JEE 2002 - SOLUTIONS - MATHEMATICS

(INDIANET GROUP)

Solution1: a, A_1, A_2, b are in arithmetic progression

$\Rightarrow A_1, A_2$ are two arithmetic means of a, b

$$\Rightarrow A_1 = a + \frac{b-a}{3} = \frac{2a+b}{3} \quad \text{(I)}$$

$$A_2 = a + \frac{2(b-a)}{3} = \frac{a+2b}{3} \quad \text{(II)}$$

a, G_1, G_2, b are in geometric progression

Let r be the common ratio.

$G_1 = ar,\ G_2 = ar^2,\ b = ar^3$

$\Rightarrow r = (b/a)^{1/3}$ (III)

$$\Rightarrow G_1 = a\left(\frac{b}{a}\right)^{1/3} = b^{1/3}a^{2/3} \quad \text{(IV)}$$

a, H_1, H_2, b are in the harmonic progression

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in AP

Let d' be the common difference.

$$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + d',\ \frac{1}{H_2} = \frac{1}{a} + 2d',\ \frac{1}{b} = \frac{1}{a} + 3d'$$

$$\Rightarrow d' = \frac{a-b}{3ab}$$

$$\Rightarrow \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{3ab} = \frac{3b+a-b}{3ab} = \frac{a+2b}{3ab}$$

$$H_1 = \frac{3ab}{a+2b} \quad \text{(V)}$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{3ab} = \frac{3b+2a-2b}{3ab} = \frac{2a+b}{3ab}$$

$$H_2 = \frac{3ab}{2a+b} \quad \text{(VI)}$$

$$\therefore \frac{G_1G_2}{H_1H_2} = \frac{(b^{1/3}a^{2/3})(a^{1/3}b^{2/3})}{\left(\frac{3ab}{a+2b}\right)\left(\frac{3ab}{2a+b}\right)}$$

$$= \frac{ab}{9a^2b^2}(a+2b)(2a+b)$$

$$= \frac{(a+2b)(2a+b)}{9ab}$$

$$\frac{A_1+A_2}{H_1+H_2} = \frac{\left(\frac{2a+b}{3}+\frac{a+2b}{3}\right)}{\frac{3ab}{a+2b}+\frac{3ab}{2a+b}}$$

$$= \frac{\frac{3(a+b)}{3}}{3ab\left(\frac{3(a+b)}{(a+2b)(2a+b)}\right)}$$

$$= \frac{(a+2b)(2a+b)}{9ab}$$

$$\Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}$$

Solution 2:
$P(n)$: $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$
LHS of $P(1)$: $(25)^2 - 24 + 5735$
$= (625 + 5735) - 24$
$= 6360 - 24$
$= 24(265 - 1)$
$= 24 \times 264$
$= 24 \times 24 \times 11$ is divisible by $(24)^2$
Hence, $P(1)$ is true
Let us assume that $P(k)$ is true
$\Rightarrow (25)^{k+1} - 24k + 5735$ divisible by $(24)^2$
Now, we have to prove that $P(k+1)$ is true.
i.e. $(25)^{k+2} - 24(k+1) + 5735$ is divisible by $(24)^2$ if $P(k)$ is true.
$(25)^{k+2} - 24(k+1) + 5735$
$= (25^{k+1}) . 25 + 25(-24k + 5735) - 25(5735 - 24k) - 24(k+1) + 5735$
$= 25[P(k)] - 24(5735) + 24 \times 25k - 24k - 24$
$= 25P(k) - 24[5735 - 24k + 1]$
$= 25P(k) - 24[5736 - 24k]$
$= 25P(k) - (24)^2[239 - k]$
$\Rightarrow P(k+1)$ is true.
Hence, proved

Solution 3: Let $E = \cos \tan^{-1} \sin \cot^{-1} x$
Let $\cot^{-1} x = \theta$
$\therefore \quad x = \cot\theta \qquad (1)$

$\Rightarrow E = \cos \tan^{-1} \sin\theta$

$x = \cot\theta$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}} \quad (2)$$

$$\Rightarrow E = \cos\tan^{-1}(\sin\theta)$$

$$= \cos\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \quad (3)$$

$$\text{Let} \tan^{-1}\frac{1}{\sqrt{1+x^2}} = y$$

$$\frac{1}{\sqrt{1+x^2}} = \tan y \quad (4)$$

To evaluate $E = \cos y$:

We have $\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$

$$\Rightarrow \cos y = \frac{1}{\sqrt{1+\tan^2 y}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{1}{\sqrt{1+x^2}}\right)^2}} \text{ (from equation (4))}$$

$$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}}$$

$$= \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$$

$$\Rightarrow E = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$$

Hence proved.

Solution 4.

Total coins = N
Number of fair coins = m
Therefore, number of biased coins = $N - m$

Case I:
Let coin drawn be fair:
Let us calculate the probability $P(A)$ of getting a head first and then a tail.
$P(A) = p(H)\,p(T)$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \quad \left[\begin{array}{l} p(H) = \text{probability of getting head from fair coin} = \frac{1}{2} \\ p(T) = \text{probability of getting tail from fair coin} = \frac{1}{2} \end{array}\right]$$

Case II:
Let the coin drawn be biased:
Let us calculate the probability $P(B)$ of getting a head first and then a tail.
$P(B) = p'(H)p'(T)$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$$

[$p'(H)$ = probability of getting a head from the biased coin.
$p'(T)$ = probability of getting a tail from the biased coin
$p'(H)$ = 2/3 (given)
$p'(T)$ = 1-$p'(H)$ = 1-2/3 = 1/3]
Let us define
$P'(A) = P(A) \times$ probability of drawing a fair coin

$$= \left(1/4\right)\left(\frac{m}{n}\right) \qquad \text{(ii)}$$

and $P'(B) = P(B) \times$ probability of drawing a biased coin.

$$= \frac{2}{9}\left(\frac{N-m}{N}\right) \qquad \text{(iii)}$$

Then from Bayes Theorem, we get,

Probability (drawing a fair coin) $= \dfrac{p'(A)}{p'(A)+p'(B)}$ (i)

From equation (i), (ii), (iii) probability (of drawing a fair coin)

$$= \frac{\frac{1}{4}\frac{m}{n}}{\frac{1}{4}\frac{m}{N}+\frac{2}{9}\frac{N-m}{N}}$$

$$= \frac{\frac{m}{4}}{\frac{m}{4}+\frac{2}{9}(N-m)}$$

$$= \frac{9m}{9m+8(N-m)}$$

$$= \frac{9m}{8N+m}$$

Solution 5:

$$Z^{p+q} - Z^p - Z^q + 1 = 0$$

$\Rightarrow (Z^p - 1)(Z^q - 1) = 0$

$\therefore$ Either α is a p^{th} root of unity or q^{th} root of unity.

Using the properties of n^{th} root of unity:
either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$
or $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$

Suppose both the equations hold simultaneously. Without loss of generalisation let $p > q$.

$\therefore 1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$

$\Rightarrow 1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} + \alpha^q + \alpha^{q+1} + \ldots + \alpha^{p-1} = 0$

$\Rightarrow 0 + \alpha^q + \alpha^{q+1} + \ldots + \alpha^{p-1} = 0$

$\Rightarrow \alpha^q \left[1 + \alpha + \ldots + \alpha^{p-q-1}\right] = 0$

Now, $\alpha^q = 1$

$\therefore$ the equation implies that

$1 + \alpha + \ldots + \alpha^{p-q-1} = 0$

Hence α should be the $(p - q)^{th}$ root of unity i.e., $\alpha^{p-1} = 1$

$\Rightarrow$ $p - q$ is a multiple of q ($\because$ q is prime)

i.e., $p - q = nq$

$\Rightarrow$ $p = (n + 1)q$

$\Rightarrow$ p is not prime which is a contradiction.

Hence proved.

Solution 6:

Let the equation of L be:

$y = mx$ (i) ($\because$ it passes through the origin)

Let us find the point of intersection of (i) and $x + y = 1$.

Substituting $y = mx$ in $x + y = 1$,

we get $x = \dfrac{1}{m + 1}$

and $y = \dfrac{m}{m + 1}$

Hence the coordinates of P are $\left(\dfrac{1}{m + 1}, \dfrac{m}{m + 1}\right)$

Similarly let us find the point of intersection of (i) with $x + y = 3$.

Substituting $y = mx$ in $x + y = 3$ we get

$x = \dfrac{3}{m + 1}$

$y = \dfrac{3m}{m + 1}$

Hence, the coordinates of Q are $\left(\dfrac{3}{m + 1}, \dfrac{3m}{m + 1}\right)$

Slope of $L_1 = 2$,

since it is parallel to $2x - y = 5$.

Slope of $L_2 = -3$,

since it is parallel to $3x + y = 5$.

$\therefore$ Equation of L_1: $\left(y - \dfrac{m}{m + 1}\right) = 2\left(x - \dfrac{1}{m + 1}\right)$ (i)

$\therefore$ Equation of L_2: $\left(y - \dfrac{3m}{m + 1}\right) = -3\left(x - \dfrac{3}{m + 1}\right)$ (ii)

Subtracting (ii) from (i), we get

$\dfrac{2m}{m + 1} = 5x - \dfrac{11}{m + 1}$

$\Rightarrow x = \dfrac{11 + 2m}{5(m + 1)}$.

$\Rightarrow$ $5mx + 5x = 11 + 2m$

$\Rightarrow$ $m(5x - 2) = 11 - 5x$

$$\Rightarrow m = \frac{11 - 5x}{5x - 2} \quad \text{(iii)}$$

Substituting this in (i) to eliminate m we get

$$y = 2x + \frac{15 - 15x}{9}$$

$\Rightarrow$ $3y = x + 5$

which is the equation of a straight line.
Hence proved.

Solution 7:
Let the equation of the straight line be:
$(y - 2) = m(x - 8)$

Substituting $x = 0$, we get, $x = \frac{(8m - 2)}{m}$

$y = 2 - 8m$
Therefore, Q ≡ (0, 2- 8m)
Substituting $y = 0$, we get,

Therefore, $P \equiv \left(\frac{8m - 2}{m}, 0\right)$

$$OP = \frac{8m - 2}{m}$$

$OQ = 2 - 8m$
$L = OP + OQ$

$$= \frac{8m - 2}{m} + 2 - 8m$$

$$= \frac{-8m^2 + 10m - 2}{m}$$

Differentiating with respect to m and setting it equal to zero for extrema:

$$\frac{dL}{dm} = \frac{m(-16m + 10) - (-8m^2 + 10m - 2)}{m^2} = 0$$

$\Rightarrow -8m^2 + 2 = 0$

$$\Rightarrow m^2 = \frac{1}{4}$$

$$\Rightarrow m = \pm\frac{1}{2}$$

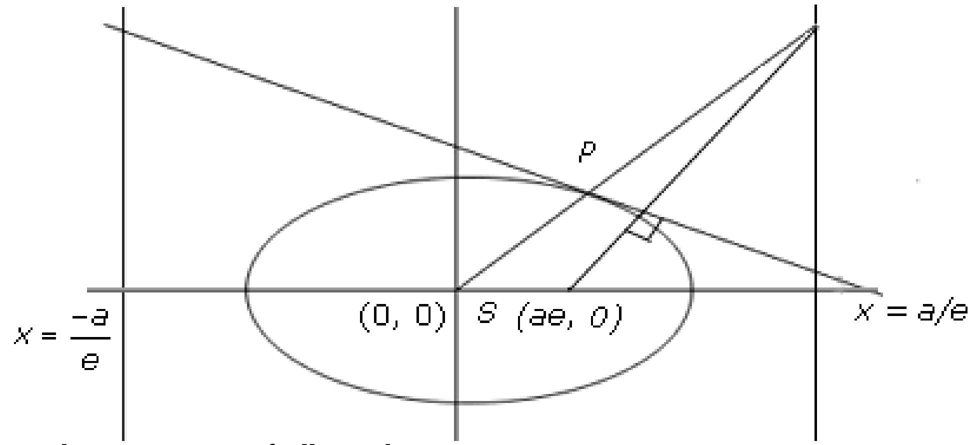
But m is given to be negative.

Therefore, $m = -\frac{1}{2}$

This m corresponds to the absolute minima (as the maxima is unbounded)
Value of absolute minima of $OP + OQ$

$$= \frac{-2 - 5 - 2}{-\frac{1}{2}} = 18$$

Solution 8:



Let the equation of ellipse be :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let a point P on the ellipse be (a cosθ, b sinθ)
Then the equation of tangent at P is :

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\Rightarrow m = \frac{-b}{a\tan\theta}$$

Equation of line L_1 joining the centre of the ellipse (0, 0) to the point P (a cosθ, b sinθ) is

$$y = \frac{b}{a}\tan\theta \cdot x \qquad (1)$$

Slope of the line L_2 perpendicular to tangent and passing through the focus S(ae, 0) is

$$m_2 = \frac{-1}{m} = \frac{a\tan\theta}{b}$$

So equation of line L_2 is

$$y - 0 = \frac{a\tan\theta}{b}(x - ae)$$

$$\Rightarrow y = \frac{a\tan\theta}{b}(x - ae) \qquad (2)$$

Solving (1) and (2) for x, we get

$$\frac{b}{a}\tan\theta . x = \frac{a}{b}\tan\theta\,(x - ae)$$

$$\Rightarrow \frac{b^2 - a^2}{ab}x = \frac{-a^2 e}{b}$$

$$\Rightarrow \frac{a^2 - b^2}{a^2}x = ae$$

But $\frac{a^2 - b^2}{a^2} = e^2$

The equation is $e^2 x = ae$

$\Rightarrow x = a/e$ which is the equation of the corresponding directrix. Hence proved

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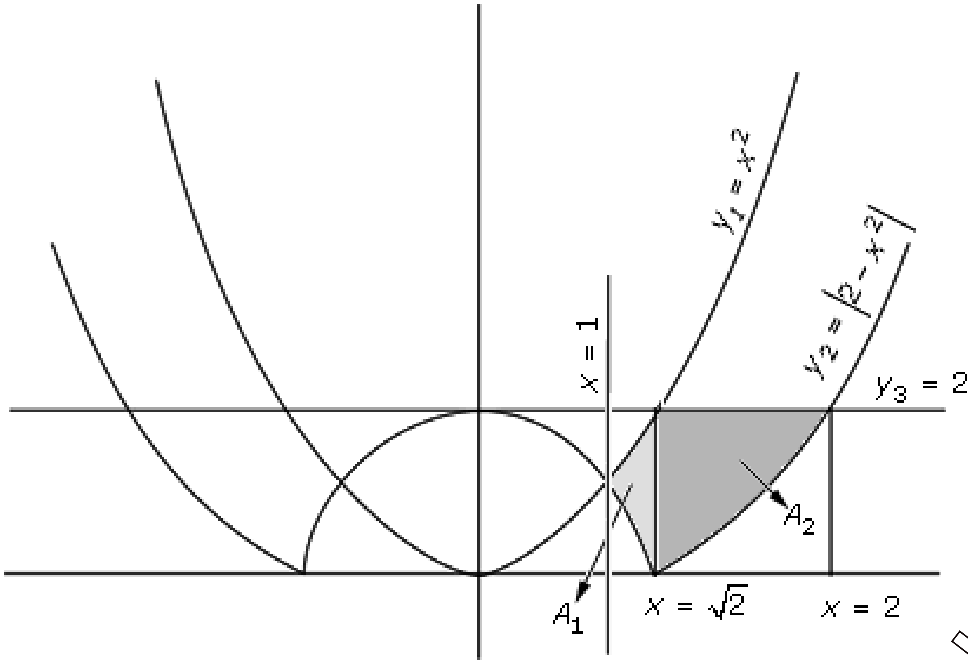
Solution 9:



Shaded area indicates the area to be calculated

$$A_1 = \int_{x=1}^{x=\sqrt{2}} [y_1 - y_2]$$

$y_2 = 2 - x^2$ for $-\sqrt{2} < x < \sqrt{2}$

So,

$$A_1 = \int_1^{\sqrt{2}} \left[x^2 - \left(2 - x^2\right)\right] dx$$

$$= \int_1^{\sqrt{2}} \left(2x^2 - 2\right) dx$$

$$= 2\int_1^{\sqrt{2}} \left(x^2 - 1\right) dx$$

$$= 2\left[\left[\frac{x^3}{3}\right]_1^{\sqrt{2}} - \left(\sqrt{2} - 1\right)\right]$$

$$= 2\left[\frac{1}{3}\left[2\sqrt{2} - 1\right] - \left[\sqrt{2} - 1\right]\right]$$

$= \frac{4}{3} + \frac{2\sqrt{2}}{3}$

$A_2 = \int_{x=\sqrt{2}}^{2} [y_3 - y_2]\, dx$

$= \int_{\sqrt{2}}^{2} [2 - (x^2 - 2)]\, dx$

$= \int_{\sqrt{2}}^{2} [4 - x^2]\, dx$

$= 4[2 - \sqrt{2}] - \frac{x^3}{3}\Big|_{\sqrt{2}}^{2}$

$= 8 - 4\sqrt{2} - \frac{8}{3} + \frac{2\sqrt{2}}{3}$

$A_2 = \frac{16}{3} - \frac{10\sqrt{2}}{3}$

$A = A_1 + A_2$

$= \frac{2\sqrt{2}}{3} + \frac{4}{3} + \frac{16}{3} - \frac{10\sqrt{2}}{3}$

$= \frac{20}{3} - \frac{8\sqrt{2}}{3}$

$= \frac{20}{3} - \frac{8}{3}\sqrt{2}$

Solution 10:

Given, $\sum_{r=1}^{3} (a_r + b_r + c_r) = 3L$

$\Rightarrow (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) + (a_3 + b_3 + c_3) = 3L$

$\Rightarrow (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3) = 3L$

Now,

$\frac{X + Y + Z}{3} \geq (XYZ)^{\frac{1}{3}}$

AM ≥ GM

$\Rightarrow \frac{(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3)}{3} = L$

If $X = a_1 + a_2 + a_3$

$Y = b_1 + b_2 + b_3$

$Z = c_1 + c_2 + c_3$

then, $\frac{X + Y + Z}{3} \geq (XYZ)^{\frac{1}{3}}$

$\Rightarrow L \geq (XYZ)^{1/3}$

$\Rightarrow L^3 \geq XYZ$

$\Rightarrow L^3 \geq (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)$

Also, $A + B + C \geq \sqrt{A^2 + B^2 + C^2}$ [since $(A + B + C)^2 - (A^2 + B^2 + C^2) = 2(AB + BC + CA) \geq 0$]

$$\Rightarrow L^3 \geq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{c_1^2 + c_2^2 + c_3^2} \qquad (1)$$

Volume of parallelopiped = $[\bar{a}\ \bar{b}\ \bar{c}]$

$= \left[\bar{a} \cdot \left(\bar{b} \times \bar{c}\right)\right] \leq |\bar{a}||\bar{b}||\bar{c}|$ [equality holds for a cuboid]

$$\Rightarrow V \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2} \cdot \sqrt{c_1^2 + c_2^2 + c_3^2} \qquad (2)$$

From (1) and (2)

V $\leq L^3$

Solution 11:

$$I = \int \left(x^{3m} + x^{2m} + x^m\right)\left(2x^{2m} + 3x^m + 6\right)^{1/m} dx,\ x > 0$$

Substitute $x^m = y$

Taking log,

$m \log x = \log y$

Differentiating,

$$m \frac{1}{x} dx = \frac{1}{y}\ dy$$

$$\Rightarrow dx = \frac{x}{my}\ dy$$

$$= \frac{y^{1/m}}{my}\ dy$$

$$\Rightarrow I = \int (y^3 + y^2 + y)(2y^2 + 3y + 6)^{1/m} \frac{y^{1/m}}{my} dy$$

$$= \frac{1}{m}\int\left[\frac{y^3 + y^2 + y}{y}\right]\left[(2y^2 + 3y + 6)^{1/m}\right] y^{\frac{1}{m}}\ dy$$

$$= \frac{1}{m}\int (y^2 + y + 1)(2y^3 + 3y^2 + 6y)^{1/m}\ dy$$

Now put $2y^3 + 3y^2 + 6y = t^m$

Differentiating both sides,

$$(6y^2 + 6y + 6)dy = mt^{m-1}\ dt$$

$$\therefore (y^2 + y + 1)\ dy = \frac{mt^{m-1}}{6}\ dt$$

$$\therefore I = \frac{1}{m}\int \frac{m}{6} t^{m-1} (t^m)^{1/m}\ dt$$

$$= \frac{1}{6}\int t^{m-1}.t \;\; dt$$

$$= \frac{1}{6}\int t^{m}dt$$

$$= \frac{1}{6}\frac{t^{m+1}}{(m+1)} + c$$

$$= \frac{(2y^3 + 3y^2 + 6y)^{\frac{m+1}{m}}}{6(m+1)} + c$$

$$\therefore \; I = \frac{1}{6(m+1)}\left[2 \; x^{3m} + 3x^{2m} + 6x^{m}\right]^{\left(\frac{m+1}{m}\right)} + c$$

Solution 12:

$$f(x) = \begin{cases} x + a, \; x < 0 \\ |x - 1|, \; x \geq 0 \end{cases}$$

$$= \begin{cases} x + a, \; x < 0 \\ x - 1, \; x \geq 1 \\ 1 - x, \; 0 \leq x < 1 \end{cases}$$

$$g(x) = \begin{cases} x + 1 \qquad , \quad x < 0 \\ (x-1)^2 + b, \text{ if } x \geq 0 \end{cases}$$

$$gof(x) = g(f(x)) = \begin{cases} f(x) + 1 \qquad , \; f(x) < 0 \\ [f(x) - 1]^2 + b, \text{ if } \; f(x) \geq 0 \end{cases}$$

Now, $f(x) < 0$

$$\Rightarrow \begin{cases} x + a < 0 \quad \text{when} \quad x < 0 \\ x - 1 < 0 \quad \text{when} \quad x \geq 1 \\ 1 - x < 0 \quad \text{when} \quad 0 \leq x < 1 \end{cases}$$

$$\Rightarrow \begin{cases} x < -a \quad \text{when} \;\; x < 0 \\ x < 1 \quad \text{when} \quad x \geq 1 \\ x > 1 \quad \text{when} \quad 0 \leq x < 1 \end{cases}$$

The last two cases are not possible

So, $f(x) < 0$ if $x < -a$

a is positive

$f(x) < 0$ if $x < -a$

$\Rightarrow f(x) \geq 0$ for $x > -a$

Now,

$$gof(x) = \begin{cases} f(x) + 1, \; x < -a, \text{ where } f(x) = x + a \\ [f(x) - 1]^2 + b \;, \; x \geq -a \end{cases}$$

$$gof(x) = \begin{cases} x + a + 1 & , \quad x < -a \\ (x + a - 1)^2 + b, & -a \le x < 0 \end{cases}$$

$$= (1 - x - 1)^2 + b,\ 0 \le x < 1$$

$$= x^2 + b,\ 0 \le x < 1$$

$gof(x) = (x - 1 - 1)^2 + b,\ x \ge 1$

$= (x - 2)^2 + b,\ x \ge 1$

Since, gof is continuous for all real x, therefore, $(a - 1)^2 + b = b$

$\Rightarrow a = 1$, b is any real number.

For $a = 1$, $b \in R$, gof is continuous

$$\Rightarrow gof(x) = \begin{cases} x + 2 & , x < -a \\ x^2 + b & , -a \le x < 1 \\ (x - 2)^2 + b, & x \ge 1 \end{cases}$$

So, gof is differentiable at $x = 0$ if $a = 1$, $b \in R$.