3\varepsilon_0} = \vec{E}$

or, $\vec{E}_0 = \vec{E} + \dfrac{1}{3}(\varepsilon - 1)\vec{E} = \dfrac{1}{3}(\varepsilon + 2)\vec{E}$

3.98 By superposition the field $\vec{E}$ inside the ball is given by

$$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{3\varepsilon_0}$$

On the other hand, if the sphere is not too small, the macroscopic equation $\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$ must hold. Thus,

$$\vec{E}\left(1 + \frac{1}{3}(\varepsilon - 1)\right) = \vec{E}_0 \quad \text{or,} \quad \vec{E} = \frac{3\vec{E}_0}{\varepsilon + 2}$$

Also
$$\vec{P} = 3\,\varepsilon_0 \frac{\varepsilon - 1}{\varepsilon + 2}\vec{E}_0$$

3.99 This is to be handled by the same trick as in 3.96. We have effectively a two dimensional situation. For a uniform cylinder full of charge with charge density ρ_0 (charge per unit volume), the electric field E at an inside point is along the (cylindrical) radius vector $\vec{r}$ and equal to,

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r}$$

$$\left(\text{div}\,\vec{E} = \frac{l}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\varepsilon_0}, \quad \text{hence,} \quad E_r = \frac{\rho}{2\varepsilon_0} r\right)$$

Therefore the polarized cylinder can be thought of as two equal and opposite charge distributions displaced with respect to each other

$$\vec{E} = \frac{1}{2\varepsilon_0}\rho\,\vec{r} - \frac{1}{2\varepsilon_0}\rho\,(\vec{r} - \delta\vec{r}) = \frac{1}{2\varepsilon_0}\rho\,\delta\,\vec{r} = -\frac{\vec{P}}{2\varepsilon_0}$$

Since $\vec{P} = -\rho\,\delta\,\vec{r}$. (direction of electric dipole moment vector being from the negative charge to positive charge .)

3.100 As in 3.98, we write $\vec{E} = \vec{E}_0 - \dfrac{\vec{P}}{2\varepsilon_0}$

using here the result of the foregoing problem.

Also
$$\vec{P} = (\varepsilon - 1)\,\varepsilon_0 \vec{E}$$

So, $\vec{E}\left(\dfrac{\varepsilon + 1}{2}\right) = \vec{E}_0$, or, $\vec{E} = \dfrac{2\vec{E}_0}{\varepsilon + 1}$ and $\vec{P} = 2\varepsilon_0 \dfrac{\varepsilon - 1}{\varepsilon + 1}\vec{E}_0$

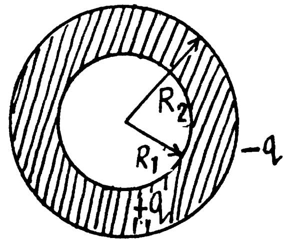
3.3 ELECTRIC CAPACITANCE ENERGY OF AN ELECTRIC FIELD

3.101 Let us mentally impart a charge q on the conductor, then

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0 \varepsilon r^2} dr + \int_{R_2}^{\infty} \frac{q}{4\pi\varepsilon_0 r^2} dr$$

$$= \frac{q}{4\pi\varepsilon_0 \varepsilon}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + \frac{q}{4\pi\varepsilon_0}\frac{1}{R_2}$$

$$= \frac{q}{4\pi\varepsilon_0 \varepsilon}\left[\frac{(\varepsilon - 1)}{R_2} + \frac{1}{R_1}\right]$$



Hence the sought capacitance,

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q\, 4\pi\varepsilon_0 \varepsilon}{q\left[\frac{(\varepsilon - 1)}{R_2} + \frac{1}{R_1}\right]} = \frac{4\pi\varepsilon_0 \varepsilon R_1}{(\varepsilon - 1)\frac{R_1}{R_2} + 1}$$

3.102 From the symmetry of the problem, the voltage across each capacitor, $\Delta\varphi = \xi/2$ and charge on each capacitor $q = C\xi/2$ in the absence of dielectric.

Now when the dielectric is filled up in one of the capacitors, the equivalent capacitance of the system,

$$C'_0 = \frac{C\varepsilon}{1 + \varepsilon}$$

and the potential difference across the capacitor, which is filled with dielectric,

$$\Delta\varphi' = \frac{q'}{\varepsilon C} = \frac{C\varepsilon}{(1+\varepsilon)}\frac{\xi}{C\varepsilon} = \frac{\xi}{(1+\varepsilon)}$$

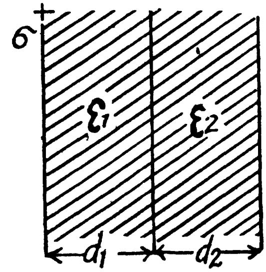
But $$\varphi \,\alpha\, E$$

So, as φ decreases $\frac{1}{2}(1 + \varepsilon)$ times, the field strength also decreases by the same factor and flow of charge, $\Delta q = q' - q$

$$= \frac{C\varepsilon}{(1+\varepsilon)}\xi - \frac{C}{2}\xi = \frac{1}{2}C\xi\frac{(\varepsilon - 1)}{(\varepsilon + 1)}$$

3.103 (a) As it is series combination of two capacitors,

$$\frac{1}{C} = \frac{d_1}{\varepsilon_0 \varepsilon_1 S} + \frac{d_2}{\varepsilon_0 \varepsilon_2 S} \quad \text{or,} \quad C = \frac{\varepsilon_0 S}{(d_1/\varepsilon_1) + (d_2/\varepsilon_2)}$$



(b) Let, σ be the initial surface charge density, then density of bound charge on the boundary plane.

$$\sigma' = \sigma\left(1 - \frac{1}{\varepsilon_1}\right) - \sigma\left(1 - \frac{1}{\varepsilon_2}\right) = \sigma\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)$$

But,
$$\sigma = \frac{q}{S} = \frac{CV}{S} = \frac{\varepsilon_0 S \varepsilon_1 \varepsilon_2}{\varepsilon_2 d_1 + \varepsilon_1 d_2} \frac{V}{S}$$

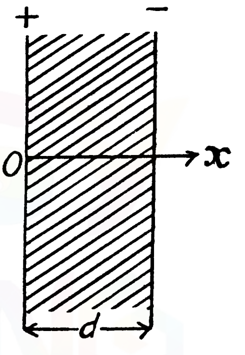
So,
$$\sigma' = \frac{\varepsilon_0 V (\varepsilon_1 - \varepsilon_2)}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

3.104 (a) We point the x-axis lowards right and place the origin on the left hand side plate. The left plate is assumed to be positively charged.

Since ε varies linearly, we can write,
$$\varepsilon(x) = a + bx$$
where a and b can be determined from the boundary condition. We have
$\varepsilon = \varepsilon_1$ at $x = 0$ and $\varepsilon = \varepsilon_2$ at $x = d$,

Thus,
$$\varepsilon(x) = \varepsilon_1 + \left(\frac{\varepsilon_2 - \varepsilon_1}{d}\right) x$$

Now potential difference between the plates



$$\varphi_+ - \varphi_- = \int_0^d \vec{E} \cdot d\vec{r} = \int_0^d \frac{\sigma}{\varepsilon_0 \, \varepsilon(x)} dx$$

$$= \int_0^d \frac{\sigma}{\varepsilon_0 \left(\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x\right)} dx = \frac{\sigma d}{(\varepsilon_2 - \varepsilon_1) \varepsilon_0} \ln \frac{\varepsilon_2}{\varepsilon_1}$$

Hence, the sought capacitance, $C = \frac{\sigma S}{\varphi_+ - \varphi_-} = \frac{(\varepsilon_2 - \varepsilon_1) \varepsilon_0 S}{(\ln \varepsilon_2/\varepsilon_1) d}$

(b) $D = \frac{q}{S}$ and $P = \frac{q}{S} - \frac{q}{S\varepsilon(x)}$

and the space density of bound charges is
$$\rho' = -\operatorname{div} P = -\frac{q(\varepsilon_2 - \varepsilon_1)}{S d \varepsilon^2(x)}$$

3.105 Let, us mentally impart a charge q to the conductor. Now potential difference between the plates,

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r}$$

$$= \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0 a/r} \frac{1}{r^2} dr = \frac{q}{4\pi\varepsilon_0 a} \ln R_2/R_1$$

Hence, the sought capacitance,

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q \, 4\pi\varepsilon_0 a}{q \ln R_2/R_1} = \frac{4\pi\varepsilon_0 a}{\ln R_2/R_1}$$

3.106 Let λ be the linear charge density then,

$$E_{1m} = \frac{\lambda}{2\pi\varepsilon_0 R_1 \varepsilon_1} \quad (1)$$

and,

$$E_{2m} = \frac{\lambda}{2\pi\varepsilon_0 R_2 \varepsilon_2} \quad (2)$$

The breakdown in either case will occur at the smaller value of r for a simultaneous breakdown of both dielectrics.

From (1) and (2)

$E_{1m} R_1 \varepsilon_1 = E_{2m} R_2 \varepsilon_2$, which is the sought relationship.

3.107 Let, λ be the linear charge density then, the sought potential difference,

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{\lambda}{2\pi\varepsilon_0\varepsilon_1 r} dr + \int_{R_2}^{R_3} \frac{\lambda}{2\pi\varepsilon_0\varepsilon_2 r} dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{\varepsilon_1}\ln R_2/R_1 + \frac{1}{\varepsilon_2}\ln R_3/R_2\right]$$

Now, as, $E_1 R_1 \varepsilon < E_2 R_2 \varepsilon_2$, so

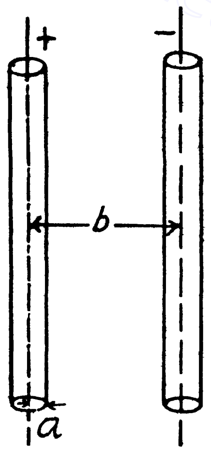
$$\frac{\lambda}{2\pi\varepsilon_0} = E_1 R_1 \varepsilon_1$$

is the maximum acceptable value, and for values greater than $E_1 R_1 \varepsilon_1$, dielectric breakdown will take place,

Hence, the maximum potential difference between the plates,

$$\varphi_+ - \varphi_- = E_1 R_1 \varepsilon_1 \left[\frac{1}{\varepsilon_1}\ln R_2/R_1 + \frac{1}{\varepsilon_2}\ln R_3/R_2\right] = E_1 R_1 \left[\ln R_2/R_1 + \frac{\varepsilon_1}{\varepsilon_2}\ln R_3/R_2\right]$$

3.108 Let us suppose that linear charge density of the wires be λ then, the potential difference, $\varphi_+ - \varphi_- = \varphi - (-\varphi) = 2\varphi$. The intensity of the electric field created by one of the wires at a distance x from its axis can be easily found with the help of the Gauss's theorem,



$$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$

Then, $\varphi = \int_a^{b-a} E\,dx = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b-a}{a}$

Hence, capacitance, per unit length,

$$\frac{\lambda}{\varphi_+ - \varphi_-} = \frac{2\pi\varepsilon_0}{\ln b/a}$$

3.109 The field in the region between the conducting plane and the wire can be obtained by using an oppositely charged wire as an image on the other side.

Then the potential difference between the wire and the plane,

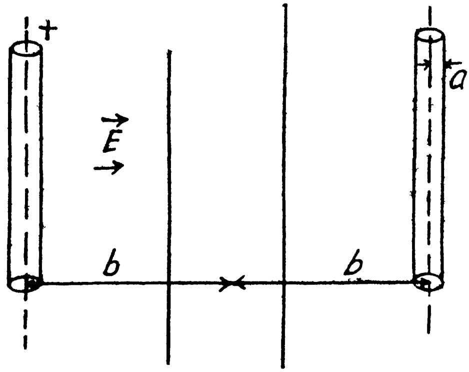
$$\Delta\varphi = \int_b \vec{E}\cdot d\vec{r}$$

$$= \int_a^b \left[\frac{\lambda}{2\pi\varepsilon_0 r} + \frac{\lambda}{2\pi\varepsilon_0(2b-r)}\right] dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} - \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{2b-a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{2b-a}{a}$$

$$\approx \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{2b}{a}, \quad \text{as} \quad b >> a$$



Hence, the sought mutual capacitance of the system per unit length of the wire

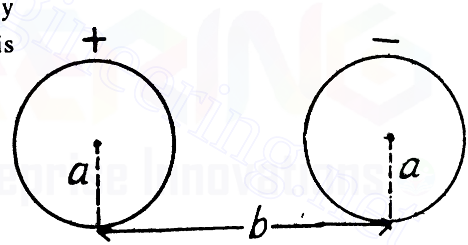
$$= \frac{\lambda}{\Delta\varphi} = \frac{2\pi\varepsilon_0}{\ln 2b/a}$$

3.110 When $b >> a$, the charge distribution on each spherical conductor is practically unaffected by the presence of the other conductor. Then, the potential φ_+ (φ_-) on the positive (respectively negative) charged conductor is

$$+\frac{q}{4\pi\varepsilon_0\varepsilon a}\left(-\frac{q}{4\pi\varepsilon_0\varepsilon a}\right)$$

Thus $\varphi_+ - \varphi_- = \dfrac{q}{2\pi\varepsilon\varepsilon_0 a}$

and $C = \dfrac{q}{\varphi_+ - \varphi_-} = 2\pi\varepsilon_0\varepsilon a.$



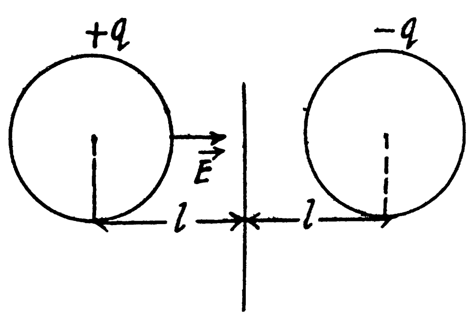
Note : if we require terms which depend on $\frac{a}{b}$, we have to take account of distribution of charge on the conductors.

3.111 As in 3.109 we apply the method of image. Then the potentical difference between the +vely charged sphere and the conducting plane is one half the nominal potential difference between the sphere and its image and is

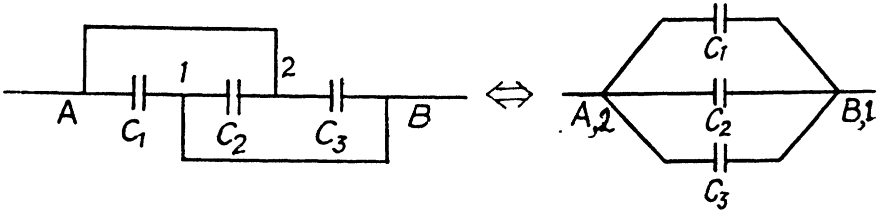
$$\Delta\varphi = \frac{1}{2}(\varphi_+ - \varphi_-) \approx \frac{q}{4\pi\varepsilon_0 a}$$

Thus

$$C = \frac{q}{\Delta\varphi} = 4\pi\varepsilon_0 a. \text{ for } l >> a.$$

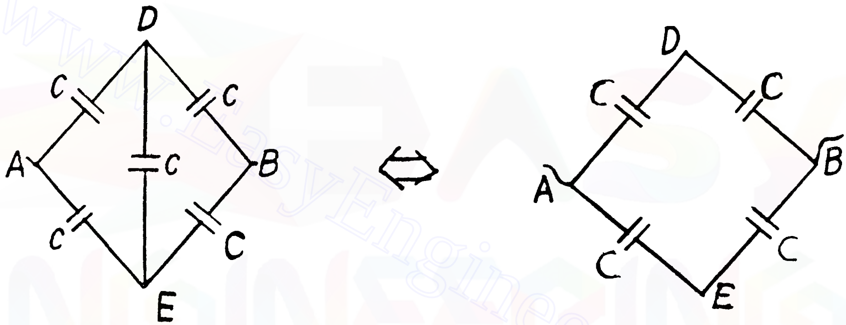


3.112



(a) Since $\varphi_1 = \varphi_B$ and $\varphi_2 = \varphi_A$

The arrangement of capacitors shown in the problem is equivalent to the arrangement shown in the Fig.



and hence the capacitance between A and B is,

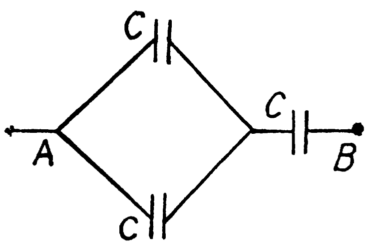
$$C = C_1 + C_2 + C_3$$

(B) From the symmetry of the problem, there is no P.d. between D and E.. So, the combination reduces to a simple arrangement shown in the Fig and hence the net capacitance,

$$C_0 = \frac{C}{2} + \frac{C}{2} = C$$

3.113 (a) In the given arrangement, we have three capacitors of equal capacitance $C = \frac{\varepsilon_0 S}{d}$ and the first and third plates are at the same potential.

Hence, we can resolve the network into a simple form using series and parallel grouping of capacitors, as shown in the figure.



Thus the equivalent capacitance

$$C_0 = \frac{(C+C)\,C}{(C+C)+C} = \frac{2}{3}C$$

(b) Let us mentally impart the charges $+q$ and $-q$ to the plates 1 and 2 and then distribute them to other plates using charge conservation and electric induction. (Fig.).

As the potential difference between the plates 1 and 2 is zero,

$$-\frac{q_1}{C}+\frac{q_2}{C}-\frac{q_1}{C}=0, \quad \left(\text{where } C=\frac{\varepsilon_0 S}{d}\right)$$

or, $\qquad q_2 = 2\,q_1,$

The potential difference between A and B,

$$\varphi = \varphi_A - \varphi_B = q_2/C$$

Hence the sought capacitane,

$$C_0 = \frac{q}{\varphi} = \frac{q_1+q_2}{q_2/C} = \frac{3\,q_1}{2q_1/C} = \frac{3}{2}C = \frac{3\,\varepsilon_0 S}{2d}$$

3.114 Amount of charge, that the capacitor of capacitance C_1 can withstand, $q_1 = C_1 V_1$ and similarly the charge, that the capacitor of capacitance C_2 can withstand, $q_2 = C_2 V_2$. But in series combination, charge on both the capacitors will be same, so, q_{max}, that the combination can withstand $= C_1 V_1$,

as $C_1 V_1 < C_2 V_2$, from the numerical data, given.

Now, net capacitance of the system,

$$C_0 = \frac{C_1 C_2}{C_1 + C_2}$$

and hence, $V_{max} = \frac{q_{max}}{C_0} = \frac{C_1 V_1}{C_1 C_2 / C_1 + C_2} = V_1\left(1+\frac{C_1}{C_2}\right) = 9\text{ kV}$

3.115 Let us distribute the charges, as shown in the figure.

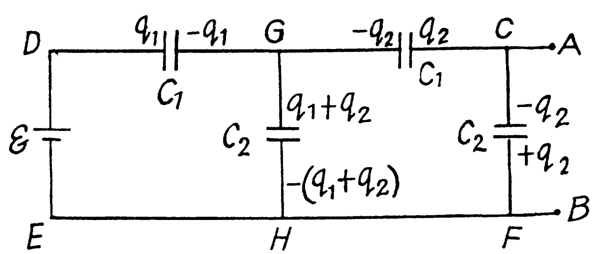
Now, we know that in a closed circuit, $-\Delta\varphi = 0$

So, in the loop, $DCFED$,

$$\frac{q_1}{C_1}-\frac{q_2}{C_1}-\frac{q_2}{C_2}=\xi \quad \text{or,} \quad q_1 = C_1\left[\xi + q_2\left(\frac{1}{C_1}+\frac{1}{C_2}\right)\right] \qquad (1)$$

Again in the loop $DGHED$,

$$\frac{q_1}{C_1}+\frac{q_1+q_2}{C_2}=\xi \qquad (2)$$



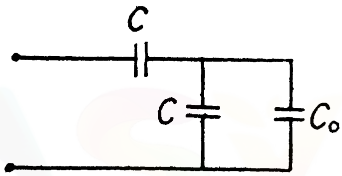
Using Eqs. (1) and (2), we get

$$q_2\left[\frac{1}{C_1}+\frac{3}{C_2}+\frac{C_1}{C_2}\right]=-\frac{\xi C_1}{C_2}$$

Now, $$\varphi_A-\varphi_B=\frac{-q_2}{C_2}=\frac{\xi}{C_2^2/C_1}\frac{1}{\left[\frac{1}{C_1}+\frac{3}{C_2}+\frac{C_1}{C_2^{\,2}}\right]}$$

or, $$\varphi_A-\varphi_B=\frac{\xi}{\left[\frac{C_2^{\,2}}{C_1^{\,2}}+\frac{3\,C_2}{C_1}+1\right]}=\frac{\xi}{\eta^2+3\,\eta+1}=10\text{ V}$$

3.116 The infinite circuit, may be reduced to the circuit, shown in the Fig. where, C_0 is the net capacitance of the combination.



So, $$\frac{1}{C+C_0}+\frac{1}{C}=\frac{1}{C_0}$$

Solving the quadratic,

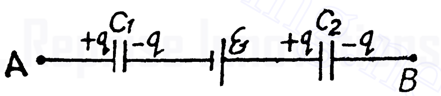
$$C\,C_0+C_0^{\,2}-C^{\,2}=0,$$

we get,

$$C_0=\frac{\left(\sqrt{5}-1\right)}{2}C,$$ taking only *+ve* value as C_0 can not be negative.

3.117 Let, us make the charge distribution, as shown in the figure.

Now, $$\varphi_A-\varphi_B=\frac{q}{C_1}-\xi+\frac{q}{C_2}$$



or, $$q=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}C_1C_2$$

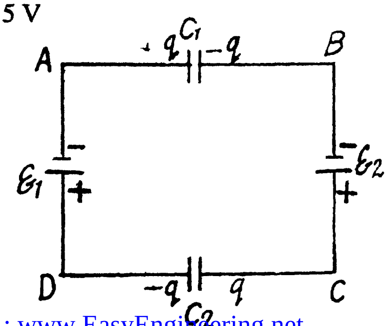
Hence, voltage across the capacitor C_1

$$=\frac{q}{C_1}=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}C_2=10\text{ V}$$

and voltage across the capacitor, C_2

$$=\frac{q}{C_2}=\frac{(\varphi_A-\varphi_B)+\xi}{C_1+C_2}C_1=5\text{ V}$$

3.118 Let $\xi_2>\xi_1$, then using $-\Delta\varphi=0$ in the closed circuit, (Fig.)



$$\frac{-q}{C_1}+\xi_2-\frac{q}{C_2}-\xi_1=0$$

or, $$q=\frac{(\xi_2-\xi_1)\,C_1C_2}{(C_1+C_2)}$$

Hence the P.D. accross the left and right plates of capacitors,

$$\varphi_1 = \frac{q}{C_1} = \frac{(\xi_2 - \xi_1) C_2}{C_1 + C_2}$$

and similarly

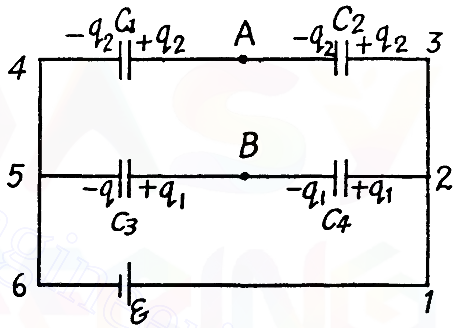
$$\varphi_2 = \frac{-q}{C_2} = \frac{(\xi_1 - \xi_2) C_1}{C_1 + C_2}$$

3.119 Taking benefit of the foregoing problem, the amount of charge on each capacitor

$$|q| = \frac{|\xi_2 - \xi_1| C_1 C_2}{C_1 + C_2}$$

3.120 Make the charge distribution, as shown in the figure. In the circuit, 12561.

$-\Delta\varphi = 0$ yields



$$\frac{q_1}{C_4} + \frac{q_1}{C_3} - \xi = 0 \quad \text{or,} \quad q_1 = \frac{\xi C_3 C_4}{C_3 + C_4}$$

and in the circuit 13461,

$$\frac{q_2}{C_2} + \frac{q_2}{C_1} - \xi = 0 \quad \text{or,} \quad q_2 = \frac{\xi C_1 C_2}{C_1 + C_2}$$

Now $\quad \varphi_A - \varphi_B = \frac{q_2}{C_1} - \frac{q_1}{C_3}$

$$= \xi \left[\frac{C_2}{C_1 + C_2} - \frac{C_4}{C_3 + C_4} \right] = \xi \left[\frac{C_2 C_3 - C_1 C_4}{(C_1 + C_2)(C_3 + C_4)} \right]$$

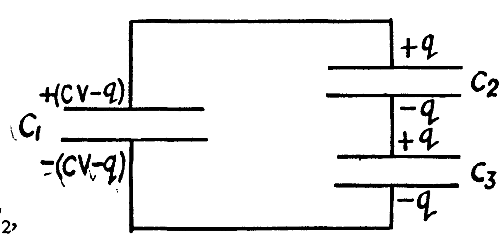
It becomes zero, when

$$(C_2 C_3 - C_1 C_4) = 0. \quad \text{or} \quad \frac{C_1}{C_2} = \frac{C_3}{C_4}$$

3.121 Let, the charge q flows through the connecting wires, then at the state of equilibrium, charge distribution will be as shown in the Fig. In the closed circuit 12341, using $-\Delta\varphi = 0$

$$-\frac{(C_1 V - q)}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = 0$$

or, $q = \dfrac{V}{(1/C_1 + 1/C_2 + 1/C_3)} = 0{\cdot}06$ mC



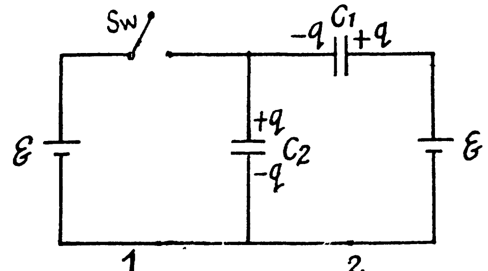
3.122 Initially, charge on the capacitor C_1 or C_2,

$q = \dfrac{\xi C_1 C_2}{C_1 + C_2}$, as they are in series combination (Fig.-a) when the switch is closed, in the circuit CDEFC from $-\Delta\varphi = 0$, (Fig. b)

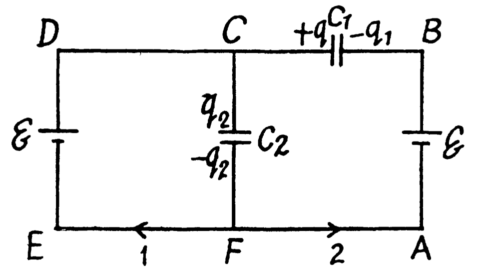
$$\xi - \frac{q_2}{C_2} = 0 \quad \text{or} \quad q_2 = C_2 \xi \tag{1}$$

And in the closed loop BCFAB from $-\Delta\varphi = 0$

$$\frac{-q_1}{C_1} + \frac{q_2}{C_2} - \xi = 0 \tag{2}$$



(a)



(b)

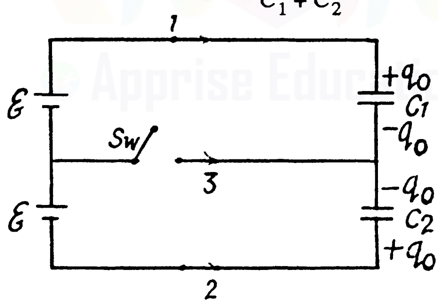
From (1) and (2) $q_1 = 0$

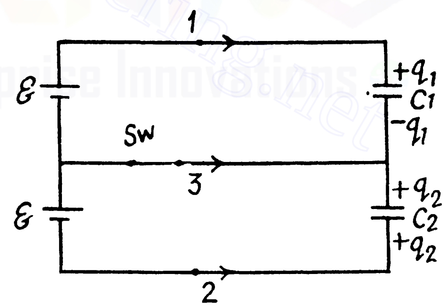
Now, charge flown through section 1 = $(q_1 + q_2) - 0 = C_2 \xi$

and charge flown through section 2 = $-q_1 - q = -\dfrac{\xi C_1 C_2}{C_1 + C_2}$

3.123 When the switch is open, (Fig-a)

$$q_0 = \frac{2\xi C_1 C_2}{C_1 + C_2}$$





and when the switch is closed,

$$q_1 = \xi C_1 \text{ and } q_2 = \xi C_2$$

Hence, the flow of charge, due to the shortening of switch,

through section 1 = $q_1 - q_0 = \xi C_1 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -24\ \mu C$

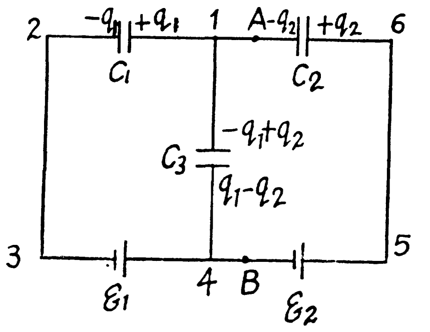
through the section 2 = $-q_2 - (q_0) = \xi C_2 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -36\ \mu C$

and through the section 3 = $q_2 - (q_2 - q_1) - 0 = \xi (C_2 - C_1) = -60\ \mu C$

3.124 First of all, make the charge distribution, as shown in the figure.

In the loop 12341, using $-\Delta\varphi = 0$

$$\frac{q_1}{C_1} - \xi_1 + \frac{q_1 - q_2}{C_3} = 0 \qquad (1)$$



Similarly, in the loop 61456, using $-\Delta\varphi = 0$

$$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \xi_2 = 0 \qquad (2)$$

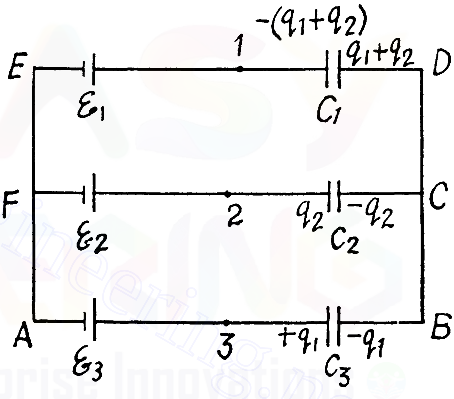
From Eqs. (1) and (2) we have

$$q_2 - q_1 = \frac{\xi_2 C_2 - \xi_1 C_1}{\frac{C_2}{C_3} + \frac{C_1}{C_3} + 1}$$

Hence, $$\varphi_A - \varphi_B = \frac{q_2 - q_1}{C_3} = \frac{\xi_2 C_2 - \xi_1 C_1}{C_1 + C_2 + C_3}$$

3.125 In the loop ABDEA, using $-\Delta\varphi = 0$

$$-\xi_3 + \frac{q_1}{C_3} + \frac{q_1 + q_2}{C_1} + \xi_1 = 0 \qquad (1)$$



Similarly in the loop ODEF, O

$$\frac{q_1 + q_2}{C_1} + \xi_1 - \xi_2 + \frac{q_2}{C_2} = 0 \qquad (2)$$

Solving Eqs. (1) and (2), we get,

$$q_1 + q_2 = \frac{\xi_2 C_2 - \xi_1 C_2 - \xi_1 C_3 + \xi_3 C_3}{\frac{C_3}{C_1} + \frac{C_2}{C_1} + 1}$$

Now, $\varphi_1 - \varphi_0 = \varphi_1 = -\dfrac{(q_1 + q_2)}{C_1}$, as $(\varphi_0 = 0)$

$$= \frac{\xi_1 (C_2 + C_3) - \xi_2 C_2 - \xi_3 C_3}{C_1 + C_2 + C_3}$$

And using the symmetry, $\varphi_2 = \dfrac{\xi_2 (C_1 + C_3) - \xi_1 C_1 - \xi_3 C_3}{C_1 + C_2 + C_3}$

and $$\varphi_3 = \frac{\xi_3 (C_1 + C_2) - \xi_1 C_1 - \xi_2 C_2}{C_1 + C_2 + C_3}$$

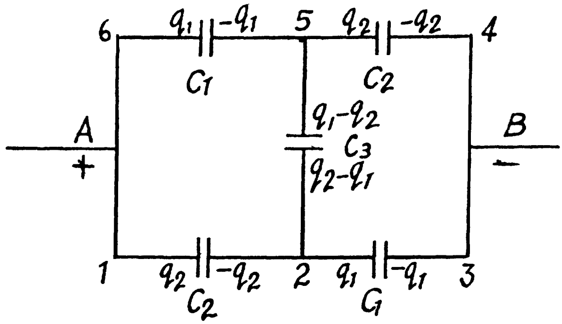
The answers have wrong sign in the book.

3.126 Taking the advantage of symmetry of the problem charge distribution may be made, as shown in the figure.

In the loop, 12561, $-\Delta\varphi = 0$

or $$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \frac{q_1}{C_1} = 0$$

or $$\frac{q_1}{q_2} = \frac{C_1 (C_3 + C_2)}{C_2 (C_1 + C_3)} \qquad (1)$$



Now, capacitance of the network,

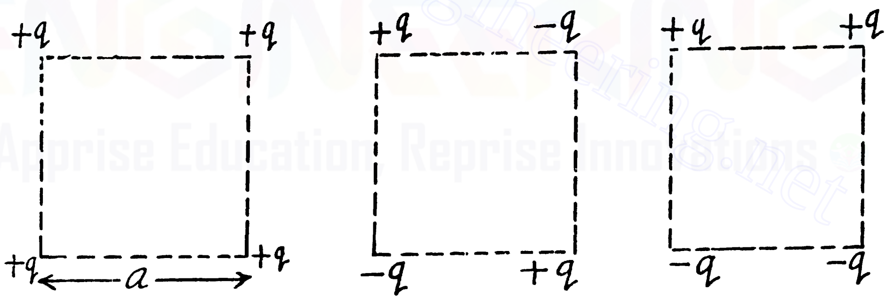
$$C_0 = \frac{q_1 + q_2}{\varphi_A - \varphi_B} = \frac{q_1 + q_2}{q_2/C_2 + q_1/C_1}$$

$$= \frac{(1 + q_1/q_2)}{\left(\frac{1}{C_2} + \frac{q_1}{q_2 C_1}\right)} \qquad (2)$$

From Eqs. (1) and (2)

$$C_0 = \frac{2 C_1 C_2 + C_3 (C_1 + C_2)}{C_1 + C_2 + 2 C_3}$$

3.127 (a) Interaction energy of any two point charges q_1 and q_2 is given by $\frac{q_1 q_2}{4 \pi \varepsilon_0 r}$ where r is the separation between the charges.



Hence, interaction energy of the system,

$$U_a = 4 \frac{q^2}{4 \pi \varepsilon_0 a} + 2 \frac{q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)}$$

$$U_b = 4 \frac{-q^2}{4 \pi \varepsilon_0 a} + 2 \frac{q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)}$$

and $$U_c = 2 \frac{q^2}{4 \pi \varepsilon_0 a} - \frac{2 q^2}{4 \pi \varepsilon_0 a} - \frac{2 q^2}{4 \pi \varepsilon_0 (\sqrt{2}\, a)} = - \frac{\sqrt{2}\, q^2}{4 \pi \varepsilon_0 a}$$

3.128 As the chain is of infinite length any two charge of same sign will occur symmetrically to any other charge of opposite sign.

So, interaction energy of each charge with all the others,

$$U = -2\frac{q^2}{4\pi\varepsilon_0 a}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ \text{......... up to } \infty\right] \quad (1)$$

But $$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \text{ .... up to } \infty$$

and putting $x = 1$ we get $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} + \text{......... up to } \infty$ (2)

From Eqs. (1) and (2),

$$U = \frac{-2q^2 \ln 2}{4\pi\varepsilon_0 a}$$

3.129 Using electrical image method, interaction energy of the charge q with those induced on the plane.

$$U = \frac{-q^2}{4\pi\varepsilon_0(2l)} = -\frac{q^2}{8\pi\varepsilon_0 l}$$

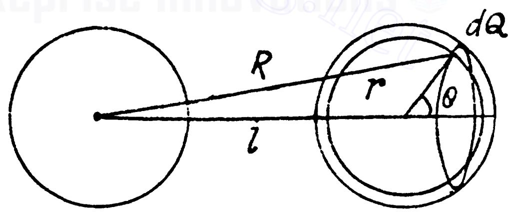
3.130 Consider the interaction energy of one of the balls (say 1) and thin spherical shell of the other. This interaction energy can be written as $\int d\varphi q$

$$= \int \frac{q_1}{4\pi\varepsilon_0 R}\rho_2(r)\,2\pi r^2 \sin\theta\, d\theta\, dr = \int_0^{\pi} \frac{\rho_2(r)\,q_1 r^2 \sin\theta\, d\theta\, dr}{2\varepsilon_0(l^2+r^2+2lr\cos\theta)^{1/2}}$$

$$= \frac{q_1 r}{2\varepsilon_0 l} dr \int_{l-r}^{l+r} dx\,\rho_2(r)$$

$$= \frac{q_1 r}{2\varepsilon_0 l} dr \cdot 2r\,\rho_2(r) \cdot 2$$

$$= \frac{q_1}{4\pi\varepsilon_0 l} 4\pi r^2\, dr\, \rho_2(r)$$



Hence finally integrating

$$U_{int} = \frac{q_1 q_2}{4\pi\varepsilon_0 l} \quad \text{where, } q_2 = \int_0^{\infty} 4\pi r^2 \rho_2(r)\, dr$$

3.131 Charge contained in the capacitor of capacitance C_1 is $q = C_1\varphi$ and the energy, stored in it :

$$U_i = \frac{q^2}{2C_1} = \frac{1}{2}C_1\varphi^2$$

Now, when the capacitors are connected in parallel, equivalent capacitance of the system, $C = C_1 + C_2$ and hence, energy stored in the system :

$U_f = \frac{C_1^2 \varphi^2}{2(C_1 + C_2)}$, as charge remains conserved during the process.

So, increment in the energy,

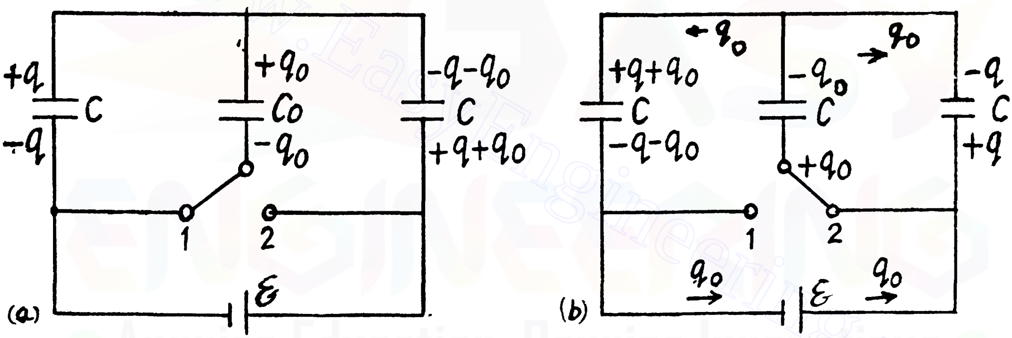
$$\Delta U = \frac{C_1^2 \varphi^2}{2}\left(\frac{1}{C_1 + C_2} - \frac{1}{C_1}\right) = \frac{-C_2 C_1 \varphi^2}{2(C_1 + C_2)} = -0{\cdot}03 \text{ mJ}$$

3.132 The charge on the condensers in position 1 are as shown. Here

$$\frac{q}{C} = \frac{q_0}{C_0} = \frac{q + q_0}{C + C_0}$$

and $$(q + q_0)\left(\frac{1}{C + C_0} + \frac{1}{C}\right) = \xi \quad \text{or,} \quad q + q_0 = \frac{C(C + C_0)\xi}{C_0 + 2C}$$

Hence, $$q = \frac{C^2 \xi}{C_0 + 2C} \quad \text{and} \quad q_0 = \frac{C C_0 \xi}{C_0 + 2C}$$



After the switch is thrown to position 2, the charges change as shown in (Fig-b).
A charge q_0 has flown in the right loop through the two condensers and a charge q_0 through the cell, Because of the symmetry of the problem there is no change in the energy stored in the condensers. Thus

H (Heat produced) = Energy delivered by the cell

$$= \Delta q \, \xi = q_0 \, \xi = \frac{C C_0 \xi^2}{C_0 + 2C}$$

3.133 Initially, the charge on the right plate of the capacitor, $q = C(\xi_1 - \xi_2)$ and finally, when switched to the position, 2. charge on the same plate of capacitor ;

$$q' = C \xi_1$$

So, $$\Delta q = q' = C \xi_2$$

Now, from energy conservation,

ΔU + Heat liberated = A_{cell}, where ΔU is the electrical energy.

$$\frac{1}{2}C\xi_1^2 - \frac{1}{2}C(\xi_1 - \xi_2)^2 + \text{Heat liberated} = \Delta q\,\xi_1$$

as only the cell with e.m.f. ξ_1 is responsible for redistribution of the charge. So,

$$C\xi_1\xi_2 - \frac{1}{2}C\xi_2^2 + \text{Heat liberated} = C\xi_2\xi_1.$$

Hence heat liberated $= \frac{1}{2}C\xi_2^2$

3.134 Self energy of each shell is given by $\frac{q\varphi}{2}$, where φ is the potential of the shell, created only by the charge q, on it.

Hence, self energy of the shells 1 and 2 are :

$$W_1 = \frac{q_1^{\,2}}{8\pi\varepsilon_0 R_1} \text{ and } W_2 = \frac{q_2^{\,2}}{8\pi\varepsilon_0 R_2}$$

The interaction energy between the charged shells equals charge q of one shell, multiplied by the potential φ, created by other shell, at the point of location of charge q.

So, $$W_{12} = q_1\frac{q_2}{4\pi\varepsilon_0 R_2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R_2}$$

Hence, total enegy of the system,

$$U = W_1 + W_2 + W_{12}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1^2}{2R_1} + \frac{q_2^2}{2R_2} + \frac{q_1 q_2}{R_2}\right]$$

3.135 Electric fields inside and outside the sphere with the help of Gauss theorem :

$$E_1 = \frac{qr}{4\pi\varepsilon_0 R^2}(r \le R), E_2 = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}(r > R)$$

Sought self energy of the ball

$$U = W_1 + W_2$$

$$= \int_0^R \frac{\varepsilon_0 E_1^{\,2}}{2}4\pi r^2\,dr + \int_R^\infty \frac{\varepsilon_0 E_2^{\,2}}{2}4\pi r^2\,dr = \frac{q^2}{8\pi\varepsilon_0 R}\left(\frac{1}{5} + 1\right)$$

Hence, $$U = \frac{3q^2}{4\pi\varepsilon_0 5R} \text{ and } \frac{W_1}{W_2} = \frac{1}{5}$$

3.136 (a) By the expression $\int \frac{1}{2}\varepsilon_0\varepsilon E^2\,dV = \int \frac{1}{2}\varepsilon\varepsilon_0 E^2 4\pi r^2\,dr$, for a spherical layer.

To find the electrostatic energy inside the dielectric layer, we have to integrate the upper expression in the limit $[a, b]$

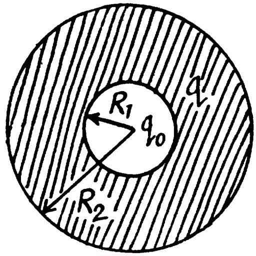
$$U = \frac{1}{2}\varepsilon_0\varepsilon\int_a^b\left(\frac{q}{4\pi\varepsilon_0\varepsilon r^2}\right)^2 4\pi r^2\,dr = \frac{q^2}{8\pi\varepsilon_0\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] = 27\text{ mJ}$$

3.137 As the field is conservative total work done by the field force,

$$A_{fd} = U_i - U_f = \frac{1}{2} q (\varphi_1 - \varphi_2)$$

$$= \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{q^2}{8 \pi \varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

3.138 Initially, energy of the system, $U_i = W_1 + W_{12}$ where, W_1 is the self energy and W_{12} is the mutual energy.



So, $$U_i = \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0 R_1} + \frac{q q_0}{4 \pi \varepsilon_0 R_1}$$

and on expansion, energy of the system,

$$U_f = W'_1 + W'_{12}$$

$$= \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0 R_2} + \frac{q q_0}{4 \pi \varepsilon_0 R_2}$$

Now, work done by the field force, A equals the decrement in the electrical energy,

i.e. $$A = (U_i - U_f) = \frac{q (q_0 + q/2)}{4 \pi \varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Alternate : The work of electric forces is equal to the decrease in electric energy of the system,

$$A = U_i - U_f$$

In order to find the difference $U_i - U_f$, we note that upon expansion of the shell, the electric field and hence the energy localized in it, changed only in the hatched spherical layer consequently (Fig.).

$$U_i - U_f = \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} (E_1^2 - E_2^2) \cdot 4 \pi r^2 \, dr$$

where E_1 and E_2 are the field intensities (in the hatched region at a distance r from the centre of the system) before and after the expansion of the shell. By using Gauss' theorem, we find

$$E_1 = \frac{1}{4 \pi \varepsilon_0} \frac{q + q_0}{r^2} \text{ and } E_2 = \frac{1}{4 \pi \varepsilon_0} \frac{q_0}{r^2}$$

As a result of integration, we obtain

$$A = \frac{q (q_0 + q/2)}{4 \pi \varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

3.139 Energy of the charged sphere of radius r, from the equation

$$U = \frac{1}{2} q \varphi = \frac{1}{2} q \frac{q}{4 \pi \varepsilon_0 r} = \frac{q^2}{8 \pi \varepsilon_0 r}$$

If the radius of the shell changes by dr then work done is

$$4 \pi r^2 F_u \, dr = -dU = q^2 / 8 \pi \varepsilon_0 r^2$$

Thus sought force per unit area,

$$F_u = \frac{q^2}{4 \pi r^2 (8 \pi \varepsilon_0 r^2)} = \frac{(4 \pi r^2 \sigma)^2}{4 \pi r^2 \times 8 \pi \varepsilon_0 r^2} = \frac{\sigma^2}{2 \varepsilon_0}$$

3.140 Initially, there will be induced charges of magnitude $-q$ and $+q$ on the inner and outer surface of the spherical layer respectively. Hence, the total electrical energy of the system is the sum of self energies of spherical shells, having radii a and b, and their mutual energies including the point charge q.

$$U_i = \frac{1}{2} \frac{q^2}{4 \pi \varepsilon_0 b} + \frac{1}{2} \frac{(-q)^2}{4 \pi \varepsilon_0 a} + \frac{-q\,q}{4 \pi \varepsilon_0 a} + \frac{q q}{4 \pi \varepsilon_0 b} + \frac{-q q}{4 \pi \varepsilon_0 b}$$

or
$$U_i = \frac{q^2}{8 \pi \varepsilon_0} \left[\frac{1}{b} - \frac{1}{a} \right]$$

Finally, charge q is at infinity hence, $U_f = 0$

Now, work done by the agent = increment in the energy

$$= U_f - U_i = \frac{q^2}{8 \pi \varepsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right]$$

3.141 (a) Sought work is equivalent to the work performed against the electric field created by one plate, holding at rest and to bring the other plate away. Therefore the required work,

$$A_{\text{agent}} = q E (x_2 - x_1),$$

where $E = \dfrac{\sigma}{2 \varepsilon_0}$ is the intensity of the field created by one plate at the location of other.

So,
$$A_{agent} = q \frac{\sigma}{2 \varepsilon_0} (x_2 - x_1) = \frac{q^2}{2 \varepsilon_0 S} (x_2 - x_1)$$

Alternate : $A_{\text{ext}} = \Delta U$ (as field is potential)

$$= \frac{q^2}{2 \varepsilon_0 S} x_2 - \frac{q^2}{2 \varepsilon_0 S} x = \frac{q^2}{2 \varepsilon_0 S} (x_2 - x_1)$$

(b) When voltage is kept const., the force acing on each plate of capacitor will depend on the distance between the plates.

So, elementary work done by agent, in its displacement over a distance dx, relative to the other,

$$dA = -F_x \, dx$$

But,
$$F_x = -\left(\frac{\sigma(x)}{2 \varepsilon_0} \right) S \, \sigma(x) \quad \text{and} \quad \sigma(x) = \frac{\varepsilon_0 V}{x}$$

Hence,
$$A = \int dA = \int_{x_1}^{x_2} \frac{1}{2} \varepsilon_0 \frac{S V^2}{x^2} dx = \frac{\varepsilon_0 S V^2}{2} \left[\frac{1}{x_1} - \frac{1}{x_2} \right]$$

Alternate : From energy Conservation,

$$U_f - U_i = A_{\text{cell}} + A_{\text{agent}}$$

or
$$\frac{1}{2}\frac{\varepsilon_0 S}{x_2}V^2 - \frac{1}{2}\frac{\varepsilon_0 S}{x_1}V^2 = \left[\frac{\varepsilon_0 S}{x_2} - \frac{\varepsilon_0 S}{x_1}\right]V^2 + A_{\text{agent}}$$

(as $A_{\text{cell}} = (q_f - q_i)\,V = (C_f - C_i)\,V^2$)

So
$$A_{\text{agent}} = \frac{\varepsilon_0 S V^2}{2}\left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

3.142 (a) When metal plate of thickness ηd is inserted inside the capacitor, capacitance of the system becomes $C_0 = \dfrac{\varepsilon_0 S}{d(1-\eta)}$

Now, initially, charge on the capacitor, $q_0 = C_0 V = \dfrac{\varepsilon_0 S V}{d(1-\eta)}$

Finally, capacitance of the capacitor, $C = \dfrac{\varepsilon_0 S}{d}$

As the source is disconnected, charge on the plates will remain same during the process. Now, from energy conservation,

$$U_f - U_i = A_{\text{agent}} \text{ (as cell does no work)}$$

or,
$$\frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{q_0^2}{C_0} = A_{\text{agent}}$$

Hence
$$A_{\text{agent}} = \frac{1}{2}\left[\frac{\varepsilon_0 S V}{d(1-\eta)}\right]^2\left[\frac{1}{C} - \frac{(1-\eta)}{C}\right] = \frac{1}{2}\frac{C V^2 \eta}{(1-\eta)^2} = 1{\cdot}5 \text{ mJ}$$

(b) Initially, capacitance of the system is given by,

$C_0 = \dfrac{C\varepsilon}{\eta(1-\varepsilon)+\varepsilon}$ (this is the capacitance of two capacitors in series)

So, charge on the plate, $q_0 = C_0 V$

Capacitance of the capacitor, after the glass plate has been removed equals C

From energy conservation,

$$A_{\text{agent}} = U_f - U_i$$
$$= \frac{1}{2}q_0^2\left[\frac{1}{C} - \frac{1}{C_0}\right] = \frac{1}{2}\frac{C V^2 \varepsilon\,\eta\,(\varepsilon - 1)}{[\varepsilon - \eta(\varepsilon - 1)]^2} = 0{\cdot}8 \text{ mJ}$$

3.143 When the capactior which is immersed in water is connected to a constant voltage source, it gets charged. Suppose σ_0 is the free charge density on the condenser plates. Because water is a dielectric, bound charges also appear in it. Let σ' be the surface density of bound charges. (Because of homogeneity of the medium and uniformity of the field when we ignore edge effects no volume density of bound charges exists.) The electric field due to free charges only $\dfrac{\sigma_0}{\varepsilon_0}$; that due to bound charges is $\dfrac{\sigma'}{\varepsilon_0}$ and the total electric field is $\dfrac{\sigma_0}{\varepsilon\varepsilon_0}$. Recalling that the sign of bound charges is opposite of the free charges, we have

$$\frac{\sigma_0}{\varepsilon\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0} \quad \text{or,} \quad \sigma' = \left(\frac{\varepsilon - 1}{\varepsilon}\right)\sigma_0$$

Because of the field that exists due to the free charges (not the total field; the field due to the bound charges must be excluded for this purpose as they only give rise to self energy effects), there is a force attracting the bound charges to the near by plates. This force is

$$\frac{1}{2}\sigma' \frac{\sigma_0}{\varepsilon_0} = \frac{(\varepsilon - 1)\,\sigma_0^2}{2\varepsilon\varepsilon_0} \text{ per unit area.}$$

The factor $\frac{1}{2}$ needs an explanation. Normally the force on a test charge is qE in an electric field E. But if the charge itself is produced by the electric filed then the force must be constructed bit by bit and is

$$F = \int_0^E q(E')\,dE'$$

if $q(E') \propto E'$ then we get

$$F = \frac{1}{2}q(E)E$$

This factor of $\frac{1}{2}$ is well known. For example the energy of a dipole of moment $\vec{p}$ in an electric field $\vec{E}_0$ is $-\vec{p}\cdot\vec{E}_0$ while the energy per unit volume of a linear dielectric in an electric field is $-\frac{1}{2}\vec{P}\cdot\vec{E}_0$ where $\vec{P}$ is the polarization vector (i.e. dipole moment per unit volume). Now the force per unit area manifests itself as excess pressure of the liquid.

Noting that $$\frac{V}{d} = \frac{\sigma_0}{\varepsilon\varepsilon_0}$$

We get $$\Delta p = \frac{\varepsilon_0\varepsilon(\varepsilon - 1)V^2}{2d^2}$$

Substitution, using $\varepsilon = 81$ for water, gives $\Delta p = 7{\cdot}17$ k Pa $= 0{\cdot}07$ atm.

3.144 One way of doing this problem will be exactly as in the previous case so let us try an alternative method based on energy. Suppose the liquid rises by a distance h. Then let us calculate the extra energy of the liquid as a sum of polarization energy and the ordinary gravitiational energy. The latter is

$$\frac{1}{2}h \cdot \rho g \cdot Sh = \frac{1}{2}\rho g S h^2$$

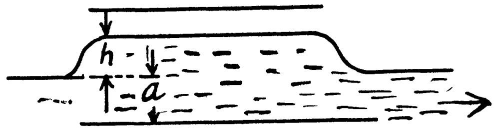
If σ is the free charge surface density on the plate, the bound charge density is, from the previous problem,

$$\sigma' = \frac{\varepsilon - 1}{\varepsilon}\sigma$$

This is also the volume density of induced dipole moment i.e. Polarization. Then the energy is, as before

$$-\frac{1}{2}\cdot\sigma' E_0 = \frac{-1}{2}\cdot\sigma'\frac{\sigma}{\varepsilon_0} = \frac{-(\varepsilon - 1)\,\sigma^2}{2\varepsilon_0\varepsilon}$$

and the total polarization energy is



$$-S(a+h)\frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\varepsilon}$$

Then, total energy is

$$U(h) = -S(a+h)\frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\varepsilon} + \frac{1}{2}\rho g S h^2$$

The actual height to which the liquid rises is determined from the formula

$$\frac{dU}{dh} = U'(h) = 0$$

This gives $$h = \frac{(\varepsilon-1)\sigma^2}{2\varepsilon_0\,\varepsilon\,\rho g}.$$

3.145 We know that energy of a capacitor, $U = \dfrac{q^2}{2C}$.

Hence, from $F_x = \left.\dfrac{\partial U}{\partial x}\right|_{q=\text{Const.}}$ we have, $F_x = \dfrac{q^2}{2}\dfrac{\partial c}{\partial x} \Big/ C^2$ (1)

Now, since $d << R$, the capacitance of the given capacitor can be calculated by the formula of a parallel plate capacitor. Therefore, if the dielectric is introduced upto a depth x and the length of the capacitor is l, we have,

$$C = \frac{2\pi\varepsilon_0\varepsilon R x}{d} + \frac{2\pi R\varepsilon_0(l-x)}{d} \quad (2)$$

From (1) and (2), we get,

$$F_x = \varepsilon_0(\varepsilon-1)\frac{\pi R V^2}{d}$$

3.146 When the capacitor is kept at a constant potential difference V, the work performed by the moment of electrostatic forces between the plates when the inner moveable plate is rotated by an angle $d\varphi$ equals the increase in the potential energy of the system. This comes about because when charges are made, charges flow from the battery to keep the potential constant and the amount of the work done by these charges is twice in magnitude and opposite in sign to the change in the energy of the capacitor Thus

$$N_z = \frac{\partial U}{\partial \varphi} = \frac{1}{2}V^2\frac{\partial C}{\partial \varphi}$$

Now the capacitor can be thought of as made up two parts (with and without the dielectric) in parallel.

Thus $C = \dfrac{\varepsilon_0 R^2\varphi}{2d} + \dfrac{\varepsilon_0\varepsilon(\pi-\varphi)R^2}{2d}$

as the area of a sector of angle φ is $\frac{1}{2}R^2\varphi$. Differentiation then gives

$$N_z = -\frac{(\varepsilon-1)\varepsilon_0 R^2 V^2}{4d}$$

The negative sign of N_z indicates that the moment of the force is acting clockwise (i.e. trying to suck in the dielectric).

3.4 ELECTRIC CURRENT

3.147 The convection current is

$$I = \frac{dq}{dt} \tag{1}$$

here, $dq = \lambda\, dx$, where λ is the linear charge density.

But, from the Gauss' theorem, electric field at the surface of the cylinder,

$$E = \frac{\lambda}{2\pi\varepsilon_0 a}$$

Hence, substituting the value of λ and subsequently of dq in Eqs. (1), we get

$$I = \frac{2E\pi\varepsilon_0\, a dx}{dt}$$

$$= 2\pi\varepsilon_0 E a v\,, \quad \text{as } \frac{dx}{dt} = v$$

3.148 Since $d << r$, the capacitance of the given capacitor can be calculated using the formula for a parallel plate capacitor. Therefore if the water (permittivity ε) is introduced up to a height x and the capacitor is of length l, we have,

$$C = \frac{\varepsilon\varepsilon_0 \cdot 2\pi r x}{d} + \frac{\varepsilon_0 (l-x)\, 2\pi r}{d} = \frac{\varepsilon_0\, 2\pi r}{d}(\varepsilon x + l - x)$$

Hence charge on the plate at that instant, $q = CV$

Again we know that the electric current intensity,

$$I = \frac{dq}{dt} = \frac{d(CV)}{dt}$$

$$= \frac{V\varepsilon_0\, 2\pi r}{d}\frac{d(\varepsilon x + l - x)}{dt} = \frac{V\, 2\pi r\,\varepsilon_0}{d}(\varepsilon - 1)\frac{dx}{dt}$$

But, $$\frac{dx}{dt} = v,$$

So, $$I = \frac{2\pi r\varepsilon_0(\varepsilon - 1)V}{d} v = 0{\cdot}11\ \mu\text{A}$$

3.149 We have, $R_t = R_0(1 + \alpha t)$,(1)

where R_t and R_0 are resistances at $t°$ C and $0°$ C respectively and α is the mean temperature coefficient of resistance.

So, $$R_1 = R_0(1 + \alpha_1 t) \text{ and } R_2 = \eta R_0 (1 + \alpha_2 t)$$

(a) In case of series combination, $R = \Sigma R_i$

so $$R = R_1 + R_2 = R_0[(1+\eta) + (\alpha_1 + \eta\alpha_2)t] \tag{1}$$

$$= R_0(1+\eta)\left[1 + 2\frac{\alpha_1 + \eta\alpha_2}{1+\eta} t\right] \tag{2}$$

Comparing Eqs. (1) and (2), we conclude that temperature co-efficient of resistance of the circuit,

$$\alpha = \frac{\alpha_1 + \eta\,\alpha_2}{1+\eta}$$

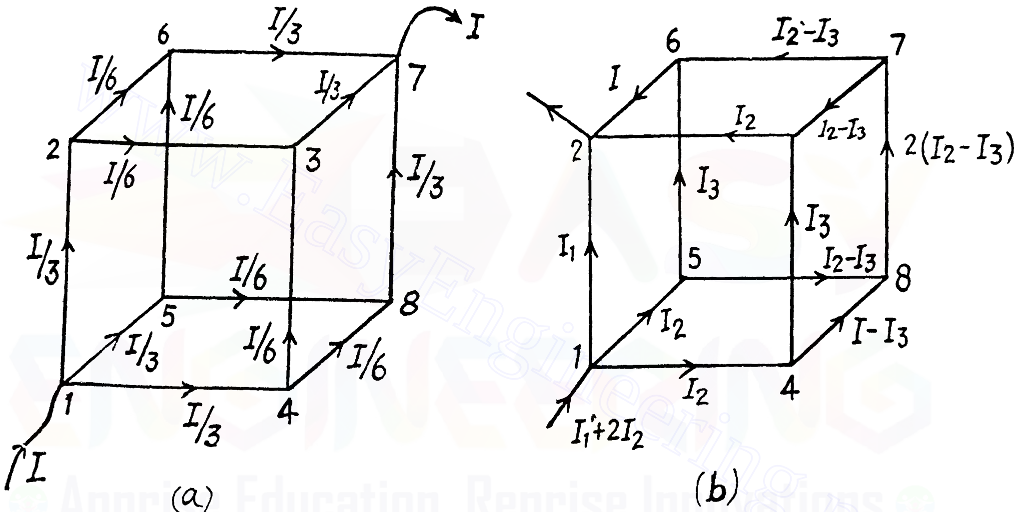
(b) In parallel combination

$$R = \frac{R_0\,(1+\alpha_1 t)\,R_0\,\eta\,(1+\alpha_2 t)}{R_0\,(1+\alpha_1 t) + \eta\,R_0\,(1+\alpha_2 t)} = R'\,(1+\alpha' t), \text{ where } R' = \frac{\eta R_0}{1+\eta}$$

Now, neglecting the terms, proportional to the product of temperature coefficients, as being very small, we get,

$$\alpha' \approx \frac{\eta\,\alpha_1 + \alpha_2}{1+\eta}$$

3.150 (a) The currents are as shown. From Ohm's law applied between 1 and 7 via 1487 (say)



$$IR_{eq} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}\,RI$$

Thus, $$R_{eq} = \frac{5R}{6}$$

(b) Between 1 and 2 from the loop 14321,

$I_1 R = 2I_2 R + I_3 R$ or, $I_1 = I_3 + 2I_2$

From the loop 48734,

$$(I_2 - I_3)\,R + 2\,(I_2 - I_3)\,R + (I_2 - I_3)\,R = I_3 R.$$

or, $$4\,(I_2 - I_3) = I_3 \text{ or } I_3 = \frac{4}{5}I_2$$

so $$I_1 = \frac{14}{5}I_2$$

Then, $(I_1 + 2I_2)\,R_{eq} = \dfrac{24}{5}I_2 R_{eq} = I_1 R = \dfrac{14}{5}I_2 R$

or $$R_{eq} = \frac{7}{12} R$$

(c) Between 1 and 3

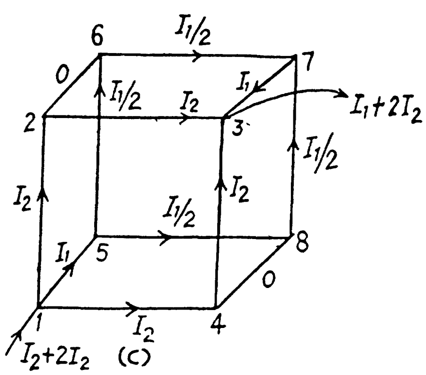
From the loop 15621

$$I_2 R = I_1 R + \frac{I_1}{2} R \quad \text{or,} \quad I_2 = 3\frac{I_1}{2}$$

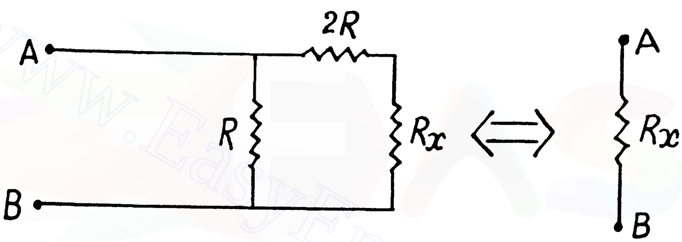
Then, $$(I_1 + 2I_2) R_{eq} = 4 I_1 R_{eq}$$

$$= I_2 R + I_2 R = 3 I_1 R$$

Hence, $$R_{eq} = \frac{3}{4} R \; .$$



3.151 Total resistance of the circuit will be independent of the number of cells,



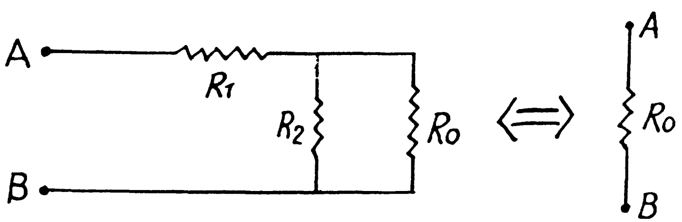
if $$R_x = \frac{(R_x + 2R) R}{R_x + 2R + R}$$

or, $$R_x^2 + 2R R_x - 2R^2 = 0$$

On solving and rejecting the negative root of the quadratic equation, we have,

$$R_x = R(\sqrt{3} - 1)$$

3.152 Let R_0 be the resistance of the network,



then, $R_0 = \dfrac{R_0 R_2}{R_0 + R_2}$ or $R_0^2 - R_0 R_1 - R_1 R_2 = 0$

On solving we get,

$$R_0 = \frac{R_1}{2}\left(1 + \sqrt{1 + 4\frac{R_2}{R_1}}\right) = 6\,\Omega$$

3.153 Suppose that the voltage V is applied between the points A and B then

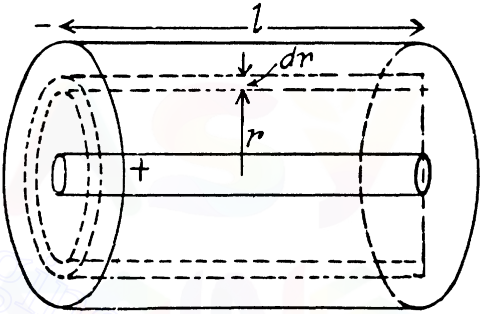
$$V = IR = I_0 R_0,$$

where R is resistance of whole the grid, I, the current through the grid and I_0, the current through the segment AB. Now from symmetry, $I/4$ is the part of the current, flowing through all the four wire segments, meeting at the point A and similarly the amount of current flowing through the wires, meeting at B is also $I/4$. Thus a current $I/2$ flows through the conductor AB, i.e.

$$I_0 = \frac{I}{2}$$

Hence,
$$R = \frac{R_0}{2}$$

3.154 Let us mentally isolate a thin cylindrical layer of inner and outer radii r and $r + dr$ respectively. As lines of current at all the points of this layer are perpendicular to it, such a layer can be treated as a cylindrical conductor of thickness dr and cross-sectional area $2\pi rl$. So, we have,



$$dR = \rho \frac{dr}{S(r)} = \rho \frac{dr}{2\pi rl}$$

and integrating between the limits, we get,

$$R = \frac{\rho}{2\pi l} \ln \frac{b}{a}$$

3.155 Let us mentally isolate a thin spherical layer of inner and outer radii r and $r + dr$. Lines of current at all the points of the this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$. So

$$dR = \rho \frac{dr}{4\pi r^2} \qquad (1)$$

And integrating (1) between the limits $[a, b]$, we get,

$$R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Now, for $b \to \infty$, we have

$$R = \frac{\rho}{4\pi a}$$

3.156 In our system, resistance of the medium $R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$,

where ρ is the resistivity of the medium

The current
$$i = \frac{\varphi}{R} = \frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Also, $i = \frac{-dq}{dt} = -\frac{d(C\varphi)}{dt} = -C\frac{d\varphi}{dt}$, as capacitance is constant. (2)

So, equating (1) and (2) we get,

$$\frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]} = -C\frac{d\varphi}{dt}$$

or,
$$-\int\frac{d\varphi}{\varphi} = \frac{\Delta t}{\frac{C\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

or,
$$\ln\eta = \frac{\Delta t\, 4\pi ab}{C\rho(b-a)}$$

Hence, resistivity of the medium,

$$\rho = \frac{4\pi\Delta t\, ab}{C(b-a)\ln\eta}$$

3.157 Let us mentally impart the charge $+q$ and $-q$ to the balls respectively. The electric field strength at the surface of a ball will be determined only by its own charge and the charge can be considered to be uniformly distributed over the surface, because the other ball is at infinite distance. Magnitude of the field strength is given by,

$$E = \frac{q}{4\pi\varepsilon_0 a^2}$$

So, current density $j = \frac{1}{\rho}\frac{q}{4\pi\varepsilon_0 a^2}$ and electric current

$$I = \int\vec{j}\cdot d\vec{S} = jS = \frac{q}{\rho\, 4\pi\varepsilon_0 a^2}\cdot 4\pi a^2 = \frac{q}{\rho\varepsilon_0}$$

But, potential difference between the balls,

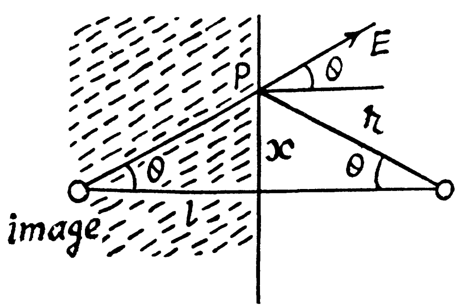
$$\varphi_+ - \varphi_- = 2\frac{q}{4\pi\varepsilon_0 a}$$

Hence, the sought resistance,

$$R = \frac{\varphi_+ - \varphi_-}{I} = \frac{2q/4\pi\varepsilon_0 a}{q/\rho\varepsilon_0} = \frac{\rho}{2\pi a}$$

3.158 (a) The potential in the unshaded region beyond the conductor as the potential of the given charge and its image and has the form

$$\varphi = A\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$



where r_1, r_2 are the distances of the point from the charge and its image. The potential has been taken to be zero on the conducting plane and on the ball

$$\varphi \approx A\left(\frac{1}{a}-\frac{1}{2l}\right) = V$$

So $A \approx Va$. In this calculation the conditions $a \ll l$ is used to ignore the variation of φ over the ball.

The electric field at P can be calculated similarly. The charge on the ball is

$$Q = 4\pi\varepsilon_0 Va$$

and
$$E_P = \frac{Va}{r^2}\; 2\cos\theta = \frac{2alV}{r^3}$$

Then $j = \frac{1}{\rho}E = \frac{2alV}{\rho r^3}$ normal to the plane.

(b) The total current flowing into the conducting plane is

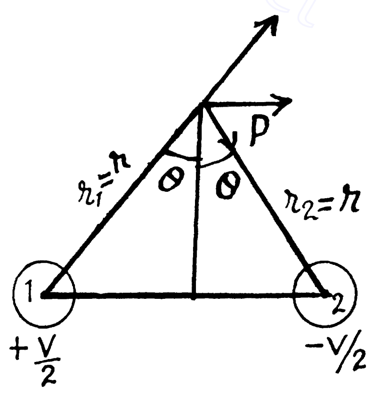
$$I = \int_0^\infty 2\pi x\, dx\, j = \int_0^\infty 2\pi x\, dx \frac{2alV}{\rho\left(\pi^2 + l^2\right)^{3/2}}$$

(On putting $y = x^2 + l^2$)

$$I = \frac{2\pi alV}{\rho}\int_{l^2}^\infty \frac{dy}{y^{3/2}} = \frac{4\pi aV}{\rho}$$

Hence
$$R = \frac{V}{I} = \frac{\rho}{2\pi a}$$

3.159 (a) The wires themselves will be assumed to be perfect conductors so the resistance is entirely due to the medium. If the wire is of length L, the resistance R of the medium is $\alpha \frac{1}{L}$ because different sections of the wire are connected in parallel (by the medium) rather than in series. Thus if R_1 is the resistance per unit length of the wire then $R = R_1/L$. Unit of R_1 is ohm-meter.



The potential at a point P is by symmetry and superposition

(for $l \gg a$)

$$\varphi = \frac{A}{2}\ln\frac{r_1}{a} - \frac{A}{2}\ln\frac{r_2}{a}$$

$$= \frac{A}{2}\ln\frac{r_1}{r_2}$$

Then $\varphi_1 = \frac{V}{2} = \frac{A}{2}\ln\frac{a}{l}$ (for the potential of 1) or, $$A = -V/\ln\frac{l}{a}$$

and $$\varphi = -\frac{V}{2\ln l/a}\ln r_1/r_2$$

We then calculate the field at a point P which is equidistant from 1 & 2 and at a distance r from both :

Then $$E = \frac{V}{2\ln l/a}\left(\frac{1}{r}\right)\times 2\sin\theta$$

$$= \frac{Vl}{2\ln l/a}\frac{1}{r^2}$$

and $$J = \sigma E = \frac{1}{\rho}\frac{V}{2\ln l/a}\frac{1}{r^2}$$

(b) Near either wire $$E = \frac{V}{2\ln l/a}\frac{1}{a}$$

and $$J = \sigma E = \frac{1}{\rho}\frac{V}{2\ln l/a}$$

Then $$I = \frac{V}{R} = L\frac{V}{R_1} = J\,2\pi aL$$

Which gives $$R_1 = \frac{\rho}{\pi}\ln l/a$$

3.160 Let us mentally impart the charges $+q$ and $-q$ to the plates of the capacitor.
Then capacitance of the network,

$$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0\int E_n\,dS}{\varphi} \quad (1)$$

Now, electric current,

$$i = \int \vec{j}\cdot d\vec{S} = \int \sigma E_n\,dS \text{ as } \vec{j}\uparrow\uparrow\vec{E}. \quad (2)$$

Hence, using (1) in (2), we get,

$$i = \frac{C\varphi}{\varepsilon\varepsilon_0}\sigma = \frac{C\varphi}{\rho\,\varepsilon\varepsilon_0} = 1{\cdot}5\,\mu\text{A}$$

3.161 Let us mentally impart charges $+q$ and $-q$ to the conductors. As the medium is poorly conducting, the surfaces of the conductors are equipotential and the field configuration is same as in the absence of the medium.

Let us surround, for example, the positively charged conductor, by a closed surface S, just containing the conductor,

then, $$R = \frac{\varphi}{i} = \frac{\varphi}{\int \vec{j}\cdot d\vec{S}} = \frac{\varphi}{\int \sigma E_n\,dS};\ \text{as } \vec{j}\uparrow\uparrow\vec{E}$$

and $$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0\int E_n\,dS}{\varphi}$$

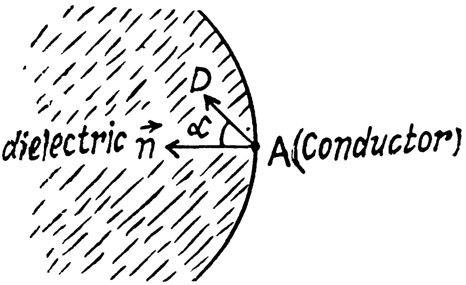
So, $$RC = \frac{\varepsilon\varepsilon_0}{\sigma} = \rho\,\varepsilon\varepsilon_0$$

3.162 The dielectric ends in a conductor. It is given that on one side (the dielectric side) the electric displacement D is as shown. Within the conductor, at any point A, there can be no normal component of electric field. For if there were such a field, a current will flow towards depositing charge there which in turn will set up countering electric field causing the normal component to vanish. Then by Gauss theorem, we easily derive

$\sigma = D_n = D\cos\alpha$ where σ is the surface charge density at A.

The tangential component is determined from the circulation theorem

$$\oint \vec{E}\cdot d\vec{r} = 0$$



It must be continuous across the surface of the conductor. Thus, inside the conductor there is a tangential electric field of magnitude,

$$\frac{D\sin\alpha}{\varepsilon_0\varepsilon} \text{ at } A .$$

This implies a current, by Ohm's law, of

$$j = \frac{D\sin\alpha}{\varepsilon_0\varepsilon\,\rho}$$

3.163 The resistance of a layer of the medium, of thickness dx and at a distance x from the first plate of the capacitor is given by,

$$dR = \frac{1}{\sigma(x)}\frac{dx}{S} \qquad (1)$$

Now, since σ varies linearly with the distance from the plate. It may be represented as,

$\sigma = \sigma_1 + \left(\frac{\sigma_2 - \sigma_1}{d}\right)x$, at a distance x from any one of the plate.

From Eq. (1)

$$dR = \frac{1}{\sigma_1 + \left(\frac{\sigma_2-\sigma_1}{d}\right)x}\frac{dx}{S}$$

or,
$$R = \frac{1}{S}\int_0^d \frac{dx}{\sigma_1 + \left(\frac{\sigma_2-\sigma_1}{d}\right)x} = \frac{d}{S(\sigma_2-\sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

Hence,
$$i = \frac{V}{R} = \frac{S\,V(\sigma_2-\sigma_1)}{d\ln\frac{\sigma_2}{\sigma_1}} = 5\text{ nA}$$

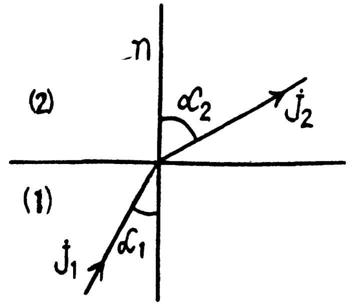
3.164 By charge conservation, current j, leaving the medium (1) must enter the medium (2). Thus

$$j_1\cos\alpha_1 = j_2\cos\alpha_2$$

Another relation follows from

$$E_{1t} = E_{2t} ,$$

which is a consequence of $\oint \vec{E} \cdot d\vec{r} = 0$



Thus $\frac{1}{\sigma_1} j_1 \sin \alpha_1 = \frac{1}{\sigma_2} j_2 \sin \alpha_2$

or, $$\frac{\tan \alpha_1}{\sigma_1} = \frac{\tan \alpha_2}{\sigma_2}$$

or, $$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\sigma_1}{\sigma_2}$$

3.165 The electric field in conductor 1 is

$$E_1 = \frac{\rho_1 I}{\pi R^2}$$

and that in 2 is $E_2 = \frac{\rho_2 I}{\pi R^2}$

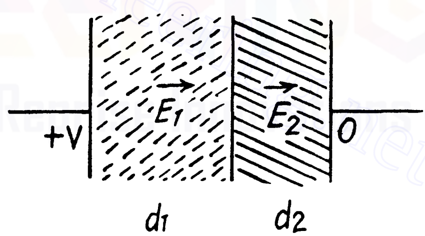
Applying Gauss' theorem to a small cylindrical pill-box at the boundary.

$$-\frac{\rho_1 I}{\pi R^2} dS + \frac{\rho_2 I}{\pi R^2} dS = \frac{\sigma\, dS}{\varepsilon_0}$$

Thus, $$\sigma = \varepsilon_0 (\rho_2 - \rho_1) \frac{1}{\pi R^2}$$

and charge at the boundary $= \varepsilon_0 (\rho_2 - \rho_1) I$

3.166 We have, $E_1 d_1 + E_2 d_2 = V$



and by current conservation

$$\frac{1}{\rho_1} E_1 = \frac{1}{\rho_2} E_2$$

Thus, $$E_1 = \frac{\rho_1 V}{\rho_1 d_1 + \rho_2 d_2},$$

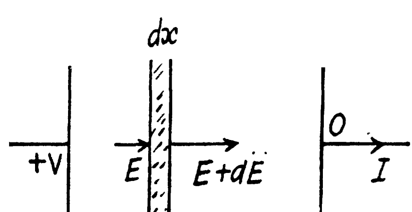
$$E_2 = \frac{\rho_2 V}{\rho_1 d_1 + \rho_2 d_2}$$

At the boundary between the two dielectrics,

$$\sigma = D_2 - D_1 = \varepsilon_0 \varepsilon_2 E_2 - \varepsilon_0 \varepsilon_1 E_1$$

$$\frac{\varepsilon_0 V}{\rho_1 d_1 + \rho_2 d_2} (\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.167 By current conservation



$$\frac{E(x)}{\rho(x)} = \frac{E(x) + dE(x)}{\rho(x) + d\rho(x)} = \frac{dE(x)}{d\rho(x)}$$

This has the solution,

$$E(x) = C \rho(x) = \frac{I \rho(x)}{A}$$

Hence charge induced in the slice per unit area

$$d\sigma = \varepsilon_0 \frac{I}{A}[\{\varepsilon(x) + d\varepsilon(x)\}\{\rho(x) + d\rho(x)\} - \varepsilon(x)\rho(x)] = \varepsilon_0 \frac{I}{A} d[\varepsilon(x)\rho(x)]$$

Thus, $$dQ = \varepsilon_0 I d[\varepsilon(x)\rho(x)]$$

Hence total charge induced, is by integration,

$$Q = \varepsilon_0 I(\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.168 As in the previous problem

$$E(x) = C\rho(x) = C(\rho_0 + \rho_1 x)$$

where $$\rho_0 + \rho_1 d = \eta\rho_0 \quad \text{or,} \quad \rho_1 = \frac{(\eta - 1)\rho_0}{d}$$

By integration $$V = \int_0^d C\rho(x)\,dx = C\rho_0 d\left(1 + \frac{\eta - 1}{2}\right) = \frac{1}{2} C\rho_0 d(\eta + 1)$$

Thus $$C = \frac{2V}{\rho_0 d(\eta + 1)}$$

Thus volume density of charge present in the medium

$$= \frac{dQ}{S dx} = \varepsilon_0 dE(x)/dx$$

$$= \frac{2\varepsilon_0 V}{\rho_0 d(\eta + 1)} \times \frac{(\eta - 1)\rho_0}{d} = \frac{2\varepsilon_0 V(\eta - 1)}{(\eta + 1)d^2}$$

3.169 (a) Consider a cylinder of unit length and divide it into shells of radius r and thickness dr Different sections are in parallel. For a typical section,

$$d\left(\frac{1}{R_1}\right) = \frac{2\pi r\,dr}{(\alpha/r^2)} = \frac{2\pi r^3\,dr}{\alpha}$$

Integrating, $$\frac{1}{R_1} = \frac{\pi R^4}{2\alpha} = \frac{S^2}{2\pi\alpha}$$

or, $$R_1 = \frac{2\pi\alpha}{S^2}, \quad \text{where} \quad S = \pi R^2$$

(b) Suppose the electric filed inside is $E_z = E_o$ (Z axis is along the axiz of the conductor). This electric field cannot depend on r in steady conditions when other components of E are absent, otherwise one violates the circulation theorem

$$\oint \vec{E} \cdot d\vec{r} = 0$$

The current through a section between radii $(r + dr,\ r)$ is

$$2\pi r dr \frac{1}{\alpha/r^2} E = 2\pi r^3 dr \frac{E}{\alpha}$$

Thus $$I = \int_0^R 2\pi r^3 dr \frac{E}{\alpha} = \frac{\pi R^4 E}{2\alpha}$$

Hence $$E = \frac{2\alpha\pi I}{S^2} \text{ when } S = \pi R^2$$

3.170 The formula is,

$$q = C V_0 (1 - e^{-t/RC})$$

or, $$V = \frac{q}{C} = V_0 (1 - e^{-t/RC}) \quad \text{or,} \quad \frac{V}{V_0} = 1 - e^{-t/RC}$$

or, $$e^{-t/RC} = 1 - \frac{V}{V_0} = \frac{V_0 - V}{V_0}$$

Hence, $t = RC \ln \dfrac{V_0}{V_0 - V} = RC \ln 10$, if $V = 0{\cdot}9\, V_0$.

Thus $t = 0.6\ \mu S$.

3.171 The charge decays according to the foumula

$$q = q_0 e^{-t/RC}$$

Here, RC = mean life = Half-life/ln 2

So, half life= $T = RC \ln 2$

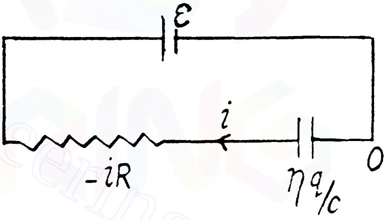
But, $$C = \frac{\varepsilon \varepsilon_0 A}{d}, \quad R = \frac{\rho d}{A}$$

Hence, $$\rho = \frac{T}{\varepsilon \varepsilon_0 \ln 2} = 1.4 \times 10^{13}\ \Omega \cdot m$$

3.172 Suppose q is the charge at time t. Initially $q = C\xi$, at $t = 0$.

Then at time t,

$$\frac{\eta q}{C} - iR - \xi = 0$$



But, $i = -\dfrac{dq}{dt}$ (- sign because charge decreases)

So $$\frac{\eta q}{C} + R\frac{dq}{dt} = \xi$$

$$\frac{dq}{dt} + \frac{\eta}{RC} q = \frac{\xi}{R}$$

or, $$\frac{d}{dt} q e^{t\eta/RC} = \frac{\xi}{R} e^{t\eta/RC}$$

or, $$q = \frac{C\xi}{\eta} + A e^{-t\eta/RC}$$

$$A = C\xi\left(1 - \frac{1}{\eta}\right), \text{ from } q = C\xi \text{ at } t = 0$$

Hence, $$q = C\xi\left(\frac{1}{\eta} + \left(1 - \frac{1}{\eta}\right) e^{-\eta t/RC}\right)$$

Finally, $$i = -\frac{dq}{dt} = \frac{\xi(\eta - 1)}{R} e^{-t\eta/RC}$$

3.173 Let r = internal resistance of the battery. We shall take the resistance of the ammeter to be = 0 and that of voltmeter to be G.

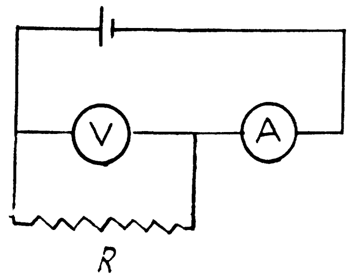
Initially, $V = \xi - Ir$, $I = \dfrac{\xi}{r + G}$

So, $$V = \xi \frac{G}{r+G} \qquad (1)$$

After the voltmeter is shunted

$$\frac{V}{\eta} = \xi - \frac{\xi r}{r + \frac{RG}{R+G}} \text{ (Voltmeter)} \qquad (2)$$

and $$\frac{\xi}{r + \frac{RG}{R+G}} = \eta \frac{\xi}{r+G} \text{ (Ammeter)} \qquad (3)$$



From (2) and (3) we have

$$\frac{V}{\eta} = \xi - \frac{\eta r \xi}{r+G} \qquad (4)$$

From (1) and (4)

$$\frac{G}{\eta} = r + G - \eta r \text{ or } G = \eta r$$

Then (1) gives the required reading

$$\frac{V}{\eta} = \frac{\xi}{\eta + 1}$$

3.174 Assume the current flow, as shown. Then potentials are as shown. Thus,

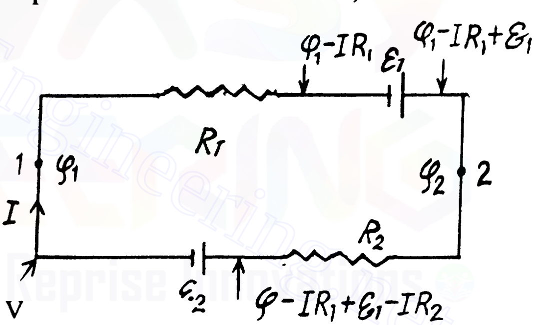
$$\varphi_1 = \varphi_1 - IR_1 + \xi_1 - IR_2 - \xi_2$$

or, $$I = \frac{\xi_1 - \xi_2}{R_1 + R_2}$$

And $$\varphi_2 = \varphi_1 - IR_1 + \xi_1$$

So, $$\varphi_1 - \varphi_2 = -\xi_1 + \frac{\xi_1 - \xi_2}{R_1 + R_2} R_1$$

$$= -(\xi_1 R_2 + \xi_2 R_1)/(R_1 + R_2) = -4 \text{ V}$$



3.175 Let, us consider the current i, flowing through the circuit, as shown in the figure. Applying loop rule for the circuit, $-\Delta\varphi = 0$

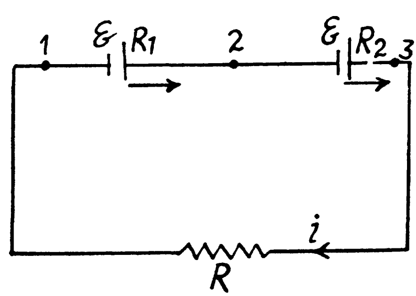
$$-2\xi + iR_1 + iR_2 + iR = 0$$

or, $$i(R_1 + R_2 + R) = 2\xi$$

or, $$i = \frac{2\xi}{R + R_1 + R_2}$$

Now, if $$\varphi_1 - \varphi_2 = 0$$

$$-\xi + iR_1 = 0$$



or, $$\frac{2\xi R_1}{R + R_1 + R_2} = \xi \text{ and } 2R_1 = R_2 + R + R_1$$

or, $R = R_1 - R_2$, which is not possible as $R_2 > R_1$

Thus, $$\varphi_2 - \varphi_3 = -\xi + iR_2 = 0$$

or,
$$\frac{2\,\xi R_2}{R + R_1 + R_2} = \xi$$

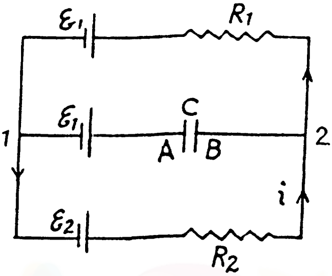
So, $R = R_2 - R_1$, which is the required resistance.

3.176 (a) Current, $i = \dfrac{N\xi}{NR} = \dfrac{N\alpha R}{NR} = \alpha$, as $\xi = \alpha R$ (given)

(b) $\varphi_A - \varphi_B = n\,\xi - ni\,R = n\,\alpha R - n\,\alpha R = 0$

3.177 As the capacitor is fully charged, no current flows through it. So, current

$$i = \frac{\xi_2 - \xi_1}{R_1 + R_2} \text{ (as } \xi_2 > \xi_1)$$



And hence, $\varphi_A - \varphi_B = \xi_1 - \xi_2 + i\,R_2$

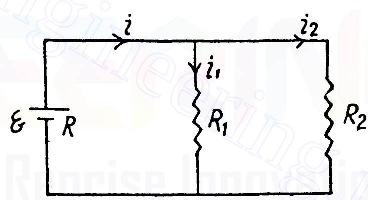
$$= \xi_1 - \xi_2 + \frac{\xi_2 - \xi_1}{R_1 + R_2} R_2$$

$$= \frac{(\xi_1 - \xi_2)\,R_1}{R_1 + R_2} = -0{\cdot}5 \text{ V}$$

3.178 Let us make the current distribution, as shown in the figure.

Current $i = \dfrac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}}$ (using loop rule)

So, current through the resistor R_1,



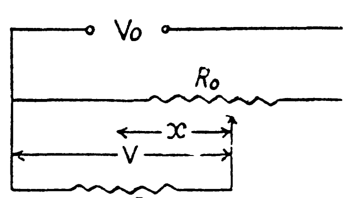
$$i_1 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_2}{R_1 + R_2}$$

$$= \frac{\xi R_2}{R\,R_1 + R\,R_2 + R_1 R_2} = 1{\cdot}2 \text{ A}$$

and similary, current through the resistor R_2,

$$i_2 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_1}{R_1 + R_2} = \frac{\xi R_1}{R\,R_1 + R_1 R_2 + R\,R_2} = 0{\cdot}8 \text{ A}$$

3.179 Total resistance $= \dfrac{l - x}{l} R_0 + \dfrac{R \cdot \dfrac{x R_0}{l}}{R + \dfrac{x R_0}{l}}$



$$= \frac{l - x}{l} R_0 + \frac{x\,R\,R_0}{l\,R + x R_0}$$

$$= R_0\left[\frac{l - x}{l} + \frac{x\,R}{lR + x\,R_0}\right]$$

Then $V = V_0 \frac{xR}{lR + xR_0} \bigg/ \left(1 - \frac{x}{l} + \frac{xR}{xR_0 + lR}\right) = V_0 R x \bigg/ \left\{lR + R_0 x \left(1 - \frac{x}{l}\right)\right\}$

For $R >> R_0, V \approx V_0 \frac{x}{l}$

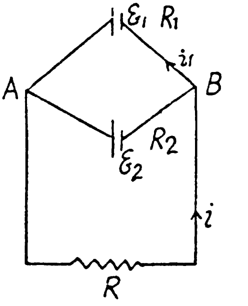
3.180 Let us connect a load of resistance R between the points A and B (Fig.)

From the loop rule, $\Delta\varphi = 0$, we obtain

$$iR = \xi_1 - i_1 R_1 \quad (1)$$

and $iR = \xi_2 - (i - i_1) R_2$

or $i(R + R_2) = \xi + i_1 R_2$ (2)



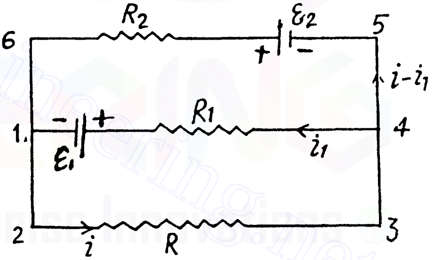
Solving Eqs. (1) and (2), we get

$$i = \frac{\xi_1 R_1 + \xi_2 R_2}{R_1 + R_2} \bigg/ R + \frac{R_1 R_2}{R_1 + R_2} = \frac{\xi_0}{R + R_0} \quad (3)$$

where $\xi_0 = \frac{\xi_1 R_1 + \xi_2 R_2}{R_1 + R_2}$ and $R_0 = \frac{R_1 R_2}{R_1 + R_2}$

Thus one can replace the given arrangement of the cells by a single cell having the emf ξ_0 and internal resistance R_0.

3.181 Make the current distribution, as shown in the diagram.



Now, in the loop 12341, applying $-\Delta\varphi = 0$

$$iR + i_1 R_1 + \xi_1 = 0 \quad (1)$$

and in the loop 23562,

$$iR - \xi_2 + (i - i_1) R_2 = 0 \quad (2)$$

Solving (1) and (2), we obtain current through the resistance R,

$$i = \frac{(\xi_2 R_1 - \xi_1 R_2)}{R R_1 + R R_2 + R_1 R_2} = 0{\cdot}02\text{ A}$$

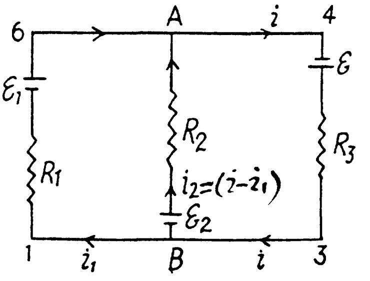
and it is directed from left to the right

3.182 At first indicate the currents in the branches using charge conservation (which also includes the point rule).

In the loops 1 BA 61 and B34AB from the loop rule, $-\Delta\varphi = 0$, we get, respectively

$$-\xi_2 + (i - i_1)R_2 + \xi_1 - i_1 R_1 = 0 \quad (1)$$

$$iR_3 + \xi_3 - (i - i_1) R_2 + \xi_2 = 0 \quad (2)$$



On solving Eqs (1) and (2), we obtain

$$i_1 = \frac{(\xi_1 - \xi_2) R_3 + R_2 (\xi_1 + \xi_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} \approx 0{\cdot}06\text{ A}$$

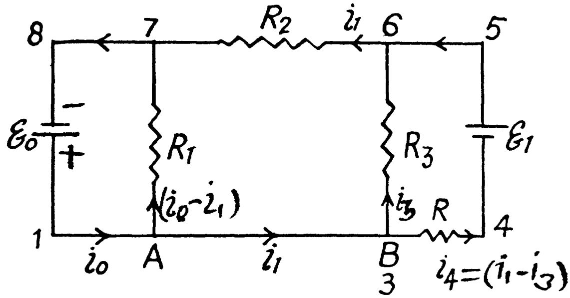
Thus $\varphi_A - \varphi_B = \xi_2 - i_2 R_2 \approx 0{\cdot}9$ V

3.183 Indicate the currents in all the branches using charge conservation as shown in the figure. Applying loop rule, $-\Delta\varphi = 0$ in the loops 1A781, 1B681 and B456B, respectively, we get

$$\xi_0 = (i_0 - i_1)\, R_1 \qquad (1)$$

$$i_3 R_3 + i_1 R_2 - \xi_0 = 0 \qquad (2) \text{ and}$$

$$(i_1 - i_3) R - \xi - i_3 R_3 = 0 \qquad (3)$$



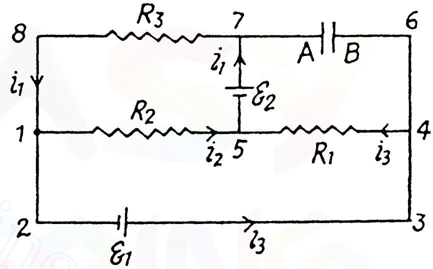
Solving Eqs. (1), (2) and (3), we get the sought current

$$(i_1 - i_3) = \frac{\xi\,(R_2 + R_3) + \xi_0 R_3}{R\,(R_2 + R_3) + R_2 R_3}$$

3.184 Indicate the currents in all the branches using charge conservation as shown in the figure. Applying the loop rule $(-\Delta\varphi = 0)$ in the loops 12341 and 15781, we get

$$-\xi_1 + i_3 R_1 - (i_1 - i_3) R_2 = 0 \qquad (1)$$

$$\text{and} \quad (i_1 - i_3) R_2 - \xi_2 + i_1 R_3 = 0 \qquad (2)$$



Solving Eqs. (1) and (2), we get

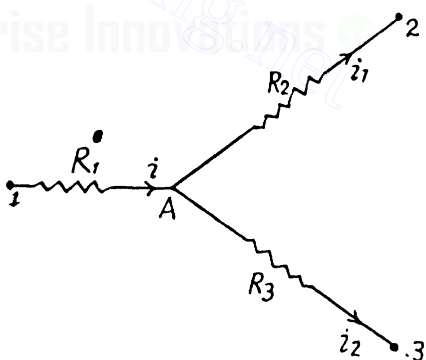
$$i_3 = \frac{\xi_1\,(R_2 + R_3) + \xi_2 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

Hence, the sought p.d.

$$\varphi_A - \varphi_B = \xi_2 - i_3 R_1$$

$$= \frac{\xi_2 R_3 (R_1 + R_{2)} - \xi_1 R_1 (R_2 + R_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} = -1\text{ V}$$

3.185 Let us distribute the currents in the paths as shown in the figure.



$$\text{Now,} \quad \varphi_1 - \varphi_2 = i R_1 + i_1 R_2 \qquad (1)$$

$$\text{and} \quad \varphi_1 - \varphi_3 = i R_1 + (i - i_1) R_3 \qquad (2)$$

Simplifying Eqs. (1) and (2) we get

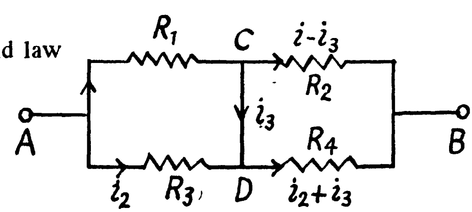
$$i = \frac{R_3\,(\varphi_1 - \varphi_2) + R_2\,(\varphi_1 - \varphi_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1} = 0{\cdot}2\text{ A}$$

3.186 Current is as shown. From Kirchhoff's Second law

$$i_1 R_1 = i_2 R_3\,,$$

$$i_1 R_1 + (\,i_1 - i_3\,) R_2 = V\,,$$

$$i_2 R_3 + (\,i_3 + i_2\,) R_4 = V$$



Eliminating i_2

$$i_1(R_1+R_2)-i_3R_2=V$$

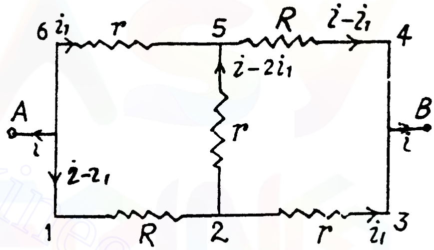
$$i_1\frac{R_1}{R_3}(R_3+R_4)+i_3R_4=V$$

Hence
$$i_3\left[R_4(R_1+R_2)+\frac{R_1R_2}{R_3}(R_3+R_4)\right]$$
$$=V\left[(R_1+R_2)-\frac{R_1}{R_3}(R_3+R_4)\right]$$

or,
$$i_3=\frac{R_3(R_1+R_2)-R_1(R_3+R_4)}{R_3R_4(R_1+R_2)+R_1R_2(R_3+R_4)}$$

On substitution we get $i_3=1{\cdot}0$ A from C to D

3.187 From the symmetry of the problem, current flow is indicated, as shown in the figure.



Now, $\varphi_A-\varphi_B=i_1r+(i-i_1)R$ (1)

In the loop 12561, from $-\Delta\varphi=0$

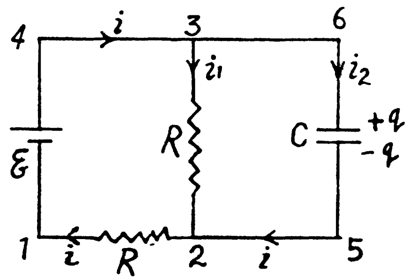
$$(i-i_1)R+(i-2i_1)r-i_1r=0$$

or, $$i_1=\frac{(R+r)}{3r+R}i \quad (2)$$

Equivalent resistance between the terminals A and B using (1) and (2),

$$R_0=\frac{\varphi_A-\varphi_B}{i}=\frac{\left[\frac{R+r}{3r+R}(r-R)+R\right]i}{i}=\frac{r(3R+r)}{3r+R}$$

3.188 Let, at any moment of time, charge on the plates be $+q$ and $-q$ respectively, then voltage across the capacitor, $\varphi=q/C$ (1)



Now, from charge conservation,

$$i=i_1+i_2,\text{ where } i_2=\frac{dq}{dt} \quad (2)$$

In the loop 65146, using $-\Delta\varphi=0$.

$$\frac{q}{C}+\left(i_1+\frac{dq}{dt}\right)R-\xi=0 \quad (3)$$

[using (1) and (2)]

In the loop 25632, using $-\Delta\varphi=0$

$$-\frac{q}{C}+i_1R=0 \quad\text{or,}\quad i_1R=\frac{q}{C} \quad (4)$$

From (1) and (2),

$$\frac{dq}{dt} R = \xi_1 - \frac{2q}{C} \quad \text{or,} \quad \frac{dq}{\xi - \frac{2q}{C}} = \frac{dt}{R} \qquad (5)$$

On integrating the expression (5) between suitable limits,

$$\int_0^q \frac{dq}{\xi - \frac{2q}{C}} = \frac{1}{R}\int_0^t dt \quad \text{or,} \quad -\frac{C}{2}\ln\frac{\xi - \frac{2q}{C}}{\xi} = \frac{t}{R}$$

Thus $$\frac{q}{C} = V = \frac{1}{2}\xi\left(1 - e^{-2t/RC}\right)$$

3.189 (a) As current i is linear function of time, and at $t = 0$ and Δt, it equals i_0 and zero respectively, it may be represented as,

$$i = i_0\left(1 - \frac{t}{\Delta t}\right)$$

Thus $$q = \int_0^{\Delta t} i\,dt = \int_0^{\Delta t} i_0\left(1 - \frac{t}{\Delta t}\right)dt = \frac{i_0\,\Delta t}{2}$$

So, $$i_0 = \frac{2q}{\Delta t}$$

Hence, $$i = \frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$

The heat generated.

$$H = \int_0^{\Delta t} i^2 R\,dt = \int_0^{\Delta t}\left[\frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)\right]^2 R\,dt = \frac{4q^2 R}{3\,\Delta t}$$

(b) Obviously the current through the coil is given by

$$i = i_0\left(\frac{1}{2}\right)^{t/\Delta t}$$

Then charge $$q = \int_0^{\infty} i\,dt = \int_0^{\infty} i_0\,2^{-t/\Delta t}\,dt = \frac{i_0\,\Delta t}{\ln 2}$$

So, $$i_0 = \frac{q\ln 2}{\Delta t}$$

And hence, heat generated in the circuit in the time interval $t\,[0, \infty]$,

$$H = \int_0^{\infty} i^2 R\,dt = \int_0^{\infty}\left[\frac{q\ln 2}{\Delta t}\,2^{-t/\Delta t}\right]^2 R\,dt = -\frac{q^2\ln 2}{2\,\Delta t}R$$

3.190 The equivalent circuit may be drawn as in the figure.

Resistance of the network $= R_0 + (R/3)$

Let, us assume that e.m.f. of the cell is ξ, then current

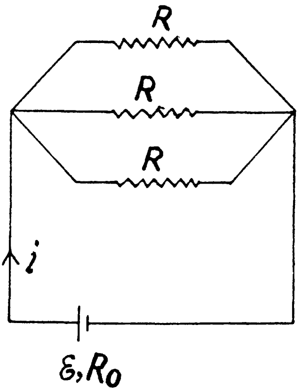
$$i = \frac{\xi}{R_0 + (R/3)}$$

Now, thermal power, generated in the circuit

$$P = i^2 R/3 = \frac{\xi^2}{(R_0 + (R/3))^2}(R/3)$$

For P to be maximum, $\frac{dP}{dR} = 0$, which yields

$$R = 3R_0$$

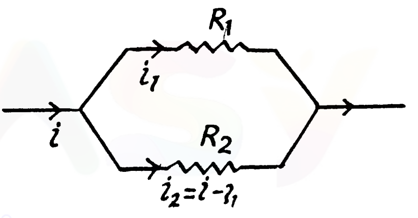


3.191 We assume current conservation but not Kirchhoff's second law. Then thermal power dissipated is

$$P(i_1) = i_1^2 R_1 + (i - i_1)^2 R_2$$

$$= i_1^2 (R_1 + R_2) - 2 i i_1 R_2 + i^2 R_2$$

$$= [R_1 + R_2]\left[i_1 - \frac{R_2}{R_1 + R_2} i\right]^2 + i^2 \frac{R_1 R_2}{R_1 + R_2}$$



The resistances being positive we see that the power dissipated is minimum when

$$i_1 = i\frac{R_2}{R_1 + R_2}$$

This corresponds to usual distribution of currents over resistance joined is parallel.

3.192 Let, internal resistance of the cell be r, then

$$V = \xi - ir \qquad (1)$$

where i is the current in the circuit. We know that thermal power generated in the battery.

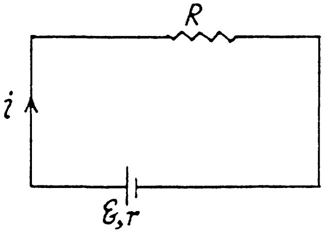
$$Q = i^2 r \qquad (2)$$

Putting r from (1) in (2), we obtain,

$$Q = (\xi - V)\, i = 0{\cdot}6 \text{ W}$$

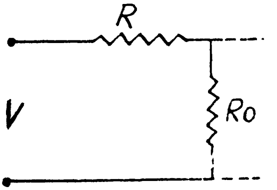
In a battery work is done by electric forces (whose origin lies in the chemical processes going on inside the cell). The work so done is stored and used in the electric circuit outside. Its magnitude just equals the power used in the electric circuit. We can say that net power developed by the electric forces is

$$P = -IV = -2\cdot 0\text{W}$$



Minus sign means that this is generated not consumed.

3.193 As far as motor is concerned the power delivered is dissipated and can be represented by a load, R_0. Thus



$$I = \frac{V}{R + R_0}$$

and $$P = I^2 R_0 = \frac{V^2 R_0}{(R_0 + R)^2}$$

This is maximum when $R_0 = R$ and the current I is then

$$I = \frac{V}{2R}$$

The maximum power delivered is

$$\frac{V^2}{4R} = P_{max}$$

The power input is $\frac{V^2}{R + R_0}$ and its value when P is maximum is $\frac{V^2}{2R}$

The efficiency then is $\frac{1}{2} = 50\%$

3.194 If the wire diameter decreases by δ then by the information given

$P =$ Power input $= \frac{V^2}{R} =$ heat lost through the surface, H.

Now, $H \propto (1 - \delta)$ like the surface area and

$$R \propto (1 - \delta)^{-2}$$

So, $$\frac{V^2}{R_0}(1 - \delta)^2 = A\,(1 - \delta) \quad \text{or,} \quad V^2 (1 - \delta) = \text{constant}$$

But $\quad V \propto 1 + \eta \quad$ so $\quad (1 + \eta)^2 (1 - \delta) = \text{Const} = 1$

Thus $$\delta = 2\eta = 2\%$$

3.195 The equation of heat balance is

$$\frac{V^2}{R} - k\,(T - T_0) = C\,\frac{dT}{dt}$$

Put $$T - T_0 = \xi$$

So, $$C\,\dot{\xi} + k\,\xi = \frac{V^2}{R} \quad \text{or,} \quad \dot{\xi} + \frac{k}{C}\,\xi = \frac{V^2}{C\,R}$$

or, $$\frac{d}{dt}\left(\xi e^{kt/c}\right) = \frac{V^2}{C\,R}\,e^{kt/c}$$

or, $$\xi\,e^{kt/c} = \frac{V^2}{kR}\,e^{kt/c} + A$$

where A is a constant. Clearly

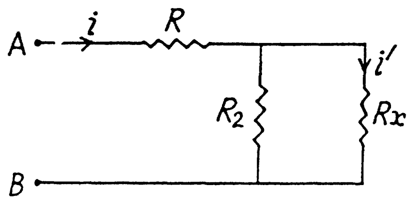
$$\xi = 0 \text{ at } t = 0, \text{ so } A = -\frac{V^2}{kR} \text{ and hence,}$$

$$T = T_0 + \frac{V^2}{kR}\left(1 - e^{-kt/C}\right)$$

3.196 Let, $\varphi_A - \varphi_B = \varphi$

Now, thermal power generated in the resistance R_x,

$$P = i'^2 R_x = \left[\frac{\varphi}{R_1 + \frac{R_2 R_x}{R_2 + R_x}} \frac{R_2}{R_2 + R_x}\right]^2 R_x,$$



For P to be independent of R_x,

$$\frac{dP}{dR_x} = 0, \text{ which yeilds}$$

$$R_x = \frac{R_1 R_2}{R_1 + R_2} = 12\ \Omega$$

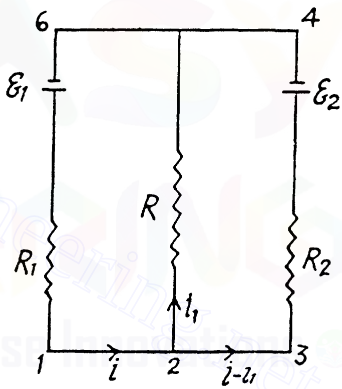
3.197 Indicate the currents in the circuit as shown in the figure.

Appying loop rule in the closed loop 12561, $-\Delta\varphi = 0$ we get

$$i_1 R - \xi_1 + i R_1 = 0 \qquad (1)$$

and in the loop 23452,

$$(i - i_1) R_2 + \xi_2 - i_1 R = 0 \qquad (2)$$



Solving (1) and (2), we get,

$$i_1 = \frac{\xi_1 R_2 + \xi_2 R_1}{R R_1 + R_1 R_2 + R R_2}$$

So, thermal power, generated in the resistance R,

$$P = i_1^2 R = \left[\frac{\xi_1 R_2 + \xi_2 R_1}{R R_1 + R_1 R_2 + R R_2}\right]^2 R$$

For P to be maximum, $\frac{dP}{dR} = 0$, which fields

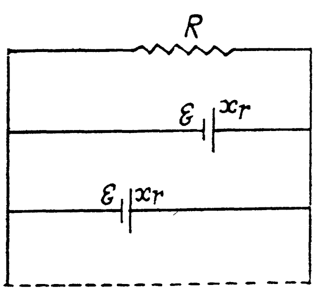
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Hence, $$P_{max} = \frac{(\xi_1 R_2 + \xi_2 R_1)^2}{4 R_1 R_2 (R_1 + R_2)}$$

3.198 Let, there are x number of cells, connected in series in each of the n parallel groups

then, $n x = N$ or, $x = \frac{N}{n}$ (1)

Now, for any one of the loop, consisting of x cells and the resistor R, from loop rule



$$i R + \frac{i}{n} xr - x \xi = 0$$

$$\text{So, } i = \frac{x \xi}{R + \frac{xr}{n}} = \frac{\frac{N}{n}\xi}{R + \frac{N r}{n^2}}, \text{ using (1)}$$

Heat generated in the resistor R,

$$Q = i^2 R = \left(\frac{N n \xi}{n^2 R + N R}\right)^2 R \tag{2}$$

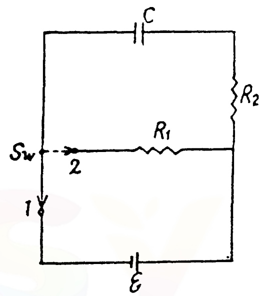
and for Q to be maximum, $\frac{dQ}{dn} = 0$, which yields

$$n = \sqrt{\frac{Nr}{R}} = 3$$

3.199 When switch 1 is closed, maximum charge accumulated on the capacitor,

$$q_{max} = C\xi, \tag{1}$$

and when switch 2 is closed, at any arbitrary instant of time,



$$(R_1 + R_2)\left(-\frac{dq}{dt}\right) = q/C,$$

because capacitor is discharging.

$$\text{or,} \quad \int_{q_{max}}^{q} \frac{1}{q}\, dq = -\frac{1}{(R_1 + R_2)C}\int_0^t dt$$

Integrating, we get

$$\ln\frac{q}{q_{max}} = \frac{-t}{(R_1 + R_2)C} \quad \text{or,} \quad q = q_{max}\, e^{\frac{-t}{(R_1 + R_2)C}} \tag{2}$$

Differentiating with respect to time,

$$i(t) = \frac{dq}{dt} = q_{max}\, e^{\frac{-t}{(R_1 + R_2)C}}\left(-\frac{1}{(R_1 + R_2)C}\right)$$

$$\text{or,} \quad i(t) = \frac{C\xi}{(R_1 + R_2)C}\, e^{\frac{-t}{(R_1 + R_2)C}}$$

Negative sign is ignored, as we are not interested in the direction of the current.

$$\text{thus,} \quad i(t) = \frac{\xi}{(R_1 + R_2)}\, e^{\frac{-1}{(R_1 + R_2)C}} \tag{3}$$

When the switch (Sw) is at the position 1, the charge (maximum) accumalated on the capacitor is,

$$q = C\xi$$

When the Sw is thrown to position 2, the capacitor starts discharging and as a result the electric energy stored in the capacitor totally turns into heat energy tho' the resistors R_1 and R_2 (during a very long interval of time). Thus from the energy conservation, the total heat liberated tho' the resistors.

$$H = U_i = \frac{q^2}{2C} = \frac{1}{2}C\xi^2$$

During the process of discharging of the capacitor, the current tho' the resistors R_1 and R_2 is the same at all the moments of time, thus

$$H_1 \propto R_1 \text{ and } H_2 \propto R_2$$

So,
$$H_1 = \frac{H\,R_1}{(R_1 + R_2)} \quad (\text{as } H = H_1 + H_2)$$

Hence
$$H_1 = \frac{1}{2}\frac{CR_1}{R_1 + R_2}\,\xi^2$$

3.200 When the plate is absent the capacity of the condenser is

$$C = \frac{\varepsilon_0 S}{d}$$

When it is present, the capacity is

$$C' = \frac{\varepsilon_0 S}{d(1-\eta)} = \frac{C}{1-\eta}$$

(a) The energy increment is clearly.

$$\Delta U = \frac{1}{2}CV^2 - \frac{1}{2}C'V^2 = \frac{C\eta}{2(1-\eta)}V^2$$

(b) The charge on the plate is

$$q_i = \frac{CV}{1-\eta} \text{ initially and } q_f = CV \text{ finally.}$$

A charge $\frac{CV\eta}{1-\eta}$ has flown through the battery charging it and withdrawing $\frac{CV^2\eta}{1-\eta}$ units of energy from the system into the battery. The energy of the capacitor has decreased by just half of this. The remaining half i.e. $\frac{1}{2}\frac{CV^2\eta}{1-\eta}$ must be the work done by the external agent in withdrawing the plate. This ensures conservation of energy.

3.201 Initially, capacitance of the system $= C\,\varepsilon$.

So, initial energy of the system : $U_i = \frac{1}{2}(C\,\varepsilon)\,V^2$

and finally, energy of the capacitor : $U_f = \frac{1}{2}C\,V^2$

Hence capacitance energy increment,

$$\Delta U = \frac{1}{2}C\,V^2 - \frac{1}{2}(C\,\varepsilon)\,V^2 = -\frac{1}{2}\,CV^2\,(\varepsilon - 1) = -0{\cdot}5 \text{ mJ}$$

From energy conservation

$$\Delta U = A_{\text{cell}} + A_{\text{agent}}$$

(as there is no heat liberation)

But $A_{\text{cell}} = (C_f - C_i)\,V^2 = (C - C\varepsilon)V^2$

Hence $A_{agent} = \Delta U - A_{cell}$

$$= \frac{1}{2}C(1-\varepsilon)V^2 = 0{\cdot}5 \text{ m J}$$

3.202 If C_0 is the initial capacitance of the condenser before water rises in it then

$$U_i = \frac{1}{2}C_0 V^2, \quad \text{where } C_0 = \frac{\varepsilon_0 2l\pi R}{d}$$

(R is the mean radius and l is the length of the capacitor plates.)

Suppose the liquid rises to a height h in it. Then the capacitance of the condenser is

$$C = \frac{\varepsilon\varepsilon_0 h 2\pi R}{d} + \frac{\varepsilon(l-h)2\pi R}{d} = \frac{\varepsilon_0 2\pi R}{d}(l + (\varepsilon-1)h)$$

and energy of the capacitor and the liquid (including both gravitational and electrosatic contributions) is

$$\frac{1}{2}\frac{\varepsilon_0 2\pi R}{d}(l+(\varepsilon-1)h)V^2 + \rho g(2\pi R hd)\frac{h}{2}$$

If the capacitor were not connected to a battery this energy would have to be minimized. But the capacitor is connected to the battery and, in effect, the potential energy of the whole system has to be minimized. Suppose we increase h by δh. Then the energy of the capacitor and the liquid increases by

$$\delta h\left(\frac{\varepsilon_0 2\pi R}{2d}(\varepsilon-1)V^2 + \rho g(2\pi R d)h\right)$$

and that of the cell diminishes by the quantity A_{cell} which is the product of charge flown and V

$$\delta h\frac{\varepsilon_0(2\pi R)}{d}(\varepsilon-1)V^2$$

In equilibrium, the two must balance; so

$$\rho g d h = \frac{\varepsilon_0(\varepsilon-1)V^2}{2d}$$

Hence $$h = \frac{\varepsilon_0(\varepsilon-1)V^2}{2\rho g d^2}$$

3.203 (a) Let us mentally islolate a thin spherical layer with inner and outer radii r and $r+dr$ respectively. Lines of current at all the points of this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$. Now, we know that resistance,

$$dR = \rho\frac{dr}{S(r)} = \rho\frac{dr}{4\pi r^2} \qquad (1)$$

Integrating expression (1) between the limits,

$$\int_0^R dR = \int_a^b \rho \frac{dr}{4\pi r^2} \quad \text{or,} \quad R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right] \tag{2}$$

Capacitance of the network, $C = \dfrac{4\pi\varepsilon_0\varepsilon}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}$ (3)

and $q = C\varphi$ [where q is the charge at any arbitrary moment] (4)

also, $\varphi = \left(\dfrac{-dq}{dt}\right)R$, as capacitor is discharging. (5)

From Eqs. (2), (3), (4) and (5) we get,

$$q = \frac{4\pi\varepsilon_0\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \frac{\left[-\frac{dq}{dt}\right]\rho\left[\frac{1}{a} - \frac{1}{b}\right]}{4\pi} \quad \text{or,} \quad \frac{dq}{q} = \frac{dt}{\rho\varepsilon\varepsilon_0}$$

Integrating $\displaystyle\int_{q_0}^{q} -\frac{dq}{q} = \frac{1}{\rho\varepsilon_0\varepsilon}\int_0^t dt = \frac{dt}{\rho\varepsilon\varepsilon_0}$

Hence $q = q_0 e^{\frac{-t}{\rho\varepsilon_0\varepsilon}}$

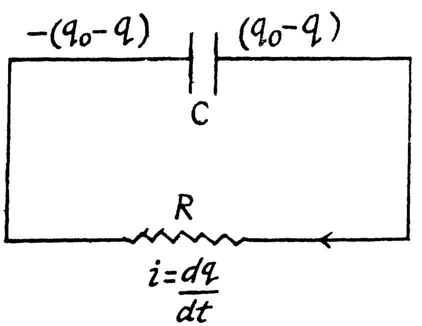
(b) From energy conservation heat generated, during the spreading of the charge,

$$H = U_i - U_f \text{ [because } A_{cell} = 0]$$

$$= \frac{1}{2}\frac{q_0^2}{4\pi\varepsilon_0\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] - 0 = \frac{q_0^2}{8\pi\varepsilon_0\varepsilon}\frac{b-a}{ab}$$

3.204 (a) Let, at any moment of time, charge on the plates be $(q_0 - q)$ then current through the resistor, $i = -\dfrac{d(q_0 - q)}{dt}$, because the capacitor is discharging.

or, $i = \dfrac{dq}{dt}$



Now, applying loop rule in the circuit,

$$iR - \frac{q_0 - q}{C} = 0$$

or, $\dfrac{dq}{dt}R = \dfrac{q_0 - q}{C}$

or, $\dfrac{dq}{q_0 - q} = \dfrac{1}{RC}dt$

At $t = 0$, $q = 0$ and at $t = \tau$, $q = q$

So, $$\ln \frac{q_0 - q}{q_0} = \frac{-\tau}{RC}$$

Thus $$q = q_0 \left(1 - e^{-\tau/RC}\right) = 0{\cdot}18 \text{ mC}$$

(b) Amount of heat generated = decrement in capacitance energy

$$= \frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{\left[q_0 - q_0\left(1 - e^{-\tau/RC}\right)\right]^2}{C}$$

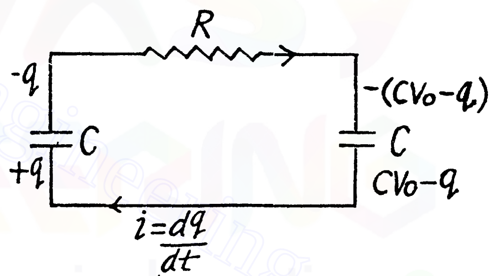
$$= \frac{1}{2}\frac{q_0^2}{C}\left[1 - e^{-\frac{2\tau}{RC}}\right] = 82 \text{ mJ}$$

3.205 Let, at any moment of time, charge flown be q then current $i = \frac{dq}{dt}$

Applying loop rule in the circuit, $-\Delta\varphi = 0$, we get :

$$\frac{dq}{dt} IR - \frac{(C V_0 - q)}{C} + \frac{q}{C} = 0$$

or, $$\frac{dq}{C V_0 - 2q} = \frac{1}{RC} dt$$



So, $$\ln \frac{C V_0 - 2q}{C V_0} = -2 \frac{t}{RC} \quad \text{for } 0 \le t \le t$$

or, $$q = \frac{C V_0}{2}\left(1 - e^{\frac{-2t}{RC}}\right)$$

Hence, $$i = \frac{dq}{dt} = \frac{C V_0}{2} \frac{2}{RC} e^{-2t/RC} = \frac{V_0}{R} e^{-2t/RC}$$

Now, heat liberaled,

$$Q = \int_0^\infty i^2 R\, dt = \frac{V_0^2}{R^2} R \int_0^\infty e^{\frac{-4t}{RC}} dt = \frac{1}{4} C V_0^2$$

3.206 In a rotating frame, to first order in ω, the main effect is a coriolis force $2m\vec{v}' \times \vec{\omega}$.

This unbalanced force will cause electrons to react by setting up a magnetic field $\vec{B}$ so that the magnetic force $e\vec{v} \times \vec{B}$ balances the coriolis force.

Thus $$-\frac{e}{2m}\vec{B} = \vec{\omega} \quad \text{or,} \quad \vec{B} = -\frac{2m}{e}\vec{\omega}$$

The flux associated with this is

$$\Phi = N\pi r^2 B = N\pi r^2 \frac{2m}{e}\omega$$

where $N = \frac{l}{2\pi r}$ is the number of turns of the ring. If ω changes (and there is time for the electron to rearrange) then B also changes and so does Φ. An emf will be induced and a current will flow. This is

$$I = N\pi r^2 \frac{2m}{e}\omega/R$$

The total charge flowing through the ballastic galvanometer, as the ring is stopped, is

$$q = N\pi r^2 / \frac{2m}{e}\omega/R$$

So, $$\frac{e}{m} = \frac{2N\pi r^2\omega}{qR} = \frac{l\omega r}{qR}$$

3.207 Let, n_0 be the total number of electorns then, total momentum of electorns,

$$p = n_0 m_e v_d \quad (1)$$

Now, $$I = \rho S_x v_d = \frac{n_0 e}{V} S_x v_d = \frac{ne}{l} v_d \quad (2)$$

Here S_x = Cross sectional area, ρ = electron charge density, V = volume of sample
From (1) and (2)

$$p = \frac{m_e}{e} Il = 0{\cdot}40\ \mu\,\text{Ns}$$

3.208 By definition
$n e v_d = j$ (where v_d is the drift velocity, n is number density of electrons.)

Then $$\tau = \frac{l}{v_d} = \frac{nel}{j}$$

So distance actually travelled

$$S = \langle v \rangle \tau = \frac{nel \langle v \rangle}{j}$$

($\langle v \rangle$ = mean velocity of thermal motion of an electron)

3.209 Let, n be the volume density of electrons, then from $I = \rho S_x v_d$,

$$I = n e S_x |\langle \vec{v} \rangle| = n e S_x \frac{l}{t}$$

So, $$t = \frac{n e S_x l}{I} = 3\ \mu\text{s}.$$

(b) Sum of electric forces

$$= |(nv) e \vec{E}| = |n S l e \rho \vec{j}|, \text{ where } \rho \text{ is resistivity of the material.}$$

$$= n S l e \rho \frac{I}{S} = n e l \rho I = 1{\cdot}0\ \mu\text{N}$$

3.210 From Gauss theorem field strength at a surface of a cylindrical shape equals, $\frac{\lambda}{2\pi\varepsilon_0 r}$, where λ is the linear charge density.

Now,
$$eV = \frac{1}{2} m_e v^2 \quad \text{or,} \quad v = \sqrt{\frac{2eV}{m_e}} \tag{1}$$

Also,
$$dq = \lambda\, dx \quad \text{so,} \quad \frac{dq}{dt} = \lambda \frac{dx}{dt}$$

or,
$$I = \lambda v \quad \text{or,} \quad \lambda = \frac{I}{v} = \frac{I}{\sqrt{\frac{2eV}{m_e}}}, \text{ using (1)}$$

Hence
$$E = \frac{I}{2\pi\varepsilon_0 r}\sqrt{\frac{m_e}{2eV}} = 32\ \text{V/m}$$

(b) For the point, inside the solid charged cylinder, applying Gauss' theorem,
$$2\pi r h E = \pi r^2 h \frac{q}{\varepsilon_0 \pi R^2 l}$$

or,
$$E = \frac{q/l}{2\pi\varepsilon_0 R^2} r = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$

So, from
$$E = -\frac{d\varphi}{dr},$$

$$\int_{\varphi_1}^{\varphi_2} -d\varphi = \int_0^R \frac{\lambda}{2\pi\varepsilon_0 R^2} r\, dr$$

or,
$$\varphi_1 - \varphi_2 = \frac{\lambda}{2\pi\varepsilon_0 R^2}\left[\frac{r^2}{2}\right]_0^R = \frac{\lambda}{4\pi\varepsilon_0}$$

Hence,
$$\varphi_1 - \varphi_2 = \frac{VI}{4\pi\varepsilon_0}\sqrt{\frac{m_e}{2eV}} = 0{\cdot}80\ \text{V}$$

3.211 Between the plates $\varphi = a\,x^{4/3}$

or,
$$\frac{\partial\varphi}{\partial x} = a \times \frac{4}{3} x^{1/3}$$

$$\frac{d^2\varphi}{\partial x^2} = \frac{4}{9} a x^{-2/3} = -\rho/\varepsilon_0$$

or,
$$\rho = -\frac{4\varepsilon_0 a}{9} x^{-2/3}$$

Let the charge on the electron be $-e$,

then $$\frac{1}{2}mv^2 - e\varphi = \text{Const.} = 0,$$

as the electron is initially emitted with neligible energy.

$$v^2 = \frac{2e\varphi}{m}, \quad v = \sqrt{\frac{2e\varphi}{m}}$$

So, $$j = -\rho v = \frac{4\varepsilon_0 a}{9}\sqrt{\frac{2\varphi}{m}}\, x^{-2/3}.$$

(j is measued from the anode to cathode, so the - ve sign.)

3.212 $E = \frac{V}{d}$

So by the definition of the mobility

$$v^+ = u^+_0 \frac{V}{d}, \; v^- = u^-_0 - \frac{V}{d}$$

and $$j = (n_+ u_0^+ + n_- u_0^-)\frac{eV}{d}$$

(The negative ions move towards the anode and the positive ion towards the cathode and the total current is the sum of the currents due to them.)

On the other hand, in equilibrium $n_+ = n_-$

So, $$n_+ = n_- = \frac{I}{S} \Big/ (u_0^+ + u_0^-)\frac{eV}{d}$$

$$= \frac{Id}{eVS(u_0^+ + u_0^-)} = 2{\cdot}3 \times 10^8 \text{ cm}^{-3}$$

3.213 Velocity = mobility × field

or, $v = u\frac{V_0}{l}\sin\omega t$, which is positive for $0 \le \omega t \le \pi$

So, maximum displacement in one direction is

$$x_{max} = \int_0^{\pi} u\frac{V_0}{l}\sin\omega t\, dt = \frac{2uV_0}{l\omega}$$

At $\omega = \omega_0$, $x_{max} = l$, so, $\frac{2uV_0}{l\omega} = l$

Thus $$u = \frac{\omega l^2}{2V_0}$$

3.214 When the current is saturated, all the ions, produced, reach the plate.

Then, $$\dot{n}_i = \frac{I_{sat}}{eV} = 6 \times 10^9 \text{ cm}^{-3}\text{ s}^{-1}$$

(Both positive ions and negative ions are counted here)

The equation of balance is, $\frac{dn}{dt} = \dot{n}_i - rn^2$

The first term on the right is the production rate and the second term is the recombination rate which by the usual statistical arguments is proportional to n^2 (= no of positive ions × no. of $-ve$ ion). In equilibrium,

$$\frac{dn}{dt} = 0$$

so, $$n_{eq} = \sqrt{\frac{\dot{n}_i}{r}} = 6 \times 10^7 \text{ cm}^{-3}$$

3.215 Initially $n = n_0 = \sqrt{\dot{n}_i / r}$

Since we can assume that the long exposure to the ionizer has caused equilibrium to be set up. Afer the ionizer is switched off,

$$\frac{dn}{dt} = -rn^2$$

or $$r\,dt = -\frac{dn}{n^2}, \quad \text{or,} \quad rt = \frac{1}{n} + \text{constant}$$

But $$n = n_0 \text{ at } t = 0, \text{ so, } rt = \frac{1}{n} - \frac{1}{n_0}$$

The concentration will decrease by a factor η when

$$r\,t_0 = \frac{1}{n_0/\eta} - \frac{1}{n_0} = \frac{\eta - 1}{n_0}$$

or, $$t_0 = \frac{\eta - 1}{\sqrt{r\,\dot{n}_i}} = 13 \text{ ms}$$

3.216 Ions produced will cause charge to decay. Clearly,

$$\eta\,CV = \text{decrease of charge} = \dot{n}_i\,e A\,dt = \frac{\varepsilon_0 A}{d} V \eta$$

or, $$t = \frac{\varepsilon_0 V \eta}{\dot{n}_i\,e\,d^2} = 4{\cdot}6 \textit{ days}$$

Note, that n_i, here, is the number of ion pairs produced.

3.217 If ν = number of electrons moving to the anode at distance x, then

$$\frac{d\nu}{dx} = \alpha\,\nu \quad \text{or} \quad \nu = \nu_0\,e^{\alpha x}$$

Assuming saturation, $I = e\,\nu_0\,e^{\alpha d}$

3.218 Since the electrons are produced uniformly through the volume, the total current attaining saturation is clearly,

$$I = \int_0^d e\,(\dot{n}_i A\,dx)\,e^{\alpha x} = e\,\dot{n}_i A \left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

Thus, $$j = \frac{I}{A} = e\,\dot{n}_i \left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

3.5 CONSTANT MAGNETIC FIELD. MAGNETICS

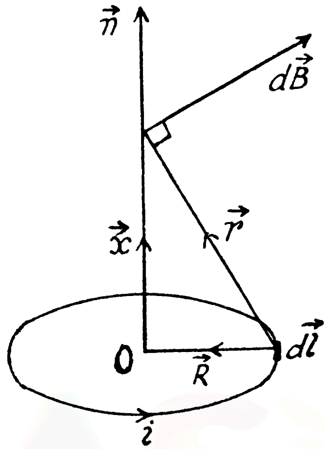
3.219 (a) From the Biot - Savart law,

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \vec{r}}{r^3}, \text{ so}$$

$$dB = \frac{\mu_0}{4\pi} i \frac{(R\, d\theta)\, R}{R^3} \left(\text{as } d\vec{l} \perp \vec{r}\right)$$

From the symmetry

$$B = \int dB = \int_0^{2\pi} \frac{\mu_0}{4\pi} \frac{i}{R} d\theta = \frac{\mu_0}{2} \frac{i}{R} = 6{\cdot}3\, \mu\text{T}$$



(b) From Biot-Savart's law :

$$\vec{B} = \frac{\mu_0}{4\pi} i \int \frac{d\vec{l} \times \vec{r}}{r^3} \left(\text{here } \vec{r} = \vec{R} + \vec{x}\right)$$

So, $$\vec{B} = \frac{\mu_0}{4\pi} i \left[\oint d\vec{l} \times \vec{R} + \oint d\vec{l} \times \vec{x}\right]$$

Since $\vec{x}$ is a constant vector and $|\vec{R}|$ is also constant

So, $\oint d\vec{l} \times \vec{x} = \left(\oint d\vec{l}\right) \times \vec{x} = 0 \left(\text{because } \oint d\vec{l} = 0\right)$

and $$\oint d\vec{l} \times \vec{R} = \oint R\, dl\, \vec{n}$$

$$= \vec{n} R \oint dl = 2\pi R^2 \vec{n}$$

Here $\vec{n}$ is a unit vector perpendicular to the plane containing the current loop (Fig.) and in the direction of $\vec{x}$

Thus we get $$\vec{B} = \frac{\mu_0}{4\pi} \frac{2\pi R^2 i}{(x^2 + R^2)^{3/2}} \vec{n}$$

3.220 As $\angle AOB = \frac{2\pi}{n}$, OC or perpendicular distance of any segment from centre equals $R \cos \frac{\pi}{n}$. Now magnetic induction at O, due to the right current carrying element AB

$$= \frac{\mu_0}{4\pi} \frac{i}{R \cos \frac{\pi}{n}} 2 \sin \frac{\pi}{n}$$

(From Biot–Savart's law, the magnetic field at O due to any section such as AB is perpendicular to the plane of the figure and has the magnitude.)

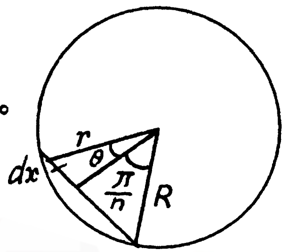
$$B = \int \frac{\mu_0}{4\pi} i \frac{dx}{r^2} \cos\theta = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \frac{\mu_0 i}{4\pi} \frac{R\cos\frac{\pi}{n}\sec^2\theta d\theta}{R^2\cos\frac{2\pi}{n}\sec^2\theta} \cos\theta = \frac{\mu_0 i}{4\pi} \frac{1}{R\cos\frac{\pi}{n}} 2\sin\frac{\pi}{n}$$

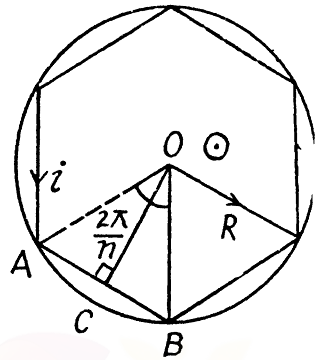
As there are n number of sides and magnetic induction vectors, due to each side at O, are equal in magnitude and direction. So,

$$B_0 = \frac{\mu_0}{4\pi} \frac{ni}{R\cos\frac{\pi}{n}} 2\sin\frac{\pi}{n} \cdot n$$

$$= \frac{\mu_0}{2\pi} \frac{ni}{R} \tan\frac{\pi}{n} \quad \text{and for } n \to \infty$$

$$B_0 = \frac{\mu_0}{2} \frac{i}{R} \underset{n\to\infty}{Lt} \left(\frac{\tan\frac{\pi}{n}}{\pi/n} \right) = \frac{\mu_0}{2} \frac{i}{R}$$





3.221 We know that magnetic induction due to a straight current carrying wire at any point, at a perpendicular distance from it is given by :

$$B = \frac{\mu_0}{4\pi} \frac{i}{r} (\sin\theta_1 + \sin\theta_2),$$

where r is the perpendicular distance of the wire from the point, considered, and θ_1 is the angle between the line, joining the upper point of straight wire to the considered point and the perpendicular drawn to the wire and θ_2 that from the lower point of the straight wire.

Here, $$B_1 = B_3 = \frac{\mu_0}{4\pi} \frac{i}{(d/2)\sin\frac{\varphi}{2}} \left\{ \cos\frac{\varphi}{2} + \cos\frac{\varphi}{2} \right\}$$

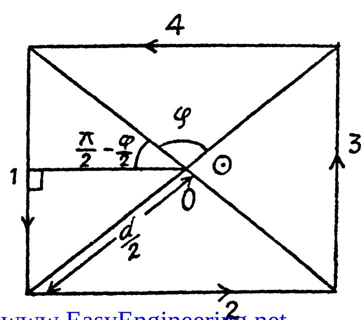
and $$B_2 = B_4 = \frac{\mu_0}{4\pi} \frac{i}{(d/2)\cos\frac{\varphi}{2}} \left(\sin\frac{\varphi}{2} + \sin\frac{\varphi}{2} \right)$$

Hence, the magnitude of total magnetic induction at O,

$$B_0 = B_1 + B_2 + B_3 + B_4$$

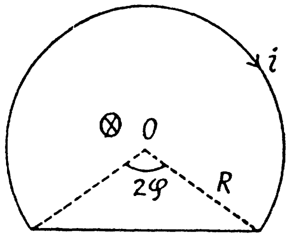
$$= \frac{\mu_0}{4\pi} \frac{4i}{d/2} \left[\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + \frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} \right]$$

$$= \frac{4\mu_0 i}{\pi d \sin\varphi} = 0{\cdot}10 \text{ mT}$$



3.222 Magnetic induction due to the arc segment at O,

$$B_{arc} = \frac{\mu_0}{4\pi}\frac{i}{R}(2\pi - 2\varphi)$$



and magnetic induction due to the line segment at O,

$$B_{line} = \frac{\mu_0}{4\pi}\frac{i}{R\cos\varphi}[2\sin\varphi]$$

So, total magnetic induction at O,

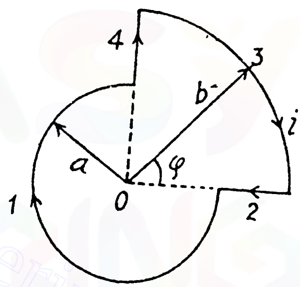
$$B_0 = B_{arc} + B_{line} = \frac{\mu_0}{2\pi}\frac{i}{R}[\pi - \varphi + \tan\varphi] = 28\ \mu\text{T}$$

3.223 (a) From the Biot-Savart law,

$$dB = \frac{\mu_0}{4\pi} i \frac{(\vec{dl} \times \vec{r})}{r^3}$$

So, magnetic field induction due to the segment 1 at O,

$$B_1 = \frac{\mu_0}{4\pi}\frac{i}{a}(2\pi - \varphi)$$



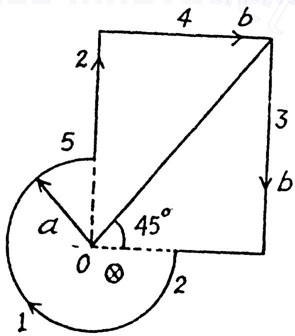
also $B_2 = B_4 = 0$, as $\vec{dl} \uparrow\uparrow \vec{r}$

and $$B_3 = \frac{\mu_0}{4\pi}\frac{i}{b}\varphi$$

Hence, $B_0 = B_1 + B_2 + B_3 + B_4$

$$= \frac{\mu_0}{4\pi} i \left[\frac{2\pi - \varphi}{a} + \frac{\varphi}{b}\right],$$

(b) Here, $B_1 = \frac{\mu_0}{4\pi}\frac{i}{a}\cdot\frac{3\pi}{a}$, $\vec{B}_2 = 0$,



$$B_3 = \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45°,$$

$$B_4 = \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45°,$$

and $B_5 = 0$

So, $B_0 = B_1 + B_2 + B_3 + B_4 + B_5$

$$= \frac{\mu_0}{4\pi}\frac{i}{a}\frac{3\pi}{2} + 0 + \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45° + \frac{\mu_0}{4\pi}\frac{i}{b}\sin 45° + 0$$

$$= \frac{\mu_0}{4\pi} i \left[\frac{3\pi}{2a} + \frac{\sqrt{2}}{b}\right]$$

3.224 The thin walled tube with a longitudinal slit can be considered equivalent to a full tube and a strip carrying the same current density in the opposite direction. Inside the tube, the former does not contribute so the total magnetic field is simply that due to the strip. It is

$$B = \frac{\mu_0}{2\pi} \frac{(I/2\pi R)\,h}{r} = \frac{\mu_0 I h}{4\pi^2 R r}$$

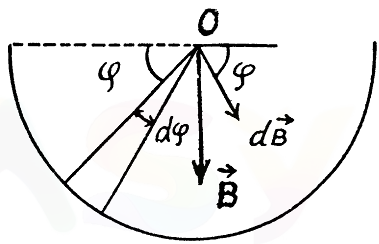
where r is the distance of the field point from the strip.

3.225 First of all let us find out the direction of vector $\vec{B}$ at point O. For this purpose, we divide the entire conductor into elementary fragments with current di. It is obvious that the sum of any two symmetric fragments gives a resultant along $\vec{B}$ shown in the figure and consequently, vector $\vec{B}$ will also be directed as shown

So, $$|\vec{B}| = \int dB \sin\varphi \qquad (1)$$

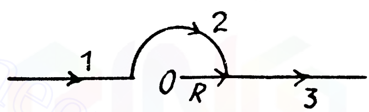
$$= \int \frac{\mu_0}{2\pi R} di \sin\varphi$$

$$= \int_0^{\pi} \frac{\mu_0}{2\pi^2 R} i \sin\varphi\, d\varphi, \quad \left(\text{as } di = \frac{i}{\pi} d\varphi\right)$$



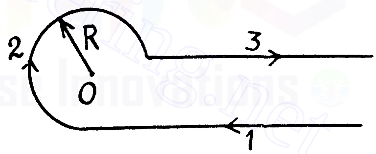
Hence $$B = \mu_0 i/\pi^2 R$$

3.226 (a) From symmetry



$$B_0 = B_1 + B_2 + B_3$$

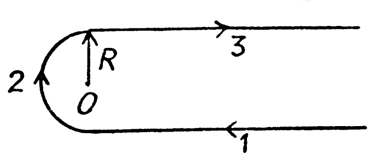
$$= 0 + \frac{\mu_0}{4\pi}\frac{i}{R}\pi + 0 = \frac{\mu_0}{4}\frac{i}{R}$$



(b) From symmetry

$$B_0 = B_1 + B_2 + B_3$$

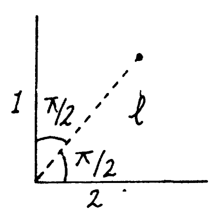
$$= \frac{\mu_0}{4\pi}\frac{i}{R} + \frac{\mu_0}{2\pi}\frac{i}{R}\frac{3\pi}{2} + 0 = \frac{\mu_0}{4\pi}\frac{i}{R}\left[1 + \frac{3\pi}{2}\right]$$



(c) From symmetry

$$B_0 = B_1 + B_2 + B_3$$

$$= \frac{\mu_0}{4\pi}\frac{i}{R} + \frac{\mu_0}{4\pi}\frac{i}{R}\pi + \frac{\mu_0}{4\pi}\frac{i}{R} = \frac{\mu_0}{4\pi}\frac{i}{R}(2+\pi)$$



3.227 $\vec{B}_0 = \vec{B}_1 + \vec{B}_2$

or, $$|\vec{B}_0| = \frac{\mu_0 i}{4\pi l}\sqrt{2} = 2{\cdot}0\,\mu\,\text{T}, \text{ (using 3.221)}$$

3.228 (a)
$$\vec{B_0} = \vec{B_1} + \vec{B_2} + \vec{B_3}$$
$$= \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}\pi(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k})$$
$$= -\frac{\mu_0}{4\pi}\frac{i}{R}[2\vec{k} + \pi\vec{i}]$$

So,
$$|\vec{B_0}| = \frac{\mu_0}{4\pi}\frac{i}{R}\sqrt{\pi^2 + 4} = 0{\cdot}30\ \mu\text{T}$$

(b) $\vec{B_0} = \vec{B_1} + \vec{B_2} + \vec{B_3}$
$$= \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}\pi(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{i})$$
$$= -\frac{\mu_0}{4\pi}\frac{i}{R}[\vec{k} + (\pi+1)\vec{i}]$$

So,
$$|\vec{B_0}| = \frac{\mu_0}{4\pi}\frac{i}{R}\sqrt{1 + (\pi+1)^2} = 0{\cdot}34\ \mu\text{T}$$

(c) Here using the law of parallel resistances
$$i_1 + i_2 = i \text{ and } \frac{i_1}{i_2} = \frac{1}{3},$$

So,
$$\frac{i_1 + i_2}{i_2} = \frac{4}{3}$$

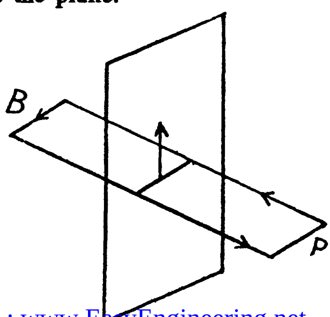
Hence
$$i_2 = \frac{3}{4}i, \text{ and } i_1 = \frac{1}{4}i$$

Thus $$\vec{B_0} = \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{k}) + \frac{\mu_0}{4\pi}\frac{i}{R}(-\vec{j}) + \left[\frac{\mu_0}{4\pi}\left(\frac{3\pi}{2}\right)\frac{i_1}{R}(-\vec{i}) + \frac{\mu_0}{4\pi}\frac{(\pi/2)\, i_2}{R}\vec{i}\right]$$
$$= \frac{-\mu_0}{4\pi}\frac{i}{R}(\vec{j} + \vec{k}) + 0$$

Thus,
$$|\vec{B_0}| = \frac{\mu_0}{4\pi}\frac{\sqrt{2}\, i}{R} = 0{\cdot}11\ \mu\text{T}$$

3.229 (a) We apply circulation theorem as shown. The current is vertically upwards in the plane and the magnetic field is horizontal and parallel to the plane.

$$\oint \vec{B}\cdot d\vec{l} = 2Bl = \mu_0 il \quad \text{or,} \quad B = \frac{\mu_0 i}{2}$$



(b) Each plane contributes $\mu_0 \frac{i}{2}$ between the planes and outside the plane that cancel.

Thus
$$B = \begin{cases} \mu_0 i \text{ between the plane} \\ 0 \text{ outside.} \end{cases}$$

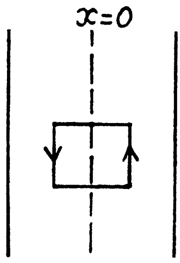
3.230 We assume that the current flows perpendicular to the plane of the paper, by circulation theorem,

$$2B\,dl = \mu_0 (2\,x\,dl)\,j$$

or, $$B = \mu_0 x j, \ |x| \le d$$

Outside, $$2\,B\,dl = \mu_0 (2d\,dl)\,j$$

or, $$B = \mu_0 d j \ \ |x| \ge d.$$

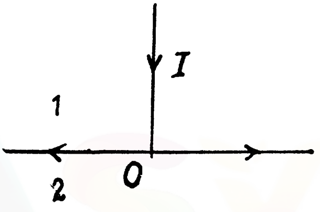


3.231 It is easy to convince oneself that both in the regions. 1 and 2, there can only be a circuital magnetic field (i.e. the component B_φ). Any radial field in region 1 or any B_Z away from the current plane will imply a violation of Gauss' law of magnetostatics, B_φ must obviously be symmetrical about the straight wire. Then in 1,

$$B_\varphi 2\pi r = \mu_0 I$$

or, $$B_\varphi = \frac{\mu_0}{2\pi}\frac{I}{r}$$

In 2, $$B_\varphi \cdot 2\pi r = 0, \quad \text{or} \quad B_\varphi = 0$$



3.232 On the axis, $B = \dfrac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}} = B_x$, along the axis.

Thus, $$\int \vec{B} \cdot d\vec{r} = \int_{-\infty}^{\infty} B_x\,dx = \frac{\mu_0 I R^2}{2}\int_{-\infty}^{\infty} \frac{dx}{(R^2 + x^2)^{3/2}}$$

$$= \frac{\mu_0 I R^2}{2}\int_{-\pi/2}^{\pi/2} \frac{R\sec^2\theta\,d\theta}{R^3\sec^3\theta}, \text{ on putting } x = R\tan\theta$$

$$= \mu_0 I \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = \mu_0 I$$

The physical interpretation of this result is that $\int_{-\infty}^{\infty} B_x\,dx$ can be thought of as the circulation of B over a closed loop by imaging that the two ends of the axis are connected, by a line at infinity (e.g. a semicircle of infinite radius).

3.233 By circulation theorem inside the conductor

$$B_\varphi 2\pi r = \mu_0 j_z \pi r^2 \quad \text{or,} \quad B_\varphi = \mu_0 j_z r/2$$

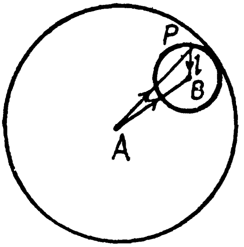
i.e., $$\vec{B} = \frac{1}{2}\mu_0 \vec{j} \times \vec{r}$$

Similarly outside the conductor,

$$B_\varphi 2\pi r = \mu_0 j_z \pi R^2 \quad \text{or,} \quad B_\varphi = \frac{1}{2}\mu_0 j_z \frac{R^2}{r}$$

So, $$\vec{B} = \frac{1}{2}\mu_0 (\vec{j} \times \vec{r}) \frac{R^2}{r^2}$$

2.234 We can think of the given current which will be assumed uniform, as arising due to a negative current, flowing in the cavity, superimposed on the true current, everywhere including the cavity. Then from the previous problem, by superposition.



$$\vec{B} = \frac{1}{2}\mu_0 \vec{j} \times (\vec{AP} - \vec{BP}) = \frac{1}{2}\mu_0 \vec{j} \times \vec{l}$$

If $\vec{l}$ vanishes so that the cavity is concentric with the conductor, there is no magnetic field in the cavity.

3.235 By Circulation theorem,

$$B_\varphi \cdot 2\pi r = \mu_0 \int_0^r j(r') \times 2\pi r' \, dr'$$

or using $B_\varphi = br^\alpha$ inside the stream,

$$br^{\alpha+1} = \mu_0 \int_0^r j(r') r' \, dr'$$

So by differentiation,

$$(\alpha + 1) br^\alpha = \mu_0 j(r) r$$

Hence, $$j(r) = \frac{b(\alpha + 1)}{\mu_0} r^{\alpha - 1}$$

3.236 On the surface of the solenoid there is a surface current density

$$\vec{j_s} = nI\hat{e}_\varphi$$

Then, $$\vec{B} = -\frac{\mu_0}{4\pi} nI \int R\, d\varphi\, dz \frac{\hat{e}_\varphi \times \vec{r_0}}{r_0^3}$$

where $\vec{r_0}$ is the vector from the current element to the field point, which is the centre of the solenoid,

Now, $$-\hat{e}_\varphi \times \vec{r_0} = R\hat{e}_z$$

$$r_0 = (z^2 + R^2)^{1/2}$$

Thus, $$B = B_z = \frac{\mu_0 nI}{4\pi} \times 2\pi R^2 \int_{-l/2}^{-l/2} \frac{dz}{(R^2 + z^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0 n I \int_{-\tan^{-1}\frac{l}{2R}}^{+\tan^{-1}\frac{l}{2R}} \cos\alpha\, d\alpha \qquad \text{(on putting } z = R\tan\alpha)$$

$$= \mu_0 n I \sin\alpha = \mu_0 nI \frac{l/2}{\sqrt{(l/2)^2 + R^2}} = \mu_0 n I \Big/ \sqrt{1 + \left(\frac{2R}{l}\right)^2}$$

3.237 We proceed exactly as in the previous problem. Then (a) the magnetic induction on the axis at a distance x from one end is clearly,

$$B = \frac{\mu_0 nI}{4\pi} \times 2\pi R^2 \int_0^\infty \frac{dz}{[R^2 + (z-x)^2]^{3/2}} = \frac{1}{2}\mu_0 n I R^2 \int_x^\infty \frac{dz}{(z^2 + R^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0 n I \int_{\tan^{-1}\frac{x}{R}}^{\pi/2} \cos\theta\, d\theta = \frac{1}{2}\mu_0 n I \left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

$x > 0$ menas that the field point is outside the solenoid. B then falls with x. $x < 0$ means that the field point gets more and more inside the solenoid. B then increases with (x) and eventually becomes constant, equal to $\mu_0 n I$. The $B - x$ graph is as given in the answer script.

(b) We have, $\dfrac{B_0 - \delta B}{B_0} = \dfrac{1}{2}\left[1 - \dfrac{x_0}{\sqrt{R^2 + x_0^2}}\right] = 1 - \eta$

or, $$-\frac{x_0}{\sqrt{R^2 + x_0^2}} = 1 - 2\eta$$

Since η is small ($\sim 1\%$), x_0 must be negative. Thus $x_0 = -|x_0|$

and $$\frac{|x_0|}{\sqrt{R^2 + |x_0|^2}} = 1 - 2\eta$$

$$|x_0|^2 = (1 - 4\eta + 4\eta^2)(R^2 + |x_0|^2)$$

$$0 = (1 - 2\eta)^2 R^2 - 4\eta(1-\eta)|x_0|^2$$

or, $$|x_0| = \frac{(1 - 2\eta)R}{2\sqrt{\eta(1-\eta)}}$$

3.238 If the strip is tightly wound, it must have a pitch of h. This means that the current will flow obliquely, partly along $\hat{e}_\varphi$ and partly along $\hat{e}_z$. Obviously, the surface current density is,

$$\vec{J}_s = \frac{I}{h}\left[\sqrt{1 - (h/2\pi R)^2}\,\hat{e}_\varphi + \frac{h}{2\pi R}\hat{e}_z\right].$$

By comparision with the case of a solenoid and a hollow straight conductor, we see that field inside the coil

$$= \mu_0 \frac{I}{h}\sqrt{1-(h/2\pi R)^2}$$

(Cf. $B = \mu_0 nI$).

Outside, only the other term contributes, so

$$B_\varphi \times 2\pi r = \mu_0 \frac{I}{h} \times \frac{h}{2\pi R} \times 2\pi R$$

or,
$$B_\varphi = \frac{\mu_0}{4\pi}\cdot\frac{2I}{r}.$$

Note - *Surface current density is defined as current flowing normally across a unit length over a surface.*

3.239 Suppose a is the radius of cross section of the core. The winding has a pitch $2\pi R/N$, so the surface current density is

$$\vec{J}_s = \frac{I}{2\pi R/N}\vec{e}_1 + \frac{I}{2\pi a}\vec{e}_2$$

where $\vec{e}_1$ is a unit vector along the cross section of the core and $\vec{e}_2$ is a unit vector along its length.

The magnetic field inside the cross section of the core is due to first term above, and is given by

$$B_\varphi \cdot 2\pi R = \mu_0 NI$$

(NI is total current due to the above surface current (first term.))

Thus, $\quad B_\varphi = \mu_0 NI/2\pi R.$

The magnetic field at the centre of the core can be obtained from the basic formula.

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}_s \times \vec{r}_0}{r_0^3} dS$ and is due to the second term.

So,
$$\vec{B} = B_z \vec{e}_z = \vec{e}_z \frac{\mu_0}{4\pi}\frac{I}{2\pi a}\int \frac{1}{R^3} R d\varphi \times 2\pi a$$

or,
$$B_z = \frac{\mu_0 I}{2R}$$

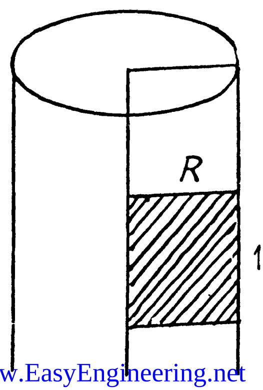
The ratio of the two magnetic field, is $= \frac{N}{\pi}$

3.240 We need the flux through the shaded area.

Now by Ampere's theorem,

$$B_\varphi 2\pi r = \mu_0 \frac{I}{\pi R^2}\cdot \pi r^2$$

or,
$$B_\varphi = \frac{\mu_0}{2\pi} I \frac{r}{R^2}$$



The flux through the shaded region is,

$$\varphi_1 = \int_0^R 1 \cdot dr \cdot B_\varphi(r)$$

$$= \int_0^R dr \frac{\mu_0}{2\pi} I \frac{r}{R^2} = \frac{\mu_0}{4\pi} I.$$

3.241 Using 3.237, the magnetic field is given by,

$$B = \frac{1}{2}\mu_0 nI\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

At the end, $B = \frac{1}{2}\mu_0 n I = \frac{1}{2}B_0$, where $B_0 = \mu_0 nI$,

is the field deep inside the solenoid. Thus,

$$\Phi = \frac{1}{2}\mu_0 nIS = \Phi_0/2, \text{ where } \Phi = \mu_0 nI S$$

is the flux of the vector B through the cross section deep inside the solenid.

3.242
$$B_\varphi 2\pi r = \mu_0 N I$$

or,
$$B_\varphi = \frac{\mu_0 N I}{2\pi r}$$

Then, $\Phi = \int_a^b B_\varphi h dr, a \le r \le b = \frac{\mu_0}{4\pi} 2N I h \ln \eta$, where $\eta = b/a$

3.243 Magnetic moment of a current loop is given by $p_m = n i S$ (where n is the number of turns and S, the cross sectional area.) In our problem, $n = 1$, $S = \pi R^2$ and $B = \frac{\mu_0}{2}\frac{i}{R}$

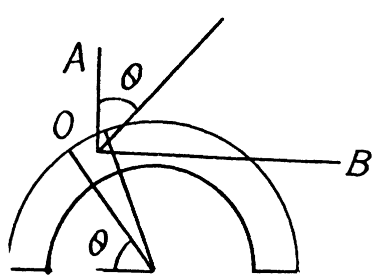
So,
$$p_m = \frac{2BR}{\mu_0}\pi R^2 = \frac{2\pi B R^3}{\mu_0}$$

3.244 Take an element of length $r d\theta$ containing $\frac{N}{\pi r} \cdot r d\theta$ turns. Its magnetic moment is

$$\frac{N}{\pi} d\theta \cdot \frac{\pi}{4} d^2 I$$

normal to the plane of cross section. We resolve it along OA and OB. The moment along OA integrates to

$$\int_0^\pi \frac{N}{4} d^2 I \, d\theta \cos\theta = 0$$



while that along OB gives

$$p_m = \int_0^\pi \frac{N d^2 I}{4} \sin\theta \, d\theta = \frac{1}{2} N d^2 I$$

3.245 (a) From Biot-Savart's law, the magnetic induction due to a circular current carrying wire loop at its centre is given by,

$$B_r = \frac{\mu_0}{2r} i$$

The plane spiral is made up of concentric circular loops, having different radii, varying from a to b. Therefore, the total magnetic induction at the centre,

$$B_0 = \int \frac{\mu_0}{2r} dN \qquad (1)$$

where $\frac{\mu_0}{2r} i$ is the contribution of one turn of radius r and dN is the number of turns in the interval $(r, r + dr)$

i.e. $$dN = \frac{N}{b-a} dr$$

Substituting in equation (1) and integrating the result over r between a and b, we obtain,

$$B_0 = \int_a^b \frac{\mu_0 i}{2r} \frac{N}{(b-a)} dr = \frac{\mu_0 \, i N}{2\,(b-a)} \ln \frac{b}{a}$$

(b) The magnetic moment of a turn of radius r is $p_m = i \pi r^2$ and of all turns,

$$p = \int p_m \, dN = \int_a^b i \pi r^2 \frac{N}{b-a} dr = \frac{\pi \, i N \,(b^3 - a^2)}{3\,(b-a)}$$

3.246 (a) Let us take a ring element of radius r and thickness dr, then charge on the ring element., $d\,q = \sigma \, 2 \pi r \, dr$

and current, due to this element, $di = \frac{(\sigma \, 2 \pi r \, dr)\, \omega}{2 \pi} = \sigma \omega r \, dr$

So, magnetic induction at the centre, due to this element : $dB = \frac{\mu_0}{2} \frac{di}{r}$

and hence, from symmetry : $B = \int dB = \int_0^R \frac{\mu_0 \sigma \omega r \, dr}{r} = \frac{\mu_0}{2} \sigma \omega R$

(b) Magnetic moment of the element, considered,

$$dp_m = (di)\, \pi r^2 = \sigma \omega \, dr \, \pi r^2 = \sigma \pi \omega r^3 \, dr$$

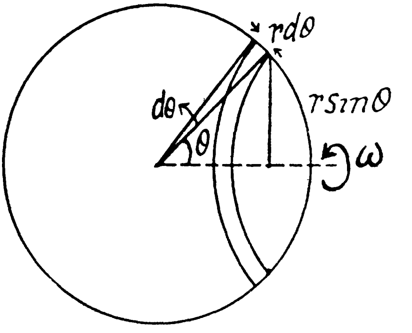
Hence, the sought magnetic moment,

$$p_m = \int dp_m = \int_0^R \sigma \pi \omega r^3 \, dr = \sigma \omega \pi \frac{R^4}{4}$$

3.247 As only the outer surface of the sphere is charged, consider the element as a ring, as shown in the figure.

The equivalent current due to the ring element,

$$di = \frac{\omega}{2\pi}(2\pi r \sin\theta \, r d\theta)\,\sigma \qquad (1)$$



and magnetic induction due to this loop element at the centre of the sphere, O,

$$dB = \frac{\mu_0}{4\pi} di \frac{2\pi r \sin\theta \, r \sin\theta}{r^3} = \frac{\mu_0}{4\pi} di \frac{\sin^2\theta}{r}$$

[Using 3.219 (b)]

Hence, the total magnetic induction due to the sphere at the centre, O,

$$B = \int dB = \int_0^{\pi/2} \frac{\mu_0}{4\pi}\frac{\omega}{2\pi}\frac{2\pi r^2 \sin\theta \, d\theta \sin^2\theta \,\sigma}{r} \text{ [using (1)]}$$

Hence, $$B = \int_0^{\pi/2} \frac{\mu_0 \sigma \omega r}{4\pi} \sin^3\theta \, d\theta = \frac{2}{3}\mu_0 \sigma \omega r = 29 \text{ pT}$$

3.248 The magnetic moment must clearly be along the axis of rotation. Consider a volume element dV. It contains a charge $\frac{q}{4\pi/3\, R^3} dV$. The rotation of the sphere causes this charge to revolve around the axis and constitute a current.

$$\frac{3q}{4\pi R^3} dV \times \frac{\omega}{2\pi}$$

Its magnetic moment will be

$$\frac{3q}{4\pi R^3} dV \times \frac{\omega}{2\pi} \times \pi r^2 \sin^2\theta$$

So the total magnetic moment is

$$p_m = \int_0^R \int_0^\pi \frac{3q}{2R^3} r^2 \sin\theta d\theta \times \frac{\omega r^2 \sin^2\theta}{2} dr = \frac{3q}{2R^3} \times \frac{\omega}{2} \times \frac{R^5}{5} \times \frac{4}{3} = \frac{1}{5} qR^2 \omega$$

The mechanical moment is

$$M = \frac{2}{5} mR^2 \omega, \quad \text{So,} \quad \frac{p_m}{M} = \frac{q}{2m}.$$

3.249 Because of polarization a space charge is present within the cylinder. It's density is

$$\rho_p = -\text{div}\,\vec{P} = -2\alpha$$

Since the cylinder as a whole is neutral a surface charge density σ_p must be present on the surface of the cylinder also. This has the magnitude (algebraically)

$$\sigma_p \times 2\pi R = 2\,\alpha\,\pi R^2 \quad \text{or,} \quad \sigma_p = \alpha R$$

When the cylinder rotates, currents are set up which give rise to magnetic fields. The contribution of ρ_p and σ_p can be calculated separately and then added.

For the surface charge the current is (for a particular element)

$$\alpha R \times 2\pi R\,dx \times \frac{\omega}{2\pi} = \alpha R^2\,\omega\,dx$$

Its contribution to the magnetic field at the centre is

$$\frac{\mu_0 R^2 (\alpha R^2 \omega\,dx)}{2\,(x^2 + R^2)^{3/2}}$$

and the total magnetic field is

$$B_s = \int_{\infty}^{-\infty} \frac{\mu_0 R^2 (\alpha R^2 \omega dx)}{2\,(x^2 + R^2)^{3/2}} = \frac{\mu_0\,\alpha\,R^4 \omega}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + R^2)^{3/2}} = \frac{\mu_0\,\alpha\,R^4\,\omega}{2} \times \frac{2}{R^2} = \mu_0\,\alpha\,R^2 \omega$$

As for the volume charge density consider a circle of radius r, radial thickness dr and length dx.

The current is $-\,2\alpha \times 2\,\pi\,r\,dr\,dx \times \dfrac{\omega}{2\pi} = -\,2\,\alpha r\,dr\,\omega\,dx$

The total magnetic field due to the volume charge distribution is

$$B_v = -\int_0^R dr \int_{-\infty}^{\infty} dx\,2\,\pi\,r\,\omega\,\frac{\mu_0 r^2}{2\,(x^2 + r^2)^{3/2}} = -\int_0^R \alpha\,\mu_0\,\omega\,r^3\,dr \int_{-\infty}^{\infty} dx\,(x^2 + r^2)^{3/2}$$

$$= -\int_0^R \alpha\,\mu_0\,\omega r\,dr \times 2 = -\,\mu_0\,\alpha\,\omega\,R^2 \quad \text{so,} \quad B = B_s + B_v = 0$$

3.250 Force of magnetic interaction, $\vec{F}_{mag} = e\,(\vec{v} \times \vec{B})$

Where, $$\vec{B} = \frac{\mu_0}{4\pi}\,\frac{e\,(\vec{v} \times \vec{r})}{r^3}$$

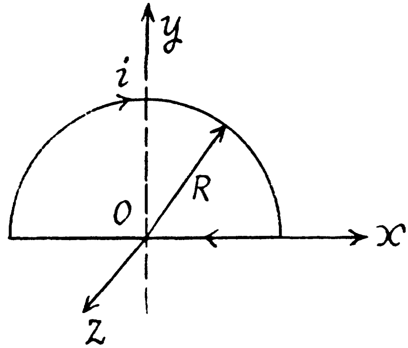
So, $$\vec{F}_{mag} = \frac{\mu_0}{4\pi}\,\frac{e^2}{r^3}\,[\vec{v} \times (\vec{v} \times \vec{r})\,]$$

$$= \frac{\mu_0}{4\pi}\,\frac{e^2}{r^3}\,[(\vec{v}\cdot\vec{r}) \times \vec{v} - (\vec{v}\cdot\vec{v}) \times \vec{r}\,] = \frac{\mu_0}{4\pi}\,\frac{e^2}{r^3}\,(-\,v^2\,\vec{r})$$

And $$\vec{F}_{ele} = e\vec{E} = e\,\frac{1}{4\,\pi\,\varepsilon_0}\,\frac{e\vec{r}}{|\,\vec{r}\,|^3}$$

Hence, $$\frac{|\vec{F}_{mag}|}{|\vec{F}_{electic}|} = -\,v^2 \mu_0\,\varepsilon_0 = \left(\frac{v}{c}\right)^2 = 1{\cdot}00 \times 10^{-6}$$

3.251 (a) The magnetic field at O is only due to the curved path, as for the line element, $d\vec{l} \uparrow\uparrow \vec{r}$.

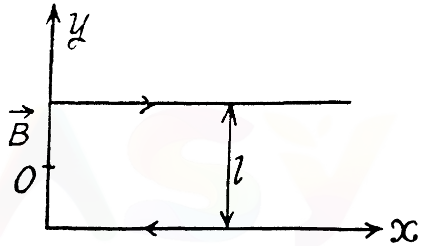


Hence, $\vec{B} = \frac{\mu_0 i}{4\pi R}\pi(-\vec{k}) = \frac{\mu_0 i}{4R}(-\vec{k})$

Thus $\vec{F}_u = iB(-\vec{j}) = \frac{\mu_0 i^2}{4R}(-\vec{j})$

So, $F_u = \frac{\pi_0 i^2}{4R} = 0{\cdot}20\ \text{N/m}$

(b) In this part, magnetic induction $\vec{B}$ at O will be effective only due to the two semi infinite segments of wire. Hence



$$\vec{B} = 2\cdot\frac{\mu_0 i}{4\pi\left(\frac{l}{2}\right)}\sin\frac{\pi}{2}(-\vec{k})$$

$$= \frac{\mu_0 i}{\pi l}(-\vec{k})$$

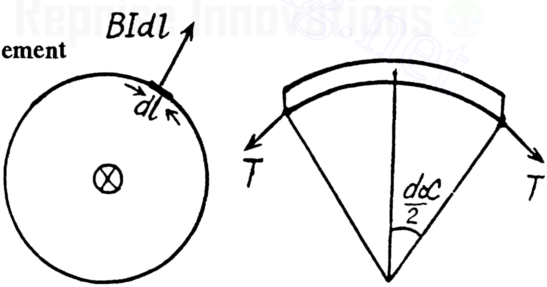
Thus force per unit length,

$$\vec{F}_u = \frac{\mu_0 l^2}{\pi l}(-\vec{i})$$

3.252 Each element of length dl experiences a force $BI\,dl$. This causes a tension T in the wire. For equilibrium,

$$T\,d\alpha = BI\,dl,$$

where $d\alpha$ is the angle subtended by the element at the centre.



Then, $$T = BI\frac{dl}{d\alpha} = BIR$$

The wire experiences a stress

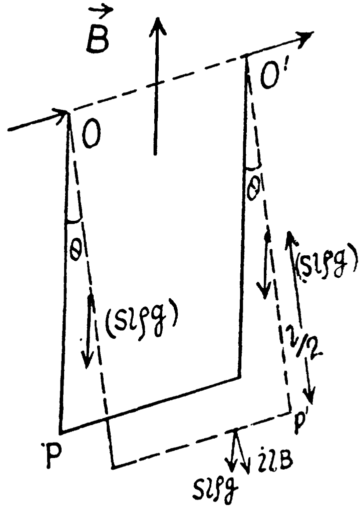
$$\frac{BIR}{\pi d^2/4}$$

This must equals the breaking stress σ_m for rupture. Thus,

$$B_{max} = \frac{\pi d^2 \sigma_m}{4IR}$$

3.253 The Ampere forces on the sides OP and $O'P'$ are directed along the same line, in opposite directions and have equal values, hence the net force as well as the net torque of these forces about the axis OO' is zero. The Ampere-force on the segment PP' and the corresponding moment of this force about the axis OO' is effective and is deflecting in nature.

In equilibrium (in the dotted position) the deflecting torque must be equal to the restoring torque, developed due to the weight of the shape.



Let, the length of each side be l and ρ be the density of the material then,

$$ilB\,(l\cos\theta) = (S\,l\,\rho)\,g\,\frac{l}{2}\sin\theta + (S\,l\,\rho)\,g\,\frac{l}{2}\sin\theta + (S\,l\,\rho)\,gl\sin\theta$$

or, $$il^2 B\cos\theta = 2\,S\,\rho\,gl^2\sin\theta$$

Hence, $$B = \frac{2S\,\rho g}{i}\tan\theta$$

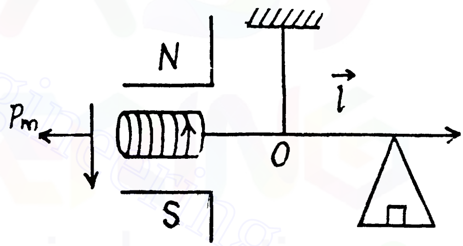
3.254 We know that the torque acting on a magnetic dipole.

$$\vec{N} = \vec{p}_m \times \vec{B}$$

But, $\vec{p}_m = i\,S\,\hat{n}$, where $\hat{n}$ is the normal on the plane of the loop and is directed in the direction of advancement of a right handed screw, if we rotate the screw in the sense of current in the loop.

On passing a current through the coil, this torque acting on the magnetic dipol, is counterbalanced by the moment of additional weight, about O.

Hence, the direction of current in the loop must be in the direction, shown in the figure.



$$\vec{p}_m \times \vec{B} = -\vec{l} \times \Delta m\vec{g}$$

or, $$N\,i\,S\,B = \Delta\,mgl$$

So, $$B = \frac{\Delta mgl}{N\,i\,S} = 0{\cdot}4\text{ T on putting the values.}$$

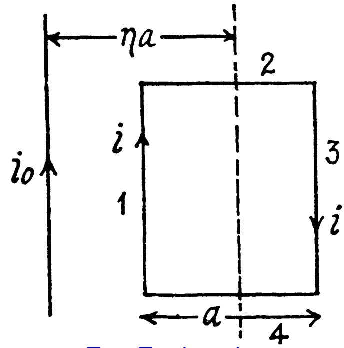
3.255 (a) As is clear from the condition, Ampere's forces on the sides (2) and (4) are equal in magnitude but opposite in direction. Hence the net effective force on the frame is the resultant of the forces, experienced by the sides (1) and (3).

Now, the Ampere force on (1),

$$F_1 = \frac{\mu_0}{2\pi}\,\frac{i i_0}{\left(\eta - \frac{1}{2}\right)}$$

and that on (3),

$$F_3 = \frac{\mu_0}{2\pi}\,\frac{i_0\,i}{\left(\eta + \frac{1}{2}\right)}$$



So, the resultant force on the frame $= F_1 - F_3$, (as they are opposite in nature.)

$$= \frac{2\,\mu_0\, i i_0}{\pi\,(4\,\eta^2 - 1)} = 0{\cdot}40\,\mu\,\text{N}.$$

(b) Work done in turning the frame through some angle, $A = \int i\, d\,\Phi = i\,(\Phi_f - \Phi_i)$, where Φ_f is the flux through the frame in final position, and Φ_i, that in the the initial position.

Here, $\quad |\Phi_f| = |\Phi_i| = \Phi$ and $\Phi_i = -\Phi_f$

so, $\quad \Delta\Phi = 2\,\Phi$ and $A = i\,2\,\Phi$

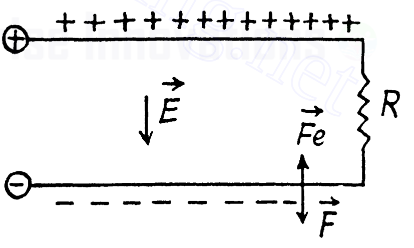
Hence, $\quad A = 2i \int \vec{B}\cdot d\,\vec{S}$

$$= 2i \int_{a\left(\eta - \frac{1}{2}\right)}^{a\left(\eta + \frac{1}{2}\right)} \frac{\mu_0}{2\pi}\frac{i_0\, a}{r}\, dr = \frac{\mu_0\, i i_0\, a}{\pi} \ln\left(\frac{2\eta + 1}{2\eta - 1}\right)$$

3.256 There are excess surface charges on each wire (irrespective of whether the current is flowing through them or not). Hence in addition to the magnetic force $\vec{F}_m$, we must take into account the electric force $\vec{F}_e$. Suppose that an excess charge λ corresponds to a unit length of the wire, then electric force exerted per unit length of the wire by other wire can be found with the help of Gauss's theorem.

$$F_e = \lambda E = \lambda \frac{1}{4\,\pi\,\varepsilon_0} \frac{2\lambda}{l} = \frac{2\lambda^2}{4\,\pi\,\varepsilon_0 l}, \tag{1}$$

where l is the distance between the axes of the wires. The magnetic force acting per unit length of the wire can be found with the help of the theorem on circulation of vector $\vec{B}$



$$F_m = \frac{\mu_0}{4\pi}\frac{2i^2}{l},$$

where i is the current in the wire. (2)

Now, from the relation,

$\lambda = C\,\varphi$, where C is the capacitance of the wires per unit lengths and is given in problem 3.108 and $\varphi = iR$

$$\lambda = \frac{\pi\varepsilon_0}{\ln\eta}\, i R \quad \text{or,} \quad \frac{i}{\lambda} = \frac{\ln\eta}{\pi\varepsilon_0 R} \tag{3}$$

Dividing (2) by (1) and then substuting the value of $\frac{i}{\lambda}$ from (3), we get,

$$\frac{F_m}{F_e} = \frac{\mu_0}{\varepsilon_0}\frac{(\ln\eta)^2}{\pi^2 R^2}$$

The resultant force of interaction vanishes when this ratio equals unity. This is possible when $R = R_0$, where

$$R_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln \eta}{\pi} = 0{\cdot}36 \text{ k}\Omega$$

3.257 Use 3.225

The magnetic field due to the conductor with semicircular cross section is

$$B = \frac{\mu_0 I}{\pi^2 R}$$

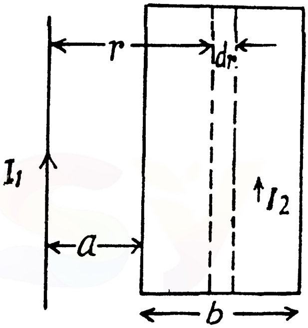
Then $$\frac{\partial F}{\partial l} = BI = \frac{\mu_0 I^2}{\pi^2 R}$$

3.258 We know that Ampere's force per unit length on a wire element in a magnetic field is given by.

$d\vec{F}_n = i(\hat{n} \times \hat{B})$ where $\hat{n}$ is the unit vector along the direction of current. (1)

Now, let us take an element of the conductor i_2, as shown in the figure. This wire element is in the magnetic field, produced by the current i_1, which is directed normally into the sheet of the paper and its magnitude is given by,

$$|\vec{B}| = \frac{\mu_0 I_1}{2\pi r} \quad (2)$$



From Eqs. (1) and (2)

$d\vec{F}_n = \frac{I_2}{b} dr\,(\hat{n} \times \vec{B})$, (because the current through the element equals $\frac{I_2}{b} dr$)

So, $d\vec{F}_n = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{b} \frac{dr}{r}$, towards left (as $\hat{n} \perp^r \vec{B}$).

Hence the magnetic force on the conductor :

$$\vec{F}_n = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{b} \int_a^{a+b} \frac{dr}{r} \text{ (towards left)} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{b} \ln \frac{a+b}{a} \text{ (towards left).}$$

Then according to the Newton's third law the magnitude of sought magnetic interaction force

$$= \frac{\mu_0 I_1 I_2}{2\pi b} \ln \frac{a+b}{a}$$

3.259 By the circulation theorem $B = \mu_0 i$,

where i = current per unit length flowing along the plane perpendicular to the paper. Currents flow in the opposite sense in the two planes and produce the given field B by superposition.

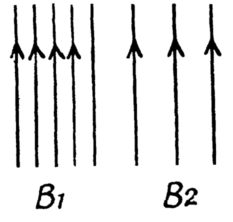
The field due to one of the plates is just $\frac{1}{2}B$. The force on the plate is,

$\frac{1}{2} B \times i \times \text{Length} \times \text{Breadth} = \frac{B^2}{2\mu_0}$ per unit area.

(Recall the formula $F = BIl$ on a straight wire)

3.260 (a) The external field must be $\frac{B_1 + B_2}{2}$, which when superposed with the internal field $\frac{B_1 - B_2}{2}$ (of opposite sign on the two sides of the plate) must give actual field. Now

$$\frac{B_1 - B_2}{2} = \frac{1}{2}\mu_0\, i$$

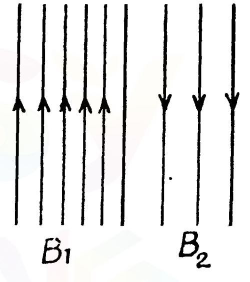


or, $$i = \frac{B_1 - B_2}{\mu_0}$$

Thus, $$F = \frac{B_1^2 - B_2^2}{2\mu_0}$$

(b) Here, the external field must be $\frac{B_1 - B_2}{2}$ upward with an internal field, $\frac{B_1 + B_2}{2}$, upward on the left and downward on the right. Thus,

$$i = \frac{B_1 + B_2}{\mu_0} \text{ and } F = \frac{B_1^2 - B_2^2}{2\mu_0}.$$



(c) Our boundary condition following from Gauss' law is, $B_1 \cos\theta_1 = B_2 \cos\theta_2$.

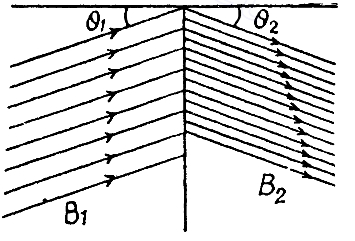
Also, $(B_1 \sin\theta_1 + B_2 \sin\theta_2) = \mu_0\, i$ where i = current per unit length.

The external field parallel to the plate must be $\frac{B_1 \sin\theta_1 - B_2 \sin\theta_2}{2}$

(The perpendicular component $B_1 \cos\theta_1$, does not matter since the corresponding force is tangential)

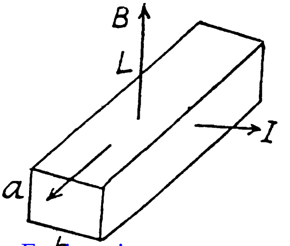
Thus, $$F = \frac{B_1^2 \sin^2\theta_1 - B_2^2 \sin^2\theta_2}{2\mu_0} \text{ per unit area}$$

$$= \frac{B_1^2 - B_2^2}{2\,\mu_0} \text{ per unit area.}$$



The direction of the current in the plane conductor is perpendicular to the paper and beyond the drawing.

3.261 The Current density is $\frac{I}{aL}$, where L is the length of the section. The difference in pressure produced must be,



$$\Delta p = \frac{1}{aL} \times B \times (abL)/ab = \frac{IB}{a}$$

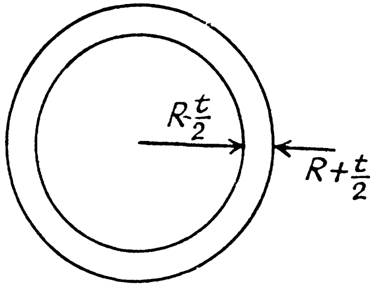
3.262 Let t = thickness of the wall of the cylinder. Then,

$J = I/2\pi R t$ along z axis. The magnetic field due to this at a distance r

$\left(R - \frac{t}{2} < r < R + \frac{t}{2}\right)$, is given by,

$$B_\varphi (2\pi r) = \mu_0 \frac{I}{2\pi R t} \pi \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\}$$

or, $$B_\varphi = \frac{\mu_0 I}{4\pi R r t} \left\{ r^2 - (R - t/2)^2 \right\}$$



Now, $\vec{F} = \int \vec{J} \times \vec{B}\, dV$

and $$p = \frac{F_r}{2\pi R L} = \frac{1}{2\pi R L} \int_{R - \frac{t}{2}}^{R + \frac{t}{2}} \frac{\mu_0 I^2}{8\pi^2 R^2 t^2 r} \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\} \times 2\pi r L\, dr$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \int_{R - \frac{t}{2}}^{R + \frac{t}{2}} \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\} dr = \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \left[\frac{\left(R + \frac{t}{2}\right)^3 - \left(R - \frac{t}{2}\right)^3}{3} - \left(R - \frac{t}{2}\right)^2 t \right]$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t} [Rt + 0(t^2)] = \frac{\mu_0 I^2}{8\pi^2 R^2}$$

3.263 When self-forces are involved, a typical factor of $\frac{1}{2}$ comes into play. For example, the force on a current carrying straight wire in a magnetic induction B is BIl. If the magnetic induction B is due to the current itself then the force can be written as,

$$F = \int_0^I B(I')\, dI'\, l$$

If $B(I') \alpha I'$, then this becomes, $F = \frac{1}{2} B(I) I l$.

In the present case, $B(I) = \mu_0 n I$ and this acts on nI ampere turns per unit length, so,

pressure $$p = \frac{F}{\text{Area}} = \frac{1}{2} \mu_0 n \frac{I \times nI \times 1 \times l}{1 \times l} = \frac{1}{2} \mu_0 n^2 I^2$$

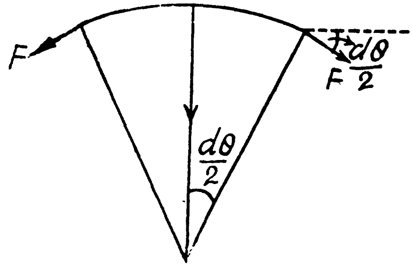
3.264 The magnetic induction B in the solenoid is given by $B = \mu_0 nI$. The force on an element dl of the current carrying conductor is,

$$dF = \frac{1}{2} \mu_0 n I I\, dl = \frac{1}{2} \mu_0 n I^2\, dl$$

This is radially outwards. The factor $\frac{1}{2}$ is explained above.

To relate dF to the tensile strength $F_{\lim}$ we proceed as follows. Consider the equilibrium of the element dl. The longitudinal forces F have a radial component equal to,

$$dF = 2F \sin \frac{d\theta}{2} = F \, d\theta$$



Thus using $dl = R \, d\theta$, $F = \frac{1}{2} \mu_0 n I^2 R$

This equals $F_{\lim}$ when, $I = I_{\lim} = \sqrt{\frac{2 F_{\lim}}{\mu_0 n R}}$

Note that $F_{\lim}$, here, is actually a force and not a stress.

3.265 Resistance of the liquid between the plates $= \frac{\rho d}{S}$

Voltage between the plates $= Ed = v Bd$,

Current through the plates $= \dfrac{vBd}{R + \frac{\rho d}{S}}$

Power, generated, in the external resistance R,

$$P = \frac{v^2 B^2 d^2 R}{\left(R + \frac{\rho d}{S}\right)^2} = \frac{v^2 B^2 d^2}{\left(\sqrt{R} + \frac{\rho d}{S\sqrt{R}}\right)^2} = \frac{v^2 B^2 d^2}{\left[\left\{R^{1/4} - \left(\frac{\rho d}{S\sqrt{R}}\right)^{1/2}\right\}^2 + 2\sqrt{\frac{\rho d}{S}}\right]^2}$$

This is maximum when $R = \frac{\rho D}{S}$ and $P_{max} = \frac{v^2 B^2 S d}{4\rho}$

3.266 The electrons in the conductor are drifting with a speed of,

$$v_d = \frac{J}{ne} = \frac{I}{\pi R^2 ne},$$

where $e =$ magnitude of the charge on the electron, $n =$ concentration of the conduction electorns.

The magnetic field inside the conductor due to this current is given by,

$$B_\varphi (2\pi r) = \pi r^2 \frac{I}{\pi R^2} \mu_0 \quad \text{or,} \quad B_\varphi = \frac{\mu_0}{2\pi} \frac{Ir}{R^2}$$

A radial electric field vB_φ must come into being in equilibrium. Its P.D. is,

$$\Delta\varphi = \int_0^R \frac{I}{\pi R^2 ne} \frac{\mu_0}{2\pi} \frac{Ir}{R^2} dr = \frac{I}{\pi R^2 ne} \left(\frac{\mu_0}{4\pi} I\right) = \frac{\mu_0 I^2}{4\pi R^2 ne}$$

3.267 Here, $v_d = \frac{E}{B}$ and $j = ne\,v_d$

$$\text{so,}\quad n = \frac{jB}{eE} = \frac{200\times10^4\,\frac{\text{A}}{\text{m}^2}\times 1\,\text{T}}{1{\cdot}6\times10^{-19}\,\text{C}\times5\times10^{-4}\,\text{V/m}}$$

$$= 2{\cdot}5\times10^{28}\ \text{per m}^3 = 2{\cdot}5\times10^{22}\ \text{per c.c.}$$

Atomic weight of Na being 23 and its density ≈ 1, molar volume is 23 c.c. Thus number of atoms per unit volume is $\frac{6\times10^{23}}{23} = 2{\cdot}6\times10^{22}$ per c.c.

Thus there is almost one conduction electron per atom.

3.268 By definition, mobility $= \frac{\text{dirft velocity}}{\text{Electric field component causing this drift}}$ or $\mu = \frac{v}{E_L}$

On other hand,

$$E_T = vB = \frac{E_L}{\eta},\ \text{as given}\qquad \text{so, } \mu = \frac{1}{\eta B} = 3{\cdot}2\times10^{-3}\ \text{m}^2/(\text{V}\cdot\text{s})$$

3.269 Due to the straight conductor, $B_\varphi = \frac{\mu_0 I}{2\pi r}$

We use the formula, $\vec{F} = (\vec{p}_m\cdot\vec{\nabla})\vec{B}$

(a) The vector $\vec{p}_m$ is parallel to the straight conductor.

$$\vec{F} = p_m\frac{\partial}{\partial Z}\vec{B} = 0,$$

because neither the direction nor the magnitude of $\vec{B}$ depends on z

(b) The vector $\vec{p}_m$ is oriented along the radius vector $\vec{r}$

$$\vec{F} = p_m\frac{\partial}{\partial r}\vec{B}$$

The direction of $\vec{B}$ at $r + dr$ is parallel to the direction at r. Thus only the φ component of $\vec{F}$ will survive.

$$F_\varphi = p_m\frac{\partial}{\partial r}\frac{\mu_0 I}{2\pi r} = -\frac{\mu_0 I p_m}{2\pi r^2}$$

(c) The vector $\vec{p}_m$ coincides in direction with the magnetic field, produced by the conductor carrying current I

$$\vec{F} = p_m\frac{\partial}{r\partial\varphi}\frac{\mu_0 I}{2\pi}\vec{e}_\varphi = \frac{\mu_0 I p_m}{2\pi r^2}\frac{\partial\vec{e}_\varphi}{\partial\varphi}$$

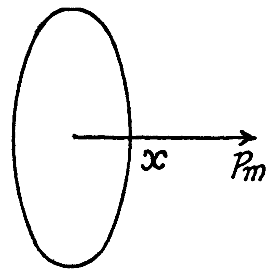
$$\text{So,}\qquad \vec{F} = -\frac{\mu_0 I p_m}{2\pi r^2}\vec{e}_r\quad \text{As, } \frac{\partial\vec{e}_\varphi}{\partial_\varphi} = -\vec{e}_r$$

3.270 $F_x = p_m \frac{\partial}{\partial x} B_x$

But, $B_x = \frac{\mu_0 I}{4\pi} \int \frac{R\,dl}{(x^2 + R^2)^{3/2}} = \frac{\mu_0 I R^2}{2\,(x^2 + R^2)^{3/2}}$

So, $F = \frac{\mu_0}{4\pi} \frac{I \cdot 2\pi R^2}{(x^2 + R^2)^{5/2}} \frac{3}{2} \cdot 2x \cdot p_m$

$= \frac{\mu_0}{4\pi} \frac{6\pi R^2 I p_m x}{(x^2 + R^2)^{5/2}}$



3.271

$$F = P_{2m} \frac{\partial}{\partial l} \left[\frac{\mu_0}{4\pi} \frac{3\,\vec{p}_{1m} \cdot \vec{r}\,\vec{r} - \vec{p}_{1m} r^2}{r^5} \right]$$

$$= P_{2m} \frac{\partial}{\partial l} \left[\frac{\mu_0}{2\pi} \frac{p_{1m}}{l^3} \right] = \frac{-3}{2} \frac{\mu_0 p_{1m} p_{2m}}{\pi l^4} = 9 \text{ nN}$$

3.272 From 3.270, for $x >> R$,

$$B_x \approx \frac{\mu_0 I' R^2}{2x^3}$$

or, $I' \approx \frac{2B_x x^3}{\mu_0 R^2} = \frac{2 \times 3 \times 10^{-5}\,T \times (10^{-1}\text{ m})^3}{1{\cdot}26 \times 10^{-6} \times (10^{-2}\text{ m})^4} \approx 0{\cdot}5$ kA

3.273

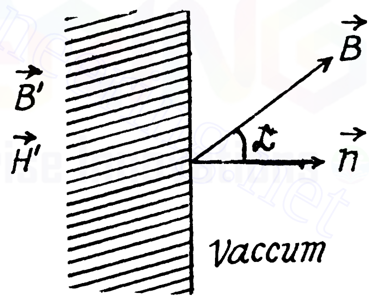
$B'_n = B \cos \alpha,$

$H'_t = \frac{1}{\mu_0} B \sin \alpha,$

$B'_t = \mu B \sin \alpha$

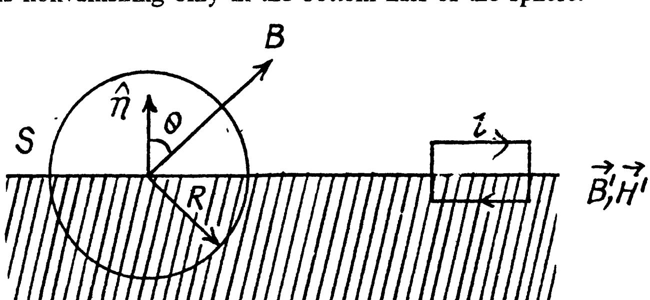
so,

$$B' = B\sqrt{\mu^2 \sin^2 \alpha + \cos^2 \alpha}$$



3.274 (a) $\oint \vec{H} \cdot d\vec{S} = \oint \left(\frac{\vec{B}}{\mu_0} - \vec{J} \right) \cdot d\vec{S} = -\oint \vec{J} \cdot d\vec{S},$ since $\oint \vec{B} \cdot d\vec{S} = 0$

Now $\vec{J}$ is nonvanishing only in the bottom half of the sphere.



Here, $B'_n = B\cos\theta,\ H'_t = \frac{1}{\mu_0} B\sin\theta,\ B'_t = \mu B\sin\theta,\ H'_n = \frac{B}{\mu\mu_0}\cos\theta$

$$J_n = \frac{B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right) \text{ and } J_t = \frac{\mu - 1}{\mu_0} B\sin\theta\ .$$

Only J_n contributes the surface integral and

$$-\oint \vec{J}\cdot d\vec{S} = -\oint_{lower} \vec{J}\cdot d\vec{S} = \oint_{lower} J_n\, dS = \frac{\pi R^2 B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right)$$

(b) $\oint_\Gamma \vec{B}\cdot d\vec{r} = (B_t - B'_t)\, l = (1 - \mu) B\, l \sin\theta$

3.275 Inside the cylindrical wire there is an external current of density $\frac{I}{\pi R^2}$. This gives a magnetic field H_φ with

$$H_\varphi 2\pi r = I\frac{r^2}{R^2} \quad \text{or,} \quad H_\varphi = \frac{Ir}{2\pi R^2}$$

From this $B_\varphi = \frac{\mu\mu_0 Ir}{2\pi R^2}$ and $J_\varphi = \frac{\mu - 1}{2\pi}\frac{Ir}{R^2} = \frac{\chi Ir}{2\pi R^2}$ = Magnetization.

Hence total volume molecular current is,

$$\oint_{r = R} \vec{J}_\varphi\cdot d\vec{r} = \int \frac{\chi I}{2\pi R}\, dl = \chi I$$

The surface current is obtained by using the equivalence of the surface current density to $\vec{J}\times\vec{n}$, this gives rise to a surface current density in the z–direction of $-\frac{\chi I}{2\pi R}$

The total molecular surface current is,

$$I'_s = -\frac{\chi I}{2\pi R}(2\pi R) = -\chi I.$$

The two currents have opposite signs.

3.276 We can obtain the form of the curves, required here, by qualitative arguments.

From $\oint \vec{H}\cdot d\vec{l} = I,$

we get $H(x >> 0) = H(x << 0) = nI$

Then $B(x >> 0) = \mu\mu_0\, nI$

$B(x < 0) = \mu_0\, nI$

Also,

$$B(x < 0) = \mu_0 H(x < 0)$$

$$J(x < 0) = 0$$

B is continuous at $x = 0$, H is not. These give the required curves as shown in the answer-sheet.

3.277 The lines of the B as well as H field are circles around the wire. Thus

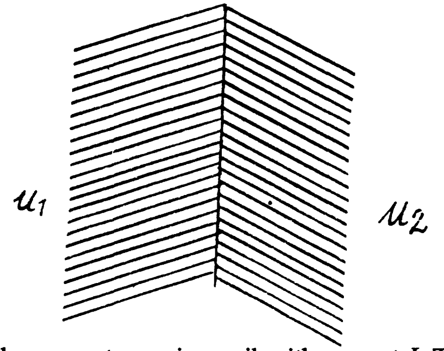
$$H_1 \pi r + H_2 \pi r = I \quad \text{or,} \quad H_1 + H_2 = \frac{I}{\pi r}$$

Also $\mu_0 \mu_1 H_1 = \mu_2 H_2 \mu_0 = B_1 = B_2 = B$

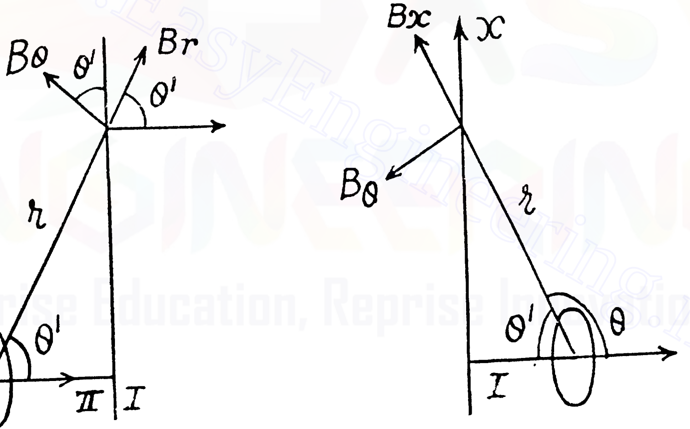
Thus
$$H_1 = \frac{\mu_2}{\mu_1 + \mu_2} \frac{I}{\pi r},$$

$$H_2 = \frac{\mu_1}{\mu_1 + \mu_2} \frac{I}{\pi r}$$

and
$$B = \mu_0 \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \frac{I}{\pi r}.$$



3.278 The medium I is vacuum and contains a circular current carrying coil with current I. The medium II is a magnetic with permeability μ. The boundary is the plane $z = 0$ and the coil is in the plane $z = l$. To find the magnetic induction, we note that the effect of the magnetic medium can be written as due to an image coil in II as far as the medium I is concerned. On the other hand, the induction in II can be written as due to the coil in I, carrying a different current. It is sufficient to consider the far away fields and ensure that the boundary conditions are satisfied there. Now for actual coil in medium I,



$$B_r = -\frac{2p_m \cos\theta'}{r^3} \cdot \left(\frac{\mu_0}{4\pi}\right), \quad B_\theta = \frac{p_m \sin\theta'}{r^3}\left(\frac{\mu_0}{4\pi}\right)$$

so, $B_2 = \dfrac{\mu_0 p_m}{4\pi}(2\cos^2\theta' - \sin^2\theta')$ and $B_x = \dfrac{\mu_0 p_m}{4\pi}(-3\sin\theta' \cos\theta')$

where $p_m = I(\pi a^2)$, a = radius of the coil.

Similarly due to the image coil,

$$B_z = \frac{\mu_0 p'_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'), \; B_x = \frac{\mu_0 p'_m}{4\pi}(3\sin\theta' \cos\theta'), \; p'_m = I'(\pi a^2)$$

As far as the medium II is concerned, we write similarly

$$B_z = \frac{\mu_0 p''_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'), \; B_x = \frac{\mu_0 p''_m}{4\pi}(-3\sin\theta' \cos\theta'), p''_m = I''(\pi a^2)$$

The boundary conditions are, $p_m + p'_m = p''_m$ (from $B_{1n} = B_{2n}$)

$-p_m + p'_m = -\frac{1}{\mu} p''_m$ (from $H_{1t} = H_{2t}$)

Thus, $$I'' = \frac{2\mu}{\mu + 1} I, \; I' = \frac{\mu - 1}{\mu + 1} I$$

In the limit, when the coil is on the boundary, the magnetic field enverywhere can be obtained by taking the current to be $\frac{2\mu}{\mu + 1} I$. Thus, $\vec{B} = \frac{2\mu}{\mu + 1} \vec{B}_0$

3.279 We use the fact that within an isolated uniformly magnetized ball,

$\vec{H}' = -\vec{J}/3$, $\vec{B}' = \frac{2\mu_0 \vec{J}}{3}$, where $\vec{J}$ is the magnetization vector. Then in a uniform magnetic field with induction $\vec{B}_0$, we have by superposition,

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0 \vec{J}}{3}, \quad \vec{H}_{in} = \frac{\vec{B}_0}{\mu_0} - \vec{J}/3$$

or, $\vec{B}_{in} + 2\mu_0 \vec{H}_{in} = 3\vec{B}_0$

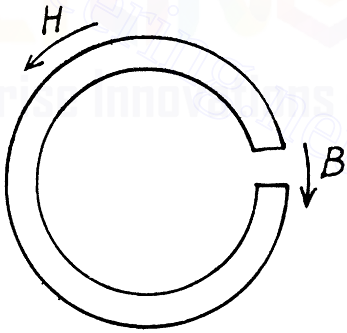
also, $\vec{B}_{in} = \mu \mu_0 \vec{H}_{in}$

Thus, $$\vec{H}_{in} = \frac{3\vec{B}_0}{\mu_0 (\mu + 2)} \text{ and } \vec{B}_{in} = \frac{3\mu \vec{B}_0}{\mu + 2}$$

3.280 The coercive force H_c is just the magnetic field within the cylinder. This is by circulation theorem, $H_c = \frac{NI}{l} = 6$ kA/m

(from $\oint \vec{H} \cdot d\vec{r} = I$, total current, considering a rectangular contour.)

3.281 We use, $\oint \vec{H} \cdot d\vec{l} = 0$



Neglecting the fringing of the lines of force, we write this as

$$H(\pi d - b) + \frac{B}{\mu_0} b = 0$$

or, $$H \approx \frac{-Bb}{\mu_0 \pi d} = 101 \text{ A/m}$$

The sense of H is opposite to B

3.282 Here, $\oint \vec{H} \cdot d\vec{l} = NI$ or, $H(2\pi R) + \frac{Bb}{\mu_0} = NI$, so, $H = \frac{NI\mu_0 - Bb}{2\pi R \mu_0}$

Hence, $$\mu = \frac{B}{\mu_0 H} = \frac{2\pi RB}{\mu_0 NI - Bb} = 3700$$

3.283 One has to draw the graph of $\mu = \frac{B}{\mu_0 H}$ versus H from the given graph. The $\mu - H$ graph starts out horizontally, and then rises steeply at about $H = 0{\cdot}04$ k A/m before falling agian. It is easy to check that $\mu_{max} \approx 10{,}000$.

3.284 From the theorem on circulation of vector $\vec{H}$.

$$H\pi d + \frac{Bb}{\mu_0} = NI \quad \text{or,} \quad B = \frac{\mu_0 NI}{b} - \frac{\mu_0 \pi d}{b} H = (1{\cdot}51 - 0{\cdot}987)\,H,$$

where B is in Tesla and H in kA/m. Besides, B and H are interrelated as in the Fig. 3.76 of the text. Thus we have to solve for B, H graphically by simultaneously drawing the two curves (the hysterisis curve and the straight line, given above) and find the point of intersection. It is at

$$H \approx 0{\cdot}26 \text{ kA/m}, \; B = 1{\cdot}25 \text{ T}$$

Then,
$$\mu = \frac{B}{\mu_0 H} \approx 4000.$$

3.285 From the formula,

$$\vec{F} = (\vec{p}_m \cdot \vec{\nabla})\vec{B} \rightarrow \vec{F} - P\int (\vec{J} \cdot \vec{\nabla})\vec{B}\,dV,$$

Thus
$$\vec{F} = \frac{\chi}{\mu\,\mu_0}\int (\vec{B}\cdot\vec{\nabla})\vec{B}\,dV$$

or since $\vec{B}$ is predominantly along the x–axis,

$$F_x = \frac{\chi}{\mu\,\mu_0}\int B_x \frac{\partial B_x}{\partial x} S dx = \frac{\chi S}{2\mu\,\mu_0}\int_{x=0}^{x=L} dB_x^2 = -\frac{\chi S B^2}{2\mu\,\mu_0} \approx \frac{\chi S B^2}{2\,\mu_0}$$

3.286 The force in question is,

$$\vec{F} = \left(\vec{p}_m \cdot \vec{\nabla}\right)\vec{B} = \frac{\chi B V}{\mu\,\mu_0}\frac{dB}{dx}$$

since B is essentiatlly in the x-direction.

So,
$$F_x \approx \frac{\chi V}{2\,\mu_0}\frac{dB^2}{dx} = \frac{\chi B_0^2 V}{2\,\mu_0}\frac{d}{dx}(e^{-2ax^2}) = -4\,ax\,e^{-2ax^2}\frac{\chi B_0^2}{2\mu_0}V$$

This is maximum when its derivative vanishes

i.e.
$$16\,a^2x^2 - 4\,a = 0, \quad \text{or,} \quad x_m = \frac{1}{\sqrt{4a}}$$

The maximum force is,

$$F_{max} = 4a\frac{1}{\sqrt{4a}}e^{-1/2}\frac{\chi B_0^2 V}{2\,\mu_0} = \frac{\chi B_0^2 V}{\mu_0}\sqrt{\frac{a}{e}}$$

So,
$$\chi \approx \left(\mu_0 F_{max}\sqrt{\frac{e}{a}}\right) \Big/ V B_0^2 = 3{\cdot}6 \times 10^{-4}$$

3.287 $F_x = (\vec{p}_m \cdot \vec{\nabla})B_x = \dfrac{\chi B V}{\mu\,\mu_0}\dfrac{dB}{dx} \approx \dfrac{\chi V}{2\,\mu_0}\dfrac{dB^2}{dx}$

This force is attractive and an equal force must be applied for balance. The work done by applied forces is,

$$A = \int_{x=0}^{x=L} -F_x\,dx = \frac{\chi V}{2\,\mu_0}(-B^2)_{x=0}^{x=L} \approx \frac{\chi V B^2}{2\,\mu_0}$$

3.6 ELECTROMAGNETIC INDUCTION. MAXWELL'S EQUATIONS

3.288 Obviously, from Lenz's law, the induced current and hence the induced e.m.f. in the loop is anticlockwise.

From Faraday's law of electromagnetic indcution,

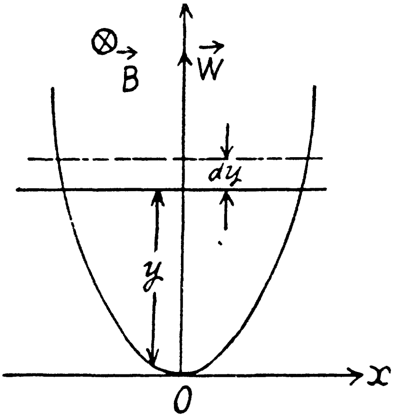
$$\xi_{in} = \left|\frac{d\,\Phi}{dt}\right|$$

Here, $d\,\Phi = \vec{B}\cdot d\,\vec{S} = -2B\,x\,dy,$

and from $y = ax^2, x = \sqrt{\frac{y}{a}}$

Hence, $\xi_{in} = 2B\sqrt{\frac{y}{a}}\,\frac{dy}{dt}$

$$= By\sqrt{\frac{8w}{a}}, \text{ using } \frac{dy}{dt} = \sqrt{2\,wy}$$



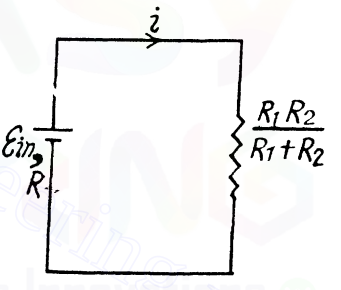
3.289 Let us assume, $\vec{B}$ is directed into the plane of the loop. Then the motional e.m.f.

$$\xi_{in} = \left|\int -(\vec{v}\times\vec{B})\cdot d\,\vec{l}\right| = v\,Bl$$

and directed in the same of $(\vec{v}\times\vec{B})$ (Fig.)

So, $$i = \frac{\xi_{in}}{R + \frac{R_1 R_2}{R_1 + R_2}} = \frac{B\,v\,l}{R + R_u}$$

As R_1 and R_2 are in parallel connections.



3.290 (a) As the metal disc rotates, any free electron also rotates with it with same angular velocity ω, and that's why an electron must have an acceleration $\omega^2 r$ directed towards the disc's centre, where r is separation of the electron from the centre of the disc. We know from Newton's second law that if a particle has some acceleration then there must be a net effecetive force on it in the direction of acceleration. We also know that a charged particle can be influenced by two fields electric and magnetic. In our problem magnetic field is absent hence we reach at the conslusion that there is an electric field near any electron and is directed opposite to the acceleration of the electron.

If E be the electric field strength at a distance r from the centre of the disc, we have from Newton's second law.

$$F_n = m\,w_n$$

$$eE = m\,r\,\omega^2, \text{ or, } E = \frac{m\,\omega^2\,r}{e},$$

and the potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a \vec{E}\cdot d\,\vec{r} = \int_0^a \frac{m\,\omega^2\,r}{e}\,dr, \text{ as } \vec{E}\uparrow\downarrow d\,\vec{r}$$

Thus $$\varphi_{cen} - \varphi_{rim} = \Delta\varphi = \frac{m\,\omega^2}{e}\frac{a^2}{2} = 3{\cdot}0\text{ nV}$$

(b) When field $\vec{B}$ is present, by definition, of motional e.m.f. :

$$\varphi_1 - \varphi_2 = \int_1^2 -(\vec{v}\times\vec{B})\cdot d\vec{r}$$

Hence the sought potential difference,

$$\varphi_{cen} - \varphi_{rim} = \int_0^a -v\,B\,dr = \int_0^a -\omega\,r\,B\,dr\,, \text{ (as } v = \omega\,r)$$

Thus $$\varphi_{rim} - \varphi_{cen} = \varphi = \frac{1}{2}\omega\,B\,a^2 = 20\text{ mV}$$

(In general $\omega < \frac{eB}{m}$ so we can neglect the effect discussed in (1) here).

3.291 By definition,

$$\vec{E} = -(\vec{v}\times\vec{B})$$

So, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_A^C -(\vec{v}\times\vec{B})\cdot d\vec{r} = \int_0^d -v\,B\,dr$$

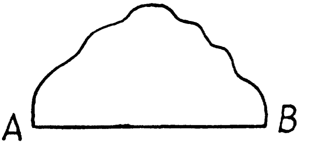
But, $v = \omega r$, where r is the perpendicular distance of the point from A.

Hence, $$\int_A^C \vec{E}\cdot d\vec{r} = \int_0^d -\omega\,B\,r\,dr = -\frac{1}{2}\omega\,B\,d^2 = -10\text{ mV}$$

This result can be generalized to a wire AC of arbitary planar shape. We have

$$\int_A^C \vec{E}\cdot d\vec{r} = -\int_A^C (\vec{v}\times\vec{B})\cdot dr = -\int_A^C ((\vec{\omega}\times\vec{r})\times\vec{B})\cdot d\vec{r}$$

$$= -\int_A^C (\vec{B}\cdot\vec{r}\,\vec{\omega} - \vec{B}\cdot\vec{\omega}\,\vec{r})\cdot d\vec{r}$$



$$= -\frac{1}{2}B\omega d^2,$$

d being AC and $\vec{r}$ being measured from A.

3.292 Flux at any moment of time,

$$|\Phi_t| = |\vec{B}\cdot d\vec{S}| = B\left(\frac{1}{2}R^2\varphi\right)$$

where φ is the sector angle, enclosed by the field.

Now, magnitude of induced e.m.f. is given by,

$$\xi_{in} = \left|\frac{d\Phi_t}{dt}\right| = \left|\frac{BR^2}{2}\frac{d\varphi}{dt}\right| = \frac{BR^2}{2}\omega,$$

where ω is the angular velocity of the disc. But as it starts rotating from rest at $t = 0$ with an angular acceleration β its angular velocity $\omega(t) = \beta t$. So,

$$\xi_{in} = \frac{BR^2}{2}\beta t.$$

According to Lenz law the first half cycle current in the loop is in anticlockwise sense, and in subsequent half cycle it is in clockwise sense.

Thus in general, $\xi_{in} = (-1)^n \frac{BR^2}{2}\beta t$, where n in number of half revolutions.

The plot $\xi_{in}(t)$, where $t_n = \sqrt{2\pi n/\beta}$ is shown in the answer sheet.

3.293 Field, due to the current carrying wire in the region, right to it, is directed into the plane of the paper and its magnitude is given by,

$B = \frac{\mu_0}{2\pi}\frac{i}{r}$ where r is the perpendicular distance from the wire.

As B is same along the length of the rod thus motional e.m.f.

$$\xi_{in} = \left| -\int_1^2 (\vec{v}\times\vec{B})\cdot d\vec{l} \right| = vBl$$

and it is directed in the sense of $(\vec{v}\times\vec{B})$

So, current (induced) in the loop,

$$i_{in} = \frac{\xi_{in}}{R} = \frac{1}{2}\frac{\mu_0 I v l}{\pi R r}$$

3.294 Field, due to the current carrying wire, at a perpendicular distance x from it is given by,

$$B(x) = \frac{\mu_0}{2\pi}\frac{i}{x}$$

Motional e.m.f is given by $\left|\int -(\vec{v}\times\vec{B})\cdot d\vec{l}\right|$

There will be no induced e.m.f. in the segments (2) and (4)

as, $\vec{v}\uparrow\uparrow d\vec{l}$ and magnitude of e.m.f. induced in 1 and 3, will be

$$\xi_1 = v\left(\frac{\mu_0}{2\pi}\frac{i}{x}\right)a \text{ and } \xi_2 = v\left(\frac{\mu_0}{2\pi}\frac{i}{(a+x)}\right)a,$$

respectively, and their sense will be in the direction of $(\vec{v}\times\vec{B})$.

So, e.m.f., induced in the network $= \xi_1 - \xi_2$ [as $\xi_1 > \xi_2$]

$$= \frac{a v \mu_0 i}{2\pi}\left[\frac{1}{x} - \frac{1}{a+x}\right] = \frac{v a^2 \mu_0 i}{2\pi x (a+x)}$$

3.295 As the rod rotates, an emf.

$$\frac{d}{dt}\frac{1}{2}a^2\theta\cdot B=\frac{1}{2}a^2 B\omega$$

is induced in it. The net current in the conductor is then $\dfrac{\xi(t)-\frac{1}{2}a^2 B\omega}{R}$

A magnetic force will then act on the conductor of magnitude BI per unit length. Its direction will be normal to B and the rod and its torque will be

$$\int_0^a \left(\frac{\xi(t)-\frac{1}{2}a^2 B\omega}{R}\right) dx\, B\, x$$

Obviously both magnetic and mechanical torque acting on the C.M. of the rod must be equal but opposite in sense. Then

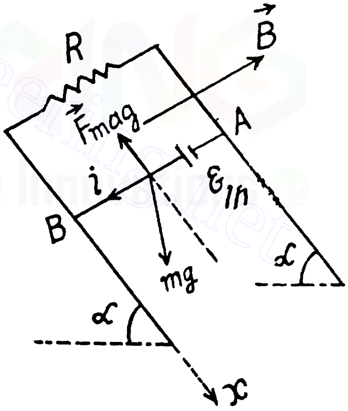
for equilibrium at constant ω

$$\frac{\xi(t)-\frac{1}{2}a^2 B\omega}{R}\cdot\frac{B\,a^2}{2}=\frac{1}{2}mga\sin\omega t$$

or, $\xi(t)=\frac{1}{2}a^2 B\omega+\frac{mgR}{aB}\sin\omega t=\frac{1}{2\,aB}(a^3B^2\omega+2\,mgR\sin\omega t)$

(The answer given in the book is incorrect dimensionally.)

3.296 From Lenz's law, the current through the connector is directed form A to B. Here $\xi_{in}=vBl$ between A and B



where v is the velocity of the rod at any moment.

For the rod, from $F_x=mw_x$

or, $mg\sin\alpha - i\,l\,B=mw$

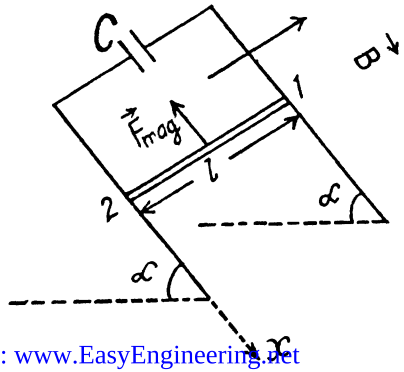
For steady state, acceleration of the rod must be equal to zero.

Hence, $mg\sin\alpha=i\,l\,B$ (1)

But, $i=\frac{\xi_{in}}{R}=\frac{v\,B\,l}{R}$

From (1) and (2) $v=\frac{mg\sin\alpha\,R}{B^2\,l^2}$

3.297 From Lenz's law, the current through the copper bar is directed from 1 to 2 or in other words, the induced crrent in the circuit is in clockwise sense.



Potential difference across the capacitor plates,

$$\frac{q}{C}=\xi_{in}\quad\text{or,}\quad q=C\,\xi_{in}$$

Hence, the induced current in the loop,

$$i = \frac{dq}{dt} = C\frac{d\xi_{in}}{dt}$$

But the variation of magnetic flux through the loop is caused by the movement of the bar. So, the induced e.m.f. $\xi_{in} = B\,lv$

and,
$$\frac{d\xi_{in}}{dt} = Bl\frac{dv}{dt} = Blw$$

Hence,
$$i = C\frac{d\xi}{dt} = C\,B\,l\,w$$

Now, the forces acting on the bars are the weight and the Ampere's force, where $F_{amp} = i\,l\,B\,(CB\,l\,w)\;B = C\,l^2 B^2 w.$

From Newton's second law, for the rod, $F_x = mw_x$

or,
$$mg\sin\alpha - C\,l^2 B^2 w = mw$$

Hence
$$w = \frac{mg\sin\alpha}{C\,l^2 B^2 + m} = \frac{g\sin\alpha}{1 + \dfrac{l^2 B^2 C}{m}}$$

3.298 Flux of $\vec{B}$, at an arbitrary moment of time t :

$$\Phi_t = \vec{B}\cdot\vec{S} = B\frac{\pi a^2}{2}\cos\omega t,$$

From Faraday's law, induced e.m.f., $\xi_{in} = -\dfrac{d\Phi}{dt}$

$$= -\frac{d\left(B\pi\dfrac{a^2}{2}\cos\omega t\right)}{dt} = \frac{B\pi a^2\omega}{2}\sin\omega t.$$

and induced current, $\quad i_{in} = \dfrac{\xi_{in}}{R} = \dfrac{B\pi a^2}{2R}\,\omega\sin\omega t.$

Now thermal power, generated in the circuit, at the moment $t = t$:

$$P(t) = \xi_{in}\times i_{in} = \left(\frac{B\pi a^2\omega}{2}\right)^2\frac{1}{R}\sin^2\omega t$$

and mean thermal power generated,

$$<P> = \frac{\left[\dfrac{B\pi a^2\omega}{2}\right]^2\dfrac{1}{R}\displaystyle\int_0^T\sin^2\omega t\,dt}{\displaystyle\int_0^T dt} = \frac{1}{2R}\left(\frac{\pi\omega a^2 B}{2}\right)^2$$

Note : The claculation of ξ_{in} which can also be checked by using motional emf is correct even though the conductor is not a closed semicircle , for the flux linked to the rectangular part containing the resistance R is not changing. The answer given in the book is off by a factor 1/4.

3.299 The flux through the coil changes sign. Initially it is BS per turn.

Finally it is $-BS$ per turn. Now if flux is Φ at an intermediate state then the current at that moment will be

$$i = \frac{-N\dfrac{d\Phi}{dt}}{R}$$

So charge that flows during a sudden turning of the coil is

$$q = \int i\,dt = -\frac{N}{R}[\Phi - (-\Phi)] = 2NBS/R$$

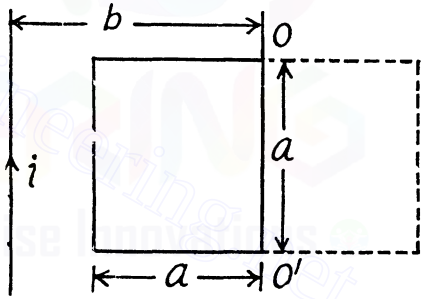
Hence, $$B = \frac{1}{2}\frac{qR}{NS} = 0{\cdot}5\ \text{T on putting the values.}$$

3.300 According to Ohm's law and Faraday's law of induction, the current i_0 appearing in the frame, during its rotation, is determined by the formula,

$$i_0 = -\frac{d\Phi}{dt} = -\frac{L\,d\,i_0}{dt}$$

Hence, the required amount of electricity (charge) is,

$$q = \int i_0\,dt = -\frac{1}{R}\int (d\Phi + L\,di_0) = -\frac{1}{R}(\Delta\Phi + L\,\Delta i_0)$$



Since the frame has been stopped after rotation, the current in it vanishes, and hence $\Delta i_0 = 0$. It remains for us to find the increment of the flux $\Delta\Phi$ through the frame ($\Delta\Phi = \Phi_2 - \Phi_1$).

Let us choose the normal $\vec{n}$ to the plane of the frame, for instance, so that in the final position, $\vec{n}$ is directed behind the plane of the figure (along $\vec{B}$).

Then it can be easily seen that in the final position, $\Phi_2 > 0$, while in the initial position, $\Phi_1 < 0$ (the normal is opposite to $\vec{B}$), and $\Delta\Phi$ turns out to be simply equal to the fulx through the surface bounded by the final and initial positions of the frame :

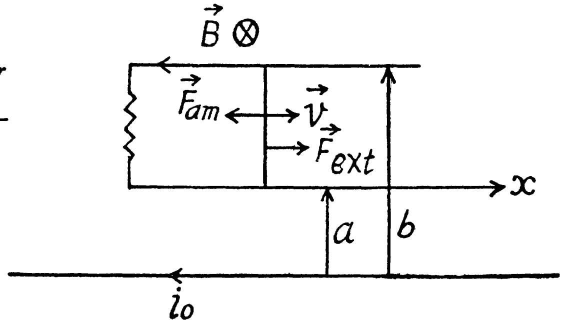
$$\Delta\Phi = \Phi_2 + |\Phi_1| = \int_{b-a}^{b+a} B\,a\,dr,$$

where B is a function of r, whose form can be easily found with the help of the theorem of circulation. Finally omitting the minus sign, we obtain,

$$q = \frac{\Delta\Phi}{R} = \frac{\mu_0\,a\,i}{2\pi R}\ln\frac{b+a}{b-a}$$

3.301 As $\vec{B}$, due to the straight current carrying wire, varies along the rod (connector) and enters linerarly so, to make the calculations simple, $\vec{B}$ is made constant by taking its average value in the range $[a, b]$.

$$< B > = \frac{\int_a^b B\,dr}{\int_a^b dr} = \frac{\int_a^b \frac{\mu_0}{2\pi}\frac{i_0}{r}\,dr}{\int_a^b dr}$$



or, $$< B > = \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}$$

(a) The flux of $\vec{B}$ changes through the loop due to the movement of the connector. According to Lenz's law, the current in the loop will be anticlockwise. The magnitude of motional e.m.f.,

$$\xi_{in} = v < B > (b-a)$$

$$= \frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}(b-a)\frac{dx}{dt} = \frac{\mu_0}{2\pi}i_0\ln\frac{b}{a}v$$

So, induced current

$$i_{in} = \frac{\xi_{in}}{R} = \frac{\mu_0}{2\pi}\frac{i_0 v}{R}\ln\frac{b}{a}$$

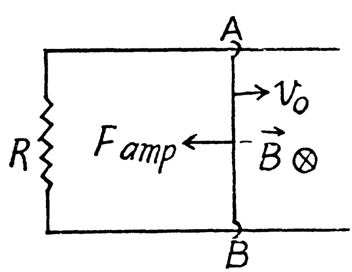
(b) The force required to maintain the constant velocity of the connector must be the magnitude equal to that of Ampere's acting on the connector, but in opposite direction.

So, $$F_{ext} = i_{in}\,l < B > = \left(\frac{\mu_0}{2\pi}\frac{i_0}{R}v\ln\frac{b}{a}\right)(b-a)\left(\frac{\mu_0}{2\pi}\frac{i_0}{(b-a)}\ln\frac{b}{a}\right)$$

$$= \frac{v}{R}\left(\frac{\mu_0}{2\pi}i_0\ln\frac{b}{a}\right)^2$$, and will be directed as shown in the (Fig.)

3.302 (a) The flux through the loop changes due to the movement of the rod AB. According to Lenz's law current should be anticlockwise in sense as we have assumed $\vec{B}$ is directed into the plane of the loop. The motion e.m.f $\xi_{in}(t) = B\,l\,v$

and induced current $i_{in} = \frac{vBl}{R}$



From Newton's law in projection form $F_x = mw_x$

$$-F_{amp} = m\frac{v\,dv}{dx}$$

But $$F_{amp} = i_{in}\,l\,B = \frac{vB^2l^2}{R}$$

So, $$-\frac{vB^2l^2}{R} = mv\frac{dv}{dx}$$

$$\text{or,} \qquad \int_0^x dx = -\frac{mR}{B^2 l^2}\int_{v_0}^0 dv \quad \text{or,} \quad x = \frac{mRv_0}{B^2 l^2}$$

(b) From equation of energy conservation; $E_f - E_i + \text{Heat liberated} = A_{cell} + A_{ext}$

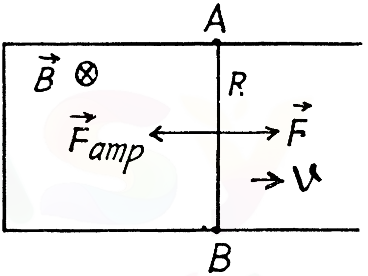
$$\left[0 - \frac{1}{2} m v_0^2\right] + \text{Heat liberated} = 0 + 0$$

So, heat liberated $= \frac{1}{2} m v_0^2$

3.303 With the help of the calculation, done in the previous problem, Ampere's force on the connector,

$\vec{F}_{amp} = \frac{v B^2 l^2}{R}$ directed towards left.

Now from Newton's second law,



$$F - F_{amp} = m\frac{dv}{dt}$$

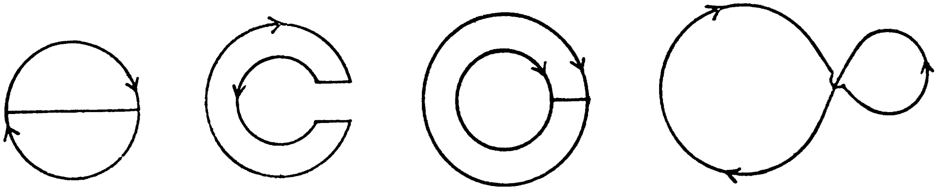
$$\text{So,} \qquad F = \frac{v B^2 l^2}{R} + m\frac{dv}{dt}$$

$$\text{or,} \qquad \int_0^t dt = m\int_0^v \frac{dv}{F - \frac{v B^2 l^2}{R}}$$

$$\text{or,} \qquad \frac{t}{m} = -\frac{R}{B^2 l^2}\ln\left(\frac{F - \frac{v B^2 l^2}{R}}{F}\right)$$

$$\text{Thus} \qquad v = \left(1 - e^{\frac{-t B^2 l^2}{Rm}}\right)\frac{RF}{B^2 l^2}$$

3.304 According to Lenz, the sense of induced e.m.f. is such that it opposes the cause of change of flux. In our problem, magnetic field is directed away from the reader and is diminishing.



So, in figure (a), in the round conductor, it is clockwise and there is no current in the connector

In figure (b) in the outside conductor, clockwise.

In figure (c) in both the conductor, clockwise; and there is no current in the connector to obey the charge conservation.

In figure (d) in the left side of the figure, clockwise.

3.305 The loops are connected in such a way that if the current is clockwise in one, it is anticlockwise in the other. Hence the e.m.f. in loop b opposes the e.m.f. in loop a.

e.m.f. in loop $a = \frac{d}{dt}(a^2 B) = a^2 \frac{d}{dt}(B_0 \sin \omega t)$

Similarly, e.m.f. in loop $b = b^2 B_0 \omega \cos \omega t$.

Hence, net e.m.f. in the circuit $= (a^2 - b^2) B_0 \omega \cos \omega t$, as both the e.m.f's are in opposite sense, and resistance of the circuit $= 4(a+b)\rho$

Therefore, the amplitude of the current

$$= \frac{(a^2 - b^2) B_0 \omega}{4(a+b)\rho} = 0{\cdot}5 \text{ A}.$$

3.306 The flat shape is made up of concentric loops, having different radii, varying from 0 to a.

Let us consider an elementary loop of radius r, then e.m.f. induced due to this loop

$$= \frac{-d(\vec{B}\cdot\vec{S})}{dt} = \pi r^2 B_0 \omega \cos \omega t.$$

and the total induced e.m.f.,

$$\xi_{ind} = \int_0^a (\pi r^2 B_0 \omega \cos \omega t)\, dN, \tag{1}$$

where $\pi r^2 \omega \cos \omega t$ is the contribution of one turn of radius r and dN is the number of turns in the interval $[r, r+dr]$.

So, $$dN = \left(\frac{N}{a}\right) dr \tag{2}$$

From (1) and (2), $\xi = \int_0^a -(\pi r^2 B_0 \omega \cos \omega t)\frac{N}{a} dr = \frac{\pi B_0 \omega \cos \omega t N a^2}{3}$

Maximum value of e.m.f. amplitude $\xi_{max} = \frac{1}{3}\pi B_0 \omega N a^2$

3.307 The flux through the loop changes due to the variation in $\vec{B}$ with time and also due to the movement of the connector.

So, $\xi_{in} = \left|\frac{d(\vec{B}\cdot\vec{S})}{dt}\right| = \left|\frac{d(BS)}{dt}\right|$ as $\vec{S}$ and $\vec{B}$ are collinear

But, B, after t sec. of beginning of motion $= \beta t$, and S becomes $= l\frac{1}{2}wt^2$, as connector starts moving from rest with a constant acceleration w.

So, $$\xi_{ind} = \frac{3}{2}\beta l w t^2$$

3.308 We use $B = \mu_0 n I$

Then, from the law of electromagnetic induction

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

So, for $r < a$

$$E_\varphi 2\pi r = -\pi r^2 \mu_0 n \dot{I} \quad \text{or,} \quad E_\varphi = -\frac{1}{2}\mu_0 n r \dot{I}. \text{ (where } \dot{I} = dI/dt\text{)}$$

For $r>a$

$$E_\varphi 2\pi r = -\pi a^2 \mu_0 n \dot{I} \quad \text{or,} \quad E_\varphi = -\mu_0 n \dot{I} a^2/2r$$

The meaning of minus sign can be deduced from Lenz's law.

3.309 The e.m.f. induced in the turn is $\mu_0 n \dot{I} \pi \frac{d^2}{4}$

The resistance is $\frac{\pi d}{S}\rho$.

So, the current is $\frac{\mu_0 n \dot{I} S d}{4\rho} = 2$ m A, where ρ is the resistivity of copper.

3.310 The changing magnetic field will induce an e.m.f. in the ring, which is obviously equal, in the two parts by symmetry (the e.m.f. induced by electromagnetic induction does not depend on resistance). The current, that will flow due to this, will be different in the two parts. This will cause an acceleration of charge, leading to the setting up of an electric field E which has opposite sign in the two parts. Thus,

$$\frac{\xi}{2} - \pi a E = r I \text{ and, } \frac{\xi}{2} + \pi a E = \eta r I,$$

where ξ is the total induced e.m.f. From this,

$$\xi = (\eta + 1) rI,$$

and
$$E = \frac{1}{2\pi a}(\eta - 1) rI = \frac{1}{2\pi a}\frac{\eta - 1}{\eta + 1}\xi$$

But by Faraday's law, $\xi = \pi a^2 b$

so,
$$E = \frac{1}{2} ab \frac{\eta - 1}{\eta + 1}$$

3.311 Go to the rotating frame with an instantaneous angular velocity $\vec{\omega}(t)$. In this frame, a Coriolis force, $2m\vec{v}' \times \vec{\omega}(t)$

acts which must be balanced by the magnetic force, $e\vec{v} \times \vec{B}(t)$

Thus,
$$\vec{\omega}(t) = -\frac{e}{2m}\vec{B}(t).$$

(It is assumed that $\vec{\omega}$ is small and varies slowly, so ω^2 and $\dot{\omega}$ can be neglected.)

3.312 The solenoid has an inductance,

$$L = \mu_0 n^2 \pi b^2 l,$$

where $n =$ number of turns of the solenoid per unit length. When the solenoid is connected to the source an e.m.f. is set up, which, because of the inductance and resistance, rises slowly, according to the equation,

$$RI + L\dot{I} = V$$

This has the well known solution,

$$I = \frac{V}{R}(1 - e^{-tR/L}).$$

Corresponding to this current, an e.m.f. is induced in the ring. Its magnetic field $B = \mu_0 n I$ in the solenoid, produces a force per unit length, $\frac{dF}{dl} = B\,i = \mu_0^2 n^2 \pi a^2 I\dot{I}/r$

$$= \frac{\mu_0^2 \pi a^2 V^2}{r}\left(\frac{n^2}{RL}\right) e^{-tR/L}(1 - e^{-tR/L}),$$

acting on each segment of the ring. This force is zero initially and zero for large t. Its maximum value is for some finite t. The maximum value of

$$e^{-tR/L}(1 - e^{-tR/L}) = \frac{1}{4} - \left(\frac{1}{2} - e^{-tR/L}\right)^2 \text{ is } \frac{1}{4}.$$

So $$\frac{dF_{max}}{dl} = \frac{\mu_0^2 \pi a^2 V^2}{r}\,\frac{n^2}{4RL} = \frac{\mu_0 a^2 V^2}{4 r R l b^2}$$

3.313 The amount of heat generated in the loop during a small time interval dt,

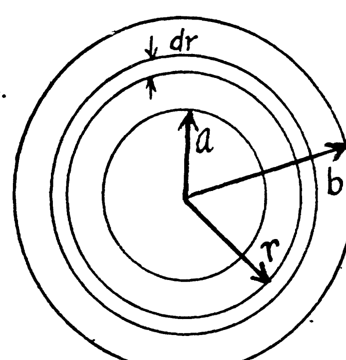
$$dQ = \xi^2/R\,dt, \text{ but, } \xi = -\frac{d\Phi}{dt} = 2at - a\tau,$$

So, $$dQ = \frac{(2at - a\tau)^2}{R}dt$$

and hence, the amount of heat, generated in the loop during the time interval 0 to τ.

$$Q = \int_0^\tau \frac{(2at - a\tau)^2}{R}dt = \frac{1}{3}\frac{a^2\tau^3}{R}$$

3.314 Take an elementary ring of radius r and width dr.



The e.m.f. induced in this elementary ring is $\pi r^2 \beta$.

Now the conductance of this ring is.

$$d\left(\frac{1}{R}\right) = \frac{h\,dr}{\rho\, 2\pi r} \text{ so } dI = \frac{h\,r\,dr}{2\rho}\beta$$

Integrating we get the total current,

$$I = \int_a^b \frac{hr\,dr}{2\rho}\beta = \frac{h\beta(b^2 - a^2)}{4\rho}$$

3.315 Given $L = \mu_0 n^2 V = \mu_0 n^2 l_0 \pi R^2$, where R is the radius of the solenoid.

Thus, $$n = \sqrt{\frac{L}{\mu_0 l_0 \pi}}\frac{1}{R}.$$

So, length of the wire required is,

$$l = n l_0 2\pi R = \sqrt{\frac{4\pi L l_0}{\mu_0}} = 0{\cdot}10\text{ km}.$$

3.316 From the previous problem, we know that,

l' = length of the wire needed $= \sqrt{\frac{L l 4\pi}{\mu_0}}$, where l = length of solenoid here.

Now, $R = \frac{\rho_0 l'}{S}$, (where S = area of crossection of the wire. Also $m = \rho S l'$)

Thus, $$l' = \frac{RS}{\rho_0} = \frac{Rm}{\rho\rho_0 l'} \quad or \quad l' = \sqrt{\frac{Rm}{\rho\rho_0}}$$

where ρ_0 = resistivity of copper and ρ = its density.

Equating, $$\frac{Rm}{\rho\rho_0} = \frac{Ll}{\mu_0/4\pi}$$

or, $$L = \frac{\mu_0}{4\pi}\frac{mR}{\rho\rho_0 l}$$

3.317 The current at time t is given by,

$$I(t) = \frac{V}{R}(1 - e^{-tR/L})$$

The steady state value is, $I_0 = \frac{V}{R}$

and $$\frac{I(t)}{I_0} = \eta = 1 - e^{-tR/L} \text{ or },\ e^{-tR/L} = 1 - \eta$$

or, $$t_o\frac{R}{L} = \ln\frac{1}{1-\eta} \quad \text{or,} \quad t_o = \frac{L}{R}\ln\frac{1}{1-\eta} = 1{\cdot}49\text{ s}$$

3.318 The time constant τ is given by

$$\tau = \frac{L}{R} = \frac{L}{\rho_0\frac{l_0}{S}},$$

where, ρ_0 = resistivity, l_0 = length of the winding wire, S = cross section of the wire.

But $$m = l\rho_0 S$$

So eliminating S, $$\tau = \frac{L}{\frac{\rho_0 l_0}{m/\rho l_0}} = \frac{mL}{\rho\rho_0 l_0^2}$$

From problem 3.315 $l_0 = \sqrt{\frac{4\pi l L}{\mu_0}}$

(note the interchange of l and l_0 because of difference in notation here.)

Thus,
$$\tau = \frac{mL}{\rho\rho_0 \frac{4\pi}{\mu_0} L\, l} = \mu_0 4\pi \frac{m}{\rho\,\rho_0\, l} = 0{\cdot}7\ \text{ms},$$

3.319 Between the cables, where $a < r < b$, the magnetic field $\vec{H}$ satisfies
$$H_\varphi 2\pi r = I \quad \text{or,} \quad H_\varphi = \frac{I}{2\pi r}$$

So
$$B_\varphi = \frac{\mu\,\mu_0 I}{2\pi r}$$

The associated flux per unit length is, $\Phi = \int\limits_{r=a}^{r=b} \frac{\mu\,\mu_0 I}{2\pi r} \times 1 \times dr = \frac{\mu\,\mu_0 I}{2\pi} \ln\frac{b}{a}$

Hence, the inductance per unit length $L_1 = \frac{\Phi}{I} = \frac{\mu\mu_0}{2\pi}\ln\eta$, where $\eta = \frac{b}{a}$

We get $L_1 = 0{\cdot}26\ \frac{\mu H}{m}$

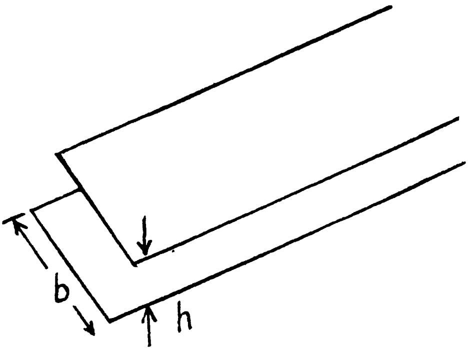
3.320 Within the solenoid, $H_\varphi \cdot 2\pi r = NI$ or $H_\varphi = \frac{NI}{2\pi r}$, $B_\varphi = \mu\mu_0 \frac{NI}{2\pi r}$

and the flux, $\Phi = N\Phi_1, = N\frac{\mu\mu_0}{2\pi} NI \int\limits_b^{a+b} \frac{a\,dr}{r}$

Finally,
$$L = \frac{\mu\,\mu_0}{2\pi} N^2 a \ln\left(1 + \frac{a}{b}\right)$$

3.321 Neglecting end effects the magnetic field B, between the plates, which is mainly parallel to the plates, is $B = \mu_0 \frac{I}{b}$

(For a derivation see 3.229 b)



Thus, the associated flux per unit length of the plates is,
$$\Phi = \mu_0 \frac{I}{b} \times h \times 1 = \left(\mu_0 \frac{h}{b}\right) \times I.$$

So, L_1 = inductance per unit length $= \mu_0 \frac{h}{b} = 25$ nH/m.

3.322 For a single current carrying wire, $B_\varphi = \frac{\mu_0 I}{2\pi r} (r > a)$. For the double line cable, with current, flowing in opposite directions, in the two conductors,

$B_\varphi \approx \frac{\mu_0 I}{\pi r}$, between the cables, by superposition. The associated flux is,

$$\Phi = \int_a^{d-a} \frac{\mu_0 I}{\pi} \frac{dr \times 1}{r} \approx \frac{\mu_0 I}{\pi} \ln \frac{d}{a} = \frac{\mu_0}{\pi} \ln \eta \times I, \text{ per unit length}$$

Hence,
$$L_1 = \frac{\mu_0}{\pi} \ln \eta$$

is the inductance per unit length.

3.323 In a superconductor there is no resistance, Hence,

$$L \frac{dI}{dt} = + \frac{d\Phi}{dt}$$

So integrating,
$$I = \frac{\Delta\Phi}{L} = \frac{\pi a^2 B}{L}$$

because
$$\Delta\Phi = \Phi_f - \Phi_i, \Phi_f = \pi a^2 B, \Phi_i = 0$$

Also, the work done is, $A = \int \xi I \, dt = \int I \, dt \frac{d\Phi}{dt} = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\pi^2 a^4 B^2}{L}$

3.324 In a solenoid, the inductance $L = \mu\mu_0 n^2 V = \mu \mu_0 \frac{N^2 S}{l}$,

where S = area of cross section of the solenoid, l = its length, $V = Sl$, $N = nl$ = total number of turns.

When the length of the solenoid is increased, for example, by pulling it, its inductance will decrease. If the current remains unchanged, the flux, linked to the solenoid, will also decrease. An induced e.m.f. will then come into play, which by Lenz's law will try to oppose the decrease of flux, for example, by increasing the current. In the superconducting state the flux will not change and so,

$$\frac{I}{l} = \text{constant}$$

Hence,
$$\frac{I}{l} = \frac{I_0}{l_0}, \quad \text{or,} \quad I = I_0 \frac{l}{l_0} = I_0 (1 + \eta)$$

3.325 The flux linked to the ring can not change on transition to the superconduction state, for reasons, similar to that given above. Thus a current I must be induced in the ring, where,

$$I = \frac{\Phi}{L} = \frac{\pi a^2 B}{\mu_0 a \left(\ln \frac{8a}{b} - 2\right)} = \frac{\pi a B}{\mu_0 \left(\ln \frac{8a}{b} - 2\right)}$$

3.326 We write the equation of the circuit as,

$$Ri + \frac{L}{\eta}\frac{di}{dt} = \xi,$$

for $t \geq 0$. The current at $t = 0$ just after inductance is changed, is

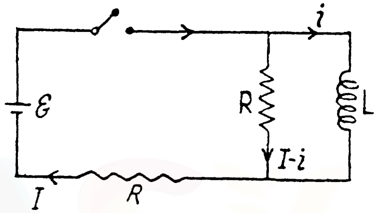
$i = \eta \frac{\xi}{R}$, so that the flux through the inductance is unchanged.

We look for a solution of the above equation in the form

$$i = A + Be^{-t/C}$$

Substituting $C = \frac{L}{\eta R}$, $B = \eta - 1, A = \frac{\xi}{R}$

Thus, $$i = \frac{\xi}{R}\left(1 + (\eta - 1)\, e^{-\eta R t/L}\right)$$



3.327 Clearly, $L\frac{di}{dt} = R(I - i) = \xi - RI$

So, $2L\frac{di}{dt} = \xi - Ri$

This equation has the solution (as in 3.312)

$i = \frac{\xi}{R}(1 - e^{-tR/2L})$

3.328 The equations are,

$$L_1\frac{di_1}{dt} = L_2\frac{di_2}{dt} = \xi - R(i_1 + i_2)$$

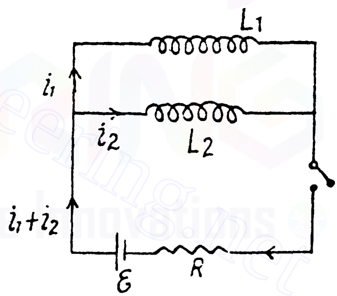
Then, $\frac{d}{dt}(L_1 i_1 - L_2 i_2) = 0$

or, $L_1 i_1 - L_2 i_2 =$ constant

But initially at $t = 0$, $i_1 = i_2 = 0$

so constant must be zero and at all times,

$$L_1 i_1 = L_2 i_2$$



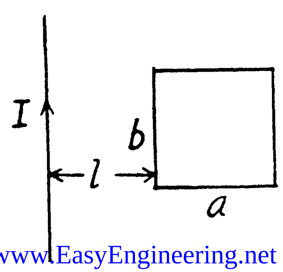
In the final steady atate, current must obviously be $i_1 + i_2 = \frac{\xi}{R}$. Thus in steady state,

$$i_1 = \frac{\xi L_2}{R(L_1 + L_2)} \text{ and } i_2 = \frac{\xi L_1}{R(L_1 + L_2)}$$

3.329 Here, $B = \frac{\mu_0 I}{2\pi r}$ at a distance r from the wire. The flux through the frame is obtained as,

$$\Phi_{12} = \int_l^{a+l} \frac{\mu_0 I}{2\pi r}\, b\,dr = \frac{\mu_0 b}{2\pi} I \ln\left(1 + \frac{a}{l}\right)$$

Thus, $L_{12} = \frac{\Phi_{12}}{I} = \frac{\mu_0 b}{2\pi}\ln\left(1 + \frac{a}{l}\right)$



3.330 Here also, $B = \dfrac{\mu_0 I}{2\pi r}$ and $\Phi = \mu_0 \mu \dfrac{I}{2\pi} \int_a^b \dfrac{h\,dr}{r} N \cdot$

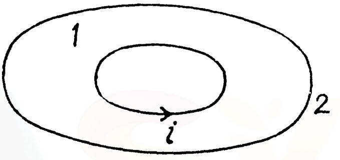
Thus, $$L_{12} = \frac{\mu \mu_0 hN}{2\pi} \ln \frac{b}{a}$$

3.331 The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current i flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.

Magnetic induction at the centre of the loop, : $B = \dfrac{\mu_0 i}{2b}$

So, flux throug loop 1, : $\Phi_{12} = \pi a^2 \dfrac{\mu_0 i}{2b}$

and from reciprocity theorem,



$$\varphi_{12} = \Phi_{21}, \; \Phi_{21} = \frac{\mu_0 \pi a^2 i}{2b}$$

So, $L_{12} = \dfrac{\Phi_{21}}{i} = \dfrac{1}{2} \mu_0 \pi a^2 / b$

3.332 Let $\vec{p}_m$ be the magnetic moment of the magnet M. Then the magnetic field due to this magnet is,

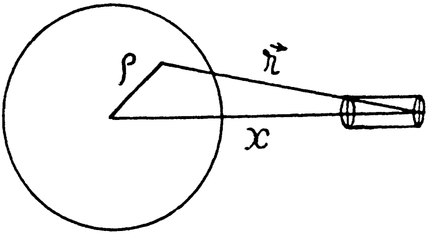
$$\frac{\mu_0}{4\pi}\left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3}\right].$$

The flux associated with this, when the magnet is along the axis at a distance x from the centre, is

$$\Phi = \frac{\mu_0}{4\pi} \int \left[\frac{3(\vec{p}_m \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_m}{r^3}\right] \cdot d\vec{S} = \Phi_1 - \Phi_2.$$

where, $$\Phi_2 = \frac{\mu_0}{4\pi} p_m \int_0^a \frac{2\pi \rho\, d\rho}{(x^2+\rho^2)^{3/2}} = \frac{\mu_0 p_m}{2}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right)$$

and $$\Phi_1 = \frac{3\mu_0 p_m x^2}{4\pi} \int_0^a \frac{2\pi \rho\, d\rho}{(x^2+\rho^2)^{5/2}}$$



$$= \frac{\mu_0 p_m x^2}{2}\left(\frac{1}{x^3} - \frac{1}{(x^2+a^2)^{3/2}}\right)$$

So, $$\Phi = \frac{-\mu_0 p_m a^2}{2(x^2+a^2)^{3/2}}$$

When the flux changes, an e.m.f. $-N\dfrac{d\Phi}{dt}$ is induced and a current $-\dfrac{N}{R}\dfrac{d\Phi}{dt}$ flows. The total charge q, flowing, as the magnet is removed to infinity from $x = 0$ is,

$$q = \frac{N}{R}\Phi(x = 0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

or,
$$p_m = \frac{2aqR}{N\mu_0}$$

3.333 If a current I flows in one of the coils, the magnetic field at the centre of the other coil is,

$$B = \frac{\mu_0 a^2 I}{2(l^2 + a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2 l^3}, \quad \text{as } l >> a.$$

The flux associated with the second coil is then approximately $\mu_0 \pi a^4 I/2 l^3$

Hence,
$$L_{12} = \frac{\mu_0 \pi a^4}{2 l^3}$$

3.334 When the current in one of the loop is $I_1 = \alpha t$, an e.m.f. $L_{12}\dfrac{dI_1}{dt} = L_{12}\alpha$, is induced in the other loop. Then if the current in the other loop is I_2 we must have,

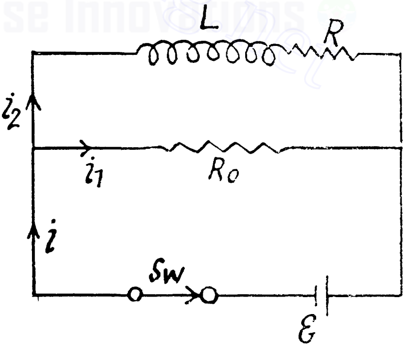
$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This familiar equation has the solution,

$I_2 = \dfrac{L_{12}\alpha}{R}\left(1 - e^{\frac{-tR}{L_2}}\right)$ which is the required current

3.335 Initially, after a steady current is set up, the current is flowing as shown.

In steady condition $i_{20} = \dfrac{\xi}{R}$, $i_{10} = \dfrac{\xi}{R_0}$.



When the switch is disconnected, the current through R_0 changes from i_{10} to the right, to i_{20} to the left. (The current in the inductance cannot change suddenly.). We then have the equation,

$$L\frac{di_2}{dt} + (R + R_0)\,i_2 = 0.$$

This equation has the solution $i_2 = i_{20}\,e^{-t(R + R_0)/L}$

The heat dissipated in the coil is,

$$Q = \int_0^\infty i_2^2 R\,dt = i_{20}^2 R\int_0^\infty e^{-2t(R + R_0)/L}\,dt$$

$$= R\,i_{20}^2 \times \frac{L}{2(R + R_0)} = \frac{L\,\xi^2}{2R(R + R_0)} = 3\,\mu\,\text{J}$$

3.336 To find the magnetic field energy we recall that the flux varies linearly with current. Thus, when the flux is Φ for current i, we can write $\phi = A\, i$. The total energy inclosed in the field, when the current is I, is

$$W = \int \xi i\, dt = \int N \frac{d\Phi}{dt} i\, d\, t$$

$$= \int N\, d\Phi\, i = \int_0^I N A\, i\, di = \frac{1}{2} N A\, I^2 = \frac{1}{2} N\, \Phi\, I$$

The characteristic factor $\frac{1}{2}$ appears in this way.

3.337 We apply circulation theorem,

$$H \cdot 2\pi b = N I, \quad \text{or,} \quad H = N I / 2\pi b.$$

Thus the total energy,

$$W = \frac{1}{2} BH \cdot 2\pi b \cdot \pi a^2 = \pi^2 a^2\, b\, BH.$$

Given N, I, b we know H, and can find out B from the $B - H$ curve. Then W can be calculated.

3.338 From $\oint \vec{H} \cdot d\vec{r} = N I$,

$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b \approx N I, \, (d >> b)$$

Also, $$B = \mu\, \mu_0 H. \text{ Thus, } H = \frac{N I}{\pi d + \mu b}.$$

Since B is continuous across the gap, B is given by,

$B = \mu\, \mu_0 \dfrac{N I}{\pi d + \mu b}$, both in the magnetic and the gap.

(a) $$\frac{W_{gap}}{W_{magnetic}} = \frac{\dfrac{B^2}{2\mu_0} \times S \times b}{\dfrac{B^2}{2\mu\mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d}.$$

(b) The flux is $N \int \vec{B} \cdot d\vec{S} = N \mu\mu_0 \dfrac{N I}{\pi d + \mu b} \cdot S = \mu_0 \dfrac{S N^2 I}{b + \dfrac{\pi d}{\mu}}$,

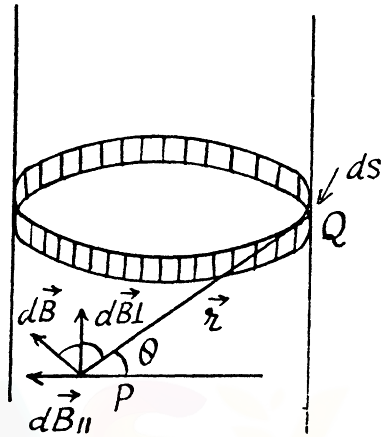
So, $$L \approx \frac{\mu_0 S N^2}{b + \dfrac{\pi d}{\mu}}.$$

Energy wise; total energy

$$= \frac{B^2}{2\mu_0} \left(\frac{\pi d}{\mu} + b \right) S = \frac{1}{2} \frac{\mu_0 N^2 S}{b + \dfrac{\pi d}{\mu}} \cdot I^2 = \frac{1}{2} L I^2$$

The L, found in the one way, agrees with that, found in the other way. Note that, in calculating the flux, we do not consider the field in the gap, since it is not linked to the winding. But the total energy includes that of the gap.

3.339 When the cylinder with a linear charge density λ rotates with a circular frequency ω, a surface current density (charge / length × time) of $i = \frac{\lambda\omega}{2\pi}$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is

$\vec{dB} = \frac{\mu_0}{4\pi}\frac{i(\vec{e}\times\vec{r})}{r^3}dS$ where $\vec{e}$ is the direction of the current. In magnitude, $|\vec{e}\times\vec{r}| = r$, since $\vec{e}$ is normal to $\vec{r}$ and the direction of $\vec{dB}$ is as shown.

It's component, $d\vec{B}_{\parallel}$ cancels out by cylindrical symmetry. The component that survives is,

$$|\vec{B}_\perp| = \frac{\mu_0}{4\pi}\int\frac{i\,dS}{r^2}\cos\theta = \frac{\mu_0\,i}{4\pi}\int d\Omega = \mu_0\,i,$$

where we have used $\frac{dS\cos\theta}{r^2} = d\Omega$ and $\int d\Omega = 4\pi$, the total solid angle around any point.

The magnetic field vanishes outside the cylinder by similar argument.

The total energy per unit length of the cylinder is,

$$W_1 = \frac{1}{2\mu_0}\mu_0^2\left(\frac{\lambda\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi}a^2\lambda^2\omega^2$$

3.340 $w_E = \frac{1}{2}\varepsilon_0 E^2$, for the electric field,

$w_B = \frac{1}{2\mu_0}B^2$ for the magnetic field.

Thus,
$$\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2,$$

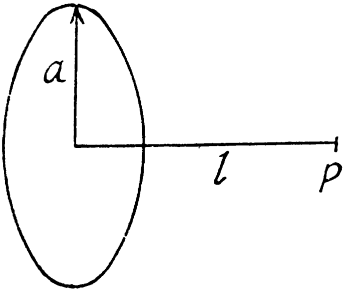
when
$$E = \frac{B}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8\ \text{V/m}$$

3.341 The electric field at P is,

$$E_p = \frac{ql}{4\pi\varepsilon_0(a^2+l^2)^{3/2}}$$

To get the magnetic field, note that the rotating ring constitutes a current $i = q\,\omega/2\,\pi$, and the corresponding magnetic field at P is,

$$B_p = \frac{\mu_0\, a^2\, i}{2\,(a^2 + l^2)^{3/2}}.$$



Thus, $\frac{w_E}{w_M} = \frac{\varepsilon_0\,\mu_0\,E^2}{B^2} = \varepsilon_0\,\mu_0\left(\frac{q l \times 2}{4\,\pi\,\varepsilon_0\,\mu_0\,a^2 i}\right)^2$

$$= \frac{1}{\varepsilon_0\,\mu_0}\left(\frac{l}{a^2\,\omega}\right)^2$$

or, $\frac{w_M}{w_E} = \varepsilon_0\,\mu_0\,\omega^2\,a^4/l^2$

3.342 The total energy of the magnetic field is,

$$\frac{1}{2}\int(\vec{B}\cdot\vec{H})\,dV = \frac{1}{2}\int\vec{B}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{J}\right)dV$$

$$= \frac{1}{2\,\mu_0}\int\vec{B}\cdot\vec{B}\,dV - \frac{1}{2}\int\vec{J}\cdot\vec{B}\,dV.$$

The second term can be interpreted as the energy of magnetization, and has the density $-\frac{1}{2}\vec{J}\cdot\vec{B}.$

3.343 (a) In series, the current I flows through both coils, and the total e.m.f. induced. when the current changes is,

$$-2L\frac{dI}{dt} = -L'\frac{dI}{dt}$$

or, $L' = 2L$

(b) In parallel, the current flowing through either coil is, $\frac{I}{2}$ and the e.m.f. induced is

$$-L\left(\frac{1}{2}\,\frac{dI}{dt}\right).$$

Equating this to $-L'\frac{dI}{dt}$, we find $L' = \frac{1}{2}L$.

3.344 We use $L_1 = \mu_0\,n_1^2\,V, L_2 = \mu_0\,n_2^2\,V$

So, $L_{12} = \mu_0\,n_1\,n_2\,V = \sqrt{L_1 L_2}$

3.345 The interaction energy is

$$\frac{1}{2\,\mu_0}\int\left|\vec{B}_1 + \vec{B}_2\right|^2 dV - \frac{1}{2\,\mu_0}\int\left|\vec{B}_1\right|^2 dV - \frac{1}{2\,\mu_0}\int\left|\vec{B}_2\right|^2 dV$$

$$= \frac{1}{\mu_0}\int\vec{B}_1\cdot\vec{B}_2\,dV$$

Here, if $\vec{B}_1$ is the magnetic field produced by the first of the current carrying loops, and $\vec{B}_2$, that of the second one, then the magnetic field due to both the loops will be $\vec{B}_1 + \vec{B}_2$.

3.346 We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and makes an angle θ with the direction of the magnetic field, due to the bigger loop. This magnetic field is,

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude,

$$|W| = \frac{\mu_0 I_1 I_2}{2b}\pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

3.347 (a) There is a radial outward conduction current. Let Q be the instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \quad \text{or,} \quad \vec{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\hat{r}.$$

On the other hand $\vec{j_d} = \frac{\partial \vec{D}}{\partial t} = \frac{d}{dt}\left(\frac{Q}{4\pi r^2}\hat{r}\right) = -\vec{j}$

(b) At the given moment, $\vec{E} = \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\hat{r}$

and by Ohm's law, $\vec{j} = \frac{\vec{E}}{\rho} = \frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$

Then,

$$\vec{j_d} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\hat{r}$$

and

$$\oint \vec{j_d}\cdot d\vec{S} = -\frac{q}{4\pi\varepsilon_0\varepsilon\rho}\int\frac{dS\cos\theta}{r^2} = -\frac{q}{\varepsilon_0\varepsilon\rho}.$$

The surface integral must be $-ve$ because $\vec{j_d}$, being opposite of $\vec{j}$, is inward.

3.348 Here also we see that neglecting edge effects, $\vec{j_d} = -\vec{j}$. Thus Maxwell's equations reduce to, div $\vec{B} = 0$, Curl $\vec{H} = 0$, $\vec{B} = \mu\vec{H}$

A general solution of this equation is $\vec{B}$ = constant = $\vec{B_0}$ · $\vec{B_0}$ can be thought of as an extraneous magnetic field. If it is zero, $\vec{B} = 0$.

3.349 Given $I = I_m \sin\omega t$. We see that

$$j = \frac{I_m}{S}\sin\omega t = -j_d = -\frac{\partial D}{\partial t}$$

or, $D = \frac{I_m}{\omega S}\cos\omega t$, so, $E_m = \frac{I_m}{\varepsilon_0\omega S}$ is the amplitude of the electric field and is 7 V/cm

3.350 The electric field between the plates can be written as,

$$E = Re \frac{V_m}{d} e^{i\omega t}, \text{ instead of } \frac{V_m}{d} \cos \omega t.$$

This gives rise to a conduction current,

$$j_c = \sigma E = \text{Re} \frac{\sigma}{d} V_m e^{i\omega t}$$

and a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \text{Re } \varepsilon_0 \varepsilon i \omega \frac{V_m}{d} e^{i\omega t}$$

The total current is,

$$j_T = \frac{V_m}{d} \sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \omega)^2} \cos (\omega t + \alpha)$$

where, $\tan \alpha = \dfrac{\sigma}{\varepsilon_0 \varepsilon \omega}$ on taking the real part of the resultant.

The corresponding magnetic field is obtained by using circulation theorem,

$$H \cdot 2 \pi r = \pi r^2 j_T$$

or, $H = H_m \cos (\omega t + \alpha)$, where, $H_m = \dfrac{r V_m}{2d} \sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \ \omega)^2}$

3.351 Inside the solenoid, there is a magnetic field,

$$B = \mu_0 n I_m \sin \omega t.$$

Since this varies in time there is an associated electric field. This is obtained by using,

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

For $r < R$, $2 \pi r E = -\dot{B} \cdot \pi r^2$, or, $E = -\dfrac{\dot{B} r}{2}$

For $r > R$, $E = -\dfrac{\dot{B} R^2}{2r}$

The associated displacement current density is,

$$j_d = \varepsilon_0 \frac{\partial E}{\partial t} = \left[\begin{array}{l} -\varepsilon_0 \ddot{B} \ r/2 \\ -\varepsilon_0 \ddot{B} \ R^2/2r \end{array} \right.$$

The answer, given in the book, is dimensionally incorrect without the factor ε_0.

3.352 In the non-relativistic limit.

$$\vec{E} = \frac{q}{4 \pi \varepsilon_0 r^3} \vec{r}$$

(a) On a straight line coinciding with the charge path,

$$\vec{j_d} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{q}{4\pi} \left[\frac{-\vec{V}}{r^3} - \frac{3 \vec{r} r}{r^4} \right], \left(\text{using, } \frac{d\vec{r}}{dt} = -\vec{v}. \right)$$

But in this case, $\dot{r} = -v$ and $v\frac{\vec{r}}{r} = \vec{v}$, so, $j_d = \frac{2q\vec{v}}{4\pi r^3}$

(b) In this case, $\dot{r} = 0$, as, $\vec{r} \perp \vec{v}$. Thus,

$$j_d = -\frac{q\vec{v}}{4\pi r^3}$$

3.353 We have, $E_p = \frac{qx}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$

then $$j_d = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{qv}{4\pi (a^2 + x^2)^{5/2}} (a^2 - 2x^2)$$

This is maximum, when $x = x_m = 0$, and minimum at some other value. The maximum displacement current density is

$$(j_d)_{\max} = \frac{qv}{4\pi a^3}$$

To check this we calculate $\frac{\partial j_d}{\partial x}$;

$$\frac{\partial j_d}{\partial x} = \frac{qv}{4\pi}[(-4x(a^2 + x^2) - 5x(a^2 - 2x^2)]$$

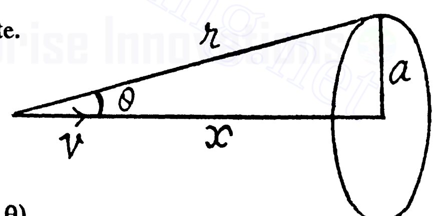
This vanishes for $x = 0$ and for $x = \sqrt{\frac{3}{2}}\, a$. The latter is easily shown to be a smaller local minimum (negative maximum).

3.354 We use Maxwell's equations in the form,

$$\oint \vec{B} \cdot d\vec{r} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S},$$

when the conduction current vanishes at the site.

We know that,



$$\int \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0} \int \frac{d\vec{S} \cdot \hat{r}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0} \int d\Omega = \frac{q}{4\pi\varepsilon_0} 2\pi (1 - \cos\theta),$$

where, $2\pi(1 - \cos\theta)$ is the solid angle, formed by the disc like surface, at the charge.

Thus, $$\oint \vec{B} \cdot d\vec{r} = 2\pi a B = \frac{1}{2}\mu_0 q \cdot \sin\theta \cdot \dot{\theta}$$

On the other hand, $x = a \cot\theta$

differentiating and using $\frac{dx}{dt} = -v$,

$$v = a \operatorname{cosec}^2\theta\, \dot{\theta}$$

Thus, $$B = \frac{\mu_0 q v r \sin\theta}{4\pi r^3}$$

This can be written as, $\vec{B} = \frac{\mu_0 q (\vec{v} \times \vec{r})}{4 \pi r^3}$

and $\vec{H} = \frac{q}{4\pi} \frac{\vec{v} \times \vec{r}}{r^3}$ (The sense has to be checked independently.)

3.355 (a) If $\vec{B} = \vec{B}(t)$, then,

Curl $\vec{E} = \frac{-\partial \vec{B}}{\partial t} \neq 0.$

So, $\vec{E}$ cannot vanish.

(b) Here also, curl $\vec{E} \neq 0$, so $\vec{E}$ cannot be uniform.

(c) Suppose for instance, $\vec{E} = \vec{a} f(t)$

where $\vec{a}$ is spatially and temporally fixed vector. Then $-\frac{\partial \vec{B}}{dt} =$ curl $\vec{E} = 0$. Generally speaking this contradicts the other equation curl $\vec{H} = \frac{\partial \vec{D}}{\partial t} \neq 0$ for in this case the left hand side is time independent but RHS. depends on time. The only exception is when $f(t)$ is linear function. Then a uniform field $\vec{E}$ can be time dependent.

3.356 From the equation Curl $\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$

We get on taking divergence of both sides

$$-\frac{\partial}{\partial t} \text{div } \vec{D} = \text{div } \vec{j}$$

But div $\vec{D} = \rho$ and hence div $\vec{j} + \frac{\partial \rho}{\partial t} = 0$

3.357 From $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

we get on taking divergence

$$0 = -\frac{\partial}{\partial t} \text{div } \vec{B}$$

This is compatible with div $\vec{B} = 0$

3.358 A rotating magnetic field can be represented by,

$B_x = B_0 \cos \omega t$; $B_y = B_0 \sin \omega t$ and $B_z = B_{zo}$

Then curl, $\vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

So, $-(\text{Curl } \vec{E})_x = -\omega B_0 \sin \omega t = -\omega B_y$

$-(\text{Curl } \vec{E})_y = \omega B_0 \cos \omega t = \omega B_x$ and $-(\text{Curl } \vec{E})_z = 0$

Hence, Curl $\vec{E} = -\vec{\omega} \times \vec{B}$,

where, $\vec{\omega} = \vec{e}_3 \omega$.

3.359 Consider a particle with charge e, moving with velocity $\vec{v}$, in frame K. It experiences a force $\vec{F} = e\vec{v} \times \vec{B}$

In the frame K', moving with velocity $\vec{v}$, relative to K, the particle is at rest. This means that there must be an electric field $\vec{E}$ in K', so that the particle experinces a force,

$$\vec{F}' = e\vec{E}' = \vec{F} = e\vec{v} \times \vec{B}$$

Thus, $$\vec{E}' = \vec{v} \times \vec{B}$$

3.360 Within the plate, there will appear a $(\vec{v} \times \vec{B})$ force, which will cause charges inside the plate to drift, until a countervailing electric field is set up. This electric field is related to B, by $E = eB$, since v & B are mutually perpendicular, and E is perpendicular to both. The charge density $\pm\,\sigma$, on the force of the plate, producing this electric field, is given by

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{or} \quad \sigma = \varepsilon_0 v B = 0{\cdot}40\ \text{pC/m}^2$$

3.361 Choose $\vec{\omega} \uparrow\uparrow \vec{B}$ along the z–axis, and choose $\vec{r}$, as the cylindrical polar radius vector of a reference point (perpendicular distance from the axis). This point has the velocity,

$$\vec{v} = \vec{\omega} \times \vec{r},$$

and experiences a $(\vec{v} \times \vec{B})$ force, which must be counterbalanced by an electric field,

$$\vec{E} = -(\vec{\omega} \times \vec{r}) \times \vec{B} = -(\vec{\omega} \cdot \vec{B})\,\vec{r}.$$

There must appear a space charge density,

$$\rho = \varepsilon_0 \operatorname{div} \vec{E} = -2\,\varepsilon_0\,\vec{\omega} \cdot \vec{B} = -8\ \text{pC/m}^3$$

Since the cylinder, as a whole is electrically neutral, the surface of the cylinder must acquire a positive charge of surface density,

$$\sigma = +\frac{2\,\varepsilon_0\,(\vec{\omega} \cdot \vec{B})\,\pi a^2}{2\pi a} = \varepsilon_0\, a\, \vec{\omega} \cdot \vec{B} = +{\cdot}2\ \text{pC/m}^2$$

3.362 In the reference frame K', moving with the particle,

$$\vec{E}' = \vec{E} + \vec{v}_0 \times \vec{B} = \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\vec{B}' = \vec{B} - \vec{v}_0 \times \vec{E}/c^2 = 0.$$

Here, $\vec{v}_0$ = velocity of K', relative to the K frame, in which the particle has velocity $\vec{v}$.

Clearly, $\vec{v}_0 = \vec{v}$. From the second equation,

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2} = \varepsilon_0 \mu_0 \times \frac{q}{4\pi\varepsilon_0} \frac{\vec{v} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{q\,(\vec{v} \times \vec{r})}{r^3}$$

3.363 Suppose, there is only electric field $\vec{E}$, in K. Then in K', considering nonrelativistic velocity $\vec{v}$, $\vec{E'} = \vec{E}$, $\vec{B} = -\frac{\vec{v}\times\vec{E}}{c^2}$,

So,
$$\vec{E'}\cdot\vec{B'} = 0$$

In the relativistic case,

$$\left.\begin{aligned}\vec{E'}_{\parallel} &= \vec{E}_{\parallel}\\ E'_{\perp} &= \frac{\vec{E}\perp}{\sqrt{1-v^2/c^2}}\end{aligned}\right\}\begin{aligned}\vec{B'}_{\parallel} &= \vec{B}_{\parallel} = 0\\ \vec{B'}_{\perp} &= \frac{-\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}\end{aligned}$$

Now,
$$\vec{E'}\cdot\vec{B'} = \vec{E'}_{\parallel}\cdot\vec{B'}_{\parallel} + \vec{E'}_{\perp}\cdot\vec{B'}_{\perp} = 0, \text{ since}$$

$$E'_{\perp}\cdot B'_{\perp} = -\vec{E}_{\perp}\cdot(\vec{v}\times\vec{E})/(1-v^2/c^2) = -\vec{E}_{\perp}\cdot(\vec{v}\times\vec{E}_{\perp})\Big/\left(1-\frac{v^2}{c^2}\right) = 0$$

3.364 In K, $\vec{B} = b\,\frac{y\,\hat{i} - x\,\hat{j}}{x^2+y^2}$, $b =$ constant.

In
$$K',\ \vec{E'} = \vec{v}\times\vec{B} = bv\,\frac{y\,\hat{j} - x\,\hat{i}}{x^2+y^2} = bv\,\frac{\vec{r}}{r^2}$$

The electric field is radial ($\vec{r} = x\,\hat{i} + y\,\hat{j}$).

3.365 In K, $\vec{E} = a\,\frac{\vec{r}}{r^2}$, $\vec{r} = (x\,\hat{i} + y\,\hat{j}\ )$

In
$$K', \vec{B'} = -\frac{\vec{v}\times\vec{E}}{c^2} = \frac{a\,\vec{r}\times\vec{v}}{c^2\,r^2}$$

The magnetic lines are circular.

3.366 In the non relativistic limit, we neglect v^2/c^2 and write,

$$\left.\begin{aligned}\vec{E'}_{\parallel} &= \vec{E}_{\parallel}\\ \vec{E}_{\perp} &\cong \vec{E}_{\perp} + \vec{v}\times\vec{B}\end{aligned}\right\}\begin{aligned}\vec{B'}_{\parallel} &= \vec{B}_{\parallel}\\ \vec{B'}_{\perp} &\cong \vec{B}_{\perp} - \vec{v}\times\vec{E}/c^2\end{aligned}$$

These two equations can be combined to give,

$$\vec{E'} = \vec{E} + \vec{v}\times\vec{B},\ \vec{B'} = \vec{B} - \vec{v}\times\vec{E}/c^2$$

3.367 Choose $\vec{E}$ in the direction of the z–axis, $\vec{E} = (0, 0, E)$. The frame K' is moving with velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$, in the $x-z$ plane. Then in the frame K',

$$\vec{E'}_{\parallel} = \vec{E}_{\parallel}\ B'_{\parallel} = 0$$

$$\vec{E'}_{\perp} = \frac{\vec{E}_{\perp}}{\sqrt{1-v^2/c^2}}\ \vec{B'}_{\perp} = \frac{-\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}$$

The vector along $\vec{v}$ is $\vec{e} = (\sin\alpha, 0, \cos\alpha)$ and the perpendicular vector in the $x-z$ plane is,

$$\vec{f} = (-\cos\alpha, 0, \sin\alpha),$$

(a) Thus using $\vec{E} = E\cos\alpha\,\vec{e} + E\sin\alpha\,\vec{f}$,

$$E'_{\parallel} = E\cos\alpha \text{ and } E'_{\parallel} = \frac{E\sin\alpha}{\sqrt{1-v^2/c^2}},$$

So

$$E' = E\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \text{ and } \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-v^2/c^2}}$$

(b) $B'_{\parallel} = 0$, $\vec{B}'_{\perp} = \dfrac{\vec{v}\times\vec{E}/c^2}{\sqrt{1-v^2/c^2}}$

$$B' = \frac{\beta E\sin\alpha}{c\sqrt{1-\beta^2}}$$

3.368 Choose $\vec{B}$ in the z direction, and the velocity $\vec{v} = (v\sin\alpha, 0, v\cos\alpha)$ in the $x-z$ plane, then in the K' frame,

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} = 0 \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \frac{\vec{v}\times\vec{B}}{\sqrt{1-v^2/c^2}} \qquad \vec{B}'_{\perp} = \frac{\vec{B}_{\perp}}{\sqrt{1-v^2/c^2}}$$

We find similarly, $E' = \dfrac{c\,\beta B\sin\alpha}{\sqrt{1-\beta^2}}$

$$B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}} \quad \tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$$

3.369 (a) We see that, $\vec{E}'\cdot\vec{B}' = \vec{E}'_{\parallel}\cdot\vec{B}'_{\parallel} + \vec{E}'_{\perp}\cdot\vec{B}'_{\perp}$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{(\vec{E}_{\perp} + \vec{v}\times\vec{B})\cdot\left(\vec{B}_{\perp} - \dfrac{\vec{v}\times\vec{E}}{c^2}\right)}{1-\dfrac{v^2}{c^2}}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B})\cdot(\vec{v}\times\vec{E})/c^2}{1-v^2/c^2}$$

$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \frac{\vec{E}_{\perp}\cdot\vec{B}_{\perp} - (\vec{v}\times\vec{B}_{\perp})\cdot(\vec{v}\times\vec{E}_{\perp})/c^2}{1-\dfrac{v^2}{c^2}}$$

But, $\vec{A}\times\vec{B}\cdot\vec{C}\times\vec{D} = A\cdot C\,B\cdot D - A\cdot D\,B\cdot C$,

so, $$\vec{E}'\cdot\vec{B}' = \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \vec{E}_{\perp}\cdot\vec{B}_{\perp}\frac{\left(1-\dfrac{v^2}{c^2}\right)}{1-\dfrac{v^2}{c^2}} = \vec{E}\cdot\vec{B}$$

(b) $E'^2 - c^2B'^2 = E'^2_{\parallel} - c^2B'^2_{\parallel} + E'^2_{\perp} - c^2B'^2_{\perp}$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1-\frac{v^2}{c^2}} \left[(\vec{E}_\perp + \vec{v} \times \vec{B})^2 - c^2 \left(\vec{B}_\perp - \frac{\vec{v} \times \vec{E}}{c^2} \right)^2 \right]$$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1-\frac{v^2}{c^2}} \left[E_\perp^2 - c^2 B_\perp^2 + (\vec{v} \times \vec{B}_\perp)^2 - \frac{1}{c^2} (\vec{v} \times E_1)^2 \right]$$

$$= E_{\|}^2 - c^2 B_{\|}^2 + \frac{1}{1-\frac{v^2}{c^2}} [E_\perp^2 - c^2 B_\perp^2] \left(1 - \frac{v^2}{c^2}\right) = E^2 - c^2 B^2,$$

since, $(\vec{v} \times \vec{A}_\perp)^2 = v^2 A_\perp^2$

3.370 In this case, $\vec{E} \cdot \vec{B} = 0$, as the fields are mutually perpendicular. Also,

$$E^2 - c^2 B^2 = -20 \times 10^8 \left(\frac{\text{V}}{\text{m}}\right)^2 \text{ is } -ve.$$

Thus, we can find a frame, in which $E' = 0$, and

$$B' = \frac{1}{c}\sqrt{c^2 B^2 - E^2} = B\sqrt{1 - \frac{E^2}{c^2 B^2}} = 0{\cdot}20\sqrt{1 - \left(\frac{4 \times 10^4}{3 \times 10^8 \times 2 \times 10^{-4}}\right)^2} = 0{\cdot}15 \text{ mT}$$

3.371 Suppose the charge q moves in the positive direction of the x–axis of the frame K. Let us go over to the moving frame K', at whose origin the charge is at rest. We take the x and x' axes of the two frames to be coincident, and the y & y' axes, to be parallel.

In the K' frame, $\vec{E}' = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\vec{r}'}{r'^3}$,

and this has the following components,

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{qx'}{r'^3},\ E_y' = \frac{1}{4\pi\varepsilon_0}\frac{qy'}{r'^3}.$$

Now let us go back to the frame K. At the moment, when the origins of the two frames coincide, we take $t = 0$. Then,

$$x = r\cos\theta = x'\sqrt{1 - \frac{v^2}{c^2}},\ y = r\sin\theta = y'$$

Also, $$E_x = E_x',\ E_y = E_y' \Big/ \sqrt{1 - v^2/c^2}$$

From these equations, $r'^2 = \dfrac{r^2 (1 - \beta^2 \sin^2\theta)}{1 - \beta^2}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^3 (1 - \beta^2 \sin^2\theta)^{3/2}} \left[(1 - \beta^2)^{3/2} \left(x'\hat{i} + \frac{y'}{\sqrt{1 - \beta^2}} \hat{j} \right) \right]$$

$$= \frac{q\vec{r}(1 - \beta^2)}{4\pi\varepsilon_0 r^3 (1 - \beta^2 \sin^2\theta)^{3/2}}$$

3.7 MOTION OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

3.372 Let the electron leave the negative plate of the capacitor at time $t = 0$

As, $$E_x = -\frac{d\varphi}{dx},\ E = \frac{\varphi}{l} = \frac{at}{l},$$

and, therefore, the acceleration of the electron,

$$w = \frac{eE}{m} = \frac{eat}{ml} \quad \text{or,} \quad \frac{dv}{dt} = \frac{eat}{ml}$$

or, $$\int_0^v dv = \frac{ea}{ml}\int_0^t t\,dt, \quad \text{or,} \quad v = \frac{1}{2}\frac{ea}{ml}t^2 \qquad (1)$$

But, from $s = \int v\,dt$,

$$l = \frac{1}{2}\frac{ea}{ml}\int_0^t t^2\,dt = \frac{eat^3}{6\,ml} \quad \text{or,} \quad t = \left(\frac{6\,ml^2}{ea}\right)^{\frac{1}{3}}$$

Putting the value of t in (1),

$$v = \frac{1}{2}\frac{ea}{ml}\left(\frac{6\,ml^2}{ea}\right)^{\frac{2}{3}} = \left(\frac{9}{2}\frac{ale}{m}\right)^{\frac{1}{3}} = 16\ \text{km/s}.$$

3.373 The electric field inside the capacitor varies with time as,

$$E = at.$$

Hence, electric force on the proton,

$$F = eat$$

and subsequently, acceleration of the proton,

$$w = \frac{eat}{m}$$

Now, if t is the time elapsed during the motion of the proton between the plates, then $t = \frac{l}{v_{\parallel}}$, as no acceleration is effective in this direction. (Here $v_{\parallel}$ is velocity along the length of the plate.)

From kinematics, $\frac{dv_{\perp}}{dt} = w$

so, $$\int_0^{v_{\perp}} dv_{\perp} = \int_0^t w\,dt,$$

(as initially, the component of velocity in the direction, $\perp$ to plates, was zero.)

or $$v_\perp = \int_0^t \frac{ea}{m}\frac{t^2}{2m} = \frac{ea}{2m}\frac{l^2}{v_\parallel^2}$$

Now, $$\tan\alpha = \frac{v_\perp}{v_\parallel} = \frac{e\,al^2}{2\,m\,v_\parallel^3}$$

$$= \frac{eal^2}{2m\left(\frac{2\,eV}{m}\right)^{\frac{3}{2}}},$$ as $v_\parallel = \left(\frac{2\,eV}{m}\right)^{\frac{1}{2}}$, from energy conservation.

$$= \frac{al^2}{4}\sqrt{\frac{m}{2\,eV^3}}$$

3.374 The equation of motion is,

$$\frac{dv}{dt} = v\,\frac{dv}{dx} = \frac{q}{m}(E_0 - ax)$$

Integrating

$$\frac{1}{2}v^2 - \frac{q}{m}\left(E_0 x - \frac{1}{2}ax^2\right) = \text{constant.}$$

But initially $v = 0$ when $x = 0$, so "constant" = 0

Thus, $$v^2 = \frac{2q}{m}\left(E_0 x - \frac{1}{2}ax^2\right)$$

Thus, $v = 0$, again for $x = x_m = \frac{2\,E_0}{a}$

The corresponding acceleration is,

$$\left(\frac{dv}{dt}\right)_{x_m} = \frac{q}{m}(E_0 - 2\,E_0) = -\frac{qE_0}{m}$$

3.375 From the law of relativistic conservation of energy

$$\frac{m_0\,c^2}{\sqrt{1-(v^2/c^2)}} - e\,Ex = m_0\,c^2.$$

as the electron is at rest ($v = 0$ for $x = 0$) initially.

Thus clearly $T = e\,Ex.$

On the other hand, $$\sqrt{1-(v^2/c^2)} = \frac{m_0\,c^2}{m_0\,c^2 + e\,Ex}$$

or, $$\frac{v}{c} = \frac{\sqrt{(m_0\,c^2 + e\,Ex)^2 - m_0^2\,c^4}}{m_0\,c^2 + eEx}$$

or, $$ct = \int c\,dt = \int \frac{(m_0\,c^2 + eEx)\,dx}{\sqrt{(m_0\,c^2 + eEx)^2 - m_0^2\,c^4}}$$

$$= \frac{1}{2\,eE}\int \frac{dy}{\sqrt{y - m_0^2\,c^4}} = \frac{1}{eE}\sqrt{(m_0\,c^2 + eEx)^2 - m_0^2\,c^4} + \text{constant}$$

The "constant" = 0, at $t = 0$, for $x = 0$,

So, $$ct = \frac{1}{eE}\sqrt{(m_0c^2 + eEx)^2 - m_0^2\,c^4}.$$

Finally, using $T = e\,E\,x$,

$$c\,e\,E\;t_0 = \sqrt{T\,(T + 2\,m_0\,c^2)} \quad \text{or,} \quad t_0 = \frac{\sqrt{T\,(T + 2\,m_0\,c^2)}}{e\,Ec}$$

3.376 As before, $T = e\,E\,x$

Now in linear motion,

$$\frac{d}{dt}\frac{m_0\,v}{\sqrt{1 - v^2/c^2}} = \frac{m_0\,w}{\sqrt{1 - v^2/c^2}} + \frac{m_0\,w}{(1 - v^2/c^2)^{3/2}}\frac{v}{c^2}\,w$$

$$= \frac{m_0}{(1 - v^2/c^2)^{3/2}}\,w = \frac{(T + m_0\,c^2)^3}{m_0^2\,c^6}\,w = e\,E,$$

So, $$w = \frac{e\,E\,m_0^2\,c^6}{(T + m_0\,c^2)^3} = \frac{eE}{m_0}\left(1 + \frac{T}{m_0\,c^2}\right)^{-3}$$

3.377 The equations are,

$$\frac{d}{dt}\left(\frac{m_0\,v_x}{\sqrt{1 - (v^2/c^2)}}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{m_0\,v_y}{\sqrt{1 - v^2/c^2}}\right) = e\,E$$

Hence, $$\frac{v_x}{\sqrt{1 - v^2/c^2}} = \text{constant} = \frac{v_0}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, by energy conservation,

$$\frac{m_0\,c^2}{\sqrt{1 - (v^2/c^2)}} = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}} + e\,E\,y$$

Dividing $$v_x = \frac{v_0\,\varepsilon_0}{\varepsilon_0 + eEy},\quad \varepsilon_0 = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, $$\frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \frac{\varepsilon_0 + e\,E\,y}{c^2}$$

Thus, $$(\varepsilon_0 + e\,Ey)\,v_y = c^2\,e\,E\,t + \text{constant}.$$

"constant" = 0 as $v_y = 0$ at $t = 0$.

Integrating again,

$$\varepsilon_0\,y + \frac{1}{2}\,e\,E\,y^2 = \frac{1}{2}\,c^2\,E\,t^2 + \text{constant}.$$

"constant" = 0, as $y = 0$, at $t = 0$.

Thus, $(ceEt)^2 = (eyE)^2 + 2\varepsilon_0 eEy + \varepsilon_0^2 - \varepsilon_0^2$

or, $$ceEt = \sqrt{(\varepsilon_0 + eEy)^2 - \varepsilon_0^2}$$

or, $$\varepsilon_0 + eEy = \sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}$$

Hence, $$v_x = \frac{v_0 \varepsilon_0}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}} \quad \text{also,} \quad v_y = \frac{c^2 eEt}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$$

and $$\tan\theta = \frac{v_y}{v_x} = \frac{eEt}{m_0 v_0}\sqrt{1 - (v_0^2/c^2)}.$$

3.378 From the figure,

$$\sin\alpha = \frac{d}{R} = \frac{dqB}{mv},$$

As radius of the arc $R = \frac{mv}{qB}$, where v is the velocity of the particle, when it enteres into the field. From initial condition of the problem,

$$qV = \frac{1}{2}mv^2 \quad \text{or,} \quad v = \sqrt{\frac{2qV}{m}}$$

Hence, $$\sin\alpha = \frac{dqB}{m\sqrt{\frac{2qV}{m}}} = dB\sqrt{\frac{q}{2mV}}$$

and $$\alpha = \sin^{-1}\left(dB\sqrt{\frac{q}{2mV}}\right) = 30°, \text{ on putting the values.}$$

3.379 (a) For motion along a circle, the magnetic force acted on the particle, will provide the centripetal force, necessary for its circular motion.

i.e. $$\frac{mv^2}{R} = evB \quad \text{or,} \quad v = \frac{eBR}{m}$$

and the period of revolution, $T = \frac{2\pi}{\omega} = \frac{2\pi R}{v} = \frac{2\pi m}{eB}$

(b) Generally, $\frac{d\vec{p}}{dt} = \vec{F}$

But, $$\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m_0\vec{v}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} + \frac{m_0}{\left(1-(v^2/c^2)\right)^{3/2}}\frac{\vec{v}(\vec{v}\cdot\dot{\vec{v}})}{c^2}$$

For transverse motion, $\vec{v}\cdot\dot{\vec{v}} = 0$ so,

$$\frac{d\vec{p}}{dt} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0}{\sqrt{1-(v^2/c^2)}}\frac{v^2}{r}, \text{ here.}$$

Thus, $$\frac{m_0 v^2}{r\sqrt{1-(v^2/c^2)}} = B\,ev \quad \text{or,} \quad \frac{v/c}{\sqrt{1-(v^2/c^2)}} = \frac{Ber}{m_0 c}$$

or, $$\frac{v}{c} = \frac{Ber}{\sqrt{B^2 e^2 r^2 + m_0^2 c^2}}$$

Finally, $$T = \frac{2\pi r}{v} = \frac{2\pi m_0}{eB\sqrt{1-v^2/c^2}} = \frac{2\pi}{cBe}\sqrt{B^2 e^2 r^2 + m_0^2 c^2}$$

3.380 (a) As before, $p = B\,qr$.

(b) $T = \sqrt{c^2 p^2 + m_0^2 c^4} = \sqrt{c^2 B^2 q^2 r^2 + m_0^2 c^4}$

(c) $$w = \frac{v^2}{r} = \frac{c^2}{r\left[1 + (m_0 c / Bqr)^2\right]}$$

using the result for v from the previous problem.

3.381 From (3.279),

$$T = \frac{2\pi\varepsilon}{c^2 eB}\text{(relativistic)}, \qquad T_0 = \frac{2\pi m_0 c^2}{c^2 eB}\text{(nonrelativistic)},$$

Here, $$m_0 c^2 \Big/ \sqrt{1-v^2/c^2} = E$$

Thus, $$\delta T = \frac{2\pi T}{c^2 eB}, \quad (T = K.E.)$$

Now, $$\frac{\delta T}{T_0} = \eta = \frac{T}{m_0 c^2}, \quad \text{so,} \quad T = \eta\, m_0 c^2$$

3.382 $T = eV = \frac{1}{2} mv^2$

(The given potential difference is not large enough to cause significant deviations from the nonrelativistic formula).

Thus, $$v = \sqrt{\frac{2eV}{m}}$$

So, $$v_{\parallel} = \sqrt{\frac{2eV}{m}}\cos\alpha, \quad v_{\perp} = \sqrt{\frac{2eV}{m}}\sin\alpha$$

Now, $$\frac{m v_{\perp}^2}{r} = Bev_{\perp} \quad \text{or,} \quad r = \frac{mv_{\perp}}{Be},$$

and $$T = \frac{2\pi r}{v_{\perp}} = \frac{2\pi m}{Be}$$

Pitch $$p = v_{\parallel} T = \frac{2\pi m}{Be}\sqrt{\frac{2eV}{m}}\cos\alpha = 2\pi\sqrt{\frac{2mV}{eB^2}}\cos\alpha$$

3.383 The charged particles will traverse a helical trajectory and will be focussed on the axis after traversing a number of turms. Thus

$$\frac{l}{v_0} = n \cdot \frac{2\pi m}{qB_1} = (n+1)\frac{2\pi m}{qB_2}$$

So,
$$\frac{n}{B_1} = \frac{n+1}{B_2} = \frac{1}{B_2 - B_1}$$

Hence,
$$\frac{l}{v_0} = \frac{2\pi m}{q(B_2 - B_1)}$$

or,
$$\frac{l^2}{2qV/m} = \frac{(2\pi)^2}{(B_2 - B_1)^2} \times \frac{1}{(q/m)^2}$$

or,
$$\frac{q}{m} = \frac{8\pi^2 V}{l^2 (B_2 - B_1)^2}$$

3.384 Let us take the point A as the origin O and the axis of the solenoid as z-axis. At an arbitrary moment of time let us resolve the velocity of electron into its two rectangular components, $\vec{v}_{\parallel}$ along the axis and $\vec{v}_{\perp}$ to the axis of solenoid. We know the magnetic force does no work, so the kinetic energy as well as the speed of the electron $|\vec{v}_{\perp}|$ will remain constant in the x-y plane. Thus $\vec{v}_{\perp}$ can change only its direction as shown in the Fig.. $\vec{v}_{\parallel}$ will remain constant as it is parallel to $\vec{B}$.

Thus at $t = t$

$$v_x = v_{\perp} \cos \omega t = v \sin \alpha \cos \omega t,$$

$$v_y = v_{\perp} \sin \omega t = v \sin \alpha \sin \omega t$$

and
$$v_z = v \cos \alpha, \quad \text{where } \omega = \frac{eB}{m}$$

As at $t = 0$, we have $x = y = z = 0$, so the motion law of the electron is.

$$\left.\begin{aligned} z &= v \cos \alpha\, t \\ x &= \frac{v \sin \alpha}{\omega} \sin \omega t \\ y &= \frac{v \sin \alpha}{\omega} (\cos \omega t - 1) \end{aligned}\right]$$

(The equation of the helix)

On the screen,
$$z = l, \text{ so } t = \frac{l}{v \cos \alpha}.$$

Then,
$$r^2 = x^2 + y^2 = \frac{2 v^2 \sin^2 \alpha}{\omega^2}\left(1 - \cos \frac{\omega l}{v \cos \alpha}\right)$$

$$r = \frac{2 v \sin \alpha}{\omega} \cdot \left| \sin \frac{\omega l}{2 v \cos \alpha} \right| = 2 \frac{mv}{eB} \sin \alpha \left| \sin \frac{leB}{2\, mv \cos \alpha} \right|$$

3.385 Choose the wire along the z–axis, and the initial direction of the electron, along the x–axis. Then the magnetic field in the $x - z$ plane is along the y–axis and outside the wire it is,

$$B = B_y = \frac{\mu_0 I}{2\pi x}, \quad (B_x = B_z = 0, \text{ if } y = 0)$$

The motion must be confined to the $x - z$ plane. Then the equations of motion are,

$$\frac{d}{dt} m v_x = -e v_z B_y$$

$$\frac{d(m v_z)}{dt} = +e v_x B_y$$

Multiplying the first equation by v_x and the second by v_z and then adding,

$$v_x \frac{dv_x}{dt} + v_z \frac{dv_z}{dt} = 0$$

or, $$v_x^2 + v_z^2 = v_0^2, \text{ say, or, } v_z = \sqrt{v_0^2 - v_x^2}$$

Then, $$v_x \frac{dv_x}{dx} = -\frac{e}{m}\sqrt{v_0^2 - v_x^2}\,\frac{\mu_0 I}{2\pi x}$$

or, $$-\frac{v_x\, dv_x}{\sqrt{v_0^2 - v_x^2}} = \frac{\mu_0 I e}{2\pi m}\frac{dx}{x}$$

Integrating, $$\sqrt{v_0^2 - v_x^2} = \frac{\mu_0 I e}{2\pi m} \ln \frac{x}{a}$$

on using, $v_x = v_0$, if $x = a$ (i.e. initially).

Now, $v_x = 0$, when $x = x_m$,

so, $$x_m = a\, e^{v_0/b}, \text{ where } b = \frac{\mu_0 I e}{2\pi m}.$$

3.386 Inside the capacitor, the electric field follows a $\frac{1}{r}$ law, and so the potential can be written as

$$\varphi = \frac{V \ln r/a}{\ln b/a}, \quad E = \frac{-V}{\ln b/a}\frac{1}{r}.$$

Here r is the distance from the axis of the capacitor.

Also, $$\frac{mv^2}{r} = \frac{qV}{\ln b/a}\frac{1}{r} \quad \text{or} \quad mv^2 = \frac{qV}{\ln b/a}$$

On the other hand,

$mv = qBr$ in the magnetic field.

Thus, $$v = \frac{V}{B r \ln b/a} \quad \text{and} \quad \frac{q}{m} = \frac{v}{Br} = \frac{V}{B^2 r^2 \ln (b/a)}$$

3.387 The equations of motion are,

$$m\frac{dv_x}{dt} = -qBv_z, \quad m\frac{dv_y}{dt} = qE \text{ and } m\frac{dv_z}{dt} = qv_xB$$

These equations can be solved easily.

First, $$v_y = \frac{qE}{m}t, \quad y = \frac{qE}{2m}t'^2$$

Then, $$v_x^2 + v_z^2 = \text{constant} = v_0^2 \text{ as before.}$$

In fact, $v_x = v_0 \cos\omega t$ and $v_z = v_0 \sin\omega t$ as one can check.

Integrating again and using $x = z = 0$, at $t = 0$

$$x = \frac{v_0}{\omega}\sin\omega t, \quad z = \frac{v_0}{\omega}(1-\cos\omega t)$$

Thus, $$x = z = 0 \text{ for } t = t_n = n\frac{2\pi}{\omega}$$

At that instant, $$y_n = \frac{qE}{2m}\times\frac{2\pi}{qB/m}\times n^2\times\frac{2\pi}{qB/m} = \frac{2\pi^2 mEn^2}{qB^2}$$

Also, $$\tan\alpha_n = \frac{v_x}{v_y}, \; (v_z = 0 \text{ at this moment})$$

$$= \frac{mv_0}{qEt_n} = \frac{mv_0}{qE}\times\frac{qB}{m}\times\frac{1}{2\pi n} = \frac{Bv_0}{2\pi En}.$$

3.388 The equation of the trajectory is,

$$x = \frac{v_0}{\omega}\sin\omega t, \quad z = \frac{v_0}{\omega}(1-\cos\omega t), \quad y = \frac{qE}{2m}t^2 \text{ as before see (3.384).}$$

Now on the screen $x = l$, so

$$\sin\omega t = \frac{\omega l}{v_0} \quad \text{or,} \quad \omega t = \sin^{-1}\frac{\omega l}{v_0}$$

At that moment,

$$y = \frac{qE}{2m\omega^2}\left(\sin^{-1}\frac{\omega l}{v_0}\right)^2$$

so, $$\frac{\omega l}{v_0} = \sin\sqrt{\frac{2m\omega^2 y}{qE}} = \sin\sqrt{\frac{2qB^2y}{Em}}$$

and $$z = \frac{v_0}{\omega}2\sin^2\frac{\omega t}{2} = l\tan\frac{\omega t}{2}$$

$$= l\tan\frac{1}{2}\left[\sin^{-1}\frac{\omega l}{v_0}\right] = l\tan\sqrt{\frac{qB^2y}{2mE}}$$

For small

$$z, \quad \frac{qB^2y}{2mE} = \left(\tan^{-1}\frac{z}{l}\right)^2 \approx \frac{z^2}{l^2}$$

or, $$y = \frac{2mE}{qB^2l^2}\cdot z^2 \text{ is a parabola.}$$

3.389 In crossed field,

$$eE = evB, \text{ so } v = \frac{E}{B}$$

Then, F = force exerted on the plate $= \frac{I}{e} \times m \frac{E}{B} = \frac{m I E}{eB}$

3.390 When the electric field is switched off, the path followed by the particle will be helical. and pitch, $\Delta l = v_{\parallel} T$, (where $v_{\parallel}$ is the velocity of the particle, parallel to $\vec{B}$, and T, the time period of revolution.)

$$= v \cos (90 - \varphi) T = v \sin \varphi T$$

$$= v \sin \varphi \frac{2 \pi m}{qB} \left(\text{as } T = \frac{2\pi}{qB}\right) \quad (1)$$

Now, when both the fields were present, $qE = qvB \sin (90 - \varphi)$, as no net force was effective on the system.

or, $$v = \frac{E}{B \cos \varphi} \quad (2)$$

From (1) and (2), $$\Delta l = \frac{E}{B} \frac{2 \pi m}{qB} \tan \varphi = 6 \text{ cm.}$$

3.391 When there is no deviation,

$$-q\vec{E} = q (\vec{v} \times \vec{B})$$

or, in scalar from, $E = vB$ (as $\vec{v} \perp \vec{B}$) or, $v = \frac{E}{B}$ (1)

Now, when the magnetic field is switched on, let the deviation in the field be x. Then,

$$x = \frac{l}{2}\left(\frac{q\, vB}{m}\right) t^2,$$

where t is the time required to pass through this region.

also, $$t = \frac{a}{v}$$

Thus $$x = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{a}{v}\right)^2 = \frac{1}{2} \frac{q}{m} \frac{a^2 B^2}{E} \quad (2)$$

For the region where the field is absent, velocity in upward direction

$$= \left(\frac{qvB}{m}\right) t = \frac{q}{m} a B \quad (3)$$

Now, $$\Delta x - x = \frac{qaB}{m} t'$$

$$= \frac{q}{m} \frac{aB^2 b}{E} \text{ when } t' = \frac{b}{v} = \frac{bB}{E} \quad (4)$$

From (2) and (4),

$$\Delta x - \frac{1}{2} \frac{q}{m} \frac{a^2 B^2}{E} = \frac{q}{m} \frac{a B^2 b}{E}$$

or, $$\frac{q}{m} = \frac{2 E \Delta x}{a B^2 (a + 2b)}$$

3.392 (a) The equation of motion is,

$$m \frac{d^2 \vec{r}}{dt^2} = q(\vec{E} + \vec{v} \times \vec{B})$$

Now, $$\vec{v} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = \vec{i} B \dot{y} - \vec{j} B \dot{x}$$

So, the equation becomes,

$$\frac{dv_x}{dt} = \frac{qBv_y}{m}, \; \frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m} v_x, \text{ and } \frac{dv_z}{dt} = 0$$

Here, $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$. The last equation is easy to integrate;

$$v_z = \text{constant} = 0,$$

since v_z is zero initially. Thus integrating again,

$$z = \text{constant} = 0,$$

and motion is confined to the $x - y$ plane. We now multiply the second equation by i and add to the first equation.

$$\xi = v_x + i v_y$$

we get the equation,

$$\frac{d\xi}{dt} = i\omega \frac{E}{B} - i\omega\xi, \; \omega = \frac{qB}{m}.$$

This equation after being multiplied by $e^{i\omega t}$ can be rewritten as,

$$\frac{d}{dt}(\xi e^{i\omega t}) = i\omega e^{i\omega t} \frac{E}{B}$$

and integrated at once to give,

$$\xi = \frac{E}{B} + C e^{-i\omega t - i\alpha},$$

where C and α are two real constants. Taking real and imaginary parts.

$$v_x = \frac{E}{B} + C\cos(\omega t + \alpha) \text{ and } v_y = -C\sin(\omega t + \alpha)$$

Since $v_y = 0$, when $t = 0$, we can take $\alpha = 0$, then $v_x = 0$ at $t = 0$ gives, $C = -\dfrac{E}{B}$

and we get,

$$v_x = \frac{E}{B}(1 - \cos\omega t) \text{ and } v_y = \frac{E}{B}\sin\omega t.$$

Integrating again and using $x = y = 0$, at $t = 0$, we get

$$x(t) = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right), \; y(t) = \frac{E}{\omega B}(1 - \cos\omega t).$$

This is the equation of a cycloid.

(b) The velocity is zero, when $\omega t = 2n\pi$. We see that

$$v^2 = v_x^2 + v_y^2 = \left(\frac{E}{B}\right)^2 (2 - 2\cos\omega t)$$

or, $$v = \frac{ds}{dt} = \frac{2E}{B}\left|\sin\frac{\omega t}{2}\right|$$

The quantity inside the modulus is positive for $0 < \omega t < 2\pi$. Thus we can drop the modulus and write for the distance traversed between two successive zeroes of velocity.

$$S = \frac{4E}{\omega B}\left(1 - \cos\frac{\omega t}{2}\right)$$

Putting $\omega t = 2\pi$, we get

$$S = \frac{8E}{\omega B} = \frac{8\,mE}{q\,B^2}$$

(c) The drift velocity is in the x-direction and has the magnitude,

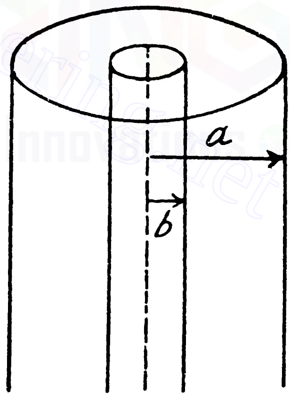
$$<v_x> = \left< \frac{E}{B}(1 - \cos\omega t) \right> = \frac{E}{B}.$$

3.393 When a current I flows along the axis, a magnetic field $B_\varphi = \frac{\mu_0 I}{2\pi\rho}$ is set up where $\rho^2 = x^2 + y^2$. In terms of components,

$$B_x = -\frac{\mu_0 I y}{2\pi\rho^2},\ B_y = \frac{\mu_0 I x}{2\pi\rho^2} \text{ and } B_z = 0$$

Suppose a p.d. V is set up between the inner cathode and the outer anode. This means a potential function of the form

$$\varphi = V\frac{\ln\rho/b}{\ln a/b},\quad a > \rho > b,$$



as one can check by solving Laplace equation.
The electric field corresponding to this is,

$$E_x = -\frac{Vx}{\rho^2\ln a/b},\ E_y = -\frac{Vy}{\rho^2\ln a/b},\ E_z = 0.$$

The equations of motion are,

$$\frac{d}{dt}mv_x = +\frac{|e|Vz}{\rho^2\ln a/b} + \frac{|e|\mu_0 I}{2\pi\rho^2}x\dot{z}$$

$$\frac{d}{dt}mv_y = +\frac{|e|Vy}{\rho^2\ln a/b} + \frac{|e|\mu_0 I}{2\pi\rho^2}y\dot{z}$$

and $$\frac{d}{dt}mv_z = -|e|\frac{\mu_0 I}{2\pi\rho^2}(x\dot{x} + y\dot{y}) = -|e|\frac{\mu_0 I}{2\pi}\frac{d}{dt}\ln\rho$$

$(-|e|)$ is the charge on the electron.
Integrating the last equation,

$$mv_z = -|e|\frac{\mu_0 I}{2\pi}\ln\rho/a = m\dot{z}.$$

since $v_z = 0$ where $\rho = a$. We now substitute this $\dot{z}$ in the other two equations to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2\right)$$

$$= \left[\frac{|e|V}{\ln a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \rho/b\right] \cdot \frac{x\dot{x} + y\dot{y}}{\rho^2}$$

$$= \left[\frac{|e|V}{\ln \frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \frac{\rho}{b}\right] \cdot \frac{1}{2\rho^2}\frac{d}{dt}\rho^2$$

$$= \left[\frac{|e|V}{\ln \frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \frac{\rho}{b}\right] \frac{d}{dt}\ln \frac{\rho}{b}$$

Integrating and using $v^2 = 0$, at $\rho = b$, we get,

$$\frac{1}{2}mv^2 = \left[\frac{|e|V}{\ln \frac{a}{b}}\ln\frac{\rho}{b} - \frac{1}{2m}|e|^2\left(\frac{\mu_0 I}{2\pi}\right)^2 \left(\ln\frac{\rho}{b}\right)\right]$$

The RHS must be positive, for all $a > \rho > b$. The condition for this is,

$$V \geq \frac{1}{2}\frac{|e|}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{a}{b}$$

3.394 This differs from the previous problem in $(a \leftrightarrow b)$ and the magnetic field is along the z-direction. Thus $B_x = B_y = 0$, $B_z = B$

Assuming as usual the charge of the electron to be $-|e|$, we write the equation of motion

$$\frac{d}{dt}mv_x = \frac{|e|V_x}{\rho^2 \ln\frac{b}{a}} - |e|B\dot{y}, \quad \frac{d}{dt}mv_y = \frac{|e|V_y}{\rho^2\ln\frac{b}{a}} + |e|B\dot{x}$$

and $$\frac{d}{dt}mv_z = 0 \Rightarrow z = 0$$

The motion is confined to the plane $z = 0$. Eliminating B from the first two equations,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{|e|V}{\ln b/a}\frac{x\dot{x} + y\dot{y}}{\rho^2}$$

or, $$\frac{1}{2}mv^2 = |e|V\frac{\ln \rho/a}{\ln b/a}$$

so, as expected, since magnetic forces do not work,

$$v = \sqrt{\frac{2|e|V}{m}}, \text{ at } \rho = b.$$

On the other hand, eliminating V, we also get,

$$\frac{d}{dt} m\,(xv_y - yv_x) = |e|B\,(x\dot{x} + y\dot{y})$$

i.e.

$$(xv_y - yv_x) = \frac{|e|B}{2m}\rho^2 + \text{constant}$$

The constant is easily evaluated, since v is zero at $\rho = a$. Thus,

$$(xv_y - yv_x) = \frac{|e|B}{2m}(\rho^2 - a^2) > 0$$

At

$$\rho = b,\ (xv_y - yv_x) \le vb$$

Thus,

$$vb \ge \frac{|e|B}{2m}(b^2 - a^2)$$

or,

$$B \le \frac{2\,mb}{b^2 - a^2}\sqrt{\frac{2|e|V}{m}} \times \frac{1}{|e|}$$

or,

$$B \le \frac{2b}{b^2 - a^2}\sqrt{\frac{2m\,B}{|e|}}$$

3.395 The equations are as in 3.392.

$$\frac{dv_x}{dt} = \frac{qB}{m}v_y,\ \frac{dv_y}{dt} = \frac{qE_m}{m}\cos\omega t - \frac{qB}{m}v_x \text{ and } \frac{dv_z}{dt}$$

with

$$\omega = \frac{qB}{m},\ \xi = v_x + iv_y, \text{ we get,}$$

$$\frac{d\xi}{dt} = i\frac{E_m}{B}\omega\cos\omega t - i\,\omega\,\xi$$

or multiplying by $e^{i\omega t}$,

$$\frac{d}{dt}(\xi\, e^{i\omega t}) = i\frac{E_m}{2B}\omega\,(e^{2i\omega t} + 1)$$

or integrating,

$$\xi\, e^{i\omega t} = \frac{E_m}{4B}e^{2i\omega t} + \frac{E_m}{2B}i\,\omega t$$

or,

$$\xi = \frac{E_m}{4B}(e^{i\omega t} + 2\,i\,\omega t\, e^{i\omega t}) + C\,e^{i\omega t}$$

since

$$\xi = 0 \text{ at } t = 0,\ C = -\frac{E_m}{4B}.$$

Thus,

$$\xi = i\frac{E_m}{2B}\sin\omega t + i\frac{E_m}{2B}\omega t\, e^{i\omega t}$$

or,

$$v_x = \frac{E_m}{2B}\omega t\sin\omega t \text{ and } v_y = \frac{E_m}{2B}\sin\omega t + \frac{E_m}{2B}\omega t\cos\omega t$$

Integrating again,

$$x = \frac{a}{2\,\omega^2}(\sin\omega t - \omega t\cos\omega t),\ y = \frac{a}{2\omega}t\sin\omega t.$$

where $a = \frac{qE_m}{m}$, and we have used $x = y = 0$, at $t = 0$.

The trajectory is an unwinding spiral.

3.396 We know that for a charged particle (proton) in a magnetic field,

$$\frac{mv^2}{r} = B\,ev \text{ or } mv = Ber$$

But,
$$\omega = \frac{eB}{m},$$

Thus
$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\,\omega^2 r^2.$$

So,
$$\Delta E = m\omega^2 r\,\Delta r = 4\,\pi^2 \nu^2 mr\,\Delta r$$

On the other hand $\Delta E = 2\,eV$, where V is the effective acceleration voltage, across the Dees, there being two crossings per revolution. So,

$$V \geq 2\,\pi^2 \nu^2 mr\,\Delta r/e$$

3.397 (a) From $\frac{mv^2}{r} = Bev$, or, $mv = Ber$

and
$$T = \frac{(Ber)^2}{2m} = \frac{1}{2}mv^2 = 12\text{ MeV}$$

(b) From $\frac{2\pi}{\omega} = \frac{2\pi\,r}{v}$

we get,
$$f_{\min} = \frac{v}{2\,\pi r} = \frac{1}{\pi r}\sqrt{\frac{T}{2m}} = 15\text{ MHz}$$

3.398 (a) The total time of acceleration is,

$$t = \frac{1}{2\,\nu}\cdot n,$$

where n is the number of passages of the Dees.

But,
$$T = neV = \frac{B^2 e^2 r^2}{2m}$$

or,
$$n = \frac{B^2\,e\,r^2}{2mV}$$

So,
$$t = \frac{\pi}{eB/m} \times \frac{B^2\,er^2}{2mV} = \frac{\pi\,B\,r^2}{2V} = \frac{\pi^2\,m\nu\,r^2}{eV} = 30\,\mu\text{ s}$$

(b) The distance covered is, $s = \sum v_n \cdot \frac{1}{2\,\nu}$

But,
$$v_n = \sqrt{\frac{2eV}{m}}\sqrt{n},$$

So,
$$s = \sqrt{\frac{eV}{2m\nu^2}}\sum\sqrt{n} = \sqrt{\frac{eV}{2m\nu^2}}\int\sqrt{n}\,dn = \sqrt{\frac{eV}{2m\nu^2}}\,\frac{2}{3}\,n^{3/2}$$

But, $$n = \frac{B^2 e^2 r^2}{2\,eV\,m} = \frac{2\,\pi^2 m\,v^2 r^2}{eV}$$

Thus, $$s = \frac{4\,\pi^3 v^2 mr^2}{3eV} = 1{\cdot}24\text{ km}$$

3.399 In the nth orbit, $\frac{2\,\pi\,r_n}{v_n} = n\;T_0 = \frac{n}{\nu}$. We ignore the rest mass of the electron and write $v_n \approx c$. Also $W \approx cp = cBer_n$.

Thus, $$\frac{2\,\pi\,W}{Bec^2} = \frac{n}{\nu}$$

or, $$n = \frac{2\,\pi\,W\nu}{Bec^2} = 9$$

3.400 The basic condition is the relativistic equation,

$$\frac{mv^2}{r} = Bqv, \quad \text{or,} \quad mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Bqr.$$

Or calling, $$\omega = \frac{Bq}{m},$$

we get, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\omega_0^2 r^2}{c^2}}}, \quad \omega_0 = \frac{Bq}{m_0}\,r$$

is the radius of the instantaneous orbit.

The time of acceleration is,

$$t = \sum_{n=1}^{N} \frac{1}{2\nu_n} = \sum_{n=1}^{N} \frac{\pi}{\omega_n} = \sum_{n}^{N} \frac{\pi\,W_n}{q\,Bc^2}.$$

N is the number of crossing of either Dee.

But, $W_n = m_0\,c^2 + \frac{n\,\Delta W}{2}$, there being two crossings of the Dees per revolution.

So, $$t = \sum \frac{\pi\,m_0 c^2}{qB\,c^2} + \sum \frac{\pi\,\Delta W_n}{2q\,B\,c^2}$$

$$= N\frac{\pi}{\omega_0} + \frac{N\,(N+1)}{4}\,\frac{\pi\,\Delta\,W}{qB\,c^2} \approx N^2\,\frac{\pi\,\Delta W}{4\,q\,Bc^2}\;(N >> 1)$$

Also, $$r = r_N\frac{v_N}{\omega_N} \approx \frac{c}{\pi}\,\frac{\partial t}{\partial N} = \frac{\Delta W}{2q\,Bc}N$$

Hence finally,
$$\omega = \frac{\omega_0}{\sqrt{1 + \frac{q^2 B^2}{m_0^2 c^2} \times \frac{\Delta W^2}{4 q^2 B^2 c^2} N^2}}$$

$$= \frac{\omega_0}{\sqrt{1 + \frac{(\Delta W)^2}{4 m_0^2 c^4} \times \frac{4 qBc^2}{\pi \Delta W} t}} = \frac{\omega_0}{\sqrt{1 + at}};$$

$$a = \frac{q B \Delta W}{\pi m_0^2 c^2}$$

3.401 When the magnetic field is being set up in the solenoid, and electric field will be induced in it, this will accelerate the charged particle. If $\dot{B}$ is the rate, at which the magnetic field is increasing, then.

$$\pi r^2 \dot{B} = 2 \pi r E \quad \text{or} \quad E = \frac{1}{2} r \dot{B}$$

Thus,
$$m \frac{dv}{dt} = \frac{1}{2} r \dot{B} q, \quad \text{or} \quad v = \frac{qBr}{2m},$$

After the field is set up, the particle will execute a circular motion of radius ρ, where

$$mv = B q \rho, \quad \text{or} \quad \rho = \frac{1}{2} r$$

3.402 The increment in energy per revolution is $e \Phi$, so the number of revolutions is,

$$N = \frac{W}{e \Phi}$$

The distance traversed is, $s = 2 \pi r N$

3.403 On the one hand,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r} \frac{d\Phi}{dt} = \frac{e}{2\pi r} \frac{d}{dt} \int_0^r 2\pi r' B(r') \, dr'$$

On the other ,

$$p = B(r) er, \quad r = \text{constant.}$$

so,
$$\frac{dp}{dt} = er \frac{d}{dt} B(r) = er \dot{B}(r)$$

Hence,
$$er \dot{B}(r) = \frac{e}{2 \pi r} \pi r^2 \frac{d}{dt} < B >$$

So,
$$\dot{B}(r) = \frac{1}{2} \frac{d}{dt} < B >$$

This equations is most easily satisfied by taking $B(r_0) = \frac{1}{2} < B >$.

3.404 The condition, $B(r_0) = \frac{1}{2} < B > = \frac{1}{2} \int_0^{r_0} B \cdot 2\pi r \, dr / \pi r_0^2$

or,
$$B(r_0) = \frac{1}{r_0^2}\int_0^{r_0} Brdr$$

This gives r_0.

In the present case,

$$B_0 - ar_0^2 = \frac{1}{r_0^2}\int_0^{r_0}(B - ar^2)\, rdr = \frac{1}{2}\left(B_0 - \frac{1}{2}ar_0^2\right)$$

or,
$$\frac{3}{4}ar_0^2 = \frac{1}{2}B_0 \quad \text{or} \quad r_0 = \sqrt{\frac{2B_0}{3a}}.$$

3.405 The induced electric field (or eddy current field) is given by,

$$E(r) = \frac{1}{2\pi r}\frac{d}{dt}\int_0^r 2\,\pi r'\,(r')\,B(r')\,dr'$$

Hence,

$$\frac{dE}{dr} = -\frac{1}{2\pi r^2}\frac{d}{dt}\int_0^r 2\pi r'\,B(r')\,dr' + \frac{dB(r)}{dt}$$

$$= -\frac{1}{2}\frac{d}{dt}\langle B\rangle + \frac{dB(r)}{dt}$$

This vanishes for $r = r_0$ by the betatron condition, where r_0 is the radius of the equilibrium orbit.

3.406 From the betatron condition,

$$\frac{1}{2}\frac{d}{dt}\langle B\rangle = \frac{dB}{dt}(r_0) = \frac{B}{\Delta t}$$

Thus,
$$\frac{d}{dt}\langle B\rangle = \frac{2B}{\Delta t}$$

and
$$\frac{d\Phi}{dt} = \pi r^2\frac{d\langle B\rangle}{dt} = \frac{2\,\pi r^2 B}{\Delta t}$$

So, energy increment per revolution is,

$$e\frac{d\Phi}{dt} = \frac{2\,\pi\, r^2 eB}{\Delta t}$$

3.407 (a) Even in the relativistic case, we know that : $p = Ber$

Thus, $W = \sqrt{c^2 p^2 + m_0^2 c^4} - m_0 c^2 = m_0 c^2\left(\sqrt{1 + (Ber/m_0 c)^2} - 1\right)$

(b) The distance traversed is,

$$2\pi r\frac{W}{e\Phi} = 2\pi r\frac{W}{2\pi r^2 eB/\Delta t} = \frac{W\Delta t}{Ber},$$

on using the result of the previous problem.